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FINITE GRAPH CERTIFICATES FOR COMPOSITE TERMS IN RATIONAL FLOOR SEQUENCES

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ABSTRACT. For every real number $\xi > 0$, we give computer-assisted proofs that $\lfloor \xi(7/5)^n \rfloor$ is divisible by at least one of $2, 3, 5, 11, 13$ for infinitely many positive integers n, and that $\lfloor \xi(5/2)^n \rfloor$ is divisible by at least one of $2, 3, 7, 11, 13, 17, 19, 23, 29, 31$ for infinitely many positive integers n, and that $\lfloor \xi(5/3)^n \rfloor$ is divisible by at least one of $2, 3, 7, 11, 13, 17, 19, 23, 29, 31, 37$ for infinitely many positive integers n. Consequently all three sequences contain infinitely many composite terms. The proofs represent an orbit avoiding the given primes by a walk in a finite labelled graph whose vertices record residues and fractional cells and whose edges carry signed carries. Components whose output depends only on a cyclic phase are excluded by an elementary nonperiodicity lemma. In the word-labelled construction the edges carry finite carry words; deterministic paths are contracted exactly, and each new prime is imposed at every intermediate time of a word. For $5/3$ chains of in-degree one are also contracted exactly, which keeps the largest graph at about $2.1 \cdot 10^7$ vertices instead of about $3.1 \cdot 10^8$. For the ratio $7/5$ we also prove an explicit waiting-time bound: if $P = \lfloor \xi \rfloor > 13$ then some term with index at most $428 + 12L(P)$, $L(P) = \min\{h \geq 0 : 5^h \geq P + 2\}$, is divisible by one of the five primes. We then determine the reach of this certificate method: contraction, prime powers, and the primes dividing $2ab$ do not enlarge the class of finitely certifiable bases; the method cannot terminate when $a \geq 2 \operatorname{rad} M$; and the product R of the auxiliary primes not dividing $2ab$ must satisfy $a \leq 2R\, J(N_0)$ for the Jacobsthal function J, whence $R \geq a^{1-o(1)}$ by an elementary bound. The base $5/3$ belongs to the finitely certifiable class, and no certificate of this kind for $5/3$ uses only primes up to 23. Every adopted finite computation, including the certificate computation for $5/3$, was performed by two implementations with disjoint scientific cores, which agree element by element on all compared finite data; the $5/3$ obstruction witness is checked by one direct program. Lean proofs cover all three divisibility theorems and their compositeness corollaries: the one-step proof for $7/5$ is kernel-checked, and the word-labelled proofs, including the one for $5/3$, combine kernel-checked soundness theorems with `native_decide` finite evaluations, which add trust in the compiler and native evaluation.

1. INTRODUCTION

The arithmetic of the integer parts of rational powers presents difficulties even for simple nonintegral bases. A useful way to prove that a sequence contains infinitely many composite terms is to exhibit a fixed finite set of primes such that infinitely many terms have a divisor in that set. Once the terms exceed all those primes, divisibility implies compositeness. Following Dubickas Dub06, §2, such a set is called *unavoidable* with respect to a base b if it has this property for $\lfloor \xi b^n \rfloor$ and every $\xi > 0$. Among integer bases, such sets exist for $b \in \{2, 3, 4, 6\}$; none exists for $b = 5$ (DN, as recalled in Dub06, §2) or for any $b \geq 3$ with $b - 1$ not squarefree (Dub06, Theorem 4), and none is expected for the other integer bases (Dub06, Conjecture 3). In this terminology, the results recalled next give unavoidable sets for the rational bases $3/2$, $4/3$, and $5/4$, and Theorem 1 gives unavoidable sets for $7/5$, $5/2$, and $5/3$.

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In 1967, Forman and Shapiro FS proved that the integer parts of the powers of $3/2$ and $4/3$ contain infinitely many composite numbers; see the historical account in DN, pp.635–636. As noted there, their proofs also apply after multiplication by any fixed positive coefficient ξ. Dubickas and Novikas treated the additional base $5/4$ and stated explicit divisibility results for all three bases: for every $\xi > 0$, infinitely many terms of each of

$$\lfloor \xi(3/2)^n \rfloor, \qquad \lfloor \xi(4/3)^n \rfloor, \qquad \lfloor \xi(5/4)^n \rfloor$$

have a divisor in the respective prime sets

$$\{2, 5, 7, 11\}, \qquad \{2, 3, 5\}, \qquad \{2, 3, 7, 11, 13\};$$

see DN, Theorem 1, p.636 and N, Theorem 1.1, p.24.

Dubickas and Novikas also proved the corresponding assertion with primes $\{2, 3, 5, 11\}$ for the *nearest integers* $\lfloor \xi(7/5)^n + 1/2 \rfloor$: this is the third assertion of DN, Theorem 4, p.637, also stated as N, Theorem 1.5(iii), p.26. For the ratio $5/2$ they proved that, for every $\xi > 0$, the *shifted* sequence $\lfloor \xi(5/2)^n \rfloor - 1$ has infinitely many terms divisible by 2, 3 or 5; see DN, Theorem 3, p.637 and N, Theorem 1.4, p.25; inside the proof, the same argument is noted to apply to $\lfloor \xi(5/2)^n \rfloor - 1 + 30k$ for every fixed integer k (DN, p.646, N, p.26). For the ratio $5/3$, Novikas N, Theorem 1.5(ii), p.26 states that, for every $\xi > 0$, the nearest integers $\lfloor \xi(5/3)^n + 1/2 \rfloor$ include infinitely many terms divisible by 2 or 3. The rounding and shift conventions matter here. The present results concern the unshifted integer parts. For $5/2$ the shift changes the alphabet of admissible carries, and our certificate uses a different prime set. For the unshifted integer parts with ratio $5/3$, avoiding 2 and 3 leaves the carry alphabet $\{-2, 2, 4\}$ (Lemma 74), and our certificate uses the further primes $7, 11, \ldots, 37$; the nearest-integer result is not used.

Theorem 1 (Main Theorem)**.** *For every $\xi \in \mathbb{R}$ with $\xi > 0$ and every $N \in \mathbb{N}$:*

(i) *there is an integer $n \geq \max(N, 1)$ such that*

$$\gcd(\lfloor \xi(7/5)^n \rfloor, 4290) > 1, \qquad 4290 = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 13; \tag{1}$$

(ii) *there is an integer $n \geq \max(N, 1)$ such that*

$$\begin{gathered} \gcd(\lfloor \xi(5/2)^n \rfloor, 40112098026) > 1, \\ 40112098026 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31; \end{gathered} \tag{2}$$

(iii) *there is an integer $n \geq \max(N, 1)$ such that*

$$\begin{gathered} \gcd(\lfloor \xi(5/3)^n \rfloor, 1484147626962) > 1, \\ 1484147626962 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37. \end{gathered} \tag{2a}$$

The indices in (i), (ii), and (iii) are quantified separately; no common index is asserted. The moduli in (2) and (2a) do not contain the prime 5.

Here and below $\mathbb{N} = \{0, 1, 2, \ldots\}$, and for an integer q we use $\gcd(q, M) = \gcd(|q|, M)$. The formulation with arbitrary N states precisely that the qualifying positive indices form an infinite set. A positive integer is *composite* if it equals uv with integers $u, v > 1$.

Corollary 2. *For every $\xi > 0$, each of the sequences*

$$\left(\lfloor \xi(7/5)^n \rfloor\right)_{n \geq 1}, \qquad \left(\lfloor \xi(5/2)^n \rfloor\right)_{n \geq 1}, \qquad \text{and} \qquad \left(\lfloor \xi(5/3)^n \rfloor\right)_{n \geq 1}$$

contains infinitely many composite terms.

For the ratio $7/5$ the certificate also yields an explicit waiting time. Write

$$L(P) = \min\{h \geq 0 : 5^h \geq P + 2\}.$$

Theorem 3 (Quantitative Theorem). *Let $\xi > 0$ with $P = \lfloor \xi \rfloor > 13$. Then there is an integer n with*

$$0 \le n \le 428 + 12L(P) \qquad \text{and} \qquad \gcd(\lfloor \xi(7/5)^n \rfloor, 4290) > 1, \tag{3}$$

and this term is composite. Consequently, for every $N \in \mathbb{N}$ with $Q_N = \lfloor \xi(7/5)^N \rfloor > 13$, some $\lfloor \xi(7/5)^n \rfloor$ with

$$N \le n \le N + 428 + 12L(Q_N)$$

is composite.

The related study of fractional parts includes Mahler's question M68, p.313 whether some $\xi > 0$ satisfies

$$0 \le \{\xi(3/2)^n\} < 1/2 \qquad \text{for all } n \ge 0.$$

Flatto, Lagarias, and Pollington FLP, Theorem 1.4, p.129 proved that, for coprime integers $p > q \ge 2$ and every $\xi > 0$,

$$\limsup_{n\to\infty} \{\xi(p/q)^n\} - \liminf_{n\to\infty} \{\xi(p/q)^n\} \ge 1/p.$$

Dubickas Dub08 obtained further results on whether all these fractional parts can lie in prescribed intervals or unions of intervals. These confinement questions provide a background for the use of fractional cells here. The range bound alone does not yield the infinitely many even terms that would suffice for compositeness, as explained in DN, p.636.

For other real bases, Alkauskas and Dubickas AD04 constructed Pisot numbers whose positive powers all have composite integer parts, as well as a transcendental number whose integer-part sequence meets every integer arithmetic progression infinitely often. Those results construct particular bases; here the base is a specified rational number and the coefficient ξ ranges over all positive reals.

Method. The proofs combine an existing nonperiodicity obstruction with explicit finite certificates. Nonperiodicity of the carry sequence $c_n = bQ_{n+1} - aQ_n$ is already contained in DN, Lemma 2, p.638 and N, Lemma 1.7, pp.28–29, and the use of restrictions on operation words to force periodicity also has precedents in DN, pp.638–639 and 647. We include an integral proof of the obstruction and make the graph construction, the all-edge phase test, and the tail-preservation arguments explicit. The one-step construction (Section 3) records one carry per edge and refines the fractional cells; it proves (1). The word-labelled construction (Section 4) labels edges by finite carry words, contracts deterministic paths without changing their output, and lifts the graph by one new prime at a time while checking the residues at every intermediate time of each word; it proves (2) and gives a second proof of (1). With the additional exact contraction of in-degree-one chains (Appendix F) it proves (2a). A finite rank certificate on the complete one-step graph, combined with a finite version of the nonperiodicity lemma, gives Theorem 3 (Section 6).

Akiyama, Frougny, and Sakarovitch AFS08 developed rational-base representations with finite transducers for arithmetic. Their nonperiodicity results use divisibility by growing powers of the denominator (AFS08, Lemma 8, p.65 and Proposition 26, p.75). Their language of integer representations is nonregular (AFS08, Corollary 7, p.65). Our finite graphs instead record necessary residue and fractional-cell constraints on a prime-avoiding orbit: every such orbit gives a walk, while a walk need not represent an orbit. This distinction allows finite certificates without requiring a finite automaton for the full representation language.

Stephan's preprint S26, dated August 15, 2026, proves the case $b = 7$ of Dub06, Conjecture 2: for every $\xi > 0$, the sequence $\lfloor \xi 7^n \rfloor$ contains infinitely many composite terms;

by the same certificate, there is no infinite right truncatable prime in base 7. Its method is closely related to the one used here. The vertices of its graph are the residues modulo $M = \prod_{p \le 31} p$ that are coprime to M, with an edge for each appended digit. The certificate states that, after the states not lying on bi-infinite paths are discarded, no strongly connected component contains two distinct cycles. Every infinite path is then ultimately periodic, and an ultimately periodic digit word gives infinitely many composite terms by a return theorem of Dubickas Dub09, Theorem 4, recalled after Theorem 46 below. The certificate is reached through a ladder of moduli that adjoins one prime at a time, using projections and rank functions, and it is verified in Lean, where the finite computation is evaluated by `native_decide` (S26, §§4–6 and Appendix A).

The present work shares several ingredients with S26: finite residue graphs in which every hypothetical prime-avoiding orbit gives an infinite walk, the lifting of a certified graph by one new prime at a time (Section 4.4), rank functions that a checker verifies in place of a formal proof of graph algorithms (Sections 6.2 and 9.4), and a Lean verification whose word-labelled finite evaluations are likewise accepted through `native_decide` (Section 9.4). The constructions differ in three respects. First, the graph of S26 consists of residues alone. For a nonintegral base not every carry word is realized by an orbit; the states here also record fractional cells, and the graphs are outer approximations: every orbit gives a walk, but a walk need not represent an orbit. Second, both certificates single out components in which every infinite walk has a periodic label sequence (simple cycles in S26; components passing the phase criterion of Lemma 7 here), but they treat them in opposite ways. In base 7, such components carry actual prime-avoiding orbits: the core of the graph of S26 at the modulus M still has 371702 states, and DJ22, Theorem 1.2, p.167 implies that every finite set of primes can be avoided by all terms for a suitable positive coefficient, so no unavoidable set exists; S26 keeps these components and applies the return theorem to their periodic words. For a nonintegral base, Lemma 4 shows that no orbit remains in such a component, so it is deleted; the certificates here reduce the graph to the empty graph and thereby yield the unavoidable sets of Theorem 1. The result of DJ22 does not apply to 7/5, 5/2, or 5/3. Third, the word-labelled contraction with prime conditions imposed at every intermediate time, the waiting-time bound of Theorem 3, and the normal-form and obstruction theorems of Sections 7 and 8 are not part of S26. The computation of S26 is a methodological comparison, not a dependency of the proofs given here. The specific contributions here are the finite certificates for the unshifted 7/5, 5/2, and 5/3 sequences with their exact verification, the word-labelled contraction with prime conditions imposed at every intermediate time, the waiting-time bound, and the structural results described next.

Scope of the method. Sections 7 and 8 study what this certificate method can and cannot do. With the operations fixed precisely (initial complete graph, phase removal, exact contraction, cell refinement, modulus enlargement, carry restriction), we prove a normal-form theorem: a finite certificate exists for some choice of parameters if and only if, for some cell count K and some finite set S of primes not dividing $2ab$, every cyclic strongly connected component of one explicit graph passes the phase test. Contraction does not enlarge the certifiable class, prime powers in the modulus can be dropped, and the primes dividing $2ab$ reduce to a restriction of the carry alphabet; the normal form changes the prime set of the conclusion, which then contains all primes dividing $2ab$. Two obstruction theorems then show that the method does not terminate when $a \ge 2 \operatorname{rad} M$, and that it does not terminate when a certain two-letter carry word admits unit residues modulo M. The second obstruction gives the necessary condition $a \le 2R\, J(N_0)$, where R is the product

of the primes of M not dividing $2ab$, N_0 is the product of the odd primes dividing ab, and J is the Jacobsthal function; an elementary bound $J(N) \le 3^{\omega(N)}$ gives $R \ge a^{1-o(1)}$. These obstruction theorems are statements about the non-termination of a fixed procedure. They are not counterexamples to any arithmetic statement, and they do not assert that any base fails to have infinitely many composite terms. Appendix F shows that 5/3 belongs to the certifiable class and that every certificate of this kind for 5/3 uses a prime at least 29.

Verification. Every adopted finite computation was carried out by two implementations whose scientific cores (edge enumeration, component decomposition, certification, contraction, lifting) were written separately from the mathematical specification; their complete graph data were compared element by element. For 5/3 the two implementations are a C++ producer with an embedded checker and a separately written extension of the second word-labelled implementation. Section 9 describes their provenance and the comparison. Lean proofs establish all three parts of Theorem 1 and Corollary 2. The one-step proof for 7/5 uses a kernel-checked finite certificate; the word-labelled proofs use kernel-checked soundness theorems and `native_decide` finite evaluations. Section 9.4 gives the precise formalization coverage and the additional compiler and native-evaluation trust of the latter proofs.

Section 2 proves the nonperiodicity lemma. Section 3 presents the one-step outer graphs and finite certification, and Section 4 the word-labelled certificates. Section 5 gives the finite calculations for (1) and (2). Section 6 proves Theorem 3. Sections 7 and 8 contain the structural and obstruction results. Section 9 describes the verification and the exact scope of the formalization. Section 10 states further results whose proofs are given in the appendices, and Section 11 lists open problems. Appendix F proves (2a), the 5/3 part of Corollary 2, and the related structural statements for 5/3.

The two finite certification procedures share the same tail-preservation argument, summarized below.

2. Carries and nonperiodicity

Throughout the general argument, let

$$a, b \in \mathbb{N}, \qquad a > b \ge 2, \qquad \gcd(a, b) = 1, \qquad \xi > 0,$$

and put $r = a/b$, with division in $\mathbb{R}$. For $n \ge 0$, define

$$x_n = \xi r^n, \qquad Q_n = \lfloor x_n \rfloor \in \mathbb{Z}, \qquad \theta_n = x_n - Q_n \in [0, 1), \tag{4}$$

and the signed integer carry

$$c_n = bQ_{n+1} - aQ_n = a\theta_n - b\theta_{n+1}. \tag{5}$$

Thus

$$1 - b \le c_n \le a - 1, \qquad bQ_{n+1} = aQ_n + c_n. \tag{6}$$

The strict real bounds $-b < c_n < a$ imply the integer bounds in (6). An alternative indexing writes $e_{n+1} = c_n$; no value of e_0 is needed. Since $x_{n+1} = rx_n > x_n$, the sequence (Q_n) is nondecreasing, and $Q_n \ge Q_0$ for all n.

We shall also use $Q_n \to +\infty$ in the explicit sense

$$\forall T \in \mathbb{Z}\ \exists N \in \mathbb{N}\ \forall n \ge N, \qquad Q_n > T. \tag{7}$$

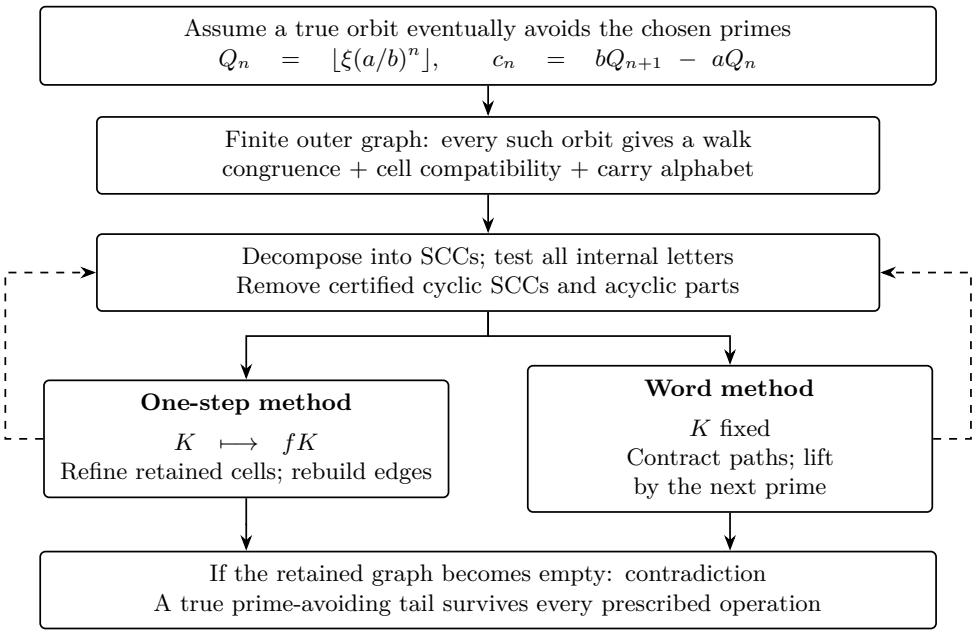


FIGURE 1. Logical flow of the one-step and word methods. Arrows from an orbit to a graph express inclusion in an outer approximation. The two branches use different operations: cell refinement in the one-step method, and contraction followed by prime lifting at fixed cell count in the word method. Dashed arrows indicate iteration.

Indeed, writing $r = 1 + \delta$ with $\delta > 0$, Bernoulli's inequality gives $r^n \geq 1 + n\delta$, and $Q_n > \xi(1 + n\delta) - 1$.

Lemma 4 (nonperiodicity)**.** *The carry sequence (c_n) is not eventually periodic: there do not exist $N \geq 0$ and $L \geq 1$ such that*

$$c_{n+L} = c_n$$

for all $n \geq N$.

Proof. First suppose an integer sequence $(D_k)_{k\geq 0}$ satisfies $bD_{k+1} = aD_k$. Induction gives

$$b^k D_k = a^k D_0.$$

Since a^k and b^k are coprime, $b^k \mid D_0$ for every k. As $b \geq 2$, this forces $D_0 = 0$, without any assumption on its sign.

Now assume that c has period L from N onward. For a fixed $m \geq N$, set

$$D_k = Q_{m+k+L} - Q_{m+k}.$$

Subtracting the two recurrences in (6) and using periodicity gives $bD_{k+1} = aD_k$. The preceding observation yields $Q_{m+L} = Q_m$. This holds for every $m \geq N$, so in particular $Q_{N+kL} = Q_N$ for every k, contrary to (7). □

This proof uses only an integral recurrence and growth. The hypotheses $b \geq 2$ and $\gcd(a, b) = 1$ are essential: integer ratios admit identically zero carries for suitable coefficients. The earlier nonperiodicity lemmas cited above also exclude eventual periodicity, since a tail can be absorbed into the new positive coefficient ξr^N.

We also record the iterated form of (6): with $C_0 = 0$ and

$$C_{i+1} = aC_i + b^i c_{u+i},$$
$$b^i Q_{u+i} = a^i Q_u + C_i \qquad (i \geq 0), \tag{8}$$

by induction, since

$$b^{i+1} Q_{u+i+1} = b^i(aQ_{u+i} + c_{u+i}) = a(a^i Q_u + C_i) + b^i c_{u+i}.$$

3. One-step outer graphs and finite certification

3.1. **The graphs.** Fix integers $M \geq 1$ and $K \geq 1$. The states recording a residue coprime to M and a fractional cell are

$$V(M,K) = \{(q,j) \in \mathbb{Z}^2 : 0 \leq q < M,\ \gcd(q,M) = 1,\ 0 \leq j < K\}. \tag{9}$$

For $M = 1$ the residue coordinate is $q = 0$. For $W \subseteq V(M,K)$, define a directed labelled graph $G_K(W)$ on W. There is an edge $(q,j) \xrightarrow{e} (z,k)$ exactly when the endpoints lie in W and the following signed integer conditions hold:

$$\begin{aligned} 1 - b \leq e \leq a - 1, &\qquad \text{(E1)} \\ bz \equiv aq + e \pmod{M}, &\qquad \text{(E2)} \\ aj - b(k+1) < eK < a(j+1) - bk. &\qquad \text{(E3)} \end{aligned} \tag{10}$$

Edges with the same endpoints and different labels are distinct. We write $G(M,K) = G_K(V(M,K))$ for the *complete* one-step graph.

Define the state of an actual orbit by

$$\sigma_K(n) = (Q_n \bmod M,\ \lfloor K\theta_n \rfloor), \tag{11}$$

where the residue is the representative in $\{0, \ldots, M-1\}$.

Proposition 5 (orbit inclusion)**.** *If $\sigma_K(n), \sigma_K(n+1) \in W$, then*

$$\sigma_K(n) \xrightarrow{c_n} \sigma_K(n+1)$$

is an edge of $G_K(W)$.

Proof. Condition (E1) is (6), and reduction of (5) modulo M gives (E2). Write $j = \lfloor K\theta_n \rfloor$ and $k = \lfloor K\theta_{n+1} \rfloor$. Then

$$j \leq K\theta_n < j+1, \qquad k \leq K\theta_{n+1} < k+1,$$

so

$$aj - b(k+1) < aK\theta_n - bK\theta_{n+1} = Kc_n < a(j+1) - bk.$$

This is (E3). Both inequalities are strict, even though a fractional part may equal zero. □

The proof uses only the real identity (5) and the half-open cells; it does not use the integrality of Q_n. This observation is used in Section 7. The graph is an outer approximation. Each orbit that eventually avoids all prime divisors of M gives a walk in the graph, but no converse realizability assertion about graph walks is used. For real numbers u, v with $j \leq u < j+1$, $k \leq v < k+1$, and $au - bv = eK$, condition (E3) is necessary and sufficient for the existence of such a pair; it expresses the compatibility of two adjacent cells only.

The conditions allow complete integer enumeration. For a fixed j, e, (E3) and $0 \leq k < K$ are equivalent to

$$\max\left(0, \left\lfloor \frac{aj - eK}{b} \right\rfloor\right) \leq k \leq \min\left(K-1, \left\lfloor \frac{a(j+1) - eK - 1}{b} \right\rfloor\right). \tag{12}$$

For example, if $A = aj - eK$, then $A < b(k+1)$ is equivalent to $\lfloor A/b \rfloor \le k$. The other bound follows by replacing the strict integer inequality $bk < B$ with $bk \le B - 1$. All floors in (12) are mathematical floors, including for negative numerators. Truncation toward zero would give an incorrect enumeration. An empty integer interval contributes no edges.

No invertibility of b modulo M is assumed. To see explicitly how all solutions of (E2) can be found, put $g = \gcd(b, M)$ and $t = aq + e$. There are no solutions unless $g \mid t$. If this divisibility holds, solve the reduced congruence

$$(b/g)z_0 \equiv t/g \pmod{M/g}, \qquad 0 \le z_0 < M/g.$$

For $M/g = 1$, use $z_0 = 0$. Otherwise the coefficient b/g is invertible modulo M/g. The full set of representatives is

$$z = z_0 + \ell(M/g), \qquad 0 \le \ell < g. \tag{13}$$

One then retains exactly those satisfying $\gcd(z, M) = 1$ and $(z, k) \in W$. Equivalently, one can enumerate all unit residues z and group them by $bz \bmod M$, as the one-step implementations do. In the case $M = 4290$, $\gcd(5, 4290) = 5$, so division by 5 modulo 4290 is unavailable.

A single compatible cell pair illustrates why both inequalities in (E3) are strict.

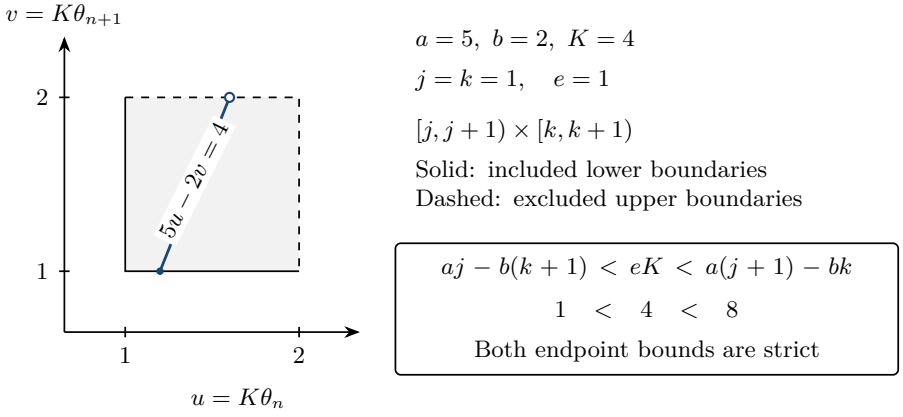


FIGURE 2. Cell compatibility for $a = 5, b = 2, K = 4, j = k = 1, e = 1$. The line meets the half-open square along $u = (4 + 2v)/5$, $1 \le v < 2$. The lower endpoint is included and the upper endpoint is excluded. The cell-pair test is local; it does not assert the existence of a full orbit.

3.2. **Components with periodic output.** A strongly connected component (SCC) is an equivalence class for mutual directed reachability, allowing paths of length zero. We call an SCC *cyclic* if it contains an internal edge. Thus a singleton is cyclic exactly when it has a self-loop.

Lemma 6. *Every infinite walk in a finite directed graph eventually remains in one cyclic SCC.*

Proof. After leaving one SCC for a different one, a walk cannot return: a return would make the two components mutually reachable. There can therefore be only finitely many changes of component. The final component contains the infinite tail and hence an internal edge. □

Let C be a cyclic SCC of $G_K(W)$. Choose a root $\rho \in C$, and let $h(v)$ be the shortest length of a path from ρ to v inside C. Define

$$d = \gcd\{\, |h(u) + 1 - h(v)| : u \xrightarrow{e} v \text{ is an internal edge of } C \,\}. \tag{14}$$

There is a nonempty closed walk in C. Summing the differences $h(u) + 1 - h(v)$ along a closed walk of length $\ell > 0$ gives ℓ. Consequently, the differences cannot all vanish, so $d > 0$. Every internal edge satisfies

$$h(v) \equiv h(u) + 1 \pmod{d}. \tag{15}$$

The same summation shows that the length of every closed walk is divisible by d. Along any nonempty closed walk, the first d departure phases therefore realize all residues modulo d. In particular, each phase has at least one outgoing internal edge.

For each phase $s \in \{0, \dots, d-1\}$, collect the labels on *all* internal edges whose source u satisfies $h(u) \equiv s \pmod{d}$. We certify C if each such collection is a singleton, written $\{\lambda_s\}$.

Lemma 7 (phase criterion)**.** *Every infinite walk confined to a certified component has a periodic label sequence, with period d.*

Proof. For a walk $v_0 \xrightarrow{\ell_0} v_1 \xrightarrow{\ell_1} \cdots$, (15) gives

$$h(v_i) \equiv h(v_0) + i \pmod{d}.$$

Therefore

$$\ell_i = \lambda_{(h(v_0)+i) \bmod d}, \qquad \ell_{i+d} = \ell_i. \qquad \square$$

The test includes parallel edges and every internal edge, not just tree edges or a selected cycle. Edges leaving C are irrelevant once a walk stays in C. Lemma 12 below proves soundness for any integer-valued h. For a complete test, h is chosen as the length of a path from a common root, as above; shortest paths are unnecessary. Theorem 21 and Corollary 22 show that the outcome is independent of the root and of these paths.

The integer d in (14) is kept separate from the primitive period p of the phase-label word. For reporting, we replace $(\lambda_0, \dots, \lambda_{d-1})$ by its shortest repeating word and then by its lexicographically least cyclic rotation. This word has length $p \mid d$, possibly $p < d$. Such normalization is for comparison and display; Lemma 7 requires neither minimality of the period nor a particular starting phase.

3.3. **Tail preservation and finite certification.** Let $R_K(W)$ be the union of all cyclic SCCs of $G_K(W)$ that fail the phase criterion. The other vertices are removed.

Lemma 8 (retained tail)**.** *If $\sigma_K(n) \in W$ for all sufficiently large n, then $\sigma_K(n) \in R_K(W)$ for all sufficiently large n.*

Proof. Proposition 5 turns the given tail into an infinite walk with labels c_n. By Lemma 6 it eventually stays in one cyclic SCC. If that component were certified, Lemma 7 would make c_n eventually periodic, contrary to Lemma 4. It must therefore be retained. $\square$

For an integer $f \geq 2$, define the refinement

$$\mathrm{Refine}_f(U) = \{(q, fj + i) : (q, j) \in U,\ 0 \leq i < f\}. \tag{16}$$

Lemma 9 (refinement)**.** *If $\sigma_K(n) \in U$, then*

$$\sigma_{fK}(n) \in \mathrm{Refine}_f(U).$$

Proof. From $j = \lfloor K\theta_n \rfloor$ one obtains

$$fj \le fK\theta_n < f(j+1).$$

Thus

$$\lfloor fK\theta_n \rfloor = fj + i$$

for a unique integer $0 \le i < f$. The residue coordinate is unchanged. □

Starting with $K_0 \ge 1$ and $W_0 = V(M, K_0)$, compute

$$K_t = K_0 f^t, \qquad U_t = R_{K_t}(W_t), \qquad W_{t+1} = \mathrm{Refine}_f(U_t). \tag{17}$$

At each new stage the edges are recomputed from (10).

Theorem 10 (finite certification)**.** *If $U_s = \varnothing$ for some finite s, then for every $\xi > 0$ and every $N \in \mathbb{N}$ there is an $n \ge \max(N, 1)$ such that*

$$\gcd(\lfloor \xi (a/b)^n \rfloor, M) > 1. \tag{18}$$

Proof. Suppose instead that for some $\xi > 0$, the integers Q_n are coprime to M for all sufficiently large n. By (9) and (11), the corresponding states lie in W_0. Lemma 8 places a further tail in U_0, and Lemma 9 places the same tail at the finer scale in W_1. Iterating these two implications over the finitely many stages puts a tail in U_s, which is empty. This contradiction proves (18). □

The beginning of the tail may increase at each stage and may depend on ξ. Neither a uniform starting time nor preservation of every finite prefix is needed. In particular, refinement is proved for true orbit states; it does not presume that every coarse graph walk has a finer lift. The graphs themselves have no ξ input. The passage to all positive real coefficients is supplied by these lemmas, rather than by sampling initial values or computing long numerical trajectories.

