UNIMODALITY OF THE NUMBER OF PARKING FUNCTIONS WITH A GIVEN MAXIMUM DISPLACEMENT

Abstract. We prove that, for every n, the numbers of parking functions of length n in which the maximum displacement of a car from its preferred spot is $0, 1, \ldots, n-1$ form a unimodal sequence. This confirms a conjecture of Celano, Elder, Hadaway, Harris, Martin, Priestley and Udell. The proof follows, for each value of the maximum displacement, the sign of the difference of two neighbouring terms as n grows. It is computer-assisted: explicit estimates are used when the displacement is large, and exact computations with integers, carried out by computer, when the displacement or the length is small.

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1. Introduction

Parking functions were introduced by Konheim and Weiss [13], who studied a scheme for the storage of records that is now called hashing with linear probing, and who applied their results, at the end of their paper, to a problem of parking cars in a street. Parking functions have since become standard objects of combinatorics: they are equinumerous with the labelled trees on one more vertex, and they occur in the study of noncrossing partitions, of hyperplane arrangements and of the diagonal harmonics; see the survey of Yan [19]. A parking function describes a way in which n cars can park one after another in a row of n spots when each car has a preferred spot and takes the first empty spot at or after it. This is the rule of hashing with linear probing, in which items are stored one after another in a table whose cells lie in a row, each item in the first empty cell at or after its hash address: the cars are the items, the spots are the cells, and the number of spots by which a car is pushed beyond the spot it prefers, its displacement, is the distance between the address of an item and its cell.

The displacements measure the cost of storing and of finding an item, and they have been studied in both settings. Their sum, the total displacement, is the cost of building the table. Knuth [12] and Flajolet, Poblete and Viola [8] find its moments, the latter also its limit laws, and Pappe [14, Theorem 3.1] shows that the generating polynomial of the total displacement of parking functions has log-concave coefficients. For the largest of the displacements, the maximum displacement, Petersson [15] determines the limit distribution in the setting of hashing, and formulas for the number of parking functions in which no displacement exceeds a given bound are given in [3] and [5]; we recall these results in Section 1.1. The theorem of Pappe is a statement about the shape of the distribution of the total displacement. The present paper is about the shape of the distribution of the maximum displacement: we ask how the numbers of parking functions of a given length whose maximum displacement is $0, 1, 2, \ldots$ compare with one another.

To fix the notation, let $n \geq 1$ be an integer. The cars are numbered $1, \ldots, n$ in the order in which they arrive, and the spots are numbered $1, \ldots, n$ along the row and are all empty at the start. Car i prefers a spot $a_i \in \{1, \ldots, n\}$ and parks in the first empty spot among $a_i, a_i + 1, \ldots, n$; if these spots are all occupied, car i cannot

park. The sequence $\alpha = (a_1, \dots, a_n)$ is called a *preference sequence* of length n, and it is called a *parking function* if every car parks; we write PF_n for the set of all parking functions of length n. For $\alpha \in \mathrm{PF}_n$ the *displacement* of car i is $d_i(\alpha) = \mathrm{pos}_i(\alpha) - a_i \geq 0$, where $\mathrm{pos}_i(\alpha)$ is the spot in which car i parks, and the *maximum displacement* of α is

$$M(\alpha) = \max_{1 \leq i \leq n} d_i(\alpha) \in \{0, 1, \dots, n-1\}.$$

Parking functions with $M(\alpha) \leq \ell$ are called *ℓ-interval parking functions* [5, Definition 2.7], and it is their number that the formulas mentioned above give. They range from the permutations of $1, \dots, n$, for $\ell = 0$, where every car parks in the spot it prefers, to all parking functions of length n, for $\ell = n-1$. Here and below we cite [5] by the numbers of its second version; the numbers that we cite are the same in the first version except in one place, which is noted in Section 1.1.

Sorting the parking functions of length n by their maximum displacement gives the numbers

$$b_{n,\ell} = \#\{\alpha \in \mathrm{PF}_n : \ M(\alpha) = \ell\}, \qquad 0 \leq \ell \leq n-1.$$

For $\ell \geq 1$ this is the number of ℓ-interval parking functions of length n that are not $(\ell-1)$-interval. The sum of these numbers is the number of all parking functions of length n, which is $(n+1)^{n-1}$ [13, Lemma 2 and Section 6]; a short proof, due to Pollak, is given in [9, Section 2]. The two ends of the sequence have closed forms: $b_{n,0} = n!$, the number of permutations, and $b_{n,n-1} = n^{n-2}$ for $n \geq 2$ [5, Example 2.8 and Section 6]. For $n \leq 7$ the numbers are

(1.1)

n	$b_{n,0},\ b_{n,1},\ \dots,\ b_{n,n-1}$
1	1
2	2, 1
3	6, 7, 3
4	24, 51, 34, 16
5	120, 421, 377, 253, 125
6	720, 3963, 4594, 3688, 2546, 1296
7	5040, 42253, 62145, 57398, 46142, 32359, 16807

The sum and the two ends of the sequence $(b_{n,0}, \dots, b_{n,n-1})$ are thus known in closed form, and the question is how the terms in between are ordered. In (1.1) each row increases up to some entry and decreases from there on. The largest entry does not stand at a fixed place, however: it is the second entry of the rows $n = 3, 4, 5$ and the third entry of the rows $n = 6, 7$. Celano, Elder, Hadaway, Harris, Martin, Priestley and Udell, who tabulate $b_{n,\ell}$ for $n \leq 9$ [5, Table 1], conjecture, among the problems with which their paper closes, that for every $n \geq 1$ the sequence $(b_{n,0}, \dots, b_{n,n-1})$ is unimodal [5, Conjecture 6.5]. Here a finite sequence $(u_0, \dots, u_m)$ of real numbers is called *unimodal* if there is some $\ell^* \in \{0, \dots, m\}$ such that $u_{\ell-1} \leq u_\ell$ holds for $1 \leq \ell \leq \ell^*$ and $u_\ell \geq u_{\ell+1}$ holds for $\ell^* \leq \ell \leq m-1$; equality is allowed. Unimodality and log-concavity are known for many sequences that arise in algebra, combinatorics and geometry, and they have been proved by methods of many kinds, which Stanley surveys in [18]. The conjecture asks for the maximum of the displacements what the theorem of Pappe quoted above answers for their sum; that theorem gives log-concavity, which is the stronger property, since a log-concave sequence of positive numbers is unimodal.

In the sentence before their conjecture, the authors of [5] state three inequalities at the two ends of a row, for all $n \geq 1$ and as consequences of their enumerative results, namely $b_{n,0} \leq b_{n,1}$, $b_{n,1} \leq b_{n,2}$ and $b_{n,n-2} \geq b_{n,n-1}$, and they add that experimentally the pattern seems to continue [5, Section 6]. As (1.1) shows, the first of these inequalities fails for $n = 2$ and the second for $n = 3, 4, 5$; both hold in the later rows of the table. For the terms between the two ends of a row, the support for the conjecture is thus the computed rows. In this paper we prove the conjecture.

Theorem 1.1. *For every integer $n \geq 1$ the sequence $(b_{n,0}, b_{n,1}, \ldots, b_{n,n-1})$ is unimodal.*

Theorem 1.1 establishes for every length the shape that is seen in the tabulated rows. Its proof gives more on the left of a row: for $n \geq 17$ the row is nondecreasing on its left half, i.e. $b_{n,\ell-1} \leq b_{n,\ell}$ whenever $1 \leq \ell \leq n/2$ (Proposition 1.2 below). In small rows this can fail, as $b_{5,1} = 421 > b_{5,2} = 377$ shows. Where the maximum of a row lies is determined neither by the theorem nor by its proof; we return to this question at the end of Section 1.2.

Our proof is analytic. It constructs no correspondence between the parking functions that are counted by two neighbouring terms, and unimodality is not obtained from a stronger property such as log-concavity: we work directly with the signs of the differences of neighbouring terms. The combinatorics of parking functions enters through a single known relation, the recursion (1.2) of Aguilar-Fraga et al. [3, Theorem 3.7], which is recalled below. It expresses the number of parking functions of length $n + 1$ in which no displacement exceeds a bound k through the same numbers for smaller lengths and the same bound. Such a recursion relates different lengths for one bound, whereas the theorem compares different values of the maximum displacement for one length. We therefore read the table (1.1) by its columns and follow, for a fixed ℓ, the sign of $b_{n,\ell} - b_{n,\ell-1}$ as n grows. The two exceptions to the inequalities of [5] that were noted above are instances of the pattern that the proof follows: in a fixed column the difference is negative for the first lengths and positive for the later ones. The theorem follows when we know that the difference, once it is positive, does not become negative again, and that it does not become positive in one column before it has done so in the column to the left. This is what we prove, for the rows and the columns that the theorem needs.

Several of the tools that we apply to the solutions of this recursion are classical, among them the binomial identity of Abel [1], the bounds of Robbins for the factorial [16] and the theorem of de Bruijn and Erdős on renewal sequences with a log-convex kernel [6]; they are cited where they are used, and the last of them is proved there in the form that we need. The steps on which the argument turns are developed in the paper. They include a linear equation that replaces the recursion, which is quadratic, in a range of rows of each column; a comparison with renewal sequences, by which a column is shown to remain positive; a property that takes the place of convexity where the recursion does not preserve convexity; and a polynomial in a real variable through which two neighbouring columns are compared. Section 1.2 describes them.

Part of the proof is carried out by computer. The estimates are made with explicit constants, and they hold for all columns from an explicit one on, namely from column 82 or from column 265, depending on the statement that is proved.

For the columns before these, and for the rows $n \le 16$, the proof relies on exact computations with integers. No parking functions are enumerated in them: the numbers that are needed are obtained from recursions, up to row 300, and the integers that occur have up to about 14700 decimal digits. In Section 1.2, after the description of the estimates, we say what is computed, why finitely many computed numbers suffice for statements about all rows of a column, and which statements of the paper depend on the computations. The computations themselves are the subject of Section 7, which contains the data that are needed to repeat them.

1.1. **Cumulative counts and related results.** We now recall the formulas for the number of ℓ-interval parking functions and the limit theorem for the maximum displacement that were mentioned at the beginning, and we fix the notation for these numbers, the cumulative counts, which is used throughout the paper. For integers $n \ge 0$ and $k \ge 0$ let

$$A_{n,k} = \#\{\alpha \in \mathrm{PF}_n : \ M(\alpha) \le k\}$$

be the number of parking functions of length n in which every car parks at most k spots beyond the spot it prefers. Here and throughout the paper we admit $n = 0$: the empty sequence is the one parking function of length 0, and its maximum displacement is 0. We also use the definition of $b_{n,\ell}$ for all integers $n \ge 0$ and $\ell \ge 0$. Thus $b_{0,0} = 1$, $b_{0,\ell} = 0$ for $\ell \ge 1$, and $b_{n,\ell} = 0$ for $\ell \ge n \ge 1$, and we have

$$A_{n,k} = \sum_{\ell=0}^{k} b_{n,\ell}, \qquad b_{n,0} = A_{n,0}, \qquad b_{n,\ell} = A_{n,\ell} - A_{n,\ell-1} \quad (\ell \ge 1).$$

In particular $A_{0,k} = 1$, while $A_{n,k}$ is nondecreasing in k and equals $(n+1)^{n-1}$ for $n \le k+1$, where the bound on the displacements is no restriction.

One description of $A_{n,k}$ is a sum over the permutations $\sigma = (\sigma_1, \dots, \sigma_n)$ of $1, \dots, n$ [5, Section 4.1, (6)–(8)]; in the first version of [5] these are the formulas (2)–(4) of Section 3. If λ_i denotes the length of the longest run of consecutive spots that ends at spot i, spot i included, and on which the values of σ are at most σ_i, then there are $\prod_{i=1}^{n} \min(k+1, \lambda_i)$ parking functions with maximum displacement at most k in which car σ_i parks in spot i for every i, and $A_{n,k}$ is the sum of these products over all σ. Another description is a recursion in n, obtained by fixing the spot of the car that arrives last [3, Theorem 3.7]. In our notation it reads

$$A_{n+1,k} = \sum_{j=0}^{n} \min(j+1, k+1) \binom{n}{j} A_{j,k}\, A_{n-j,k} \qquad (n \ge 0,\ k \ge 0), \tag{1.2}$$

with $A_{0,k} = 1$. Below, (1.2) is called the *last-car recursion*; it is restated as Proposition 3.1. Our proof starts from it: this recursion, the value $A_{0,k} = 1$ and the monotonicity of $A_{n,k}$ in k are all that the proof takes from the definition of a parking function.

The limit theorem is the theorem of Petersson on hashing with linear probing [15, Theorem 5.1], in which the hash addresses of the items are independent and uniformly distributed over the cells. For a table with m cells and $m-1$ items, conditioned on its last cell remaining empty, it says that the maximum displacement $d_{\max}$ of the items satisfies

$$\frac{m - d_{\max}}{m^{2/3}} \longrightarrow Y \quad \text{in distribution,} \qquad \Pr(Y \le z) = 1 - \psi(z^{3/2}),$$

where $\psi(s) = \mathbb{E}\, e^{-s\mathcal{A}}$ and $\mathcal{A} = \int_0^1 \max_{0\le s\le t} B_s\, dt$ for a standard Brownian motion B; an explicit series for ψ is given in [11, Corollary 2.4]. Let $m = n+1$. Since an item that reaches an empty cell occupies it, the last cell remains empty exactly when the hash addresses of the n items form a parking function of length n; the displacements of the items are then those of the cars, and all parking functions of length n are equally likely. In our terms the theorem therefore says that all but an arbitrarily small fraction of the sum $(n+1)^{n-1}$ of the sequence $(b_{n,0}, \dots, b_{n,n-1})$ lies at a distance of order $n^{2/3}$ from its end $\ell = n-1$. It is a statement of convergence in distribution, and it compares no two neighbouring terms of the sequence.

Among related results on special classes of parking functions, Selig and Zhu show that for nondecreasing parking functions the maximum displacement is the height of the corresponding Łukasiewicz path [17, Theorem 2.1], and Ager-Hart, Beerbower, Harris, Jakovleski and McClinton count, for each ℓ, the ℓ-interval Fubini rankings, a subfamily of the ℓ-interval parking functions [2, Theorem 4.2 and Corollary 4.1].

1.2. **Outline of the proof.** We describe the proof of Theorem 1.1 in the order of the argument. The terms that we introduce on the way are used throughout the paper; the numerical constants of the proof are left, with few exceptions, to the statements in the body. We arrange the numbers $b_{n,k}$ in a table as in (1.1), with rows indexed by the length n and columns indexed by the displacement k. For integers $k \ge 1$ and $n \ge 0$ the *step* of column k in row n is

$$d_{n,k} = b_{n,k} - b_{n,k-1},$$

and we say that column k is *positive*, *nonpositive* or *negative* in row n according to the sign of this step. Row n has a *strict descent* at i if $b_{n,i} < b_{n,i-1}$, and a *strict ascent* at j if $b_{n,j-1} < b_{n,j}$. A finite sequence is unimodal as soon as no strict descent is followed, further to the right, by a strict ascent. So the theorem follows once we show that $d_{n,i} < 0 < d_{n,j}$ is impossible for $1 \le i < j \le n-1$ (Proposition 20.2), and everything else in the paper is about the signs of the $d_{n,k}$.

These signs compare the entries of one row, i.e. counts for different displacements and one length, whereas the last-car recursion (1.2), as was said after Theorem 1.1, relates different lengths for one bound on the displacements; we have no recursion for the $b_{n,\ell}$ themselves. For $k \ge 2$ the step is a second difference in the bound, $d_{n,k} = A_{n,k} - 2A_{n,k-1} + A_{n,k-2}$, of three sequences in n each of which satisfies its own recursion. The proof therefore fixes a column k, follows the sign of $d_{n,k}$ as n grows, and returns to the rows only at the end. A column $k \ge 2$ starts at row $n = k$, where $b_{k,k} = 0$ and the step $d_{k,k} = -b_{k,k-1} = -k^{k-2}$ is negative. The picture behind the proof is that the column stays nonpositive for a while, turns positive at some row, and does not turn back. The first row $n > k$ with $d_{n,k} > 0$ is called the *crossing row* of column k, and the rows n with $k+1 \le n \le 2k-1$ form its *chain range*.

Theorem 1.1 rests on three propositions about columns, which make precise as much of this picture as the theorem needs. The first is the inequality on the left of a row that was mentioned after the theorem: in a row $n \ge 17$ no column k with $2k \le n$ is negative, so that in these rows a column is not negative beyond its chain range.

Proposition 1.2. *Let n and k be integers with $n \ge 17$, $k \ge 1$ and $2k \le n$. Then $b_{n,k-1} \le b_{n,k}$.*

The second proposition says that a column which is positive in one row of its chain range is positive in the next row of this range as well, so that there it changes its sign at most once, from nonpositive to positive.

Proposition 1.3. *Let $k \geq 2$ and n be integers with $k+1 \leq n \leq 2k-2$. If $b_{n,k-1} < b_{n,k}$, then $b_{n+1,k-1} < b_{n+1,k}$.*

Applied in consecutive rows, Proposition 1.3 shows that a column which is positive in a row of its chain range is positive in every later row of this range: if $k \geq 2$, m and n are integers with $k+1 \leq m \leq n \leq 2k-1$ and $b_{m,k-1} < b_{m,k}$, then $b_{n,k-1} < b_{n,k}$.

The third says that as long as column k has not turned positive, column $k+1$ has not turned positive either.

Proposition 1.4. *Let $k \geq 9$ and $n \geq k+2$ be integers. If $b_{j,k} \leq b_{j,k-1}$ for every integer j with $k+1 \leq j \leq n$, then $b_{n,k+1} \leq b_{n,k}$.*

Propositions 1.2, 1.3 and 1.4 are proved in Sections 15, 12 and 19. The theorem follows from them in a few lines, which are given in Section 20: in a row $n \geq 17$, Proposition 1.2 excludes a strict descent at $i \leq n-1$ unless the row lies in the chain range of column i, and in that case Propositions 1.3 and 1.4 carry the nonpositive step from column i to every later column of the row, so that no strict ascent follows. The rows $n \leq 16$ are computed (Lemma 7.1); they have to be treated apart, since the inequality of Proposition 1.2 fails in some of them and Proposition 1.4 starts at column 9. Each of the three propositions, in turn, is proved by estimates for the large columns and by exact computation, carried out by computer, for the small ones: the estimates apply to the columns $k \geq 82$ in Propositions 1.2 and 1.3, and to the columns $k \geq 265$ in Proposition 1.4. We describe the estimates first and the computations after them.

The estimates work with the counts divided by $n!$. The sequence $n \mapsto b_{n,\ell}$ of the entries of column ℓ of the table is called *level* ℓ, and we write $\beta_n^{(\ell)} = b_{n,\ell}/n!$; for a column $k \geq 2$ we write $\delta_n = d_{n,k}/n!$. The last-car recursion is quadratic, but in the chain range it can be replaced by a linear relation, the column equation:

$$\delta_n = \frac{k+1}{n}\,\delta_{n-1} + \frac{2}{n}\,\beta_{n-1}^{(k-1)} + \sum_{m=1}^{n-k} \varepsilon_m\,\delta_{n-m} \qquad (k+1 \leq n \leq 2k-1),$$

where $\varepsilon_1 = 1$ and $\varepsilon_m = (m-1)^{m-1}/m!$ for $m \geq 2$ (Proposition 4.5). The step in row n is fed by a nonnegative source, which comes from level $k-1$, and by the earlier steps of the same column, with positive weights. The negative start is fed back into the following rows, the source works against it, and the crossing row is the first row in which the source prevails. Summed over the first rows of the chain range, together with the corresponding equation for level $k-1$, the column equation shows that a large column k cannot remain nonpositive in all the rows $k+1, \ldots, k+r$ unless r is at most a certain multiple of $k^{2/3}$ (Proposition 9.2). Beyond the chain range there is no exact linear equation, and the proof works with inequalities that come from the generating series of the counts (Section 5).

The difficulty of the proof lies at three places, one in the proof of each of the three propositions. The first, in Proposition 1.3, is that a column has to stay positive after its crossing row. The column equation keeps feeding the rows before the crossing, in which the step is nonpositive and of unknown size, into every later row,

so that the positivity of the source does not decide the sign. On the *near rows*, the rows of the chain range up to $k+0.47k$, we compare the column equation at row n with the column equation at the crossing row, where the outcome was positive, and this controls what the rows before the crossing contribute (Proposition 10.4). The comparison requires that the numbers $(k+m)\,\varepsilon_m\,\gamma_k^{-m}$, where $\gamma_k = (1+1/k)^{k-2}$, do not increase with m, and the estimate by which we show this reaches $m \le 0.47k$; this is the reason for the cut. On the rows of the chain range beyond these, the *far rows*, we use the column equation at row n alone, together with a run of positive rows before row n, and bound the step from below by a combination of values of level $k-1$ (Proposition 12.3). Up to row $k+0.6k$ the coefficients of this combination are nonnegative, and the claim reduces to a comparison of explicit numbers. Beyond row $k+0.6k$ some of the coefficients may be negative, and we bound the terms that carry them by a comparison with a renewal sequence. The comparison is proved in Section 11 for arbitrary sequences, without reference to parking functions. Let $f = (f_i)_{i\ge 1}$ be a positive log-convex sequence, the kernel, and let G be its renewal sequence, given by $G_0 = 1$ and $G_n = \sum_{i=1}^{n} f_i\, G_{n-i}$ for $n \ge 1$. If a sequence v satisfies $\sum_{i=1}^{s} f_i\, v_{s-i} \le v_s$ for all s up to some ρ, then for every $M \ge 1$ the sum $\sum_{m=M}^{\rho} \theta_m\, f_m\, v_{\rho-m}$, taken with nonnegative nondecreasing weights θ_m, is largest in proportion to v_ρ when $v = G$ (Proposition 11.4). This bound rests on the classical theorem that the renewal sequence of a log-convex kernel is itself log-convex, which goes back to de Bruijn and Erdős [6, Theorem 1].

The second place is in Proposition 1.2 for a large column, in the rows beyond its chain range, where only inequalities are available. The rows before the crossing are still present in every later row, and we outweigh them through a single weighted sum of the steps of the column, which Proposition 14.7 bounds from below. This bound needs a convexity of the cumulative counts, suitably normalised, along a diagonal of their table, that is, when the length and the bound on the displacements are raised together. One would prove such a convexity by induction on the last-car recursion, but convexity does not pass through this induction: the recursion multiplies the terms by factors that increase with the bound on the displacements, and a factor of this kind does not preserve the convexity $2w_0 \le w_- + w_+$ of three numbers w_-, w_0, w_+ that belong to three consecutive bounds. We therefore replace convexity by a family of inequalities, which we call property K (Definition 13.1):

$$2\,e^{\mu/k}\,w_0 \le e^{\mu}\,w_+ + e^{-\mu}\,w_- \qquad \text{for every real number } \mu \ge 0,$$

where k is the index of the column. Convexity is the case $\mu = 0$. Like convexity, property K passes to sums and to nonnegative multiples of triples; unlike convexity, it also survives the multiplication by the factors that the recursion produces. Such a factor turns the inequality for one value of μ into the inequality for a larger value, so that the family as a whole passes through the induction although its single members do not (Lemma 13.2). One part of the recursion is not covered by this and needs an estimate of its own (Proposition 13.8).

The third place is in Proposition 1.4 for a large column k, where two ranges have to meet. At a row $n = k + r$ of the chain range the steps of columns k and $k+1$, divided by $n!$, are each written as a second difference of one polynomial, in which a real variable takes the place of the bound on the displacements, minus a sum over the compositions of an integer, in which a part of size p carries the weight $p^{p-1}/p!$ and each cut between two parts carries a factor (Proposition 18.6). When the cuts carry no factor, the number of parts of a composition of m that is drawn

in proportion to its weight follows the law of the birthday problem, that is, the law of the number of draws from m equally likely values before the first repetition (Proposition 16.3), and the two expressions are compared through estimates for this law. In this way we show that a nonpositive step of column k forces a negative step of column $k+1$, and we can carry this through as long as r is at most a certain multiple of $k^{2/3}$ (Proposition 19.3). The hypothesis of Proposition 1.4, however, says only that column k has not turned positive up to row $k+r$. That this confines r to the range just named is Proposition 9.2, in which the multiple is the same. From Proposition 9.2, or from a computation for the columns that it does not reach, the proofs of Propositions 1.3 and 1.2 also take the early crossing row from which they start.

It remains to describe the part of the proof that is carried out by computer. For the columns below those at which the estimates start, that is, below column 82 in Propositions 1.2 and 1.3 and below column 265 in Proposition 1.4, and for the rows $n \leq 16$, we rely on three computations, which are the subject of Section 7. They are made in exact arithmetic with integers. No parking function is enumerated: the cumulative counts $A_{n,k}$ are obtained from the last-car recursion (1.2) or, in the chain range, from a linear recursion that follows from it, and each computation ends in finitely many comparisons between integers that are formed from these counts. The first computation gives all entries of the rows $n \leq 16$ and shows that each of these rows is unimodal (Lemma 7.1). The second gives two signs for each of the columns $9 \leq k \leq 264$: column k is positive in a certain row of its chain range, and column $k+1$ is negative in the row before it (Lemma 7.3). It reaches row 300, and its integers have up to about 740 decimal digits. The third gives, for each of the columns $2 \leq k \leq 81$, the three counts $A_{n,k}$, $A_{n,k-1}$ and $A_{n,k-2}$, of which the step $d_{n,k}$ is the second difference, in all rows n below a certain row; this row depends on the column and is at most row 175. With these counts it verifies Proposition 1.3 for the column, verifies the inequality of Proposition 1.2 in those of the computed rows to which the proposition applies, and verifies that finitely many sums which are formed from the counts are nonnegative (Lemma 7.5). The integers that occur in these sums have up to about 14700 decimal digits.

Finitely many computed numbers suffice for a different reason in each case. The rows $n \leq 16$ are finitely many, and Proposition 1.3 concerns only the rows of the chain range, which are finitely many for each column. Proposition 1.2, by contrast, concerns all rows $n \geq 2k$ of column k. Here a propagation theorem (Proposition 15.5) shows that, when the sums of the third computation are nonnegative, the step of the column is nonnegative in every row beyond the computed ones. Proposition 1.4 concerns infinitely many rows of each column as well, and for it the two signs of the second computation suffice because of Proposition 1.3. By the first sign, the hypothesis of Proposition 1.4, that column k is nonpositive in all the rows $k+1, \ldots, n$, can hold only for the rows n up to the row of the second sign. If column $k+1$ were positive in such a row n, then by Proposition 1.3 it would still be positive in the row of the second sign, where it is negative. For the columns $82 \leq k \leq 264$ the first sign has a second use, which was mentioned above: it supplies the positive row early in the chain range that the proofs of Propositions 1.3 and 1.2 need, and that Proposition 9.2 provides from column 265 on (Corollary 9.3).

The statements of the paper that are obtained by computer are thus Lemmas 7.1, 7.3 and 7.5. Propositions 1.2, 1.3 and 1.4 depend on them only for the columns

$k \le 264$; for the columns $k \ge 265$ they are proved without them. Theorem 1.1 depends on them through these propositions and through Lemma 7.1. Elsewhere in the paper the numerical verifications are comparisons between explicit numbers, which arise from the estimates and are made where they are needed.

The proof leaves open where the maximum of a row lies. In the picture described above, the columns $k \ge 2$ that are positive in row n are those whose crossing row is at most n, and the maximum of the row stands at the last of them. Our argument, however, uses only bounds for the crossing row of a column and does not determine it; and, as was said in Section 1.1, the theorem of Petersson shows where most of the sum of a row lies and not which term is the largest. Still, the statements that we prove on the way confine the maximum from one side. In a row $n \ge 335$ the first column p at which the maximum is attained stands at most $(121/25)^{1/3}\, n^{2/3}$ columns before the last column of the row, and the row increases strictly from column 81 to column p and decreases strictly from column $p+1$ on (Proposition 20.3 and the paragraph after its proof, which the proof of Theorem 1.1 does not use). The order $n^{2/3}$ of this bound is that of the distance from the end of the row at which, by the theorem of Petersson, most of the sum of the row lies. The constant is that of the range of rows which the estimates of Propositions 9.2 and 19.3 reach; it has no meaning beyond these estimates, and we have no bound for the distance from below.

1.3. **Organisation of the paper.** Propositions 1.2, 1.3 and 1.4 are stated in Section 1.2. Sections 2–7 are preparatory: they treat the explicit sequences that occur, the last-car recursion, the column equation, the inequalities that hold in all rows, a monotonicity of the normalised counts on which the proof of Proposition 1.2 rests, and the facts that are obtained by exact computation. Sections 8 and 9 bound the crossing row. Proposition 1.3 is proved in Sections 10–12, Proposition 1.2 in Sections 13–15, and Proposition 1.4 in Sections 16–19. Section 20 proves Theorem 1.1 and closes with a bound for the position of the maximum of a row (Proposition 20.3).

2. Preliminaries

Throughout the paper sequences are indexed by the integers $n \ge 0$ unless another range is stated, and we use the convention $0^0 = 1$, so that $n^n/n!$ has the value 1 at $n = 0$. By a *power series* we mean a formal power series in one variable X with real coefficients; for a power series F we write $[X^n]F$ for the coefficient of X^n in F and F' for the formal derivative of F.

The proofs work with three explicit sequences, (a_j), (c_n) and (ε_m), and the rest of this section collects what is needed about them: the ratios of their consecutive terms, the identities that connect them, bounds for their size, and two sums.

2.1. **The numbers a_j, c_n and ε_m.** The *Cayley coefficients* are the numbers

$$a_0 = 0, \qquad a_j = \frac{j^{\,j-1}}{j!} \quad (j \ge 1);$$

by Cayley's formula, j^{j-1} is the number of rooted labelled trees on j vertices. The preferences of a parking function are also written a_i, as in the Introduction; no confusion will arise. Shifted by one, the Cayley coefficients are the numbers

$$c_n = a_{n+1} = \frac{(n+1)^{n-1}}{n!} \qquad (n \ge 0),$$

so that $c_0 = 1$. Since there are $(n+1)^{n-1}$ parking functions of length n [9, Section 2], c_n is the number of parking functions of length n divided by $n!$. The third sequence is the *free kernel*

$$\varepsilon_0 = 0, \qquad \varepsilon_m = \frac{(m-1)^{m-1}}{m!} \quad (m \geq 1),$$

in which $\varepsilon_1 = 0^0/1! = 1$; it is the kernel of a renewal equation satisfied by the c_n (Corollary 2.7). The first values are $a_1 = a_2 = 1$, $a_3 = \frac{3}{2}$, $a_4 = \frac{8}{3}$ and $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{1}{2}$, $\varepsilon_3 = \frac{2}{3}$, $\varepsilon_4 = \frac{9}{8}$. We often use the definitions in the forms

$$(n+1)\, c_n = \frac{(n+1)^n}{n!}, \qquad (m+1)\, \varepsilon_{m+1} = \frac{m^m}{m!}, \tag{2.1}$$

which hold for all integers $n \geq 0$ and $m \geq 0$. In particular, $a_j > 0$ for $j \geq 1$, $c_n > 0$ for $n \geq 0$, and $\varepsilon_m > 0$ for $m \geq 1$.

The ratios of consecutive terms of these sequences are expressed by $(1 + 1/i)^i$ and $(1+1/i)^{i-1}$, and both are nondecreasing in $i \geq 1$. For the first this is classical: for every integer $i \geq 1$,

$$\Big(1 + \frac{1}{i}\Big)^i \leq \Big(1 + \frac{1}{i+1}\Big)^{i+1}, \tag{2.2}$$

by Bernoulli's inequality. For the second it is the following lemma.

Lemma 2.1. *For all integers i and j with $1 \leq i \leq j$,*

$$\Big(1 + \frac{1}{i}\Big)^{i-1} \leq \Big(1 + \frac{1}{j}\Big)^{j-1}.$$

Proof. By induction on j it suffices to treat $j = i + 1$. By the inequality stated before the lemma and since $1 + \frac{1}{i+1} \leq 1 + \frac{1}{i}$,

$$\Big(1 + \frac{1}{i}\Big)^{i-1}\Big(1 + \frac{1}{i}\Big) \leq \Big(1 + \frac{1}{i+1}\Big)^{i+1} \leq \Big(1 + \frac{1}{i+1}\Big)^{i}\Big(1 + \frac{1}{i}\Big),$$

and dividing by $1 + \frac{1}{i}$ gives the claim. □

Since $a_{j+1} = (j+1)^j/(j+1)! = (j+1)^{j-1}/j!$, the ratio of two consecutive Cayley coefficients is

$$a_{j+1} = \Big(1 + \frac{1}{j}\Big)^{j-1} a_j \qquad (j \geq 1). \tag{2.3}$$

By Lemma 2.1 and (2.3) the ratio a_{j+1}/a_j is nondecreasing in $j \geq 1$, that is, the Cayley coefficients a_j with $j \geq 1$ form a log-convex sequence. We use this property in the following cross-multiplied form, which for $s = 1$ and $v = u+1$ reads $a_{u+1}^2 \leq a_u\, a_{u+2}$.

Lemma 2.2. *For all integers u, v and s with $1 \leq u \leq v$ and $s \geq 0$,*

$$a_{u+s}\, a_v \leq a_u\, a_{v+s}.$$

Proof. We argue by induction on s; the case $s = 0$ is trivial. If the inequality holds for s, then (2.3), applied with $j = u + s$ and with $j = v + s$, gives

$$\begin{aligned} a_{u+s+1}\, a_v &= \Big(1 + \frac{1}{u+s}\Big)^{u+s-1} a_{u+s}\, a_v \\ &\leq \Big(1 + \frac{1}{v+s}\Big)^{v+s-1} a_u\, a_{v+s} = a_u\, a_{v+s+1}, \end{aligned}$$

where the inequality is obtained by multiplying the induction hypothesis by the inequality of Lemma 2.1 for $u + s \le v + s$. □

2.2. **The binomial identity of Abel.** The identities among the numbers a_j, c_n and ε_m that we use all come from one classical identity, the binomial identity of Abel [1]. It has the form of the binomial theorem for $(x+y)^n$, with $(y-k)^{n-k}$ in place of y^{n-k} and with the following weights in place of the powers x^k: for a real number x and an integer $k \ge 0$ let

$$w_0(x) = 1, \qquad w_k(x) = x\,(x+k)^{k-1} \quad (k \ge 1).$$

Note that $c_k = w_k(1)/k!$ for every $k \ge 0$. We use the identity with $x = 1$ in this section and with general x in Section 11.

