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A uniform spectral gap and winding-flux tension on the SU(2) plaquette ladder

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Abstract

For the periodic one-plaquette-wide SU(2) Kogut–Susskind ladder at fixed lattice spacing and every even length $L \geq 2$, we give a computer-assisted argument for a full spectral gap exceeding one fundamental-link electric energy throughout $0 \leq x = \lambda/e \leq 12.25$, and exceeding 0.46 through $x = 16$. All link representations are retained. The winding-sector ground-energy difference per plaquette has an even-length thermodynamic limit. It is at least 0.5584 through $x = 12$, at least 0.20 through $x = 30$, and at least 0.1425 through $x = 32$. A plaquette-product trial gives an exact variational bound at arbitrary length. The spectral estimates combine a split-link reflection construction, finite-ring interval enclosures and Lean certificates. The physical realization of the formal hypotheses and the cutoff limit are paper arguments.

1 Model and claims

A ring of L squares has bottom and top links b_i, t_i from rung i to rung $i+1$ and rung links r_i directed upward. Indices are periodic. Take normalized Haar measure on $\mathrm{SU}(2)^{3L}$ and restrict L^2 to the functions invariant under gauge transformations at all $2L$ vertices. In units of the electric energy $e > 0$ of one fundamental link, set $x = \lambda/e$ and

$$H(x) = H_0 - xW, \qquad H_0 = \tfrac{4}{3}\sum_{\ell} C_\ell, \qquad W = \tfrac{1}{2}\sum_i \mathrm{tr}(b_i r_{i+1} t_i^{-1} r_i^{-1}).$$

The link Casimir has eigenvalue $j_\ell(j_\ell+1)$, the plaquette trace is fundamental and $\|W\| \leq L$. There is one gauge-invariant spin-network vector per admissible labeling because the vertices are trivalent. The three spins at each vertex satisfy the triangle inequalities and have integer sum. Hence $c = (-1)^{2j_{b_i}+2j_{t_i}}$ is independent of i. A fundamental plaquette changes both horizontal spins on its square by half a unit, so H preserves c. Let $E_\pm(L,x)$ denote the two sector minima and $\Delta_+(L,x)$ the difference of the first two ordered levels, with multiplicity, in $c = +1$.

Theorem 1. *For every even $L \geq 2$ and $0 \leq x \leq 12$ the even-winding ground state is simple, $\Delta_+(L,x) \geq g_+(x) > 1$, and $E_-(L,x) - E_+(L,x) \geq L\sigma(x)$. The interval certificates give the bounds in Table 1. The full gauge-invariant ground state is therefore simple, lies in $c = +1$, and has gap at least $\min\{g_+(x), L\sigma(x)\} > 1$.*

At $x = 0$ an odd-winding labeling has at least one half-integer horizontal spin per column. A spin-1/2 loop along one rail attains the electric lower bound L. The vacuum has energy zero. On two plaquettes winding can thus cost less than an elementary plaquette loop of energy four. Theorem 1 controls excitations in the vacuum sector and the competing winding sector separately. The term winding tension below refers to a closed flux on a periodic strip of fixed transverse width.

x	g_+	σ	x	g_+	σ
$[0, 0.5]$	2.82	0.990	$[4, 5]$	1.70	0.700
$[0.5, 1]$	2.58	0.963	$[5, 6]$	1.55	0.664
$[1, 2]$	2.26	0.887	$[6, 8]$	1.38	0.614
$[2, 3]$	2.03	0.810	$[8, 10]$	1.21	0.582
$[3, 4]$	1.86	0.748	$[10, 12]$	1.03	0.5584

Table 1: Downward-rounded uniform lower bounds in electric units. At shared endpoints either adjacent estimate applies.

Earlier work. Ground-state simplicity at a fixed finite length also follows directly in the untruncated model at every real x. The heat kernel on the connected compact group $\mathrm{SU}(2)^{3L}$ is strictly positive. By Feynman–Kac the bounded real plaquette potential multiplies each path weight by a strictly positive factor bounded below by $e^{-t|x|L}$. Thus the compact heat semigroup improves positivity and its leading eigenfunction is unique up to normalization. Gauge transformations and the simultaneous center flip of the two horizontal links at any one cut preserve H and positivity. They fix the normalized positive ground function. In the spin-network basis this center flip acts by c, so the ground function is physical and even. The quantitative question is how far other levels stay above it as L grows.