Theorem 10 assumes a finite successful calculation. It does not assert termination for arbitrary choices of a, b, M, K_0, f.

The modulus may also grow from stage to stage. For $U \subseteq V(M, K)$ and integers $h, f \ge 1$, put $M' = hM$, $K' = fK$, and

$$U' = \{(q + \ell M,\ fj + i) : (q, j) \in U,\ 0 \le \ell < h,\ 0 \le i < f,$$
$$\gcd(q + \ell M, hM) = 1\} \subseteq V(M', K').$$

Lemma 11 (joint enlargement)**.** *If $\sigma_K(n) \in U$ as a state modulo M and $\gcd(Q_n, hM) = 1$, then*

$$\sigma_{K'}(n) \in U'$$

as a state modulo M'.

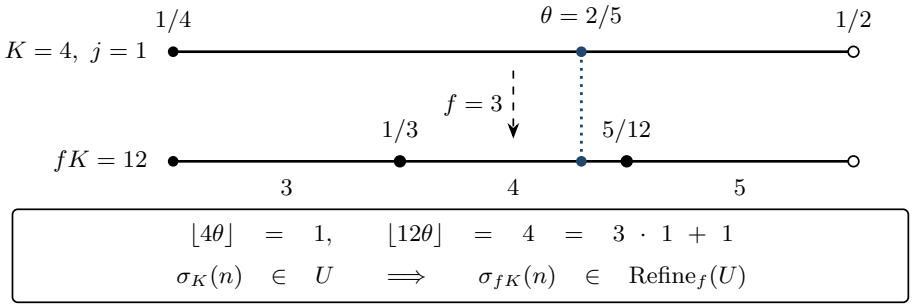
Proof. Let $q = Q_n \bmod M$ and $q' = Q_n \bmod hM$. Then $q' \equiv q \pmod{M}$ and $0 \le q' < hM$, so $q' = q + \ell M$ with $0 \le \ell < h$, and

$$\gcd(q', hM) = \gcd(Q_n, hM) = 1.$$

The cell coordinate is handled by Lemma 9. □

Since every intermediate modulus divides the final one, a tail coprime to the final modulus is coprime to each intermediate modulus, and the induction of Theorem 10 goes through with Lemma 11 in place of Lemma 9.

Cell refinement preserves the cell containing each true orbit point, as the following example shows.



The same true orbit point lies in exactly one finer cell.
Edges are rebuilt; an arbitrary coarse walk need not lift.

FIGURE 3. The cell $[1/4, 1/2)$ at $K = 4$ splits into cells $3, 4, 5$ at $fK = 12$. The point $\theta = 2/5$ belongs to cell 4. The residue coordinate is unchanged. This is an inclusion for true orbit states, not a lifting assertion for arbitrary graph walks.

4. Word-labelled certificates

4.1. **Word-labelled graphs and coverings.** Let $\mathcal{C} \subseteq [1-b, a-1] \cap \mathbb{Z}$ be a finite set of carries, the *alphabet*. A *word-labelled graph* $G = (V, E)$ consists of a finite vertex set V and a finite set E of edges $e = (u, w, v)$ with $u, v \in V$ and $w \in \mathcal{C}^+$ a nonempty finite word; $|w| \geq 1$ is the length of the edge. Edges are identified as triples: the same triple is one edge, and edges with the same endpoints and different words are distinct. An infinite walk is

$$v_0 \xrightarrow{w_0} v_1 \xrightarrow{w_1} \cdots$$

with each $(v_i, w_i, v_{i+1}) \in E$; its *output* is the concatenation

$$w_0 w_1 w_2 \cdots \in \mathcal{C}^{\mathbb{N}},$$

an infinite word because no edge has length zero. A one-step graph of Section 3 is the special case in which every word has length one.

Let (c_n) be the carry sequence of a true orbit and $n_0 \geq 0$. We say that G *covers the tail from* n_0, written $G \models n_0$, if there are an infinite walk $(v_i, w_i)_{i \geq 0}$ of G and times $t_0 = n_0 < t_1 < t_2 < \cdots$ with

$$t_{i+1} = t_i + |w_i|, \qquad w_i = (c_{t_i}, c_{t_i+1}, \ldots, c_{t_{i+1}-1}) \qquad (i \geq 0);$$

that is, the output of the walk equals $(c_n)_{n \geq n_0}$. For each operation Φ below (removal, contraction, lifting) we prove *tail preservation*: if $G \models n_0$ (together with an arithmetic hypothesis specific to the operation), then $\Phi(G) \models n_1$ for some $n_1 \geq n_0$. The starting time may increase at each operation and may depend on ξ; no finite prefix is preserved.

4.2. **The weighted phase criterion and removal.** Let C be a cyclic SCC of G, that is, an SCC containing at least one internal edge (u, w, v) with $u, v \in C$. Let $h : C \to \mathbb{Z}$ be *any* integer-valued function, and for an internal edge $e = (u, w, v)$ put

$$\delta(e) = h(u) + |w| - h(v),$$
$$d = d_C = \gcd\{|\delta(e)| : e \text{ an internal edge of } C\}.$$

Lemma 12 (weighted phase criterion)**.** *(a) $d \geq 1$, and every internal edge satisfies*

$$h(v) \equiv h(u) + |w| \pmod{d};$$

the total word length of every nonempty closed walk in C is a positive multiple of d.

(b) *Suppose there is a map $\lambda : \mathbb{Z}/d \to \mathcal{C}$ such that for every internal edge $e = (u, w, v)$ and every letter position $0 \leq i < |w|$,*

$$w_i = \lambda\big((h(u) + i) \bmod d\big). \tag{19}$$

Then every infinite walk (v_i, w_i) inside C has output $y = w_0 w_1 \cdots$ with

$$y_t = \lambda((h(v_0) + t) \bmod d)$$

for all $t \geq 0$; in particular $y_{t+d} = y_t$ for all t.

We call C *certified* (with respect to h) when (19) holds for some λ.

Proof. (a) Take an internal edge $e_1 = (u, w, v)$ and, by strong connectivity, a path $e_2 \cdots e_m$ from v to u inside C (empty if $u = v$). Along the closed walk $e_1 \cdots e_m$ the terms of h telescope, so

$$\sum_{k \leq m} \delta(e_k) = \sum_{k \leq m} |w_k| \geq 1.$$

Hence not all $\delta(e)$ vanish and $d \geq 1$. The congruence is the definition of the gcd, and the total length of a nonempty closed walk equals the sum of its δ, a positive multiple of d.

(b) Put $\tau_i = \sum_{k<i} |w_k|$. Induction along the walk with (a) gives

$$h(v_i) \equiv h(v_0) + \tau_i \pmod{d}.$$

The output letter at position $t = \tau_i + k$ with $0 \leq k < |w_i|$ is

$$(w_i)_k = \lambda((h(v_i) + k) \bmod d) = \lambda((h(v_0) + t) \bmod d). \qquad \square$$

Lemma 12 is a sufficient condition for any integer-valued h. For the algorithms below, choose a root $\rho \in C$ and, for each $v \in C$, a path π_v from ρ to v inside C, and set

$$h(v) = |\operatorname{out}(\pi_v)|.$$

Thus $h(v)$ is the total word length, rather than the number of edges. The path to ρ may be empty, and the other paths need not be shortest. The implementations use the first root paths found by their graph traversals.

Unless an arbitrary h is explicitly specified, *certified* and *uncertified* below refer to the full test (19) with these root path lengths. With this convention the test is complete for eventual periodicity of all internal infinite outputs, and its outcome is independent of the root and paths (Theorem 21 and Corollary 22). A test left unfinished because of a resource limit is inconclusive.

By Lemma 12(a), every phase $s \in \mathbb{Z}/d$ carries at least one letter of some internal edge, since a closed walk of total length $\ell \geq d$ occupies the phases $h(v_0), \ldots, h(v_0) + \ell - 1$. With all words of length one, (19) is the phase criterion of Section 3.2.

Removal. Let $\mathcal{U}$ be the family of uncertified cyclic SCCs of G. Define

$$R(G) = \Big(\bigcup_{C \in \mathcal{U}} C,\ \{(u, w, v) \in E : \exists C \in \mathcal{U},\ u, v \in C\} \Big) :$$

only the internal edges of uncertified components are kept; edges between components, acyclic components, and certified components are dropped.

Proposition 13 (tail preservation under removal)**.** *If $G \models n_0$, then $R(G) \models n_1$ for some $n_1 \geq n_0$.*

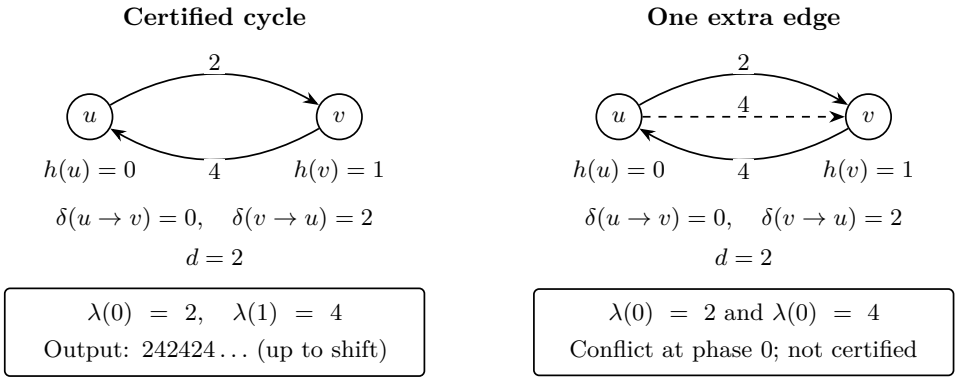
Proof. The covering walk is an infinite walk in a finite graph, so by Lemma 6 there are i_1 and a cyclic SCC C with $v_i \in C$ for all $i \geq i_1$; the edges (v_i, w_i, v_{i+1}), $i \geq i_1$, are internal to C. If C were certified, Lemma 12(b) would give

$$c_{n+d} = c_n$$

for all $n \geq t_{i_1}$, with $d \geq 1$, contrary to Lemma 4. Hence $C \in \mathcal{U}$, and $(v_i, w_i)_{i \geq i_1}$ is a walk of $R(G)$; take $n_1 = t_{i_1}$. □

Keeping only internal edges is sound because Lemma 6 makes every edge of the tail internal to one component. Dropping an internal edge of an uncertified component would not be sound; keeping more edges would be sound but would change the reported counts.

The phase test can be read as a consistency check on letters assigned to cyclic positions.



In both panels, h is the total word length of a chosen path from the root u.

FIGURE 4. Two abstract word graphs with one-letter labels. In both panels, $h(u) = 0, h(v) = 1$ are total word lengths of paths from root u, and $d = 2$. The left cycle is certified. Adding the dashed edge on the right creates two different letters at phase zero, so the full test fails. These examples illustrate the criterion, not the particular carry alphabet for 5/2.

4.3. **Exact contraction of deterministic paths.** Let $G' = R(G)$. Identical triples are merged first; this does not change the set of walks. Out-degrees $\deg^+(u)$ are counted in G' after merging.

Lemma 14. *(a) Every vertex of G' has $\deg^+ \geq 1$. (b) If every vertex of a component C of G' has $\deg^+ = 1$, then C is a simple directed cycle (a single vertex with a self-loop included), and C is certified in the sense of Lemma 12 for the root path lengths along the cycle.*

Proof. (a) If $|C| \geq 2$, strong connectivity gives every vertex an internal out-edge; if $|C| = 1$, cyclicity gives a self-loop.

(b) With unique successors the internal edges form a functional graph, and a strongly connected functional graph is one cycle

$$v_1 \xrightarrow{w_1} \cdots \xrightarrow{w_m} v_1.$$

Put $L = \sum |w_k|$, $h(v_1) = 0$,

$$h(v_{k+1}) = h(v_k) + |w_k|.$$

Then $\delta(e_k) = 0$ for $k < m$, $\delta(e_m) = L$, and $d = L$; each phase $0 \le s < L$ carries exactly one letter, the s-th letter of $w_1 \cdots w_m$, so (19) holds. □

Hence, when removal uses the full test with root path lengths, no component consisting only of out-degree-one vertices reaches the contraction step. The implementations nevertheless keep one *anchor* vertex in such a component as a safeguard (used when a phase test is skipped because of a resource bound); in all computations of Section 5 this rule never fired.

Definition (macro vertices and contraction)**.** Let B consist of all vertices of G' with $\deg^+ \ge 2$, together with one anchor (the least vertex in the vertex order) in every component that contains no such vertex. For $u \in B$ and each out-edge $e = (u, w, v)$ of u: while $v \notin B$, follow the unique out-edge (v, w', v') of v and replace $w \leftarrow ww'$, $v \leftarrow v'$. When first $v \in B$, record the *macro edge* (u, w, v). The contraction $P(G')$ has vertex set B and edge set the set of macro edges (identical triples merged).

Lemma 15. *The iteration in the definition terminates, and every infinite walk of G' visits B infinitely often.*

Proof. If the iteration did not terminate, the vertices visited would all have out-degree one and would be finitely many, so the iteration would enter a cycle Z of out-degree-one vertices. The unique out-edges of Z stay in Z, so Z has no edge to its complement in its component C; strong connectivity gives $C = Z$, a component of out-degree-one vertices, whose anchor lies in $B \cap Z$, and the iteration stops there: a contradiction. The same argument applies to an infinite walk that avoids B from some point on. □

Proposition 16 (exactness of contraction)**.** *Every infinite walk of G' starting at a vertex of B determines an infinite walk of $P(G')$ whose vertex sequence is the sequence of visits to B of the former, whose output is the same infinite word, and whose k-th macro edge has length equal to the total length of the corresponding segment; conversely, every infinite walk of $P(G')$ arises in this way from at least one infinite walk of G'. Consequently, if $G' \models n_1$ then*

$$P(G') \models n_2,$$

where n_2 is the first time at or after n_1 at which the covering walk visits B.

Proof. At a vertex $u \in B$ the walk chooses an edge $e = (u, w, v)$; while $v \notin B$ it is forced through the unique out-edge, following the same vertices and words as the iteration in the definition, until it reaches the next vertex of B after finitely many steps (Lemma 15).

Thus each segment between consecutive visits to B is exactly one macro edge, and conversely each macro edge expands to at least one path of G' (after identical triples are merged, two segments with the same start, word, and end give the same macro edge, so the correspondence is surjective but need not be injective).

The correspondence is segment by segment, and outputs are concatenations of the same words; only the forward direction is used for tail preservation. The covering statement follows, the time n_2 existing by Lemma 15. □

Contraction is used here only between removal and lifting; the cell count K is fixed at the initial stage of the word method, and cell coordinates enter only through the initial edges. We do not use, and do not claim, any refinement of cells after a contraction. For the ratio 5/3 the in-direction counterpart of this contraction is also used; it is defined and proved exact in Appendix F.2 (Lemma 75).

Contraction changes the segmentation of the output word while preserving every letter and its order.

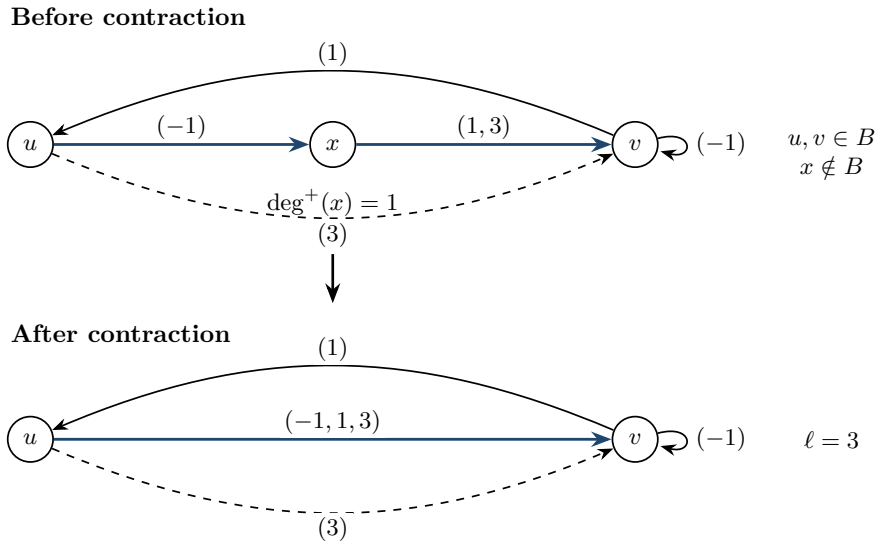


FIGURE 5. An exact contraction. The vertex x has one outgoing edge; u and v are macro vertices because each has two outgoing edges. The words (-1) and $(1,3)$ concatenate to $(-1,1,3)$, whose length remains three. All other edges are retained. Line styles identify corresponding edges.

4.4. **Lifting by a new prime.** Let p be a prime with $p \nmid b$. The *lift* $\mathrm{Lift}_p(G)$ has vertices (u,q) with $u \in V(G)$ and $q \in \mathbb{F}_p^\times$. For an edge $(u,w,v) \in E(G)$ with $w = (c_0,\dots,c_{\ell-1})$ and $q_0 = q$, compute

$$q_{i+1} = b^{-1}(aq_i + c_i) \bmod p \qquad (0 \le i < \ell).$$

There is an edge $((u,q_0),w,(v,q_\ell))$ if and only if $q_0, q_1, \dots, q_\ell$ are all nonzero. The condition on the endpoint residue q_ℓ is included, since q_ℓ is the starting residue of the next edge. Vertices incident to no edge may be deleted: a vertex without out-edges never lies on an infinite walk, and a vertex without in-edges lies only at the start of one, so deleting them preserves coverings after shifting the start by one edge.

Proposition 17 (tail preservation under lifting)**.** *Let $p \nmid b$, $G \models n_0$, and suppose $p \nmid Q_n$ for all $n \ge n'$. Then*

$$\mathrm{Lift}_p(G) \models n_1$$

for $n_1 = t_{i_0}$, where i_0 is the least index with $t_{i_0} \ge \max(n_0, n')$.

Proof. For $i \ge i_0$ put

$$q^{(i)} = Q_{t_i} \bmod p \neq 0.$$

Reducing $bQ_{n+1} = aQ_n + c_n$ modulo p gives

$$Q_{n+1} \equiv b^{-1}(aQ_n + c_n).$$

Hence the forward iteration along $w_i = (c_{t_i}, \dots, c_{t_{i+1}-1})$ starting at $q^{(i)}$ produces the residues of $Q_{t_i}, Q_{t_i+1}, \dots, Q_{t_{i+1}}$, all nonzero, ending at $q^{(i+1)}$.

Thus

$$((v_i, q^{(i)}), w_i, (v_{i+1}, q^{(i+1)}))$$

is an edge of the lift, and $(v_i, q^{(i)})_{i \geq i_0}$ is a covering walk. □

Enumerating all starting residues $q \in \mathbb{F}_p^\times$ is what makes the argument independent of the size of the initial integer: whatever unit $Q_{t_i} \bmod p$ is, the corresponding vertex exists.

Lemma 18 (root formula)**.** *Let $p \nmid ab$, and let*

$$C_0 = 0, \qquad C_{i+1} = aC_i + b^i c_i$$

for a word $w = (c_0, \dots, c_{\ell-1})$, so that (8) holds. Put

$$R_i = -C_i a^{-i} \bmod p,$$

so that $R_0 = 0$ and

$$R_{i+1} = R_i - c_i b^i a^{-i-1}.$$

For a starting residue q, the forward iteration has $q_0, \dots, q_\ell$ all nonzero if and only if

$$q \notin \{R_0, \dots, R_\ell\},$$

and then

$$q_\ell = \alpha q + \beta \qquad \text{with} \qquad \alpha = (ab^{-1})^\ell,$$
$$\beta_0 = 0, \qquad \beta_{i+1} = ab^{-1}\beta_i + b^{-1}c_i.$$

Proof. By (8) applied to any integer lift of q,

$$b^i q_i \equiv a^i q + C_i.$$

Since $p \nmid b$, $q_i \equiv 0$ if and only if $a^i q + C_i \equiv 0$, that is, $q \equiv -C_i a^{-i} = R_i$ ($p \nmid a$).

The recurrence for R_i is

$$-(aC_i + b^i c_i)a^{-i-1} = R_i - c_i b^i a^{-i-1},$$

and the affine form of q_ℓ is the expansion of

$$q_{i+1} = ab^{-1}q_i + b^{-1}c_i.$$ □

Thus enumeration by forbidden starting residues and forward iteration give the same edge set; implementation A uses the former and implementation B the latter (Section 9). If $p \mid b$, no inverse of b exists and

$$0 \equiv aQ_n + c_n \pmod{p},$$

so $Q_n \bmod p$ is determined by c_n alone and the condition $p \nmid Q_n$ is equivalent to $c_n \not\equiv 0 \pmod{p}$, a condition on the alphabet. If $p \mid a$, then

$$Q_{n+1} \equiv b^{-1}c_n,$$

independent of the starting residue; provided every letter of the alphabet is coprime to p (as for the alphabets of Lemmas 23 and 24, whose letters are coprime to ab), the lift multiplies vertices by $p-1$ without removing anything, while an edge whose word contains a letter divisible by p would be removed for every starting residue. This is how the primes 2 and 5 are handled in Section 5.

Checking an entire word under a prime lift requires every intermediate residue, not just its endpoints.

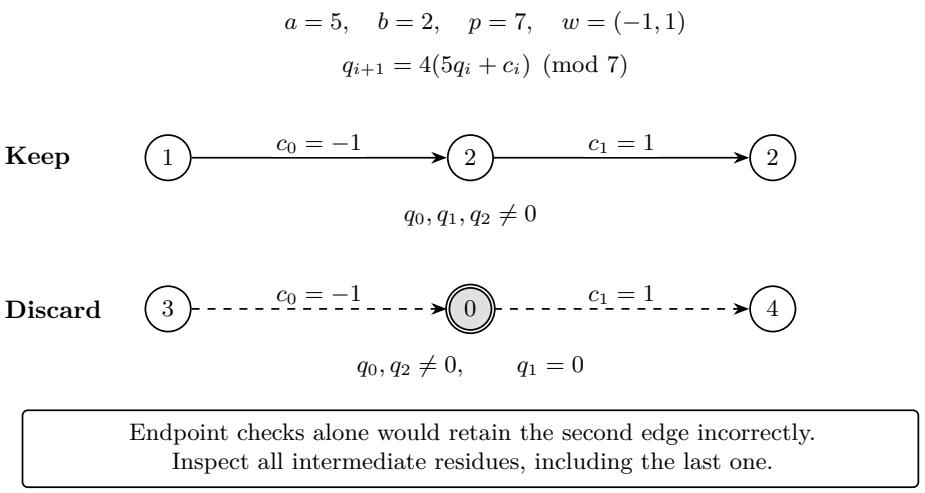


FIGURE 6. Two lifts of the same word $(-1, 1)$ for $a = 5, b = 2, p = 7$, where $2^{-1} = 4 \pmod 7$. Starting at residue 1 gives $1, 2, 2$, so the edge survives. Starting at 3 gives $3, 0, 4$, so it is discarded even though both endpoints are nonzero. Every displayed residue follows from integer arithmetic modulo seven.

4.5. **The finite success theorem. Hypothesis (H).** A finite set S_0 of primes with

$$M_0 = \prod_{p \in S_0} p$$

and an alphabet $\mathcal{C}$ satisfy: for every $\xi > 0$ and every n,

$$\gcd(Q_n, M_0) = \gcd(Q_{n+1}, M_0) = 1$$

implies $c_n \in \mathcal{C}$.

Theorem 19 (word-labelled finite success)**.** *Let $a > b \geq 2$, $\gcd(a, b) = 1$, $K \geq 1$, let $S_0, \mathcal{C}$ satisfy (H), and let $p_1, \ldots, p_s$ be distinct primes with $p_k \nmid ab$ and $p_k \notin S_0$. Let G_0^{in} be the graph with vertices $\{0, \ldots, K-1\}$ and edges $(j, (c), k)$ for all $c \in \mathcal{C}$ and $0 \leq j, k < K$ with*

$$aj - b(k+1) < cK < a(j+1) - bk.$$

Define

$$G_k^{\text{out}} = P(R(G_k^{\text{in}})) \quad (0 \leq k \leq s),$$
$$G_k^{\text{in}} = \text{Lift}_{p_k}(G_{k-1}^{\text{out}}) \quad (1 \leq k \leq s).$$

If $G_s^{\text{out}} = \varnothing$, then for $M = M_0 \prod_{k=1}^{s} p_k$,

$$\forall \xi > 0 \ \forall N \in \mathbb{N} \ \exists n \geq \max(N, 1) : \ \gcd(Q_n, M) > 1. \tag{20}$$

Proof. Fix $\xi > 0$ and suppose (20) fails: there is $n_0 \geq 1$ with

$$\gcd(Q_n, M) = 1$$

for all $n \geq n_0$. By (H), $c_n \in \mathcal{C}$ for all $n \geq n_0$.

The cell $j_n = \lfloor K\theta_n \rfloor$ lies in $\{0, \dots, K-1\}$, and Proposition 5 with $M = 1$ (where (E2) is trivial) shows that

$$j_n \xrightarrow{c_n} j_{n+1}$$

is an edge of G_0^{in}. Hence

$$G_0^{\mathrm{in}} \models n_0$$

with all words of length one.

Proposition 13 and Proposition 16 give

$$G_0^{\mathrm{out}} \models n_1.$$

For $k \geq 1$, since $p_k \mid M$ we have $p_k \nmid Q_n$ for $n \geq n_0$, so Proposition 17 gives

$$G_k^{\mathrm{in}} \models n_k',$$

and again Propositions 13 and 16 give

$$G_k^{\mathrm{out}} \models n_{k+1}.$$

After finitely many steps

$$G_s^{\mathrm{out}} \models n_{s+1},$$

but the empty graph has no infinite walk. □

The quantification in (20) is the same as in (18); the graphs take no ξ as input. The primes of S_0 are not lifted; they enter only through the alphabet, and (H) is an arithmetic statement to be verified by hand for each base (Lemmas 23 and 24).

Corollary 20. *Under the hypotheses of Theorem 19, with*

$$P_{\max} = \max(S_0 \cup \{p_1, \dots, p_s\}),$$

for every $\xi > 0$ and every N there is $n \geq \max(N, 1)$ with Q_n composite.

Proof. By (7), $Q_n > P_{\max}$ for all large n. Apply (20) beyond that threshold and beyond N. A prime p dividing $\gcd(Q_n, M)$ satisfies

$$p \leq P_{\max} < Q_n,$$

so $Q_n = pv$ with $v > 1$. □

4.6. **The all-edge test and commuting closed words.**

Theorem 21. *Let C be a cyclic SCC of a word-labelled graph and $\rho \in C$. The following are equivalent.*

(a) *There exist an integer-valued h on C and λ satisfying (19); moreover, if this holds for some h, then it holds when $h(v)$ is the total word length of any chosen path from ρ to v inside C.*
(b) *Every infinite walk inside C has eventually periodic output.*
(c) *Any two nonempty closed output words at ρ (outputs of nonempty closed walks from ρ to ρ inside C) commute.*

Proof. (a)⇒(b) is Lemma 12(b).

(b)⇒(c). Suppose closed words u, v at ρ with $uv \neq vu$. Let U, V be the corresponding closed walks; UV and VU are closed walks with outputs $x_0 = uv$ and $x_1 = vu$ of the same length L, $x_0 \neq x_1$. For $\varepsilon \in \{0,1\}^{\mathbb{N}}$ the walk $X_{\varepsilon_0} X_{\varepsilon_1} \cdots$ stays in C and has output

$$z_\varepsilon = x_{\varepsilon_0} x_{\varepsilon_1} \cdots .$$

Reading z_ε in blocks of length L recovers ε. If z_ε were eventually periodic with period P, it would have period PL, and block decoding would make ε eventually periodic. Choosing ε not eventually periodic (for instance $0\,1\,00\,1\,000\,1\cdots$) contradicts (b).

(c)⇒(a). Two commuting nonempty words are powers of a common word (induction on length, removing the shorter word from the front of the longer), and the primitive root of a nonempty word is unique, so all closed words at ρ are powers of one primitive word z; put $p = |z|$. For each $v \in C$ fix a path π_v from ρ to v with

$$h(v) = |\operatorname{out}(\pi_v)|$$

and a return path σ_v from v to ρ. The paths at $v = \rho$ are allowed to be empty.

For an internal edge $e = (u, w, v)$ the words

$$\omega_1 = \operatorname{out}(\pi_u)\, w\, \operatorname{out}(\sigma_v) \qquad \text{and}$$
$$\omega_2 = \operatorname{out}(\pi_v)\operatorname{out}(\sigma_v)$$

are closed words, with $\omega_1 \neq \varepsilon$. Both are powers of z, allowing the zeroth power when ω_2 is empty. Therefore p divides

$$|\omega_1| - |\omega_2| = h(u) + |w| - h(v);$$

hence $p \mid \delta(e)$ for every internal edge and $p \mid d$. Define

$$\lambda(s) = z_{s \bmod p}$$

for $s \in \mathbb{Z}/d$, well defined since $p \mid d$.

As $\omega_1 = z^k$, its letter at position t is $z_{t \bmod p}$, and w_i sits at position $h(u) + i$ of ω_1, so

$$w_i = z_{(h(u)+i) \bmod p} = \lambda((h(u) + i) \bmod d).$$

This is (19) for the root path lengths h. □

Corollary 22. *Choose a root $\rho \in C$ and paths $\pi_v : \rho \to v$ inside C, and put*

$$h(v) = |\operatorname{out}(\pi_v)|.$$

Whether (19) holds is independent of the root and of these paths. For every such choice,

$$d = g_C := \gcd\{|\operatorname{out}(\gamma)| : \gamma \text{ is a nonempty closed walk in } C\}.$$

In particular, two complete tests using root path lengths agree on which components are certified and on the value of d.

Proof. By Theorem 21, the test holds for any such choice exactly when every internal infinite walk has eventually periodic output. This property does not involve the root or paths.

For the equality of gcds, summing $\delta(e)$ around a closed walk gives its total word length. Hence $d \mid g_C$. Conversely, for $e = (u, w, v)$, choose a return path $\sigma_v : v \to \rho$. Then

$$\delta(e) = |\operatorname{out}(\pi_u e \sigma_v)| - |\operatorname{out}(\pi_v \sigma_v)|.$$

Both terms are lengths of closed walks at ρ. Each is divisible by g_C, including a zero length if the second walk is empty. Thus $g_C \mid \delta(e)$ for every internal edge, so $g_C \mid d$. This proves $d = g_C$, whether or not the component is certified. □

The restriction on h is essential. Consider the two-edge cycle

$$u \xrightarrow{(2)} v, \qquad v \xrightarrow{(4)} u.$$

Its infinite outputs are the two rotations of $(2,4)^\infty$. The root path lengths $h(u)=0$, $h(v)=1$ give

$$(\delta(u\to v),\delta(v\to u))=(0,2),\qquad d=2,$$

and (19) holds with $\lambda(0)=2$, $\lambda(1)=4$. The arbitrary choice $h(u)=h(v)=0$ instead gives

$$(\delta(u\to v),\delta(v\to u))=(1,1),\qquad d=1.$$

Both letters would then occupy phase zero, so (19) fails. For a general integer-valued h, the telescoping argument gives only $d \mid g_C$; Lemma 12 still guarantees soundness, but failure of that test need not imply a nonperiodic output.

With root path lengths, Theorem 21 does give the converse: if the full test (19) fails, the component contains an internal infinite walk whose output is not eventually periodic. No sound test that removes a component solely because all its infinite outputs are eventually periodic can remove such a component from the same finite graph. Further removal requires additional arithmetic or realizability information.

5. The rational bases

This section treats 7/5 and 5/2. The certificate for 5/3, which also uses the contraction of in-degree-one chains, is given in Appendix F.

5.1. **The ratio 7/5 by the one-step method.** Use

$$(a,b,M,K_0,f)=(7,5,4290,32,4). \tag{21}$$

The initial graph has $\varphi(4290)K_0 = 960\cdot 32 = 30720$ vertices. The two one-step implementations described in Section 9.2 produce the four stages in Table 1. The retained set is empty at $K=2048$.