Lemma 2.3 (Abel [1, p. 159])**.** *For every integer $n \ge 0$ and all real numbers x and y,*

$$\sum_{k=0}^{n} \binom{n}{k} w_k(x)\,(y-k)^{n-k} = (x+y)^n.$$

In the notation of [1] the identity reads $(x+\alpha)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} \alpha\,(\alpha - \mu\beta)^{\mu-1}(x + \mu\beta)^{n-\mu}$ for arbitrary x, α and β and every positive integer n, the term with $\mu = 0$ being x^n; the lemma is the case $\beta = -1$, with our x and y in the places of α and x, and for $n = 0$ it is trivial.

Abel's identity yields the following recursion for the c_n. It is the last-car recursion of the Introduction, divided by $n!$, in the case in which the bound on the displacements is no restriction.

Lemma 2.4. *For every integer $n \ge 0$,*

$$(n+1)\,c_{n+1} = \sum_{j=0}^{n} (j+1)\,c_j\,c_{n-j}.$$

Proof. We put $j = n - k$ on the right-hand side. By (2.1) and since $c_k = w_k(1)/k!$, it becomes

$$\begin{aligned}\sum_{k=0}^{n} \frac{(n+1-k)^{n-k}}{(n-k)!} \cdot \frac{w_k(1)}{k!} &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} w_k(1)\,(n+1-k)^{n-k} \\ &= \frac{(n+2)^n}{n!},\end{aligned}$$

where the last step is Lemma 2.3 with $x = 1$ and $y = n + 1$. This is the left-hand side, since $c_{n+1} = (n+2)^n/(n+1)!$. □

Written with the Cayley coefficients themselves, the recursion is the following convolution identity, on which the comparison of the weights of compositions in Section 16 rests.

Lemma 2.5. *For every integer $t \ge 0$,*

$$\sum_{i=0}^{t-1} i\,a_i\,a_{t-i} = (t-1)\,a_t.$$

Proof. Since $a_0 = 0$, the cases $t \le 1$ are trivial. Let $t = n+2$ with $n \ge 0$. The term with $i = 0$ vanishes, and with $i = j + 1$ the others add up to $\sum_{j=0}^{n} (j+1)\,c_j\,c_{n-j}$, which by Lemma 2.4 equals $(n+1)\,c_{n+1} = (t-1)\,a_t$. □

2.3. **The series C and the free kernel.** Let

$$C = \sum_{n\geq 0} c_n X^n = 1 + X + \tfrac{3}{2}X^2 + \tfrac{8}{3}X^3 + \cdots .$$

Its constant coefficient is $c_0 = 1$, so C has an inverse $1/C$. Since $c_n = a_{n+1}$ and $a_0 = 0$, we have $[X^j](XC) = a_j$ for every $j \geq 0$. With these series the recursion of Lemma 2.4 becomes the differential equation

$$C' = (XC)'\, C, \tag{2.4}$$

as one sees by comparing the coefficients of X^n, the coefficient of X^j in $(XC)'$ being $(j+1)\, a_{j+1} = (j+1)\, c_j$. Classically, XC is the tree function T, the exponential generating series of the rooted labelled trees, which satisfies $T = Xe^T$, and $C = e^T$; the differential equation is the derivative form of the second relation.

In Section 4 power series are divided by C. Since $(1/C)' = -C'/C^2$, the differential equation (2.4) gives

$$\Big(\frac{1}{C}\Big)' = -\frac{1}{C}\,(XC)', \tag{2.5}$$

and the coefficients of a quotient by C are given by the following lemma.

Lemma 2.6. *Let F be a power series. Then, for every integer $n \geq 0$,*

$$[X^n]\Big(\frac{1}{C}\,F\Big) = [X^n]F - \sum_{m=1}^{n} \varepsilon_m\, [X^{n-m}]F.$$

Proof. We determine the coefficients of $1/C$. By the product rule $(XC)' = C + XC'$ and by the differential equation in the form $C'/C = (XC)'$,

$$\frac{1}{C}\,(XC)' = 1 + X\,(XC)' = \sum_{n\geq 0} \frac{n^n}{n!}\, X^n,$$

since the coefficient of X^{n-1} in $(XC)'$ is $n\, a_n = n^n/n!$ for $n \geq 1$. By (2.5), $(1/C)'$ is the negative of this series. Comparing the coefficients of X^m and using (2.1), we find that

$$(m+1)\,[X^{m+1}]\frac{1}{C} = -\frac{m^m}{m!} = -(m+1)\,\varepsilon_{m+1} \qquad (m \geq 0).$$

Hence $[X^m](1/C) = -\varepsilon_m$ for every $m \geq 1$, while $[X^0](1/C) = 1$, and the assertion is the formula for the coefficients of the product $\frac{1}{C}\,F$. □

It follows that the c_n satisfy a renewal equation whose kernel is the free kernel.

Corollary 2.7. *For every integer $n \geq 1$,*

$$\sum_{m=1}^{n} \varepsilon_m\, c_{n-m} = c_n.$$

Proof. This is Lemma 2.6 with $F = C$, since the coefficient of X^n in $\frac{1}{C}\,C = 1$ is 0 for $n \geq 1$. □

It remains to record the log-convexity of the free kernel, on which the arguments of Sections 10 and 12 rest. For an integer $m \geq 1$ let

$$g_1 = \frac{1}{2}, \qquad g_m = \frac{m}{m+1}\Big(1 + \frac{1}{m-1}\Big)^{m-1} \quad (m \geq 2).$$

A direct computation from the definition of ε shows that g_m is the ratio of two consecutive terms of the free kernel:

$$\varepsilon_{m+1} = g_m\, \varepsilon_m \qquad (m \geq 1).$$

Lemma 2.8. *For every integer $m \geq 1$,*

$$\varepsilon_{m+1}^2 \leq \varepsilon_m\, \varepsilon_{m+2}.$$

Proof. Since $\varepsilon_{m+1} = g_m\, \varepsilon_m$ and the ε_m with $m \geq 1$ are positive, the assertion is that $g_m \leq g_{m+1}$. We have $g_1 = \frac{1}{2} \leq \frac{4}{3} = g_2$. For $m \geq 2$ neither of the two positive factors of g_m decreases when m is replaced by $m+1$, the second by the inequality (2.2) with $i = m-1$. □

2.4. **Stirling bounds.** For an integer $n \geq 0$ the number $e^{-n}n^n/n!$ is the probability that a Poisson variable with mean n takes the value n; it is called the *Poisson mass* at n, and its value at $n = 0$ is 1. The size of our sequences is estimated through the Poisson mass, since by their definitions

$$e^{-j}a_j = \frac{1}{j} \cdot e^{-j}\frac{j^j}{j!} \qquad \text{and} \qquad e^{-(m+1)}\varepsilon_{m+1} = \frac{e^{-1}}{m+1} \cdot e^{-m}\frac{m^m}{m!}$$

for all integers $j \geq 1$ and $m \geq 0$. By Stirling's formula $n! \sim \sqrt{2\pi n}\,(n/e)^n$, the Poisson mass at n is asymptotic to $c_*/\sqrt{n}$, where

$$c_* = \frac{1}{\sqrt{2\pi}} = 0.398942\ldots, \qquad \text{so that} \qquad 0.39894 \leq c_* \leq 0.39895.$$

We use Stirling's formula with the bounds of Robbins [16, p. 26, (1) and (2)]: for every integer $n \geq 1$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n)}. \tag{2.6}$$

Since $e^{-n}n^n/n! = (n/e)^n/n!$ and $e^{-1/(12n)} \geq 1 - \frac{1}{12n}$, these bounds enclose the Poisson mass at n between $\left(1 - \frac{1}{12n}\right) c_*/\sqrt{n}$ and $c_*/\sqrt{n}$:

$$\left(1 - \frac{1}{12n}\right)\frac{1}{\sqrt{2\pi n}} \leq e^{-n}\,\frac{n^n}{n!} \leq \frac{1}{\sqrt{2\pi n}} \qquad (n \geq 1). \tag{2.7}$$

2.5. **Two sums.** We finally need the values of two series: $\sum_j e^{-j}a_j = 1$ and $\sum_m e^{-m}\varepsilon_m = 1 - e^{-1}$. The first is classical. The numbers $e^{-j}a_j = e^{-j}j^{j-1}/j!$ with $j \geq 1$ are the probabilities of the Borel distribution with parameter 1, which is the law of the total progeny of a Galton–Watson process whose offspring law is the Poisson law with mean 1, and this total progeny is finite with probability 1 [4, Section 2.2 and (10)]. Hence the series of nonnegative real numbers $\sum_{j\geq 0} e^{-j}a_j$ converges, and

$$\sum_{j\geq 0} e^{-j}\, a_j = 1. \tag{2.8}$$

The second sum is the one that the later sections use, and we deduce it from the first.

Both sums are values at $X = e^{-1}$ of series that Section 2.3 treats formally. By (2.3) the ratio a_{j+1}/a_j tends to e, so that e^{-1} is the radius of convergence of XC. The second value is therefore not obtained by substituting e^{-1} for X in an identity that holds inside the disc of convergence, and we argue with partial sums instead, by means of the following elementary fact about a renewal equation.

Let $(f_m)_{m\ge 0}$ and $(u_n)_{n\ge 0}$ be sequences of nonnegative real numbers such that $f_0 = 0$, $u_0 = 1$ and the renewal equation

$$u_n = \sum_{m=0}^{n} f_m\, u_{n-m} \qquad (n \ge 1)$$

holds, and write $F_M = \sum_{m=0}^{M} f_m$ and $U_M = \sum_{n=0}^{M} u_n$ for the partial sums. Then, for all integers $M \ge 0$ and $k \ge 0$,

$$1 - \frac{1}{U_M} \le F_M \qquad \text{and} \qquad F_M\, U_k \le U_{M+k} - 1. \tag{2.9}$$

To see this, we sum the renewal equation over $1 \le n \le M$ and exchange the order of summation. Since $f_0 = 0$, this gives

$$U_M - 1 = \sum_{m=0}^{M} f_m\, U_{M-m} \qquad (M \ge 0).$$

The sequence (U_M) is nondecreasing, and $U_0 = 1$. Bounding U_{M-m} by U_M gives the first of the two inequalities. For the second we use the last display with $M + k$ in place of M, discard the terms with $m > M$, and bound U_{M+k-m} from below by U_k for $m \le M$.

Lemma 2.9. *The series of nonnegative real numbers*

$$\sum_{m\ge 0} e^{-m}\, \varepsilon_m$$

converges, and its sum is $1 - e^{-1}$.

Proof. We apply the inequalities (2.9) to $f_m = e^{-m}\varepsilon_m$ and $u_n = e^{-n}c_n$, for which $f_0 = 0$ and $u_0 = 1$; here the renewal equation is the equation of Corollary 2.7 multiplied by e^{-n}. Since $c_n = a_{n+1}$ and $a_0 = 0$, the sum (2.8) says that $\sum_{n\ge 0} u_n = e$. Hence $U_k \le e$ for every k, and $U_k \to e$.

The second of the two inequalities that we apply now gives $F_M\, U_k \le U_{M+k} - 1 \le e - 1$ for all M and k, and letting $k \to \infty$ we get $F_M \le 1 - e^{-1}$ for every $M \ge 0$. The first inequality gives $F_M \ge 1 - 1/U_M$, and the right-hand side tends to $1 - e^{-1}$ as $M \to \infty$. Hence $F_M \to 1 - e^{-1}$. □

3. The last-car recursion

In this section we bring the last-car recursion (1.2), the one relation that the proof of Theorem 1.1 takes from the parking process, into the form in which it is used: a recursion for the counts $A_{n,k}$ divided by $n!$. The last subsection shows that the ratio of these normalised counts to the numbers c_n of Section 2 does not decrease along the diagonal (Proposition 3.5).

3.1. The recursion.

The last-car recursion expresses $A_{n+1,k}$ through the numbers $A_{j,k}$ with $j \le n$, for the same bound k on the displacements.

Proposition 3.1 ([3, Theorem 3.7])**.** *For all integers $n \ge 0$ and $k \ge 0$,*

$$A_{n+1,k} = \sum_{j=0}^{n} \min(j+1, k+1) \binom{n}{j} A_{j,k}\, A_{n-j,k}.$$

The recursion is obtained by sorting the parking functions of length $n+1$ by the spot $s = j+1$ in which car $n+1$ parks. The j earlier cars that park on the left of s can be chosen in $\binom{n}{j}$ ways. Their preferences form a parking function of length j, and those of the other $n-j$ earlier cars, reduced by s, form a parking function of length $n-j$; each of these cars has the same displacement there as in the given parking function. Car $n+1$ prefers one of the spots $s-t$ with $0 \le t \le j$, and its displacement is t, so that $\min(j+1, k+1)$ of its possible preferences are compatible with the bound k.

3.2. **The step of a column and the normalised counts.** The recursion concerns the cumulative counts $A_{n,k}$, whereas Theorem 1.1 is a statement about the signs of the steps $d_{n,k}$. Since $b_{n,\ell} = A_{n,\ell} - A_{n,\ell-1}$ for $\ell \ge 1$, a step is a second difference of the cumulative counts:

$$b_{n,k} - b_{n,k-1} = A_{n,k} - 2A_{n,k-1} + A_{n,k-2} \qquad (k \ge 2,\ n \ge 0). \tag{3.1}$$

The analytic arguments of the paper work with the counts divided by $n!$. An integer $y \ge 1$ is called a *threshold*; it stands for the condition that all displacements are smaller than y, i.e. $M(\alpha) \le y-1$. For a threshold y and an integer $n \ge 0$ let

$$z_n^{(y)} = \frac{A_{n,y-1}}{n!}$$

be the number of parking functions of length n all of whose displacements are smaller than y, divided by $n!$. We call $z_n^{(y)}$ the *normalised count* of threshold y at length n. Since $A_{0,k} = 1$ and $A_{n,k}$ is nondecreasing in k,

$$z_0^{(y)} = 1 \qquad \text{and} \qquad 0 \le z_n^{(y)} \le z_n^{(y')} \quad \text{for } 1 \le y \le y' \text{ and } n \ge 0. \tag{3.2}$$

In the same way, for integers $\ell \ge 0$ and $n \ge 0$ let

$$\beta_n^{(\ell)} = \frac{b_{n,\ell}}{n!} \ge 0$$

be the normalised count of level ℓ, and for integers $k \ge 1$ and $n \ge 0$ let

$$\delta_n^{(k)} = \beta_n^{(k)} - \beta_n^{(k-1)} = \frac{d_{n,k}}{n!}$$

be the *normalised step* of column k in row n, which has the sign of $d_{n,k}$. When the column is fixed, we write δ_n for $\delta_n^{(k)}$. Since $b_{n,0} = A_{n,0}$ and $b_{n,\ell} = A_{n,\ell} - A_{n,\ell-1}$ for $\ell \ge 1$, we have

$$\beta_n^{(0)} = z_n^{(1)} \qquad \text{and} \qquad \beta_n^{(\ell)} = z_n^{(\ell+1)} - z_n^{(\ell)} \quad \text{for } \ell \ge 1, \tag{3.3}$$

and (3.1), divided by $n!$, reads

$$\delta_n^{(k)} = z_n^{(k+1)} - 2\, z_n^{(k)} + z_n^{(k-1)} \qquad (k \ge 2).$$

Thus a column $k \ge 2$ involves the three thresholds $k+1$, k and $k-1$ together.

Divided by $n!$, the recursion of Proposition 3.1 loses its binomial coefficient and becomes a recursion for the sequence $(z_n^{(y)})_{n \ge 0}$, into which the threshold enters only through the factor $\min(j+1, y)$. Without a bound on the displacements this factor is $j+1$, and the recursion is that of the c_n (cf. Lemma 2.4). We shall compare solutions of the two recursions, and such a comparison uses of the factor only the range in which it agrees with $j+1$; we therefore state the recursion for a general weight.

Definition 3.2. Let $\omega = (\omega(t))_{t\geq 1}$ and $W = (W_n)_{n\geq 0}$ be sequences of real numbers. The sequence W is a *solution of the weighted last-car recursion with weight* ω if $W_0 = 1$ and, for every integer $n \geq 0$,

$$(n+1)\, W_{n+1} = \sum_{j=0}^{n} \omega(j+1)\, W_j\, W_{n-j}.$$

The recursion determines W_{n+1} from $W_0, \ldots, W_n$ and $\omega(1), \ldots, \omega(n+1)$, so that every weight has exactly one solution. By Lemma 2.4 and $c_0 = 1$, the sequence $(c_n)_{n\geq 0}$ is the solution with weight $\omega(t) = t$.

Corollary 3.3. *Let $y \geq 1$ be an integer. Then the sequence $z^{(y)} = (z_n^{(y)})_{n\geq 0}$ is a solution of the weighted last-car recursion with weight $\omega(t) = \min(t, y)$, i.e. $z_0^{(y)} = 1$ and, for every integer $n \geq 0$,*

$$(n+1)\, z_{n+1}^{(y)} = \sum_{j=0}^{n} \min(j+1, y)\, z_j^{(y)}\, z_{n-j}^{(y)}. \tag{3.4}$$

Proof. Since $z_0^{(y)} = A_{0,y-1} = 1$, this is Proposition 3.1 with $k = y - 1$, divided by $n!$. □

The two factors $z_j^{(y)}$ and $z_{n-j}^{(y)}$ of a term of this recursion belong to the cars on the left, resp. on the right, of the last car. Their lengths j and $n - j$ add up to n, one less than the length on the left-hand side.

Let x be a real number. The *tilted form* of a sequence $(s_n)_{n\geq 0}$ at the *tilt* x is the sequence $(s_n x^n)_{n\geq 0}$; in particular, the numbers $z_n^{(y)} x^n$ are the tilted counts of threshold y at tilt x. For $x > 0$ a tilt changes no sign, so that a comparison within one row is the same statement before and after tilting, and the tilted counts are nonnegative and nondecreasing in the threshold, as the $z_n^{(y)}$ themselves are. What a tilt changes is the comparison of different rows, since the ratio of two consecutive terms is multiplied by x; this is what the tilts of Sections 6 and 8 are chosen for. The recursion of Corollary 3.3 keeps its form under a tilt: multiplying (3.4), with N in place of n, by x^{N+1}, we see that for every integer $y \geq 1$, every real number x and every integer $N \geq 0$,

$$(N+1)\, z_{N+1}^{(y)} x^{N+1} = x \sum_{i=0}^{N} \min(i+1, y)\; z_i^{(y)} x^i\; z_{N-i}^{(y)} x^{N-i}. \tag{3.5}$$

3.3. **The free range.** The lengths $n \leq y$ form the *free range* of threshold y. Recall that for these lengths the condition that all displacements are smaller than y is no restriction, so that $A_{n,y-1}$ is the number $(n+1)^{n-1}$ of all parking functions of length n [9, Section 2]. In normalised form this is the following lemma, which also follows directly from the recursions of Corollary 3.3 and Lemma 2.4; we give this short proof.

Lemma 3.4. *Let $y \geq 1$ and $n \geq 0$ be integers with $n \leq y$. Then*

$$z_n^{(y)} = c_n.$$

Proof. By Corollary 3.3 the sequences $z^{(y)}$ and $(c_n)_{n\geq 0}$ are the solutions of the weighted last-car recursion with weights $\min(t, y)$ and t, which agree for $1 \leq t \leq y$. Since the recursion determines W_{n+1} from $W_0, \ldots, W_n$ and $\omega(1), \ldots, \omega(n+1)$, induction on n gives $z_n^{(y)} = c_n$ for all $n \leq y$. □

The lengths $y+1 \le n \le 2y+1$ form the *first layer* of threshold y. For n in the first layer the lengths of the two factors of a term in the recursion (3.4) for $z_n^{(y)}$ add up to $n-1 \le 2y$, so that at most one of the two factors lies outside the free range; the other factor is one of the c_m.

In the free range the normalised counts of a level and the normalised steps of a column vanish: by the relations (3.3) and Lemma 3.4,

$$\beta_n^{(\ell)} = 0 \quad (\ell \ge 1,\ n \le \ell) \qquad \text{and hence} \qquad \delta_n^{(k)} = 0 \quad (k \ge 2,\ n < k).$$

3.4. **Monotonicity along the diagonal.** Since $n!\,c_n$ is the number of parking functions of length n, the ratio $z_n^{(y)}/c_n$ is the probability that all displacements of a uniformly chosen parking function of length n are smaller than y. One more car can push the maximum displacement up. The following proposition says that allowing one more unit of displacement at least compensates for this: the probability does not decrease when n and y are both raised by one. The diagonal lower bound of Section 5 (Proposition 5.9) is derived from it. The proof of the proposition is an induction on n, in which the last-car recursion writes each of the probabilities as an average over the spot in which the last car parks. The averages for the two sides of the inequality differ in their weights and in their values, and the second has one term more, so that the proof compares each term of the first with two neighbouring terms of the second.

Proposition 3.5. *For all integers $y \ge 1$ and $n \ge 0$,*

$$\frac{z_n^{(y)}}{c_n} \le \frac{z_{n+1}^{(y+1)}}{c_{n+1}}.$$

Proof. For integers $y \ge 1$ and $0 \le j \le n$ put

$$F_n^{(y)} = \frac{z_n^{(y)}}{c_n}, \qquad q_n(j) = \frac{(j+1)\,c_j\,c_{n-j}}{(n+1)\,c_{n+1}},$$
$$v_n^{(y)}(j) = \frac{\min(j+1,y)}{j+1}\,F_j^{(y)}\,F_{n-j}^{(y)}.$$

We prove that $F_n^{(y)} \le F_{n+1}^{(y+1)}$ by induction on n, for all $y \ge 1$ at once. The recursion of Corollary 3.3 expresses $F_{n+1}^{(y)}$ as an average of the $v_n^{(y)}(j)$ with weights $q_n(j)$. Intuitively, $q_n(j)$ is the probability that the last car of a uniformly chosen parking function of length $n+1$ parks in spot $j+1$, and $v_n^{(y)}(j)$ is the probability, given this, that all displacements are smaller than y.

Since the c_i are positive, the inequalities (3.2) give

$$0 \le F_i^{(y)} \le F_i^{(y+1)} \qquad (y \ge 1,\ i \ge 0),$$

so that the ratios $F_i^{(y)}$ are nonnegative and nondecreasing in the threshold.

We begin with the representation of $F_{n+1}^{(y)}$ as an average: for $y \ge 1$ and $n \ge 0$,

$$F_{n+1}^{(y)} = \sum_{j=0}^{n} q_n(j)\,v_n^{(y)}(j) \qquad \text{and} \qquad \sum_{j=0}^{n} q_n(j) = 1.$$

Indeed, $q_n(j)\,v_n^{(y)}(j) = \min(j+1,y)\,z_j^{(y)}\,z_{n-j}^{(y)}/((n+1)\,c_{n+1})$, so that the first identity is the recursion (3.4) divided by $(n+1)\,c_{n+1}$; the second is Lemma 2.4 divided by the same number.

Next we compare the weights of two consecutive averages. For $0 \le j \le n$,

$$q_{n+1}(j) \le q_n(j) \qquad \text{and} \qquad q_{n+1}(j+1) \le q_n(j).$$

After cancelling $(j+1)\,c_j$ and clearing the denominators, the first inequality reads $(n+1)\,c_{n+1-j}\,c_{n+1} \le (n+2)\,c_{n-j}\,c_{n+2}$. Since $c_i = a_{i+1}$ for every $i \ge 0$, this follows from the inequality $a_{n-j+2}\,a_{n+2} \le a_{n-j+1}\,a_{n+3}$, which is Lemma 2.2 with $u = n-j+1$, $v = n+2$ and $s = 1$. In the same way, with c_{n-j} cancelled, the second inequality reads

$$(j+2)\,c_{j+1} \cdot (n+1)\,c_{n+1} \le (j+1)\,c_j \cdot (n+2)\,c_{n+2}.$$

By (2.3), we have $c_i = (1+\frac{1}{i})^{i-1} c_{i-1}$ for $i \ge 1$, and hence $(i+1)\,c_i = i\,(1+\frac{1}{i})^i\,c_{i-1}$. We insert the second formula for $i = j+1$ on the left of the displayed inequality and the first formula for $i = n+2$ on the right, and cancel $(j+1)\,c_j\,c_{n+1}$. This turns it into the inequality between the first and the last member of

$$\begin{aligned}(n+1)\Big(1+\frac{1}{j+1}\Big)^{j+1} &\le (n+1)\Big(1+\frac{1}{n+1}\Big)^{n+1} \\ &= (n+2)\Big(1+\frac{1}{n+1}\Big)^{n} \le (n+2)\Big(1+\frac{1}{n+2}\Big)^{n+1}.\end{aligned}$$

Here the first step holds because $j \le n$ and $(1+\frac{1}{i})^i$ is nondecreasing in $i \ge 1$ (Section 2), and the last step is Lemma 2.1.

The two averages will be compared by means of a general inequality between two weighted sums, which we prove before the induction. Let $m \ge 0$ be an integer. Let $q(j)$ and $v(j)$, for $0 \le j \le m$, and $q'(j)$ and $v'(j)$, for $0 \le j \le m-1$, be real numbers such that

$$\sum_{j=0}^{m} q(j) = \sum_{j=0}^{m-1} q'(j)$$

and, for every $0 \le j \le m-1$,

$$q(j) \le q'(j), \qquad q(j+1) \le q'(j), \qquad v'(j) \le v(j), \qquad v'(j) \le v(j+1).$$

We claim that then

$$\sum_{j=0}^{m-1} q'(j)\,v'(j) \le \sum_{j=0}^{m} q(j)\,v(j).$$

For the proof let $\sigma_t = \sum_{j<t} q(j)$ for $0 \le t \le m+1$ and $\sigma'_t = \sum_{j<t} q'(j)$ for $0 \le t \le m$. These partial sums interlace:

$$\sigma_t \le \sigma'_t \le \sigma_{t+1} \qquad (0 \le t \le m).$$

The first inequality follows from $q(j) \le q'(j)$, term by term. For the second, the inequalities $q(j+1) \le q'(j)$ give

$$\sigma'_m - \sigma'_t = \sum_{t \le j < m} q'(j) \ge \sum_{t \le j < m} q(j+1) = \sigma_{m+1} - \sigma_{t+1},$$

and $\sigma'_m = \sigma_{m+1}$ by the hypothesis on the two total sums. We now show by induction on t that, for $0 \le t \le m$,

$$\sum_{j<t} q'(j)\,v'(j) \le \sum_{j<t} q(j)\,v(j) + (\sigma'_t - \sigma_t)\,v(t).$$

The case $t = 0$ is trivial. For the step from t to $t+1 \le m$ we split $q'(t) = (\sigma_{t+1} - \sigma'_t) + (\sigma'_{t+1} - \sigma_{t+1})$ into two parts, which are nonnegative because the

partial sums interlace, and we use $v'(t) \le v(t)$ on the first part and $v'(t) \le v(t+1)$ on the second:

$$q'(t)\,v'(t) \le (\sigma_{t+1} - \sigma'_t)\,v(t) + (\sigma'_{t+1} - \sigma_{t+1})\,v(t+1).$$

When this is added to the induction hypothesis, the terms with $v(t)$ combine to $(\sigma_{t+1} - \sigma_t)\,v(t) = q(t)\,v(t)$, and the result is the inequality for $t+1$. For $t = m$ the last term of the inequality so proved is $(\sigma_{m+1} - \sigma_m)\,v(m) = q(m)\,v(m)$, so that the case $t = m$ is the claim.

We now carry out the induction on n. For $n = 0$ both sides of the inequality $F_n^{(y)} \le F_{n+1}^{(y+1)}$ are 1, by Lemma 3.4. Let $n \ge 0$, and suppose that $F_i^{(y')} \le F_{i+1}^{(y'+1)}$ for all $i \le n$ and all $y' \ge 1$. We show that $F_{n+1}^{(y)} \le F_{n+2}^{(y+1)}$ for every $y \ge 1$. Recall that $F_{n+1}^{(y)}$ is the average of the values $v_n^{(y)}(j)$, $0 \le j \le n$, with the weights $q_n(j)$, and that $F_{n+2}^{(y+1)}$ is the average of the values $v_{n+1}^{(y+1)}(j)$, $0 \le j \le n+1$, with the weights $q_{n+1}(j)$. We apply the general inequality with $m = n+1$, with $q = q_{n+1}$ and $q' = q_n$, and with $v = v_{n+1}^{(y+1)}$ and $v' = v_n^{(y)}$. The two total sums are 1, because the weights of each of the two averages add up to 1, as we saw when we wrote $F_{n+1}^{(y)}$ as an average, and the two conditions on the weights are the inequalities $q_{n+1}(j) \le q_n(j)$ and $q_{n+1}(j+1) \le q_n(j)$ proved above. It remains to verify the two conditions on the values, for $0 \le j \le n$. Each value is a product of three nonnegative factors, and we compare the factors one by one.

The first condition is $v_n^{(y)}(j) \le v_{n+1}^{(y+1)}(j)$. Here $\min(j+1, y) \le \min(j+1, y+1)$, we have $F_j^{(y)} \le F_j^{(y+1)}$ by the monotonicity in the threshold, and we have $F_{n-j}^{(y)} \le F_{n+1-j}^{(y+1)}$ by the induction hypothesis for $i = n - j$.

The second condition, $v_n^{(y)}(j) \le v_{n+1}^{(y+1)}(j+1)$, is verified similarly. With $t = \min(j+1, y)$ we have $\min(j+2, y+1) = t+1$, so that the first factors of the two values are $\frac{t}{j+1}$ and $\frac{t+1}{j+2}$, and $\frac{t}{j+1} \le \frac{t+1}{j+2}$ because $t \le j+1$. Moreover $F_j^{(y)} \le F_{j+1}^{(y+1)}$ by the induction hypothesis for $i = j$, and $F_{n-j}^{(y)} \le F_{n-j}^{(y+1)}$ by the monotonicity in the threshold.

Hence the general inequality gives $F_{n+1}^{(y)} \le F_{n+2}^{(y+1)}$, which completes the induction. □

4. The chain range and the column equation

The last-car recursion gives each normalised count as a sum of products of two earlier ones. Up to a certain length, however, it can be replaced by a linear relation, and this relation, combined for the three thresholds $k+1$, k and $k-1$, is the column equation: an exact linear recursion for the normalised step $\delta_n^{(k)}$ of a column $k \ge 2$ on its chain range $k+1 \le n \le 2k-1$ (Proposition 4.5). The crossing criterion and the proof that a positive column stays positive in its chain range (Proposition 1.3) start from this equation.

4.1. **The single-chain formula.** Recall that, by Lemma 2.4 and $c_0 = 1$, the sequence (c_n) is the solution of the weighted last-car recursion (Definition 3.2) with weight $\omega(t) = t$. We consider a solution W whose weight ω agrees with this weight up to some integer $k \ge 0$, i.e. $\omega(t) = t$ for $1 \le t \le k$. The recursion is quadratic in W, but for $n \le 2k+1$ the deviation $c_n - W_n$ satisfies a linear relation with explicit coefficients, the *single-chain formula*. We state it in two forms. The first

gives $c_n - W_n$ in terms of the earlier values of W; for $n \le k$ its sum is empty, and it says that $W_n = c_n$.

Proposition 4.1. *Let $\omega = (\omega(t))_{t\ge1}$ and $W = (W_n)_{n\ge0}$ be sequences of real numbers such that W is a solution of the weighted last-car recursion with weight ω (Definition 3.2), and let $k \ge 0$ be an integer such that $\omega(t) = t$ for every integer t with $1 \le t \le k$. Then, for every integer n with $0 \le n \le 2k+1$,*

$$c_n - W_n = \sum_{t=k+1}^{n} c_{n-t}\Big(c_{t-1} - \frac{\omega(t)}{t}\,W_{t-1}\Big).$$

Proof. By Definition 3.2 we have $W_0 = 1$ and

$$(n+1)\,W_{n+1} = \sum_{j=0}^{n} \omega(j+1)\,W_j\,W_{n-j} \qquad (n \ge 0).$$

We write

$$\mathcal{W} = \sum_{n\ge0} W_n X^n, \qquad U = \frac{\mathcal{W}}{C} = \sum_{i\ge0} U_i X^i, \qquad L = \sum_{t\ge1} L_t X^t,$$

where $C = \sum_{n\ge0} c_n X^n$ is the series of Section 2, and where

$$L_t = c_{t-1} - \frac{\omega(t)}{t}\,W_{t-1} \qquad (t \ge 1).$$

Thus $t\,L_t$ is the difference between the factor $t\,c_{t-1}$ of the recursion of Lemma 2.4 and the corresponding factor $\omega(t)\,W_{t-1}$ of the recursion for W.

We first show that

$$W_n = c_n \quad (0 \le n \le k) \qquad \text{and} \qquad L_t = 0 \quad (1 \le t \le k).$$

For $n < k$ every weight that occurs in the recursion for W_{n+1} is $\omega(j+1) = j+1$, so that $W_0, \dots, W_k$ satisfy the recursion of Lemma 2.4; since $W_0 = c_0 = 1$, the first assertion follows by induction on n. The second follows from the first by the definition of L_t, since $\omega(t) = t$ for $1 \le t \le k$.

Next we show that $U' = -L'\,U$. In terms of series, the recursion for W reads

$$\mathcal{W}' = \Big(\sum_{j\ge0} \omega(j+1)\,W_j\,X^j\Big)\,\mathcal{W},$$

and since $(XC)' = \sum_{j\ge0}(j+1)\,c_j X^j$, the definition of L_t gives

$$L' = (XC)' - \sum_{j\ge0} \omega(j+1)\,W_j\,X^j.$$

Since $(1/C)' = -(1/C)\,(XC)'$ by (2.5), the product rule and the last two displays give

$$\begin{aligned} U' &= \frac{1}{C}\,\mathcal{W}' + \mathcal{W}\Big(\frac{1}{C}\Big)' \\ &= \frac{1}{C}\Big(\sum_{j\ge0} \omega(j+1)\,W_j\,X^j\Big)\,\mathcal{W} - \frac{1}{C}\,(XC)'\,\mathcal{W} = -L'\,U. \end{aligned}$$

Comparing the coefficients of X^i on the two sides, we obtain

$$(i+1)\,U_{i+1} = -\sum_{j=0}^{i}(j+1)\,L_{j+1}\,U_{i-j} \qquad (i \ge 0).$$

Moreover $U_0 = W_0/c_0 = 1$.