The two-purity criterion originates in the periodic spin-one gap work [1]; we use Lu's reflection-positive generalization and its Lean library [2]. Lattice gauge reflection positivity is classical [3]. General perturbation methods yield gaps near gapped classical Hamiltonians [4]. Here the specific contributions are the sector construction and explicit bounds on a finite coupling interval. The operator matrices and truncated plaquette chain are studied in Refs. [5, 6, 7, 8]. Zache et al. use a quantum-group regularization [9]. Earlier work includes fixed-electric-flux free energies [11] and real-time string breaking on a truncated ladder [13]. Siew et al. analyzed static-quark tension in a qubit regularization and gave the transverse-field Ising Hamiltonian in its odd-flux sector (their Eq. 4) [12]. Here the quantitative sector inequalities retain every link representation.

2 A reflection-positive ring for each winding sector

Fix a half-integer cutoff $J \geq 1/2$ on every link. Split $b_i = b_i'' b_{i+1}'$ and $t_i = t_i'' t_{i+1}'$. A cell carries (b', b'', t', t'', r) and is invariant at its two original vertices. Before midpoint constraints the split space is a tensor product of such cells. Each half-link receives half the original electric coefficient. Write N for the sum over midpoint generators of $(S_a^R \otimes 1 + 1 \otimes S_a^L)^2$, on outgoing and incoming halves. Translation conventions are $L(h)f(u) = f(h^{-1}u)$ and $R(h)f(u) = f(uh)$.

Lemma 2. *For $x, \mu \geq 0$ the cutoff split Hamiltonian with penalty μN is a reflection-positive ring in the sense of Ref. [2]. Each physical winding sector inherits the three spatial moment inequalities.*

Proof. We use the cell criterion of Lu [2, Supplement, Eq. (S10) and Sec. SI.B]. In a real basis the cell reflection is a symmetric involution, the on-site operator is real symmetric and reflection invariant, and each oriented bond is $-\sum D \otimes RDR$ with each real D symmetric or antisymmetric. At length two the periodic sum contains both orientations. We verify these hypotheses before taking the midpoint penalty limit.

Complex conjugation K commutes with translations, hence $KS_aK = -S_a$ and $K(iS_a)K = iS_a$. It preserves the Casimirs, vertex invariance and cutoff projection Π. In a real orthonormal basis of the cutoff cell, iS_a at a free midpoint end is real antisymmetric. These end translations commute

with the vertex action at the opposite end. The reflection R is pullback by the Haar-preserving involution

$$\rho(b', b'', t', t'', r) = ((b'')^{-1}, (b')^{-1}, (t'')^{-1}, (t')^{-1}, r).$$

It commutes with the vertex actions and exchanges paired Casimirs. Thus R is real symmetric and commutes with Π and the on-site operator $(2/3 + \mu) \sum_{\text{halves}} C + (4/3)C_r$. Conjugating right translation on b'' by R gives left translation by the same element on b', and likewise on the top rail. Each of the six penalty cross terms is $-D \otimes RDR$ for $D = \sqrt{2\mu}\, i S_a^R$.

Set $P = (b'')^{-1} r t''$, $Q = b' r (t')^{-1}$. Both are vertex invariant, $P \circ \rho = Q$, and the plaquette trace across adjacent cells is $\operatorname{tr}(Q_{i+1} P_i^{-1})$. In unit quaternion coordinates this is $2 \sum_{k=0}^{3} p_{i,k} q_{i+1,k}$. Multiplication by each real p_k is self-adjoint. The magnetic bond gives four more terms $-D \otimes RDR$, with D multiplication by $\sqrt{x}\, p_k$. Compression replaces D by $\Pi D \Pi$ in each factor because $\Pi R = R \Pi$. This yields the required ring, including both oriented bonds at $L = 2$.