Table 1: The complete stages for (21). Edges are counted with their labels. The certified column counts certified cyclic SCCs, and the final column is $|U_t|$.

K_t	Vertices	Edges	Cyclic SCCs	Certified SCCs	Retained vertices
32	30720	55440	17	16	1363
128	5452	9512	3	2	952
512	3808	6281	1	0	944
2048	3776	5951	15	15	0

The numbers of all SCCs, including noncyclic singletons, are respectively 29312, 4440, 2865, and 3658. At the first three stages there is one uncertified cyclic SCC, of respective size 1363, 952, and 944. Their internal edge counts are 2332, 1550, and 1505, and all three have $d=1$. Their removal is not justified by the phase test; they are carried into the next subdivision. In particular, the vertex counts of the next stages are exactly four times these retained sizes.

Table 2 groups certified components by K, d, and normalized primitive word. Component sizes are listed with multiplicity. There are no certified components at $K=512$.

Table 2: All nine groups of certified components. The parameter d is (14), and p is the length of the displayed primitive word.

K	Number of SCCs	Component sizes	d	p	Primitive word, up to rotation
32	2	1, 1	1	1	(2)
32	6	2, 2, 2, 2, 2, 2	2	2	$(-2, 4)$
32	8	6, 6, 6, 6, 6, 6, 6, 6	6	2	$(-2, 4)$
128	1	24	8	8	$(-4, 2, 2, 2, 2, 2, 2, 2)$
128	1	39	13	13	$(-4, 2, 2, 2, 2, 2, -4, 2, 2, 2, 2, 2, 2)$
2048	3	1, 1, 1	1	1	(2)
2048	3	4, 4, 8	4	4	$(-2, -2, 4, 4)$
2048	3	6, 6, 6	6	1	(2)
2048	6	12, 12, 12, 12, 24, 24	12	4	$(-2, -2, 4, 4)$

Thus five distinct primitive words occur. The rows with $d = 6, p = 1$ and $d = 12, p = 4$, for example, show why d must not be reported as the primitive word length. The words describe possible output within the outer graph components; an infinite true orbit cannot remain in any of these components, by Lemma 4.

As additional checks of the procedure, Table 3 applies the same construction at $K_0 = 32$ to the three floor results of DN, Theorem 1. Each stops after its first stage. These are reproductions of the stated divisibility conclusions, not new claims about those three bases.

Table 3: Known cases reproduced by both one-step implementations; all retained sets are empty.

a/b	Prime set	M	Vertices	Edges	Cyclic SCCs	Certified SCCs
3/2	$\{2, 5, 7, 11\}$	770	7680	8640	6	6
4/3	$\{2, 3, 5\}$	30	256	480	3	3
5/4	$\{2, 3, 7, 11, 13\}$	6006	46080	82170	18	18

5.2. **The ratio 7/5 by the word method.**

Lemma 23. *Let $a = 7$, $b = 5$. If Q_n and Q_{n+1} are odd and $5 \nmid Q_n$, then*

$$c_n \in \{-4, -2, 2, 4, 6\}.$$

Hence (H) holds with

$$S_0 = \{2, 5\} \qquad \text{and} \qquad \mathcal{C} = \{-4, -2, 2, 4, 6\}.$$

Proof. By (6), $-4 \le c_n \le 6$. Since $5 \equiv 7 \equiv 1 \pmod 2$,

$$c_n = 5Q_{n+1} - 7Q_n \equiv Q_{n+1} - Q_n \pmod 2,$$

which is even when both terms are odd; so

$$c_n \in \{-4, -2, 0, 2, 4, 6\}.$$

If $c_n = 0$ then $5Q_{n+1} = 7Q_n$ and, as $\gcd(5,7) = 1$, $5 \mid Q_n$. □

The prime 2 could also be lifted by Proposition 17, but $\mathbb{F}_2^\times = \{1\}$ and the lift condition reduces to "all letters even", already contained in the alphabet. The prime 5 divides b and cannot be lifted; by the remark after Lemma 18, $5 \nmid Q_n$ is equivalent to $c_n \not\equiv 0 \pmod 5$, which within the range $[-4, 6]$ and with even letters means $c_n \neq 0$. Thus the alphabet expresses the avoidance of 2 and 5 as a necessary condition, which is all that Theorem 19 requires.

Apply Theorem 19 with $K = 2048$, the alphabet of Lemma 23, and the primes $p_1, p_2, p_3 = 3, 11, 13$. The value $K = 2048$ is a free parameter, chosen to match the final cell count of Table 1; soundness does not depend on it. Table 4 lists the four stages. Both implementations of Section 9.3 agree on every vertex and every word-labelled edge of the eight graphs; the output graph of the last stage is empty. In these four stages no macro edge is produced by more than one contracted chain, so the number of distinct edges equals the number counted with multiplicity.

Table 4: The word method for 7/5 with $K = 2048$, $\mathcal{C} = \{-4, -2, 2, 4, 6\}$, primes $3, 11, 13$. Input graphs are G_k^{in} and output graphs G_k^{out} of Theorem 19. "Uncertified vertices" counts the vertices of uncertified cyclic SCCs.

k	p_k	Input vertices	Input edges	Cyclic SCCs	Certified	Uncertified vertices	Output vertices	Output edges	Max. word length (output)
0	—	2048	8366	1	0	2048	2048	8366	1
1	3	4096	9010	10	9	691	593	1292	13
2	11	5928	11276	4	3	89	62	131	8
3	13	744	1342	2	2	0	0	0	—

The total numbers of SCCs at the four stages are 1, 3387, 5840, and 704. The final prime set is $\{2, 3, 5, 11, 13\}$, as in (21). This is not a theorem about a new base but a second proof of (1) by the method of Section 4, and a consistency check between the two methods.

5.3. **The ratio 5/2.**

Lemma 24. *Let $a = 5$, $b = 2$. If Q_n is odd, then*

$$c_n \in \{-1, 1, 3\}.$$

Hence (H) holds with

$$S_0 = \{2\} \qquad \text{and} \qquad \mathcal{C} = \{-1, 1, 3\}.$$

Proof. By (6), $-1 \le c_n \le 4$, and

$$c_n = 2Q_{n+1} - 5Q_n \equiv Q_n \pmod 2.$$ □

Only the oddness of Q_n is used; neither the oddness of Q_{n+1} nor avoidance of 5 is needed. The prime 5 is not included in the modulus of (2), for three independent reasons: $5 \mid a$, so Lemma 18 does not apply; by the remark after Lemma 18, a lift by 5 would give

$$Q_{n+1} \equiv 2^{-1} c_n = 3c_n \pmod 5,$$

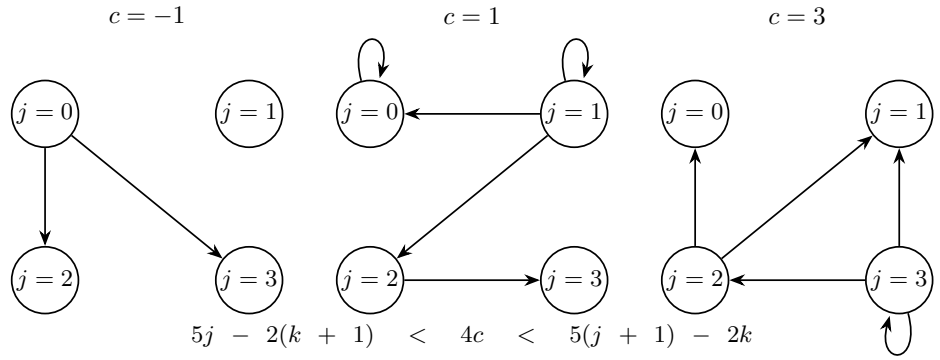
which is nonzero for $c_n \in \{-1, 1, 3\}$ (residues $2, 3, 4$), so on a tail with odd carries $5 \nmid Q_{n+1}$ holds automatically and the lift would remove nothing; and the statement without 5 is stronger.

With $K = 4$, the initial graph G_0^{in} has the four vertices $j \in \{0, 1, 2, 3\}$ and the twelve edges $(j, (c), k)$ with $c \in \{-1, 1, 3\}$ and

$$5j - 2(k + 1) < 4c < 5(j + 1) - 2k,$$

listed in Table 5. By Theorem 19 every tail avoiding the primes of (2) walks in this graph; this is a covering of all cells, not a sampling of initial values, and no converse realizability is asserted.

The initial four-cell graph is small enough to display completely, grouped by carry label.



The three panels together contain all 12 labelled edges.

FIGURE 7. All twelve edges of the initial 5/2 graph at $K = 4$, separated into three panels by carry c. Each panel uses the same four vertices; the panels are overlaid to recover the full graph. The edges are enumerated by the strict integer inequality shown below them and agree with Table 5.

Table 5: The twelve initial edges for 5/2, $K = 4$, as pairs (carry, target cell).

Source cell j	Edges (c, k)
0	$(-1, 2), (-1, 3), (1, 0)$
1	$(1, 0), (1, 1), (1, 2)$
2	$(1, 3), (3, 0), (3, 1)$
3	$(3, 1), (3, 2), (3, 3)$

Apply Theorem 19 with $K = 4$, the alphabet of Lemma 24, and the nine primes

$$p_1, \ldots, p_9 = 3, 7, 11, 13, 17, 19, 23, 29, 31.$$

Table 6 lists the ten stages. Both implementations of Section 9.3 agree on every vertex and every word-labelled edge of the twenty graphs, and the output graph of stage 9 is empty: all 1084 cyclic SCCs of G_9^{in} are certified. Edges are counted as sets of triples (Section 4.1); the multiplicities with which contracted chains produce the same triple carry no proof content and are recorded in Appendix A for comparison with the original research notes.

Table 6: The word method for 5/2 with $K = 4$, $\mathcal{C} = \{-1, 1, 3\}$. Columns as in Table 4.

k	p_k	Input vertices	Input edges	Cyclic SCCs	Certified	Uncertified vertices	Output vertices	Output edges	Max. word length (output)
0	—	4	12	1	0	4	4	12	1
1	3	8	17	1	0	7	5	12	2
2	7	30	58	4	2	16	9	19	4
3	11	90	159	3	0	68	39	84	8
4	13	462	812	4	1	342	250	524	15
5	17	3985	7020	11	8	2913	2059	4058	38
6	19	36843	57660	73	70	15933	8890	16696	56
7	23	194242	293181	23	21	84824	40051	73286	98
8	29	1101968	1542910	311	310	269245	100401	171256	210
9	31	2856938	3256019	1084	1084	0	0	0	—

The total numbers of SCCs at the ten stages are 1, 2, 16, 25, 123, 1060, 20800, 109409, 832175, and 2850774. The maximal word length of an input graph can be smaller than that of the preceding output graph (207 at stage 9 against 210 at stage 8), because a lift discards every edge whose word forces a zero residue for all starting residues. The anchor rule of Section 4.3 never fired. A direct product of all residues modulo the modulus of (2) with four cells would have

$$4\varphi(40112098026) = 30656102400$$

vertices; the staged construction never forms it.

The vertex counts show how growth under lifting alternates with reduction inside each stage.

5.4. **Proofs of Theorem 1 and Corollary 2.**

Proof of Theorem 1. (i) The finite calculation verifies the graphs, retained sets, and refinements in (17) for the parameters (21), with $s = 3$ and $U_3 = \varnothing$. Verification includes the full edge sets and component conditions, as detailed in Section 9; the summary counts alone are not the certificate. Theorem 10 gives (1).

Alternatively, Table 4 verifies the hypotheses of Theorem 19 with $S_0 = \{2, 5\}$, $\mathcal{C} = \{-4, -2, 2, 4, 6\}$ (Lemma 23), $K = 2048$, primes $3, 11, 13$, and $G_3^{\text{out}} = \varnothing$; then (20) with

$$M = 10 \cdot 3 \cdot 11 \cdot 13 = 4290$$

is (1).

(ii) Table 6 verifies the hypotheses of Theorem 19 with $S_0 = \{2\}$, $\mathcal{C} = \{-1, 1, 3\}$ (Lemma 24), $K = 4$, the nine primes $3, 7, 11, 13, 17, 19, 23, 29, 31$, each distinct, coprime to 10, and not in S_0, and $G_9^{\text{out}} = \varnothing$. Then (20) with

$$M = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 = 40112098026$$

is (2).

(iii) Part (iii) is proved in Appendix F.3 from Theorem 76 and Table 17. □

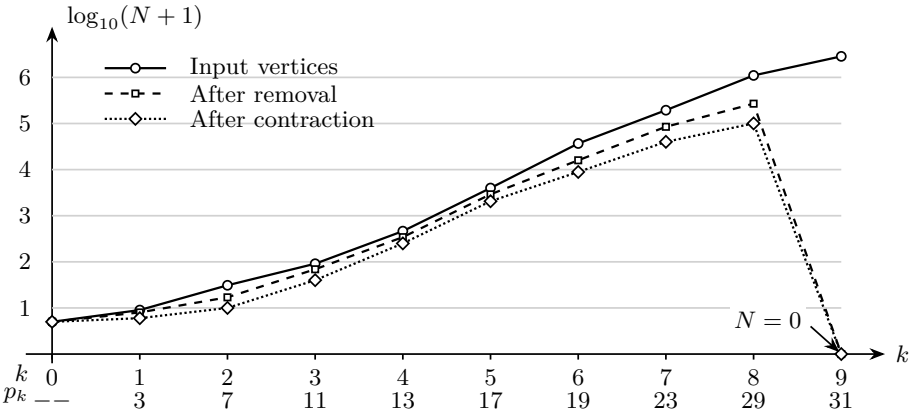


FIGURE 8. Vertex counts from Table 6. The ordinate is $\log_{10}(N+1)$, so the final empty output $N = 0$ is visible at height zero. Solid circles show input counts, dashed squares counts after removal, and dotted diamonds counts after contraction. Connecting segments guide the eye; they do not assert monotone decay across stages. All input cyclic SCCs are certified at stage nine.

Proof of Corollary 2. By (7), eventually $Q_n > 13$ for the sequence with ratio $7/5$, $Q_n > 31$ for the sequence with ratio $5/2$, and $Q_n > 37$ for the sequence with ratio $5/3$. Given any lower bound on the index, apply Theorem 1 beyond both that bound and the growth threshold. A prime divisor of the gcd in (1) lies in $\{2, 3, 5, 11, 13\}$, one of the gcd in (2) lies in $\{2, 3, 7, 11, 13, 17, 19, 23, 29, 31\}$, and one of the gcd in (2a) lies in $\{2, 3, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$; in each case it is a proper divisor of Q_n. Hence Q_n is composite, at arbitrarily large indices. This is Corollary 20 in the first two cases, and the same argument with Theorem 76 in the third. □

6. Quantitative waiting time for $7/5$

6.1. **A finite version of nonperiodicity.**

Lemma 25. *Let $u \in \mathbb{N}$, $1 \le d \le \ell$, and suppose*

$$c_{u+i+d} = c_{u+i}$$

for $0 \le i < \ell - d$. Put

$$D_i = Q_{u+i+d} - Q_{u+i}.$$

Then

$$b^{\ell-d} \mid D_0. \tag{22}$$

If $D_0 > 0$, then with $\nu_b(D_0) = \max\{k \ge 0 : b^k \mid D_0\}$,

$$\begin{aligned} \ell &\le d + \nu_b(D_0), \\ b^{\ell-d} &\le D_0 < r^d(Q_u + 1) - Q_u. \end{aligned} \tag{23}$$

Proof. For $0 \le i < \ell - d$, subtracting the recurrences (6) gives

$$bD_{i+1} = (aQ_{u+i+d} + c_{u+i+d}) - (aQ_{u+i} + c_{u+i}) = aD_i.$$

Induction gives

$$b^k D_k = a^k D_0$$

for $0 \le k \le \ell - d$, and $\gcd(a^k, b^k) = 1$ gives (22).

If $D_0 > 0$, the positive divisor $b^{\ell-d}$ is at most D_0, so $\ell - d \le \nu_b(D_0)$.

Finally

$$Q_{u+d} \le x_{u+d} = r^d x_u < r^d(Q_u + 1).$$

The integer b need not be prime. □

Lemma 26. *If $x_n \ge b/(a-b)$, then $Q_{n+1} \ge Q_n + 1$. In particular, if $Q_u \ge b/(a-b)$, then Q_n is strictly increasing for $n \ge u$, and*

$$D_0 = Q_{u+d} - Q_u \ge d \ge 1$$

in Lemma 25.

Proof.

$$rx_n - x_n = x_n(a-b)/b \ge 1,$$

so

$$\lfloor rx_n \rfloor \ge \lfloor x_n + 1 \rfloor = Q_n + 1.$$ □

For $7/5$, $b/(a-b) = 5/2$, so $Q_u \ge 3$ suffices. A positive difference must not be assumed when the integer parts stay at 0 or 1.

6.2. **From a rank certificate to a waiting time.** Let H be a finite labelled graph with edges of length one, together with a partition of $V(H)$ into *blocks* and the following data.

- (Q-rank) An integer $\rho(B) \in [1, \rho_{\max}]$ for every block B such that every edge $u \to v$ between distinct blocks satisfies
$$\rho(B(v)) \ge \rho(B(u)) + 1.$$
Put $R = \rho_{\max} - 1$.
- (Q-block) Every block either has no internal edge, or is *periodic*: there are $d_B \in [1, D]$, $h_B : B \to \mathbb{Z}$, and $\lambda_B : \mathbb{Z}/d_B \to \mathcal{C}$ such that every internal edge $u \xrightarrow{c} v$ satisfies
$$h_B(v) \equiv h_B(u) + 1 \pmod{d_B} \qquad \text{and}$$
$$c = \lambda_B(h_B(u) \bmod d_B).$$
- (Q-cyc) An integer $\eta(B) \in [0, m]$ for every block with
$$\eta(B) \ge \mathbf{1}[B \text{ periodic}], \qquad \text{and}$$
$$\eta(B(v)) \ge \eta(B(u)) + \mathbf{1}[B(v) \text{ periodic}]$$
for every edge between distinct blocks.

No block with internal edges may remain uncertified; blocks need not be SCCs. An SCC decomposition supplies such data through its acyclic quotient, with ρ the length of a longest block sequence ending at the block and η the largest number of periodic blocks on such a sequence.

By (Q-rank), every finite walk in H has at most R edges between distinct blocks, visits each block during one contiguous interval, and by (Q-cyc) visits at most m periodic blocks.

An *orbit segment of length T lies in H* if the states $\sigma(0), \dots, \sigma(T)$ of the orbit (in the coordinates of H) are vertices of H and $\sigma(n) \xrightarrow{c_n} \sigma(n+1)$ is an edge of H for $0 \le n < T$.

Theorem 27 (waiting time from a rank certificate)**.** *Let*

$$\kappa = \log_b r > 0, \qquad \gamma \ge 1 + \kappa,$$
$$P = Q_0 \ge b/(a-b), \qquad \text{and} \qquad A \ge \log_b(P+2).$$

If an orbit segment of length T lies in H, then

$$T \le R\gamma^m + (A + \gamma D) \sum_{j=0}^{m-1} \gamma^j. \tag{24}$$

For $m = 0$ the sum is empty and $T \le R$.

Proof. Step 1: stay in one periodic block. Suppose the segment enters a periodic block B at time u and the states at times $u, \dots, u+\ell$ lie in B, so the edges at times $u, \dots, u+\ell-1$ are internal. Induction with (Q-block) gives

$$c_{u+i} = \lambda_B((h_B(\sigma(u)) + i) \bmod d_B),$$

hence

$$c_{u+i+d_B} = c_{u+i}$$

for $0 \le i < \ell - d_B$.

If $\ell \ge d_B$, Lemma 25 applies, and Lemma 26 (with $Q_u \ge P \ge b/(a-b)$) gives $D_0 > 0$; so

$$b^{\ell - d_B} \le D_0 < r^{d_B}(Q_u + 1) \qquad \text{and}$$
$$\ell - d_B < \log_b(Q_u + 1) + d_B\kappa.$$

Since

$$Q_u \le x_u = r^u \xi < r^u(P+1),$$

we have

$$Q_u + 1 < r^u(P+1) + 1 \le r^u(P+2),$$

so

$$\log_b(Q_u + 1) < u\kappa + A.$$

Together,

$$\ell < A + \kappa u + (1+\kappa)d_B \le A + (\gamma - 1)u + \gamma D.$$

If $\ell < d_B$ the same bound holds trivially, since

$$\ell < d_B \le D \le \gamma D.$$

Thus the exit time (or the end of the segment) satisfies

$$u + \ell < \gamma u + A + \gamma D.$$

Step 2: bookkeeping. Let $B_0, \dots, B_s$ ($s \le R$) be the blocks visited, with entry times $u_0 = 0 < u_1 < \cdots < u_s$, let ℓ_k be the number of internal edges traversed in B_k, so that

$$u_{k+1} = u_k + \ell_k + 1 \qquad \text{and} \qquad T = u_s + \ell_s;$$

edgeless blocks have $\ell_k = 0$.

Let $k_1 < \cdots < k_{m'}$ ($m' \le m$) be the indices of the periodic blocks, $k_0 = 0$, $x_0 = 0$,

$$x_j = u_{k_j} + \ell_{k_j}$$

the exit time from the j-th periodic block,

$$\delta_j = k_j - k_{j-1} \qquad \text{for} \qquad 1 \le j \le m', \qquad \text{and}$$
$$\delta_{m'+1} = s - k_{m'}.$$

Between periodic blocks only edgeless blocks occur, so time advances by one per inter-block edge:

$$u_{k_j} = x_{j-1} + \delta_j, \qquad T = x_{m'} + \delta_{m'+1}, \qquad \text{and}$$
$$\sum_{j=1}^{m'+1} \delta_j = s \le R.$$

Step 3: expansion. By Step 1,

$$x_j < \gamma u_{k_j} + A + \gamma D = \gamma(x_{j-1} + \delta_j) + A + \gamma D.$$

Induction gives

$$x_{m'} < \sum_{j=1}^{m'} \gamma^{m'-j+1}\delta_j + (A + \gamma D) \sum_{j=0}^{m'-1} \gamma^j.$$

Since $\gamma \ge 1$, $\gamma^{m'-j+1} \le \gamma^{m'}$ and $1 \le \gamma^{m'}$, so

$$\begin{aligned} T &= x_{m'} + \delta_{m'+1} \\ &< \gamma^{m'} \sum_{j=1}^{m'+1} \delta_j + (A + \gamma D) \sum_{j<m'} \gamma^j \\ &\le R\gamma^m + (A + \gamma D) \sum_{j<m} \gamma^j. \end{aligned}$$

If $m' = 0$ then

$$T = s \le R. \qquad \square$$

Each inter-block edge is amplified by a factor of at most γ^m, once for every periodic block visited afterwards. The proof handles finite walks only and uses neither infinite walks nor samples of ξ. If any cyclic block is uncertified the theorem does not apply, since a walk may stay in it forever.

The ranks count blocks and periodic blocks along paths, whereas the period records the output inside a block.

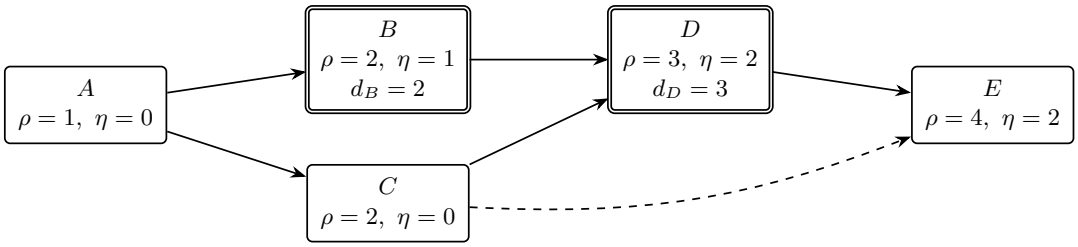
6.3. **The certificate for 7/5.**

Lemma 28 (reduction of the modulus)**.** *Suppose* $\gcd(Q_n, 4290) = 1$ *for* $0 \le n \le T$*. Then* $c_n \in \mathcal{C} = \{-4, -2, 2, 4, 6\}$ *for* $0 \le n < T$*, and with* $M = 429 = 3 \cdot 11 \cdot 13$ *and* $K = 2048$*, the orbit segment of length* T *lies in the graph*

$$H = G(429, 2048) \text{ restricted to labels in } \mathcal{C}.$$

Proof. Lemma 23 gives the alphabet. Since $\gcd(Q_n, 429) = 1$, the states $(Q_n \bmod 429, \lfloor 2048\theta_n \rfloor)$ lie in $V(429, 2048)$, and Proposition 5 makes consecutive states adjacent with label c_n. $\square$

Illustrative condensation DAG (not the full computed graph)



Double border: periodic block. Single border: no internal edge.

ρ: maximum number of blocks η: maximum number of periodic blocks

d_B: internal output period

Computed 7/5 certificate (Table 7)								
491 275 blocks, 35 cyclic blocks								
$\max \rho$	=	65,	$\max \eta$	=	6,	$\max d_B$	=	13

FIGURE 9. An illustrative condensation DAG distinguishes three quantities: ρ is the maximum number of blocks on a path ending at a block, η is the maximum number of periodic blocks on such a path, and d_B is an internal output period. The five-block diagram is schematic. The separate box records the actual 7/5 certificate values from Table 7; those values do not describe the five-block example.

Because $\gcd(5, 429) = 1$, the congruence (E2) has the unique solution

$$z \equiv 5^{-1}(7q + c) \pmod{429};$$

this is a consequence of reducing the modulus from 4290 to 429, where 5 is a non-unit. The avoidance of 2 and 5 is carried by the alphabet, and the prime 7 is not assumed.

The graph H has

$$\varphi(429) \cdot 2048 = 240 \cdot 2048 = 491520$$

vertices. Its SCC decomposition and the ranks ρ, η computed on the acyclic quotient give the data of Table 7. Both implementations of Section 9.3 enumerate all vertices and edges of H, agree on the block partition, on the attributes of every block, and on the phase labels of every periodic block, and verify the inequalities of (Q-rank), (Q-block), (Q-cyc) on every edge.

Table 7: The rank certificate for 7/5 at $M = 429$, $K = 2048$.

Quantity	Value
Legal vertices	491520
Labelled edges	891990
Blocks (SCCs)	491275
Cyclic blocks	35
Certified cyclic blocks	35
Uncertified cyclic blocks	0
$\max \rho$	65
$\max \eta$	6

Quantity	Value
$\max d_B$	13

Thus (Q-rank) holds with $\rho_{\max} = 65$, $R = 64$; (Q-cyc) with $m = 6$; (Q-block) with $D = 13$. Since $7^4 = 2401 < 3125 = 5^5$, we have $(7/5)^4 < 5$, $\kappa = \log_5(7/5) < 1/4$, and $\gamma = 5/4$ is admissible. With

$$A = L(P) = \min\{h \geq 0 : 5^h \geq P + 2\} \geq \log_5(P + 2),$$

the constants in (24) are, exactly in rational arithmetic,

$$\sum_{j=0}^{5} \left(\tfrac{5}{4}\right)^j = \frac{11529}{1024},$$

$$64\left(\tfrac{5}{4}\right)^6 + \tfrac{5}{4} \cdot 13 \cdot \frac{11529}{1024} = \frac{15625}{64} + \frac{749385}{4096} = \frac{1749385}{4096},$$

$$\text{and} \quad 1749385 < 428 \cdot 4096 = 1753088, \qquad 11529 < 12 \cdot 1024 = 12288.$$

Hence, for an orbit segment of length T lying in H with $P \geq 3$,

$$T \leq \frac{1749385}{4096} + \frac{11529}{1024} L(P) < 428 + 12L(P). \tag{25}$$

Proof of Theorem 3. Let $P = \lfloor \xi \rfloor > 13$ and $T_0 = 428 + 12L(P)$. If $\gcd(Q_n, 4290) = 1$ for all $0 \leq n \leq T_0$, then by Lemma 28 the orbit segment of length T_0 lies in H, and Theorem 27 with the certificate of Table 7 (Lemma 26 applies since $P > 13 \geq 3$) gives

$$T_0 < 428 + 12L(P) = T_0,$$

a contradiction.

Hence some $0 \leq n \leq T_0$ has $\gcd(Q_n, 4290) > 1$.

For such n, a prime $p \in \{2, 3, 5, 11, 13\}$ divides

$$Q_n \geq Q_0 = P > 13 \geq p,$$

so $Q_n = pv$ with $v > 1$ is composite.

For the last assertion, apply the result to $\xi' = \xi(7/5)^N$, for which

$$\lfloor \xi'(7/5)^n \rfloor = Q_{N+n}$$

and $\lfloor \xi' \rfloor = Q_N$; the graph H does not depend on ξ, and P enters only through the bound $Q_u < r^u(P + 1)$. □

The bound (25) itself holds for $P \geq 3$; the hypothesis $P > 13$ is used only to conclude compositeness. The integer definition of $L(P)$ avoids rounding questions; it equals $\lceil \log_5(P + 2) \rceil$ for every $P \geq 0$.

7. Scope of the certificate method

This section and the next study the reach of the procedure of Sections 3–4 when its operations are fixed precisely. Throughout,

$$\Sigma = \{1 - b, \ldots, a - 1\},$$

and $\operatorname{rad} M$ is the product of the distinct primes dividing M.

7.1. **The general carry alphabet.** For a reduced base a/b define

$$\mathcal{C}(a,b) = \{c \in \mathbb{Z} : 1-b \le c \le a-1,\ c \equiv b-a \pmod 2,\ \gcd(c,ab) = 1\}. \tag{26}$$

Lemma 29. *Let* $\mathrm{rad}(2ab) \mid M$. *Every edge* $(q,j) \xrightarrow{e} (z,k)$ *of* $G(M,K)$ *has* $e \in \mathcal{C}(a,b)$. *In particular* $c_n \in \mathcal{C}(a,b)$ *whenever* $\gcd(Q_nQ_{n+1}, 2ab) = 1$. *Conversely, for such* M, *the label conditions imposed by the primes dividing* $2ab$ *are exactly* $e \in \mathcal{C}(a,b)$.

Proof. Let $p \mid a$; then $p \nmid b$ and (E2) gives

$$bz \equiv e \pmod p,$$

so $p \mid e$ if and only if $p \mid z$, which is excluded as z is a unit.

Let $p \mid b$; then

$$0 \equiv aq + e \pmod p,$$

so $p \mid e$ if and only if $p \mid q$, excluded.

If a, b are odd and $2 \mid M$, then

$$z \equiv q + e \pmod 2$$

with q, z odd, so e is even, that is, $e \equiv b - a \pmod 2$; if one of a, b is even the parity condition $e \equiv b - a \pmod 2$ is the same as $\gcd(e, ab)$ being odd, which was shown. The converse is the same computation read backwards. □

For $7/5$,

$$\mathcal{C}(7,5) = \{-4, -2, 2, 4, 6\},$$

and for $5/2$,

$$\mathcal{C}(5,2) = \{-1, 1, 3\} :$$

the alphabets of Lemmas 23 and 24 are exactly the general ones. When $\mathcal{C}(a,b)$ is used to express avoidance of all primes dividing $2ab$, the conclusion of a certificate is a divisibility statement whose prime set contains all primes dividing $2ab$; the individual reductions of Section 5 avoid some of these primes (the prime 7 for $7/5$, the prime 5 for $5/2$).

7.2. **The procedure and its three soundness notions. Word-labelled graphs with expansions.** In this section a word-labelled graph has vertex set contained in some $V(M,K)$, and each edge $u \xrightarrow{w} v$ records an *expansion path*

$$u = u_0 \xrightarrow{w_0} u_1 \cdots \xrightarrow{w_{|w|-1}} u_{|w|} = v$$

of edges of $G(M,K)$. One-step graphs are the case of length one. For $M \mid M'$ and $K' = fK$, the *projection* is

$$\pi(q,J) = (q \bmod M, \lfloor J/f \rfloor).$$

Pseudo-orbits. A *right-directed* M*-pseudo-orbit* is a sequence $(q_n, \theta_n, c_n)_{n \ge n_0}$ with $q_n \in (\mathbb{Z}/M)^\times$, $\theta_n \in [0,1)$, $c_n \in \mathbb{Z}$, and

$$a\theta_n - b\theta_{n+1} = c_n \text{ (exactly, in } \mathbb{R}), \qquad bq_{n+1} \equiv aq_n + c_n \pmod M.$$

A *left-directed* pseudo-orbit satisfies the same conditions for $n \le n_0$. A true orbit whose tail avoids the primes of M gives a right-directed pseudo-orbit with $q_n = Q_n \bmod M$; a pseudo-orbit gives, for every K, a walk $(q_n, \lfloor K\theta_n \rfloor)$ of $G(M,K)$, by the proof of Proposition 5, which uses only the real identity and the half-open cells. A pseudo-orbit modulo M reduces to one modulo any divisor of M. What a true orbit has and a pseudo-orbit lacks is a single integer sequence Q_n linking residues and fractional parts.