From this recursion we now determine the coefficients of U up to degree $2k+1$. We claim that

$$U_i = 0 \quad (1 \le i \le k) \qquad \text{and} \qquad U_i = -L_i \quad (1 \le i \le 2k+1). \tag{4.1}$$

With $i-1$ in place of i, the recursion reads

$$i\,U_i = -\sum_{j=0}^{i-1} (j+1)\,L_{j+1}\,U_{i-1-j} \qquad (i \ge 1).$$

For $1 \le i \le k$ every coefficient L_{j+1} on the right has $j+1 \le k$ and vanishes, since $L_t = 0$ for $1 \le t \le k$, so that $U_i = 0$. Now let $1 \le i \le 2k+1$. The term with $j = i-1$ is $i\,L_i\,U_0 = i\,L_i$. In a term with $j < i-1$, either $j+1 \le k$, and then $L_{j+1} = 0$ as before; or $j \ge k$, and then $1 \le i-1-j \le k$, since $i \le 2k+1$, so that $U_{i-1-j} = 0$ by the first part of the claim. Hence $i\,U_i = -i\,L_i$, which proves the claim. For $i = 2k+2$ the argument no longer applies: the term with $j = k$ contains $L_{k+1}U_{k+1} = -L_{k+1}^2$, which need not vanish.

It remains to return from U to W. Let $0 \le n \le 2k+1$. Comparing the coefficients of X^n in $\mathcal{W} = U\,C$ and using $U_0 = 1$ and the claim just proved, we find that

$$W_n = \sum_{t=0}^{n} U_t\,c_{n-t} = c_n - \sum_{t=k+1}^{n} L_t\,c_{n-t},$$

which is the assertion by the definition of L_t. □

The second form states the same relation for the coefficients of the quotient by C, in which W is combined with the free kernel of Section 2.

Proposition 4.2. *Let ω, W and k be as in Proposition 4.1. Then, for every integer n with $1 \le n \le 2k+1$,*

$$W_n - \sum_{m=1}^{n} \varepsilon_m\,W_{n-m} = \begin{cases} 0, & n \le k, \\ \dfrac{\omega(n)}{n}\,W_{n-1} - c_{n-1}, & n > k. \end{cases}$$

Proof. We keep the notation $\mathcal{W}$, U and L of the proof of Proposition 4.1. By Lemma 2.6, applied to $\mathcal{W}$, the coefficient of X^n in $U = \mathcal{W}/C$ is

$$U_n = W_n - \sum_{m=1}^{n} \varepsilon_m\,W_{n-m}.$$

By the relations (4.1) this is 0 for $1 \le n \le k$, and for $k < n \le 2k+1$ it is $-L_n = \frac{\omega(n)}{n}\,W_{n-1} - c_{n-1}$. □

4.2. **The level equation and the column equation.** We now apply the single-chain formula to the normalised counts of Section 3. Let $y \ge 1$ be a threshold. By Corollary 3.3, $z^{(y)}$ is a solution of the weighted last-car recursion with weight $\omega(t) = \min(t, y)$, which equals t for $1 \le t \le y$. Hence Propositions 4.1 and 4.2 apply to $W = z^{(y)}$ with y in place of k, and the second of them gives, for every integer n with $y \le n \le 2y+1$,

$$z_n^{(y)} - \sum_{m=1}^{n} \varepsilon_m\,z_{n-m}^{(y)} = \frac{y}{n}\,z_{n-1}^{(y)} - c_{n-1}. \tag{4.2}$$

Indeed, for $n > y$ this is the second case of the proposition with $\omega(n) = y$, and for $n = y$ both sides vanish, the right-hand side by Lemma 3.4.

This relation is linear in $z^{(y)}$, and the term c_{n-1} on its right-hand side does not depend on y. It can therefore be subtracted for neighbouring thresholds, and the differences that arise are the normalised counts of the levels and the normalised steps of the columns. Recall from Section 3 that

$$\begin{aligned}
&\beta_n^{(\ell)} = z_n^{(\ell+1)} - z_n^{(\ell)} && (\ell \geq 1,\ n \geq 0),\\
&\beta_n^{(\ell)} = 0 && (\ell \geq 1,\ 0 \leq n \leq \ell),\\
&\delta_n^{(k)} = z_n^{(k+1)} - 2\, z_n^{(k)} + z_n^{(k-1)} && (k \geq 2,\ n \geq 0),\\
&\delta_n^{(k)} = 0 && (k \geq 2,\ 0 \leq n < k);
\end{aligned}$$

here the third line is (3.1) divided by $n!$.

For two neighbouring thresholds $\ell + 1$ and ℓ, the difference of the relations (4.2) is the *level equation*, which determines $\beta_n^{(\ell)}$ row by row for $\ell + 1 \leq n \leq 2\ell + 1$.

Lemma 4.3. *Let $\ell \geq 1$ and n be integers with $\ell + 1 \leq n \leq 2\ell + 1$. Then*

$$\beta_n^{(\ell)} = \frac{z_{n-1}^{(\ell+1)}}{n} + \frac{\ell}{n}\,\beta_{n-1}^{(\ell)} + \sum_{m=1}^{n-\ell-1} \varepsilon_m\, \beta_{n-m}^{(\ell)}.$$

Proof. We subtract (4.2) for $y = \ell$ from the same relation for $y = \ell + 1$; both apply in the stated range of n. The terms c_{n-1} cancel, and since $\beta_m^{(\ell)} = z_m^{(\ell+1)} - z_m^{(\ell)}$ for every $m \geq 0$, we obtain

$$\beta_n^{(\ell)} - \sum_{m=1}^{n} \varepsilon_m\, \beta_{n-m}^{(\ell)} = \frac{\ell+1}{n}\, z_{n-1}^{(\ell+1)} - \frac{\ell}{n}\, z_{n-1}^{(\ell)} = \frac{z_{n-1}^{(\ell+1)}}{n} + \frac{\ell}{n}\,\beta_{n-1}^{(\ell)}.$$

The terms with $m \geq n - \ell$ vanish, because $\beta_{n-m}^{(\ell)} = 0$ for $n - m \leq \ell$, and the assertion follows. □

We turn to the step of a column $k \geq 2$. As recalled above, it vanishes in rows $n < k$, and Proposition 4.1 gives its value in row k.

Lemma 4.4. *For every integer $k \geq 2$,*

$$\delta_k^{(k)} = -\frac{a_k}{k}.$$

Proof. We apply Proposition 4.1 to $W = z^{(k-1)}$, with $k - 1$ in place of k and with $n = k \leq 2(k-1) + 1$. The sum consists of the single term $t = k$, in which $\omega(k) = k - 1$, and since $c_0 = 1$ and $z_{k-1}^{(k-1)} = c_{k-1}$ (Lemma 3.4), it gives

$$c_k - z_k^{(k-1)} = c_{k-1} - \frac{k-1}{k}\, c_{k-1} = \frac{c_{k-1}}{k}.$$

On the other hand, $\beta_k^{(k)} = 0$, as recalled above, and $\beta_k^{(k-1)} = z_k^{(k)} - z_k^{(k-1)} = c_k - z_k^{(k-1)}$ by Lemma 3.4. Since $c_{k-1} = a_k$, we conclude that

$$\delta_k^{(k)} = \beta_k^{(k)} - \beta_k^{(k-1)} = -\big(c_k - z_k^{(k-1)}\big) = -\frac{a_k}{k}.$$ □

Multiplied by $k!$, the lemma reads $d_{k,k} = -k^{k-2}$, so that every column $k \geq 2$ begins with a negative step. From row $k + 1$ on, the step satisfies a linear recursion as long as the relation (4.2) is available for the three thresholds $k + 1$, k and $k - 1$,

i.e. on the chain range $k+1 \le n \le 2k-1$ of the column. This recursion is the column equation stated in Section 1.

Proposition 4.5. *Let $k \ge 2$ and n be integers with $k+1 \le n \le 2k-1$. Then the normalised step of column k satisfies*

$$\delta_n = \frac{k+1}{n}\,\delta_{n-1} + \frac{2}{n}\,\beta_{n-1}^{(k-1)} + \sum_{m=1}^{n-k} \varepsilon_m\,\delta_{n-m}.$$

Proof. We argue as in the proof of Lemma 4.3, now with three thresholds. For $k+1 \le n \le 2k-1$ the relation (4.2) holds for each of the thresholds $y = k+1$, k and $k-1 \ge 1$. We take it for $y = k+1$, subtract twice the relation for $y = k$, and add the relation for $y = k-1$. The terms c_{n-1} cancel, and we obtain

$$\begin{aligned}\delta_n - \sum_{m=1}^{n} \varepsilon_m\,\delta_{n-m} &= \frac{k+1}{n}\,z_{n-1}^{(k+1)} - \frac{2k}{n}\,z_{n-1}^{(k)} + \frac{k-1}{n}\,z_{n-1}^{(k-1)} \\ &= \frac{k+1}{n}\,\delta_{n-1} + \frac{2}{n}\left(z_{n-1}^{(k)} - z_{n-1}^{(k-1)}\right) = \frac{k+1}{n}\,\delta_{n-1} + \frac{2}{n}\,\beta_{n-1}^{(k-1)},\end{aligned}$$

where the first two equalities follow from the expression of the step as a second difference, $\delta_m = z_m^{(k+1)} - 2\,z_m^{(k)} + z_m^{(k-1)}$, and the third from $\beta_{n-1}^{(k-1)} = z_{n-1}^{(k)} - z_{n-1}^{(k-1)}$. The terms with $m > n-k$ vanish, because $\delta_{n-m} = 0$ for $n-m < k$, and the assertion follows. □

5. The series of a threshold and three inequalities on all rows

The column equation of Proposition 4.5 is confined to the chain range of a column. Beyond it no exact linear equation is available, and the proof of Theorem 1.1 rests there on three inequalities that hold in every row: the column inequality (Proposition 5.7), the one-tree bound (Proposition 5.8) and the diagonal lower bound (Proposition 5.9). To prove them we pass to the generating series of the $z_n^{(y)}$, for which the last-car recursion is a differential equation, and compare the series of adjacent thresholds.

We use the notation of Section 2 for formal power series in X with real coefficients, which we call series. For series F and G we write $F \succeq 0$ if every coefficient of F is nonnegative, and $F \succeq G$, or $G \preceq F$, if $F - G \succeq 0$. Sums and products of series that are $\succeq 0$ are $\succeq 0$.

5.1. The series of a threshold.

For a threshold $y \ge 1$ let

$$Z_y = \sum_{n \ge 0} z_n^{(y)}\,X^n,$$

and

$$\Gamma_y = \sum_{j \ge 0} \min(j+1, y)\,z_j^{(y)}\,X^j.$$

The constant term of Z_y is $z_0^{(y)} = 1$. With these two series the last-car recursion becomes a differential equation.

Lemma 5.1. *For every integer $y \ge 1$,*

$$Z_y' = \Gamma_y\,Z_y.$$

Proof. The coefficients of X^n in Z_y' and in $\Gamma_y\, Z_y$ are $(n+1)\, z_{n+1}^{(y)}$ and $\sum_{j=0}^{n} \min(j+1, y)\, z_j^{(y)}\, z_{n-j}^{(y)}$, and these are equal for every $n \geq 0$ by Corollary 3.3. □

5.2. **The prime kernel.** The proofs of the three inequalities divide by the series Z_y of a threshold, and this subsection describes the coefficients of its inverse. Let $y \geq 1$ be a threshold. Since the constant term of Z_y is 1, the inverse $1/Z_y$ exists. We put

$$K_y = 1 - \frac{1}{Z_y},$$

and we call the sequence of the coefficients of this series, $\kappa_m^{(y)} = [X^m]K_y$ for $m \geq 0$, the *prime kernel* of threshold y. By definition

$$Z_y\,(1 - K_y) = 1.$$

The constant term of $1/Z_y$ is 1, so that $\kappa_0^{(y)} = 0$, and for every series $F = \sum_{n\geq 0} F_n X^n$ we have

$$[X^n]\big(F\,(1-K_y)\big) = F_n - \sum_{m=1}^{n} \kappa_m^{(y)}\, F_{n-m} \qquad (n \geq 0).$$

For $F = Z_y$ the product is 1, and this formula becomes

$$z_n^{(y)} = \sum_{m=1}^{n} \kappa_m^{(y)}\, z_{n-m}^{(y)} \qquad (n \geq 1). \tag{5.1}$$

This equation has the form of the renewal equation $c_n = \sum_{m=1}^{n} \varepsilon_m\, c_{n-m}$ of Corollary 2.7, with $z^{(y)}$ in the place of (c_n) and the prime kernel in the place of the free kernel.

Lemma 5.2. *For every integer $y \geq 1$,*

$$K_y' = (1 - K_y)\,\Gamma_y.$$

Proof. Differentiating the identity $Z_y\,(1-K_y) = 1$ and using Lemma 5.1, we get $Z_y\, K_y' = Z_y'\,(1-K_y) = \Gamma_y\, Z_y\,(1-K_y) = \Gamma_y$, and multiplying by $1 - K_y$ gives the claim. □

Lemma 5.3. *Let $y \geq 1$ and m be integers with $1 \leq m \leq y$. Then*

$$\kappa_m^{(y)} = \varepsilon_m.$$

Proof. For $1 \leq n \leq y$ Lemma 3.4 turns the equation (5.1) into $c_n = \sum_{i=1}^{n} \kappa_i^{(y)}\, c_{n-i}$. Subtracting the equation $c_n = \sum_{i=1}^{n} \varepsilon_i\, c_{n-i}$ of Corollary 2.7, we obtain

$$\sum_{i=1}^{n} \big(\kappa_i^{(y)} - \varepsilon_i\big)\, c_{n-i} = 0 \qquad (1 \leq n \leq y),$$

and since $c_0 = 1$ the assertion follows by induction on m. □

Beyond the free range the prime kernel satisfies a recursion with positive coefficients.

Lemma 5.4. *Let $y \geq 2$ and $N \geq y$ be integers. Then*

$$N\, \kappa_N^{(y)} = \sum_{j=1}^{y-1} (y-j)\, a_j\, \kappa_{N-j}^{(y)}.$$

Proof. Let $H_y = \sum_{i=0}^{y-2} (y-1-i)\, c_i\, X^i$. Since $z_i^{(y)} = c_i$ for $i \le y-2$ by Lemma 3.4, a comparison of coefficients shows that $\Gamma_y = y\, Z_y - H_y$. By Lemma 5.2, and since $Z_y\,(1-K_y) = 1$, it follows that

$$K_y' = (1-K_y)\,\Gamma_y = y\, Z_y\,(1-K_y) - (1-K_y)\, H_y = y - (1-K_y)\, H_y.$$

We compare the coefficients of X^{N-1}. On the left the coefficient is $N\,\kappa_N^{(y)}$. On the right the constant y contributes nothing, because $N-1 \ge 1$, and since H_y has degree at most $y-2 \le N-2$, only the coefficients $-\kappa_m^{(y)}$ of $1-K_y$ with $m \ge 1$ enter the product. Hence

$$N\,\kappa_N^{(y)} = \sum_{i=0}^{y-2} (y-1-i)\, c_i\, \kappa_{N-1-i}^{(y)},$$

which is the assertion, with $j = i+1$ and $c_{j-1} = a_j$. □

For example, for $y = 2$ the recursion reads $N\,\kappa_N^{(2)} = \kappa_{N-1}^{(2)}$ for $N \ge 2$, because $a_1 = 1$, and with $\kappa_1^{(2)} = \varepsilon_1 = 1$ it gives $\kappa_N^{(2)} = 1/N!$ for all $N \ge 1$. In particular $\kappa_3^{(2)} = 1/6$, whereas $\varepsilon_3 = 2/3$, so that the range $m \le y$ of Lemma 5.3 cannot be extended.

The recursion of Lemma 5.4 gives the sign of the prime kernel. We state the result for the tilted form $\kappa_m^{(y)}\, x^m$ of the kernel at a tilt $x \ge 0$; for $x = 1$ it says that $\kappa_m^{(y)} \ge 0$.

Lemma 5.5. *Let $y \ge 2$ be an integer and $x \ge 0$ a real number. Then*

$$\kappa_m^{(y)}\, x^m \ge 0 \qquad \textit{for every integer } m \ge 0.$$

Proof. We use induction on m. We have $\kappa_0^{(y)} = 0$, and $\kappa_m^{(y)}\, x^m = \varepsilon_m\, x^m \ge 0$ for $1 \le m \le y$ by Lemma 5.3. For $m > y$ the recursion of Lemma 5.4 with $N = m$, multiplied by x^m, reads

$$m\,\kappa_m^{(y)}\, x^m = \sum_{j=1}^{y-1} (y-j)\, a_j\, x^j \cdot \kappa_{m-j}^{(y)}\, x^{m-j},$$

and every term on the right is nonnegative, by the induction hypothesis and since $x \ge 0$. □

5.3. **The column inequality and the one-tree bound.** Let $k \ge 2$ be an integer. Recall from Section 3 that $\beta_n^{(\ell)} = z_n^{(\ell+1)} - z_n^{(\ell)}$ for $\ell \ge 1$, so that

$$\delta_n^{(k)} = \beta_n^{(k)} - \beta_n^{(k-1)} = z_n^{(k+1)} - 2\, z_n^{(k)} + z_n^{(k-1)} \qquad (n \ge 0).$$

Thus the normalised steps of column k are the coefficients of the series

$$\mathsf{D} = Z_{k+1} - 2\, Z_k + Z_{k-1},$$

the second difference of Z_y with respect to y at $y = k$. The first two inequalities bound these steps from below in terms of a second sequence. For a column $k \ge 2$ let $(\xi_t)_{t \ge 0}$ be the sequence defined by $\xi_0 = 0$ and, for $t \ge 1$, by

$$t\,\xi_t = \min(t, k+1)\, z_{t-1}^{(k+1)} - 2\min(t,k)\, z_{t-1}^{(k)} + \min(t, k-1)\, z_{t-1}^{(k-1)}.$$

We write $\xi_t^{(k)}$ for ξ_t when the column has to be shown. We have $\xi_t = 0$ for $t < k$: for $1 \le t \le k-1$ the three minima in the definition are equal to t, and

$z_{t-1}^{(k+1)} = z_{t-1}^{(k)} = z_{t-1}^{(k-1)} = c_{t-1}$ by Lemma 3.4. From index $k+1$ on, ξ_t is expressed by the step and by level $k-1$, in the combination that occurs in the column equation.

Lemma 5.6. *Let $k \geq 2$ and $n \geq k$ be integers. Then*

$$(n+1)\,\xi_{n+1}^{(k)} = (k+1)\,\delta_n^{(k)} + 2\,\beta_n^{(k-1)}.$$

Proof. For $t = n+1 \geq k+1$ the three minima in the definition of ξ_t are $k+1$, k and $k-1$, so that the left-hand side is $(k+1)\,z_n^{(k+1)} - 2k\,z_n^{(k)} + (k-1)\,z_n^{(k-1)}$. Since $\delta_n^{(k)} = z_n^{(k+1)} - 2\,z_n^{(k)} + z_n^{(k-1)}$ and $\beta_n^{(k-1)} = z_n^{(k)} - z_n^{(k-1)}$, so is the right-hand side. □

The first inequality extends the column equation of Proposition 4.5 to all rows $n \geq k+1$: the free kernel is replaced by the prime kernel of threshold k, and the equation becomes an inequality that bounds the step from below. In the chain range $k+1 \leq n \leq 2k-1$ it holds with equality: there $m \leq n-k \leq k-1$, so that $\kappa_m^{(k)} = \varepsilon_m$ by Lemma 5.3, and the inequality is the column equation multiplied by n. The proof compares the series of two adjacent thresholds. Applied to the thresholds k and $k+1$ and to the thresholds $k-1$ and k, the comparison bounds $\mathsf{D}\,(1-K_k)$ from below, and the proposition is the coefficient of X^n in this bound.

Proposition 5.7. *Let $k \geq 2$ and $n \geq k+1$ be integers. Then*

$$(k+1)\,\delta_{n-1}^{(k)} + 2\,\beta_{n-1}^{(k-1)} \leq n\left(\delta_n^{(k)} - \sum_{m=1}^{n-k} \kappa_m^{(k)}\,\delta_{n-m}^{(k)}\right).$$

Proof. We write $\delta_j = \delta_j^{(k)}$ and $\xi_t = \xi_t^{(k)}$.

We begin with a positivity principle. For series F, G and S,

$$\text{if} \quad F' = G\,F + S, \quad G \succeq 0, \quad S \succeq 0 \quad \text{and} \quad [X^0]F \geq 0, \quad \text{then} \quad F \succeq 0.$$

Indeed, in terms of the coefficients F_n, G_n and S_n of the three series the equation reads $(n+1)\,F_{n+1} = \sum_{j=0}^{n} G_j\,F_{n-j} + S_n$, and $F_n \geq 0$ follows by induction on n. For $G = 0$ the principle says that a series with nonnegative constant term whose derivative is $\succeq 0$ is $\succeq 0$.

Next we introduce the series with which two adjacent thresholds are compared, and we show that it is $\succeq 0$. Let $y \geq 2$ be an integer, and let Ω_y be the series with constant term 0 and derivative $\Omega_y' = \Gamma_y - \Gamma_{y-1}$. Its coefficients are

$$[X^t]\Omega_y = \frac{1}{t}\Big(\min(t,y)\,z_{t-1}^{(y)} - \min(t,y-1)\,z_{t-1}^{(y-1)}\Big) \qquad (t \geq 1). \tag{5.2}$$

Let further $\mathcal{R}_y = \sum_{j \geq y-1} z_j^{(y-1)} X^j$ be the sum of the terms of Z_{y-1} of degree $j \geq y-1$. Then

$$\Omega_y' = y\,(Z_y - Z_{y-1}) + \mathcal{R}_y.$$

Indeed, the coefficient of X^j on the left is $\min(j+1,y)\,z_j^{(y)} - \min(j+1,y-1)\,z_j^{(y-1)}$. For $j \leq y-2$ the two minima are both $j+1$, and $z_j^{(y)} = z_j^{(y-1)} = c_j$ by Lemma 3.4, so that the coefficients on both sides vanish. For $j \geq y-1$ the minima are y and $y-1$, and $y\,z_j^{(y)} - (y-1)\,z_j^{(y-1)} = y\,\big(z_j^{(y)} - z_j^{(y-1)}\big) + z_j^{(y-1)}$. Recall that

$0 \le z_n^{(y-1)} \le z_n^{(y)}$ for all $n \ge 0$. The two series $y\,(Z_y - Z_{y-1})$ and $\mathcal{R}_y$ are therefore $\succeq 0$, and with the positivity principle for $G = 0$ we obtain

$$\Omega_y' \succeq 0 \qquad \text{and} \qquad \Omega_y \succeq 0.$$

We now compare the thresholds $y-1$ and y. For $y \ge 2$ let

$$\mathcal{Q}_y = Z_y\,(1 - K_{y-1}) \qquad \text{and} \qquad \Theta_y = 1 - Z_{y-1}\,(1 - K_y).$$

By the definition of K_y we have $\mathcal{Q}_y = Z_y/Z_{y-1}$ and $1 - \Theta_y = Z_{y-1}/Z_y$. The constant term of $\mathcal{Q}_y$ is 1 and that of Θ_y is 0, and by the product rule and Lemmas 5.1 and 5.2,

$$\begin{aligned}\mathcal{Q}_y' &= \Gamma_y\,Z_y\,(1 - K_{y-1}) - Z_y\,(1 - K_{y-1})\,\Gamma_{y-1} = \mathcal{Q}_y\,\Omega_y',\\ \Theta_y' &= -\Gamma_{y-1}\,Z_{y-1}\,(1 - K_y) + Z_{y-1}\,(1 - K_y)\,\Gamma_y = (1 - \Theta_y)\,\Omega_y'.\end{aligned}$$

We show that

$$\Theta_y \preceq \Omega_y \preceq \mathcal{Q}_y - 1 \qquad (y \ge 2). \tag{5.3}$$

Formally, the differential equations for $\mathcal{Q}_y$ and Θ_y say that $\mathcal{Q}_y = e^{\Omega_y}$ and $1 - \Theta_y = e^{-\Omega_y}$, and the claim is the analogue for coefficients of the inequalities $1 - e^{-s} \le s \le e^s - 1$ for a real number s.

For the right-hand inequality let $Y = \mathcal{Q}_y - 1 - \Omega_y$. The constant term of Y is 0, and since $\mathcal{Q}_y' = \mathcal{Q}_y\,\Omega_y'$,

$$Y' = \mathcal{Q}_y\,\Omega_y' - \Omega_y' = (Y + \Omega_y)\,\Omega_y' = \Omega_y'\,Y + \Omega_y\,\Omega_y'.$$

Here $\Omega_y' \succeq 0$ and $\Omega_y\,\Omega_y' \succeq 0$, because Ω_y and Ω_y' are $\succeq 0$, and the positivity principle gives $Y \succeq 0$, i.e. $\Omega_y \preceq \mathcal{Q}_y - 1$.

For the left-hand inequality we first show that $\Theta_y \succeq 0$. The equation $\Theta_y' = (1 - \Theta_y)\,\Omega_y'$ does not show this: written as $\Theta_y' = -\Omega_y'\,\Theta_y + \Omega_y'$, it multiplies Θ_y by a series whose coefficients are nonpositive, and the positivity principle does not apply. We therefore rewrite its right-hand side. Since $Z_y\,(1 - K_y) = 1$, we have $\Theta_y = (Z_y - Z_{y-1})\,(1 - K_y)$, hence $Z_y - Z_{y-1} = \Theta_y\,Z_y$ and

$$(1 - \Theta_y)\,(Z_y - Z_{y-1}) = Z_{y-1}\,(1 - K_y)\cdot \Theta_y\,Z_y = Z_{y-1}\,\Theta_y.$$

Inserting the expression $y\,(Z_y - Z_{y-1}) + \mathcal{R}_y$ for Ω_y', we obtain

$$\Theta_y' = y\,(1 - \Theta_y)\,(Z_y - Z_{y-1}) + (1 - \Theta_y)\,\mathcal{R}_y = \big(y\,Z_{y-1} - \mathcal{R}_y\big)\,\Theta_y + \mathcal{R}_y.$$

In this linear equation the series that multiplies Θ_y is $\succeq 0$: the coefficient of X^j in $y\,Z_{y-1} - \mathcal{R}_y$ is $y\,z_j^{(y-1)}$ for $j \le y-2$ and $(y-1)\,z_j^{(y-1)}$ for $j \ge y-1$. Since $\mathcal{R}_y \succeq 0$ as well and the constant term of Θ_y is 0, the positivity principle gives $\Theta_y \succeq 0$. Now $\Omega_y - \Theta_y$ has constant term 0, and since $\Theta_y' = (1 - \Theta_y)\,\Omega_y'$ its derivative is

$$\Omega_y' - (1 - \Theta_y)\,\Omega_y' = \Theta_y\,\Omega_y',$$

a product of the two series $\Theta_y \succeq 0$ and $\Omega_y' \succeq 0$. By the positivity principle with $G = 0$ we get $\Omega_y - \Theta_y \succeq 0$, which completes the proof of (5.3).

We return to column k and deduce the proposition. Since $Z_k\,(1 - K_k) = 1$, the definitions of $\mathcal{Q}_{k+1}$ and Θ_k give

$$\mathsf{D}\,(1 - K_k) = Z_{k+1}\,(1 - K_k) - 2 + Z_{k-1}\,(1 - K_k) = (\mathcal{Q}_{k+1} - 1) - \Theta_k,$$

and the comparison (5.3), applied with $y = k+1$ to the first term and with $y = k$ to the second, gives

$$\mathsf{D}\,(1 - K_k) \succeq \Omega_{k+1} - \Omega_k = \sum_{t \geq 0} \xi_t\, X^t. \tag{5.4}$$

The equality holds by the formula (5.2) for the coefficients of Ω_y and by the definition of ξ_t. We compare the coefficients of X^n. Since $\kappa_0^{(k)} = 0$, the coefficient on the left is $\delta_n - \sum_{m=1}^{n} \kappa_m^{(k)}\, \delta_{n-m}$, in which the terms with $n - m < k$ vanish, because $\delta_j = 0$ for $j < k$ (Section 3). Hence

$$\delta_n - \sum_{m=1}^{n-k} \kappa_m^{(k)}\, \delta_{n-m} \geq \xi_n.$$

Since $n - 1 \geq k$, Lemma 5.6 gives $n\,\xi_n = (k+1)\,\delta_{n-1} + 2\,\beta_{n-1}^{(k-1)}$, and the assertion follows on multiplying by n. □

The second inequality bounds the step of column k from below by the convolution of the sequence (ξ_t) with the sequence $z^{(k)}$.

Proposition 5.8. *Let $k \geq 2$ and $n \geq 0$ be integers. Then*

$$\sum_{t=0}^{n} \xi_t^{(k)}\, z_{n-t}^{(k)} \leq \delta_n^{(k)}.$$

Proof. We multiply the inequality (5.4), which was shown in the proof of Proposition 5.7, by $Z_k \succeq 0$. Since $(1 - K_k)\, Z_k = 1$, this gives

$$\mathsf{D} \succeq \Big(\sum_{t \geq 0} \xi_t^{(k)}\, X^t\Big)\, Z_k,$$

and the proposition is the coefficient of X^n in this inequality. □

5.4. **The diagonal lower bound.** For an integer $k \geq 2$ let

$$\gamma_k = \Big(1 + \frac{1}{k}\Big)^{k-2}.$$

The sequence $(\gamma_k)_{k \geq 2}$ is positive and nondecreasing: for $2 \leq i \leq j$ Lemma 2.1 and $1/j \leq 1/i$ give

$$\gamma_i \Big(1 + \frac{1}{i}\Big) \leq \gamma_j \Big(1 + \frac{1}{j}\Big) \leq \gamma_j \Big(1 + \frac{1}{i}\Big).$$

The third inequality compares two entries of the table that are neighbours along a diagonal; in terms of the counts it reads $\gamma_k\, n\, b_{n-1,k-1} \leq b_{n,k}$. For $n = k+1$ it is an equality: since $\beta_k^{(k)} = \beta_{k+1}^{(k+1)} = 0$, Lemma 4.4, applied to columns k and $k+1$, gives $\beta_k^{(k-1)} = a_k/k$ and $\beta_{k+1}^{(k)} = a_{k+1}/(k+1)$, and $a_{k+1} = (1+1/k)^{k-1}\, a_k$ by (2.3).

Proposition 5.9. *Let $k \geq 2$ and $n \geq 1$ be integers. Then*

$$\gamma_k\, \beta_{n-1}^{(k-1)} \leq \beta_n^{(k)}.$$

Proof. Fix $k \geq 2$. Let $E_0 = 0$ and $E_n = \beta_n^{(k)} - \gamma_k\, \beta_{n-1}^{(k-1)}$ for $n \geq 1$. We show by induction on n that $E_n \geq 0$, and we keep the notation of the proof of Proposition 5.7. By the definition of the $\beta_n^{(\ell)}$, the E_n are the coefficients of the series

$$E = (Z_{k+1} - Z_k) - \gamma_k\, X\, (Z_k - Z_{k-1}).$$

We first divide E by Z_k, that is, we multiply it by $1-K_k$. Since $Z_k\,(1-K_k)=1$, the definitions $\mathcal{Q}_{k+1}=Z_{k+1}\,(1-K_k)$ and $\Theta_k=1-Z_{k-1}\,(1-K_k)$ give $(Z_{k+1}-Z_k)\,(1-K_k)=\mathcal{Q}_{k+1}-1$ and $(Z_k-Z_{k-1})\,(1-K_k)=\Theta_k$. The comparison (5.3), applied with $y=k+1$ and with $y=k$, gives

$$E\,(1-K_k)=(\mathcal{Q}_{k+1}-1)-\gamma_k\,X\,\Theta_k\succeq\Omega_{k+1}-\gamma_k\,X\,\Omega_k,$$

and, since $\kappa_0^{(k)}=0$, a comparison of the coefficients of X^n shows that

$$E_n-\sum_{m=1}^{n}\kappa_m^{(k)}\,E_{n-m}\geq[X^n]\big(\Omega_{k+1}-\gamma_k\,X\,\Omega_k\big)\qquad(n\geq 0). \tag{5.5}$$

Next we bound the right-hand side of this inequality from below. We show that

$$(i+1)\,[X^{i+1}]\big(\Omega_{k+1}-\gamma_k\,X\,\Omega_k\big)\geq k\,E_i\qquad(i\geq k). \tag{5.6}$$

Let $i\geq k$. By the formula (5.2), in which all the minima are now the thresholds themselves,

$$\begin{aligned}(i+1)\,[X^{i+1}]\Omega_{k+1}&=(k+1)\,z_i^{(k+1)}-k\,z_i^{(k)},\\ i\,[X^i]\Omega_k&=k\,z_{i-1}^{(k)}-(k-1)\,z_{i-1}^{(k-1)}.\end{aligned}$$

Inserting this and the definition of E_i, we find by a direct computation that

$$\begin{aligned}&(i+1)\,[X^{i+1}]\big(\Omega_{k+1}-\gamma_k\,X\,\Omega_k\big)-k\,E_i\\ &\qquad=\Big(z_i^{(k+1)}-\gamma_k\,\frac{i+1}{i}\,z_{i-1}^{(k)}\Big)+\gamma_k\,\frac{i+1-k}{i}\,\Big(z_{i-1}^{(k)}-z_{i-1}^{(k-1)}\Big),\end{aligned}$$

and we show that the two terms on the right are nonnegative. For the second this holds because $i\geq k$ and $z_{i-1}^{(k-1)}\leq z_{i-1}^{(k)}$. For the first, Proposition 3.5 with threshold k and index $i-1$ gives $z_{i-1}^{(k)}/c_{i-1}\leq z_i^{(k+1)}/c_i$. Since $c_i=(1+1/i)^{i-1}\,c_{i-1}=\gamma_i\,\frac{i+1}{i}\,c_{i-1}$ by (2.3), it follows that

$$z_i^{(k+1)}\geq\frac{c_i}{c_{i-1}}\,z_{i-1}^{(k)}=\gamma_i\,\frac{i+1}{i}\,z_{i-1}^{(k)}\geq\gamma_k\,\frac{i+1}{i}\,z_{i-1}^{(k)},$$

where the last step holds because $\gamma_k\leq\gamma_i$ for $2\leq k\leq i$. This proves (5.6).