Let C_{out} simultaneously multiply b'', t'' by $-I$, and define C_{in} using b', t'. A gauge transformation by $-I$ at both original vertices changes all four horizontal signs and fixes r. Consequently $C_{\text{in}} C_{\text{out}} = 1$ on invariant functions. The reflection exchanges these commuting involutions, hence preserves each eigenspace of C_{out}. The center flip is real and commutes with the cutoff, the Casimirs and all midpoint translations. It also fixes $P = (b'')^{-1} r t''$ since its two signs cancel. Every bond factor and the on-site term therefore restrict to the same real cell eigenspace. Both choices of sign occur for $J \geq 1/2$, as witnessed by 1 and $\operatorname{tr}(b' b'')$.

Use one chosen eigenspace at every cell. Midpoint transformations preserve the rejoined plaquette and commute with the cutoff Hamiltonian. The kernel of N consists of functions of the paired products: on two half-links (u, v) with fixed uv, the action $(u, v) \mapsto (uh, h^{-1} v)$ is transitive. Multiplication pullback is isometric for normalized Haar measure. Acting on $f(uv)$, either half Casimir is the physical Casimir; their half coefficients add to the original one, and the common representation label obeys the same cutoff. On this kernel C_{out} acts as the simultaneous physical rail-center flip at that cut. Thus its selected eigenvalue is precisely the winding sector, in both directions of the identification. The vacuum and a fundamental rail loop witness the two nonempty kernels. At fixed J, length and inverse temperature, letting $\mu \to \infty$ in the finite-dimensional trace suppresses the positive eigenspaces of N. Continuity passes the three spatial inequalities to the physical trace in each chosen sector. □

In either sector let $Z(n) = \operatorname{Tr} e^{-\beta H_n}$ at fixed $\beta > 0, J$. For even $n \geq 2$ and even $2 \leq p \leq q$ the inequalities are

$$Z(2n) \leq Z(n)^2, \quad 2[Z(2n)^2 - Z(4n)] \leq [Z(n)^2 - Z(2n)]^2, \quad Z(q) \leq Z(p)^{q/p}. \tag{1}$$

Fixed transverse width. The same moment construction applies to a strip of any fixed number of rails with open transverse boundaries and translation-invariant nonnegative plaquette couplings. Take a cell to be a whole column with longitudinal half-links h'_a, h''_a at each vertex a. For a transverse edge r_{ab} the two quaternion variables are $(h''_a)^{-1} r_{ab} h''_b$ and $h'_a r_{ab} (h'_b)^{-1}$. The same reflection exchanges them. Applying the central gauge element at every vertex fixes all transverse edges, hence the products of incoming and outgoing center signs agree. Every plaquette factor preserves their common sign. The preceding factorization and midpoint limit now give (1) separately in both sectors for each fixed width. Quantitative bounds on wider strips would require their own seed inequalities and the corresponding intertwiner multiplicities.

3 From finite matrices to uniform bounds

Now let P project onto spins through j_* and Q' onto the additional shell at $j_* + 1/2$. Work within either sector at $J \geq j_* + 1/2$, put $Q = \Pi_J - P$ and choose $\alpha > 0$. Completing a square yields

$$H_J \geq A \oplus B, \qquad A = PHP - (x^2/\alpha)PWQ'WP, \qquad B = QH_JQ - \alpha Q.$$

The difference is X^*X for $X = \sqrt{\alpha}\,Q - (x/\sqrt{\alpha})QWP$, since the plaquette selection rule gives $QWP = Q'WP$. On Q, $B \geq H_0 - (xL + \alpha)$. If $S(j)$ sums $e^{-\beta H_0}$ over admissible labels through j, min–max therefore gives

$$Z_J(L, \beta) \leq \mathrm{Tr}_P\, e^{-\beta A} + e^{\beta(xL+\alpha)}[S(\infty) - S(j_*)].$$

At $L = 2$ the computation includes labels through $j = 8$, then bounds the remainder by $6tS_1^5$. Here t bounds the one-link tail by a geometric series and S_1 bounds the unrestricted one-link weight sum. Dropping vertex constraints and choosing a link above 8 gives this overcount. The odd-sector minimum is bounded below by the lesser of $\lambda_{\min}(A)$ and the minimum electric energy on Q minus $2x + \alpha$.