Operations. An *execution* is a finite sequence $\Gamma_0, \ldots, \Gamma_s$ of word-labelled graphs with parameters (M_t, K_t), $M_t \mid M_{t+1}$, $K_t \mid K_{t+1}$, built by the following operations.

- (I) *Initial graph.* Either $\Gamma_0 = G(M_0, K_0)$ on all legal states, with all labels in Σ; or, when the final modulus is to contain $\mathrm{rad}(2ab)$, $\Gamma_0 = G'(R_0, K_0)$, the graph with vertices (q, j), $q \in (\mathbb{Z}/R_0)^\times$ ($R_0 = 1$ allowed, $\gcd(R_0, 2ab) = 1$), edges labelled by $e \in \mathcal{C}(a, b)$ with
$$bz \equiv aq + e \pmod{R_0}$$
and (E3); in this form we set $M_0 = \mathrm{rad}(2ab)R_0$, and the residues modulo the primes of $2ab$ are represented by the alphabet (Lemma 29 and Proposition 36).
- (P) *Phase removal.* Replace Γ_t by the union of its uncertified cyclic SCCs with their internal edges. In each component, use the total word lengths of paths from a common root and apply (19) to every internal edge and letter (Section 4.2; Lemma 7 for length one). The outcome is independent of these choices by Corollary 22.
- (C) *Contraction.* The operation P of Section 4.3; expansion paths are concatenated.
- (R) *Cell refinement.* $K_{t+1} = fK_t$; a vertex (q, j) is replaced by $(q, fj+i)$, $0 \le i < f$, and each edge by all refinements of its expansion path that satisfy the edge conditions of $G(M_t, K_{t+1})$ at every step.
- (L) *Modulus enlargement.* $M_{t+1} = hM_t$; each edge is replaced by all sequences of residues modulo M_{t+1} along its expansion path that are units and satisfy (E2) at every step (all solutions, with no inverse of b assumed). Lifting by a prime $p \nmid ab$ (Section 4.4) is the case $h = p$, and in the second form of (I) the new primes do not divide $2ab$.
- (A) *Carry restriction.* If $\mathrm{rad}(2ab) \mid M_t$, restrict labels to $\mathcal{C}(a, b)$.

Let $\mathfrak{M}_0$ be the procedure allowing exactly these operations. An execution *succeeds* if $\Gamma_s = \varnothing$. The executions of Section 5 are instances: Table 1 uses (I), (P), (R); Tables 4 and 6 use the second form of (I) with $R_0 = 1$, then (P), (C), (L).

Soundness. Let ω be an infinite walk (right- or left-directed) of $G(M_*, K_*)$ with $M_t \mid M_*$ and $K_t \mid K_*$, whose output is not eventually periodic (in the direction of the walk). We say that *a tail of ω becomes a walk of Γ_t* if there is an infinite walk of Γ_t whose successive expansion paths coincide with the projection of the state sequence of ω to (M_t, K_t), and whose output is a tail of the carry sequence of ω. An operation is

- *walk-sound* if it preserves this property for every such ω;
- *pseudo-orbit-sound* if it preserves it for every M-pseudo-orbit with non-eventually-periodic carries, $M_t \mid M$, in place of ω;
- *orbit-sound* if it preserves it for every true orbit tail avoiding the primes of the final modulus.

Walk-soundness implies pseudo-orbit-soundness, which implies orbit-soundness, by the inclusions above. Orbit-soundness of the operations is the content of Theorems 10 and 19.

Lemma 30 (preservation)**.** *The operations (P), (C), (R), (L), (A) are walk-sound, for right- and for left-directed walks.*

Proof. (R) By Lemma 31 below, the projection of ω to (M_t, K_{t+1}) is a walk of $G(M_t, K_{t+1})$ whose further projection to (M_t, K_t) is the current expansion path; so it is one of the refined edges.

(L) Likewise, the residues of ω modulo M_{t+1} are units satisfying (E2), hence among all solutions.

(A) By Lemma 29, every label of ω lies in $\mathcal{C}(a, b)$ once $\mathrm{rad}(2ab) \mid M_t \mid M_*$.

(P) A right-directed infinite walk of Γ_t eventually stays in one cyclic SCC C (Lemma 6). For a left-directed walk, the set of times spent in each SCC is an interval, since an SCC cannot be re-entered; finitely many intervals cover $\mathbb{Z}_{\le n_0}$, so the leftmost is unbounded below, and that SCC contains an internal edge. If C were certified, the output of the tail would be periodic: for right-directed walks by Lemma 12(b); for left-directed walks, the congruence of Lemma 12(a) propagated backwards shows that the letter at time $t \le -1$ (relative to the last vertex v_0) equals $\lambda((h(v_0)+t) \bmod d)$. This contradicts the non-periodicity of the tail of ω. Hence C is retained and the tail is a walk of the retained subgraph.

(C) For right-directed walks this is Proposition 16. For a left-directed walk, if it avoided the macro vertex set B from some point to the left, all its vertices there would have out-degree one; by finiteness some vertex repeats, $v_m = v_{m'}$ with $m < m'$, and $Z = \{v_m, \dots, v_{m'-1}\}$ is a cycle of out-degree-one vertices closed under successors. Its SCC is then Z itself, all of whose vertices have out-degree one, so its anchor lies in $B \cap Z$, contradicting avoidance of B. Hence the walk visits B infinitely often to the left, and its segments between visits are macro edges, as in Proposition 16. □

Any further operation that is pseudo-orbit-sound in both directions may be adjoined to $\mathfrak{M}_0$ without affecting the obstruction theorems of Section 8. An example, the whole-word interval test, is treated in Appendix E.4 (Proposition 73); it is pseudo-orbit-sound but not walk-sound, and it is excluded from Theorem 33.

7.3. **Projection and inheritance of certification.**

Lemma 31 (projection)**.** *Let $M \mid M'$ and $K' = fK$. The projection π maps every edge $(q, J) \xrightarrow{e} (z, H)$ of $G(M', K')$ to an edge of $G(M, K)$ with the same label.*

Proof. (E1) is unchanged and (E2) reduces modulo M. For (E3), write $J = fj + i$, $H = fk + \ell$ with $0 \le i, \ell < f$. Then $eK' = efK$ and

$$\begin{aligned} f\big(aj - b(k+1)\big) &\le afj + ai - bfk - b(\ell+1) \\ &< efK < afj + a(i+1) - bfk - b\ell \\ &\le f\big(a(j+1) - bk\big), \end{aligned}$$

the outer inequalities using $i \ge 0$, $\ell + 1 \le f$ and $i + 1 \le f$, $\ell \ge 0$. Dividing by f gives (E3) for (j, k). □

Lemma 32 (inheritance)**.** *In the setting of Lemma 31, let $W \subseteq V(M, K)$ and $W' \subseteq V(M', K')$ with $\pi(W') \subseteq W$, and let C' be a cyclic SCC of $G_{K'}(W')$ with root ρ' and root path lengths h'. Then $\pi(C')$ lies in one cyclic SCC C of $G_K(W)$, with root path lengths h and gcd d; $d \mid d'$; and if C is certified then so is C'.*

Proof. By Lemma 31 and $\pi(W') \subseteq W$, the projection of every internal edge of C' is an edge of the induced subgraph $G_K(W)$; hence the image of C' is strongly connected in $G_K(W)$ and contains an internal edge, so it lies in a cyclic SCC C.

Projecting the chosen path from ρ' to x and using (15) along it gives

$$h(\pi x) \equiv h(\pi\rho') + h'(x) \pmod{d}.$$

For an internal edge $u' \to v'$ of C',

$$h'(u') + 1 - h'(v') \equiv h(\pi u') + 1 - h(\pi v') \equiv 0 \pmod{d},$$

so $d \mid d'$.

If C is certified, the label of $u' \to v'$ is

$$\lambda_{h(\pi u') \bmod d} = \lambda_{(h(\pi\rho')+h'(u')) \bmod d},$$

which depends only on $h'(u') \bmod d'$; hence C' is certified. □

7.4. **Normal form of finite certificates.**

Theorem 33 (contraction does not enlarge the class)**.** *Let $\Gamma_0, \dots, \Gamma_s$ be a successful execution of $\mathfrak{M}_0$, and let $M_* = M_s$ (with $\mathrm{rad}(2ab) \mid M_*$ in the second form of (I)) and $K_* = K_s$. Then every cyclic SCC of the complete one-step graph $G(M_*, K_*)$ passes the phase test of Section 3.2. Conversely, a one-step certificate as in Theorem 10 (or Lemma 11) is an execution of $\mathfrak{M}_0$ without (C).*

Proof. Suppose $G(M_*, K_*)$ has an infinite walk ω with non-eventually-periodic output. Its projection to (M_0, K_0) is a walk of Γ_0: in the first form of (I) by Lemma 31; in the second form, its labels lie in $\mathcal{C}(a, b)$ by Lemma 29 and its residues modulo R_0 satisfy (E2).

By Lemma 30, inductively, a tail of ω becomes a walk of every Γ_t, including $\Gamma_s = \varnothing$: a contradiction.

Hence every infinite walk of $G(M_*, K_*)$ has eventually periodic output, in particular every infinite walk inside a cyclic SCC, and Theorem 21 ((b)⇒(a)) shows that every cyclic SCC is certified. The converse is clear. □

Consequently, for a fixed K and a fixed set of primes, the order in which primes are added does not affect whether an execution can succeed; it affects only the intermediate sizes. The conclusion "no walk of $G(M_*, K_*)$ has non-eventually-periodic output" does not use Theorem 21; only its restatement as a phase test does.

Theorem 34 (projection monotonicity)**.** *Let a one-step execution with operations (P), (R), (L) start from $W_0 = V(M_0, K_0)$, with $U_t = R_{K_t}(W_t)$ and W_{t+1} the full enlargement of U_t (Lemmas 9 and 11), and suppose $U_s = \varnothing$. Then*

$$R_{K_s}(V(M_s, K_s)) = \varnothing :$$

the complete graph at the final parameters certifies in one stage. Consequently, if $M_s \mid M'$ and $K_s \mid K'$, then

$$R_{K'}(V(M', K')) = \varnothing$$

as well; in particular the pairs $(s!, s!)$, $s = 1, 2, \dots$, miss no one-step certificate. The certificate at (M', K') proves only $\gcd(Q_n, M') > 1$ infinitely often; it does not recover the conclusion for the original prime set.

Proof. We show

$$\pi_t(R_{K_s}(V(M_s, K_s))) \subseteq U_t$$

by induction on t, where π_t projects to (M_t, K_t). For $t = 0$, Lemma 32 applies since W_0 is the full vertex set.

For the step, let v lie in an uncertified cyclic SCC C' of the final graph. By induction

$$\pi_t(C') \subseteq U_t,$$

hence

$$\pi_{t+1}(C') \subseteq W_{t+1}.$$

The set $\pi_{t+1}(C')$ is strongly connected inside $G_{K_{t+1}}(W_{t+1})$ (an induced subgraph keeps the edges) and lies in one cyclic SCC C; if C were certified, Lemma 32 would certify C'. So $C \subseteq U_{t+1}$ and $\pi_{t+1}(v) \in U_{t+1}$.

At $t = s$, π_s is the identity.

The last assertion follows from Lemma 32 applied to the projection from (M', K') to (M_s, K_s) with $W = V(M_s, K_s)$, $W' = V(M', K')$. □

This is relative completeness only: a search over the pairs $(s!, s!)$ finds a certificate whenever one exists at some parameters, but with the weaker conclusion $\gcd(Q_n, s!) > 1$, and no termination is asserted.

Proposition 35 (prime powers)**.** *For every M and K, the sets of infinite output words (together with their cell sequences) of $G(M, K)$ and of $G(\operatorname{rad} M, K)$ coincide.*

Proof. One inclusion is Lemma 31. Conversely, let $(\bar{q}_n, j_n)_{n\geq 0}$ be an infinite walk of $G(\operatorname{rad} M, K)$ with output (c_n). For each $p^e \parallel M$ we construct units q_n modulo p^e with

$$q_n \equiv \bar{q}_n \pmod{p} \qquad \text{and} \qquad bq_{n+1} \equiv aq_n + c_n \pmod{p^e}.$$

If $p \nmid b$: take any lift q_0 of $\bar{q}_0$ and

$$q_{n+1} = b^{-1}(aq_n + c_n) \bmod p^e;$$

modulo p this is the same recurrence with the same initial value, so $q_n \equiv \bar{q}_n$.

If $p \mid b$ (so $p \nmid a$): put

$$q_n = -\sum_{j=0}^{e-1} b^j a^{-j-1} c_{n+j} \bmod p^e.$$

Then

$$\begin{aligned} bq_{n+1} - aq_n &= -\sum_{j=1}^{e} b^j a^{-j} c_{n+j} + \sum_{j=0}^{e-1} b^j a^{-j} c_{n+j} \\ &= c_n - b^e a^{-e} c_{n+e} \\ &\equiv c_n \pmod{p^e}, \end{aligned}$$

and modulo p,

$$q_n \equiv -a^{-1} c_n \equiv \bar{q}_n$$

because the original walk satisfies $0 \equiv a\bar{q}_n + c_n \pmod{p}$.

Combine by the Chinese remainder theorem; (E3) involves only cells and labels. □

Proposition 36 (primes dividing $2ab$)**.** *Let $M_0 = \operatorname{rad}(2ab)$ and $\gcd(R, M_0) = 1$. The sets of infinite output words (with cell sequences) of $G(M_0 R, K)$ and of $G'(R, K)$ (defined in (I)) coincide.*

Proof. A walk of $G(M_0 R, K)$ has labels in $\mathcal{C}(a, b)$ by Lemma 29 and residues modulo R satisfying (E2); this gives a walk of $G'(R, K)$.

Conversely, given a walk $(\bar{q}_n, j_n)$ of $G'(R, K)$ with output $c_n \in \mathcal{C}(a, b)$, define residues for each prime $p \mid M_0$: if $p \mid a$, let q_0 be any unit and

$$q_{n+1} \equiv b^{-1} c_n$$

(then $bq_{n+1} \equiv c_n \equiv aq_n + c_n$); if $p \mid b$, let

$$q_n \equiv -a^{-1} c_n$$

(then $bq_{n+1} - aq_n \equiv c_n$); if $p = 2$ and a, b are odd, let $q_n = 1$ (then $b \equiv a + c_n \pmod 2$ since c_n is even). All are units since $p \nmid c_n$. Combine with $\bar{q}_n$ by the Chinese remainder theorem. □

Define the class of *finitely certifiable bases*

$$\mathcal{F}_{\mathrm{cert}} = \Big\{ \tfrac{a}{b} : \ a > b \geq 2, \ \gcd(a,b) = 1, \\ \exists K \ \exists S \text{ finite}, \ p \nmid 2ab \ (p \in S) : \\ \text{every cyclic SCC of } G'(\textstyle\prod_{p\in S} p, K) \\ \text{satisfies Theorem 21(c)} \Big\}.$$

Theorem 37 (normal form)**.** *For a reduced base a/b the following are equivalent.*

1. *Some execution of $\mathfrak{M}_0$ succeeds.*
2. *Some one-step execution (operations (I), (P), (R), (L), first form) succeeds for some modulus M and some sequence of cell counts.*
3. *There are K and a finite set S of primes not dividing $2ab$ such that every cyclic SCC of $G'(\prod_{p\in S} p, K)$ passes the phase test.*
4. *$a/b \in \mathcal{F}_{\mathrm{cert}}$.*

Proof. (3)⇔(4) is Theorem 21.

(3)⇒(2): by Proposition 36 and Theorem 21, every cyclic SCC of $G(M_0 R, K)$ is certified, which is a one-step execution with $s = 0$.

(2)⇒(1) is clear.

(1)⇒(3): by Theorem 33 every cyclic SCC of $G(M_*, K_*)$ is certified. By Lemma 31 and Theorem 21, the same holds for $G(M_0 M_*, K_*)$ with $M_0 = \mathrm{rad}(2ab)$: every infinite walk of the finer graph projects to one with eventually periodic output.

By Proposition 35 it holds for

$$G(\mathrm{rad}(M_0 M_*), K_*) = G(M_0 R, K_*),$$

where R is the product of the primes of M_* not dividing $2ab$; by Proposition 36 and Theorem 21 it holds for $G'(R, K_*)$. □

The equivalence concerns the *existence* of some certificate, not the prime set of the conclusion. A certificate in (2) proves $\gcd(Q_n, M) > 1$ infinitely often; a certificate in (3) proves $\gcd(Q_n, 2abR) > 1$ infinitely often, a conclusion whose prime set contains every prime dividing $2ab$. For example, (1) uses $M = 4290$, which does not contain 7, while the normal form for $7/5$ uses $\{2, 5, 7\} \cup \{3, 11, 13\}$. Membership of $3/2, 4/3, 5/4, 7/5, 5/2$ in $\mathcal{F}_{\mathrm{cert}}$ follows from the certificates of Section 5 (Tables 1, 3, 4, 6), and membership of $5/3$ is Corollary 77 in Appendix F; no other base is claimed to belong to it, and no simple inequality or congruence in a, b characterizes the class.

8. Obstructions and the external-prime budget

The results of this section have the following common form: for a given modulus M, *no execution of $\mathfrak{M}_0$ all of whose moduli M_t divide M succeeds*; the same holds when $\mathfrak{M}_0$ is extended by operations that are pseudo-orbit-sound in both directions. They are statements about the non-termination of a procedure. They do *not* imply

$$\exists\, \xi > 0 \ \exists\, n_0 \ \forall n \geq n_0 : \ \gcd(\lfloor \xi r^n \rfloor, M) = 1,$$

and they are not counterexamples to any statement about composite terms. The witnesses are pseudo-orbits, which lack an integer sequence linking residues and fractional parts; Lemma 41 shows that every finite window of such a witness is realized by a true orbit, in the form $\forall N \ \exists \xi_N$, which cannot be exchanged for $\exists \xi \ \forall N$. Procedures that use information

available only for true orbits (integrality of Q_n, forward realizability, or the recurrence hypothesis of Section 10) are not covered by these theorems. Corollary 78 in Appendix F.5 applies Lemma 69 to 5/3 with the primes up to 23.

Lemma 38 (type of the obstruction)**.** *Let $M \geq 2$. If there is a right- or left-directed M-pseudo-orbit whose carry sequence is not eventually periodic (in its direction), and whose carries lie in $\mathcal{C}(a,b)$ whenever the second form of (I) is used, then no execution of $\mathfrak{M}_0$ (or of an extension by pseudo-orbit-sound operations) with all $M_t \mid M$ succeeds.*

Proof. The pseudo-orbit yields, for every K, a walk of $G(M,K)$ with the same carries; its projection to (M_0, K_0) is a walk of Γ_0 (in the second form of (I), the labels lie in $\mathcal{C}(a,b)$ by hypothesis and the residues modulo R_0 satisfy (E2)). By Lemma 30, applied inductively, a tail of it becomes a walk of every Γ_t; hence $\Gamma_s \neq \varnothing$. □

8.1. **Large numerators.**

Theorem 39. *Let $M \geq 2$, $R = \operatorname{rad} M$, and*

$$a \geq 2R.$$

Then no execution of $\mathfrak{M}_0$ (or of an extension by pseudo-orbit-sound operations) with all $M_t \mid M$ and Γ_0 as in (I) succeeds. Hence a successful execution requires

$$a < 2\operatorname{rad}(M). \tag{27}$$

Proof. Step 1: a left-directed nonperiodic carry sequence. Put $\theta_0 = 0$. Given $\theta_{n+1} \in [0,1)$, choose

$$c_n \in [-b\theta_{n+1},\, a - b\theta_{n+1}) \cap (b - a + R\mathbb{Z}),$$
$$\theta_n = \frac{b\theta_{n+1} + c_n}{a}.$$

The half-open interval has length $a \geq 2R$, so it contains at least $\lfloor a/R \rfloor \geq 2$ terms of the progression. Then

$$0 \leq b\theta_{n+1} + c_n < a$$

gives $\theta_n \in [0,1)$, the identity

$$a\theta_n - b\theta_{n+1} = c_n$$

is exact, and $-b < c_n < a$ gives $1 - b \leq c_n \leq a - 1$. Using only two ordered choices at each step, the map $\{0,1\}^{\mathbb{N}} \to \Sigma^{\mathbb{Z}_{\leq 0}}$ is injective (at the first index where two choice sequences differ, the values θ_{n+1} agree and the c_n differ), so uncountably many sequences arise. A sequence $(c_n)_{n \leq 0}$ that is *left-eventually periodic* (there are $N \leq 0$ and $L \geq 1$ with $c_{n-L} = c_n$ for all $n \leq N$) is determined by $(N, L, c_{N-L+1}, \ldots, c_0)$, so there are only countably many. Choose a sequence that is not left-eventually periodic. Modulo R, $q_n = 1$ satisfies $bq_{n+1} \equiv aq_n + c_n$ because $c_n \equiv b - a \pmod{R}$.

Step 2: units modulo prime powers. For each $p^k \parallel M$ we define $q_n \equiv 1 \pmod{p}$ with

$$bq_{n+1} \equiv aq_n + c_n \pmod{p^k}.$$

If $p \nmid a$: $q_0 = 1$ and

$$q_n = a^{-1}(bq_{n+1} - c_n) \bmod p^k$$

for $n < 0$; since $c_n \equiv b - a \pmod{p}$, $q_{n+1} \equiv 1$ gives

$$q_n \equiv a^{-1}(b - (b - a)) = 1.$$

If $p \mid a$ (so b is invertible modulo p^k): choose L with $p^k \mid a^L$ and put

$$q_n = \sum_{j=0}^{L-1} a^j b^{-j-1} c_{n-1-j} \bmod p^k.$$

Then

$$\begin{aligned} bq_{n+1} - aq_n &= \sum_{j=0}^{L-1} a^j b^{-j} c_{n-j} - \sum_{j=1}^{L} a^j b^{-j} c_{n-j} \\ &= c_n - a^L b^{-L} c_{n-L} \\ &\equiv c_n \pmod{p^k}, \end{aligned}$$

and modulo p,

$$q_n \equiv b^{-1} c_{n-1} \equiv b^{-1}(b-a) \equiv 1.$$

All indices used are ≤ -1.

The Chinese remainder theorem gives $\gcd(q_n, M) = 1$ and

$$bq_{n+1} \equiv aq_n + c_n \pmod{M}.$$

Step 3. $(q_n, \theta_n, c_n)_{n \leq 0}$ is a left-directed M-pseudo-orbit with non-left-eventually-periodic carries. If $\mathrm{rad}(2ab) \mid M$, then $c_n \equiv b - a \pmod{p}$ for every $p \mid 2ab$ gives $c_n \in \mathcal{C}(a, b)$ (for $p \mid a$, $c_n \equiv b$; for $p \mid b$, $c_n \equiv -a$; for $p = 2$ with a, b odd, c_n is even). Lemma 38 applies. □

The hypothesis on the initial graph is essential: a procedure that starts from a different sound initial restriction derived from properties of true orbits does not belong to $\mathfrak{M}_0$, and the witness need not pass it.

Corollary 40. *(a) For a fixed finite set $\mathcal{P}$ of primes, the reduced bases a/b for which some execution of $\mathfrak{M}_0$ with moduli composed of primes in $\mathcal{P}$ (with any exponents) succeeds are finitely many:*

$$a < 2 \prod_{p \in \mathcal{P}} p$$

and $b < a$.

(b) *The bases $r_t = (2t+7)/(2t+5)$, $t \geq 0$, are reduced. A successful execution for r_t with final modulus M_t requires*

$$\mathrm{rad}(M_t) > (2t+7)/2,$$

that is,

$$\mathrm{rad}(M_t) \geq t + 4.$$

With the primes $\{2, 3, 5, 11, 13\}$ of (1) alone, no execution succeeds for $t \geq 4287$. The same holds for $(14k+1)/(10k+1)$, which tend to $7/5$, for $k \geq 613$.

Proof. (a) is (27). (b) The pairs are coprime: their differences are 2 and their members are odd for the first family; for the second, a common divisor divides

$$5(14k+1) - 7(10k+1) = -2$$

and both members are odd. The thresholds are

$$2t + 7 \geq 8580 \iff t \geq 4287$$

(as $2 \cdot 4286 + 7 = 8579$) and

$$14k + 1 \geq 8580 \iff k \geq 613.$$ □

Part (b) says that the certificate for 7/5 does not transfer to nearby bases by the same prime set; it says nothing about composite terms for those bases.

8.2. **Finite windows are realized by true orbits.**

Lemma 41. *Let integers q_i with $\gcd(q_i, M) = 1$, reals $\theta_i \in [0, 1)$ $(0 \le i \le L)$, and integers c_i $(0 \le i < L)$ satisfy*

$$a\theta_i - b\theta_{i+1} = c_i \qquad \text{and}$$
$$bq_{i+1} \equiv aq_i + c_i \pmod{M}.$$

Then for every T_0 there is $\xi > T_0$ with $\lfloor \xi r^i \rfloor \equiv q_i \pmod{M}$ and $\{\xi r^i\} = \theta_i$ for $0 \le i \le L$.

Proof. Write $Q_i = q_i + Mt_i$ with integers t_i to be found, and

$$d_i = (aq_i + c_i - bq_{i+1})/M \in \mathbb{Z}.$$

The requirement $bQ_{i+1} = aQ_i + c_i$ is

$$bt_{i+1} = at_i + d_i.$$

Put $S_0 = 0$, $S_{j+1} = aS_j + b^j d_j$, so that

$$b^j t_j = a^j t_0 + S_j$$

by induction.

Since $\gcd(a, b^L) = 1$, choose

$$t_0 \equiv -a^{-L} S_L \pmod{b^L}.$$

For $j \le L$,

$$S_L \equiv a^{L-j} S_j \pmod{b^j},$$

hence

$$a^j t_0 + S_j \equiv -a^j a^{-L} a^{L-j} S_j + S_j \equiv 0 \pmod{b^j},$$

so

$$t_j = (a^j t_0 + S_j)/b^j$$

is an integer, and $bt_{j+1} = at_j + d_j$ holds as an identity of rationals, hence of integers.

Adding $b^L T$ to t_0 adds $a^j b^{L-j} T$ to t_j; taking T large makes all Q_i positive and as large as desired.

Finally $x_i = Q_i + \theta_i$ satisfies

$$bx_{i+1} = aQ_i + c_i + b\theta_{i+1} = ax_i,$$

so with $\xi = x_0$ we get $x_i = \xi r^i$ and $\lfloor x_i \rfloor = Q_i$. □

8.3. **The two-carry obstruction.** Let $\Delta = a - b$ (not to be confused with the phase gcd d).

Theorem 42. *Let $M \ge 2$ and let R be the product of the primes dividing M that do not divide $2ab$ ($R = 1$ allowed). Suppose there is an integer $m \ge 1$ with*

$$h = 2Rm \le a - 1, \qquad c = \Delta - h, \qquad \gcd(c, ab) = 1. \tag{28}$$

Then no execution of $\mathfrak{M}_0$ (or of an extension by pseudo-orbit-sound operations) with all $M_t \mid M$ succeeds.

Proof. *Carry conditions.* We have

$$1-b \le c < \Delta \le a-1, \qquad c \equiv \Delta \pmod 2 \quad (h \text{ is even}),$$
$$c \equiv \Delta \pmod R,$$
$$\text{and} \quad \gcd(\Delta, ab) = \gcd(a-b, a)\gcd(a-b, b) = 1;$$

so both c and Δ lie in $\mathcal{C}(a,b)$. (The value $c=0$ is excluded by $\gcd(c,ab)=1$.)

Fixed points and an invariant interval. Put

$$\psi_\Delta(x) = (bx+\Delta)/a = 1 + \tfrac{b}{a}(x-1) \qquad \text{and}$$
$$\psi_c(x) = (bx+c)/a = \psi_\Delta(x) - h/a;$$

these are the backward maps $\theta_{n+1} \mapsto \theta_n$ for the carries Δ and c.

For the word $W_L = \Delta^L c$ (in forward time order), the backward composition is

$$\Phi_L = \psi_\Delta^L \circ \psi_c,$$
$$\Phi_L(x) = 1 + \left(\tfrac{b}{a}\right)^{L+1}(x-1) - \tfrac{h}{a}\left(\tfrac{b}{a}\right)^L,$$

with fixed point

$$x_L = 1 - hb^L/(a^{L+1} - b^{L+1}).$$

Since $a > b$, $x_L < x_{L+1}$, and $x_L \uparrow 1$ with $x_L < 1$. As $c \ge 1-b$, $\max(0, -c/b) < 1$, so there is a least L with

$$\max(0, -c/b) < x_L;$$

put $I = [x_L, x_{L+1}]$. The maps Φ_L, Φ_{L+1} are increasing affine maps with slopes in $(0,1)$ and fixed points at the left and right endpoints of I respectively, so both map I into I.

For $x \in I$, $\psi_c(x) > 0$ because $x > -c/b$, and

$$\psi_c(x) < \psi_\Delta(x) < 1$$

because $c < \Delta$ and $x < 1$; ψ_Δ maps $(0,1)$ into $(\Delta/a, 1)$. Hence all intermediate values of a backward application of W_L or W_{L+1} starting in I lie in $(0,1)$.

Infinite concatenation. Let $U = W_L$, $V = W_{L+1}$, and let $B_1 B_2 \cdots$ be any sequence of blocks in $\{U, V\}$, block i starting at time n_i ($n_1 = 0$). For $k \ge i$ the sets

$$I_i^{(k)} = \Phi_{B_i} \circ \cdots \circ \Phi_{B_k}(I)$$

are closed subintervals of I, decreasing in k, of length at most

$$(b/a)^{(L+1)(k-i+1)}|I| \to 0;$$

their intersection is a single point $\theta_{n_i} \in I$, and

$$\Phi_{B_i}(\theta_{n_{i+1}}) = \theta_{n_i}$$

by construction.

Inside each block the values θ_n are defined by the backward recursion from $\theta_{n_{i+1}}$, whose composition is Φ_{B_i}. Thus $0 < \theta_n < 1$ and

$$a\theta_n - b\theta_{n+1} = c_n$$

exactly for all $n \ge 0$.

Nonperiodic coding. The words UV and VU have the same length $2L+3$ and differ at position L (letters c and Δ). Encode a binary sequence that is not eventually periodic (for instance $1\,0\,1\,00\,1\,000\cdots$) by $0 \mapsto UV$, $1 \mapsto VU$. If the carry sequence were eventually periodic with period P, it would have period $P(2L+3)$ and block decoding would make the binary sequence eventually periodic.

Unit residues modulo M. Every prime of M divides $2abR$. For $p^e \parallel M$: if $p \mid a$, take $q_0 = 1$ and

$$q_{n+1} = b^{-1}(aq_n + c_n) \bmod p^e;$$

modulo p,

$$q_{n+1} \equiv b^{-1}c_n \neq 0$$

since $c_n \in \{c, \Delta\}$ is coprime to a.

If $p \mid b$, take

$$q_n = -\sum_{j=0}^{e-1} b^j a^{-j-1} c_{n+j} \bmod p^e;$$

as in the proof of Proposition 35,

$$bq_{n+1} - aq_n \equiv c_n \pmod{p^e}, \qquad \text{and}$$
$$q_n \equiv -a^{-1}c_n \neq 0 \pmod{p}.$$

If $p \mid R$, take $q_0 \equiv -1 \pmod{p^e}$ and

$$q_{n+1} = b^{-1}(aq_n + c_n) \bmod p^e$$

(b is a unit modulo p^e since $p \nmid 2ab$); reducing modulo p, $c_n \equiv \Delta$ gives

$$q_{n+1} \equiv b^{-1}(-a + a - b) = -1,$$

so every q_n is a unit.

If $p = 2$ and a, b are odd, then c_n is even; take q_0 odd, and

$$q_{n+1} \equiv q_n + c_n \equiv q_n \pmod{2}.$$

Combine by the Chinese remainder theorem.