It remains to carry out the induction. For $n\leq k$ we have $E_n=0$, because $\beta_j^{(\ell)}=0$ for $j\leq\ell$ (Section 3). Let $i\geq k$, and suppose that $E_j\geq 0$ for all $j\leq i$. By (5.5) for $n=i+1$ and by (5.6),

$$E_{i+1}\geq\sum_{m=1}^{i+1}\kappa_m^{(k)}\,E_{i+1-m}+\frac{k}{i+1}\,E_i.$$

Here $\kappa_m^{(k)}\geq 0$ by Lemma 5.5 with $x=1$, and the E_j on the right are nonnegative by the induction hypothesis. Hence $E_{i+1}\geq 0$. □

6. The critical tilt

Recall from Section 3 that a tilt is a real number x and that the tilted form of a sequence (s_n) is the sequence $(s_n x^n)$. The proof of the inequality $d_{n,k}\geq 0$ for $2k\leq n$ (Proposition 1.2) works, for each column $k\geq 2$, at one particular tilt, the critical tilt x_k. The main statement at this tilt is Proposition 6.5: for $k\geq 2$ the tilted counts $z_N^{(k)}x_k^N$ of threshold k never increase in N, so that they can serve as monotone weights in a summation by parts.

We begin with the definition. Since $c_j = a_{j+1}$, the ratio formula (2.3) gives $c_{j+1} = \big(1 + \frac{1}{j+1}\big)^j c_j$, and hence, for every real number x and every integer $j \ge 0$,

$$c_{j+1}\, x^{j+1} = x\Big(1 + \frac{1}{j+1}\Big)^j\, c_j\, x^j. \tag{6.1}$$

For an integer $k \ge 1$, the *critical tilt* of column k is the tilt at which the ratio $x\big(1 + \frac{1}{j+1}\big)^j$ of two consecutive terms of the tilted sequence $(c_j x^j)$ equals 1 for $j = k-1$, i.e.

$$x_k = \Big(\frac{k}{k+1}\Big)^{k-1}, \qquad \text{so that} \qquad x_k\Big(1 + \frac{1}{k}\Big)^{k-1} = 1.$$

We have $0 < x_k \le 1$, and $c_k\, x_k^k = c_{k-1}\, x_k^{k-1}$ by the choice of x_k. The first values are $x_1 = 1$, $x_2 = \frac{2}{3}$, $x_3 = \frac{9}{16}$ and $x_4 = \frac{64}{125}$, and $x_k > e^{-1}$ for every k, since $(1 + 1/k)^{k-1} < e$. The tilt is chosen with threshold k in view. On the free range $N \le k$ the tilted counts of threshold k are $z_N^{(k)} x^N = c_N x^N$ (Lemma 3.4), and for $x > x_k$ the count at $N = k$ is larger than the count at $N = k-1$; so x_k is the largest tilt at which the tilted counts of threshold k can be nonincreasing in N.

Here and in the rest of the paper, when a column k is fixed, a hat denotes the tilted form of a sequence at the critical tilt x_k. Thus

$$\hat{c}_n = c_n\, x_k^n, \qquad \hat{a}_n = a_n\, x_k^n, \qquad \hat{\varepsilon}_m = \varepsilon_m\, x_k^m,$$
$$\hat{z}_N^{(y)} = z_N^{(y)}\, x_k^N, \qquad \hat{\beta}_N^{(\ell)} = \beta_N^{(\ell)}\, x_k^N, \qquad \hat{\delta}_n = \delta_n^{(k)}\, x_k^n,$$

and $\hat{\kappa}_m = \kappa_m^{(k)}\, x_k^m$ for the prime kernel of threshold k itself.

The numbers $\hat{c}_n$ do not increase up to index k and do not decrease from index $k-1$ on, so that $\hat{c}_{k-1} = \hat{c}_k$ is their smallest value: for all integers $k \ge 1$ and $n \ge 0$,

$$\begin{aligned} c_{n+1}\, x_k^{n+1} &\le c_n\, x_k^n \qquad \text{if } n+1 \le k, \\ c_n\, x_k^n &\le c_{n+1}\, x_k^{n+1} \qquad \text{if } n+1 \ge k. \end{aligned} \tag{6.2}$$

Indeed, by the identity (6.1) with $x = x_k$ and $j = n$ the two terms differ by the factor $x_k\big(1 + \frac{1}{n+1}\big)^n$, and by Lemma 2.1 and the definition of x_k this factor is at most $x_k\big(1 + \frac{1}{k}\big)^{k-1} = 1$ if $n+1 \le k$ and at least 1 if $n+1 \ge k$.

The tilted free kernel does not increase up to index k either.

Lemma 6.1. *Let k and m be integers with $1 \le m < k$. Then*

$$\varepsilon_{m+1}\, x_k^{m+1} \le \varepsilon_m\, x_k^m.$$

Proof. For $m = 1$ the claim is $\frac{1}{2}x_k^2 \le x_k$, since $\varepsilon_1 = 1$ and $\varepsilon_2 = \frac{1}{2}$, and it holds because $0 < x_k \le 1$. Let $m \ge 2$. By the definitions of a_{i+1} and ε_{i+1},

$$a_{i+1} = \Big(1 + \frac{1}{i}\Big)^i\, \varepsilon_{i+1} \qquad (i \ge 1).$$

Moreover, $\hat{a}_{m+1} \le \hat{a}_m$: indeed, $\hat{a}_{n+1} = x_k\, \hat{c}_n$ for every $n \ge 0$, and $\hat{c}_m \le \hat{c}_{m-1}$ by (6.2), since $m \le k$. Hence

$$\big(1 + \tfrac{1}{m}\big)^m\, \hat{\varepsilon}_{m+1} = \hat{a}_{m+1} \le \hat{a}_m = \big(1 + \tfrac{1}{m-1}\big)^{m-1}\, \hat{\varepsilon}_m \le \big(1 + \tfrac{1}{m}\big)^m\, \hat{\varepsilon}_m,$$

where the two equalities are the identity for $i = m$ and for $i = m-1$, and the last inequality holds because $(1 + 1/i)^i$ is nondecreasing in $i \ge 1$ (Section 2). Dividing by $\big(1 + \frac{1}{m}\big)^m$ gives the claim. □

We turn to the prime kernel $\kappa^{(y)}$ of a threshold $y \geq 2$, which was introduced in Section 5. For $1 \leq m \leq y$ it is the free kernel (Lemma 5.3), and beyond this range it is no longer explicit. The next lemma and its proof show that it nevertheless inherits the monotonicity of the free kernel: if the tilted free kernel does not increase up to index y, then the tilted prime kernel of threshold y does not increase at all, and beyond the threshold it decays at least like $1/m$. For $y = k$ and the critical tilt x_k the hypothesis is Lemma 6.1.

Lemma 6.2. *Let $y \geq 2$ be an integer and let $x \geq 0$ be a real number such that*

$$\varepsilon_{m+1}\, x^{m+1} \leq \varepsilon_m\, x^m \qquad \text{for all integers } m \text{ with } 1 \leq m < y.$$

Then

$$(m+1)\, \kappa^{(y)}_{m+1}\, x^{m+1} \leq m\, \kappa^{(y)}_m\, x^m \qquad \text{for all integers } m \geq y.$$

Proof. We prove the following two statements together, by induction on $m \geq 1$:

(A_m) $\kappa^{(y)}_{m+1}\, x^{m+1} \leq \kappa^{(y)}_m\, x^m$;

(B_m) if $m \geq y$, then $(m+1)\, \kappa^{(y)}_{m+1}\, x^{m+1} \leq m\, \kappa^{(y)}_m\, x^m$.

The lemma is (B_m) for $m \geq y$. Let $m \geq 1$, and suppose that (A_i) holds for all i with $1 \leq i < m$.

Suppose first that $m + 1 \leq y$. Then (A_m) is the hypothesis of the lemma for this m, because at the indices m and $m + 1$ the prime kernel is the free kernel (Lemma 5.3).

Suppose now that $m \geq y$. Multiplying the recursion of Lemma 5.4 by x^N, we obtain

$$N\, \kappa^{(y)}_N\, x^N = \sum_{j=1}^{y-1} (y-j)\, a_j x^j\; \kappa^{(y)}_{N-j}\, x^{N-j} \qquad (N \geq y).$$

We compare the right-hand sides for $N = m + 1$ and for $N = m$. They have the same weights $(y - j)\, a_j x^j$, which are nonnegative. For $1 \leq j \leq y - 1$ we have $1 \leq m - j < m$, so (A_i) with $i = m - j$ gives $\kappa^{(y)}_{m+1-j}\, x^{m+1-j} \leq \kappa^{(y)}_{m-j}\, x^{m-j}$. Hence each term of the sum for $N = m + 1$ is at most the corresponding term of the sum for $N = m$, and (B_m) follows. Finally, $\kappa^{(y)}_{m+1}\, x^{m+1} \geq 0$ by Lemma 5.5, so that

$$m\, \kappa^{(y)}_{m+1}\, x^{m+1} \leq (m+1)\, \kappa^{(y)}_{m+1}\, x^{m+1} \leq m\, \kappa^{(y)}_m\, x^m,$$

and division by m gives (A_m). This completes the induction. □

The statements (A_m) of this induction are the monotonicity of the tilted prime kernel: under the hypothesis of Lemma 6.2, that is, for an integer $y \geq 2$ and a real number $x \geq 0$ with $\varepsilon_{m+1}\, x^{m+1} \leq \varepsilon_m\, x^m$ for $1 \leq m < y$, we have

$$\kappa^{(y)}_{m+1}\, x^{m+1} \leq \kappa^{(y)}_m\, x^m \qquad \text{for all integers } m \geq 1. \tag{6.3}$$

We now come to the tilted counts $\hat{z}^{(k)}_N = z^{(k)}_N\, x_k^N$ of threshold k. Besides the first inequality of (6.2), the proof of their monotonicity (Proposition 6.5 below) requires the inequality

$$\sigma_k \leq 1, \qquad \text{where} \qquad \sigma_k = x_k \sum_{j=1}^{k} c_j\, x_k^j = \sum_{n=2}^{k+1} a_n\, x_k^n, \tag{6.4}$$

between explicit numbers; the two sums agree because $c_j = a_{j+1}$. We prove this inequality for $k \geq 12$ by estimating $a_n\, x_k^n$ with Stirling's formula (Lemma 6.4).

This estimate does not reach the smallest columns, and for $2 \le k \le 11$ we evaluate σ_k, in the proof of the proposition. The estimate uses the following bound for the excess of x_k over e^{-1}.

Lemma 6.3. *For every integer $k \ge 1$,*

$$e\,x_k \le \exp\Big(\frac{3}{2k}\Big).$$

Proof. The elementary inequality $\log(1+t) \ge \frac{2t}{2+t}$ for $t \ge 0$, applied with $t = 1/k$, gives $\log\big(1+\frac{1}{k}\big) \ge \frac{2}{2k+1}$. Multiplying by $k-1$, we obtain

$$(k-1)\log\Big(1+\frac{1}{k}\Big) \ge \frac{2(k-1)}{2k+1} \ge 1 - \frac{3}{2k},$$

where the second inequality is checked directly. Hence $\big(1+\frac{1}{k}\big)^{k-1} \ge \exp\big(1-\frac{3}{2k}\big)$, and by the definition of x_k

$$x_k = \Big(1+\frac{1}{k}\Big)^{-(k-1)} \le \exp\Big(-1+\frac{3}{2k}\Big). \qquad \square$$

Lemma 6.4. *For every integer $k \ge 12$,*

$$x_k \sum_{j=1}^{k} c_j\, x_k^j \le 1.$$

Proof. Write $x = x_k$ and $\lambda = 3/(2k)$. Since $c_j = a_{j+1}$, the left-hand side is $\sum_{n=2}^{k+1} a_n x^n$.

Let $2 \le n \le k+1$. By the definition of a_n and by the upper bound in (2.7),

$$a_n\, x^n = \frac{(e\,x)^n}{n} \cdot e^{-n}\,\frac{n^n}{n!} \le \frac{(e\,x)^n}{\sqrt{2\pi}\,n^{3/2}}.$$

By Lemma 6.3 we have $(e\,x)^n \le e^{\lambda n}$, and $0 \le \lambda n \le \frac{3(k+1)}{2k} \le \frac{13}{8}$, because $n \le k+1$ and $k \ge 12$. Since $e^t \le 1 + t + t^2$ for $0 \le t \le 1.7$, as one easily checks, this gives with $t = \lambda n$

$$\sqrt{2\pi}\; a_n\, x^n \le \frac{1+\lambda n+\lambda^2 n^2}{n^{3/2}} = n^{-3/2} + \lambda\, n^{-1/2} + \lambda^2\, n^{1/2} \qquad (2 \le n \le k+1).$$

We sum this over $2 \le n \le k+1$ and compare each of the three sums with an integral. Since $t^{-3/2}$ is convex, $t^{-1/2}$ decreases and $t^{1/2}$ increases, the value of these functions at n is at most their mean over $[n-\frac{1}{2}, n+\frac{1}{2}]$, resp. over $[n-1, n]$, resp. over $[n, n+1]$. Therefore

$$\begin{aligned}
\sum_{n=2}^{k+1} n^{-3/2} &\le \int_{3/2}^{k+3/2} t^{-3/2}\,dt = \frac{2}{\sqrt{3/2}} - \frac{2}{\sqrt{k+3/2}},\\
\sum_{n=2}^{k+1} n^{-1/2} &\le \int_{1}^{k+1} t^{-1/2}\,dt = 2\sqrt{k+1} - 2,\\
\sum_{n=2}^{k+1} n^{1/2} &\le \int_{2}^{k+2} t^{1/2}\,dt \le \frac{2}{3}\,(k+2)^{3/2},
\end{aligned}$$

and with $\lambda = 3/(2k)$ we arrive at

$$\sqrt{2\pi}\sum_{n=2}^{k+1} a_n\, x^n \le \frac{2}{\sqrt{3/2}} - \frac{2}{\sqrt{k+3/2}} + \frac{3\,(\sqrt{k+1}-1)}{k} + \frac{3\,(k+2)^{3/2}}{2k^2}.$$

It remains to show that the right-hand side of this inequality is at most $\sqrt{2\pi}$. Put $r = \sqrt{k+2}$, so that $r \ge 3.74$, because $3.74^2 < 14 \le k+2$. Since $\sqrt{k+1} \le r$ and $\sqrt{k+3/2} \le r$, the right-hand side is at most

$$\frac{2}{\sqrt{3/2}} + \Big(-\frac{2}{r} + \frac{3(r-1)}{k} + \frac{3r^3}{2k^2}\Big).$$

We claim that the expression in the bracket is at most 0.87 for every $k \ge 12$. To prove this, we multiply the expression by $2rk^2 > 0$ and use the relation $r^2 = k+2$ to remove the powers of r higher than the first:

$$\begin{aligned} 2rk^2\Big(-\frac{2}{r} + \frac{3(r-1)}{k} + \frac{3r^3}{2k^2}\Big) &= -4k^2 + 6\,(r^2 - r)\,k + 3r^4 \\ &= -4k^2 + 6\,(k+2)\,k - 6rk + 3\,(k+2)^2 \\ &= 5k^2 + 24k + 12 - 6rk. \end{aligned}$$

The claim is therefore equivalent to the inequality $5k^2 + 24k + 12 - 6rk \le 1.74\, rk^2$, that is, to

$$5k^2 + 24k + 12 \le rk\,(1.74\,k + 6).$$

Since $r \ge 3.74$, the right-hand side of this inequality is at least $3.74\,k\,(1.74\,k+6) = 6.5076\,k^2 + 22.44\,k$. Subtracting the left-hand side from this lower bound leaves

$$1.5076\,k^2 - 1.56\,k - 12 = k\,(1.5076\,k - 1.56) - 12 \ge 12 \cdot 16 - 12 > 0,$$

where we have used that $k \ge 12$ and hence $1.5076\,k - 1.56 \ge 16$. This proves the claim.

Since $2/\sqrt{3/2} < 1.6331$ and $\sqrt{2\pi} > 2.5066$, we conclude that $\sqrt{2\pi}\sum_{n=2}^{k+1} a_n\, x^n \le 1.6331 + 0.87 = 2.5031 < \sqrt{2\pi}$, and the lemma follows. □

In terms of the counts, the main statement of the section says that

$$A_{N+1,k-1} \le (N+1)\left(1 + \frac{1}{k}\right)^{k-1} A_{N,k-1} \qquad (N \ge 0,\ k \ge 2),$$

and the factor $\left(1+\frac{1}{k}\right)^{k-1}$ cannot be replaced by a smaller one, since for $N = k-1$ both sides are equal to $(k+1)^{k-1}$.

Proposition 6.5. *Let $k \ge 2$ be an integer. Then, for every integer $N \ge 0$,*

$$z_{N+1}^{(k)}\, x_k^{N+1} \le z_N^{(k)}\, x_k^N.$$

Proof. We derive the proposition from a statement about an arbitrary solution of a tilted last-car recursion.

For the general statement, let $y \ge 1$ be an integer, let $x \ge 0$ be a real number, and let $(W_N)_{N\ge 0}$ be a solution of the weighted last-car recursion with weight $\omega(t) = x\min(t, y)$ in the sense of Definition 3.2, i.e. $W_0 = 1$ and

$$(N+1)\,W_{N+1} = x\sum_{i=0}^{N} \min(i+1, y)\, W_i\, W_{N-i} \qquad (N \ge 0).$$

Suppose that

(a) $W_{N+1} \le W_N$ for all $N < y$, and
(b) $x \sum_{j=1}^{y} W_j \le 1$.

We claim that $W_{N+1} \le W_N$ for all $N \ge 0$.

Since $W_0 = 1$ and $x \ge 0$, the recursion shows by induction on N that $W_N \ge 0$ for all N.

Put $w_N = W_N - W_{N+1}$. For $N \ge y$ we have the identity

$$(N+1)\, w_N = (1-xy)\, W_N + x \sum_{i=0}^{N-1} \min(i+1, y)\, W_i\, w_{N-1-i}.$$

To see this, write $(N+1)\, w_N = W_N + N\, W_N - (N+1)\, W_{N+1}$ and insert the recursion, with $N-1$ in place of N for the second term and as it stands for the third. In the latter the term with $i = N$ equals $xy\, W_N$, because $N+1 > y$ and $W_0 = 1$, and the terms with $i \le N-1$ combine with those of the former, since $W_{N-1-i} - W_{N-i} = w_{N-1-i}$.

The first term on the right of this identity may be negative. We show that it is outweighed by those terms of the sum that contain the decreases $w_0, \dots, w_{y-1}$ on the free range, and prove in this way that $w_N \ge 0$, by induction on N. For $N < y$ this is condition (a). Let $N \ge y$, and suppose that $w_j \ge 0$ for all $j < N$. Then $W_0 \ge W_1 \ge \cdots \ge W_N \ge 0$, and every term of the sum in the identity is nonnegative. Substituting $j = N-1-i$, we write this sum as $\sum_{j=0}^{N-1} \min(N-j, y)\, W_{N-1-j}\, w_j$, and we keep only the terms with $j \le y-1$; they are all present, because $y \le N$. For these terms

$$\min(N-j, y) \ge y - j \qquad \text{and} \qquad W_{N-1-j} \ge W_N,$$

the first because $N \ge y$. Hence

$$\sum_{i=0}^{N-1} \min(i+1, y)\, W_i\, w_{N-1-i} \ge W_N \sum_{j=0}^{y-1} (y-j)\, w_j.$$

Since $w_j = W_j - W_{j+1}$ and $W_0 = 1$, summation by parts gives

$$\sum_{j=0}^{y-1} (y-j)\, w_j = y\, W_0 - \sum_{j=1}^{y} W_j = y - \sum_{j=1}^{y} W_j.$$

Inserting the last two displays into the identity for $(N+1)\, w_N$, where $x \ge 0$, we see that the terms $xy\, W_N$ cancel:

$$\begin{aligned}(N+1)\, w_N &\ge (1-xy)\, W_N + x\, W_N \Big(y - \sum_{j=1}^{y} W_j \Big) \\ &= W_N \Big(1 - x \sum_{j=1}^{y} W_j \Big) \ge 0,\end{aligned}$$

by condition (b) and by $W_N \ge 0$. This completes the induction and proves the claim.

We now return to the tilted counts of threshold $k \ge 2$, and apply the claim with $y = k$ and $x = x_k$ to the sequence $W_N = z_N^{(k)}\, x_k^N$. By (3.5) this sequence satisfies the recursion of the general statement, and $W_0 = z_0^{(k)} = 1$. On the free range $N \le k$ we have $W_N = c_N\, x_k^N = \hat{c}_N$ by Lemma 3.4. Condition (a) therefore reads $\hat{c}_{N+1} \le \hat{c}_N$ for $N < k$, which is the first inequality of (6.2), and condition (b) is the inequality

$\sigma_k \le 1$ of (6.4). For $k \ge 12$ it is Lemma 6.4. For $2 \le k \le 11$ it is checked directly: the number $\sigma_k = \sum_{n=2}^{k+1} a_n\, x_k^n$, with $a_n = n^{n-1}/n!$ and $x_k = (k/(k+1))^{k-1}$, is a sum of at most eleven rational numbers, and evaluating the ten numbers $\sigma_2, \dots, \sigma_{11}$ exactly we find that they decrease from $\sigma_2 = \frac{8}{9}$ to $\sigma_{11} = 0.7791\ldots$, so that $\sigma_k < 1$ in each of the ten cases. Hence (a) and (b) hold for every $k \ge 2$, and the claim gives $z_{N+1}^{(k)}\, x_k^{N+1} \le z_N^{(k)}\, x_k^N$ for all $N \ge 0$. □

Finally, we carry the diagonal lower bound of Proposition 5.9 over to the critical tilt, where its constant becomes rational. Recall from Section 3 that $\beta_N^{(\ell)} = z_N^{(\ell+1)} - z_N^{(\ell)}$ for $\ell \ge 1$; the corollary is thus an inequality between the tilted counts of thresholds $k-1$, k and $k+1$.

Corollary 6.6. *Let $k \ge 2$ and $n \ge 1$ be integers. Then*

$$k\, \beta_{n-1}^{(k-1)}\, x_k^{n-1} \le (k+1)\, \beta_n^{(k)}\, x_k^n.$$

Proof. By Proposition 5.9 we have $\gamma_k\, \beta_{n-1}^{(k-1)} \le \beta_n^{(k)}$, where $\gamma_k = \left(1 + \frac{1}{k}\right)^{k-2}$, and the definition of x_k can be written as $(k+1)\, x_k\, \gamma_k = k$. Hence

$$k\, \beta_{n-1}^{(k-1)}\, x_k^{n-1} = (k+1)\, x_k^n\, \gamma_k\, \beta_{n-1}^{(k-1)} \le (k+1)\, x_k^n\, \beta_n^{(k)}.$$ □

7. Exact computations for the small rows and the small columns

In this section we establish the three facts in the proof of Theorem 1.1 that are obtained by exact computation. The first concerns all rows $n \le 16$ (Lemma 7.1); the second, Proposition 1.4 for the columns $9 \le k \le 264$ (Lemma 7.3); the third, Propositions 1.3 and 1.2 for the columns $2 \le k \le 81$ (Lemma 7.5). These are the rows and the columns that our estimates do not reach.

The computations are carried out by computer and involve integers only, so that all of the arithmetic is exact. No parking function is enumerated: the numbers computed are cumulative counts $A_{n,\ell}$, or the differences $(n+1)^{n-1} - A_{n,\ell}$, and they are obtained from a recursion in n, either the last-car recursion (Proposition 3.1) or the single-chain formula (Proposition 4.1). Each of the three facts is then a finite number of comparisons between integers that are formed from these numbers. The second and the third computation depend on one number for each column, $\zeta(k)$ and $n_0(k)$, which locates the rows that are examined. We list these numbers in full, in Definitions 7.2 and 7.4, and the proof of each lemma gives the recursion, the rows up to which it is run and the comparisons that are made, so that each computation can be repeated from what is written here.

7.1. **Rows up to 16.** The inequality $d_{n,k} \ge 0$ for $2k \le n$ is proved for the rows $n \ge 17$ only (Proposition 1.2), and in smaller rows it can fail. We therefore compute all rows $n \le 16$. Each of them turns out to be unimodal, and the following lemma states this in the form needed for Proposition 20.2.

Lemma 7.1. *Let n, i and j be integers with $0 \le n \le 16$ and $1 \le i < j \le n-1$. Then row n does not have both a strict descent at i and a strict ascent at j.*

Proof. For $0 \le \ell \le 15$ and $0 \le m \le 16$ we compute $A_{m,\ell}$, starting from $A_{0,\ell} = 1$, by the last-car recursion of Proposition 3.1,

$$A_{m+1,\ell} = \sum_{j=0}^{m} \min(j+1, \ell+1) \binom{m}{j} A_{j,\ell}\, A_{m-j,\ell}.$$

The entries of row n are then $b_{n,0} = A_{n,0}$ and $b_{n,\ell} = A_{n,\ell} - A_{n,\ell-1}$ for $1 \le \ell \le n-1$. We find that the rows with $1 \le n \le 11$ are

n	$b_{n,0},\ b_{n,1},\ \ldots,\ b_{n,n-1}$
1	1
2	2, 1
3	6, 7, 3
4	24, 51, 34, 16
5	120, 421, 377, 253, 125
6	720, 3963, 4594, 3688, 2546, 1296
7	5040, 42253, 62145, 57398, 46142, 32359, 16807
8	40320, 505515, 929856, 979430, 865970, 702292, 497442, 262144
9	362880, 6724381, 15298809, 18289811, 17520519, 15455851, 12587507, 8977273, 4782969
10	3628800, 98618763, 275075236, 371706836, 385131546, 359973796, 318092120, 259627152, 186093442, 100000000
11	39916800, 1582715773, 5373200117, 8177419578, 9160454598, 9008899440, 8390168112, 7411908040, 6056786384, 4357947691, 2357947691

and that the rows with $12 \le n \le 16$ are

n	$b_{n,0},\ b_{n,1},\ \ldots,\ b_{n,n-1}$
12	479001600, 27612565995, 113407725802, 193804329368, 234483247248, 242766518158, 234882851708, 218509882050, 192828376414, 157676318044, 113792213426, 61917364224
13	6227020800, 520631327581, 2573638729605, 4926465506021, 6428493655749, 7014734252870, 7025563491106, 6770961415170, 6285825855182, 5539649733253, 4531383858509, 3278177135413, 1792160394037
14	87178291200, 10554164679243, 62521925061642, 133777256144940, 188000138187750, 216295074328842, 224520420787986, 222211003267142, 213687649761928, 197900551632256, 174165050709924, 142490660510968, 103290082620258, 56693912375296
15	1307674368000, 228975516609853, 1619488339989969, 3866365396216526, 5844430081201758, 7086949855748155, 7637297774350499, 7747121093904120, 7638524284820008, 7325291684300523, 6767645976331523, 5947866448755034, 4866510228104930, 3533624614867663, 1946195068359375
16	20922789888000, 5294731892093355, 44569555672124464, 118532872437716726, 192529294913883426, 245884738565040108, 275378436591129714, 286698620154512400, 288176016598878112, 283408731647719814, 271000724330122940, 249788273565590526, 219249309938712778, 179389782285766308, 130443446088709186, 72057594037927936

In each of these sixteen rows the entries do not decrease up to the largest entry of the row and do not increase after it. Hence a strict descent at an index i stands to the right of the largest entry and is not followed by a strict ascent at an index $j > i$. This is the assertion of the lemma. □

7.2. **Two signs for each column from 9 to 264.** For $9 \le k \le 264$, Proposition 1.4 is derived in Section 19 from Proposition 1.3 and from two signs: column k is positive in some row of its chain range, and column $k+1$ is negative one row earlier. The two rows are $k+\zeta(k)+1$ and $k+\zeta(k)$, with $\zeta(k)$ defined as follows.

Definition 7.2. For an integer k with $9 \le k \le 264$, the number $\zeta(k)$ is given by the following table: $\zeta(k)$ is the value ζ in whose range of columns k lies.

ζ	k	ζ	k	ζ	k
7	9	17	67–74	27	163–174
8	10–14	18	75–83	28	175–185
9	15–20	19	84–91	29	186–197
10	21–25	20	92–101	30	198–210
11	26–31	21	102–110	31	211–222
12	32–37	22	111–120	32	223–235
13	38–44	23	121–130	33	236–248
14	45–51	24	131–141	34	249–261
15	52–58	25	142–151	35	262–264
16	59–66	26	152–162		

Lemma 7.3. *Let k be an integer with $9 \le k \le 264$, and let $\zeta = \zeta(k)$. Then*

(1) $\zeta + 2 \le k$;
(2) $b_{k+\zeta+1,\,k-1} < b_{k+\zeta+1,\,k}$;
(3) $b_{k+\zeta,\,k+1} < b_{k+\zeta,\,k}$.

Proof. Part (1) is read from the table of Definition 7.2, in each line of which $\zeta+2$ is at most the first column of the range.

Write $p_n = (n+1)^{n-1}$ for $n \ge 0$, so that $p_0 = 1$ and $p_n = n!\,c_n$, and for a threshold $y \ge 1$ and $n \ge 0$ let

$$U_n^{(y)} = p_n - A_{n,y-1}.$$

Recall that $A_{n,\ell}$ is nondecreasing in ℓ and equals p_n for $n \le \ell + 1$. Hence $U_n^{(y)}$ is a nonnegative integer, the number of parking functions of length n in which some displacement is at least y, and $U_n^{(y)} = 0$ for $n \le y$. We claim that, for $y \ge 1$ and $n \le 2y+1$,

$$U_n^{(y)} = \sum_{t=y+1}^{n} \binom{n}{t} p_{n-t}\left((t-y)\,p_{t-1} + y\,U_{t-1}^{(y)}\right). \tag{7.1}$$

Indeed, by Corollary 3.3 the sequence $z^{(y)}$ is a solution of the weighted last-car recursion with weight $\omega(t) = \min(t, y)$, so that Proposition 4.1 applies to $W = z^{(y)}$ with y in place of k. Since $\omega(t) = y$ for $t \ge y+1$, it gives, for $n \le 2y+1$,

$$c_n - z_n^{(y)} = \sum_{t=y+1}^{n} c_{n-t}\Big(c_{t-1} - \frac{y}{t}\,z_{t-1}^{(y)}\Big).$$

Multiplying this identity by $n!$ and using $c_m = p_m/m!$, $z_m^{(y)} = A_{m,y-1}/m!$ and $n!/((n-t)!\,(t-1)!) = t\binom{n}{t}$, we obtain the claim, because $t\,p_{t-1} - y\,A_{t-1,y-1} = (t-y)\,p_{t-1} + y\,U_{t-1}^{(y)}$.

For $n \ge y+1$ the right-hand side of the recursion (7.1) contains, besides binomial coefficients and the p_t, only the $U_t^{(y)}$ with $y \le t \le n-1$, and $U_y^{(y)} = 0$. So the

recursion gives $U_{y+1}^{(y)}, U_{y+2}^{(y)}, \dots$ one after another, up to row $2y+1$, and no row of the free range has to be computed.

The steps are expressed by these numbers: since $A_{n,\ell} = p_n - U_n^{(\ell+1)}$, the identity (3.1) gives

$$d_{n,j} = 2\,U_n^{(j)} - U_n^{(j+1)} - U_n^{(j-1)} \qquad (j \ge 2,\ n \ge 0).$$

With $m = k + \zeta$ we have in particular

$$\begin{aligned} d_{k+\zeta+1,\,k} &= 2\,U_{m+1}^{(k)} - U_{m+1}^{(k+1)} - U_{m+1}^{(k-1)}, \\ d_{k+\zeta,\,k+1} &= 2\,U_m^{(k+1)} - U_m^{(k+2)} - U_m^{(k)}. \end{aligned}$$

Rows m and $m+1$ lie in the range $n \le 2y+1$ of the recursion (7.1) for each of the four thresholds y that occur: the strongest of the conditions is $m+1 \le 2(k-1)+1$, for threshold $k-1$, and this is part (1). For each of the 256 columns we compute the six numbers on the right-hand sides by this recursion. The largest row that occurs is row 300, for $k = 264$, and the integers that occur have up to about 740 decimal digits. For $k = 9$, for instance, where $\zeta = 7$ and the two rows are 17 and 16, we find that

$$\begin{aligned} d_{17,9} &= 18882703175184212 > 0, \\ d_{16,10} &= -12408007317596874 < 0. \end{aligned}$$

The second of these numbers agrees with the proof of Lemma 7.1: by definition $d_{16,10} = b_{16,10} - b_{16,9}$, and the two entries of row 16 displayed there, which are computed by the last-car recursion, have this difference. In the same way we find in each of the 256 columns that $d_{k+\zeta+1,k} > 0$ and that $d_{k+\zeta,k+1} < 0$. These are parts (2) and (3). □

By part (1) and since $\zeta(k) \ge 7$, the rows $k + \zeta(k) + 1$ and $k + \zeta(k)$ lie in the chain range of column k, resp. of column $k+1$.

7.3. **Columns 2 to 81.** For $2 \le k \le 81$ two statements are supplied by computation: Proposition 1.3, which concerns the finitely many rows $k+1, \dots, 2k-1$ of the chain range, and the inequality $d_{n,k} \ge 0$ of Proposition 1.2, which concerns all rows $n \ge \max(2k, 17)$. For the latter, column k is computed up to a row $n_0(k)$, and at this row we verify a condition from which the propagation theorem (Proposition 15.5) derives the inequality in all later rows. These rows are the following.