A normalized four-plaquette trial vector gives an affine Rayleigh upper bound $T(x)$ on its sector ground energy. The Gibbs variational principle makes $\log Z(2, \beta)$ convex in x. Interpolating its endpoint upper enclosures gives an affine majorant $\ell(x)$. Consequently

$$\frac{Z(4,\beta)}{Z(2,\beta)^2} \geq e^{-\beta T(x) - 2\ell(x)}, \qquad \frac{Z(4,2\beta)}{Z(4,\beta)^2} \geq e^{-2\beta T(x) - 4\ell(x)}.$$

The latter uses $Z(4, \beta) \leq Z(2, \beta)^2$. The smaller endpoint bound in each expression works throughout the interval. Take J above or equal to the maximum of the block shell and *all* trial cutoffs, including the two-plaquette trial at $j_* + 1/2$.

Even sector. The two seed bounds above supply the inputs of Lu's chain-certificate theorem at seed length two. For 760 intervals the rational chains in the companion tables return a simple ground state and gap greater than $g_+ = \log(r)/\beta$, uniformly in J, L. Each row satisfies $2.7182818286^{\,p} < r^q$ for $\beta = p/q$, proving $g_+ > 1$. The smallest tabulated rate is approximately 1.03822845.

Odd sector relative to the even ground state. Use inverse temperatures $\beta > \beta_1 > 0$ and inputs $\omega \leq E_-(2)$, $\log Z_+(2, \beta_1) \leq \ell_1$, $E_+(2) \leq \theta$ and $Z_+(4, \beta)/Z_+(2, \beta)^2 \geq 1 - \varepsilon$. Write $f = \varepsilon^2/[2(1-\varepsilon)^2]$, $0 \leq \varepsilon < 1$, $f \leq 1/2$, and $D = -[\log(1-\varepsilon) + \log(1-f)]/4$. Then

$$E_-(L)/L \geq \omega/2, \qquad E_+(L)/L \leq U = \frac{\ell_1/2 + \beta\theta/2 + D}{\beta - \beta_1}. \tag{2}$$

For the first bound take the zero-temperature limit of $Z_-(L) \leq Z_-(2)^{L/2}$ at finite cutoff. For the second, put $P_n = Z_+(2n, \beta)/Z_+(n, \beta)^2$. The squaring inequality gives $1 - P_{2n} \leq (1 - P_n)^2/(2P_n^2)$. The function $u^2/[2(1-u)^2]$ is increasing for $0 \leq u < 1$ and at most u on $[0, 1/2]$. Hence $P_2 \geq 1 - \varepsilon$ and $P_{2^k} \geq 1 - f$ for $k \geq 2$. Iterate $\log Z_+(2n) = 2\log Z_+(n) + \log P_n$ on dyadic lengths and divide by length. The first defect has weight $1/4$ and the later weights sum to at most $1/4$. Compare L to a larger dyadic length by (1). This gives $\log Z_+(L, \beta)/L \geq \frac{1}{2}\log Z_+(2, \beta) - D \geq -\beta\theta/2 - D$. At β_1, comparison with length two gives $\log Z_+(L, \beta_1)/L \leq \ell_1/2$. Combining with $Z_+(L, \beta_1) \geq e^{(\beta-\beta_1)E_+(L)} Z_+(L, \beta)$ proves (2).

On each interval interpolate ω from below by concavity of the odd ground energy, ℓ_1 from above by convexity, and θ using its trial line. One defect bound holds throughout the interval. In 180 rational rows we use $b(u) = u + u^2/2 + u^3/3 + u^4/4 + u^5/(1-u) \geq -\log(1-u)$ in the evaluation of D. Both endpoints satisfy $\sigma \leq \omega/2 - U$, and the substituted expression is affine between them. The smallest σ is $349/625$.

Removing the cutoff. At fixed L, x, H_0 has compact resolvent and finite spin-network sums form a core. Adding bounded $-xW$ preserves these properties. The nested cutoff spaces are form dense in each sector. Each ordered Ritz value therefore converges by min–max to the corresponding eigenvalue, with multiplicity. The difference of the first two even levels retains the fixed interval margin $g_+ > 1$. The two sector minima converge separately, retaining their separation $L\sigma$. This proves Theorem 1.