The result is a right-directed M-pseudo-orbit with carries in $\mathcal{C}(a, b)$ that are not eventually periodic. Lemma 38 applies. □

Corollary 43 (external primes are necessary)**.** *For every reduced base $a > b \geq 2$, no execution of $\mathfrak{M}_0$ whose moduli involve only primes dividing $2ab$ succeeds.*

Proof. Take $R = 1$ in Theorem 42. If Δ is odd and ≥ 3, take

$$c = 1, \qquad h = \Delta - 1 \geq 2.$$

If Δ is even and ≥ 4, then a, b are odd, $c = 2$ is coprime to ab, and

$$h = \Delta - 2 \geq 2.$$

If $\Delta = 1$, take

$$c = -1 \geq 1 - b, \qquad h = 2 \leq a - 1$$

(as $a \geq 3$).

If $\Delta = 2$, then a, b are odd, $b \geq 3$, $a \geq 5$, and

$$c = -2 \geq 1 - b, \qquad h = 4 \leq a - 1.$$

In every case $h \leq a - 1$ is even and positive, so (28) holds with $m = h/2$. □

For 7/5 with $M = 4290$ we have $R = 3 \cdot 11 \cdot 13 = 429$ and $h = 858m \leq 6$ is impossible, consistent with Theorem 1(i).

8.4. **The external-prime budget.** Let $J(N)$ denote the Jacobsthal function: the least positive integer J such that every J consecutive integers contain one coprime to N; $J(1) = 1$. Equivalently, $J(N)$ is the largest gap between consecutive integers coprime to N. Let

$$N_0 = \prod\{p \mid ab : p \text{ odd}\}$$

and $k = \omega(N_0)$ its number of prime factors; since $a > b \geq 2$ are coprime, $N_0 \geq 3$.

Theorem 44. *Let R be as in Theorem 42. If some execution of $\mathfrak{M}_0$ with all $M_t \mid M$ succeeds, then*

$$a \leq 2R\, J(N_0). \tag{29}$$

Moreover, for every odd squarefree N with $\omega(N) = k$,

$$J(N) \leq 3^k;$$

hence

$$R \geq \frac{a}{2 \cdot 3^k}, \qquad 2^k k! \leq N_0 < a^2, \tag{30}$$

so that $3^k = a^{o(1)}$ and

$$\liminf \log R / \log a \geq 1$$

along any sequence of successful certificates with $a \to \infty$. If the primes available for R are restricted to a fixed finite set with product R_0, then

$$a \leq 2R_0\, 3^{k_0 - 1},$$

where $k_0 \geq 4$ is any integer with

$$2^{k_0} k_0! > 4R_0^2\, 9^{k_0};$$

in particular only finitely many reduced bases admit a successful execution with that set.

Proof. By Theorem 42, success implies

$$\gcd(\Delta - 2Rm, ab) > 1$$

for every

$$1 \leq m \leq H := \lfloor (a-1)/(2R) \rfloor.$$

Since $2R$ is invertible modulo N_0, let

$$u \equiv \Delta(2R)^{-1} \pmod{N_0};$$

for an odd prime $p \mid ab$, $p \mid \Delta - 2Rm$ if and only if $m \equiv u \pmod{p}$. The prime 2 never divides $\Delta - 2Rm$ when $2 \mid ab$ (then Δ is odd), and is not a factor of ab otherwise.

Hence

$$\gcd(\Delta - 2Rm, ab) = 1 \iff \gcd(m - u, N_0) = 1.$$

If $H \geq J(N_0)$, the H consecutive integers $1 - u, \ldots, H - u$ contain one coprime to N_0, giving a witness m. So success forces $H \leq J(N_0) - 1$, that is,

$$(a-1)/(2R) < J(N_0),$$

which is (29).

For the bound on J: in a window of H consecutive integers, the number coprime to N is

$$\sum_{t \mid N} \mu(t)\, \#\{x : t \mid x\}, \qquad \text{and} \qquad \#\{x : t \mid x\} = H/t + \epsilon_t$$

with $|\epsilon_t| < 1$, $\epsilon_1 = 0$; so the count is at least

$$H\varphi(N)/N - (2^k - 1).$$

As N is odd,

$$\varphi(N)/N = \prod(1 - 1/p) \geq (2/3)^k,$$

and $H = 3^k$ gives a count ≥ 1.

The first inequality in (30) follows from (29); the second from

$$N_0 \geq \prod_{i \leq k}(2i + 1) > 2^k k! \qquad \text{and} \qquad N_0 \leq ab < a^2.$$

From

$$k! < a^2 \qquad \text{and} \qquad k! \geq (k/e)^k$$

one gets

$$k \log k \ll \log a,$$

so $k = O(\log a / \log\log a)$ and

$$3^k = a^{O(1/\log\log a)}.$$

For the effective bound, the ratio $2^k k!/(4R_0^2\, 9^k)$ is multiplied by $2(k+1)/9 > 1$ when $k \geq 4$, so it exceeds 1 for all $k \geq k_0$.

If $k \geq k_0$, then

$$a^2 > N_0 \geq 2^k k! > 4R_0^2\, 9^k \geq (2R_0\, 3^k)^2 \geq a^2$$

by (29) and $J(N_0) \leq 3^k$, a contradiction; so $k \leq k_0 - 1$ and

$$a \leq 2R_0\, 3^{k_0 - 1}.$$ □

The condition (29) is necessary, not sufficient: closing every $m \leq H$ does not guarantee a certificate.

Corollary 45. *(a) If $a = p$ is an odd prime and $b = 2^s < p$, then $N_0 = p$, $J(p) = 2$, and a successful execution requires*

$$R \geq p/4.$$

(b) If a, b are distinct odd primes, then $J(ab) = 3$ and a successful execution requires

$$R \geq a/6.$$

Proof. Among two consecutive integers at most one is divisible by p, so $J(p) = 2$. Among three consecutive integers at most one is divisible by a and at most one by b, so $J(ab) \leq 3$; by the Chinese remainder theorem there are consecutive $x \equiv 0 \pmod a$, $x + 1 \equiv 0 \pmod b$, so $J(ab) = 3$. Apply (29). □

These are constraints on any certificate for the bases named; they are not statements about composite terms of those bases.

Remark (Iwaniec's bound). Let $C(k)$ be the maximal number of consecutive integers each divisible by one of k arbitrarily chosen primes. Iwaniec proved

$$C(k) \ll k^2 \log^2 k$$

IW78, Corollary, p.226; see also CW13, §1, where the same statement is written as $g(n) \leq X(k \log k)^2$ for $k = \omega(n)$. Since $J(N) \leq C(\omega(N)) + 1$ for $\omega(N) \geq 2$ and $J(N) = 2$ for $\omega(N) = 1$, this gives

$$J(N_0) \ll (k \log 2k)^2$$

with an absolute implied constant, and with $k \log 2k \ll \log a$ from (30), the condition (29) yields

$$R \gg \frac{a}{(\log a)^2}.$$

This sharpens (30) quantitatively. It is an external input, not a result of this paper; the constant is absolute but not specified here, and no theorem above depends on it. For explicit finite inputs, (29) and the gcd test of Theorem 42 are used directly.

The explicit certificates for the bases 101/2 and 101/100, the selection of primes that eliminate persistent return words, and the whole-word interval test are given in Appendix E.4.

9. Verification and reproducibility

The paper sources, computation code, Lean formal proofs, reference data, and reproduction instructions are publicly available at:

`https://github.com/ixixi/rational-floor-certificates`

The repository README describes the contents; REPRODUCE.md and ENVIRONMENT.md explain how to reproduce the results and set up the required environment. Appendix A gives the main commands.

9.1. **What the finite computations establish.** The table entries of Section 5, Table 7, and Table 17 are summaries, not the certificates. The mathematical obligations discharged by the finite computations, and checked element by element by two implementations, are the following.

1. *Initial coverage.* The initial vertex set is the full set of legal states (all of $V(M, K_0)$ for Table 1 and for Table 7; all K cells for Tables 4, 6, and 17).
2. *Complete edge sets.* Every edge satisfying (E1)–(E3), or the initial inequalities of Theorem 19, is present; the lift of every edge is computed for every starting residue.
3. *Residues at every intermediate time.* In a lift, all $\ell + 1$ residues along a word of length ℓ, including the endpoint, are nonzero.
4. *All-letter phase conditions.* For every certified component, (19) holds at every letter position of every internal edge, with a single label per phase.
5. *Rank monotonicity.* For Table 7, the inequalities of (Q-rank) and (Q-cyc) hold on every edge between distinct blocks, and the bounds $\rho \le 65$, $\eta \le 6$, $d_B \le 13$ hold.
6. *Retention.* Every vertex and every internal edge of every uncertified cyclic component is kept, and nothing else.
7. *Output-preserving contraction.* Macro vertices are exactly the vertices of out-degree at least two (plus anchors), each chain of out-degree-one vertices ends at a macro vertex, and macro words are the concatenations of the chain words; for the backward passes of Table 17, the same holds with in-degrees and backward chains.
8. *Final emptiness.* The last retained set (Table 1) or the last output graph (Tables 4, 6, and 17) is empty; for Table 7, no cyclic block is uncertified.

The strongly connected components themselves are verified by sufficient conditions: the proposed blocks are nonempty, disjoint, cover all vertices, are internally reachable forwards and backwards, and admit a strictly monotone integer rank on all edges between distinct blocks (acyclic quotient). These conditions determine the SCC partition. Agreement of counts or of hashes is recorded but is not the criterion; the criterion is the element-wise comparison of the complete data.

9.2. **The one-step computation for 7/5.** The one-step calculation of Tables 1–3 was performed by two implementations in Python's arbitrary-precision integer arithmetic, using only the standard library and no floating point in the proof computation. Implementation A enumerates target cells with (12) and computes SCCs by iterative Kosaraju traversal. Implementation B tests (E3) directly for every $k \in \{0, \dots, K-1\}$, caching the resulting lists by (j, e), and computes SCCs by an iterative form of Tarjan's algorithm T, §4. Both classify all unit residues by $bz \bmod M$ to solve (E2). B was developed from the mathematical specification and a common output schema, without reading or importing A or the original baseline program; its design, source, and outputs were fixed before the comparison with A.

The validation checks every vertex and labelled edge. It recomputes shortest distances, all edge differences, phase label collections, certification flags, and primitive words, checks that the retained set is exactly the union of uncertified cyclic components, and checks that every child in (16) is present in the next stage. For a rejected phase test, two internal edges with the same departure phase and distinct labels provide a checkable witness. Across the four main stages and the three first stages of Table 3, the comparison covers 97772 vertices, 168474 labelled edges, and 94006 SCCs; no mismatch in the scientific data was found. The recorded executions used CPython 3.11.13 for A and CPython 3.10.12 for B on Linux, each case with a 600-second wall-time limit and a 4 GiB address-space limit. Reaching a limit is recorded as `inconclusive` and supplies neither a counterexample nor a minimality claim; all adopted cases completed. The programs are included under `supplement/computations/implementation_a/` and `supplement/computations/implementation_b/`. The public verification procedure and the all-element comparison are described in the computation guide and the reproduction guide.

9.3. **The word-labelled computations.** The computations of Tables 4, 6, and 7 were performed by two implementations with disjoint scientific cores.

Implementation A (`word_a`) is a provenance-tracked working copy of the programs written in the original research on which this paper is based (C++ constructions driven by Python, with a Python third check for the first five stages of Table 6), together with a driver that fixes resource limits, records inputs and outputs, and writes the graphs in a normalized text format. These programs contain two constructions labelled "A" and "B"; the two share a common origin, file formats, and conventions, and their mutual agreement is a consistency check within one lineage, not the independent verification described next. The scientific cores are retained in the public sources. The provenance and packaging changes are described in the computation guide; the source manifest identifies every distributed source by SHA-256.

Implementation B (`word_b`) was written by a different programmer from a mathematical specification equivalent to Section 4 and the comparison contract `word-schema.md`, which fixes the definitions, the text format, and the normalization only, without reading the original programs, implementation A, or the tables of the original research notes. It consists of a C++17 production program and a separately written literal Python reference program (direct inequality tests for all (j, c, k), forward iteration for all starting residues, Kosaraju's algorithm, and Kahn's algorithm for acyclicity), whose normalized outputs coincide byte for byte on all graphs. B computes h as the word length of the first root path found, uses forward iteration rather than Lemma 18, and re-verifies its SCC partitions by the sufficient conditions of Section 9.1. Its outputs were fixed, with sizes and hashes recorded, before A's outputs were opened.

Comparison. B's comparator compares, for each of the 29 file pairs (ten input and ten output graphs for Table 6, four and four for Table 4, one graph for Table 7), the header

parameters, the vertex sets, and the sets of word-labelled triples (u, w, v), and checks the sort order, absence of duplicates, and consistency of each file. For Table 7 it compares the block partition as a family of member sets (independent of block numbering), the attributes of each block (size, internal edges, d_B, ρ, η), the phase-label sequences, and the absence of uncertified cyclic blocks. All 29 pairs agree element by element. The comparison procedure and reference data are given in the computation guide and checks/reference.json; Appendix A gives the public reproduction entry point. Both implementations compute h from total word lengths of paths inside each component from a common root. Corollary 22 therefore implies agreement of the reported d_B values, even when their roots or paths differ.

Environment and limits. A ran under Python 3.10.12 with g++ 11.4.0 on Linux, with wall-time and address-space limits of 600 s and 4 GiB per construction for Table 6 (1500 s overall) and 300 s and 4 GiB otherwise. B ran its C++ program under limits of 3600 s and 8 GiB (Table 6), 300 s and 4 GiB (Table 4), and 600 s and 4 GiB (Table 7), its Python reference under 1800 s and 8 GiB, and the comparison under 1800 s and 8 GiB. No limit was reached; a limit reached would have been recorded as `inconclusive`. Wall times are recorded in the run records and are not performance claims.

Meaning of independence. The independence concerns the scientific cores of the two implementations: edge enumeration, lifting, component decomposition, certification, and contraction were coded separately from the specification. It does not extend to the mathematical lemmas, which are common, to the shared text format, or to human refereeing, and it does not mean clean-room development in a legal sense. Agreement of the two implementations is evidence that they computed the same finite objects; the soundness of the operations (Section 4) and the passage to all $\xi > 0$ (Theorem 19) are mathematical statements proved above, not outputs of the programs.

9.4. **The Lean formalization and its boundary.** A Lean formalization covers all three parts of Theorem 1 and Corollary 2, with different trust bases for the finite evaluations. It uses Lean 4.29.1 and Mathlib at commit `5e932f97dd25535344f80f9dd8da3aab83df0fe6`, with transitive dependencies pinned in `supplement/lean/lake-manifest.json`.

The one-step certificate. The one-step formalization follows the integer-difference proof of Lemma 4, orbit inclusion (Proposition 5), a rank-and-phase version of tail preservation, and the finite iteration of Theorem 10.

The formal certificate avoids a formal proof of the SCC algorithms. At each stage it assigns each vertex to a block, and each block a natural number rank. On every edge between distinct blocks the rank must strictly decrease (the opposite orientation to the strictly increasing ρ of (Q-rank) in Section 6.2; either convention yields a well-founded order). Each block is designated as one of the following:

1. **Retained:** every vertex of the block is included in the retained set.
2. **Edgeless:** the block contains no internal edge.
3. **Periodic:** a positive integer d, a phase in $\{0, \dots, d-1\}$ for every vertex, and a label for every phase satisfy the phase-increment and departure-label conditions on every internal edge.

These are sufficient conditions for Lemma 8. Along an infinite walk the rank cannot strictly decrease infinitely often, so the walk eventually remains in one block. That block cannot be edgeless. If it is periodic, its label sequence contradicts Lemma 4. It must therefore be retained. An SCC decomposition supplies such blocks and ranks through its acyclic quotient, but the formal implication does not require the blocks to be identified as SCCs.

The abstract structure `FiniteSuccess` requires initial coverage of all legal states, the rank/phase condition on **every** edge satisfying (E1)–(E3) between current states, coverage of all retained-block vertices, inclusion of all refined states at the next stage, and an empty final retained set. The following declarations, all with namespace prefix `MathPaper.`, identify the principal formal dependencies.

Table 8: Principal Lean declarations. All names except those beginning with `WordExec.` have prefix `MathPaper.`; `Pipe.` denotes the executable word-labelled pipeline, and `WordExec.` the precompiled two-direction pipeline.

Mathematical content	Lean declaration
Growth (7)	`floor_orbit_eventually_gt`
Lemma 4	`floor_carry_not_eventually_periodic`
Proposition 5 and legal orbit states	`orbit_edge`, `orbit_state_legal`
Phase-periodic labels	`phase_labels_periodic`
Tail preservation from rank/phase data	`rank_certificate_retained_tail`, `certified_orbit_retained_tail`
Lemma 9	`floor_refinement_bounds`, `refined_state_mem`
Theorem 10 in certificate form	`finite_success_visits`
Conditional compositeness conclusion	`finite_success_composite`
Signed interval enumeration (12)	`edge_interval_iff`
Concrete four-stage certificate	`mainFiniteSuccess`
Theorem 1(i)	`floor_seven_fifths_visits`
Corollary 2 for 7/5	`floor_seven_fifths_composite`
Word-labelled covering and phase output, Lemma 12(b)	`Covers`, `phase_output_periodic`
Word-labelled tail preservation, Proposition 13	`WordCert.stage_tail`
Reaching macro vertices and preserving the output, Lemma 15 and the forward direction of Proposition 16	`macro_visited`, `chain_realized`, `contract_tail`
Prime lifting, Proposition 17	`runWord_orbit`, `lift_tail`
Theorem 19 and Corollary 20 in certificate form	`word_finite_success_visits`, `word_finite_success_composite`
Lemmas 23–24	`carry_alphabet_75`, `carry_alphabet_52`
Initial coverage, stage checking, lifting, and pipeline soundness	`Pipe.initialStage_sound`, `Pipe.checkStage_sound`, `Pipe.liftArr_sound`, `Pipe.wordPipeline_sound`
Finite word-labelled evaluations	`five_halves_ok`, `seven_fifths_ok`
Theorem 1(ii) and Corollary 2 for 5/2	`floor_five_halves_visits`, `floor_five_halves_composite`
Theorem 1(i), alternative word-labelled proof	`floor_seven_fifths_visits_word`
Lemma 75(a), in-degree-one contraction	`macroIn_recurrent`, `contractIn_tail`, `WordCertIn.stage_tail`
Theorem 76 in certificate form	`StageReduce.tail`, `word_finite_success_visits_B`, `word_finite_success_composite_B`
Lemma 74	`carry_alphabet_53`

Mathematical content	Lean declaration
Two-direction pipeline soundness	`WordExec.checkStageIn_sound`, `WordExec.compactStage_sem`, `WordExec.wordPipelineBidir_sound`
Finite two-direction evaluation	`five_thirds_ok`
Theorem 1(iii) and Corollary 2 for 5/3	`floor_five_thirds_visits`, `floor_five_thirds_composite`

The closed term `mainFiniteSuccess : FiniteSuccess 7 5 4290` discharges the certificate argument of the general reduction. The two one-step final theorems take only a real coefficient ξ, a proof of $\xi > 0$, and an arbitrary $N \in \mathbb{N}$, and return an $n \geq \max(N, 1)$ satisfying the respective conclusion, with no unproved finite-computation hypothesis. The finite checker does not assume that the edge list exported by Python is complete: proved enumeration lemmas place every edge satisfying (E1)–(E3) among three carry candidates, five residue lifts, and three target-cell candidates, and the checker tests the rank/phase condition on an outer approximation of these edges. The Boolean checks are proved by `decide +kernel`, split into smaller statements joined by proved composition lemmas. The compositeness proof uses the eventually valid bound $Q_n > M$ and defines compositeness as $Q_n = uv$ with integers $u, v > 1$, without dependence on prime factorization.

The complete one-step certificate, main module, and aggregate build succeeded. The axiom audit of 30 designated declarations, including `mainFiniteSuccess` and both final theorems, shows dependencies confined to Lean's standard `propext`, `Classical.choice`, and `Quot.sound`; no `sorryAx`, custom axiom, or external-computation axiom occurs (the formalization guide). An independent audit rebuilt all 79 modules of the one-step project from a fresh project cache and checked the types and axiom dependencies of 60 declarations (58 from the project and two independently written restatements of Theorem 1(i) and its corollary), with the same three standard axioms only. The declared scope and commands for repeating the type and axiom checks are given in the formalization guide. The rebuild used compiled library caches for the pinned Mathlib and its dependencies, whose revisions were checked against the lock file; it did not rebuild every dependency from source. For this one-step proof the trust base comprises the fixed Lean kernel, its standard axioms, and those fixed libraries. The declaration-level coverage is summarized in the formalization guide. The public build procedure is given in the reproduction guide.

The word-labelled certificates. The added formalization in `MathPaper/Word/` extends the block/rank method to nonempty edge words. Its covering relation records a walk and strictly increasing boundary times, with each edge word equal to the corresponding segment of the carry sequence. In a periodic block the phase advances by the full word length, and every letter must agree with its phase label. In a retained block the checker verifies a macro-vertex set and a remaining word and destination for each non-macro vertex. The consistency conditions make the remaining word length decrease along a non-macro path and preserve the concatenated output on passage to macro edges. Thus the formal proof uses the forward direction of Proposition 16; it need not show that every walk of the output graph expands to an input walk.

The kernel-checked theorem `Pipe.wordPipeline_sound` proves that `Pipe.wordPipeline a b K alphabet primes = true` implies `WordFiniteSuccess a b K alphabet primes`. The pipeline enumerates the initial cell edges using the strict inequalities (E3). At each stage, auxiliary algorithms compute proposed blocks, ranks, phases, contraction data, and

output edges. Their correctness is not assumed: the proved checker verifies the required conditions on every input edge. The proved lifting function considers every nonzero starting residue and every letter, including the residue at the final endpoint; its inverse of b is checked modulo the newly added prime. Vertex renumbering uses preservation of covering under an arbitrary vertex map, so no unproved injectivity assumption is needed. The pipeline accepts only after the final output is empty. Together with the two proved carry alphabet lemmas, its soundness theorem discharges the finite-certificate arguments of the general visitation and compositeness theorems.

The two accepted finite evaluations are

```
Pipe.wordPipeline 5 2 4 [-1, 1, 3] [3, 7, 11, 13, 17, 19, 23, 29, 31] = true
Pipe.wordPipeline 7 5 2048 [-4, -2, 2, 4, 6] [3, 11, 13] = true
```

They are proved by `native_decide`. In the fixed Lean version this mechanism asserts the native evaluation through a generated axiom for each theorem; it does not use `Lean.ofReduceBool`. The actual additional axioms, with namespace prefix `MathPaper.`, are `five_halves_ok._native.native_decide.ax_1_1` and `seven_fifths_ok._native.native_decide.ax_1_1`, respectively. Accordingly, the 5/2 final theorems depend on the first generated axiom in addition to the three standard axioms, and the alternative 7/5 theorem depends on the second. The word-labelled theory and checker soundness depend only on the standard axioms. No `sorryAx` occurs in the recorded dependencies. The added trust is in the compiler and native evaluation of the Lean pipeline, not in a Python or C++ success flag, exported graph, or hash. These final theorems take only $\xi > 0$ and N, with no certificate argument, and give exactly the quantifiers and moduli of Theorem 1(i), (ii) and the 5/2 part of Corollary 2.

The added modules, aggregate build, and axiom display were accepted. The formalization guide records their coverage and axiom boundary. A separate independent audit of the word-labelled proof rebuilt all 14 added Word modules, reran both native finite evaluations, and checked the types and axiom dependencies of 46 project declarations and five independent restatements or checks. It confirmed the trust boundary described above. That audit reused compiled one-step project dependencies and the fixed library caches; it was not a fresh recompilation of all one-step modules or of the libraries. An attempted `decide +kernel` evaluation of a smaller pipeline instance stopped at opaque hashing operations, so neither word-labelled evaluation is claimed as a kernel-only finite computation.

The certificate for 5/3. The in-degree-one contraction is formalized in the original direction, without reversal. In a retained block the checker verifies a macro-vertex set and, for every other vertex, a backward chain: a word read from a macro vertex and that vertex. The recorded chain lengths are bounded by a stated bound. Along a walk avoiding macro vertices the chain length increases strictly with every edge, so macro vertices recur, and consecutive visits delimit macro edges of the output graph with the same words. The formal proof uses neither SCCs nor the finiteness of the graph (`macroIn_recurrent`, `contractIn_tail`). A stage may consist of any finite sequence of checked forward and backward passes (`StageReduce`); between passes the words are compacted and the vertices renumbered, which preserves coverings through an arbitrary vertex map.

The executable two-direction pipeline is the module `WordExec`. It contains copies of the data types, checker, auxiliary computations, lift, and initial graph of the pipeline above, and adds the in-direction checker and computation, the compaction, and the iteration rule of Appendix F.3. It does not import Mathlib, and it is precompiled, so that the finite evaluation runs compiled code. The kernel-checked theorem `WordExec.wordPipelineBidir_sound` proves that `WordExec.wordPipelineBidir a b K alphabet primes = true`

implies `WordFiniteSuccessB a b K alphabet primes`, the certificate form of Theorem 76. The accepted finite evaluation is

```
WordExec.wordPipelineBidir 5 3 3 [-2, 2, 4] [7, 11, 13, 17, 19, 23, 29, 31, 37] =
↪  true
```

proved by `native_decide` with the generated axiom `five_thirds_ok._native.native_decide.ax_1_1`. With Lemma 74 (`carry_alphabet_53`), the final theorems `floor_five_thirds_visits` and `floor_five_thirds_composite` take only $\xi > 0$ and N and give the quantifiers and modulus of (2a) and the 5/3 part of Corollary 2. Their axiom dependencies are the three standard axioms and this generated axiom; the theory and checker soundness use only the standard axioms. The trust base is therefore of the same kind as for 5/2, with the compiled code of `WordExec` evaluated natively. The evaluation took 657 s with a peak resident set of 12.7 GiB on the machine of Section 9.5, more than the harness limit of the other modules; the separate limit for this module is stated in the formalization guide. The Lean pipeline uses the same iteration rule as the C++ implementations, and its per-stage counts agree with Table 17; they are diagnostics and not part of the proof. The independent audit of the word-labelled proof described below was made before this addition and does not cover the 5/3 modules.

Boundary. The formalization establishes the arithmetic conclusions of Theorem 1 and Corollary 2 with the trust bases just stated. It does not formally identify the reported blocks as SCCs, prove that h is a shortest distance or that d is the specified gcd, or establish the reported table counts and words. Those data are checked by the A/B computations; the Lean checker uses sufficient rank, phase, and contraction conditions. Within Section 4, the gcd positivity argument of Lemma 12(a), Lemma 14, the reverse correspondence in Proposition 16, the root formula of Lemma 18, and Theorem 21 with Corollary 22 are not formalized. Positive periods are required by the checker, and lifting uses the forward recurrence. Theorem 3, Section 6 (including the waiting-time certificate of Table 7), Sections 7–8, and the further results of Section 10 and Appendices B–E also remain outside the formalization. In Appendix F, Lemma 75(b), Corollaries 77 and 78, and the counts of Tables 17 and 18 are not formalized. Their evidence consists of the proofs in this paper and, where finite computation is required, the recorded A/B comparisons.

9.5. **The computation for 5/3.** The calculation of Table 17 was performed by two implementations with disjoint scientific cores. The obligations of Section 9.1 apply to it, with the backward passes of Appendix F.3 added to the contraction conditions.

The producer (`supplement/computations/five_thirds/`) is the program with which the certificate for 5/3 was first computed. It is a C++17 program that enumerates the initial edges, lifts every edge for every starting residue, computes SCCs, applies the phase test, and contracts in both directions, the backward pass being computed by reversal as in Lemma 75(b). An embedded checker, compiled into the same program, verifies during the full run the initial edges by direct inequalities, all lifted edges by a forward calculation of the surviving residue sets, sufficient rank and all-letter phase conditions on every edge (without assuming that the blocks are SCCs), and the reconstruction of every contracted edge. Counted with stage multiplicity, the full run checked 44536828 lifted edges, 126567962 rank, retention, and phase edges, 1508656 periodic-phase letters, and 86280584 contracted edges. The producer and its checker share representations and development lineage. A separate literal Python program of the same lineage reconstructs the first five stages, through the prime 17, and agrees on all ten of their input and output graphs (5834 vertices, 13260 edges, and 52757 letters in total).

Implementation B of Section 9.3 was extended by an option for backward passes: every edge and every word is reversed, removal and out-contraction are applied, and the result is reversed back, with the iteration rule of Appendix F.3. The extension was written from the statement of Lemma 75 and the iteration rule, without reading the producer, its checker, or its Python program; until the comparison was complete, these were only compiled and run. The default pipeline of B is unchanged: without the new option, all its graphs for Tables 4 and 6 are byte for byte identical to those of the previous version.

Comparison. The producer writes the input and output graphs of each stage with its own vertex numbering, recording only the cells. B names a vertex by its residue and cell; its vertices are ordered by cell and then by the residues modulo the added primes in the order of addition. In this order, both graphs of each pair are written in a common canonical text (the vertex cells and the sorted word-labelled triples), and the two texts are compared byte for byte. All 20 graphs of Table 17 agree, 35289089 vertices, 48786290 edges, and 1245585922 letters in total, as do the per-stage numbers of cyclic and certified components, retained vertices, letters, and maximal word lengths. B's graph files and the digests of the canonical texts are recorded in checks/reference.json. B without backward passes gives the first columns of Table 18.

Environment and limits. Both full runs used g++ 11.4.0 under Linux (WSL2) on one machine with 62 GiB of memory; B took 178 s with a peak resident set of 2.4 GiB, the producer 185 s with 2.5 GiB, and the comparison 223 s. The public entry point runs each command under the limits stated in the reproduction guide. Wall times are recorded in the run records and are not performance claims.

Scope. The independence concerns the scientific cores, as in Section 9.3. B's extension was written in a separate AI-assisted session from the same mathematical specification as the producer; no independent human review of the 5/3 computation is claimed. The companion obstruction of Appendix F.5 is checked by exact rational arithmetic from the recorded words, without a search graph, by one direct program; it has no second implementation. Section 9.4 describes the Lean formalization of Theorem 1(iii); its pipeline, which follows the same iteration rule, reproduces the per-stage counts of Table 17.

10. Further results deferred to the appendices

Two groups of further results are stated here without proof; the proofs are given in Appendices B and D, and the supporting computations were performed by the two implementations of Section 9.3 and compared element by element (Sections D.5 and E.2). They are not used in Sections 1–9.

Recurrent carries. An infinite word $w = w_0 w_1 \cdots$ over a finite alphabet is *recurrent* if for every $L \geq 1$ and every $B \geq 1$ there is $n \geq B$ with

$$w_{n+i} = w_i$$

for $0 \leq i < L$; every tail of a recurrent word is recurrent.

Theorem 46 (recurrent carries force composites; proof in Appendix B.3)**.** *Let $a > b \geq 2$, $\gcd(a, b) = 1$, $\xi > 0$, and* suppose that the carry sequence $c_0 c_1 \cdots$ of the orbit has a recurrent tail $c_{j_0} c_{j_0+1} \cdots$. *Then (Q_n) contains infinitely many composite terms. Moreover, if $j \geq j_0$ satisfies $Q_j > 1$ and $\gcd(Q_j, ab) = 1$, then Q_j divides infinitely many later terms.*

The recurrence assumption concerns the carry word itself. Confinement of a walk to a strongly connected component does not imply recurrence of its output.

Theorem 46 is a return argument in the spirit of a theorem of Dubickas Dub09, Theorem 4: if integers x_n satisfy $x_{n+d} = kx_n + F(x_{n+1}, \ldots, x_{n+d-1})$ with an integer $k \neq 0$ and

$F \in \mathbb{Z}[z_1, \ldots, z_{d-1}]$, then (x_n) is purely periodic modulo every q with $\gcd(k, q) = 1$, and it contains infinitely many composite terms if $|x_n| \to \infty$. Stephan S26, Proposition 3.1 applies the case $d = 1$ to ultimately periodic digit words in an integer base. For a nonintegral base the carry word is never eventually periodic (Lemma 4). In Theorem 46 the letters act on $\mathbb{Z}/M$ by the affine permutations $x \mapsto b^{-1}(ax + c)$, the word is only assumed to be recurrent, and the finite-prefix copying lemma (Lemma 49) replaces periodicity in producing a return to the initial residue (Theorem 50).

Finite-height results. For the bases 6/5 and 9/7, a sieve over carry words combined with exact interval and divisor tests covers all initial integers up to an explicit height.