Definition 7.4. For an integer k with $2 \le k \le 81$, the number $n_0(k)$ is given by the following table.

k	2	3	4	5	6	7	8	9	10	11
$n_0(k)$	7	10	13	15	18	20	23	25	28	30
k	12	13	14	15	16	17	18	19	20	21
$n_0(k)$	32	35	37	39	41	44	46	48	50	52
k	22	23	24	25	26	27	28	29	30	31
$n_0(k)$	55	57	59	61	63	66	68	70	72	74
k	32	33	34	35	36	37	38	39	40	41
$n_0(k)$	76	78	80	82	85	87	89	91	93	95
k	42	43	44	45	46	47	48	49	50	51
$n_0(k)$	97	99	101	103	105	107	109	111	114	116
k	52	53	54	55	56	57	58	59	60	61
$n_0(k)$	118	120	122	124	126	128	130	132	134	136
k	62	63	64	65	66	67	68	69	70	71
$n_0(k)$	138	140	142	144	146	148	150	152	154	156
k	72	73	74	75	76	77	78	79	80	81
$n_0(k)$	158	160	162	164	166	168	170	171	173	175

Recall that $x_k = (k/(k+1))^{k-1}$ is the critical tilt of column k (Section 6) and that the sequence $(\xi_t^{(k)})_{t\ge 0}$ of column k is defined in Section 5.

Lemma 7.5. *Let k be an integer with $2 \le k \le 81$, and let $n_0 = n_0(k)$. Then*

(1) $2k \le n_0$;
(2) *for every integer n with $k+1 \le n \le 2k-2$: if $b_{n,k-1} < b_{n,k}$, then $b_{n+1,k-1} < b_{n+1,k}$;*
(3) *for every integer n with $2k \le n$, $17 \le n$ and $n < n_0$: $b_{n,k-1} \le b_{n,k}$;*
(4) *for every integer s with $k \le s \le n_0$: $\sum_{t=s}^{n_0} \xi_t^{(k)}\, x_k^t \ge 0$.*

Proof. Part (1) is read from the table of Definition 7.4, in which $n_0(k) \ge 2k+3$ for every k. We write $\xi_t = \xi_t^{(k)}$.

For $\ell = k$, $k-1$ and $k-2$ we compute $A_{n,\ell}$ for $0 \le n \le n_0 - 1$, starting from $A_{0,\ell} = 1$, by the last-car recursion of Proposition 3.1, as in the proof of Lemma 7.1. By (3.1) these numbers give the steps

$$d_{n,k} = A_{n,k} - 2A_{n,k-1} + A_{n,k-2} \qquad (0 \le n \le n_0 - 1).$$

They also give the numbers $\Xi_t = t!\,\xi_t$ for $1 \le t \le n_0$: by the definition of ξ_t and by $z_{t-1}^{(y)} = A_{t-1,y-1}/(t-1)!$, these are the integers

$$\Xi_t = \min(t, k+1)\, A_{t-1,k} - 2\min(t,k)\, A_{t-1,k-1} + \min(t, k-1)\, A_{t-1,k-2}.$$

Parts (2) and (3) are statements about the signs of the steps in rows $k+1, \dots, 2k-1$, resp. in rows $\max(2k, 17), \dots, n_0 - 1$. No row beyond $n_0 - 1$ occurs in them, because $2k - 1 \le n_0 - 1$ by part (1).

For part (4) we clear the denominators. For $k \le s \le n_0$ let

$$\mathsf{S}_s = \sum_{t=s}^{n_0} \Xi_t\, \frac{n_0!}{t!}\, (k+1)^{(k-1)(n_0-t)}\, k^{(k-1)(t-s)}.$$

These numbers are integers, and they are computed one after another for $s = n_0, n_0-1, \dots, k$, from $\mathsf{S}_{n_0+1} = 0$ and

$$\mathsf{S}_s = \Xi_s \, \frac{n_0!}{s!} \, (k+1)^{(k-1)(n_0-s)} + k^{k-1} \, \mathsf{S}_{s+1}.$$

Since $\Xi_t \, n_0!/t! = n_0! \, \xi_t$ and $x_k = k^{k-1}/(k+1)^{k-1}$, the definition of S_s reads

$$\mathsf{S}_s = \frac{n_0! \, (k+1)^{(k-1)n_0}}{k^{(k-1)s}} \sum_{t=s}^{n_0} \xi_t \, x_k^t.$$

The factor in front of the sum is positive, so part (4) says that $\mathsf{S}_s \geq 0$ for every s with $k \leq s \leq n_0$.

We carry out this computation for each of the eighty columns. The largest row for which the numbers $A_{n,\ell}$ are computed is row 174, for $k = 81$, and in that column the integers S_s have up to about 14700 decimal digits. For $k = 2$, for instance, where $n_0 = 7$ and no row occurs in parts (2) and (3), the values of Ξ_t for $1 \leq t \leq 7$ and of S_t for $2 \leq t \leq 7$ are

t	1	2	3	4	5	6	7
Ξ_t	0	-1	-1	2	51	710	9819
S_t		3912	308136	188088	88374	34548	9819

and the six numbers S_t are positive. In each of the eighty columns we find the following. For every n with $k+1 \leq n \leq 2k-2$ we have $d_{n,k} \leq 0$ or $d_{n+1,k} > 0$, which is part (2). For every n with $\max(2k, 17) \leq n \leq n_0 - 1$ we have $d_{n,k} \geq 0$, which is part (3). And $\mathsf{S}_s \geq 0$ for every s with $k \leq s \leq n_0$, which is part (4). □

8. The crossing criterion

For the columns $k \geq 265$, every positive row from which the later arguments start is obtained from one statement, the crossing criterion (Proposition 8.2). For a column $k \geq 2$ and an integer R with $1 \leq R \leq k-1$ it names two inequalities between finite sums that depend on k and R only, and it asserts that, when both hold, column k is positive in some row j with $k < j \leq k+R$. The sums are replaced by bounds in closed form in Section 9, where the criterion is applied with R of order $k^{2/3}$. The sums are formed at the diagonal tilt of the column, which we introduce first.

8.1. **The diagonal tilt.** Let $k \geq 2$ be an integer. The *diagonal tilt* of column k is the positive number

$$\vartheta_k = \frac{1}{\gamma_k} = \Big(\frac{k}{k+1}\Big)^{k-2},$$

where $\gamma_k = (1+1/k)^{k-2}$ is the factor in the diagonal lower bound, Proposition 5.9; recall from Section 5 that γ_k is nondecreasing in k. When a column k is fixed, a tilde denotes the tilted form of a sequence at this tilt:

$$\tilde{\varepsilon}_m = \varepsilon_m \vartheta_k^m, \qquad \tilde{\beta}_j^{(\ell)} = \beta_j^{(\ell)} \vartheta_k^j, \qquad \tilde{\delta}_n = \delta_n^{(k)} \vartheta_k^n.$$

Since $\vartheta_k > 0$, each $\tilde{\delta}_n$ has the sign of $d_{n,k}$. The diagonal tilt is not the critical tilt x_k of Section 6; the two are related by $x_k = \frac{k}{k+1} \, \vartheta_k$.

The tilt is chosen so that $\vartheta_k \gamma_k = 1$, which removes the factor γ_k from the diagonal lower bound. Indeed, this bound says that $\gamma_k \, \beta_n^{(k-1)} \leq \beta_{n+1}^{(k)}$ for every $n \geq 0$, and

$\vartheta_k^n = \vartheta_k^{n+1}\gamma_k$. Hence, whenever column k is nonpositive in row $n+1$, i.e. whenever $\beta_{n+1}^{(k)} \le \beta_{n+1}^{(k-1)}$, we have

$$\tilde\beta_n^{(k-1)} = \vartheta_k^{n+1}\gamma_k\,\beta_n^{(k-1)} \le \vartheta_k^{n+1}\beta_{n+1}^{(k)} \le \vartheta_k^{n+1}\beta_{n+1}^{(k-1)} = \tilde\beta_{n+1}^{(k-1)}. \tag{8.1}$$

Thus the tilted counts of level $k-1$ do not decrease as long as column k has not turned positive. Since γ_i is nondecreasing in i, we also have

$$\vartheta_k\gamma_i \ge \vartheta_k\gamma_k = 1 \qquad \text{for } i \ge k.$$

The following lemma locates the tilt relative to e^{-1}. For a column k we put

$$\bar a = \frac{5}{2k}, \qquad \underline{a} = \frac{5}{2k} - \frac{4}{3k^2};$$

the proof of the lemma shows that these two numbers enclose $1 + \log\vartheta_k$.

Lemma 8.1. *Let $k \ge 2$ be an integer. Then the diagonal tilt of column k satisfies*

$$e^{-1} \le \vartheta_k \le e^{-1}e^{\bar a}.$$

Proof. We show that

$$\underline{a} \le 1 + \log\vartheta_k \le \bar a,$$

which gives the lemma because $\underline{a} = (15k-8)/(6k^2) \ge 0$. By Taylor's formula with remainder,

$$u - \frac{u^2}{2} \le \log(1+u) \le u - \frac{u^2}{2} + \frac{u^3}{3}$$

for every real number $u \ge 0$. We apply this with $u = 1/k$ to

$$1 + \log\vartheta_k = 1 - (k-2)\,\log\Bigl(1 + \frac{1}{k}\Bigr).$$

Expanding the products, we find that

$$\frac{5}{2k} - \frac{4}{3k^2} + \frac{2}{3k^3} \le 1 + \log\vartheta_k \le \frac{5}{2k} - \frac{1}{k^2},$$

where the left-hand side is at least $\underline{a}$ and the right-hand side is at most $\bar a$. □

8.2. **Statement and proof of the criterion.** We fix a column $k \ge 2$ and introduce the notation in which the criterion is stated. For an integer $M \ge 0$ we set

$$\chi_M = 1 - \vartheta_k - \sum_{m=1}^{M} \tilde\varepsilon_m.$$

Since $\varepsilon_m \ge 0$ and $\varepsilon_1 = 1$, we have $\tilde\varepsilon_m \ge 0$ and $\tilde\varepsilon_1 = \vartheta_k$; hence

$$\chi_0 = 1 - \vartheta_k, \qquad \chi_1 = 1 - 2\vartheta_k, \qquad \chi_0 \ge \chi_1 \ge \chi_2 \ge \cdots.$$

For an integer $R \ge 0$ let

$$S_R = \sum_{i=0}^{R-1} \frac{(i\,\vartheta_k)^i}{i!},$$

in which the term with $i = 0$ is 1. Since $m\,\varepsilon_m = (m-1)^{m-1}/(m-1)!$ for $m \ge 1$, we have

$$\sum_{m=1}^{R} m\,\tilde\varepsilon_m = \vartheta_k S_R,$$

so that $\vartheta_k S_R$ is the first moment of the tilted kernel up to index R.

The remaining quantities are built from the numbers

$$\pi_j = \frac{\vartheta_k^j \, a_j}{j} \qquad (j \geq 1),$$

which are positive. The ratio formula (2.3) and the definition of γ_i give

$$a_{i+1} = \Big(1 + \frac{1}{i}\Big)^{i-1} a_i = \gamma_i \, \frac{i+1}{i} \, a_i \qquad (i \geq 2).$$

Hence, since $\vartheta_k \gamma_j \geq \vartheta_k \gamma_k = 1$ for $j \geq k$,

$$\pi_{j+1} = \vartheta_k \gamma_j \, \pi_j \quad (j \geq 2), \qquad \pi_k = \pi_{k+1} \leq \pi_{k+2} \leq \pi_{k+3} \leq \cdots .$$

Thus the numbers π_n do not decrease from $n = k$ on. For an integer i let

$$\Pi_i = \sum_{n=k}^{i} \pi_n$$

be the partial sums of the numbers π_n from index k on, so that $\Pi_i = 0$ for $i < k$. If these numbers were constant, the quotient Π_{j+1}/π_{j+1} would be the number $j-k+2$ of terms of Π_{j+1}. For $j \geq k$ we denote by e_j the amount by which the quotient falls short of this number,

$$\mathsf{e}_j = (j-k+2) - \frac{\Pi_{j+1}}{\pi_{j+1}} = \sum_{n=k}^{j+1} \Big(1 - \frac{\pi_n}{\pi_{j+1}}\Big).$$

Since $\pi_n \leq \pi_{j+1}$ for $k \leq n \leq j+1$, every term of the last sum is nonnegative, so that $\mathsf{e}_j \geq 0$, and $\mathsf{e}_k = 0$ because $\pi_k = \pi_{k+1}$. We put

$$\mathsf{E} = \mathsf{E}(k,R) = \sum_{j=k}^{k+R-2} \frac{\mathsf{e}_j}{j+1},$$

which is nonnegative and vanishes for $R \leq 1$.

Finally, for $i \geq 2$ we set

$$\phi_i = 1 - \frac{i}{(i+1)\,\vartheta_k \gamma_i} = 1 - \frac{\gamma_k \, c_{i-1}}{c_i};$$

the two expressions agree because $c_i/c_{i-1} = a_{i+1}/a_i = \gamma_i \, (i+1)/i$ and $\gamma_k = 1/\vartheta_k$. For $i \geq k$ we have $0 < \phi_i < 1$, since $\vartheta_k \gamma_i \geq 1$. We also put

$$\Sigma_\phi = \Sigma_\phi(k,R) = \sum_{i=k+1}^{k+R-1} (i-k+1)\,\phi_i,$$

which is likewise nonnegative and vanishes for $R \leq 1$.

The proof of the criterion argues by contradiction. If column k is nonpositive in rows $k+1, \ldots, k+R$, the tilted counts $\tilde{\beta}_j^{(k-1)}$ of level $k-1$ are largest in row $k+R$, and the level equation at the diagonal tilt bounds this largest count by multiples of one and the same term of the equation, from above when it is read in row $k+R$ and from below when it is summed over rows $k, \ldots, k+R$; the lower bound contains a correction, which the proof controls with the column equation. When the two bounds are combined, the sizes of this term and of the largest count, which are unknown, cancel, and an inequality remains that contradicts the second of the two inequalities that the criterion assumes. In this notation the criterion reads as follows.

Proposition 8.2. *Let $k \geq 2$ and R be integers with $1 \leq R \leq k-1$, and put $N = k+R$. Suppose that the quantities χ_R, S_R, e_{N-1}, $\mathsf{E} = \mathsf{E}(k,R)$ and $\Sigma_\phi = \Sigma_\phi(k,R)$ of column k satisfy*

$$\chi_R \geq 0$$

and

$$\Big(\chi_R + \frac{\vartheta_k\,(R+1)}{N}\Big)\big(\mathsf{e}_{N-1} + \Sigma_\phi\big) < \vartheta_k\,\Big(\frac{R(R+1)}{N} - 1 - S_R - \mathsf{E}\Big). \tag{8.2}$$

Then there is an integer j with $k < j \leq N$ and $d_{j,k} > 0$.

Proof. We write $\delta_j = \delta_j^{(k)}$ and $\lambda = \chi_R + \vartheta_k(R+1)/N$, the first factor on the left of (8.2). We assume that the conclusion fails, i.e.

$$\delta_j \leq 0 \qquad \text{for all } k < j \leq N,$$

and we derive

$$\vartheta_k\,\Big(\frac{R(R+1)}{N} - 1 - S_R - \mathsf{E}\Big) \leq \lambda\,\big(\mathsf{e}_{N-1} + \Sigma_\phi\big), \tag{8.3}$$

which contradicts the second hypothesis.

We first record two consequences of $\chi_R \geq 0$. Since $R \geq 1$ and the numbers χ_M do not increase with M, we have

$$\chi_M \geq \chi_R \geq 0 \quad (0 \leq M \leq R) \qquad \text{and} \qquad \vartheta_k \leq \tfrac{1}{2},$$

the latter because $1 - 2\vartheta_k = \chi_1 \geq \chi_R$. In particular $\lambda > 0$.

Next we normalise by the number of all parking functions of length n, which, we recall, is $c_n\,n!$. For $n \geq 0$ we put

$$F_n = \frac{z_n^{(k)}}{c_n}, \qquad F_n^- = \frac{z_n^{(k-1)}}{c_n}, \qquad P_n = \frac{\beta_n^{(k-1)}}{c_n}.$$

These are the proportions, among all parking functions of length n, of those whose maximum displacement is at most $k-1$, at most $k-2$, and exactly $k-1$. Thus $P_n \geq 0$ and

$$F_n = F_n^- + P_n.$$

By Lemma 3.4, $z_n^{(y)} = c_n$ for $n \leq y$, and therefore

$$F_n = 1 \quad (n \leq k), \qquad F_n^- = 1 \ \text{ and } \ P_n = 0 \quad (n \leq k-1).$$

For $i \geq 1$ let $J_i = F_i - F_{i-1}^-$; this is the increase of the cumulative proportion when the length and the bound on the displacements are both raised by one. We have $J_i \geq 0$ by Proposition 3.5, applied with threshold $k-1$, and $J_k = 0$ because $F_k = F_{k-1}^- = 1$. Since $F_n = F_n^- + P_n$, it follows that

$$F_n - F_{n+1} = P_n - J_{n+1} \qquad (n \geq 0).$$

We multiply the level equation of Lemma 4.3 for level $k-1$ by ϑ_k^n. Since $a_n = c_{n-1}$, its first term becomes $\vartheta_k^n\, z_{n-1}^{(k)}/n = \pi_n F_{n-1}$, and the equation reads

$$\tilde\beta_n^{(k-1)} = \pi_n\,F_{n-1} + \frac{(k-1)\,\vartheta_k}{n}\,\tilde\beta_{n-1}^{(k-1)} + \sum_{m=1}^{n-k} \tilde\varepsilon_m\,\tilde\beta_{n-m}^{(k-1)} \tag{8.4}$$

for $k \le n \le 2k-1$, where $\tilde\beta_n^{(k-1)} = 0$ for $n \le k-1$ because $P_n = 0$ for these n. In the same way the column equation of Proposition 4.5, whose source term is $\frac{2}{n}\,\beta_{n-1}^{(k-1)}$, becomes

$$\tilde\delta_n = 2\,\pi_n\,P_{n-1} + \frac{(k+1)\,\vartheta_k}{n}\,\tilde\delta_{n-1} + \sum_{m=1}^{n-k} \tilde\varepsilon_m\,\tilde\delta_{n-m} \tag{8.5}$$

for $k+1 \le n \le 2k-1$, and the initial value $\delta_k = -a_k/k$ of Lemma 4.4 becomes $\tilde\delta_k = -\pi_k$. Since $R \le k-1$, we have $N \le 2k-1$, so that the tilted level equation holds in rows $k,\dots,N$ and the tilted column equation in rows $k+1,\dots,N$.

Our assumption implies that the tilted counts of level $k-1$ are largest in row N. Indeed, for $k \le n < N$, column k is nonpositive in row $n+1$, and the chain of inequalities (8.1) gives $\tilde\beta_n^{(k-1)} \le \tilde\beta_{n+1}^{(k-1)}$. Since the tilted counts are nonnegative and vanish for $j \le k-1$, it follows that

$$0 \le \tilde\beta_j^{(k-1)} \le \tilde\beta_N^{(k-1)} \qquad \text{for every } j \le N.$$

An upper bound for $\tilde\beta_N^{(k-1)}$ now follows from the equation of row N. In the tilted level equation (8.4) with $n = N$ the coefficients are nonnegative, and, as we have just seen, every tilted count on the right is at most $\tilde\beta_N^{(k-1)}$. Hence

$$\tilde\beta_N^{(k-1)} \le \pi_N\,F_{N-1} + \Big(\frac{(k-1)\,\vartheta_k}{N} + \sum_{m=1}^{R} \tilde\varepsilon_m\Big)\,\tilde\beta_N^{(k-1)}.$$

By the definition of χ_R and since $N-(k-1) = R+1$, the bracket equals $1-\lambda$. Therefore

$$\lambda\,\tilde\beta_N^{(k-1)} \le \pi_N\,F_{N-1}. \tag{8.6}$$

The lower bound for $\tilde\beta_N^{(k-1)}$ comes from an identity that involves all rows $k,\dots,N$. We claim that, without any assumption on signs,

$$\begin{aligned} F_{N-1}\,\Pi_N = \tilde\beta_N^{(k-1)} &+ \Big(1 - \vartheta_k - \frac{(k-1)\,\vartheta_k}{N}\Big)\,\tilde\beta_{N-1}^{(k-1)} \\ &+ \sum_{j=k}^{N-2} \Big(\chi_{N-j} + \frac{\vartheta_k\,\mathsf{e}_j}{j+1}\Big)\,\tilde\beta_j^{(k-1)} + \sum_{i=k}^{N-1} \Pi_i\,J_i. \end{aligned} \tag{8.7}$$

To see this, we add the tilted level equations (8.4) for $n = k,\dots,N$ and collect the coefficient of each $\tilde\beta_j^{(k-1)}$ on the right. Since $\tilde\beta_{k-1}^{(k-1)} = 0$, this gives

$$\sum_{n=k}^{N} \pi_n\,F_{n-1} = \tilde\beta_N^{(k-1)} + \sum_{j=k}^{N-1} \Big(1 - \frac{(k-1)\,\vartheta_k}{j+1} - \sum_{m=1}^{N-j} \tilde\varepsilon_m\Big)\,\tilde\beta_j^{(k-1)}.$$

We sum the left-hand side by parts: since $F_{i-1} - F_i = P_{i-1} - J_i$,

$$\sum_{n=k}^{N} \pi_n\,F_{n-1} = F_{N-1}\,\Pi_N + \sum_{i=k}^{N-1} \Pi_i\,(P_{i-1} - J_i).$$

The terms with P are multiples of the tilted counts: $P_{k-1} = 0$, and for $j \ge k$ we have $\pi_{j+1}P_j = \vartheta_k\,\tilde\beta_j^{(k-1)}/(j+1)$, again by $a_{j+1} = c_j$, and $\Pi_{j+1} = \pi_{j+1}\,(j-k+2-\mathsf{e}_j)$

by the definition of e_j. So

$$\sum_{i=k}^{N-1} \Pi_i \, P_{i-1} = \sum_{j=k}^{N-2} \frac{\vartheta_k \, (j-k+2-\mathsf{e}_j)}{j+1} \, \tilde{\beta}_j^{(k-1)}.$$

Equating the two expressions for $\sum_n \pi_n F_{n-1}$, we find that the coefficient of $\tilde{\beta}_j^{(k-1)}$ is $\chi_{N-j} + \vartheta_k \, \mathsf{e}_j/(j+1)$ for $k \le j \le N-2$, because $(k-1)+(j-k+2) = j+1$, and $1 - \vartheta_k - (k-1)\vartheta_k/N$ for $j = N-1$, because $\tilde{\varepsilon}_1 = \vartheta_k$. This is the claimed identity.

Besides multiples of the tilted counts, the right-hand side of this identity contains the numbers J_i, and we bound them next: for every $i \ge k$,

$$J_i \le \phi_i \, (1 - F_{i-1}^-).$$

For the proof we compare the parking functions that the two cumulative counts leave out. Since $\beta_n^{(\ell)} = z_n^{(\ell+1)} - z_n^{(\ell)}$ for $\ell \ge 1$, and since $z_n^{(n)} = c_n$ for $n \ge 1$ by Lemma 3.4, we have the telescoping sums

$$c_{i-1} - z_{i-1}^{(k-1)} = \sum_{s=0}^{i-k-1} \beta_{i-1}^{(k-1+s)}, \qquad c_i - z_i^{(k)} = \sum_{s=0}^{i-k-1} \beta_i^{(k+s)}.$$

The diagonal lower bound, Proposition 5.9, in column $k+s$ gives $\gamma_{k+s} \, \beta_{i-1}^{(k-1+s)} \le \beta_i^{(k+s)}$. Here $\gamma_k \le \gamma_{k+s}$, and the normalised counts are nonnegative. Summing over s, we obtain

$$\gamma_k \left(c_{i-1} - z_{i-1}^{(k-1)} \right) \le c_i - z_i^{(k)}.$$

Dividing by c_i, we obtain $(1-\phi_i)\,(1 - F_{i-1}^-) \le 1 - F_i$, because $\gamma_k \, c_{i-1}/c_i = 1 - \phi_i$, and hence $J_i = (1 - F_{i-1}^-) - (1 - F_i) \le \phi_i \, (1 - F_{i-1}^-)$.

The identity (8.7) also contains the proportion F_{N-1}, and the bound for J_i contains the proportions F_{i-1}^-. We bound them by means of the column equation, through the inequality

$$\sum_{j=k}^{N-1} P_j \le \frac{1}{2},$$

which says that the proportions of parking functions with maximum displacement exactly $k-1$, summed over rows $k, \dots, N-1$, are at most $\frac{1}{2}$ as long as column k has not turned positive. To prove it, we add the tilted column equations (8.5) for $n = k+1, \dots, N$ and collect the terms as before:

$$2 \sum_{n=k+1}^{N} \pi_n \, P_{n-1} = \tilde{\delta}_N - \tilde{\delta}_k + \sum_{i=k}^{N-1} \Big(1 - \frac{(k+1)\,\vartheta_k}{i+1} - \sum_{m=1}^{N-i} \tilde{\varepsilon}_m \Big) \, \tilde{\delta}_i.$$

For $k \le i \le N-1$ the coefficient in the last sum is at least $1 - \vartheta_k - \sum_{m=1}^{R} \tilde{\varepsilon}_m = \chi_R \ge 0$, because $(k+1)/(i+1) \le 1$ and $N - i \le R$. The values $\tilde{\delta}_i$ with $k \le i \le N$ are nonpositive, by $\tilde{\delta}_k = -\pi_k$ and by our assumption. So the last sum and $\tilde{\delta}_N$ are nonpositive, whence $2 \sum_{n=k+1}^{N} \pi_n P_{n-1} \le -\tilde{\delta}_k = \pi_k$. Since $\pi_n \ge \pi_k$ for $n \ge k$ and $P_{n-1} \ge 0$, the left-hand side is at least $2\pi_k \sum_{j=k}^{N-1} P_j$, and the asserted inequality follows.

Since $F_n - F_{n+1} = P_n - J_{n+1}$ and $J_{n+1} \ge 0$, we have $F_n - F_{n+1} \le P_n$, and $F_k = 1$; hence $1 - F_n \le \sum_{j=k}^{n-1} P_j$ and, because $F_n = F_n^- + P_n$, also $1 - F_n^- =$

$1 - F_n + P_n \le \sum_{j=k}^{n} P_j$ for $n \ge k$. Since $\sum_{j=k}^{N-1} P_j \le \frac{1}{2}$, we obtain

$$F_{N-1} \ge \frac{1}{2} \qquad \text{and} \qquad 1 - F_n^- \le \frac{1}{2} \quad (k \le n \le N-1).$$

We now combine these estimates, beginning with an upper bound for the right-hand side of the identity (8.7). The coefficients of the tilted counts there are nonnegative: by the two consequences of $\chi_R \ge 0$ recorded at the beginning, $1 - \vartheta_k - (k-1)\vartheta_k/N \ge 1 - 2\vartheta_k \ge 0$ and $\chi_{N-j} \ge 0$ for $k \le j \le N-2$, and $\mathsf{e}_j \ge 0$. Since the tilted counts are largest in row N, we may replace every $\tilde\beta_j^{(k-1)}$ by $\tilde\beta_N^{(k-1)}$, and the terms that contain the tilted counts are together at most $L\,\tilde\beta_N^{(k-1)}$, where

$$L = 1 + \Big(1 - \vartheta_k - \frac{(k-1)\,\vartheta_k}{N}\Big) + \sum_{M=2}^{R} \chi_M + \vartheta_k\,\mathsf{E}$$

is nonnegative. In the sum $\sum_{i=k}^{N-1} \Pi_i J_i$ the summand with $i = k$ vanishes because $J_k = 0$. For $k < i < N$ the bound $J_i \le \phi_i\,(1 - F_{i-1}^-)$ and the inequality $1 - F_{i-1}^- \le \frac{1}{2}$ give $0 \le J_i \le \phi_i/2$, and $\Pi_i \le (i-k+1)\,\pi_i \le (i-k+1)\,\pi_N$ because the numbers π_n do not decrease from $n = k$ on. Hence $\sum_{i=k}^{N-1} \Pi_i J_i \le \frac{1}{2}\pi_N \Sigma_\phi$, and altogether

$$F_{N-1}\,\Pi_N \le L\,\tilde\beta_N^{(k-1)} + \frac{\pi_N}{2}\,\Sigma_\phi.$$

Now we eliminate $\tilde\beta_N^{(k-1)}$ and F_{N-1}. By the definition of e_{N-1} we have $\Pi_N = \pi_N\,(R + 1 - \mathsf{e}_{N-1})$. We multiply the last display by λ, use the upper bound (8.6) in the form $L\,\lambda\,\tilde\beta_N^{(k-1)} \le L\,\pi_N F_{N-1}$, and divide by π_N:

$$F_{N-1}\,\big[(R + 1 - \mathsf{e}_{N-1})\,\lambda - L\big] \le \frac{\lambda\,\Sigma_\phi}{2}.$$

If the bracket is not positive, it is at most $\lambda\Sigma_\phi$, because $\lambda\Sigma_\phi \ge 0$. If it is positive, then $F_{N-1} \ge \frac{1}{2}$ shows that half of it is at most $\frac{1}{2}\lambda\Sigma_\phi$. In both cases

$$(R+1)\,\lambda - L \le \lambda\,\big(\mathsf{e}_{N-1} + \Sigma_\phi\big). \tag{8.8}$$

It remains to evaluate the left-hand side. For $M \le R$ we have $\chi_M - \chi_R = \sum_{m=M+1}^{R} \tilde\varepsilon_m$, and in the sum of these differences over $0 \le M \le R$ the number $\tilde\varepsilon_m$ occurs m times; hence, since $\sum_{m=1}^{R} m\,\tilde\varepsilon_m = \vartheta_k S_R$,

$$\sum_{M=0}^{R} \chi_M = (R+1)\,\chi_R + \vartheta_k\,S_R.$$

Removing $\chi_0 = 1 - \vartheta_k$ and $\chi_1 = 1 - 2\vartheta_k$ and inserting the result into the definition of L, we get

$$L = (R+1)\,\chi_R + \vartheta_k\,S_R + 2\vartheta_k - \frac{(k-1)\,\vartheta_k}{N} + \vartheta_k\,\mathsf{E},$$

and, since $(R+1)^2 + (k-1) - 2N = R(R+1) - N$,

$$(R+1)\,\lambda - L = \vartheta_k\,\Big(\frac{R(R+1)}{N} - 1 - S_R - \mathsf{E}\Big).$$

Inserted into (8.8), this is the inequality (8.3), which contradicts the second hypothesis of the proposition. Hence our assumption is false, and there is an integer j with $k < j \le N$ and $d_{j,k} > 0$. □

Remark. Since $\chi_R \le \chi_1 = 1-2\vartheta_k$, the first hypothesis of the criterion requires $\vartheta_k \le \frac{1}{2}$. Since $\vartheta_k = 1/\gamma_k$ does not increase with k and $\vartheta_7 = (7/8)^5 = 16807/32768 > \frac{1}{2}$, the criterion says nothing about columns $k \le 7$. For a given column the same hypothesis limits how large R may be taken: at tilt e^{-1} the number χ_M would be the positive tail $\sum_{m>M} \varepsilon_m e^{-m}$, by Lemma 2.9, and since $\vartheta_k \ge e^{-1}$ by Lemma 8.1, it is smaller than this tail by an excess that grows with M. In the second hypothesis $R(R+1)/N$ is of order R^2/k, while S_R is at most of order $\sqrt{R}$ and the other terms are corrections; so this hypothesis can hold once $R^{3/2}$ exceeds a multiple of k.

9. The crossing row

The hypotheses of the crossing criterion, Proposition 8.2, are inequalities between finite sums whose number of terms grows with R. We now replace each of these sums by a bound in closed form and verify the resulting inequalities for R of order $k^{2/3}$. The outcome is that every column $k \ge 265$ is positive in some row $j > k$ with $25\,(j-k-1)^3 \le 121\,k^2$ (Proposition 9.2) and, together with the computed signs for the columns below 265, that every column $k \ge 82$ is positive in some row $k+r$ with $1 \le r \le 27k/100$ (Corollary 9.3). The proofs of Propositions 1.3 and 1.2 start from such an early positive row, and the first bound confines the rows before the crossing to the range that the proof of Proposition 1.4 treats.

9.1. **The crossing criterion in closed form.** Let $k \ge 2$ and $R \ge 1$ be integers, and put $N = k + R$. We use the quantities of column k that are introduced in Section 8: the diagonal tilt $\vartheta_k = 1/\gamma_k$, the numbers χ_R and S_R, the π_j with their partial sums Π_i, the e_j with their sum E, the ϕ_i with their sum Σ_ϕ, and the numbers $\bar a = 5/(2k)$ and $\underline{a} = 5/(2k) - 4/(3k^2)$; the proof of Proposition 9.1 recalls each definition where it is used. Recall also that $c_* = 1/\sqrt{2\pi}$. For $R \ge 2$ we put

$$
\begin{aligned}
\bar S &= 1 + c_*\Big[2\sqrt{R-1} - 1 + \frac{2\bar a}{3}\,R\sqrt{R} + \frac{2}{5}\cdot\frac{633}{1000}\,\bar a^2 R^2\sqrt{R}\Big],\\
\bar{\mathsf{E}} &= \frac{5\,(R-2)(R-1)\,R\,(R+1)}{24\,k^3},\\
\bar H &= (R-1) - k\,\log\frac{k+R+1}{k+2},\\
\bar G &= \frac{R^2-1}{2} - (k-1)(R-1) + k\,(k-1)\,\log\frac{k+R}{k+1},
\end{aligned}
\tag{9.1}
$$

and

$$
\begin{aligned}
\underline{t} = {}& \frac{2e^{-1}c_*\big(1-\frac{1}{12R}\big)}{\sqrt{R+1}} - e^{-1}\,\bar a\,\Big[e^{\bar a} + 1 + c_*\big(2\sqrt{R-1}-1\big)\Big]\\
& - e^{-1}\,\frac{633}{1000}\,\bar a^2\,\Big[1 + c_*\Big(\frac{2}{3}\,R\sqrt{R} + 2\sqrt{R-1} - 1\Big)\Big],\\
\bar t = {}& \frac{2e^{-1}c_*}{\sqrt{R-1}} - e^{-1}\,\underline{a}\,\Big[2 + c_*\Big(2\sqrt{R} - 2 - \frac{1}{4}\Big)\Big].
\end{aligned}
\tag{9.2}
$$

We shall show that $\bar S$ bounds S_R, that $\bar{\mathsf{E}}$ bounds E and that $\bar H + \frac{5}{k}\bar G$ bounds $\mathsf{e}_{N-1} + \Sigma_\phi$, all from above, and that $\underline{t} \le \chi_R \le \bar t$. We also write

$$
\mathsf{A} = e\,\bar t + e^{\bar a}\,\frac{R+1}{N}, \qquad \mathsf{B} = \bar H + \frac{5}{k}\,\bar G, \qquad \mathsf{C} = \frac{R(R+1)}{N} - 1 - \bar S - \bar{\mathsf{E}}.
\tag{9.3}
$$

With these bounds the criterion takes the following form, in which the last hypothesis is the inequality $\mathsf{A}\,\mathsf{B} < \mathsf{C}$ multiplied by e^{-1}.