4 Thermodynamic winding tension and continuation in coupling

Set $\mathcal{E}_\pm(L,x) = E_\pm(L,x)/L$ and $d_L(x) = \mathcal{E}_-(L,x) - \mathcal{E}_+(L,x)$.

Proposition 3. *At fixed $x \geq 0$ each density is nondecreasing along even lengths and converges to a finite limit. The winding tension $\tau(x) = \lim_{L\to\infty,\ L \text{ even}} d_L(x)$ exists, satisfies $\tau(0) = 1$, and is at least $\sigma(x)$ on $[0, 12]$.*

Proof. At fixed J, take $-\log/(\beta q)$ in the last inequality of (1) and let $\beta \to \infty$. It gives $\mathcal{E}_\pm(q,x) \geq \mathcal{E}_\pm(p,x)$ at that cutoff. Remove J for these two fixed lengths. The vacuum and the spin-1/2 loop along one rail have zero diagonal plaquette expectation, because each plaquette changes its horizontal link labels. They give upper bounds 0 and 1 for the even and odd densities, respectively. Both densities are bounded below by $-x$. Taking their separate monotone limits proves existence, and the finite-ring lower bound passes to their difference. □

Proposition 4. *Write $U_i = b_i r_{i+1} t_i^{-1} r_i^{-1}$ and let χ_j be the trace in spin j. For a finite real vector $v \neq 0$ define*

$$f = \sum_{n=0}^{k} v_n \chi_{n/2}, \qquad u_v(x) = \frac{\frac{4}{3}\sum_{n=0}^{k} n(n+2)v_n^2 - x\sum_{n=0}^{k-1} v_n v_{n+1}}{\sum_{n=0}^{k} v_n^2}.$$

For every even $L \geq 2$, $\mathcal{E}_+(L,x) \leq u_v(x)$ and $\mathcal{E}_-(L,x) \leq 1 + u_v(x)$.

Proof. For the even trial take $\Psi = \prod_i f(U_i)$. Characters are class functions, so Ψ is gauge invariant. A simultaneous center flip of the two rail links at a cut fixes every plaquette. Condition on the bottom links and rungs. Integration over the independent top links makes the U_i independent Haar variables. Character orthogonality gives $\|\Psi\|^2 = N^L$ with $N = \sum v_n^2$. Fundamental character fusion gives $\int f^2(\operatorname{tr} U/2) = \sum v_n v_{n+1}$.

For the electric quadratic form differentiate in an orthonormal Lie algebra basis on each link. A derivative on a rung hits two different plaquettes, including at $L = 2$. Its mixed term vanishes after integration over the two private top links, since for each factor $\int f(U_i) D_r f(U_i)\, dt_i = \frac{1}{2} D_r \int f(U_i)^2\, dt_i = 0$. On each incidence the sum of the squared derivatives integrates to the group Dirichlet form of f, by bi-invariance of the metric (translations and inversion are isometries). There are four incidences per plaquette and the character of spin $n/2$ has Casimir $n(n+2)/4$. With the electric coefficient $4/3$ these terms give the numerator above.

For the odd trial multiply Ψ by $B = \mathrm{tr}(b_1 \cdots b_L)$. The trace closes the loop gauge invariantly and the center flip changes its sign. Conditioned on bottoms and rungs the same top integrations apply. A derivative on a bottom link produces one new mixed term whose conditional integral vanishes by $\int f D_b f = 0$. The remaining new term is the electric energy of B, namely L times its squared norm, since it is fundamental on each bottom link. Orthogonality gives $\int B^2 = 1$. Thus the normalized expectation increases by one per plaquette. □

Plaquette-product variational states have a long history in Hamiltonian lattice gauge theory [10]. Here their energy on the ladder can be evaluated exactly, and an upper bound in one winding sector can be paired with a lower bound in the other.