Theorem 47 (finite height; proof in Appendix D.4)**.** *(a) If* $6 < \lfloor \xi \rfloor \le 5^{40}$*, then* $\lfloor \xi(6/5)^j \rfloor$ *is composite for some* $0 \le j \le 40$*. (b) If* $9 < \lfloor \xi \rfloor \le 7^{18}$*, then* $\lfloor \xi(9/7)^j \rfloor$ *is composite for some* $0 \le j \le 18$*.*

For 6/5 the shifted sequence $\lfloor \xi(6/5)^n \rfloor - 1$ is treated in DN, Theorem 3, p.637; Theorem 47(a) concerns the unshifted sequence on a finite range and is a different statement.

Appendix C treats the family $(6t + 3)/(6t + 1)$, for which the all-coefficient statement reduces to an exact integer halting problem (Theorem 58), and Appendix E treats arbitrarily long prime prefixes along that family (Theorem 68, using known results on narrow arithmetic progressions of primes as external inputs), explicit return-word obstructions for 9/7 and 6/5 (Corollary 70), and the items deferred from Section 8. Appendix F contains the 5/3 certificate used for Theorem 1(iii), together with Corollaries 77 and 78.

11. Open problems

1. *Further bases and infinite families.* It remains to determine which further rational bases admit finite certificates and to construct a uniform certificate for an infinite family. Appendix F adds 5/3 to the bases known to belong to $\mathcal{F}_{\mathrm{cert}}$. Theorem 44 gives a necessary budget of external primes, $R \ge a^{1-o(1)}$, but no construction meeting it. Eliminating specific return words (Appendix E.3) and covering all recurrent components are different tasks.
2. *Finite height versus all coefficients.* Theorem 47 leaves the initial integers above the stated heights untreated. For an initial integer $s + kb^N$ with the same first N carries as s, the terms are $Q_j(s) + ka^j b^{N-j}$ by (8), and a divisor found for $k = 0$ need not divide the terms for other k.
3. *Recurrence.* Theorem 46 excludes recurrent carry tails on an orbit all of whose terms are prime, but it is open whether a non-recurrent all-prime tail can exist (Corollary 51); confinement to a strongly connected component of an outer graph does not imply recurrence (Appendix B.1).
4. *Additional operations.* The whole-word interval test (Appendix E.4) is pseudo-orbit-sound but not walk-sound; whether adjoining it changes the class $\mathcal{F}_{\mathrm{cert}}$ is open. Whether the halting program of Appendix C halts for every t (Theorem 58) is open.
5. *Minimality and subsequences.* The certificates give finite prime sets without establishing their minimality; for 5/3, Corollary 78 shows only that some prime at least 29 is needed. Whether the conclusions hold along prescribed arithmetic progressions or sparse subsequences of indices, and what bounds hold for the density of prime terms, remain separate questions.

Appendix A. Reproducing the computations and the formal proof

The paper sources, scientific programs, and Lean project are distributed in the public repository rational-floor-certificates. Run commands from the root of the downloaded distribution. The reproduction guide gives the dependencies, commands, resource limits, and output locations. The common entry point is:

```
python3 -B reproduce.py verify
python3 -B reproduce.py check --output ../certificate-preflight
python3 -B reproduce.py recompute --output ../certificate-replay
```

The distribution identifies its sources by the `source_sha256` field of the source manifest, computed from the SHA-256 values of the distributed source and check files. `SHA256SUMS` lists the individual file hashes. checks/reference.json records the finite reference data used for comparison. Generated graphs and logs are written to the chosen output location. Reaching a resource limit is recorded as `inconclusive`; an incomplete run is not an accepted certificate.

Word-labelled computations, implementation A. The public driver `supplement/computations/word_a/run_word.py` applies resource limits, runs the original scientific constructions, writes normalized graphs, and runs the same-lineage post-checks. The cases `five-halves`, `cross-75`, and `q75` regenerate the data of Tables 6, 4, and 7, respectively. The reproduction guide gives the complete commands; the computation guide describes the source provenance and comparison scope. The C++ constructions and the Python normalization code are included. Regeneration therefore uses the distributed sources.

Edge multiplicities. In Tables 4 and 6 edges are counted as distinct triples. Counting contracted chains with multiplicity, and propagating the multiplicity through lifting as the original research notes do, gives for the ten stages of Table 6 the input edge counts 12, 17, 58, 167, 820, 7083, 60242, 315483, 1685458, 3754612 and the output edge counts 12, 12, 20, 85, 529, 4256, 18069, 80704, 201199, 0; for Table 4 all multiplicities are one. Both implementations report these numbers and agree on them; they carry no proof content.

Word-labelled computations, implementation B. The separate C++ program, Python reference, and comparator are included under `supplement/computations/word_b/`. The following commands show the mathematical inputs and use an initially absent output directory outside the distribution. The public entry point applies the limits and performs the complete comparison described in Section 9.3:

```
mkdir -p ../certificate-manual/build \
  ../certificate-manual/OUT52 ../certificate-manual/OUT75 \
  ../certificate-manual/OUTQ75 ../certificate-manual/REF52
g++ -O2 -std=c++17 -o ../certificate-manual/build/wgb
↪  supplement/computations/word_b/wgb.cpp
../certificate-manual/build/wgb pipeline --a 5 --b 2 --K 4    --carries -1,1,3
↪  --primes 3,7,11,13,17,19,23,29,31 --out ../certificate-manual/OUT52
../certificate-manual/build/wgb pipeline --a 7 --b 5 --K 2048 --carries
↪  -4,-2,2,4,6 --primes 3,11,13 --out ../certificate-manual/OUT75
../certificate-manual/build/wgb q75      --a 7 --b 5 --M 429 --K 2048 --carries
↪  -4,-2,2,4,6 --out ../certificate-manual/OUTQ75
python3 supplement/computations/word_b/wgb_ref.py pipeline --a 5 --b 2 --K 4
↪  --carries -1,1,3 --primes 3,7,11,13,17,19,23,29,31 --out
↪  ../certificate-manual/REF52
python3 supplement/computations/word_b/compare_wordgraph.py batch --out
↪  ../certificate-manual/CMPDIR LABEL=FILE_A:FILE_B ...
```

The shared normalization contracts are comparison-schema.md and word-schema.md. The reference data and artifact hashes are recorded in checks/reference.json and the source manifest. Hashes identify finite artifacts; they do not replace element-wise comparison or the mathematical completeness arguments.

The ratio 5/3. The common entry points include the computation of Section 9.5: the preflight runs implementation B through the prime 17 and compares it with the producer's recorded small graphs, and the full run executes both implementations through all ten stages, compares all 20 stage graphs, and records in its receipt that both final output graphs are empty. The companion obstruction of Appendix F.5 is checked in both the preflight and the full run. The following commands show the inputs; the output directories must not exist:

```
../certificate-manual/build/wgb pipeline --a 5 --b 3 --K 3 --carries -2,2,4
↪  --primes 7,11,13,17,19,23,29,31,37 --bidir --out ../certificate-manual/OUT53
python3 -B supplement/computations/five_thirds/reproduce.py all --out
↪  ../certificate-manual/PRODUCER53
python3 -B
↪  supplement/computations/additional_obstructions/check_return_obstruction.py
↪  supplement/computations/additional_obstructions/obstruction_53/return_obstruc⌋
↪  tion.json
```

One-step computation. The public reproduction procedure also regenerates the four cases of Tables 1–3 with both one-step implementations and compares their complete outputs. Run without Python optimization, since the independent verifier uses assertions for checks. The source programs, reference checks, and generated Lean certificate sources are included; the computation guide specifies the compared data and the reproduction guide gives the commands.

Formal proof. The generated Lean data and proof sources are included, so checking the one-step proof does not require rerunning their generator. The word-labelled modules define and execute their finite pipeline inside Lean; they do not import the A/B graph dumps. Follow the reproduction guide to obtain the pinned library dependencies and build the project modules sequentially with a fresh project cache. After the one-step finite modules and before the final `MathPaper` target and aggregate build, check the word-labelled modules sequentially:

```
python3 supplement/lean/tools/check_finite.py \
  MathPaper.Word.Graph MathPaper.Word.Cert MathPaper.Word.Lift \
  MathPaper.Word.Chain MathPaper.Word.Alphabet \
  MathPaper.Word.Pipeline.Data MathPaper.Word.Pipeline.Check \
  MathPaper.Word.Pipeline.Compute MathPaper.Word.Pipeline.LiftArr \
  MathPaper.Word.Pipeline.Initial MathPaper.Word.Pipeline.Main \
  MathPaper.Word.FiveHalves MathPaper.Word.SevenFifths \
  MathPaper.Word.InContract MathPaper.Word.Bidir WordExec \
  MathPaper.Word.ExecSound MathPaper.Word.FiveThirds MathPaper.Word
```

The `FiveHalves`, `SevenFifths`, and `FiveThirds` targets rerun the three finite `native_decide` evaluations when built from a fresh project cache; the last needs about 13 GiB of memory and 11 minutes on the machine of Section 9.5. Then run the final targets, aggregate build, and `Audit.lean` axiom display as instructed in the reproduction guide. Compare the dependencies with the declared axiom boundary in the formalization guide. The additional evaluation axioms are the trust boundary described in Section 9.4; these evaluations are not kernel-only computations. Section 9.4 also specifies the scope of the completed independent rebuilds and type and axiom checks.

Appendix B. Recurrent carries

This appendix proves Theorem 46 and the integer copying condition for the family $(6t+3)/(6t+1)$. Its results are general lemmas about integer recurrences and finite permutations; they do not depend on Sections 3–9 and are not used there.

B.1. **Recurrent words.** Recall from Section 10 that an infinite word $w = w_0 w_1 \cdots$ over a finite alphabet is *recurrent* if for every $L \geq 1$ and $B \geq 1$ there is $n \geq B$ with

$$w[n : n + L] = w[0 : L],$$

where

$$w[i : j] = w_i w_{i+1} \cdots w_{j-1}.$$

Equivalently, every finite prefix occurs infinitely often, or again every factor that occurs at all occurs infinitely often (a factor lies in some prefix).

Lemma 48 (tails)**.** *If w is recurrent, then so is $w[k :]$ for every $k \geq 0$.*

Proof. Given $L, B \geq 1$, apply recurrence of w to the prefix length $k + L$ and the bound B: there is $n \geq B$ with

$$w[n : n + k + L] = w[0 : k + L].$$

Comparing positions k onward gives

$$w[n + k : n + k + L] = w[k : k + L],$$

that is,

$$w'[n : n + L] = w'[0 : L]$$

for $w' = w[k :]$. □

Uniform recurrence is not required. For example, let $w_0 = 0$ and

$$w_{k+1} = w_k \, 1 \, 0^{k+1} \, 1 \, w_k;$$

each w_k is a prefix of w_{k+1}, so a limit word w_∞ exists, in which every w_k occurs infinitely often (it occurs in w_{j+1} at position $|w_j| + j + 3$ for every $j \geq k$), while the runs of zeros are unbounded and 1 occurs infinitely often, so w_∞ is not eventually periodic. Conversely, the word $1\,0\,1\,00\,1\,000\cdots$, in which the runs of zeros have lengths $1, 2, 3, \ldots$, has no recurrent tail: a prefix of any tail that contains two letters 1 contains a factor $1\,0^k\,1$ which, the run lengths being strictly increasing, occurs only once in the whole word. This word is generated by a one-vertex graph with two self-loops, so confinement of a walk to a strongly connected component does not imply recurrence of its output.

Neither example is claimed to be the carry sequence of a true orbit.

B.2. **The finite-prefix copying lemma.**

Lemma 49 (finite-prefix copying)**.** *Let X be a set with m elements and let each letter c act by a permutation f_c of X. Let w be a finite word, $t_1, \dots, t_m$ positive integers, $S_0 = 0$, $S_k = t_1 + \cdots + t_k$, and $L \geq 0$, such that*

$$w[t_k : t_k + S_{k-1} + L] = w[0 : S_{k-1} + L] \qquad (k = 1, \dots, m) \tag{31}$$

and $|w| \geq S_m + L$. Put

$$G_n = f_{w_{n-1}} \circ \cdots \circ f_{w_0}$$

($G_0 = \mathrm{id}$; maps act from the right, so G_n applies the first letter first). Then for every $x \in X$ there are $0 \leq i < j \leq m$ such that

$$n = t_{i+1} + \cdots + t_j > 0$$

satisfies

$$G_n(x) = x \qquad \text{and} \qquad w[n : n + L] = w[0 : L].$$

Proof. *(1) Composition rule.* If $0 \leq s \leq S_{k-1}$, then

$$G_{t_k+s} = G_s \circ G_{t_k} :$$

indeed

$$G_{t_k+s} = f_{w_{t_k+s-1}} \circ \cdots \circ f_{w_{t_k}} \circ G_{t_k},$$

and by (31), $w_{t_k+i} = w_i$ for $0 \leq i < s$, so the left factor is G_s.

(2) Copies compose. For indices $i_1 < \cdots < i_h$ and $n = t_{i_1} + \cdots + t_{i_h}$,

$$G_n = G_{t_{i_1}} \circ G_{t_{i_2}} \circ \cdots \circ G_{t_{i_h}},$$
$$w[n : n + L] = w[0 : L].$$

By induction on h; $h = 0$ is $G_0 = \mathrm{id}$. For $h \geq 1$ put

$$n' = t_{i_1} + \cdots + t_{i_{h-1}}$$

and $k = i_h$; since all $i_j < k$, $n' \leq S_{k-1}$. By (1),

$$G_n = G_{t_k+n'} = G_{n'} \circ G_{t_k},$$

and by induction

$$G_{n'} = G_{t_{i_1}} \circ \cdots \circ G_{t_{i_{h-1}}}.$$

For the word, $n' + L \leq S_{k-1} + L$, so (31) gives

$$w[t_k + n' : t_k + n' + L] = w[n' : n' + L] = w[0 : L].$$

(3) Pigeonhole. Put $H_0 = \mathrm{id}$ and

$$H_k = G_{t_1} \circ \cdots \circ G_{t_k};$$

by (2) applied to $\{1, \dots, k\}$, $H_k = G_{S_k}$, the actual state at time S_k.

Among the $m+1$ values $H_0(x), \dots, H_m(x)$ in the m-element set X two coincide: $H_i(x) = H_j(x)$ with $i < j$.

With

$$B = G_{t_{i+1}} \circ \cdots \circ G_{t_j}$$

we have $H_j = H_i \circ B$, so

$$H_i(x) = H_i(B(x)),$$

and injectivity of H_i (a composition of permutations) gives $B(x) = x$.

By (2) applied to $\{i+1,\dots,j\}$,

$$B = G_n \qquad \text{with} \qquad n = t_{i+1} + \cdots + t_j > 0 \qquad \text{and}$$
$$w[n : n+L] = w[0 : L].$$

□

Injectivity is used exactly once, in step (3); the conclusion fails for non-injective actions. Applying the pigeonhole principle directly to the states along the orbit would only give a return to *some* state, not to the initial state; the copies align the compositions so that the common left factor can be cancelled.

Theorem 50 (residue recurrence)**.** *Let integer sequences $(Q_n)_{n\ge 0}$, $(c_n)_{n\ge 0}$ satisfy*

$$bQ_{n+1} = aQ_n + c_n,$$

let the word $c = c_0c_1\cdots$ be recurrent, and let $M \ge 2$ with $\gcd(M, ab) = 1$. Then for every $B \ge 1$ and $L \ge 0$ there is $n \ge B$ with

$$Q_n \equiv Q_0 \pmod{M} \qquad \textit{and} \qquad c[n : n+L] = c[0 : L].$$

Proof. Let $X = \mathbb{Z}/M$ and

$$f_c(x) = b^{-1}(ax + c);$$

since a, b are units modulo M, each f_c is an affine permutation, and

$$Q_n \equiv G_n(Q_0).$$

Choose copy times successively: for $k = 1,\dots,M$, the required prefix length $S_{k-1} + L$ is finite, so by recurrence (when this length is positive; otherwise any $t_k \ge B$) there is $t_k \ge B$ satisfying (31), S_{k-1} being determined by $t_1,\dots,t_{k-1}$.

Lemma 49 with $m = M$ and $x = Q_0 \bmod M$ gives

$$n = t_{i+1} + \cdots + t_j \ge t_j \ge B,$$

$G_n(Q_0) \equiv Q_0$, and the return of the prefix. □

The hypothesis $\gcd(M, ab) = 1$ cannot be dropped. If a prime $p \mid M$ divides a, then $f_c(x) \equiv b^{-1}c \pmod{p}$ is constant, not injective, and $Q_n \bmod p$ is determined by c_{n-1} alone; if $Q_0 \bmod p$ is not of the form $b^{-1}c$ for a letter c of the word (for instance $p \mid Q_0$ and no letter is divisible by p), then $Q_n \equiv Q_0 \pmod{p}$ never holds for $n \ge 1$, however recurrent the word is.

B.3. **Proof of Theorem 46.**

Proof of Theorem 46. Let $c[j_0 :]$ be recurrent.

Case 1: some $j \ge j_0$ has $Q_j > 1$ and $\gcd(Q_j, ab) = 1$. By Lemma 48, $c[j :]$ is recurrent. Apply Theorem 50 to the sequences $Q'_n = Q_{j+n}$, $c'_n = c_{j+n}$ with $M = Q_j \ge 2$ and $L = 0$: for every B there is $n \ge B$ with

$$Q_{j+n} \equiv Q_j \equiv 0 \pmod{Q_j}.$$

By (7),

$$Q_{j+n} > Q_j > 1$$

for all large n, so Q_j is a proper divisor and Q_{j+n} is composite; there are infinitely many such n. This also proves the last assertion of the theorem.

Case 2: no such j. By (7), $Q_j \le 1$ for only finitely many j, so

$$\gcd(Q_j, ab) > 1$$

for all large j. A prime $p \mid \gcd(Q_j, ab)$ satisfies $p \le ab$, so once $Q_j > ab$ we have $Q_j = pv$ with $v > 1$. □

Case 2 does not use recurrence, but the conclusion of Case 1, that Q_j divides infinitely many later terms, does; in general $Q_j \mid Q_n$ need not occur. This is why the recurrence hypothesis is part of the statement.

Corollary 51 (refutation form)**.** *If all terms Q_n with $n \ge n_0$ are prime, then the carry word has no recurrent tail.*

Proof. Suppose $c[j_0 :]$ is recurrent. By (7) choose $j \ge \max(j_0, n_0)$ with $P = Q_j > a$. Then P is a prime larger than $a > b$, so $\gcd(P, ab) = 1$, and Theorem 50 with $M = P$ gives arbitrarily large n with $P \mid Q_n$ and $Q_n > P$, hence composite, a contradiction. □

The condition $P > a$ serves only to guarantee $\gcd(P, ab) = 1$. Theorem 46 excludes recurrent carry tails on an eventually-all-prime orbit; it does not exclude orbits all of whose carry tails are non-recurrent (Section B.1 shows that such words exist as words), and this remaining case is open (Section 11).

B.4. **The integer copying condition for the family** $(6t+3)/(6t+1)$**.** Let

$$\begin{aligned} &t \ge 1, \qquad a = 6t+3, \qquad b = 6t+1, \\ &\delta = t \bmod 2, \qquad H = 3t - 1 - \delta \end{aligned}$$

(an odd integer), and

$$\begin{aligned} T_t(Q) &= Q + 2\Big\lfloor \frac{Q + 3t + 1}{b} \Big\rfloor, \\ F_t(z) &= \Big\lceil \frac{az - \delta}{b} \Big\rceil, \\ e &= bF_t(z) - az + \delta. \end{aligned} \tag{32}$$

Appendix C shows that a tail of a true orbit all of whose terms are coprime to 6 follows the map T_t; here we study T_t as an integer map.

Lemma 52 (conjugation and digits)**.** *For an odd integer $Q = 2z + H$: (a) $T_t(Q) = 2F_t(z) + H$, and T_t preserves parity; (b)*

$$0 \le e \le b - 1 \qquad \textit{and} \qquad e \equiv \delta - az \pmod{b};$$

(c) the carry $c = bT_t(Q) - aQ$ equals $2(e - (3t-1))$; (d)

$$F_t(s + bv) = F_t(s) + av$$

for all integers s, v.

Proof. (a)

$$Q + 3t + 1 = 2z + (b-1) - \delta,$$

so

$$\lfloor (Q + 3t + 1)/b \rfloor = 1 + \lfloor (2z - 1 - \delta)/b \rfloor.$$

With $a = b + 2$,

$$\begin{aligned} &(az - \delta)/b = z + (2z - \delta)/b, \qquad \text{and} \\ &\lceil n/b \rceil = \lfloor (n-1)/b \rfloor + 1 \end{aligned}$$

for integers n gives

$$F_t(z) = z + 1 + \lfloor (2z - 1 - \delta)/b \rfloor.$$

Hence

$$2F_t(z) + H = 2z + H + 2 + 2\lfloor (2z - 1 - \delta)/b \rfloor = T_t(Q).$$

Parity is preserved since $T_t(Q) - Q$ is even.

(b) is the definition of the ceiling.

(c)

$$c = 2(bF_t(z) - az) + (b-a)H = 2(e-\delta) - 2H = 2(e-(3t-1)).$$

(d)

$$\lceil (a(s+bv)-\delta)/b \rceil = \lceil (as-\delta)/b + av \rceil. \qquad \square$$

Proposition 53 (digit words and residues)**.** *Let $L \geq 1$. The map sending an integer z_0 to the first L digits $e_0 \cdots e_{L-1}$ of its F_t-orbit induces a bijection*

$$\mathbb{Z}/b^L \to \{0, \ldots, b-1\}^L.$$

Consequently, for odd Q_0, Q_0' and their T_t-orbits with carry words c, c',

$$c[0:L] = c'[0:L] \iff Q_0 \equiv Q_0' \pmod{b^L},$$

and in particular, along one orbit,

$$c[k:k+L] = c[0:L] \iff Q_k \equiv Q_0 \pmod{b^L}.$$

Proof. Induction on L. For $L = 1$,

$$e_0 \equiv \delta - az_0 \pmod{b}$$

with $\gcd(a, b) = 1$ ($a = b + 2$, b odd), so $z_0 \bmod b \mapsto e_0$ is a bijection.

For the step, if $z_0' = z_0 + b^{L+1}v$, then $e_0 = e_0'$ and by Lemma 52(d)

$$z_1' = F_t(z_0') = z_1 + ab^L v \equiv z_1 \pmod{b^L},$$

so the remaining L digits agree by induction (the map is well defined).

Conversely, if the digit words agree, then e_0 determines $z_0 \bmod b$; write

$$z_0 = s + bq, \qquad z_0' = s + bq', \qquad \text{so}$$
$$z_1 = F_t(s) + aq, \qquad z_1' = F_t(s) + aq';$$

by induction

$$z_1 \equiv z_1' \pmod{b^L},$$

and since a is a unit modulo b^L, $q \equiv q' \pmod{b^L}$, that is,

$$z_0 \equiv z_0' \pmod{b^{L+1}}$$

(injectivity).

Both sets have b^{L+1} elements, so the map is a bijection.

For carries, Lemma 52(c) makes $c_n \leftrightarrow e_n$ bijective, and since b is odd, $Q = 2z + H$ gives

$$Q \equiv Q' \pmod{b^L} \iff z \equiv z' \pmod{b^L}. \qquad \square$$

Proposition 54 (P-stage congruence certificate)**.** *Let $P > a$ be prime, $Q_n = T_t^n(P)$, and let positive integers $k_1, \ldots, k_P$ with $s_0 = 0$, $s_i = s_{i-1} + k_i$ satisfy*

$$Q_{k_i} \equiv P \pmod{b^{s_{i-1}}} \qquad (i = 1, \ldots, P). \tag{33}$$

Then $P \mid Q_n$ for some $0 < n \leq s_P$, and $Q_n > P$, so Q_n is composite.

Proof. By Proposition 53 with $L = s_{i-1}$ and $k = k_i$, (33) gives

$$c[k_i : k_i + s_{i-1}] = c[0 : s_{i-1}],$$

which is (31) with $t_i = k_i$ and $L = 0$.

Apply Lemma 49 on $X = \mathbb{Z}/P$ with

$$f_c(x) = b^{-1}(ax + c)$$

(permutations, since $P > a > b$ gives $\gcd(P, ab) = 1$) to $x = 0$: some $n = k_{i+1} + \cdots + k_j \le s_P$ has

$$Q_n \equiv G_n(0) = 0.$$

If $Q \ge 3t$ then

$$\lfloor (Q + 3t + 1)/b \rfloor \ge 1 \qquad \text{and}$$
$$T_t(Q) \ge Q + 2;$$

as T_t is nondecreasing and $P > a > 3t$, the orbit is strictly increasing and $Q_n > P$. □

Application to true orbits. Let $\lfloor \xi \rfloor = P$. If all terms $\lfloor \xi r^n \rfloor$, $0 \le n \le s_P$, are coprime to 6, then by Proposition 55 (Appendix C) they coincide with the T_t-orbit of P, and Proposition 54 gives a composite term at some $n \le s_P$. Otherwise some term $\ge P > 3$ is divisible by 2 or 3 and is composite. In either case a composite term occurs at a time $\le s_P$, and no interval-consistency check is needed. The existence of such certificates for all t and P is not proved.

Example. For $9/7$ ($t = 1$) and $\xi = 11$, the carries of the true orbit $\lfloor 11(9/7)^n \rfloor$ satisfy $c_0 = c_{43} = -1$, so the copy condition (31) holds with $t_1 = 1$, $t_2 = 43$, and $L = 0$; step (1) of the proof of Lemma 49 then gives $G_{44} = G_1 \circ G_{43}$ on $\mathbb{Z}/11$, and one checks directly that the residue returns:

$$11 \mid Q_{44} = 697829 = 11 \cdot 63439.$$

(Lemma 49 itself would require eleven copy times; here the return is verified directly.)

This certificate lives on the true orbit rather than on the T_1-orbit (since $\lfloor 99/7 \rfloor = 14 \ne T_1(11) = 15$), and $P = 11 > a = 9$ is admissible for Lemma 49 itself because $\gcd(11, 63) = 1$.

Remark (quantitative form). Let $R(L)$ be the first positive return time of the prefix $w[0 : L]$. If $R(L) \le KL$ for all $L \ge 1$, then copies with $L = 1$ can be chosen with

$$t_k = R(S_{k-1} + 1) \le K(S_{k-1} + 1),$$

so

$$S_k + 1 \le (K + 1)^k,$$

and a return modulo M to the initial state occurs by time $(K + 1)^M - 1$. No such K is proved for the family, and this bound is not used.

Appendix C. Non-branching families and the exact integer halting problem

This appendix concerns the family $r_t = (6t+3)/(6t+1)$, $t \ge 1$, that is $9/7, 15/13, 21/19, \ldots$ (the members $j = 3t - 2$ of the family $(2j + 7)/(2j + 5)$ of Corollary 40). It proves that a tail avoiding the primes 2 and 3 follows the integer map T_t of (32), that the all-coefficient statement for a fixed t is equivalent to the halting of an exact integer program, and it constructs further families with a unique prime successor. None of this proves halting for the family, and no all-coefficient theorem for any r_t is asserted.

C.1. **The non-branching lemma.**

Proposition 55 (non-branching)**.** *Let*

$$t \geq 1, \qquad a = 6t+3, \qquad b = 6t+1,$$
$$x > 0, \qquad Q = \lfloor x \rfloor, \qquad Q' = \lfloor rx \rfloor.$$

If Q is odd and $\gcd(Q', 6) = 1$*, then $Q' = T_t(Q)$.*

Proof. The carry $c = bQ' - aQ$ satisfies

$$-6t \leq c \leq 6t+2$$

by (6). Since a, b, Q, Q' are odd, $c = 2s$ is even with

$$-3t \leq s \leq 3t+1.$$

As $3 \mid a$, $3 \nmid b$, $3 \nmid Q'$, we have

$$c \equiv bQ' \not\equiv 0 \pmod 3,$$

so $3 \nmid s$; this excludes $s = -3t$, leaving

$$-3t+1 \leq s \leq 3t+1,$$

an interval of exactly $b = 6t+1$ integers.

Modulo b, $a \equiv 2$, so $2s = c \equiv -2Q$ and

$$s \equiv -Q \pmod b;$$

the interval contains exactly one integer in each residue class, so s is unique.

With $m = \lfloor (Q+3t+1)/b \rfloor$,

$$0 \leq Q + 3t + 1 - bm \leq 6t$$

gives

$$Q - 3t + 1 \leq bm \leq Q + 3t + 1,$$

so $s = bm - Q$ lies in the interval and is $\equiv -Q$. Hence

$$Q' = (aQ + 2s)/b = (a-2)Q/b + 2m = Q + 2m = T_t(Q). \qquad \square$$

Only the oddness of Q and $\gcd(Q', 6) = 1$ are used; $3 \nmid Q$ is not needed. The conclusion is uniqueness of the candidate: if $T_t(Q)$ is divisible by 3, no admissible successor exists.

C.2. **Conjugation to a ceiling map and digit restriction.** By Lemma 52, the map T_t on odd integers $Q = 2z + H$ is conjugate to

$$F_t(z) = \lceil (az - \delta)/b \rceil,$$

with digits

$$e = bF_t(z) - az + \delta \in \{0, \ldots, b-1\}$$

and carries

$$c = 2(e - (3t-1)).$$

For $t = 2u$ even, $\delta = 0$, $H = 6u - 1$, and $F_t(z) = \lceil az/b \rceil$ is the ordinary ceiling map.

Lemma 56 (digit restriction)**.** *(a)*

$$Q' = T_t(Q) \equiv 2e - 1 \pmod 3;$$

hence $3 \mid Q'$ *if and only if* $e \equiv 2 \pmod 3$. *(b) If* Q *and* Q' *are primes larger than* a*, then*

$$e \in D_t = \{0 \le e \le 6t : \ \gcd(e - (3t-1), (6t+3)(6t+1)) = 1\}.$$

(c)

$$d_t = |D_t| \le 4t + 1 < b.$$

Proof. (a) $e \equiv F_t(z) + \delta \pmod 3$ since $b \equiv 1$, $a \equiv 0$; and $H \equiv -1 - \delta$, so

$$Q' = 2F_t(z) + H \equiv 2(e - \delta) - 1 - \delta \equiv 2e - 1 \pmod 3.$$

(b) By Lemma 29, $\gcd(c, ab) = 1$, and ab is odd, so

$$\gcd(e - (3t-1), ab) = 1.$$

(c) Since $3t - 1 \equiv 2 \pmod 3$, every $e \equiv 2 \pmod 3$ is excluded, and there are $2t$ such e in $[0, 6t]$; so

$$d_t \le 6t + 1 - 2t.$$ □

Table 9: Admissible digit sets for $t \le 3$.

t	a/b	D_t
1	9/7	$\{0, 1, 3, 4, 6\}$
2	15/13	$\{1, 3, 4, 6, 7, 9, 12\}$
3	21/19	$\{0, 3, 4, 6, 7, 9, 10, 12, 13, 16, 18\}$

Remark (real consistency). For a true orbit $x_n = 2z_n + H + \theta_n$, so

$$x_n/2 - z_n \in [H/2, (H+1)/2).$$

From $bz_{n+1} = az_n + e_n - \delta$,

$$r^{-(n+1)} z_{n+1} - r^{-n} z_n = r^{-(n+1)}(e_n - \delta)/b$$

with $|e_n - \delta| \le b$, so

$$\Lambda = \lim r^{-n} z_n$$

exists and

$$z_n = \Lambda r^n - \sum_{j \ge 0} (e_{n+j} - \delta) b^j / a^{j+1};$$

for $z_0 > b/2$, $\Lambda > 0$, and for a true orbit $\xi = 2\Lambda$ is determined. Membership of all digits in D_t implies neither the interval condition of width $1/2$ nor the primality of the terms.

Remark. The iterates of the ceiling map $x \mapsto \lceil (p/q)x \rceil$ on integers are the object of a conjecture on equidistribution of residues modulo q^k in AEV26, Conjecture 1.6; that paper concerns integer seeds and states conjectures, not theorems, and nothing here depends on it.

C.3. **The exact integer halting program.** Fix $t \geq 1$ and an initial prime $P > a$. Starting from $(Q, L, U, B) = (P, 0, 1, 1)$, repeat:

$$\begin{aligned} Q^+ &= T_t(Q), \quad c = bQ^+ - aQ, \quad B^+ = bB, \\ L^+ &= \max(0, aL - cB), \\ U^+ &= \min(B^+, aU - cB), \end{aligned}$$

and *halt* if Q^+ is composite or $L^+ \geq U^+$; otherwise replace (Q, L, U, B) by (Q^+, L^+, U^+, B^+) and continue. Primality is decided exactly. Reaching a bound on time, memory, or iterations is recorded as undetermined and is not a halt.