Proposition 9.1. *Let $k \geq 2$ and R be integers with $2 \leq R \leq k-1$, and put $N = k + R$. Let $\bar{S}$, $\bar{\mathsf{E}}$, $\bar{H}$, $\bar{G}$, $\underline{t}$ and $\bar{t}$ be the numbers defined in* (9.1) *and* (9.2) *for these k and R. Suppose that*

$$\bar{a}\,R \leq \frac{6679}{10000}, \qquad \underline{t} \geq 0,$$

and

$$\Big(\bar{t} + e^{-1}e^{\bar{a}}\,\frac{R+1}{N}\Big)\Big(\bar{H} + \frac{5}{k}\,\bar{G}\Big) < e^{-1}\Big(\frac{R(R+1)}{N} - 1 - \bar{S} - \bar{\mathsf{E}}\Big). \tag{9.4}$$

Then there is an integer j with $k < j \leq N$ and $d_{j,k} > 0$.

Proof. By Proposition 8.2 it suffices to show that $\chi_R \geq 0$ and that

$$\Big(\chi_R + \frac{\vartheta_k\,(R+1)}{N}\Big)\big(\mathsf{e}_{N-1} + \Sigma_\phi\big) < \vartheta_k\Big(\frac{R(R+1)}{N} - 1 - S_R - \mathsf{E}\Big). \tag{9.5}$$

We first prove the bounds

$$\begin{gathered} S_R \leq \bar{S}, \qquad \mathsf{E} \leq \bar{\mathsf{E}}, \\ 0 \leq \mathsf{e}_{N-1} + \Sigma_\phi \leq \bar{H} + \frac{5}{k}\,\bar{G}, \qquad \underline{t} \leq \chi_R \leq \bar{t}. \end{gathered} \tag{9.6}$$

We write

$$\sigma = 1 + \log\vartheta_k, \qquad \text{so that} \qquad \vartheta_k = e^{\sigma-1},$$

and we use throughout that for real $t \geq 0$ the Taylor polynomial of $\log(1+t)$ of even (resp. odd) degree bounds it from below (resp. above).

The bounds that we have listed rest on two estimates for the tilt, which we prove first. By Lemma 8.1 we have $e^{-1} \leq \vartheta_k \leq e^{-1}e^{\bar{a}}$, and its proof shows more precisely that $\underline{a} \leq 1 + \log\vartheta_k \leq \bar{a}$. Since $\underline{a} = (15k-8)/(6k^2) \geq 0$, this gives

$$0 \leq \underline{a} \leq \sigma \leq \bar{a}.$$

We also need a bound for the products $\vartheta_k\gamma_i$: for every integer $i \geq k$,

$$0 \leq \log(\vartheta_k\gamma_i) \leq \frac{5\,(i-k)}{2\,k\,i}. \tag{9.7}$$

The lower bound holds because $\vartheta_k\gamma_k = 1$ and the sequence $(\gamma_i)_{i\geq 2}$ is nondecreasing (Section 5). For the upper bound we write $\log(\vartheta_k\gamma_i) = \log\gamma_i - \log\gamma_k$ as the sum of the differences $\log\gamma_{m+1} - \log\gamma_m$ over $k \leq m < i$. For $m \geq 2$ we bound $\log\gamma_{m+1} = (m-1)\log(1+\frac{1}{m+1})$ from above by the Taylor polynomial of degree 3 and $\log\gamma_m = (m-2)\log(1+\frac{1}{m})$ from below by the Taylor polynomial of degree 4. A direct computation with these two bounds gives

$$\begin{aligned} \log\gamma_{m+1} - \log\gamma_m &\leq \frac{5}{2m(m+1)} - \frac{29m^4 + 21m^3 + m^2 + 7m + 6}{12\,m^4(m+1)^3} \\ &\leq \frac{5}{2}\Big(\frac{1}{m} - \frac{1}{m+1}\Big), \end{aligned}$$

and summing over $k \leq m < i$ we obtain the asserted upper bound.

We turn to the first of the bounds, the inequality $S_R \leq \bar{S}$. Recall that $S_R = \sum_{i=0}^{R-1}(i\vartheta_k)^i/i!$. Since $\vartheta_k = e^{\sigma-1}$, the term with index i is $e^{\sigma i}\cdot e^{-i}i^i/i!$, the Poisson

mass at i multiplied by $e^{\sigma i}$. To bound the exponential we use the elementary inequality

$$e^s \le 1 + s + \frac{633}{1000}\, s^2 \qquad \Big(0 \le s \le \frac{6679}{10000}\Big),$$

which holds because the factor $\sum_{j\ge 0} s^j/(j+2)!$ of s^2 in the exponential series is smaller than 0.6328 on this interval. For a real number y with $0 \le y \le R$ we have $0 \le \sigma y \le \bar a y \le \bar a R \le \frac{6679}{10000}$, because $0 \le \sigma \le \bar a$ and by the hypothesis on $\bar a R$, so that the inequality for e^s gives

$$e^{\sigma y} \le e^{\bar a y} \le 1 + \bar a\, y + \frac{633}{1000}\, \bar a^2 y^2 \qquad (0 \le y \le R). \tag{9.8}$$

The Poisson mass at 0 is 1, and the Poisson mass at $i \ge 1$ is at most $c_*/\sqrt{i}$ by (2.7). Therefore

$$\begin{aligned} \sum_{i=0}^{R-1} \frac{e^{-i} i^i}{i!} &\le 1 + c_* \sum_{i=1}^{R-1} \frac{1}{\sqrt{i}} \le 1 + c_*\big(2\sqrt{R-1} - 1\big), \\ \sum_{i=0}^{R-1} i\, \frac{e^{-i} i^i}{i!} &\le c_* \sum_{i=1}^{R-1} \sqrt{i} \le \frac{2}{3}\, c_*\, R\sqrt{R}, \\ \sum_{i=0}^{R-1} i^2\, \frac{e^{-i} i^i}{i!} &\le c_* \sum_{i=1}^{R-1} i\sqrt{i} \le \frac{2}{5}\, c_*\, R^2\sqrt{R}, \end{aligned} \tag{9.9}$$

where the second inequality in each line is a comparison of the sum with an integral. By the bound for $e^{\sigma y}$ with $y = i$, the sum S_R is at most the first of these three sums, plus $\bar a$ times the second, plus $\frac{633}{1000}\bar a^2$ times the third, and the three bounds add up to $\bar S$.

Next we prove that $\mathsf{E} \le \bar{\mathsf{E}}$. Recall that $\pi_j = \vartheta_k^j\, a_j/j$ and $\Pi_i = \pi_k + \cdots + \pi_i$, that $\mathsf{e}_j = (j-k+2) - \Pi_{j+1}/\pi_{j+1}$, and that $\mathsf{E} = \sum_{j=k}^{N-2} \mathsf{e}_j/(j+1)$. The numbers π_j are positive, and for $n \ge 2$,

$$\pi_{n+1} = \vartheta_k \gamma_n\, \pi_n,$$

because $a_{n+1} = (1 + \frac{1}{n})^{n-1} a_n = \gamma_n\, \frac{n+1}{n}\, a_n$ by (2.3). Since $\vartheta_k \gamma_k = 1$, we get $\pi_{k+1} = \pi_k$ and hence $\mathsf{e}_k = 0$. For $j \ge k$, the relations $\Pi_{j+2} = \Pi_{j+1} + \pi_{j+2}$ and $\pi_{j+2} = \vartheta_k \gamma_{j+1}\, \pi_{j+1}$ give

$$\mathsf{e}_{j+1} = (j-k+2)\Big(1 - \frac{1}{\vartheta_k \gamma_{j+1}}\Big) + \frac{\mathsf{e}_j}{\vartheta_k \gamma_{j+1}}.$$

Since $\vartheta_k \gamma_{j+1} \ge \vartheta_k \gamma_k = 1$, it follows by induction that $\mathsf{e}_j \ge 0$ for all $j \ge k$, and then, with $1 - 1/t \le \log t$ and $1/t \le 1$ for real $t \ge 1$, that

$$\mathsf{e}_{j+1} \le (j-k+2)\, \log(\vartheta_k \gamma_{j+1}) + \mathsf{e}_j \qquad (j \ge k). \tag{9.10}$$

We claim that $\mathsf{e}_{k+m} \le \frac{5}{6k^2}\, m(m+1)(m+2)$ for every integer $m \ge 0$. This is clear for $m = 0$, and the induction step follows from the last display with $j = k+m$ and from the bound (9.7) with $i = k+m+1$, where $1/i \le 1/k$:

$$\begin{aligned} \mathsf{e}_{k+m+1} &\le (m+2)\, \frac{5\,(m+1)}{2k^2} + \frac{5}{6k^2}\, m(m+1)(m+2) \\ &= \frac{5}{6k^2}\, (m+1)(m+2)(m+3). \end{aligned}$$

Finally $\mathsf{e}_j/(j+1) \le \mathsf{e}_j/k$ for $j \ge k$, and $\sum_{m=0}^{R-2} m(m+1)(m+2) = \frac{1}{4}(R-2)(R-1)R(R+1)$. Hence

$$\mathsf{E} = \sum_{j=k}^{N-2} \frac{\mathsf{e}_j}{j+1} \le \frac{5}{6k^3} \sum_{m=0}^{R-2} m(m+1)(m+2) = \bar{\mathsf{E}}.$$

We come to the two bounds for $\mathsf{e}_{N-1} + \Sigma_\phi$. Recall that $\phi_i = 1 - i/((i+1)\,\vartheta_k\gamma_i)$ and $\Sigma_\phi = \sum_{i=k+1}^{N-1}(i-k+1)\,\phi_i$. We have just seen that $\mathsf{e}_j \ge 0$ for all $j \ge k$; in particular $\mathsf{e}_{N-1} \ge 0$. Moreover $\phi_i \ge 0$ for $i \ge k$ because $\vartheta_k\gamma_i \ge 1$, so that $\Sigma_\phi \ge 0$. For the upper bound we write $\bar{H}(k,m)$ and $\bar{G}(k,m)$ for the expressions that define $\bar{H}$ and $\bar{G}$, with an integer $m \ge 1$ in place of R, so that $\bar{H}(k,1) = \bar{G}(k,1) = 0$. The increments of these expressions dominate two families of summands: for every integer $m \ge 1$,

$$\frac{m+1}{k+m+1} \le \bar{H}(k,m+1) - \bar{H}(k,m),$$
$$\frac{m(m+1)}{k+m} \le \bar{G}(k,m+1) - \bar{G}(k,m).$$

Both inequalities follow by a direct computation from the bounds for $\log(1+t)$: the first increment is $1 - k\log\bigl(1+\frac{1}{k+m+1}\bigr) \ge 1 - \frac{k}{k+m+1}$, and the second is $m + \frac{3}{2} - k + k(k-1)\log\bigl(1+\frac{1}{k+m}\bigr)$, which by the Taylor polynomial of degree 2 exceeds $\frac{m(m+1)}{k+m} = m+1-k+\frac{k(k-1)}{k+m}$ by at least $\frac{1}{2} - \frac{k(k-1)}{2(k+m)^2} \ge 0$. Now let $1 \le m \le R-1$. By the bound (9.7) with $i = k+m$ and by the inequality for the increment of $\bar{G}$,

$$(m+1)\,\log(\vartheta_k\gamma_{k+m}) \le \frac{5}{2k} \cdot \frac{m(m+1)}{k+m} \le \frac{5}{2k}\bigl(\bar{G}(k,m+1) - \bar{G}(k,m)\bigr).$$

We use this twice. First, the inequality (9.10) with $j = k+m-1$ gives $\mathsf{e}_{k+m} - \mathsf{e}_{k+m-1} \le \frac{5}{2k}\bigl(\bar{G}(k,m+1) - \bar{G}(k,m)\bigr)$, and summing over $1 \le m \le R-1$, with $\mathsf{e}_k = 0$ and $\bar{G}(k,1) = 0$, we get $\mathsf{e}_{N-1} \le \frac{5}{2k}\,\bar{G}$. Second,

$$\phi_i = \frac{1}{i+1} + \frac{i}{i+1}\Bigl(1 - \frac{1}{\vartheta_k\gamma_i}\Bigr) \le \frac{1}{i+1} + \log(\vartheta_k\gamma_i)$$

for $i \ge k$, so that the term of Σ_ϕ with index $i = k+m$ satisfies

$$(m+1)\,\phi_{k+m} \le \bigl(\bar{H}(k,m+1) - \bar{H}(k,m)\bigr) + \frac{5}{2k}\bigl(\bar{G}(k,m+1) - \bar{G}(k,m)\bigr).$$

Summing over $1 \le m \le R-1$ we get $\Sigma_\phi \le \bar{H} + \frac{5}{2k}\,\bar{G}$. The two bounds add up to $\mathsf{e}_{N-1} + \Sigma_\phi \le \bar{H} + \frac{5}{k}\,\bar{G}$.

Finally, we prove the bounds $\underline{t} \le \chi_R \le \bar{t}$. Recall that $\chi_R = 1 - \vartheta_k - \sum_{m=1}^{R} \varepsilon_m \vartheta_k^m$. Since $\vartheta_k^m = e^{\sigma m} e^{-m}$ and $\sum_{m\ge1} e^{-m}\varepsilon_m = 1 - e^{-1}$ by Lemma 2.9,

$$\chi_R = \sum_{m>R} e^{-m}\varepsilon_m - e^{-1}\bigl(e^{\sigma} - 1\bigr) - \sum_{m=1}^{R} \bigl(e^{\sigma m} - 1\bigr)\, e^{-m}\varepsilon_m.$$

Thus χ_R is the tail of the series minus the excess that is caused by $\sigma > 0$, i.e. minus the sum of the two terms that are subtracted in this formula, and we bound the tail and the excess from both sides. The terms are Poisson masses: since $m\,\varepsilon_m = (m-1)^{m-1}/(m-1)!$,

$$e^{-m}\varepsilon_m = \frac{e^{-1}}{m} \cdot \frac{e^{-(m-1)}(m-1)^{m-1}}{(m-1)!} \qquad (m \ge 1).$$

For the tail we use that for real $y > 0$ the integral of $t^{-3/2}$ over $[y, y+1]$ lies between the values of the integrand at the two ends,

$$\frac{1}{(y+1)\sqrt{y+1}} \le 2\Big(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+1}}\Big) \le \frac{1}{y\sqrt{y}}.$$

Let $m \ge R+1$, so that $m-1 \ge R \ge 2$. By the expression of $e^{-m}\varepsilon_m$ through the Poisson mass at $m-1$ and the upper bound in (2.7),

$$e^{-m}\varepsilon_m \le \frac{e^{-1}c_*}{m\sqrt{m-1}} \le \frac{e^{-1}c_*}{(m-1)\sqrt{m-1}} \le 2e^{-1}c_*\Big(\frac{1}{\sqrt{m-2}} - \frac{1}{\sqrt{m-1}}\Big).$$

By the same expression and the lower bound in (2.7), in which the factor $1-\frac{1}{12(m-1)}$ is at least $1-\frac{1}{12R}$,

$$\begin{aligned} e^{-m}\varepsilon_m &\ge \frac{e^{-1}c_*\big(1-\frac{1}{12R}\big)}{m\sqrt{m-1}} \ge \frac{e^{-1}c_*\big(1-\frac{1}{12R}\big)}{m\sqrt{m}} \\ &\ge 2e^{-1}c_*\Big(1-\frac{1}{12R}\Big)\Big(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{m+1}}\Big). \end{aligned}$$

Both bounds telescope when they are summed over $m \ge R+1$, and we obtain

$$\frac{2e^{-1}c_*\big(1-\frac{1}{12R}\big)}{\sqrt{R+1}} \le \sum_{m>R} e^{-m}\varepsilon_m \le \frac{2e^{-1}c_*}{\sqrt{R-1}}.$$

For the lower bound of χ_R we bound the excess from above. From $1-\sigma \le e^{-\sigma}$ we get $e^{\sigma}-1 \le \sigma e^{\sigma} \le \bar{a}\, e^{\bar{a}}$. For $1 \le m \le R$, the bound (9.8) with $y = m$ and the expression of $e^{-m}\varepsilon_m$ through the Poisson mass at $m-1$ give

$$\big(e^{\sigma m}-1\big)\, e^{-m}\varepsilon_m \le e^{-1}\Big(\bar{a} + \frac{633}{1000}\,\bar{a}^2\, m\Big)\frac{e^{-(m-1)}(m-1)^{m-1}}{(m-1)!}.$$

We sum over $1 \le m \le R$, write $i = m-1$, and use the bounds (9.9): the sum of the Poisson masses at $i < R$ is at most $1 + c_*(2\sqrt{R-1}-1)$, and the sum of these masses with weights $i+1$ is at most $1 + c_*\big(\frac{2}{3}R\sqrt{R} + 2\sqrt{R-1} - 1\big)$. Subtracting the resulting bounds for the excess from the lower bound for the tail, we obtain $\underline{t} \le \chi_R$.

For the upper bound of χ_R we bound the excess from below by its part of the first order: $e^{\sigma}-1 \ge \sigma$ and $e^{\sigma m}-1 \ge \sigma m$, so that, since $m\, e^{-m}\varepsilon_m$ is e^{-1} times the Poisson mass at $m-1$, the excess is at least $e^{-1}\sigma\big(1 + \sum_{i=0}^{R-1} e^{-i} i^i/i!\big)$. By the lower bound in (2.7),

$$\sum_{i=0}^{R-1} \frac{e^{-i}i^i}{i!} \ge 1 + c_* \sum_{i=1}^{R-1}\Big(\frac{1}{\sqrt{i}} - \frac{1}{12\, i\sqrt{i}}\Big) \ge 1 + c_*\Big(2\sqrt{R} - 2 - \frac{1}{4}\Big),$$

since $\sum_{i=1}^{n} i^{-1/2} \ge 2\sqrt{n+1} - 2$ and $\sum_{i=1}^{n} i^{-3/2} \le 3$ for $n \ge 1$, again by comparison with integrals. As $\sigma \ge \underline{a} \ge 0$, the excess is at least $e^{-1}\underline{a}\,\big[2 + c_*(2\sqrt{R} - 2 - \frac{1}{4})\big]$, and subtracting this from the upper bound for the tail we obtain $\chi_R \le \bar{t}$.

It remains to deduce the two conditions from the bounds (9.6). Since $\underline{t} \le \chi_R$ and $\underline{t} \ge 0$ by hypothesis, we have $\chi_R \ge 0$. By the bounds for χ_R and for $\mathsf{e}_{N-1} + \Sigma_\phi$ and by $0 < \vartheta_k \le e^{-1}e^{\bar{a}}$, the two factors on the left of the inequality (9.5) that is to be proved are nonnegative, the first is at most $e^{-1}\mathsf{A}$, and the second is at most

B. In particular $e^{-1}\mathsf{A}\,\mathsf{B} \geq 0$, so that the last hypothesis of the proposition, which reads $e^{-1}\mathsf{A}\,\mathsf{B} < e^{-1}\mathsf{C}$, implies $\mathsf{C} > 0$. Since $\vartheta_k \geq e^{-1}$, $S_R \leq \bar{S}$ and $\mathsf{E} \leq \bar{\mathsf{E}}$, we obtain

$$\Big(\chi_R + \frac{\vartheta_k\,(R+1)}{N}\Big)\big(\mathsf{e}_{N-1} + \Sigma_\phi\big) \leq e^{-1}\mathsf{A}\,\mathsf{B} < e^{-1}\mathsf{C} \\ \leq \vartheta_k\,\mathsf{C} \leq \vartheta_k\Big(\frac{R(R+1)}{N} - 1 - S_R - \mathsf{E}\Big),$$

which is the inequality to be proved. This completes the proof. □

9.2. **The choice of R.** We shall apply Proposition 9.1 to a column $k \geq 265$, and the difficulty lies in the choice of R, because the hypotheses pull in opposite directions. The inequality (9.4) needs R large: on its right-hand side the term $R(R+1)/N$, which is of order R^2/k, has to exceed $\bar{S}$, which is of order $\sqrt{R}$, and this requires R of order at least $k^{2/3}$. The other two hypotheses need R small: $\bar{a}R$ grows with R, and in $\underline{t}$ the first term decreases with R while the terms that are subtracted grow. For $k = 265$ a numerical evaluation of the closed forms shows that the three hypotheses hold together exactly for $64 \leq R \leq 70$.

Let $k \geq 265$ be an integer. We take for R the least positive integer with $121\,k^2 < 25\,R^3$, which exists because $121\,k^2 < 25\,k^3$; it is $\lfloor (11k/5)^{2/3} \rfloor + 1$ and equals 70 for $k = 265$. By the minimality of R,

$$25\,(R-1)^3 \leq 121\,k^2 < 25\,R^3.$$

To verify the hypotheses for all $k \geq 265$ at once, we work with the real number $u > 0$ defined by

$$k\,u^3 = \frac{5}{11}, \qquad \text{so that} \qquad 121\,k^2u^6 = 25.$$

Multiplying the two inequalities for R by u^6 and taking cube roots, we get $1 < Ru^2 \leq 1 + u^2$. Since $k \geq 265$, we have $265\,u^3 \leq \frac{5}{11} < 265 \cdot (0.119705)^3$, hence $u < 0.119705 < \frac{3}{25}$. In particular $R \geq 2$, because $Ru^2 > 1 > u^2$. Since $\frac{1}{k} = \frac{11}{5}u^3$, the numbers $\bar{a} = 5/(2k)$ and $\underline{a} = 5/(2k) - 4/(3k^2)$ are $\bar{a} = \frac{11}{2}u^3$ and $\underline{a} = \frac{11}{2}u^3 - \frac{484}{75}u^6$.

Thus R is about u^{-2}, and the square roots of $R-1$, R and $R+1$, which occur in the closed forms $\bar{S}$, $\underline{t}$ and $\bar{t}$, are about u^{-1}. More precisely, the bounds for Ru^2 give

$$1 - u^2 < (R-1)\,u^2 \leq 1, \qquad 1 < Ru^2 \leq 1 + u^2, \qquad (R+1)\,u^2 \leq 1 + 2u^2,$$

and since $1 + u^2 \leq (1 + \frac{u^2}{2})^2$ and $1 + 2u^2 \leq (1 + u^2)^2$, taking square roots we get $\sqrt{R-1}\,u \leq 1$, $1 < \sqrt{R}\,u \leq 1 + \frac{u^2}{2}$ and $\sqrt{R+1}\,u \leq 1 + u^2$. For $\sqrt{R-1}$ we also need a bound from below. Since $(R-1)\,u^2 > 1 - u^2 > 0$, we have

$$(R-1)\,u^2\Big(1 + \frac{51}{100}\,u^2\Big)^2 > (1-u^2)\Big(1 + \frac{51}{50}\,u^2\Big) \\ = 1 + \frac{u^2}{50}\,(1 - 51\,u^2) \geq 1,$$

where the last step holds because $u < \frac{3}{25}$ gives $u^2 < \frac{9}{625} < \frac{1}{51}$. Therefore $1 \leq \sqrt{R-1}\,u\,\big(1 + \frac{51}{100}u^2\big)$. Finally, $k \geq 265$ gives $0 \leq \bar{a} \leq \frac{5}{530} = \frac{1}{106}$, and $e^s \leq \frac{1}{1-s}$ for $0 \leq s < 1$; together with the enclosure $0.39894 \leq c_* \leq 0.39895$ of Section 2 this gives $1 \leq e^{\bar{a}} \leq \frac{106}{105}$ and $\frac{19947}{50000} \leq c_* \leq \frac{7979}{20000}$.

For later reference we collect the relations that have been obtained:

$$
\begin{aligned}
k\,u^3 = \frac{5}{11}, \qquad 0 < u < 0.119705 < \frac{3}{25}, \qquad 1 < Ru^2 \le 1 + u^2,\\
\bar a = \frac{11}{2}\,u^3, \qquad \underline{a} = \frac{11}{2}\,u^3 - \frac{484}{75}\,u^6, \qquad 1 \le e^{\bar a} \le \frac{106}{105},\\
\sqrt{R-1}\,u \le 1 \le \sqrt{R-1}\,u\Big(1 + \frac{51}{100}\,u^2\Big), \qquad 1 < \sqrt{R}\,u \le 1 + \frac{u^2}{2},\\
\sqrt{R+1}\,u \le 1 + u^2, \qquad \frac{19947}{50000} \le c_* \le \frac{7979}{20000}.
\end{aligned}
\tag{9.11}
$$

By these relations, each closed form is a power of u times a polynomial in u, up to errors that can be controlled. To leading order, $e\,\underline{t}$ is $c_* u\,(2 - 11u)$, and the numbers A, B and C of (9.3) are

$$
\Big(2c_* + \frac{11}{5}\Big)u, \qquad \frac{11}{10\,u}, \qquad \frac{1}{u}\Big(\frac{11}{5} - 2c_*\Big).
$$

So $\mathsf{A}\,\mathsf{B}$ stays near 3.3 while C grows like $1.40/u$, i.e. like $k^{1/3}$, and both $\underline{t} \ge 0$ and $\mathsf{A}\,\mathsf{B} < \mathsf{C}$ hold when k is large. The work is therefore at the columns just above 265, where u is close to $\frac{3}{25}$ and the terms of first order in u are not small.

It remains to prove, for every $k \ge 265$ and for this R, the two inequalities

$$
\underline{t} \ge 0 \qquad \text{and} \qquad \mathsf{A}\,\mathsf{B} < \mathsf{C}
\tag{9.12}
$$

themselves, keeping track of these terms. This is done in the next two subsections, in which $k \ge 265$ is an integer, R and u are the numbers just defined, and the relations (9.11) are used throughout. We write

$$
\mathfrak{c}_- = \frac{19947}{50000}, \qquad \mathfrak{c}_+ = \frac{7979}{20000}
$$

for the two rational bounds of c_* in (9.11), so that $0 < \mathfrak{c}_- \le c_* \le \mathfrak{c}_+ < 1$.

9.3. **The inequality $\underline{t} \ge 0$.** By the definition (9.2) of $\underline{t}$,

$$
\begin{aligned}
e\,\underline{t} = \frac{2c_*\big(1 - \frac{1}{12R}\big)}{\sqrt{R+1}} - \bar a\,\big(e^{\bar a} + 1 + c_*\,(2\sqrt{R-1} - 1)\big)\\
- \frac{633}{1000}\,\bar a^2\Big(1 + c_*\Big(\frac{2}{3}\,R\sqrt{R} + 2\sqrt{R-1} - 1\Big)\Big).
\end{aligned}
\tag{9.13}
$$

We bound the three terms on the right-hand side by polynomials in u and c_*. Note that $u^2 \le \frac{9}{625}$.

For the first term, $Ru^2 > 1$ and $\sqrt{R+1}\,u \le 1 + u^2$ give $1 - \frac{1}{12R} \ge 1 - \frac{u^2}{12} \ge 0$ and $\frac{1}{\sqrt{R+1}} \ge \frac{u}{1+u^2} \ge u\,(1 - u^2) \ge 0$, so that

$$
\frac{2c_*\big(1 - \frac{1}{12R}\big)}{\sqrt{R+1}} \ge 2c_*\,u\Big(1 - \frac{u^2}{12}\Big)(1 - u^2).
$$

For the second term, $\bar a = \frac{11}{2}u^3$, $e^{\bar a} \le \frac{106}{105}$ and $\sqrt{R-1}\,u \le 1$ give

$$
\bar a\,\big(e^{\bar a} + 1 + c_*\,(2\sqrt{R-1} - 1)\big) \le \frac{11}{2}\,u^3\Big(\frac{211}{105} - c_*\Big) + 11\,c_*\,u^2.
$$

For the third term, $\bar a^2 = \frac{121}{4}u^6$, $R\sqrt{R}\,u^3 = Ru^2\cdot\sqrt{R}\,u \le (1+u^2)\big(1+\frac{u^2}{2}\big)$ and $\sqrt{R-1}\,u \le 1$ give

$$\bar a^2\Big(1+c_*\Big(\frac{2}{3}\,R\sqrt{R}+2\sqrt{R-1}-1\Big)\Big) \le \frac{121}{4}\Big[(1-c_*)\,u^6+\frac{2}{3}\,c_*\,u^3\,(1+u^2)\Big(1+\frac{u^2}{2}\Big)+2c_*\,u^5\Big].$$

We subtract the second bound and $\frac{633}{1000}$ times the third from the first and expand. The right-hand side of (9.13) is at least

$$u\Big[c_*\Big(2-11u-\frac{56593}{6000}\,u^2-\frac{687337}{12000}\,u^4+\frac{76593}{4000}\,u^5-\frac{25531}{4000}\,u^6\Big) -\frac{2321}{210}\,u^2-\frac{76593}{4000}\,u^5\Big].$$

For $0\le u\le\frac{3}{25}$ the polynomial that multiplies c_* is positive: since $u^2\le\frac{9}{625}$, it is at least $2-11u-10.3\,u^2\ge 0.5$. Since $c_*\ge\mathfrak{c}_-$, the bracket is therefore at least its value with $\mathfrak{c}_-$ in the place of c_*. This value is a polynomial in u with constant term $2\mathfrak{c}_-=0.79788$ and with no term in u^3. Since $0<\mathfrak{c}_-<1$, its coefficients of u, u^2, u^4, u^5 and u^6 are negative; here the two terms with u^5 combine to $-\frac{76593}{4000}\,(1-\mathfrak{c}_-)\,u^5$. On $[0,\frac{3}{25}]$ the polynomial is therefore at least its value at $u=\frac{3}{25}$, and a direct evaluation shows that this value is $0.0529\ldots$. Hence $e\,\underline{t}\ge 0$, which is the first of the inequalities (9.12).

9.4. **The inequality $\mathsf{A}\,\mathsf{B}<\mathsf{C}$.** By their definitions (9.3), (9.1) and (9.2), and since $N=k+R$, the numbers A, B and C are

$$\mathsf{A}=\frac{2c_*}{\sqrt{R-1}}-\underline{a}\Big(2+c_*\Big(2\sqrt{R}-2-\frac{1}{4}\Big)\Big)+\frac{e^{\bar a}\,(R+1)}{k+R},$$

$$\mathsf{B}=(R-1)-k\log\frac{k+R+1}{k+2} +\frac{5}{k}\Big(\frac{R^2-1}{2}-(k-1)(R-1)+k(k-1)\log\frac{k+R}{k+1}\Big),$$

$$\mathsf{C}=\frac{R\,(R+1)}{k+R}-1-\Big(1+c_*\Big(2\sqrt{R-1}-1+\frac{2\bar a}{3}\,R\sqrt{R}+\frac{2}{5}\cdot\frac{633}{1000}\,\bar a^2R^2\sqrt{R}\Big)\Big) -\frac{5\,(R-2)(R-1)\,R\,(R+1)}{24\,k^3}.$$

We write

$$h=R\,u^2,\qquad w=\frac{11}{5}\,u,$$

so that $1<h\le 1+u^2$ and $0<w\le\frac{33}{125}$. The relation $ku^3=\frac{5}{11}$, i.e. $ku^2w=1$, serves to eliminate k:

$$(k+R)\,u^3=\frac{5}{11}+hu,\qquad \frac{5}{k}=11\,u^3,\qquad k^3u^9=\frac{125}{1331}.$$

We define three polynomials in u,

$$\begin{aligned}
\mathsf{A}^*(u) &= 2\mathfrak{c}_+\Big(1+\frac{51}{100}\,u^2\Big) - \Big(2-\frac{9}{4}\,\mathfrak{c}_+\Big)\Big(\frac{11}{2}\,u^2-\frac{484}{75}\,u^5\Big)\\
&\qquad - 2\mathfrak{c}_+\Big(\frac{11}{2}\,u-\frac{484}{75}\,u^4\Big)+\frac{106}{105}\cdot\frac{11}{5}\,(1+2u^2)(1-w+w^2),\\
\mathsf{B}^*(u) &= \frac{11}{10}+\frac{484}{75}\,u+\frac{26}{25}\,u^2,\\
\mathsf{C}_*(u) &= \frac{11}{5}\,(1-w)\Big(1+\frac{146}{121}\,w^2\Big)-2u-\frac{1331}{600}\,u^2\,(1+u^2)^2\\
&\qquad - \mathfrak{c}_+\Big[2-u+\Big(\frac{11}{3}\,u+\frac{121}{10}\cdot\frac{633}{1000}\,u^2\,(1+u^2)\Big)(1+u^2)\Big(1+\frac{u^2}{2}\Big)\Big],
\end{aligned}$$

and prove successively

$$\begin{aligned}
0\le \mathsf{A}\le u\,\mathsf{A}^*(u), \qquad u\,\mathsf{B}\le \mathsf{B}^*(u),\\
\mathsf{C}_*(u)\le u\,\mathsf{C}, \qquad u\,\mathsf{A}^*(u)\,\mathsf{B}^*(u)<\mathsf{C}_*(u).
\end{aligned}$$

Since $\mathsf{B}^*(u)\ge 0$, these four statements give

$$u\,\mathsf{A}\,\mathsf{B}\le \mathsf{A}\,\mathsf{B}^*(u)\le u\,\mathsf{A}^*(u)\,\mathsf{B}^*(u)<\mathsf{C}_*(u)\le u\,\mathsf{C},$$

and the second of the inequalities (9.12) follows on division by u.