Theorem 5. *For every even $L \geq 2$, the full spectral gap exceeds 1 through $x = 12.25$, exceeds 0.76 throughout $[12.25, 14]$, and exceeds 0.46 throughout $[14, 16]$. Both d_L and τ satisfy the six lower bounds in Table 2.*

Interval	Floor for d_L, τ	Trial coefficients v	$\omega(a)$	$\omega(b)$
$[12, 16]$	0.42	$(30, 30, 15, 4, 1)$	-9.555414	-15.023872
$[16, 20]$	0.40	$(30, 32, 18, 6, 1)$	-15.023872	-20.791110
$[20, 24]$	0.36	$(30, 34, 21, 8, 2)$	-20.791110	-26.800203
$[24, 28]$	0.20	$(30, 36, 23, 10, 3)$	-26.800203	-33.172360
$[28, 30]$	0.20	$(40, 49, 33, 15, 5, 1)$	-33.172360	-36.161741
$[30, 32]$	0.1425	$(40, 50, 34, 16, 5, 1)$	-36.161741	-39.523661

Table 2: Lower enclosures ω for $E_-(2, x)$ and one even-sector trial per interval. Each floor follows by checking $\omega/2 - u_v$ at the two displayed endpoints.

Proof. Length comparison gives $\mathcal{E}_-(L, x) \geq E_-(2, x)/2$. At the endpoints we enclose this two-plaquette energy by the odd block with $j_* = 3$ and penalties $\alpha = 12$ through 28, 16 at 30 and 15523/1000 at 32 (the endpoint at 12 comes from the original tension table). For example, the new block P has dimension 1,734. In doubled-spin labels $n = 2j$ the electric numerator is $\sum n(n{+}2)$ with denominator three. Enumeration of the first omitted shell $\max n = 7$ gives minimum numerator 120. For $\max n = m \geq 8$ the other two links at one endpoint of a maximal link have labels summing to at least m. Their contributions with the maximal link alone give at least $8(8{+}2){+}2{\cdot}4(4{+}2) = 128$. Thus the complementary lower bound is $40 - 2x - \alpha$. Each reported ω takes account of both blocks. The function $E_-(2, x)/2 - u_v(x)$ is concave, so the endpoint checks prove the floors on the closed intervals. Proposition 3 gives the same floors in the limit.

A sharper bound near 12 distinguishes gaps above one. At 11.3 the length-two enclosure $\log Z_+(2, 3) \leq 29.333221$ and length comparison give $\mathcal{E}_+(L, 11.3) \geq -29.333221/6$. At 12 the last right endpoint of (2) gives $\mathcal{E}_+(L, 12) \leq -5.33611$. By concavity its secant slope from 11.3 to 12 is an upper bound on all subsequent slopes, giving $\mathcal{E}_+(L, x) \leq V(x) = -5.33611 - 0.6389(x - 12)$ for $x \geq 12$. The odd enclosures at 12 and 14 are -9.555414 and -12.245748. The chord for $E_-(2, x)/2 - V(x)$ gives endpoint floors 0.5584 and 0.4910, hence at least 0.549975 throughout $[12, 12.25]$. Odd levels therefore cost more than one for every $L \geq 2$ in this range.

Eleven additional even-sector intervals tile $[12, 16]$. They use $\beta = 33/20$, $j_* = 3$, four-plaquette trial cutoff 5/2 and $\alpha = 12$. Both implementations independently bound the two defects on each whole interval. The six-step rational chains yield $\log r/\beta > 1,\ 0.76,\ 0.46$ in the three respective bands. For a target g the same rational exponential test as in Section 3 uses $p/q = g\beta$. The even-sector rates pass through the cutoff limit. The odd sector costs at least $0.42L$ across $[12, 16]$,

with the stronger near-12 estimate above. Taking the smaller sector bound proves the full-gap statements. □

Proposition 4 also gives a two-sided benchmark. At $x = 12$ take $v = (30, 26, 11, 3)$, giving $u_v = -13520/2559$. The endpoint $\log Z_+(2, 3) \le 32.043171$ gives $\mathcal{E}_+(L, 12) \ge -32.043171/6$. Consequently $0.5584 \le d_L(12) \le 1.05722$, also for $\tau(12)$. At 32 the complementary floor $40-64-\alpha$ equals -40 at $\alpha = 16$. Choosing $\alpha = 15523/1000$ gives -39.523 on the complement and a common downward-rounded lower bound -39.523661 on the retained block. The two implementations independently clear that threshold. The same trial has $u_v(30) = -152359/8307$ and $u_v(32) = -165346/8307$. Both endpoint differences exceed $57/400$, proving the final row by concavity. The working spin cutoff is unchanged.