Proposition 57 (exactness)**.** *After n accepted steps, $B = b^n$ and $[L/B, U/B)$ is exactly the set of fractional parts θ_n of those $\xi \in [P, P+1)$ whose orbit has integer parts $P = Q_0, \ldots, Q_n$. In particular $L^+ \geq U^+$ means that the finite sequence $Q_0, \ldots, Q_{n+1}$ is realized by no real ξ.*

Proof. The map

$$\theta \mapsto \theta' = (a\theta - c)/b$$

is an increasing affine bijection, so the image of $[L/B, U/B)$ is

$$[(aL - cB)/(bB), (aU - cB)/(bB)),$$

whose intersection with $[0, 1)$ is

$$[L^+/B^+, U^+/B^+).$$

Initially θ_0 ranges over $[0, 1)$. Inductively, θ_{n+1} lies in the image intersected with $[0, 1)$ if and only if the θ_n producing it lies in the previous set. □

C.4. **Equivalence with the all-coefficient statement.**

Theorem 58 (halting equivalence)**.** *For fixed $t \geq 1$ the following are equivalent.*

(A) *For every $\xi > 0$, the sequence $\lfloor \xi r_t^n \rfloor$ contains infinitely many composite terms.*
(B) *The program of Section C.3 halts for every initial prime $P > a$.*

Proof. (B)⇒(A). If (A) fails, there are ξ and n_0 with Q_n prime for all $n \geq n_0$. By (7) choose $n_1 \geq n_0$ with $Q_{n_1} > a$ and replace ξ by ξr^{n_1}; then $P = \lfloor \xi \rfloor > a$ is prime and all terms are primes $> a > 3$, coprime to 6. By Proposition 55,

$$Q_{n+1} = T_t(Q_n)$$

for all n, so the program's Q coincides with the true terms and never becomes composite, and the true θ_n lies in every interval by Proposition 57, so no interval is empty. The program does not halt, contradicting (B).

(A)⇒(B). Suppose the program does not halt for some $P > a$. Then all $Q_n = T_t^n(P)$ are prime (a non-composite integer $> a \geq 9$ is prime) and all intervals are nonempty, so by Proposition 57 there is, for each n, some $\xi_n \in [P, P + 1)$ with

$$\lfloor \xi_n r^j \rfloor = Q_j$$

for $0 \leq j \leq n$.

Let

$$\begin{aligned} I_j &= [Q_j r^{-j}, (Q_j + 1) r^{-j}] \qquad \text{and} \\ J_n &= I_0 \cap \cdots \cap I_n. \end{aligned}$$

Then

$$\xi_n \in J_n \neq \varnothing,$$

$J_{n+1} \subseteq J_n$, and each J_n is a closed subset of the compact interval $I_0 = [P, P+1]$; by Cantor's intersection theorem $\bigcap_n J_n \neq \varnothing$, and since the widths are at most $r^{-n} \to 0$, the intersection is a single point $\xi_* > 0$. (With half-open intervals the finite intersection property would not yield a common point; this is why closed intervals are used and the endpoints are treated separately.)

Thus

$$Q_n \leq \xi_* r^n \leq Q_n + 1$$

for all n.

If $\xi_* r^n = Q_n + 1$ for some n, then

$$\xi_* = (Q_n + 1) b^n / a^n$$

is rational, $\xi_* = u/v$ in lowest terms; $\xi_* r^m \in \mathbb{Z}$ requires $v b^m \mid u a^m$, hence $v \mid a^m$ and $b^m \mid u$, which holds for only finitely many m as $u \neq 0$.

So for all large n,

$$Q_n \leq \xi_* r^n < Q_n + 1,$$

that is,

$$Q_n = \lfloor \xi_* r^n \rfloor$$

is prime, contradicting (A). □

Whether (B) holds for every t is open. The orbit of ξ_* may differ from the T_t-orbit at finitely many indices; agreement of the tails suffices.

C.5. **Removing prime forks for general bases.** Let $a = b + d$, $\gcd(a, b) = 1$, $1 < a/b < 2$ (so $d < b$). For $x \in [Q, Q+1)$ the possible values of $\lfloor rx \rfloor$ are the integers m for which $[m, m+1)$ meets $[aQ/b, a(Q+1)/b)$. The latter interval has length $a/b < 2$, so the candidates consist of at most three consecutive integers and include at most two odd integers, necessarily of the form $L, L+2$ when both occur.

Lemma 59 (two odd candidates)**.** *If Q is odd and both L and $L+2$ are candidates, then with $C = L + 1$,*

$$\begin{aligned} aQ &= bC - k, \quad 1 \leq k < d, \quad k \equiv a \pmod 2, \\ bL &= a(Q-1) + (d+k), \\ b(L+2) &= a(Q+1) - (d-k). \end{aligned}$$

Proof. L is a candidate iff $aQ/b < L + 1$ and $L < a(Q+1)/b$; $L+2$ is a candidate iff $aQ/b < L + 3$ and $L + 2 < a(Q+1)/b$.

Together, $aQ < bC$ and $bC + b < aQ + a$, that is,

$$bC - d < aQ < bC,$$

so

$$k = bC - aQ \in \{1, \ldots, d-1\}.$$

As L is odd, C and bC are even, and $aQ \equiv a \pmod 2$, so $k \equiv a \pmod 2$.

The identities follow from

$$\begin{aligned} bL &= bC - b = aQ + k - b \qquad \text{and} \\ b(L+2) &= bC + b = aQ + k + b. \end{aligned}$$

□

Corollary 60 (no prime fork)**.** *If for every k with $1 \le k < d$ and $k \equiv a \pmod 2$ one has*

$$\gcd(k, b) > 1, \qquad or \qquad \gcd(d + k, a) > 1, \qquad or \qquad \gcd(d - k, a) > 1,$$

then there is no odd prime $Q > a$ with two prime candidates $L, L + 2 > a$. Moreover, if $L, L + 2 > 3$ are prime then $3 \mid C$, so for a prime $Q > 3$ a fork requires

$$3 \mid a \iff 3 \mid k;$$

values of k violating this equivalence are also excluded.

Proof. If $p \mid \gcd(k, b)$ then $p \mid aQ$, $p \nmid a$, $p \mid Q$, and Q prime forces $Q = p \le b < a$.

If $p \mid \gcd(d + k, a)$ then $p \mid bL$, $p \nmid b$, $p \mid L$, $L = p \le a$.

If $p \mid \gcd(d - k, a)$, similarly $p \mid L + 2$.

For the last assertion, among $L, L + 1, L + 2$ the multiple of 3 is $C = L + 1$, and

$$aQ = bC - k \equiv -k \pmod 3$$

with $3 \nmid Q$. □

Example. For $a = 462u + 77$, $b = 462u + 71$ ($u \ge 0$): $d = 6$, a odd, $k \in \{1, 3, 5\}$; $7 \mid a$ and $11 \mid a$ give $\gcd(d + k, a) > 1$ for $k = 1, 5$; for $k = 3$, $3 \mid k$ but $a \equiv 2 \pmod 3$, so the equivalence fails. The pairs are reduced ($\gcd(a, b) \mid 6$, a odd, $3 \nmid a$) with $a < 2b$. Hence a prime $Q > a$ has at most one prime successor in this family; the surviving candidate need not be the larger one.

C.6. **Families with a prescribed prime-power difference.**

Proposition 61 (families for every prime-power difference)**.** *Let $d = p^h \ge 2$ and*

$$K_d = \{k : 1 \le k < d,\ k \equiv d + 1 \pmod 2\}.$$

For each $k \in K_d$ there is a prime $\ell_k \ne p$ dividing $d + k$; let P_d be the product of the distinct ℓ_k. Then $\gcd(P_d, 2d) = 1$, and every $a > 2d$ with

$$a \equiv 0 \pmod{P_d} \qquad and \qquad a \equiv d + 1 \pmod{2d}$$

(an arithmetic progression modulo $2dP_d$), with $b = a - d$, satisfies $\gcd(a, b) = 1$, $1 < a/b < 2$, and the hypothesis of Corollary 60 in the form

$$\gcd(d + k, a) > 1$$

for all relevant k.

Proof. For $k \in K_d$, $d + k \equiv 2d + 1$ is odd and $d < d + k < 2d$; there is no power of p in $(d, 2d)$, so $d + k \ge 3$ has a prime factor $\ell_k \ne p$, necessarily odd.

Hence P_d is odd and coprime to p, so $\gcd(P_d, 2d) = 1$ and the two congruences are compatible.

From $a \equiv 1 \pmod d$,

$$\gcd(a, b) = \gcd(a, d) = 1;$$
$$a > 2d \iff a < 2b;$$

and $a \equiv d + 1 \pmod 2$ makes $k \equiv a \pmod 2$ equivalent to $k \in K_d$, with

$$\ell_k \mid \gcd(d + k, a).$$ □

Table 10: Families from Proposition 61 for small d.

d	a	b	Range
2	$12u+3$	$12u+1$	$u \geq 1$
3	$30u+10$	$30u+7$	$u \geq 0$
4	$280u+245$	$280u+241$	$u \geq 0$
5	$210u+126$	$210u+121$	$u \geq 0$

For odd d the numerator a is even; Lemma 59 handles this through the parity condition $k \equiv a \pmod 2$. These are families with a uniform prime-successor rule, not families for which compositeness has been established.

Proposition 62 (ceiling form)**.** *In the setting of Proposition 61, write $b = 2dh+1$, $H = 2h-1$, and $Q = 2z+H$. The largest odd integer below $r(Q+1)$ is $2\lceil az/b\rceil + H$; hence, if an odd candidate for the successor of Q exists, the largest one is*

$$Q' = 2\lceil az/b\rceil + H,$$

and a prime successor $> a$ of a prime $Q > a$ can only be this value. With $z' = \lceil az/b\rceil$ and $e = bz' - az \in [0, b)$,

$$\gcd(Q, b) = \gcd(2e+d+1, b),$$
$$\gcd(Q', a) = \gcd(2e+2d+1, a),$$

so a tail of primes $> a$ has all digits in

$$D_{a,b} = \{0 \leq e < b : \gcd(2e+d+1, b) = \gcd(2e+2d+1, a) = 1\}.$$

Proof. Put $y = (Q+1)/2 = z+h$; the largest odd integer below $2ry$ is $2y'-1$ with $y' = \lceil ry - 1/2\rceil$.

Since b is odd, ry is not a half-integer, so

$$y' = \lfloor ry + 1/2\rfloor = \lfloor (2ay+b)/(2b)\rfloor = \lfloor ay + (b-1)/2)/b\rfloor.$$

With $(b-1)/2 = dh$ and $a = b+d$,

$$ay + dh = az + bh + (b-1),$$

so

$$y' = h + \lfloor (az+b-1)/b\rfloor = h + \lceil az/b\rceil \qquad \text{and}$$
$$Q' = 2y' - 1 = 2\lceil az/b\rceil + H.$$

For the gcd identities,

$$dQ = 2dz + b - 1 - d \equiv -2e - 1 - d \pmod b$$

(as $2e \equiv -2az \equiv -2dz$), and $\gcd(d, b) = 1$; similarly $dQ' = 2dz' + b - 1 - d$ with $bz' \equiv e \pmod a$ and $b \equiv -d \pmod a$ give

$$dQ' \equiv -(2e+2d+1) \pmod a. \qquad \square$$

A uniform halting theorem for these families, showing that every sufficiently large initial value eventually violates the digit restriction or the interval condition, is not proved.

Appendix D. Prime-prefix counting, the word sieve, and finite height

This appendix proves Theorem 47. Section D.1 bounds the number of initial primes whose orbit stays prime for N steps; Section D.2 describes a sieve over carry words for a general base and the resulting power-type bound; Sections D.3–D.4 turn the sieve into a complete treatment of all initial integers up to an explicit height for 6/5 and 9/7. None of these results is an all-coefficient theorem.

D.1. **A counting bound along the family.** For $t \geq 1$, $N \geq 1$ and $X \geq a = 6t + 3$, let $S_t(X, N)$ be the set of primes $a < P \leq X$ for which the program of Section C.3 accepts N steps.

Proposition 63.

$$|S_t(X, N)| \leq d_t^N(\lfloor X/(2b^N)\rfloor + 1),$$

where $d_t = |D_t| \leq 4t + 1$. *If* $N = N(X)$ *is the least integer with* $b^N \geq X$, *then*

$$|S_t(X, N(X))| \leq d_t\, X^{\alpha_t}$$

with

$$\alpha_t = \log d_t / \log b < 1.$$

The same bounds hold for the set of initial primes for which the program never halts.

Proof. Each $P \in S_t(X, N)$ determines a digit word of length N with all digits in D_t (Lemma 56), hence one of at most d_t^N words; by Proposition 53 the word determines $z_0 \bmod b^N$, hence $P = 2z_0 + H$ modulo $2b^N$; and a residue class modulo $2b^N$ has at most $\lfloor X/(2b^N)\rfloor + 1$ elements in $(a, X]$.

For $N = N(X)$, $2b^N > X$ gives at most one element per word, and $b^{N-1} < X$ gives

$$d_t^N \leq d_t \cdot d_t^{\log_b X} = d_t X^{\alpha_t};$$

$\alpha_t < 1$ since $d_t < b$. A non-halting initial prime lies in $S_t(X, N)$ for every N. □

A bound of the form CX^α with $\alpha < 1$ does not imply that the set is empty: the tail of a single non-halting orbit has only $O(\log X)$ starting points below X.

D.2. **The word sieve for a general base.** Let $a > b \geq 2$ be coprime and let $c_0 \cdots c_{\ell-1}$ be a carry word, with $C_0 = 0$, $C_{j+1} = aC_j + b^j c_j$, so that

$$b^j Q_j = a^j Q_0 + C_j$$

by (8).

Lemma 64. *(i)* Integrality. *The congruence*

$$a^\ell Q_0 \equiv -C_\ell \pmod{b^\ell}$$

determines $Q_0 \bmod b^\ell$ *and implies*

$$b^j \mid a^j Q_0 + C_j$$

for all $j \leq \ell$*; hence for* $X \leq b^\ell$ *at most one integer* $Q_0 \in [1, X]$ *has the given word. (ii)* Exact interval. *With*

$$\theta_j = (a^j \theta_0 - C_j)/b^j,$$

the word is realized by some real orbit segment if and only if

$$[0,1) \cap \bigcap_{j=1}^{\ell} \Big[\frac{C_j}{a^j}, \frac{C_j + b^j}{a^j}\Big) \neq \varnothing.$$

(iii) Fixed divisors. *For a prime* $p \nmid ab$, $p \mid Q_j$ *if and only if* $Q_0 \equiv R_j \pmod p$*, where* $R_0 = 0$ *and*

$$R_{j+1} = R_j - c_j b^j a^{-j-1} \pmod p.$$

If

$$\{R_0, \ldots, R_\ell\} = \mathbb{F}_p,$$

then every initial integer has $p \mid Q_j$ *for some* $j \leq \ell$*; since there are at most* $\ell + 1$ *roots, only primes* $p \leq \ell + 1$ *can cover* $\mathbb{F}_p$*.*

Proof. (i)

$$C_\ell = a^{\ell-j} C_j + \sum_{i \geq j} a^{\ell-1-i} b^i c_i \equiv a^{\ell-j} C_j \pmod{b^j},$$

so $b^\ell \mid a^\ell Q_0 + C_\ell$ gives

$$b^j \mid a^{\ell-j}(a^j Q_0 + C_j)$$

and, as $\gcd(a, b) = 1$,

$$b^j \mid a^j Q_0 + C_j.$$

(ii) $\theta_j \in [0, 1)$ if and only if

$$\theta_0 \in [C_j/a^j, (C_j + b^j)/a^j);$$

the map $\theta_0 \mapsto \theta_j$ is exact, as in Proposition 57.

(iii)

$$p \mid Q_j \iff a^j Q_0 + C_j \equiv 0 \iff Q_0 \equiv -C_j a^{-j},$$
$$\text{and} \quad -(aC_j + b^j c_j) a^{-j-1} = R_j - c_j b^j a^{-j-1}.$$

□

Let A_ℓ be the number of words of length ℓ such that (1) every letter lies in $\mathcal{C}(a, b)$, (2) the interval in (ii) is nonempty, and (3) no prime $p \leq \ell+1$ with $p \nmid 2ab$ has $\{R_0, \ldots, R_\ell\} = \mathbb{F}_p$. If $Q_0, \ldots, Q_\ell$ are primes larger than $\max(a, \ell+1)$, then the carry word of the segment satisfies (1) by Lemma 29, (2) with the true θ_0, and (3) because a covering prime would divide some

Q_j, forcing $Q_j = p \le \ell + 1$. Absence of a fixed divisor is a necessary condition only; no prime-tuple conjecture is used to make it sufficient.

Proposition 65 (power-type bound)**.** *Fix ℓ, put $\lambda = A_\ell^{1/\ell}$ and $\alpha = \log\lambda/\log b$, and suppose*

$$A_s \le C\lambda^{s-1}$$

for $0 \le s < \ell$. Then the number of primes P with $\max(a, \ell+1) < P \le X$ such that some $\xi \in [P, P+1)$ produces $N(X)+1$ prime terms is at most CX^α, where $N(X)$ is the least integer with $b^{N(X)} \ge X$.

Proof. Write $N = k\ell + s$ with $0 \le s < \ell$. The carry word of an all-prime segment of $N+1$ terms, all larger than $\max(a, \ell+1)$, splits into k blocks of length ℓ and one of length s, each satisfying (1)–(3) with the fractional part at the start of the block as initial value; ignoring compatibility between blocks overcounts, so there are at most

$$A_\ell^k A_s \le C\lambda^{N-1}$$

words.

For $N = N(X)$, $b^{N-1} < X$ gives

$$\lambda^{N-1} = b^{\alpha(N-1)} < X^\alpha,$$

and by Lemma 64(i) each word has at most one initial integer in $[1, X]$, distinct P giving distinct words. □

The hypothesis $A_s \le C\lambda^{s-1}$ is equivalent to the integer inequalities $A_\ell \le C^\ell$ ($s = 0$) and $A_s^\ell \le C^\ell A_\ell^{s-1}$ ($1 \le s < \ell$). Table 11 gives the complete counts for the two bases, computed as described in Section D.5; the exponents are certified by

$$18035747 < 7^9 = 40353607 \qquad \text{and} \qquad 3955763 < 5^{10} = 9765625,$$

that is, $\alpha < 1/2$ for 9/7 and $\alpha < 1/4$ for 6/5, and the constants $C = 13$ and $C = 10$ are the least integers satisfying the inequalities. For 6/5 the alphabet is $\mathcal{C}(6,5) = \{-1, 1\}$, and a prime successor requires $Q \equiv 1 \pmod 5$ (then $c = -1$) or $Q \equiv 4 \pmod 5$ (then $c = 1$), since $c \equiv -Q \pmod 5$; for 9/7 it is $\mathcal{C}(9,7) = \{-4, -2, 2, 4, 8\}$.

Table 11: The sieve counts A_s for $0 \le s \le \ell$.

Base	ℓ	$A_0, \ldots, A_\ell$	C	Range
9/7	18	1, 5, 19, 67, 199, 530, 1315, 3130, 7250, 16511, 37155, 82942, 183838, 404586, 883163, 1909362, 4083475, 8631875, 18035747	13	$P > 19$
6/5	40	1, 2, 4, 8, 15, 28, 50, 79, 124, 191, 285, 417, 601, 864, 1233, 1757, 2493, 3522, 4950, 6965, 9717, 13582, 18786, 26085, 35937, 49661, 67971, 93166, 126734, 172576, 233228, 315696, 423604, 568869, 757642, 1010562, 1335866, 1766366, 2315828, 3038864, 3955763	10	$P > 41$

D.3. **Recovering the initial integer.**

Lemma 66. *Let $N \geq 2$ and let c be a word of length N. (a)* (Seed.) *Put*

$$s(c) = (-C_N(a^N)^{-1}) \bmod b^N \in [0, b^N).$$

Every prime $P \leq b^N$ whose orbit has carry word c equals $s(c)$; in particular, if $s(c)$ is even or $s(c) \leq \max(a, N+1)$, no such prime exceeds $\max(a, N+1)$. (b) (Interval.) *The word is realized by a real orbit segment with $\lfloor \xi \rfloor = s(c)$ if and only if the interval of Lemma 64(ii) is nonempty. (c)* (Roots.) *If a prime $p \leq N+1$, $p \nmid 2ab$, covers $\mathbb{F}_p$, or if*

$$s(c) \bmod p \in \{R_0, \ldots, R_N\},$$

then every orbit with initial integer $P = s(c) > N+1$ and carry word c has a term Q_j with $p \mid Q_j$, $Q_j \geq P > p$, hence composite.

Proof. (a) By Lemma 64(i), $P \equiv s(c) \pmod{b^N}$; since $0 < P \leq b^N$ and b^N is composite, $P = s(c)$ (and $s(c) = 0$ would give $P = b^N$, not prime).

(b) is Lemma 64(ii).

(c) is Lemma 64(iii) together with $Q_j \geq Q_0 = P$. □

D.4. **Proof of Theorem 47.**

Proof of Theorem 47. (a) Let $6 < P = \lfloor \xi \rfloor \leq 5^{40}$ and suppose that none of $Q_0, \ldots, Q_{40}$ is composite; since all are $\geq P \geq 7$, all are prime. Put $N = 40$. If $P \leq 41 = \max(6, N+1)$, then $P \in \{7, 11, \ldots, 41\}$, and each of these ten primes is excluded individually (Table 13).

If $P \geq 43$, then all terms are primes larger than $\max(a, N+1)$, so the carry word c of length 40 satisfies (1)–(3) of Section D.2 with $a/b = 6/5$, and by Lemma 66(a) $P = s(c)$, which is odd and larger than 41, and by Lemma 66(c)

$$s(c) \bmod p \notin \{R_j\}$$

for every prime $p \in \{7, 11, \ldots, 41\}$.

The complete enumeration of all $A_{40} = 3955763$ words satisfying (1)–(3) (Table 12) finds 1978668 with even seed and 1977095 rejected by the root test, none with seed ≤ 41, and no survivor. This contradiction proves (a).

(b) Let $9 < P \leq 7^{18}$ and suppose none of $Q_0, \ldots, Q_{18}$ is composite. With $N = 18$, the primes $P \leq 19 = \max(9, N+1)$, namely $11, 13, 17, 19$, are excluded individually (Table 13).

For $P \geq 23$ the same sieve with $a/b = 9/7$, primes $p \in \{5, 11, 13, 17, 19\}$, and $A_{18} = 18035747$ words finds 9018851 even seeds and 9012304 root rejections, none with seed ≤ 19, and 4592 surviving candidates P.

For each survivor, all terms are primes, hence coprime to 6, so by Proposition 55 the orbit follows

$$T(Q) = Q + 2\lfloor (Q+4)/7 \rfloor$$

(the map T_1), and the carries of this T-orbit are those of the sieve word: the carry is determined by

$$c \equiv -2Q \pmod 7,$$

and the letters $-4, -2, 2, 4, 8$ have distinct residues $3, 5, 2, 4, 1$ modulo 7.

Each of the 4592 candidates has a proper divisor at some time $j \leq 7$: there is an integer $1 < d < T^j(P)$ with $d \mid T^j(P)$, and the recorded witnesses use divisors at most 9923, each checked as an exact equality

$$T^j(P) = d \cdot e.$$

Hence Q_j is composite for some $j \le 7 \le 18$, a contradiction. □

Table 12: The finite-height sieve. "Words" is A_N; "even" and "roots" count words whose seed is even or is rejected by Lemma 66(c); "small" counts seeds $\le \max(a, N+1)$.

Base	N	Height b^N	Words	Even seed	Root rejection	Small seed	Survivors
6/5	40	5^{40}	3955763	1978668	1977095	0	0
9/7	18	7^{18}	18035747	9018851	9012304	0	4592

For the 4592 survivors the maximal time at which a divisor is used is 7. The distribution of these times over the candidates depends on the order and range of the trial divisions and is not reported; the statement used in the proof is the existence of a proper divisor at some time $j \le 7$ for every candidate.

Table 13: Small initial primes. For $x \in [P, P+1)$ the successor $\lfloor rx \rfloor$ has at most two candidate values for 6/5 and at most three for 9/7 (Section C.5); the rows list every candidate where it matters. For 6/5, "$Q \equiv s \pmod 5$" with $s \notin \{1, 4\}$ means that no candidate successor of Q is coprime to 30 (Section D.2), so every candidate, being at least 8, is composite; where a unique candidate coprime to 30 exists, it is named. "Empty interval" means that the exact interval of Section C.3 for the all-prime T-orbit becomes empty at the indicated step, so that orbit is realized by no ξ, while any deviation from it is a term divisible by 2 or 3, hence composite.

Base	P	Witness	Composite by time
6/5	7	$7 \equiv 2 \pmod 5$	1
6/5	11	the only candidate coprime to 30 is 13; $13 \equiv 3 \pmod 5$	2
6/5	13	$13 \equiv 3 \pmod 5$	1
6/5	17	$17 \equiv 2 \pmod 5$	1
6/5	19	the only candidate coprime to 30 is 23; $23 \equiv 3 \pmod 5$	2
6/5	23	$23 \equiv 3 \pmod 5$	1
6/5	29	both candidates 34 and $35 = 5 \cdot 7$ are composite	1
6/5	31	the only candidate coprime to 30 is 37; $37 \equiv 2 \pmod 5$	2
6/5	37	$37 \equiv 2 \pmod 5$	1
6/5	41	both candidates $49 = 7 \cdot 7$ and 50 are composite	1
9/7	11	both candidates 14 and $15 = 3 \cdot 5$ are composite	1
9/7	13	empty interval after step 4 ($L = 2450 > U = 2401$, $B = 7^4$)	4
9/7	17	empty interval after step 3 ($L = 350 > U = 343$, $B = 7^3$)	3
9/7	19	both candidates 24 and $25 = 5 \cdot 5$ are composite	1

Theorem 47 says nothing about $\lfloor \xi \rfloor > 5^{40}$ or $\lfloor \xi \rfloor > 7^{18}$. For an initial integer $s + kb^N$ with the same first N carries as s, the terms are

$$Q_j(s) + ka^j b^{N-j},$$

and a divisor found for $k = 0$ need not divide the terms for $k \neq 0$.

D.5. **Verification of the finite-height computation.** The counts of Tables 11–13 and the list of 4592 survivors were produced by implementation A of Section 9.3, that is, by the two same-lineage constructions of the original programs, which track the exact interval in different coordinates (the current fractional interval with the seed lifted digit by digit, and the initial interval with common denominator a^N with the seed solved at the leaves) and agree on all depth counts, all classification counts, and the survivor list; on six small instances

$$(a, b, N) \in \{(3, 2, 5), (4, 3, 5), (5, 3, 5), (6, 5, 6), (7, 5, 5), (9, 7, 5)\}$$

with $9, 11, 37, 50, 54, 530$ words, they also agree with a Python oracle enumerating all words by direct product and exact rational arithmetic. The sieve counts of Table 11 were recomputed in full. The finite-height computations and their reproduction are described in the computation guide.

Independently of these, implementation B of Section 9.3 (a different programmer, code written from the specification of Lemmas 64–66 and not from the original programs) re-enumerated the sieve for both bases, recomputed the seeds, the root rejections and the survivor list, checked every survivor's divisor equality and every small-prime witness, and recomputed the constants of Table 11 as least integers from its own counts. All quantities compared element by element (the count arrays, the classification counts, the survivor set, the 4592 witness equalities, the fourteen small-prime witnesses, the six small instances, and the constants) agree. The comparison is described in the computation guide, with reference data in checks/reference.json. Agreement is evidence that both implementations computed the same finite objects; the passage from these objects to Theorem 47 is the argument of Sections D.2–D.4.

Appendix E. Long prime prefixes and return-word obstructions

E.1. **Arbitrarily long prime prefixes along the family.**

Lemma 67 (transplanting a prime progression)**.** *Let $L \geq 3$, let $P, P+D, \ldots, P+(L-1)D$ be primes larger than 3, and let*

$$P > 48L^2D^2.$$

Let b be an integer with $b \equiv 1 \pmod 6$ and $|b - 2P/D| \leq 3$, and put $a = b+2$, $t = (b-1)/6$, $\xi = P + 1/2$. Then $t \geq 1$, $P > a$, and

$$\lfloor \xi(a/b)^j \rfloor = P + jD$$

for $0 \leq j < L$.

Proof. $6 \mid D$. If D were odd, one of $P, P + D$ would be even; if $3 \nmid D$, the three terms $P, P + D, P + 2D$ would cover all residues modulo 3.

Existence of b. Every real number is within 3 of an integer $\equiv 1 \pmod 6$. Since

$$2P/D > 96L^2D \geq 96 \cdot 9 \cdot 6,$$

we get

$$b \geq 2P/D - 3 \geq P/D \geq 7,$$

so $t \geq 1$.

First estimate. Write $b = 2P/D + \varepsilon$ with $|\varepsilon| \leq 3$; then

$$(2P + 1)/b - D = (1 - D\varepsilon)/b$$

and $|1 - D\varepsilon| \leq 4D$, so

$$|(2P + 1)/b - D| \leq 4D^2/P.$$

Binomial expansion. With $u = 2/b$,

$$(1+u)^j = 1 + ju + R_j$$

where

$$0 \le R_j \le \sum_{k\ge 2} (ju)^k/k! \le \tfrac{1}{2}(ju)^2 e^{ju};$$

here

$$ju \le 2L/b \le 2LD/P < 1,$$

so $e^{ju} < 3$ and

$$R_j \le \tfrac{3}{2}(ju)^2 = 6j^2/b^2 \le 6j^2D^2/P^2.$$

Therefore

$$\begin{aligned}
&\Big|(P+\tfrac{1}{2})(1+u)^j - (P + jD + \tfrac{1}{2})\Big| \\
&\le j\Big|\frac{2P+1}{b} - D\Big| + (P+\tfrac{1}{2})R_j \\
&\le \frac{4jD^2}{P} + \frac{8j^2D^2}{P} \\
&\le \frac{12L^2D^2}{P} < \frac{1}{4},
\end{aligned}$$

using $(P + 1/2)/P \le 4/3$ and $4j + 8j^2 \le 12L^2$ for $j < L$. Hence

$$\xi(a/b)^j \in (P + jD + 1/4, P + jD + 3/4)$$

and the integer part is $P + jD$.

$P > a$.

$$a = b + 2 \le 2P/D + 5 \le P/3 + 5 < P$$

since $D \ge 6$. □

Theorem 68 (arbitrarily long prime prefixes)**.** *For every $L \ge 3$ and every T there are $t \ge T$ and a rational $\xi > 0$ such that $\lfloor \xi((6t+3)/(6t+1))^j \rfloor$ is prime for $0 \le j < L$ and the first term exceeds $6t + 3$. Consequently there is no bound, uniform in t and in the initial prime, on the number of steps after which the program of Section C.3 halts.*

Proof. This uses as an external input the theorem on narrow arithmetic progressions in the primes: for $k \ge 2$ and $\delta > 0$, every subset A of the primes in $[1, Y]$ with

$$|A| \ge \delta Y/\log Y$$

contains, for all sufficiently large Y, a nontrivial k-term arithmetic progression whose common difference D satisfies

$$|D| \le C_{k,\delta}(\log Y)^{(k-1)2^{k-2}}$$

SHAO15, Theorem 1.3; the qualitative form, with an exponent depending on k and δ only, is TZ14, Theorem 5 specialized to arithmetic progressions. (In [TZ14] and [SHAO15] the letter L denotes the exponent; here L is the length k.)

Take $k = L$ and

$$A = \{p \text{ prime} : Y/2 < p \le Y\};$$

by the prime number theorem

$$|A| \ge \tfrac{1}{3} Y/\log Y$$

for large Y, so $\delta = 1/3$ is admissible.

The progression obtained lies in $(Y/2, Y]$, so its least term P satisfies $P > Y/2$, all terms exceed 3 for large Y, and after reversing the order if necessary $D > 0$ with

$$D \le C(\log Y)^{(L-1)2^{L-2}}.$$

Hence

$$P/(48L^2D^2) \to \infty$$

as $Y \to \infty$, and Lemma 67 applies for large Y, giving

$$t = (b-1)/6 \ge (P/D - 1)/6 \to \infty. \qquad \square$$

The statement is

$$\forall L\, \forall T\, \exists t \ge T\, \exists \xi;$$

it does not assert an infinite tail of primes for any base, and it does not contradict any halting statement for a fixed t. The implied constants and the threshold for Y in the external theorem are not explicit, so no explicit t is obtained this way. A narrow-progression theorem is essential: a progression with D comparable to P would not satisfy the hypothesis of Lemma 67.