We begin with the bounds for A, and treat its three terms one by one. From $\sqrt{R-1}\,u\le 1$ and $1\le\sqrt{R-1}\,u\,(1+\frac{51}{100}u^2)$ we get

$$2c_*\,u\le\frac{2c_*}{\sqrt{R-1}}\le 2c_*\,u\,\Big(1+\frac{51}{100}\,u^2\Big).$$

For the second term note that $\underline{a}=u\,\big(\frac{11}{2}u^2-\frac{484}{75}u^5\big)$ and $0\le\underline{a}\le\frac{11}{2}u^3$. From $\sqrt{R}\,u\ge 1$ we get

$$\underline{a}\,\Big(2+c_*\,\Big(2\sqrt{R}-2-\frac{1}{4}\Big)\Big)\ge\underline{a}\,\Big(2-\frac{9}{4}\,c_*\Big)+2c_*\,\Big(\frac{11}{2}\,u^2-\frac{484}{75}\,u^5\Big),$$

and from $\sqrt{R}\,u\le 1+\frac{u^2}{2}$ we get

$$\underline{a}\,\Big(2+c_*\,\Big(2\sqrt{R}-2-\frac{1}{4}\Big)\Big)\le\frac{11}{2}\,u^3\,(2+2c_*\,\sqrt{R})\le 11\,u^3+11\,c_*\,u^2\Big(1+\frac{u^2}{2}\Big).$$

For the third term we have $\frac{R+1}{k+R}=u\,(h+u^2)/(\frac{5}{11}+hu)$, and

$$h+u^2\le\frac{11}{5}\,(1+2u^2)(1-w+w^2)\Big(\frac{5}{11}+hu\Big).$$

Indeed, since $h\ge 1$ and $\frac{11}{5}\big(\frac{5}{11}+u\big)=1+w$, the right-hand side is at least $(1+2u^2)(1-w+w^2)(1+w)=(1+2u^2)(1+w^3)$, and this is at least $1+2u^2\ge h+u^2$. Since $e^{\bar a}\le\frac{106}{105}$, it follows that

$$0\le\frac{e^{\bar a}\,(R+1)}{k+R}\le\frac{106}{105}\cdot\frac{11}{5}\,u\,(1+2u^2)(1-w+w^2).$$

The lower bounds of the first and the third term and the upper bound of the second give

$$\mathsf{A}\ge u\,\Big[c_*\,\Big(2-11u-\frac{11}{2}\,u^3\Big)-11\,u^2\Big],$$

and for $c_* \geq \mathfrak{c}_-$ and $0 < u \leq \frac{3}{25}$ the bracket is at least $0.39 \cdot 0.67 - 0.16 > 0$. The upper bounds of the first and the third term and the lower bound of the second give an upper bound for A that is affine in c_*, and in which the coefficient of c_* is

$$u\Big[2\Big(1+\frac{51}{100}\,u^2\Big)+\frac{9}{4}\Big(\frac{11}{2}\,u^2-\frac{484}{75}\,u^5\Big)-2\Big(\frac{11}{2}\,u-\frac{484}{75}\,u^4\Big)\Big] \geq u\,(2-11u) > 0.$$

Since $c_* \leq \mathfrak{c}_+$, this upper bound is therefore at most its value with $\mathfrak{c}_+$ in the place of c_*, which is $u\,\mathsf{A}^*(u)$.

We turn to the bound for B. As in the proof of Proposition 9.1, we bound $\log(1+t)$ for $t \geq 0$ from below by its Taylor polynomial of degree 4 and from above by that of degree 5. Put $\tau = \frac{R-1}{k+2}$ and $\tau' = \frac{R-1}{k+1}$. Then $0 \leq \tau \leq \tau' \leq w$, the last inequality because $(k+1)\,\tau' u^2 = h - u^2 \leq 1 = ku^2 w \leq (k+1)\,u^2 w$. Since $\frac{k+R+1}{k+2} = 1+\tau$ and $R-1 = (k+2)\,\tau$, the Taylor polynomial of degree 4 gives

$$(R-1) - k\log\frac{k+R+1}{k+2} = 2\tau + k\big(\tau - \log(1+\tau)\big) \leq 2\tau + k\Big(\frac{\tau^2}{2}-\frac{\tau^3}{3}+\frac{\tau^4}{4}\Big).$$

The right-hand side does not decrease in $\tau \geq 0$, as its derivative shows. We may therefore replace τ by w. Multiplying by u and using $ku^2w = 1$, we obtain

$$\begin{aligned} u\Big[(R-1) - k\log\frac{k+R+1}{k+2}\Big] &\leq 2uw + \frac{w}{2u} - \frac{w^2}{3u} + \frac{w^3}{4u} \\ &= \frac{11}{10} - \frac{121}{75}\,u + \frac{3531}{500}\,u^2. \end{aligned}$$

In the same way $\frac{k+R}{k+1} = 1+\tau'$ and $R-1 = (k+1)\,\tau'$, hence $\frac{R^2-1}{2} = (k+1)\,\tau' + \frac{(k+1)^2}{2}\,\tau'^2$. Since $k(k-1) \geq 0$, the Taylor polynomial of degree 5 gives

$$\begin{aligned} &\frac{R^2-1}{2} - (k-1)(R-1) + k(k-1)\log\frac{k+R}{k+1} \\ &\qquad\qquad \leq 2\tau' + \frac{3k+1}{2}\,\tau'^2 + k(k-1)\Big(\frac{\tau'^3}{3}-\frac{\tau'^4}{4}+\frac{\tau'^5}{5}\Big). \end{aligned}$$

The right-hand side does not decrease in $\tau' \geq 0$ either, so we may replace τ' by w, and we may then omit the term $-k\big(\frac{w^3}{3}-\frac{w^4}{4}+\frac{w^5}{5}\big) \leq 0$. Multiplying by $\frac{5u}{k} = 11\,u^4$ and using $ku^2w = 1$, we obtain

$$\begin{aligned} &\frac{5u}{k}\Big[\frac{R^2-1}{2} - (k-1)(R-1) + k(k-1)\log\frac{k+R}{k+1}\Big] \\ &\qquad\qquad \leq \frac{121}{15}\,u - \frac{1331}{100}\,u^2 + \frac{74657}{1250}\,u^3 + \frac{242}{5}\,u^5 + \frac{1331}{50}\,u^6. \end{aligned}$$

We have thus bounded u times each of the two lines in the definition of B by a polynomial in u. The sum of the two polynomials is

$$\mathsf{B}^*(u) - u^2\Big(\frac{911}{125} - \frac{74657}{1250}\,u - \frac{242}{5}\,u^3 - \frac{1331}{50}\,u^4\Big),$$

and for $0 \leq u \leq \frac{3}{25}$ the last bracket is positive: it decreases in u, and at $u = \frac{3}{25}$ it is larger than 0.03. Hence $u\,\mathsf{B} \leq \mathsf{B}^*(u)$.

For the bound on C we treat the four terms of $u\,\mathsf{C}$ one by one. First, $u\,\frac{R(R+1)}{k+R} = h\,(h+u^2)/(\frac{5}{11}+hu)$, and

$$h\,(h+u^2) \geq \lambda\Big(\frac{5}{11}+hu\Big), \qquad \text{where} \quad \lambda = \frac{11}{5}\,(1-w)\Big(1+\frac{146}{121}\,w^2\Big).$$

For $h = 1$ the right-hand side is $(1 - w^2)\big(1 + \frac{146}{121}w^2\big) = 1 + u^2 - \frac{146}{121}w^4 \le 1 + u^2$, because $\frac{25}{121}w^2 = u^2$. For $h > 1$ the difference $h\,(h + u^2) - \lambda\,(\frac{5}{11} + hu)$ of the two sides exceeds its value at $h = 1$ by $(h-1)(h+1+u^2-\lambda u)$, which is positive because $\lambda u \le \frac{11}{5}u < 1$. Hence the first term of $u\,\mathsf{C}$ is at least λ. Second, the term -1 of C contributes $-u$. Third, with $\bar a = \frac{11}{2}u^3$,

$$u\left(1 + c_*\left(2\sqrt{R-1} - 1 + \frac{2\bar a}{3}\,R\sqrt{R} + \frac{2}{5}\cdot\frac{633}{1000}\,\bar a^2 R^2\sqrt{R}\right)\right)$$
$$= u + c_*\left(2\sqrt{R-1}\,u - u + \frac{11}{3}\,u\,h\,(\sqrt{R}\,u) + \frac{121}{10}\cdot\frac{633}{1000}\,u^2h^2\,(\sqrt{R}\,u)\right),$$

and by $\sqrt{R-1}\,u \le 1$, $h \le 1 + u^2$, $\sqrt{R}\,u \le 1 + \frac{u^2}{2}$ and $c_* \le \mathfrak{c}_+$ this is at most u plus $\mathfrak{c}_+$ times the square bracket in the definition of $\mathsf{C}_*(u)$. Fourth,

$$u\cdot\frac{5\,(R-2)(R-1)\,R\,(R+1)}{24\,k^3} = \frac{1331}{600}\,u^2\,(h-2u^2)(h-u^2)\,h\,(h+u^2)$$
$$\le \frac{1331}{600}\,u^2\,(1+u^2)^2,$$

because $(h - 2u^2)(h + u^2) \le h\,(h - u^2)$ and $0 \le h\,(h - u^2) \le 1 + u^2$. Subtracting the last three bounds from the first, we obtain $\mathsf{C}_*(u) \le u\,\mathsf{C}$.

We are left with the inequality between the polynomials. Since $\mathfrak{c}_+$ is a rational number, the difference $\mathsf{C}_*(u) - u\,\mathsf{A}^*(u)\,\mathsf{B}^*(u)$ is a polynomial of degree 8 in u with rational coefficients. Expanding it in exact arithmetic, we find that it is

$$\frac{14021}{10000} - \frac{7857223}{700000}\,u - \frac{21491467261}{12600000000}\,u^2 + \frac{39724085447}{2520000000}\,u^3$$
$$- \frac{149002190689}{2800000000}\,u^4 + \frac{72772583951}{3150000000}\,u^5 - \frac{124571939481}{700000000}\,u^6$$
$$- \frac{14494390471}{196875000}\,u^7 - \frac{10711569649}{1200000000}\,u^8,$$

which, with its coefficients rounded, is

$$1.4021 - 11.22\,u - 1.706\,u^2 + 15.76\,u^3 - 53.22\,u^4$$
$$+ 23.10\,u^5 - 178.0\,u^6 - 73.62\,u^7 - 8.926\,u^8.$$

Its constant term is $\frac{11}{5} - 2\mathfrak{c}_+$, its coefficients of u^3 and u^5 are positive, and its other six coefficients are negative. For $0 \le u \le \frac{3}{25}$ the polynomial is therefore at least its constant term plus the values at $u = \frac{3}{25}$ of the six terms with a negative coefficient. These six values are $-1.34695\ldots$, $-0.02456\ldots$, $-0.01103\ldots$, $-0.00053\ldots$, $-0.00002\ldots$ and a number of absolute value less than 10^{-6}, and they add up to more than -1.38315. Hence

$$\mathsf{C}_*(u) - u\,\mathsf{A}^*(u)\,\mathsf{B}^*(u) > 1.4021 - 1.38315 > 0.018 \qquad \Big(0 \le u \le \frac{3}{25}\Big).$$

This proves the last of the four statements, and with it the inequality $\mathsf{A}\,\mathsf{B} < \mathsf{C}$.

9.5. **The crossing in the strip.** The *strip* of a column $k \ge 2$ consists of the rows $k + r$ with $r \ge 1$ and $25\,r^3 \le 121\,k^2$, i.e. with $1 \le r \le (11k/5)^{2/3}$. We now show that a column $k \ge 265$ is positive in a row of its strip or in the first row beyond it.

Proposition 9.2. *Let $k \ge 265$ be an integer. Then there is an integer j with*

$$k < j, \qquad 25\,(j-k-1)^3 \le 121\,k^2, \qquad d_{j,k} > 0.$$

Proof. Let R be the least positive integer with $121\,k^2 < 25\,R^3$ and let $u > 0$ be the real number with $k\,u^3 = \frac{5}{11}$, as in Section 9.2, so that the relations (9.11) hold. We apply Proposition 9.1 to k and R.

First, $2 \le R \le k-1$. We have seen that $R \ge 2$, and $R+1 < k$ because $Ru^2 \le 1+u^2$ and $u < \frac{3}{25}$ give

$$(R+1)\,u^3 = u \cdot Ru^2 + u^3 \le u + 2u^3 < \frac{5}{11} = k\,u^3.$$

Second, $\bar{a}\,R \le \frac{6679}{10000}$, because $\bar{a} = \frac{11}{2}u^3$, $Ru^2 \le 1+u^2$ and $u < 0.119705$ give

$$\begin{aligned}\bar{a}\,R = \frac{11}{2}\,u \cdot Ru^2 &\le \frac{11}{2}\,u\,(1+u^2)\\ &< \frac{11}{2} \cdot 0.119705 \cdot \bigl(1 + (0.119705)^2\bigr) < 0.66782 < \frac{6679}{10000}.\end{aligned}$$

Third, $\underline{t} \ge 0$, and fourth, the inequality (9.4), which is the inequality $\mathsf{A}\,\mathsf{B} < \mathsf{C}$ multiplied by e^{-1}: these are the two inequalities (9.12), which were proved in Sections 9.3 and 9.4.

Proposition 9.1 now gives an integer j with $k < j \le k+R$ and $d_{j,k} > 0$, and $25\,(j-k-1)^3 \le 25\,(R-1)^3 \le 121\,k^2$ by the minimality of R. □

Consequently, if $k \ge 265$, $n > k$ and column k is nonpositive in all rows $k+1, \dots, n$, then row n lies in the strip of the column: the integer j of the proposition is larger than n, so that $n-k \le j-k-1$.

Remark. The bound of Proposition 9.2 is not sharp. For $k = 265$ a computation with the last-car recursion shows that the crossing row is row $k+36$, while the proposition gives a positive row $j \le k+70$. The criterion behind the proposition replaces the tilted counts of level $k-1$ in the rows before the crossing by the largest of them, and it is not designed to find the exact row.

Together with the computed signs of Lemma 7.3 for the columns below 265, the proposition gives a bound for the crossing row of every column $k \ge 82$.

Corollary 9.3. *For every integer $k \ge 82$ there is an integer r with*

$$1 \le r, \qquad 100\,r \le 27\,k, \qquad d_{k+r,k} > 0.$$

Proof. Let first $82 \le k \le 264$, and let $\zeta = \zeta(k)$ be the number of Definition 7.2. By Lemma 7.3 we have $b_{k+\zeta+1,k-1} < b_{k+\zeta+1,k}$, i.e. $d_{k+r,k} > 0$ for $r = \zeta+1 \ge 1$. It remains to check that $100\,(\zeta(k)+1) \le 27\,k$. The table of Definition 7.2 shows that ζ does not decrease and that $\zeta(92) = 20$, $\zeta(133) = 24$ and $\zeta(264) = 35$. Hence $100\,(\zeta(k)+1)$ is at most 2100 for $82 \le k \le 92$, at most 2500 for $93 \le k \le 133$ and at most 3600 for $134 \le k \le 264$, and the claim follows from

$$2100 \le 27 \cdot 82, \qquad 2500 \le 27 \cdot 93, \qquad 3600 \le 27 \cdot 134.$$

Let now $k \ge 265$. Proposition 9.2 gives an integer $j > k$ with $d_{j,k} > 0$ and $25\,s^3 \le 121\,k^2$, where $s = j-k-1 \ge 0$. We take $r = j-k = s+1$. If s were larger than $0.266\,k$, we would have $25\,s^3 > 25 \cdot (0.266)^3 \cdot 265\,k^2 > 121\,k^2$, because $k \ge 265$, $(0.266)^3 \cdot 265 > 4.98$ and $\frac{121}{25} = 4.84$. Hence $s \le 0.266\,k$, and $100\,r \le 26.6\,k + 100 \le 27\,k$, because $0.4\,k \ge 106$. □

10. Persistence on the near rows

We turn to persistence: a column that has turned positive in its chain range stays positive there (Proposition 1.3). On the near rows this is Proposition 10.4: if a column $k \geq 42$ is nonpositive in rows $k, \dots, q-1$ and positive in rows $q, \dots, n-1$, where $k < q < n \leq k + 0.47k$, then it is positive in row n as well. The rows beyond $k + 0.47k$ need a different argument, which is given in Sections 11 and 12.

10.1. **A lemma on two rows.** The argument is contained in the following lemma about real sequences; in the proof of Proposition 10.4 it is applied to the column equation at the diagonal tilt. The lemma concerns a sequence η that satisfies, at a row j, an equation of the form

$$\eta_j = \sigma_j U_{j-1} + \psi_j \eta_{j-1} + \sum_{m=1}^{j-k} p_m \eta_{j-m},$$

which is the form of the column equation (cf. Proposition 4.5). In the first term U is a nonnegative sequence, and this term acts as a source at row j. The sum, a convolution of η with the kernel p, feeds the values of η in rows $k, \dots, j-1$ back into row j, the value in row $j-m$ with weight $p_m > 0$. If η is nonpositive in rows $k, \dots, q-1$ and positive in rows $q, \dots, n-1$, the feedback at row n still contains the rows before q, with values that are nonpositive and about whose size, row by row, nothing is known, so that the positivity of the source does not decide the sign of η_n. The lemma uses the equation at rows $q = k + r$ and $n = q + t$ only. For a log-convex kernel the weight of every row before q changes between these two rows by at most the factor p_{r+t}/p_r by which the weight of row k changes, and so the feedback of the rows before q at row n is controlled by their feedback at row q, where the source outweighed it.

Lemma 10.1. *Let $k \geq 0$, $r \geq 1$ and $t \geq 1$ be integers, and put $q = k + r$ and $n = q + t$. Let $\eta = (\eta_j)_{j \geq 0}$, $U = (U_j)_{j \geq 0}$ and $p = (p_m)_{m \geq 0}$ be sequences of real numbers, and let σ_q, σ_n, ψ_q and ψ_n be real numbers with $\sigma_q \geq 0$, $\psi_q \geq 0$ and $\psi_n > 0$. Suppose that the following hold.*

(1) *(equation at the two rows) We have*

$$\eta_q = \sigma_q U_{q-1} + \psi_q \eta_{q-1} + \sum_{m=1}^{r} p_m \eta_{q-m},$$
$$\eta_n = \sigma_n U_{n-1} + \psi_n \eta_{n-1} + \sum_{m=1}^{r+t} p_m \eta_{n-m}.$$

(2) *(initial value) $\eta_k = -U_k$.*
(3) *(diagonal bound) $U_{j-1} - U_j \leq \eta_j$ for every j with $k+1 \leq j \leq n-1$.*
(4) *(nonnegativity) $U_j \geq 0$ for every $j \geq 0$.*
(5) *(kernel) $p_m > 0$ for $1 \leq m \leq r+t$, and $p_{m+1} \leq p_m$ for $1 \leq m \leq r+t-1$.*
(6) *(cross inequality) $p_{m+t}\, p_r \leq p_{r+t}\, p_m$ for $1 \leq m \leq r$.*
(7) *(two-point condition) $p_{r+t}\, \sigma_q \leq p_r\, \sigma_n$.*
(8) *(signs) $\eta_j \leq 0$ for $k \leq j \leq q-1$, and $\eta_j > 0$ for $q \leq j \leq n-1$.*

Then $\eta_n > 0$.

Proof. Suppose, for a contradiction, that $\eta_n \leq 0$. Let

$$E = \sum_{j=q}^{n-1} \eta_j \qquad \text{and} \qquad H = \sum_{j=k}^{q-1} \eta_j,$$

so that $E > 0$ and $H \leq 0$ by (8). Let further

$$L = \sum_{m=1}^{t} p_m \, \eta_{n-m}, \qquad B_n = \sum_{m=1}^{r} p_{m+t} \, \eta_{q-m}, \qquad B_q = \sum_{m=1}^{r} p_m \, \eta_{q-m}.$$

Thus L is the feedback at row n of rows $q, \ldots, n-1$, and B_n (resp. B_q) is the feedback of rows $k, \ldots, q-1$ at row n (resp. at row q). Splitting the sum in the second equation of (1) at $m = t$, we may write the two equations as

$$\eta_n = \sigma_n \, U_{n-1} + \psi_n \, \eta_{n-1} + L + B_n, \qquad \eta_q = \sigma_q \, U_{q-1} + \psi_q \, \eta_{q-1} + B_q.$$

We need three comparisons of these sums:

$$L \geq p_t \, E, \qquad B_n \geq p_{t+1} \, H, \qquad p_{r+t} \, B_q \leq p_r \, B_n.$$

The first two hold because the kernel does not increase, by (5): we have $p_m \geq p_t$ and $\eta_{n-m} > 0$ for $1 \leq m \leq t$, and $p_{m+t} \leq p_{t+1}$ and $\eta_{q-m} \leq 0$ for $1 \leq m \leq r$. The third is the cross inequality (6), multiplied by $\eta_{q-m} \leq 0$ and summed over $1 \leq m \leq r$.

We also need two consequences of the diagonal bound (3):

$$U_{q-1} - U_{n-1} \leq E \qquad \text{and} \qquad -U_{q-1} \leq H.$$

The first is the sum of (3) over $q \leq j \leq n-1$, which telescopes. In the same way the sum of (3) over $k+1 \leq j \leq q-1$ is $U_k - U_{q-1} \leq H - \eta_k$, and since $\eta_k = -U_k$ by (2), this is the second. Adding the two inequalities we get

$$-(E + H) \leq U_{n-1}.$$

We first bound σ_n from above. Since $\eta_n \leq 0$ and $\psi_n \, \eta_{n-1} > 0$, the equation at row n, in the form $\eta_n = \sigma_n \, U_{n-1} + \psi_n \, \eta_{n-1} + L + B_n$, gives

$$\sigma_n \, U_{n-1} + L < -B_n.$$

Hence

$$\begin{aligned} \sigma_n \, U_{n-1} &< -B_n - L \leq -p_{t+1} \, H - p_t \, E \\ &\leq -p_{t+1} \, (E + H) \leq p_{t+1} \, U_{n-1}, \end{aligned}$$

by the comparisons $L \geq p_t \, E$ and $B_n \geq p_{t+1} \, H$, by $p_{t+1} \leq p_t$, and because $-(E + H) \leq U_{n-1}$. This excludes $U_{n-1} = 0$, so that $U_{n-1} > 0$ by (4), and

$$\sigma_n < p_{t+1}.$$

A lower bound for σ_n comes from row q. There we have $\eta_q > 0$, $\eta_{q-1} \leq 0$ and $\psi_q \geq 0$, so the equation at row q, in the form $\eta_q = \sigma_q \, U_{q-1} + \psi_q \, \eta_{q-1} + B_q$, gives $-B_q < \sigma_q \, U_{q-1}$. We transfer this inequality to row n. By the comparison $p_{r+t} \, B_q \leq p_r \, B_n$ and the inequality just obtained,

$$-p_r \, B_n \leq -p_{r+t} \, B_q < p_{r+t} \, \sigma_q \, U_{q-1},$$

and by the bound $U_{q-1} - U_{n-1} \leq E$ and the two-point condition (7),

$$p_{r+t} \, \sigma_q \, U_{q-1} \leq p_{r+t} \, \sigma_q \, (U_{n-1} + E) \leq p_r \, \sigma_n \, (U_{n-1} + E).$$

Dividing by p_r, we obtain $-B_n < \sigma_n\,(U_{n-1}+E)$. Since $\sigma_n\,U_{n-1}+L<-B_n$, as we saw in the first step, this gives $L<\sigma_n\,E$, and with $L\ge p_t\,E$ it gives $p_t\,E<\sigma_n\,E$. Since $E>0$, we conclude that

$$p_t<\sigma_n.$$

The two bounds for σ_n and (5) give $p_t<\sigma_n<p_{t+1}\le p_t$, which is impossible. Hence $\eta_n>0$. □

10.2. **The free kernel at the diagonal tilt.** Beyond its log-convexity, the application of Lemma 10.1 requires of the kernel p two monotonicity properties, which the free kernel at the diagonal tilt has on an initial range of indices. Recall from Section 2 that the free kernel ε satisfies $\varepsilon_{m+1}=g_m\,\varepsilon_m$ for $m\ge 1$, where $g_1=\frac12$ and $g_m=\frac{m}{m+1}\big(1+\frac{1}{m-1}\big)^{m-1}$ for $m\ge 2$, and from Section 8 that $\tilde\varepsilon_m=\varepsilon_m\,\vartheta_k^m$, where $\vartheta_k=1/\gamma_k$ is the diagonal tilt of column $k\ge 2$ and $\gamma_k=(1+1/k)^{k-2}$.

The two properties are that $\tilde\varepsilon_{m+1}\le\tilde\varepsilon_m$ as long as $m+1\le\frac35 k$, and, for $k\ge 42$, that $(k+\rho)\,\tilde\varepsilon_\rho$ does not increase in ρ on the shorter range $1\le\rho\le 0.47k$; the second gives the two-point condition of Lemma 10.1. We state the first for ε itself, as a bound for its growth in one step by γ_k; multiplied by ϑ_k^{m+1}, it takes the form just given.

Lemma 10.2. *Let $k\ge 2$ and $m\ge 1$ be integers with $5\,(m+1)\le 3\,k$. Then*

$$\varepsilon_{m+1}\le\gamma_k\,\varepsilon_m.$$

Proof. Since $\varepsilon_{m+1}=g_m\,\varepsilon_m$ and $\varepsilon_m\ge 0$, it suffices to show that $g_m\le\gamma_k$. For $m=1$ this holds because $g_1=\frac12$ and $\gamma_k=(1+1/k)^{k-2}\ge 1$. For $m\ge 2$ we compare the two numbers through the bounds

$$g_m\le\exp\Big(1-\frac{3}{2\,(m+1)}\Big)\qquad(m\ge 2),\tag{10.1}$$

$$\gamma_k\ge\exp\Big(1-\frac{5}{2k}\Big)\qquad(k\ge 2).\tag{10.2}$$

The second of these is the upper bound $\vartheta_k\le e^{-1}e^{5/(2k)}$ of Lemma 8.1, because $\gamma_k=1/\vartheta_k$. To prove the first, let $m\ge 2$ and put $i=m-1$, so that $i\ge 1$ and

$$\log g_m=i\,\log\Big(1+\frac1i\Big)-\log\Big(1+\frac{1}{i+1}\Big).$$

By the elementary inequalities $u-\frac{u^2}{2}\le\log(1+u)\le u-\frac{u^2}{2}+\frac{u^3}{3}$ for $u\ge 0$, of which we use the upper one at $u=1/i$ and the lower one at $u=1/(i+1)$,

$$\log g_m\le 1-\frac{1}{2i}+\frac{1}{3i^2}-\frac{1}{i+1}+\frac{1}{2\,(i+1)^2}\le 1-\frac{3}{2\,(i+2)},$$

where the second inequality holds for every $i\ge 1$: a direct computation shows that it is equivalent to $7i^3+4i^2\ge 4i+4$. Since $i+2=m+1$, this is the bound for g_m.

Finally, the hypothesis $5\,(m+1)\le 3\,k$ says that $\frac{5}{2k}\le\frac{3}{2\,(m+1)}$, so that for $m\ge 2$ the two bounds give $g_m\le\gamma_k$. □

Lemma 10.3. *Let $k\ge 42$, r and s be integers with $1\le r\le s$ and $100\,s\le 47\,k$. Then the free kernel at the diagonal tilt of column k satisfies*

$$(k+s)\,\tilde\varepsilon_s\le(k+r)\,\tilde\varepsilon_r.$$

Proof. It suffices to prove that, for every integer $\rho \geq 1$ with $100\,(\rho+1) \leq 47\,k$,

$$(k+\rho+1)\,\tilde{\varepsilon}_{\rho+1} \leq (k+\rho)\,\tilde{\varepsilon}_{\rho}.$$

Indeed, the lemma follows by applying this inequality for $\rho = r, r+1, \ldots, s-1$.

Let ρ be such an integer. Since $\varepsilon_{\rho+1} = g_\rho\,\varepsilon_\rho$ and $\vartheta_k\gamma_k = 1$, the inequality to be proved is

$$(k+\rho+1)\,g_\rho \leq (k+\rho)\,\gamma_k,$$

multiplied by $\varepsilon_\rho\,\vartheta_k^{\rho+1} \geq 0$. For $\rho = 1$ the displayed inequality holds because $g_1 = \frac{1}{2}$ and $\gamma_k \geq 1$.

Let $\rho \geq 2$. We first show that

$$\frac{1}{k+\rho} \leq \frac{3}{2\,(\rho+1)} - \frac{5}{2k}.$$

A direct computation shows that this inequality is equivalent to $5\rho^2 + (4k+5)\,\rho + 7k \leq 3k^2$. The left-hand side of the latter increases with ρ, and $\rho \leq 0.47k - 1$ by the hypothesis on ρ. At $\rho = 0.47k - 1$ it equals $2.9845\,k^2 + 0.65\,k$, which is at most $3k^2$ because $0.0155\,k \geq 0.651$ for $k \geq 42$.

By the inequality $1 + y \leq e^y$ for real y, by the inequality just proved, and by the bounds (10.1) for g_ρ and (10.2) for γ_k from the proof of Lemma 10.2, we obtain

$$\begin{aligned}\Big(1 + \frac{1}{k+\rho}\Big)\,g_\rho &\leq \exp\Big(\frac{1}{k+\rho}\Big)\,g_\rho \leq \exp\Big(\frac{3}{2\,(\rho+1)} - \frac{5}{2k}\Big)\exp\Big(1 - \frac{3}{2\,(\rho+1)}\Big)\\ &= \exp\Big(1 - \frac{5}{2k}\Big) \leq \gamma_k.\end{aligned}$$

Multiplied by $k+\rho$, this is the inequality $(k+\rho+1)\,g_\rho \leq (k+\rho)\,\gamma_k$ that remained to be proved. □

10.3. **Rows up to $k + 0.47k$.** We now apply Lemma 10.1 to a column at its diagonal tilt. In the following proposition q is the crossing row of column k; the case $t = 1$ is included, so that the proposition applies from the first row after the crossing row on.

Proposition 10.4. *Let $k \geq 42$, $r \geq 1$ and $t \geq 1$ be integers with $100\,(r+t) \leq 47\,k$, and put $q = k + r$ and $n = q + t$. Suppose that*

(1) *$d_{j,k} \leq 0$ for every j with $k \leq j < q$, and*
(2) *$d_{j,k} > 0$ for every j with $q \leq j < n$.*

Then $d_{n,k} > 0$.

Proof. Let $\delta_j = d_{j,k}/j!$ be the normalised step of column k, and let $\vartheta_k = 1/\gamma_k$ be its diagonal tilt. We apply Lemma 10.1, with the given integers k, r and t, to the sequences

$$\eta_j = \tilde{\delta}_j = \delta_j\,\vartheta_k^j, \qquad U_j = \tilde{\beta}_j^{(k-1)} = \beta_j^{(k-1)}\,\vartheta_k^j, \qquad p_m = \tilde{\varepsilon}_m = \varepsilon_m\,\vartheta_k^m$$

and to the numbers

$$\sigma_q = \frac{2\vartheta_k}{q}, \qquad \sigma_n = \frac{2\vartheta_k}{n}, \qquad \psi_q = \frac{(k+1)\,\vartheta_k}{q}, \qquad \psi_n = \frac{(k+1)\,\vartheta_k}{n},$$

all of which are positive. Each η_j has the sign of $d_{j,k}$, so that hypothesis (8) of the lemma consists of the two assumptions of the proposition, and its conclusion $\eta_n > 0$ is the claim. It remains to verify hypotheses (1) to (7) of the lemma.

For (1), note that rows q and n lie in the chain range $k+1 \le j \le 2k-1$ of column k, because $r+t \le 0.47k < k$. For a row j of the chain range, the equation of Proposition 4.5, multiplied by ϑ_k^j, reads

$$\tilde{\delta}_j = \frac{2\vartheta_k}{j}\,\tilde{\beta}_{j-1}^{(k-1)} + \frac{(k+1)\,\vartheta_k}{j}\,\tilde{\delta}_{j-1} + \sum_{m=1}^{j-k} \tilde{\varepsilon}_m\,\tilde{\delta}_{j-m},$$

and for $j=q$ and $j=n$ these are the two equations of (1).

Hypothesis (2) holds because $\beta_k^{(k)} = 0$ (Section 3), so that $\delta_k = -\beta_k^{(k-1)}$. For (3), Proposition 5.9 gives $\gamma_k\,\beta_{j-1}^{(k-1)} \le \beta_j^{(k)}$ for every $j \ge 1$. Multiplying by ϑ_k^j, which removes the factor γ_k since $\vartheta_k\gamma_k = 1$, and subtracting U_j on both sides, we get

$$U_{j-1} - U_j \le \bigl(\beta_j^{(k)} - \beta_j^{(k-1)}\bigr)\,\vartheta_k^j = \eta_j.$$

Hypothesis (4) holds because $\beta_j^{(k-1)} = b_{j,k-1}/j! \ge 0$.

For (5), the p_m with $m \ge 1$ are positive because the ε_m are. For $1 \le m \le r+t-1$ we have $5\,(m+1) \le 5\,(r+t) \le 3k$, because $100\,(r+t) \le 47\,k$, so that Lemma 10.2 gives $\varepsilon_{m+1} \le \gamma_k\,\varepsilon_m$; multiplied by ϑ_k^{m+1}, this is $p_{m+1} \le p_m$.

For (6), the sequence $(\varepsilon_m)_{m\ge1}$ is log-convex (Lemma 2.8), so that the ratios $\varepsilon_{m+1}/\varepsilon_m$ do not decrease in m. Hence $\varepsilon_{m+t}/\varepsilon_m \le \varepsilon_{r+t}/\varepsilon_r$ for $1 \le m \le r$, since each side is a product of t consecutive ratios, and multiplying by $\varepsilon_m\,\varepsilon_r\,\vartheta_k^{m+t+r}$ we obtain $p_{m+t}\,p_r \le p_{r+t}\,p_m$.