Multiplying every bottom link by $-I$ is a Haar-preserving involution commuting with the vertex actions, Casimirs, cutoff and winding charge. Each plaquette contains exactly one bottom link, so this unitary interchanges $H(x)$ and $H(-x)$ sector by sector. The sector densities are therefore even in x and concave, hence nonincreasing for $x \ge 0$. An increment $h \ge 0$ changes each by a number in $[-h, 0]$. Thus d_L and its even-length limit τ are 1-Lipschitz. The ordering in Proposition 3 concerns the two densities separately. Their convergence, and that of d_L to τ, is uniform on each compact coupling interval. Indeed the sector densities and their limits have a common Lipschitz constant one because $\|W\| \le L$. On a finite mesh choose a single length giving the desired accuracy at every mesh point; the Lipschitz estimate controls the intervening values.

5 Computational evidence and proof status

For loop spins j_i changing to J_i, write a_i for the spectator between i and $i+1$ in cyclic order. The trace matrix element in Eq. (34) of Ref. [6] is

$$(-1)^{\sum_i (a_i + j_i + J_i)} \prod_{i=1}^{4} \sqrt{(2j_i + 1)(2J_i + 1)} \begin{Bmatrix} a_i & j_i & j_{i+1} \\ 1/2 & J_{i+1} & J_i \end{Bmatrix}.$$

The implemented corner factors regroup to this formula by symmetry of the $6j$ symbol. The four extra $1/2$ terms in their sign exponents sum to two. Each term in W carries a further factor $1/2$. In the archived calculation sign and rational square agree exactly across 1,759,172 local configurations counted over three sizes. The separate cutoff-$1/2$ check reproduces the even-sector spin-chain spectrum of Ref. [6] on two through eight plaquettes.

The Julia sector computations use interval arithmetic and exact rational Racah sums. The Python replay builds its own sector matrices from SymPy $6j$ symbols, with integer FLINT matrix products and scalar Arb enclosures. Floating eigenvectors supply coordinates: in a dyadic basis V the replay bounds $\|V^*V - I\| \le \eta < 1$ and bounds V^*MV above by a diagonal matrix of Gershgorin upper endpoints. Min–max with the two Gram bounds gives upper enclosures on the ordered eigenvalues of M, sufficing to bound $\operatorname{Tr} e^M$. Both implementations share the operator inequalities and recoupling conventions. A separate direct computation before sector resolution provides a 240-interval regression on $[0, 6]$.

Proof component	What is established and where
Physical ring	Lemma 2 identifies the gauge-sector trace with the penalty limit and verifies Lu's cell hypotheses.
Finite numerical inputs	Interval and integer enclosures for energies and partition functions supply the hypotheses of the scalar certificates.
Formal implications	Lean checks interval coverage, rational chains and gap propagation, including `gapGT_of_chainCertificate` in the pinned library and the last tension chord in `RefinedTension.lean`.
Removal of cutoffs	Fixed-volume Ritz convergence preserves the spectral bounds. The two density limits and the character-trial expectation are proved in Sections 3–4.

The companion files include interval records, replay programs and rational comparisons with the Lean tables. We rebuilt the ladder theorems on Lean 4.36.0-rc1 with revision `77b5588` of Lu's library. The full-gap declarations concern finite spectral families satisfying moment and seed hypotheses. We recomputed the last original winding interval, the new odd anchors and eleven additional even-sector intervals. Lean checks the new rational chains, their tiling and their rate tests. The trial-state calculation and its physical interpretation are on paper. The companion checks every rational endpoint inequality in Table 2.

Provenance. The computational companion is available from GitHub release r2. Its Zenodo version identifier is 10.5281/zenodo.24519435. The file `su2-ladder-companion-r2.zip` has the following SHA-256 digest, split across two lines.

```
a762a6747125f033a1bc7914ca4adc56
f538dbc7b06a7014788d5acbe6c6dd95
```

Claude produced the initial mathematical note, implementations and formalizations in collaboration with Will Strinz. Codex prepared the review draft, checked certificate data and developed the thermodynamic, variational and extended-gap arguments. A subsequent computational review suggested the direct center restriction and the rational penalty at 32. The physical operator argument remains open to specialist scrutiny.

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