Example (nine prime terms). With

$$t = 1654617, \qquad a = 9927705, \qquad b = 9927703,$$
$$\xi = 2084817839/2, \qquad P = 1042408919, \qquad D = 210,$$

one has

$$\lfloor \xi (a/b)^j \rfloor = P + 210j$$

for $0 \le j \le 8$, and these nine integers are prime, while

$$\lfloor \xi (a/b)^9 \rfloor = 1042410809 = 11 \cdot 94764619.$$

The hypothesis

$$P > 48 \cdot 81 \cdot 210^2 = 171460800$$

holds, and b is the integer $\equiv 1 \pmod 6$ nearest to $2P/D$. The integer parts were computed in exact rational arithmetic and the primes checked by trial division. This is not a counterexample to anything.

E.2. **Return words as an obstruction.** The following lemma is of the same type as Theorem 42: it constructs a right-directed pseudo-orbit with non-eventually-periodic carries and applies Lemma 38.

Lemma 69 (return-word obstruction)**.** *Let* $\gcd(m, ab) = 1$*,* $q \in (\mathbb{Z}/m)^\times$*, and let* U, V *be carry words such that*
(1) *following*

$$bz \equiv aq' + c \pmod{m}$$

from q *along each word, all residues are units and the last residue is* q*;*
(2) *there is a closed interval* $I \subset (0,1)$ *such that the backward maps*

$$\theta \mapsto (b\theta + c)/a$$

applied along each word from a point of I *keep all values in* $(0,1)$ *and map* I *into* I*;*
(3) $UV \neq VU$*;*
(4) *every letter satisfies*

$$\gcd(c, ab) = 1 \qquad \text{and} \qquad c \equiv b - a \pmod{2}.$$

Then for every M *whose prime factors divide* $2abm$ *there is a right-directed* M*-pseudo-orbit with non-eventually-periodic carries, and no execution of* $\mathfrak{M}_0$ *(or of an extension by pseudo-orbit-sound operations) with all* $M_t \mid M$ *succeeds.*

Proof. (i) Prime powers of m*.* Let $\tilde{m}$ be the least common multiple of m and the part of M composed of primes of m. For a word w of length ℓ, the residue map

$$\pi_w : x \mapsto (a/b)^\ell x + C(w)/b^\ell$$

modulo $\tilde{m}$ is an affine permutation ($p \nmid ab$). Since $\pi_U(q) \equiv q \pmod{m}$, π_U permutes the finite fibre

$$F = \{x \in \mathbb{Z}/\tilde{m} : x \equiv q \pmod{m}\},$$

and likewise π_V; so for any $\hat{x} \in F$ there are $s, t \geq 1$ with

$$\pi_U^s(\hat{x}) = \hat{x} \qquad \text{and} \qquad \pi_V^t(\hat{x}) = \hat{x}.$$

Intermediate residues reduce modulo m to the units of (1), hence are units modulo $\tilde{m}$.

(ii) Non-commutation. If $U^s V^t = V^t U^s$, then U^s and V^t are powers of a common primitive word z; since the primitive root of w^k is that of w, U and V are both powers of z and commute, contradicting (3).

(iii) Fractional parts. U, V map I into I, hence so do U^s, V^t, with all intermediate values in $(0,1)$. Concatenate the blocks $X_0 = U^s V^t$ and $X_1 = V^t U^s$ (of equal length and distinct) according to a binary sequence that is not eventually periodic; the nested closed intervals of the proof of Theorem 42 (each backward block map is a contraction with slope $(b/a)^{|X_i|}$) define θ_n for all $n \geq 0$ with

$$a\theta_n - b\theta_{n+1} = c_n$$

exactly, and block decoding shows that the carries are not eventually periodic.

(iv) Primes dividing $2ab$*.* For $p \mid a$, $p \mid b$, and $p = 2$ with a, b odd, construct unit residues modulo p^e as in the proof of Theorem 42, using (4). Combine all residues by the Chinese remainder theorem to a pseudo-orbit modulo $\tilde{m} \cdot$ (the rest of M) and reduce to M. Lemma 38 gives the conclusion. □

Corollary 70 (explicit witnesses)**.** *(a) For* $9/7$*, with*

$$m = 715 = 5 \cdot 11 \cdot 13, \qquad q = 713, \qquad I = [1/5, 3/10],$$

and the words U, V *of Table 14, no execution of* $\mathfrak{M}_0$ *whose moduli involve only the primes* $S_{13} = \{2, 3, 5, 7, 11, 13\}$ *succeeds; in particular none with modulus dividing a power of* 4290*. (b) For* $6/5$*, with*

$$m = 17017 = 7 \cdot 11 \cdot 13 \cdot 17, \qquad q = 12854, \qquad I = [9/10, 49/50],$$

and the words U, V *of Table 15, no execution whose moduli involve only the primes* $S_{17} = \{2, 3, 5, 7, 11, 13, 17\}$ *succeeds.*

Proof. The hypotheses of Lemma 69 are verified in exact arithmetic: the residue sequences of Tables 14–15 return to q through units; the backward images of the endpoints of I (sufficient, since the maps are increasing and affine) stay in $(0, 1)$ and land in I; $UV \neq VU$ since the words have different lengths and neither is a prefix of the other's period structure (for $9/7$, U and V differ at position 2; for $6/5$, at position 19); and all letters lie in $\mathcal{C}(9, 7) = \{-4, -2, 2, 4, 8\}$, respectively $\mathcal{C}(6, 5) = \{-1, 1\}$. □

Table 14: The $9/7$ witness ($m = 715$, $q = 713$). Residues are listed from q to the return.

Word	Letters	Residues
U (length 18)	$2, -2, 2, -2, 4, -4, 2, 2, 2, 2, -4, 4, -2, 4, 4, -4, 2, 4$	713, 202, 668, 42, 258, 128, 164, 109, 549, 604, 164, 6, 519, 667, 41, 564, 316, 713, 713
V (length 17)	$2, -2, -4, 8, -4, 4, -2, 4, -4, 4, 2, 2, 2, -4, 2, 4, 4$	713, 202, 668, 654, 127, 367, 166, 111, 654, 636, 614, 279, 359, 564, 316, 713, 713, 713

Table 15: The $6/5$ witness ($m = 17017$, $q = 12854$); $+$ and $-$ denote the carries $+1$ and -1.

Word	Letters	Residues
U (length 33)	$+^{15} - +^{3} -^{3} +^{9} -^{2}$	12854, 15425, 8300, 16767, 6507, 11212, 16858, 13423, 9301, 7758, 2503, 13214, 15857, 5415, 13305, 5756, 6907, 11692, 417, 3904, 8088, 6302, 14369, 226, 13885, 6452, 11146, 9972, 15370, 8234, 9881, 8454, 13548, 12854
V (length 39)	$+^{15} - +^{12} -^{2} +^{3} - +^{3} -^{2}$	12854, 15425, 8300, 16767, 6507, 11212, 16858, 13423, 9301, 7758, 2503, 13214, 15857, 5415, 13305, 5756, 6907, 11692, 417, 3904, 4685, 12429, 14915, 7688, 2419, 2903, 13694, 16433, 12913, 12092, 4300, 11967, 747, 4300, 15370, 8234, 9881, 8454, 13548, 12854

For $6/5$ a shorter witness modulo 1001 (root 380, $I = [4/5, 19/20]$, two words of length 27) also exists; the witness modulo 17017 suffices for (b). Examples of the lifting orders in step (i): for $9/7$ modulo $715 \cdot 55$ one may take $(s, t) = (10, 20)$, and modulo $715 \cdot 169$, $(26, 52)$; for $6/5$ modulo $17017 \cdot 77$, $(14, 14)$, and modulo $17017 \cdot 289$, $(136, 136)$.

As in Section 8, these are statements about the non-termination of the procedure. The residues q_n form a sequence of residue classes, not the integer parts of a single orbit. By Lemma 41 every finite window of the witness is realized by a true orbit with arbitrarily large coefficient, in the form $\forall N \, \exists \xi_N$; this does not give a single ξ realizing the whole witness,

and the realizing integers are not claimed to be prime, only coprime to the specified primes. Unlike Theorem 58, no initial integer is fixed here, so the compactness argument does not apply.

E.3. **Eliminating one return-word language with a further prime.**

Proposition 71. *Let $a = 9$, $b = 7$, and let U, V be the words of Table 14. Regard U and V as single edges and form the graph on the residues $q \in \{1, \dots, 16\}$ modulo 17 with an edge $q \xrightarrow{w} z$ whenever following w from q modulo 17 keeps all residues, including the last, nonzero. The graph has exactly the twelve edges*

$$4 \xrightarrow{U} 16,\ 4 \xrightarrow{V} 16,\ 5 \xrightarrow{U} 1,$$
$$6 \xrightarrow{V} 4,\ 8 \xrightarrow{V} 9,\ 9 \xrightarrow{U} 9,$$
$$10 \xrightarrow{V} 14,\ 11 \xrightarrow{U} 13,\ 11 \xrightarrow{V} 8,$$
$$14 \xrightarrow{U} 2,\ 15 \xrightarrow{U} 4,\ 16 \xrightarrow{V} 12,$$

and with the ranks

$$\operatorname{rank}(1, \dots, 16) = (0, 0, 0, 2, 1, 3, 0, 1, 0, 2, 2, 0, 0, 1, 3, 1)$$

every edge other than the self-loop $9 \xrightarrow{U} 9$ strictly decreases the rank. Consequently there is no true orbit with ratio $9/7$ all of whose terms are prime and whose carry sequence is an infinite concatenation of U and V.

Proof. The edge list and the rank inequalities are finite verifications. For an all-prime orbit, $17 \nmid Q_n$ once $Q_n > 17$, so the residues Q_n mod 17 at block boundaries form an infinite walk in the graph. Since the rank strictly decreases on every edge except the self-loop, after finitely many blocks only $9 \xrightarrow{U} 9$ is used, and the carry sequence is eventually U^∞, periodic with period 18, contradicting Lemma 4. □

The edge table depends only on a mod 17 and b mod 17, so the same twelve edges arise for every pair $a \equiv 9$, $b \equiv 7$ (mod 17); but U and V are return words only for $9/7$ (for $a = 111$, $b = 109$, the word U from 713 modulo 715 does not return and meets a non-unit), so nothing is asserted for other members of the family. Proposition 71 eliminates the language generated by U and V; it does not cover all carry sequences of $9/7$, and it does not imply that adding the prime 17 yields a certificate for $9/7$.

E.4. **Explicit two-carry certificates, prime selection, and the whole-word interval test.** *Instances of Theorem 42.* For $a/b = 101/2$ with $R = 15$ (auxiliary primes $3, 5$):

$$\Delta = 99, \qquad m = 1, \qquad h = 30,$$
$$c = 69, \qquad \gcd(69, 202) = 1;$$

the least admissible L is 0, with

$$x_0 = 23/33, \qquad x_1 = 3379/3399,$$
$$I = [23/33, 3379/3399], \quad \text{and} \quad U = (69), \quad V = (99, 69).$$

Hence no execution of $\mathfrak{M}_0$ whose moduli involve only the primes $\{2, 3, 5, 101\}$ succeeds.

For $a/b = 101/100$ with $R = 21$ (primes $3, 7$):

$$\Delta = 1, \qquad m = 1, \qquad h = 42,$$
$$c = -41, \qquad \gcd(41, 10100) = 1;$$

the threshold is $-c/b = 41/100$, and the least L with $x_L > 41/100$, where

$$x_L = 1 - 42 \cdot 100^L/(101^{L+1} - 100^{L+1}),$$

is $L = 54$, so $I = [x_{54}, x_{55}]$ with the words $U = 1^{54}(-41)$ and $V = 1^{55}(-41)$. This example shows that negative carries are handled exactly.

For $5/2$ with $R = 1$:

$$\Delta = 3, \qquad m = 1, \qquad h = 2, \qquad c = 1, \qquad L = 0,$$
$$U = (1), \qquad V = (3, 1),$$

which is the witness of Corollary 43 for this base.

Lemma 72 (primes producing a common nonzero translation)**.** *Let $W = \{w_1, \dots, w_s\}$ be nonempty words with lengths ℓ_i and $C_i = C(w_i)$ as in (8), $g = \gcd(\ell_1, \dots, \ell_s)$, and*

$$D = \gcd\big(a^g - b^g,\ C_i b^{\ell_1} - C_1 b^{\ell_i}\ (2 \le i \le s)\big).$$

For a prime $p \nmid ab$, the residue maps

$$F_i(q) = (a^{\ell_i} q + C_i)/b^{\ell_i}$$

are all congruent modulo p to one and the same translation $q \mapsto q + \tau$ with $\tau \neq 0$ if and only if $p \mid D$ and $p \nmid C_1$. In that case, no infinite concatenation of words of W survives the lift by p: the residues at block boundaries are $q, q+\tau, q+2\tau, \dots$ and vanish within p blocks.

Proof. The slope of F_i is $(a/b)^{\ell_i}$; all slopes are 1 if and only if the order of a/b modulo p divides every ℓ_i, hence divides g, that is, $p \mid a^g - b^g$.

The translations C_i/b^{ℓ_i} coincide if and only if

$$C_i b^{\ell_1} \equiv C_1 b^{\ell_i} \pmod{p},$$

and the common translation is nonzero if and only if $p \nmid C_1$ (b being a unit). □

This eliminates only concatenations of W; it removes a component only when all return words of the component lie in W. For the two-carry words $U = \Delta^L c$, $V = \Delta^{L+1} c$ of Theorem 42, $g = 1$ and $D \mid a - b$; if $p \mid \Delta$ and $p \nmid c$, both words act as $q \mapsto q + c/b$. For $101/2$: $D = 99$, $C_1 = 69$; the prime 3 divides C_1 and does not work, while 11 gives the translation

$$\tau = 69 \cdot 2^{-1} \equiv 7 \pmod{11},$$

so the two-word branching that survives 3 and 5 is eliminated by adding 11. For 5/2: $D = 3$, and 3 plays the same role. When $\Delta \in \{1, 2\}$, $D \mid 2$ and no odd prime is obtained this way; this is a limitation of this particular mechanism.

Proposition 73 (whole-word interval test)**.** *For a macro edge with starting cell $[j/K, (j+1)/K)$ and word $c_0 \cdots c_{\ell-1}$, define*

$$\begin{aligned}(L_0, U_0, B_0) &= (j, j+1, K) \qquad \text{and}\\ B_{i+1} &= bB_i,\\ L_{i+1} &= \max(0, aL_i - c_iB_i),\\ U_{i+1} &= \min(B_{i+1}, aU_i - c_iB_i).\end{aligned}$$

Then $[L_i/B_i, U_i/B_i)$ is exactly the set of fractional parts after i letters of the real orbit segments starting in the cell with these carries. Hence rejecting a macro edge when $L_i \geq U_i$ for some i (or when the final interval misses the target cell) is pseudo-orbit-sound: every pseudo-orbit, and in particular every true orbit, passing through the edge has exact fractional parts witnessing nonemptiness. The operation is not walk-sound, since (E3) expresses only the compatibility of two adjacent cells, and a path of length ≥ 2 in $G(M, K)$ need not admit real fractional parts.

Proof. The map

$$\theta \mapsto (a\theta - c)/b$$

is an increasing affine bijection, so the image of $[L/B, U/B)$ is

$$[(aL - cB)/(bB), (aU - cB)/(bB)),$$

and intersecting with $[0, 1)$ gives the update; exactness follows by induction as in Proposition 57. □

Whether adjoining this test changes the class of Theorem 37 is not determined here; an exploratory computation with it is not adopted, and the certificates of Section 5 and Appendix F do not use it. The witnesses of Theorem 42 and Lemma 69 pass this test at every intermediate time, which distinguishes them from spurious local paths.

Appendix F. The ratio 5/3

This appendix proves Theorem 1(iii). The certificate is a word-labelled certificate as in Section 4 with one additional operation, the exact contraction of in-degree-one chains (Section F.2). Section F.4 shows that 5/3 belongs to the class $\mathcal{F}_{\text{cert}}$ of Section 7.4, and Section F.5 records a companion obstruction for the primes up to 23. The verification of the finite calculation is described in Section 9.5.

F.1. **The carry alphabet and the initial graph.**

Lemma 74. *Let $a = 5$, $b = 3$. If Q_n and Q_{n+1} are odd and $3 \nmid Q_n$, then*

$$c_n \in \{-2, 2, 4\}.$$

Hence (H) holds with

$$S_0 = \{2, 3\} \qquad \text{and} \qquad \mathcal{C} = \{-2, 2, 4\}.$$

Proof. By (6), $-2 \leq c_n \leq 4$. Since $c_n = 3Q_{n+1} - 5Q_n \equiv Q_{n+1} + Q_n \pmod 2$, the carry is even. If $c_n = 0$, then $3Q_{n+1} = 5Q_n$ and $3 \mid Q_n$. □

No condition modulo 5 is used, and the modulus of (2a) does not contain the prime 5. The general alphabet of Section 7.1 is the same set: $\mathcal{C}(5,3) = \{-2, 2, 4\}$.

With $K = 3$, the initial graph G_0^{in} of Theorem 19 has the vertices $j \in \{0, 1, 2\}$ and the six edges $(j, (c), k)$ with $c \in \{-2, 2, 4\}$ and

$$5j - 3(k+1) < 3c < 5(j+1) - 3k,$$

listed in Table 16.

Table 16: The six initial edges for 5/3, $K = 3$, as pairs (carry, target cell).

Source cell j	Edges (c, k)
0	$(-2, 2)$
1	$(2, 0), (2, 1)$
2	$(2, 1), (2, 2), (4, 0)$

F.2. **Exact contraction of in-degree-one chains.** Let $G' = R(G)$, with identical triples merged, and count in-degrees $\deg^-(v)$ in G'. Let B^- consist of all vertices of G' with $\deg^- \geq 2$, together with one anchor (the least vertex) in every component that contains no such vertex. For $v \in B^-$ and each in-edge $e = (u, w, v)$ of v: while $u \notin B^-$, take the unique in-edge (u', w', u) of u and replace $w \leftarrow w'w$, $u \leftarrow u'$. When first $u \in B^-$, record the macro edge (u, w, v). The *in-contraction* $P^-(G')$ has vertex set B^- and the set of these macro edges. For a word-labelled graph H, let $\mathrm{rev}(H)$ reverse every edge and the letter order of every word.

Lemma 75 (exact in-degree-one contraction)**.** *(a) The backward iteration terminates, every infinite walk of G' visits B^- infinitely often, and each segment of the walk between consecutive visits to B^- is a macro edge of $P^-(G')$, with the same word. Consequently, if $G' \models n_1$, then $P^-(G') \models n_2$, where n_2 is the first time at or after n_1 at which the covering walk visits B^-.*

(b) If a cyclic SCC of $\mathrm{rev}(G)$ *satisfies (19) with h, λ, d, then it satisfies (19) in G with h', λ', d,*

$$h'(x) = -h(x), \qquad \lambda'_s = \lambda_{(-s-1) \bmod d}.$$

Hence, by Theorem 21(a), a cyclic SCC of G passes the phase test if and only if it passes in $\mathrm{rev}(G)$*, $R(\mathrm{rev}(G)) = \mathrm{rev}(R(G))$, and*

$$\mathrm{rev}(P(R(\mathrm{rev}(G)))) = P^-(R(G)).$$

Proof. (a) Every vertex of G' has $\deg^- \geq 1$, by the argument of Lemma 14(a), so a vertex outside B^- has exactly one in-edge. If the backward iteration did not terminate, it would visit only finitely many vertices of in-degree one and would enter a cycle Z of such vertices. The unique in-edge of every vertex of Z starts in Z, so no edge enters Z from the rest of its component C; strong connectivity gives $C = Z$, and the anchor of C lies in $B^- \cap Z$: a contradiction. Similarly, suppose that an infinite walk (v_i, w_i) avoids B^- for $i \geq i_0$. Then (v_i, w_i, v_{i+1}) is the unique in-edge of v_{i+1} for $i \geq i_0$. Some vertex repeats, $v_i = v_j$ with $i_0 < i < j$, and $v_i, \ldots, v_{j-1}$ form a cycle of in-degree-one vertices whose in-edges lie on the cycle; as before it is a whole component and contains an anchor, a contradiction.

Let $i < j$ be consecutive visits to B^-. For $i < k < j$, the edge (v_{k-1}, w_{k-1}, v_k) is the unique in-edge of v_k. Starting from the in-edge (v_{j-1}, w_{j-1}, v_j) of $v_j \in B^-$, the backward

iteration therefore passes through $v_{j-1}, \dots, v_{i+1}$ and stops at v_i, recording the macro edge $(v_i, w_i \cdots w_{j-1}, v_j)$. The walk of macro edges between the successive visits has the same output, and the covering statement follows as in Proposition 16.

(b) Let $(v, \mathrm{rev}(w), u)$ be an internal edge of the SCC in $\mathrm{rev}(G)$, so that $h(u) \equiv h(v) + |w| \pmod d$ and $w_{|w|-1-i} = \lambda_{(h(v)+i) \bmod d}$. Then $h'(v) \equiv h'(u) + |w|$, and for $0 \le i < |w|$,

$$w_i = \lambda_{(h(v)+|w|-1-i) \bmod d} = \lambda_{(h(u)-1-i) \bmod d} = \lambda'_{(h'(u)+i) \bmod d}.$$

The converse is the same computation applied to $\mathrm{rev}(G)$, since rev is an involution. By Theorem 21(a), passing the phase test does not depend on the choice of h among functions satisfying (19), so the test gives the same result in both graphs. Reversal preserves the strongly connected components and their cyclicity. Hence the retained components of G and $\mathrm{rev}(G)$ coincide, and $R(\mathrm{rev}(G)) = \mathrm{rev}(R(G))$. In $\mathrm{rev}(R(G))$ out-degrees are in-degrees of $R(G)$, the same anchors are chosen, and the forward iteration of Section 4.3 is the backward iteration above with reversed words. □

Part (a) is the in-direction counterpart of Lemma 15 and Proposition 16. By part (b), the implementations compute $P^-(R(G))$ as "reverse, remove, out-contract, reverse back"; the graphs of Table 17 therefore rely on part (b) as well. The proofs below use part (a) for right-directed walks, and the Lean proof of Theorem 1(iii) checks the backward passes directly, without part (b).

F.3. **The finite calculation.** For 5/3 the stage map $G_k^{\mathrm{out}} = P(R(G_k^{\mathrm{in}}))$ of Theorem 19 is replaced by a finite sequence of passes. A *forward pass* maps H to $P(R(H))$, and a *backward pass* maps H to $P^-(R(H))$. Each stage applies one forward pass and then repeats a backward pass followed by a forward pass, at most ten times or until such a round leaves the number of vertices unchanged. The stopping rule is a choice of computation, not part of the soundness argument.

Theorem 76 (two-direction finite success)**.** *Theorem 19 remains valid when each G_k^{out} is obtained from G_k^{in} by any finite sequence of forward and backward passes.*

Proof. In the proof of Theorem 19, Proposition 13 and Proposition 16 give the passage from a covering of G_k^{in} to a covering of $P(R(G_k^{\mathrm{in}}))$ from a later time. Proposition 13 and Lemma 75(a) give the same passage for a backward pass. Composing finitely many passes preserves a covering from a later time, and the remaining steps of the proof are unchanged. □

The adopted execution uses $K = 3$, the alphabet of Lemma 74, and the nine primes

$$p_1, \dots, p_9 = 7, 11, 13, 17, 19, 23, 29, 31, 37.$$

Table 17 lists its ten stages. The cyclic and certified columns refer to the first removal of each stage; the output columns refer to the graph after all passes of the stage. Edges are counted as distinct word-labelled triples. At stage 9 all 14920 cyclic SCCs of G_9^{in} are certified, and the output graph is empty. No cell refinement, interval propagation, or floating-point pruning is used; the cell count remains 3.

Table 17: The word method for 5/3 with $K = 3$, $\mathcal{C} = \{-2, 2, 4\}$, forward and backward passes. Uncertified vertices refer to the first removal; rounds counts the backward–forward rounds.

k	p_k	Input vertices	Input edges	Cyclic SCCs	Certified	Uncertified vertices	Output vertices	Output edges	Max. word length (output)	Rounds
0	—	3	6	1	0	3	2	5	2	1
1	7	12	24	2	1	9	7	17	5	2
2	11	70	141	2	1	57	38	93	10	2
3	13	456	903	1	0	387	218	573	29	4
4	17	3482	7203	1	0	2873	1546	4295	47	6
5	19	27603	57768	78	77	21797	11127	32604	87	7
6	23	244045	503983	247	246	176295	67809	215777	122	10
7	29	1882118	3735784	117	116	1187270	374123	1283571	243	10
8	31	10869041	18298921	300	299	4232762	729480	2712521	702	9
9	37	21077909	21932101	14920	14920	0	0	0	—	0

Without backward passes, that is, with the stage map of Theorem 19, the same parameters give the input sizes of Table 18, computed by implementation B only as a diagnostic, not as part of the certificate. The lift to the prime 37 was counted without being formed: it would have 313651296 vertices and 416687195 lifted edges (counted before identical triples are merged). With backward passes the largest input graph has 21077909 vertices. A direct product of the unit residues modulo R_{53}, the part of the modulus of (2a) prime to 6, with three cells would have

$$3\varphi(247357937827) = 413857382400$$

vertices; neither staged construction forms it.

Table 18: Input sizes for 5/3 with and without backward passes. Entries marked * were counted but not formed; that edge count includes repeated triples.

k	p_k	Input vertices (forward only)	Input edges (forward only)	Input vertices (both directions)	Input edges (both directions)
0	—	3	6	3	6
1	7	12	24	12	24
2	11	80	152	70	141
3	13	600	1084	456	903
4	17	5868	10376	3482	7203
5	19	59041	102690	27603	57768
6	23	678409	1152740	244045	503983
7	29	7592055	12381638	1882118	3735784
8	31	67194454	102086150	10869041	18298921
9	37	313651296*	416687195*	21077909	21932101

Proof of Theorem 1(iii). Table 17 verifies the hypotheses of Theorem 76 with $S_0 = \{2, 3\}$, $\mathcal{C} = \{-2, 2, 4\}$ (Lemma 74), $K = 3$, the nine primes $7, 11, 13, 17, 19, 23, 29, 31, 37$, each

distinct, coprime to 15, and not in S_0, and an empty output graph at stage 9. Then (20) with

$$M = 6 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 = 1484147626962$$

is (2a). □

F.4. **Membership in the certifiable class.** Put

$$R_{53} = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 = 247357937827.$$

Corollary 77. $5/3 \in \mathcal{F}_{\mathrm{cert}}$*: every cyclic SCC of* $G'(R_{53}, 3)$ *passes the phase test.*

Proof. The graph $G'(R_{53}, 3)$ uses the alphabet $\mathcal{C}(5, 3) = \{-2, 2, 4\}$. Suppose that some infinite walk of it has an output that is not eventually periodic. Its cells give a walk of G_0^{in} with the same output. Its residues modulo each new prime are units at every time, so, by induction over the stages of Table 17, each lift keeps a tail of the projected walk. Lemma 30 (for phase removal, contraction, and lifting, which are walk-sound) and Lemma 75(a) keep a tail through every pass and every lift. The walk would survive in the empty output graph of stage 9, which is impossible. Hence every infinite walk inside a cyclic SCC of $G'(R_{53}, 3)$ has eventually periodic output, and Theorem 21 shows that each such SCC passes the phase test. □

This argument does not add in-contraction to the operations of Section 7.2; it proves a property of the complete graph of the normal form directly, without forming its 413857382400 vertices. By the remark after Theorem 37, the normal form yields the conclusion $\gcd(Q_n, 2 \cdot 3 \cdot 5 \cdot R_{53}) > 1$ infinitely often, whose prime set contains 5; Theorem 1(iii) is the stronger statement obtained from Lemma 74 and Table 17.

F.5. **The prime pool through 23.** The return-word witness used here has

$$m = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 7436429, \qquad q = 3684006, \qquad I = [2/5, 1/2],$$

and two words U, V over $\{-2, 2, 4\}$ of lengths 119 and 134, recorded without abbreviation in `supplement/computations/additional_obstructions/`.

Corollary 78. *For* $5/3$*, no execution of* $\mathfrak{M}_0$*, or of an extension by pseudo-orbit-sound operations, succeeds when all moduli are composed of primes in* $\{2, 3, 5, 7, 11, 13, 17, 19, 23\}$*, for any cell counts and prime powers. The same holds for executions that also use backward passes.*

Proof. The data satisfy the hypotheses of Lemma 69 with $(a, b) = (5, 3)$. A direct check, independent of any search graph, verifies condition (1) at 255 residues, including the starting residue of each word, condition (2) at the endpoints of I for all 253 suffixes (the backward maps are increasing affine maps), $UV \neq VU$, and $\gcd(c, 15) = 1$, $c \equiv 0 \pmod 2$ for every letter. Lemma 69 gives the first statement. The pseudo-orbit it constructs gives a right-directed walk whose output is not eventually periodic at every stage, and Lemma 75(a) keeps a tail of such a walk through a backward pass. □

Thus any certificate of this kind for $5/3$ uses at least one prime $p \geq 29$. The corollary does not show that 29 suffices, that 37 is necessary, or that the prime set of (2a) is minimal. Like the other results of Section 8, it concerns the non-termination of a procedure and is not a counterexample to any arithmetic statement.

Acknowledgments and disclosure of AI involvement

This project began with the author's interest in Mills' constant, the least real number $A > 1$ for which $\lfloor A^{3^n} \rfloor$ is prime for every positive integer n. After learning that this least constant is irrational [SA25], the author asked GPT-6 Pro in ChatGPT (powered by GPT-6 Astra) to prove the stronger assertion that every Mills number is irrational. The author estimates having issued roughly five hundred prompts urging the system to continue and not give up. That exploration did not produce the requested proof. The author subsequently asked the AI to identify mathematically meaningful results suitable for a paper among the material obtained, and to write that paper. The present work grew out of that request; it does not resolve the original question about all Mills numbers.

The author thanks GPT-6 Astra, accessed as GPT-6 Pro in ChatGPT and also used in Codex, as the principal AI system in this work. The author also thanks Claude Fable 5.1 for limited assistance through Claude Code with some tasks during preparation of the paper when Codex's weekly usage limit was reached. After the initial conjecture, all mathematical development, identification of results suitable for publication, proof attempts, computation programs, exposition, and Lean formalization were performed by AI systems. The author supplied no mathematically substantive instructions or advice for these tasks. The author's requests for formalization and for an automated workflow in Codex, in which drafting agents and critic agents iterated before typesetting, specified tasks and a procedure without contributing mathematical content. These exchanges were automated critiques, not independent review by human mathematicians.

The author's sole mathematical contribution to this process was the initial conjecture that every Mills number, not only the least one, is irrational. Beyond that conjecture, the author made no mathematical contribution and supplied no ideas for proofs, mathematical arguments, calculations, or mathematical assessment of the results. The author's subsequent involvement consisted of nonmathematical requests to continue working, identify publishable material, write the paper, and produce Lean proofs, together with the decision to release the work. The author has personally confirmed that the Lean proofs are accepted by Lean; their coverage and trust assumptions are specified in Section 9.4. Readers should assess the arguments, reproducible computations, and formal artifacts themselves.

Version 3.0.0 adds the base 5/3. For this addition, the author proposed reversing the direction of the search: rather than trying to prove the statement for preselected bases, to look for bases that admit a finite certificate of the kind described here, and asked for further bases to be sought. GPT-6 Astra, accessed as GPT-6 Pro in ChatGPT, carried out this search and found the certificate for 5/3. It also developed the mathematics of Appendix F, including the contraction of chains of in-degree one, and wrote the first implementation of the computation. The second implementation, the Lean formalization of the 5/3 theorem, and the integration of the addition into this paper were carried out by AI through Claude Code. Apart from the proposal above, the author's requests for this addition specified tasks without contributing mathematical content.

The author expects that research in which a human poses an initial question and AI carries out all subsequent mathematical or technical work will become increasingly common in mathematics and other disciplines, including cases in which the initiating human provides no substantive guidance. This is the author's personal expectation, not a conclusion established by the mathematical results here. This account makes the provenance of the work, the contributions of the AI systems, and the limits of the author's contribution explicit.

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