Finally, the two-point condition (7), multiplied by $qn/(2\vartheta_k)$, becomes $(k+r+t)\,\tilde{\varepsilon}_{r+t} \le (k+r)\,\tilde{\varepsilon}_r$. This is Lemma 10.3 with $s = r+t$, whose hypotheses $k \ge 42$ and $100\,s \le 47\,k$ are those of the proposition. Thus Lemma 10.1 applies, and it gives $\eta_n > 0$, i.e. $d_{n,k} > 0$. □

11. A comparison with renewal sequences

On the far rows of a column, the lower bound for the step with which Section 12 works contains terms whose coefficients may be negative. They are bounded there by a comparison which we prove here for arbitrary sequences: for a sequence v about which we know only that $\sum_{i=1}^{s} f_i\,v_{s-i} \le v_s$ for $s \le \rho$, where f is a log-convex kernel, a weighted sum of terms $f_m v_{\rho-m}$ is at most a multiple of v_ρ that does not depend on v (Proposition 11.4). We then make this multiple explicit for the kernel to which the comparison is applied.

11.1. **Renewal sequences of log-convex kernels.** The sequence with which v is compared is the renewal sequence of the kernel.

Definition 11.1. Let $f = (f_i)_{i\ge0}$ and $u = (u_n)_{n\ge0}$ be sequences of real numbers. We say that u is *the renewal sequence of the kernel* f if

$$u_0 = 1 \qquad \text{and} \qquad u_n = \sum_{i=1}^{n} f_i\,u_{n-i} \quad \text{for every } n \ge 1.$$

The second condition is the *renewal equation* at index n. The term f_0 does not occur in it, and the two conditions determine u from f term by term: $u_1 = f_1$, $u_2 = f_1^2 + f_2$, and so on. For example, (c_n) is the renewal sequence of the free kernel (ε_m), by Corollary 2.7 and since $c_0 = 1$. We shall use repeatedly that the renewal sequence u of a kernel f with positive terms is positive:

(11.1) if $f_i > 0$ for every $i \ge 1$, then $u_n > 0$ for every $n \ge 0$,

as the renewal equation shows by induction on n.

For a kernel f and a sequence $v = (v_s)_{s\ge 0}$ of real numbers, the numbers

$$\iota_s = v_s - \sum_{i=1}^{s} f_i\, v_{s-i} \qquad (s \ge 0)$$

are called the *innovations* of v with respect to f. Thus ι_s is what v_s contains beyond the feedback of the earlier values through the kernel, and $\iota_0 = v_0$. For an integer $\rho \ge 0$ we say that v *has nonnegative innovations with respect to f up to ρ* if $\iota_s \ge 0$ for every $s \le \rho$, i.e. if

$$\sum_{i=1}^{s} f_i\, v_{s-i} \le v_s \qquad \text{for every integer } s \text{ with } 0 \le s \le \rho.$$

In these terms the renewal sequence of f is the sequence whose innovations are $\iota_0 = 1$ and $\iota_s = 0$ for $s \ge 1$.

Lemma 11.2. *Let f, G, v and ι be sequences of real numbers, indexed by the integers $s \ge 0$. Suppose that G is the renewal sequence of the kernel f (Definition 11.1), and that*

$$v_s = \iota_s + \sum_{i=1}^{s} f_i\, v_{s-i} \qquad \textit{for every } s \ge 0.$$

Then

$$v_s = \sum_{t=0}^{s} G_{s-t}\, \iota_t \qquad \textit{for every } s \ge 0.$$

Proof. We argue by induction on s and assume the formula for all indices smaller than s. Inserting it for v_{s-i} with $1 \le i \le s$ and interchanging the two summations, we obtain

$$\sum_{i=1}^{s} f_i\, v_{s-i} = \sum_{t=0}^{s-1}\Big(\sum_{i=1}^{s-t} f_i\, G_{s-t-i}\Big)\iota_t = \sum_{t=0}^{s-1} G_{s-t}\, \iota_t,$$

where the inner sum equals G_{s-t} by the renewal equation at index $s-t \ge 1$. Adding $\iota_s = G_0\, \iota_s$ gives $v_s = \sum_{t=0}^{s} G_{s-t}\, \iota_t$. □

We use the log-convexity of a sequence of positive numbers in cross-multiplied form, and we need it also for a sequence of which only an initial segment is known to be log-convex. Let $(q_n)_{n\ge 0}$ be a sequence of positive real numbers, and let $N \ge 0$ be an integer such that $q_{n+1}^2 \le q_n\, q_{n+2}$ for every $n < N$. Then

$$q_{i+1}\, q_j \le q_i\, q_{j+1} \qquad (0 \le i \le j \le N). \tag{11.2}$$

Indeed, the hypothesis says that q_{n+1}/q_n does not decrease for $0 \le n \le N$, and the inequality compares these ratios at $n = i$ and $n = j$. If $q_{n+1}^2 \le q_n\, q_{n+2}$ holds for every $n \ge 0$, then moreover

$$q_{i+m}\, q_j \le q_i\, q_{j+m} \qquad (0 \le i \le j,\ m \ge 0), \tag{11.3}$$

because q_{i+m}/q_i is the product of the m ratios q_{i+t+1}/q_{i+t} with $0 \le t < m$, and each of them is at most the corresponding ratio q_{j+t+1}/q_{j+t}.

The comparison rests on the classical theorem that log-convexity passes from a kernel to its renewal sequence. For a kernel of sum 1 this is Theorem 1 of de Bruijn and Erdős [6, pp. 161–162], who state it with strict inequalities and remark that it holds with inequalities that are not strict as well. We need it for kernels whose

sum is not 1, such as the kernel of Section 11.2. The proof in [6] does not use that the sum is 1, and since the theorem is not stated without this condition there, we repeat the proof in our notation.

Proposition 11.3. *Let $f = (f_i)_{i\geq 0}$ and $u = (u_n)_{n\geq 0}$ be sequences of real numbers such that u is the renewal sequence of the kernel f (Definition 11.1). Suppose that $f_i > 0$ for every $i \geq 1$ and that $f_{j+2}^2 \leq f_{j+1}\, f_{j+3}$ for every $j \geq 0$, i.e. that the sequence $f_1, f_2, f_3, \ldots$ is log-convex. Then*

$$u_{n+1}^2 \leq u_n\, u_{n+2} \qquad \text{for every } n \geq 0.$$

Proof. By (11.1) all u_n are positive. We prove the inequality by induction on n.

For $n = 0$ the claim is $f_1^2 \leq f_1^2 + f_2$, since $u_1 = f_1$ and $u_2 = f_1^2 + f_2$.

Let $n \geq 1$, and suppose that $u_{j+1}^2 \leq u_j\, u_{j+2}$ for every $j < n$. We apply the cross-multiplied form (11.2) of log-convexity twice. For the sequence $q_j = f_{j+1}$, which is positive and log-convex by the hypothesis on the kernel, it gives

$$f_{i+1}\, f_{n+1} \leq f_i\, f_{n+2} \qquad (1 \leq i \leq n).$$

For u and $N = n$, where the hypothesis on the sequence is the induction hypothesis, it gives

$$u_{n-i+1}\, u_n \leq u_{n-i}\, u_{n+1} \qquad (1 \leq i \leq n).$$

Hence both factors of every term of the sum

$$S = \sum_{i=1}^{n} \big(f_i\, f_{n+2} - f_{i+1}\, f_{n+1}\big)\big(u_{n-i}\, u_{n+1} - u_{n-i+1}\, u_n\big)$$

are nonnegative, and $S \geq 0$.

On the other hand, when the product is multiplied out, four sums over i arise, and the renewal equation at indices n, $n+1$ and $n+2$ evaluates them:

$$\sum_{i=1}^{n} f_i\, u_{n-i} = u_n, \qquad \sum_{i=1}^{n} f_i\, u_{n-i+1} = u_{n+1} - f_{n+1},$$

$$\sum_{i=1}^{n} f_{i+1}\, u_{n-i} = u_{n+1} - f_1 u_n, \qquad \sum_{i=1}^{n} f_{i+1}\, u_{n-i+1} = u_{n+2} - f_1 u_{n+1} - f_{n+2}.$$

Inserting these values, we find that

$$S = f_{n+1}\big(u_n\, u_{n+2} - u_{n+1}^2\big),$$

since all other terms cancel in pairs. As $S \geq 0$ and $f_{n+1} > 0$, we conclude that $u_{n+1}^2 \leq u_n\, u_{n+2}$. □

We now come to the main result of this subsection. Let f be a kernel as in Proposition 11.3, let G be its renewal sequence, and let v have nonnegative innovations with respect to f up to ρ. For an integer $M \geq 1$ and weights θ_m, the sum $\sum_{m=M}^{\rho} \theta_m f_m v_{\rho-m}$ is the part with $m \geq M$ of the feedback $\sum_{i=1}^{\rho} f_i v_{\rho-i}$ into v_ρ, taken with these weights. The following proposition says that, if the weights are nonnegative and do not decrease, this part is largest in proportion to v_ρ when $v = G$, that is, for the sequence whose only innovation occurs at index 0.

Proposition 11.4. *Let f, G, v and θ be sequences of real numbers, indexed by the integers $n \geq 0$, and let $\rho \geq 0$ and $M \geq 1$ be integers. Assume that*

(1) *$f_i > 0$ and $f_{i+1}^2 \leq f_i\, f_{i+2}$ for every $i \geq 1$;*

(2) *G is the renewal sequence of the kernel f (Definition 11.1);*
(3) *$\theta_M \geq 0$, and $\theta_m \leq \theta_{m'}$ whenever $M \leq m \leq m' \leq \rho$;*
(4) *$\sum_{i=1}^{s} f_i\, v_{s-i} \leq v_s$ for every integer s with $0 \leq s \leq \rho$.*

Then

$$\Big(\sum_{m=M}^{\rho} \theta_m\, f_m\, v_{\rho-m}\Big)\, G_\rho \leq \Big(\sum_{m=M}^{\rho} \theta_m\, f_m\, G_{\rho-m}\Big)\, v_\rho.$$

Proof. By (1) and (2), the sequence G is positive, by (11.1), and it satisfies $G_{n+1}^2 \leq G_n\, G_{n+2}$ for every $n \geq 0$, by Proposition 11.3. Hence the inequality (11.3), applied to G with indices $s - m \leq \rho - m$ and shift m, gives

$$G_s\, G_{\rho-m} \leq G_{s-m}\, G_\rho \qquad (0 \leq m \leq s \leq \rho).$$

For $0 \leq s \leq \rho$ let

$$\mathsf{F}_s = \sum_{m=M}^{s} \theta_m\, f_m\, G_{s-m},$$

so that $\mathsf{F}_s = 0$ for $s < M$ and the sum on the right-hand side of the claim is F_ρ. By (3) we have $\theta_m \geq \theta_M \geq 0$ for $M \leq m \leq \rho$, and therefore $\mathsf{F}_s \geq 0$ for every s.

We begin with the inequality in which the weights θ_m are left out: for $M \leq s \leq \rho$ we have

$$\Big(\sum_{m=M}^{s} f_m\, G_{s-m}\Big)\, G_\rho \leq \Big(\sum_{m=M}^{\rho} f_m\, G_{\rho-m}\Big)\, G_s. \tag{11.4}$$

Indeed, by the renewal equation at indices s and ρ, the two sums in brackets equal $G_s - \sum_{m=1}^{M-1} f_m G_{s-m}$ and $G_\rho - \sum_{m=1}^{M-1} f_m G_{\rho-m}$, so that the inequality to be shown is equivalent to

$$\sum_{m=1}^{M-1} f_m\, G_s\, G_{\rho-m} \leq \sum_{m=1}^{M-1} f_m\, G_{s-m}\, G_\rho,$$

which holds term by term, because $G_s\, G_{\rho-m} \leq G_{s-m}\, G_\rho$ for $0 \leq m \leq s$, as we saw above.

Next we show that the same holds with the weights, i.e. that

$$\mathsf{F}_s\, G_\rho \leq \mathsf{F}_\rho\, G_s \qquad (0 \leq s \leq \rho). \tag{11.5}$$

For $s < M$ the left-hand side is 0 and the right-hand side is nonnegative. Let $M \leq s \leq \rho$. We split F_ρ into the terms with $m \leq s$ and those with $m > s$, which gives

$$\begin{aligned}\mathsf{F}_\rho\, G_s - \mathsf{F}_s\, G_\rho = &\sum_{m=s+1}^{\rho} \theta_m\, f_m\, G_{\rho-m}\, G_s \\ &- \sum_{m=M}^{s} \theta_m\, f_m \big(G_{s-m}\, G_\rho - G_s\, G_{\rho-m}\big).\end{aligned}$$

In the second sum we have $\theta_m \le \theta_s$ for $M \le m \le s$ by (3), and every difference in brackets is nonnegative, again because $G_s\, G_{\rho-m} \le G_{s-m}\, G_\rho$ for $0 \le m \le s$. Hence

$$\sum_{m=M}^{s} \theta_m\, f_m \big(G_{s-m}\, G_\rho - G_s\, G_{\rho-m}\big) \le \theta_s \sum_{m=M}^{s} f_m \big(G_{s-m}\, G_\rho - G_s\, G_{\rho-m}\big) \le \theta_s \sum_{m=s+1}^{\rho} f_m\, G_{\rho-m}\, G_s,$$

where the second inequality is (11.4), with the terms $M \le m \le s$ of its right-hand side moved to the left. In the first sum we have $\theta_m \ge \theta_s$ for $s < m \le \rho$, again by (3), so that

$$\sum_{m=s+1}^{\rho} \theta_m\, f_m\, G_{\rho-m}\, G_s \ge \theta_s \sum_{m=s+1}^{\rho} f_m\, G_{\rho-m}\, G_s.$$

The last two displays together show that $\mathsf{F}_\rho\, G_s - \mathsf{F}_s\, G_\rho \ge 0$.

It remains to pass from G to v. Let $\iota_s = v_s - \sum_{i=1}^{s} f_i v_{s-i}$ for $s \ge 0$ be the innovations of v. By (4) we have $\iota_s \ge 0$ for $s \le \rho$, and by Lemma 11.2 we have $v_s = \sum_{t=0}^{s} G_{s-t}\, \iota_t$ for every $s \ge 0$. Inserting this formula for $v_{\rho-m}$ and interchanging the two summations, we obtain

$$\sum_{m=M}^{\rho} \theta_m\, f_m\, v_{\rho-m} = \sum_{t=0}^{\rho} \iota_t \sum_{m=M}^{\rho-t} \theta_m\, f_m\, G_{\rho-t-m} = \sum_{t=0}^{\rho} \iota_t\, \mathsf{F}_{\rho-t}.$$

Multiplying by G_ρ and applying the inequality (11.5) with $s = \rho - t$ to each term, where $\iota_t \ge 0$, we obtain

$$\Big(\sum_{m=M}^{\rho} \theta_m\, f_m\, v_{\rho-m}\Big)\, G_\rho = \sum_{t=0}^{\rho} \iota_t\, \mathsf{F}_{\rho-t}\, G_\rho \le \mathsf{F}_\rho \sum_{t=0}^{\rho} G_{\rho-t}\, \iota_t = \mathsf{F}_\rho\, v_\rho. \qquad \square$$

11.2. **An explicit kernel and its renewal sequence.** We now turn to the kernel to which Section 12 applies Proposition 11.4. The *frozen kernel* is the sequence

$$f_0 = 0, \qquad f_1 = \tfrac{3}{2}\, e^{-1}, \qquad f_i = e^{-i}\, \varepsilon_i \quad (i \ge 2),$$

where (ε_i) is the free kernel of Section 2. Since $\varepsilon_0 = 0$ and $\varepsilon_1 = 1$, this can be written as

$$f_i = e^{-i}\, \varepsilon_i + \begin{cases} \frac{1}{2}\, e^{-1} & \text{if } i = 1, \\ 0 & \text{otherwise} \end{cases} \qquad (i \ge 0).$$

The factor e^{-i} is there because $\sum_m e^{-m} \varepsilon_m = 1 - e^{-1}$ by Lemma 2.9, so that the frozen kernel has sum $1 - \frac{1}{2} e^{-1} < 1$; the summand $\frac{1}{2} e^{-1}$ at $i = 1$ takes the place of a coefficient of the level equation (Lemma 4.3) that is at least $\frac{1}{2}$ in the rows to which Section 12 applies the bound. From here to the end of the section f denotes the frozen kernel. We have $f_i > 0$ for every $i \ge 1$, because $\varepsilon_i > 0$ for $i \ge 1$.

The renewal sequence of f is given by a closed formula, from which the lower bound of Lemma 11.7 will be obtained, and we therefore define a sequence G by this formula. For integers $j \ge 0$ and $s \ge 0$ let

$$h_{j,s} = [X^s]\, \big(X^j C^{j+1}\big)$$

be the coefficient of X^s in the power series $X^j C^{j+1}$, where $C = \sum_{n\ge 0} c_n X^n$ is the series of Section 2, and for $s \ge 0$ let

$$H_s = \sum_{j=0}^{s} 2^{-j}\, h_{j,s}, \qquad G_s = e^{-s} H_s = e^{-s} \sum_{j=0}^{s} 2^{-j}\, [X^s]\, X^j C^{j+1}. \tag{11.6}$$

From here to the end of the section G denotes the sequence $(G_s)_{s\ge 0}$ defined in this way. Since $X^j C^{j+1}$ is divisible by X^j, and since $[X^s]C = c_s$, we have

$$h_{j,s} = 0 \quad \text{for } s < j, \qquad \text{and} \qquad h_{0,s} = c_s.$$

In particular the sum over j that defines H_s may be extended to any range $0 \le j \le N$ with $N \ge s$ without changing its value. The first values are $G_0 = 1$, $G_1 = \frac{3}{2}e^{-1}$ and $G_2 = \frac{11}{4}e^{-2}$.

The formula for G_s is found from generating series. By Lemma 2.6 with $F = 1$ we have $1/C = 1 - \sum_{m\ge 1} \varepsilon_m X^m$. Before the scaling by e^{-s}, the renewal equation of the frozen kernel reads $H_s = \sum_{m=1}^{s} \varepsilon_m H_{s-m} + \frac{1}{2} H_{s-1}$ for $s \ge 1$, with $H_0 = 1$; it says that the series $\sum_s H_s X^s$ is the inverse of $1/C - X/2$, and

$$\frac{1}{1/C - X/2} = \frac{C}{1 - XC/2} = \sum_{j\ge 0} \Big(\frac{X}{2}\Big)^j C^{j+1},$$

where only the terms with $j \le s$ contribute to the coefficient of X^s. In the proof of the following lemma we verify the renewal equation for the coefficients directly.

Lemma 11.5. *The sequence G defined by* (11.6) *is the renewal sequence of the frozen kernel f in the sense of Definition 11.1: we have $G_0 = 1$ and, for every integer $n \ge 1$,*

$$G_n = \sum_{i=1}^{n} f_i\, G_{n-i}.$$

Proof. Let $h_{j,s}$ and H_s be as in the definition of G. Since $h_{0,s} = c_s$, we have $H_0 = h_{0,0} = c_0 = 1$, and hence $G_0 = 1$.

Let $j \ge 0$ and $s \ge 1$. Lemma 2.6, applied to $F = X^j C^{j+1}$, gives

$$[X^s]\Big(\frac{1}{C}\, X^j C^{j+1}\Big) = h_{j,s} - \sum_{m=1}^{s} \varepsilon_m\, h_{j,s-m}.$$

The series on the left is $X^j C^j$. Its coefficient of X^s is 0 for $j = 0$, because $s \ge 1$, and it is $h_{j-1,s-1}$ for $j \ge 1$, because $X^j C^j = X \cdot X^{j-1} C^j$. Thus

$$h_{0,s} = \sum_{m=1}^{s} \varepsilon_m\, h_{0,s-m} \qquad \text{and} \qquad h_{j,s} = \sum_{m=1}^{s} \varepsilon_m\, h_{j,s-m} + h_{j-1,s-1} \quad (j \ge 1).$$

We multiply the identity for j by 2^{-j} and sum over $0 \le j \le s$. On the left we obtain H_s. On the right, since $h_{j,s-m} = 0$ for $j > s-m$, we have $\sum_{j=0}^{s} 2^{-j} h_{j,s-m} = H_{s-m}$ for $1 \le m \le s$; the remaining terms add up to

$$\sum_{j=1}^{s} 2^{-j}\, h_{j-1,s-1} = \frac{1}{2} \sum_{j=0}^{s-1} 2^{-j}\, h_{j,s-1} = \frac{1}{2}\, H_{s-1}.$$

Hence

$$H_s = \sum_{m=1}^{s} \varepsilon_m\, H_{s-m} + \tfrac{1}{2}\, H_{s-1} \qquad (s \ge 1).$$

Multiplied by e^{-s}, this equation is $G_s = \sum_{i=1}^{s} f_i\, G_{s-i}$, because $f_1 = e^{-1}\varepsilon_1 + \frac{1}{2}e^{-1}$ and $f_i = e^{-i}\varepsilon_i$ for $i \geq 2$. □

We turn to the two estimates for G through which the bound of Proposition 11.4 becomes explicit: an upper bound for the partial sums of G, and a lower bound of order $s^{-3/2}$ for a single term G_s. The lower bound is needed because in that proposition G_ρ is a factor on the side of the sum that is to be bounded.

Lemma 11.6. *For every integer $N \geq 0$, the renewal sequence G of the frozen kernel satisfies*

$$\sum_{s=0}^{N-1} G_s \leq 2e.$$

Proof. We apply Lemma 11.2 to the constant sequence 1. For this we first bound the partial sums of the frozen kernel f: for every integer $s \geq 0$,

$$\sum_{i=1}^{s} f_i \leq 1 - \tfrac{1}{2}\, e^{-1}.$$

Indeed, by the definition of the frozen kernel the sum on the left is at most $\frac{1}{2}e^{-1} + \sum_{i=1}^{s} e^{-i}\varepsilon_i$, and the partial sums of the series of the nonnegative numbers $e^{-i}\varepsilon_i$ are at most its sum, which is $1 - e^{-1}$ by Lemma 2.9.

Without loss of generality, $N \geq 1$. The constant sequence $v_s = 1$ satisfies $v_s = \iota_s + \sum_{i=1}^{s} f_i v_{s-i}$ for every $s \geq 0$, with $\iota_s = 1 - \sum_{i=1}^{s} f_i$. Since G is the renewal sequence of f by Lemma 11.5, Lemma 11.2 applies, and at index $N-1$ it gives

$$1 = \sum_{t=0}^{N-1} G_{N-1-t}\, \iota_t.$$

Here $\iota_t \geq \frac{1}{2}e^{-1}$ by the bound for the partial sums of f, and $G_{N-1-t} > 0$ by (11.1), since $f_i > 0$ for $i \geq 1$. Hence

$$1 \geq \tfrac{1}{2}\, e^{-1} \sum_{t=0}^{N-1} G_{N-1-t} = \tfrac{1}{2}\, e^{-1} \sum_{s=0}^{N-1} G_s,$$

which is the claim. □

Lemma 11.7. *For every integer $s \geq 6$, the renewal sequence G of the frozen kernel satisfies*

$$G_s \geq \frac{e}{s+1} \cdot \frac{1 - \dfrac{1}{12(s+1)}}{\sqrt{2\pi(s+1)}} \cdot \Big(4 - \frac{21}{s+1}\Big).$$

Proof. Let $h_{j,s}$ and H_s be as in the definition (11.6) of G, so that $G_s = e^{-s}H_s$. Recall that the weights of Lemma 2.3 are $w_0(x) = 1$ and $w_n(x) = x\,(x+n)^{n-1}$ for $n \geq 1$.

We first bring the identity of Abel into convolution form: for every integer $n \geq 1$ and all real numbers x and y,

$$\sum_{i=0}^{n} \binom{n}{i} w_i(x)\, w_{n-i}(y) = (x+y)\,(x+y+n)^{n-1}.$$

To see this, put $L = \sum_{i=0}^{n-1} \binom{n}{i} w_i(x)\,(y+n-i)^{n-1-i}$. Since $w_{n-i}(y) = y\,(y+n-i)^{n-1-i}$ for $i \le n-1$ and $w_0(y) = 1$, the left-hand side of the identity to be proved is $yL + w_n(x)$. On the other hand, Lemma 2.3 with $y+n$ in the place of y gives

$$\sum_{i=0}^{n} \binom{n}{i} w_i(x)\,(y+n-i)^{n-i} = (x+y+n)^n.$$

The term with $i = n$ on the left is again $w_n(x)$. In the terms with $i \le n-1$ we write $(y+n-i)^{n-i} = \big(y+(n-i)\big)(y+n-i)^{n-1-i}$ and use that $(n-i)\binom{n}{i} = n\binom{n-1}{i}$. The identity then reads

$$yL + n\sum_{i=0}^{n-1} \binom{n-1}{i} w_i(x)\,(y+n-i)^{n-1-i} + w_n(x) = (x+y+n)^n,$$

and the sum in the middle equals $(x+y+n)^{n-1}$ by Lemma 2.3 for $n-1$, again with $y+n$ in the place of y. Therefore

$$yL + w_n(x) = (x+y+n)^n - n\,(x+y+n)^{n-1} = (x+y)\,(x+y+n)^{n-1}.$$

With the convolution form we can compute the coefficients $h_{j,s}$. We claim that, for all integers $j \ge 0$ and $n \ge 0$,

$$[X^n]\,C^{j+1} = \frac{w_n(j+1)}{n!}.$$

The proof is by induction on j. For $j = 0$ the claim is the identity $c_n = w_n(1)/n!$ of Section 2. If the claim holds for j, then the formula for the coefficients of a product gives

$$[X^n]\,C^{j+2} = \sum_{i=0}^{n} [X^i]\,C^{j+1} \cdot c_{n-i} = \frac{1}{n!}\sum_{i=0}^{n} \binom{n}{i} w_i(j+1)\,w_{n-i}(1).$$

For $n \ge 1$ this is $(j+2)\,(j+2+n)^{n-1}/n! = w_n(j+2)/n!$ by the convolution form of the identity of Abel with $x = j+1$ and $y = 1$, and for $n = 0$ it is $1 = w_0(j+2)$. This proves the claim.

For $j \le s$ we have $h_{j,s} = [X^s]\,(X^j C^{j+1}) = [X^{s-j}]\,C^{j+1}$, and the formula just proved gives

$$h_{j,s} = \frac{w_{s-j}(j+1)}{(s-j)!} = \frac{(j+1)\,(s+1)^{s-j-1}}{(s-j)!} \qquad (0 \le j \le s),$$

where for $j = s$ both expressions are 1. Dividing by $c_s = (s+1)^{s-1}/s!$ and writing $s!/(s-j)! = \prod_{i=1}^{j}(s+1-i)$, we obtain the closed form

$$h_{j,s} = c_s\,(j+1)\prod_{i=1}^{j}\Big(1 - \frac{i}{s+1}\Big) \qquad (0 \le j \le s).$$

We now bound the $h_{j,s}$ from below. For real numbers $t_1, \dots, t_j$ in $[0,1]$ we have $\prod_{i=1}^{j}(1-t_i) \ge 1 - \sum_{i=1}^{j} t_i$, by a routine induction on j. Applied to $t_i = i/(s+1)$ with $1 \le i \le j$, whose sum is $j(j+1)/(2(s+1))$, this turns the closed form into

$$h_{j,s} \ge c_s\,(j+1)\Big(1 - \frac{j(j+1)}{2(s+1)}\Big) \qquad (0 \le j \le s).$$

For the summation over j we use the two formulas

$$\sum_{j=0}^{s} \frac{j+1}{2^j} = 4 - \frac{s+3}{2^s}, \qquad \sum_{j=0}^{s} \frac{j\,(j+1)^2}{2^j} = 40 - \frac{s^3+8s^2+27s+40}{2^s},$$

which hold for every integer $s \geq 0$ and are easily proved by induction on s. Multiplying the lower bound for $h_{j,s}$ by 2^{-j} and summing over $0 \leq j \leq s$, we obtain

$$H_s \geq c_s\Big(4 - \frac{s+3}{2^s}\Big) - \frac{c_s}{2(s+1)}\Big(40 - \frac{s^3+8s^2+27s+40}{2^s}\Big).$$

A direct computation shows that the right-hand side equals

$$c_s\Big(4 - \frac{21}{s+1}\Big) + \frac{c_s}{s+1}\Big(1 + \frac{s^3+6s^2+19s+34}{2^{s+1}}\Big),$$

and the second term of this expression is positive. Hence

$$H_s \geq c_s\Big(4 - \frac{21}{s+1}\Big).$$

Finally, we insert the lower Stirling bound for c_s. Since $c_s = (s+1)^{s-1}/s! = (s+1)^s/(s+1)!$, the lower bound in (2.7) at $n = s+1$ gives

$$e^{-s}c_s = \frac{e}{s+1} \cdot e^{-(s+1)}\,\frac{(s+1)^{s+1}}{(s+1)!} \geq \frac{e}{s+1} \cdot \frac{1 - \dfrac{1}{12(s+1)}}{\sqrt{2\pi(s+1)}}.$$

Since $s \geq 6$, the factor $4 - \frac{21}{s+1}$ is positive. So we may multiply the inequality $H_s \geq c_s\left(4 - \frac{21}{s+1}\right)$ by e^{-s} and insert the last display, which gives

$$G_s = e^{-s}H_s \geq e^{-s}c_s\Big(4 - \frac{21}{s+1}\Big) \geq \frac{e}{s+1} \cdot \frac{1 - \dfrac{1}{12(s+1)}}{\sqrt{2\pi(s+1)}} \cdot \Big(4 - \frac{21}{s+1}\Big). \qquad \square$$

12. Persistence on the far rows and the single crossing

In this section we prove Proposition 1.3, one of the three propositions about columns from which Theorem 1.1 is derived: a column that is positive in a row of its chain range is positive in the next row of that range. What remains to be treated are the far rows, those beyond row $k+0.47k$, which Proposition 10.4 does not reach. This is done in Proposition 12.3 for the columns $k \geq 82$. Together with Proposition 10.4 it gives the single crossing of every column $k \geq 82$ (Proposition 12.5), and with the computed columns $k \leq 81$ it completes the proof of Proposition 1.3.

12.1. **Auxiliary results.** We begin with three auxiliary results for the proof of Proposition 12.3. The first is a lemma about arbitrary sequences. From an inequality of the form of the column equation, which is hypothesis (3) below, and a lower bound of the form of Proposition 5.9, which is hypothesis (4), it derives a lower bound for one term of a sequence η in terms of a second sequence u alone, provided that a run of terms of η before it is nonnegative.

Lemma 12.1. *Let $(h_m)_{m\geq 0}$, $(u_j)_{j\geq 0}$ and $(\eta_j)_{j\geq 0}$ be sequences of real numbers, let $\gamma > 0$ be a real number, and let $k \geq 1$, n and τ be integers with $k+1 \leq n$ and $2 \leq \tau \leq n-k$. Assume that*

(1) *$h_m > 0$ for every $m \geq 1$;*
(2) *$u_{k-1} = 0$;*

(3) $\dfrac{2}{n}\,u_{n-1} + \dfrac{k+1}{n}\,\eta_{n-1} + \displaystyle\sum_{m=1}^{n-k} h_m\,\eta_{n-m} \le \eta_n$;
(4) $\gamma\,u_{j-1} - u_j \le \eta_j$ *for every* j *with* $k \le j \le n-1$;
(5) $h_{m+1} \le \gamma\,h_m$ *for every* m *with* $1 \le m < \tau$;
(6) $\eta_j \ge 0$ *for every* j *with* $n-\tau+1 \le j \le n-1$.

Then

$$\Big(\frac{2}{n} - \frac{h_\tau}{\gamma^{\tau-1}}\Big)u_{n-1} + \sum_{m=\tau}^{n-k-1} \big(\gamma\,h_m - h_{m+1}\big)\,u_{n-1-m} \le \eta_n.$$

Proof. Put $\psi = h_\tau/\gamma^{\tau-1}$, which is positive by (1). We first note that

$$\psi\,\gamma^{m-1} \le h_m \qquad (1 \le m \le \tau).$$

For $m = \tau$ this is an equality, and for smaller m it follows from (5) by downward induction on m. Since $\tau \ge 2$, we have $\eta_{n-1} \ge 0$ by (6), and (3) gives

$$\frac{2}{n}\,u_{n-1} + \sum_{m=1}^{n-k} h_m\,\eta_{n-m} \le \eta_n. \tag{12.1}$$

For $1 \le m \le n-k$, hypothesis (4) at $j = n-m$ reads $\gamma\,u_{n-1-m} - u_{n-m} \le \eta_{n-m}$.

Let $1 \le m \le \tau-1$. Then $\eta_{n-m} \ge 0$ by (6), and the bound $\psi\,\gamma^{m-1} \le h_m$ and (4) give

$$h_m\,\eta_{n-m} \ge \psi\gamma^{m-1}\,\eta_{n-m} \ge \psi\,\big(\gamma^m\,u_{n-1-m} - \gamma^{m-1}\,u_{n-m}\big).$$

The right-hand sides telescope, and since $\psi\gamma^{\tau-1} = h_\tau$ we obtain

$$\sum_{m=1}^{\tau-1} h_m\,\eta_{n-m} \ge h_\tau\,u_{n-\tau} - \psi\,u_{n-1}.$$

Let now $\tau \le m \le n-k$. Here we use only that $h_m > 0$, so that $h_m\,\eta_{n-m} \ge h_m\,(\gamma\,u_{n-1-m} - u_{n-m})$ by (4). Summing over these m and shifting the index by one in the terms $h_m\,u_{n-m}$, we get

$$\begin{aligned}\sum_{m=\tau}^{n-k} h_m\,\eta_{n-m} &\ge \sum_{m=\tau}^{n-k} \gamma\,h_m\,u_{n-1-m} - \sum_{m=\tau-1}^{n-k-1} h_{m+1}\,u_{n-1-m}\\ &= -h_\tau\,u_{n-\tau} + \sum_{m=\tau}^{n-k-1} \big(\gamma\,h_m - h_{m+1}\big)\,u_{n-1-m} + \gamma\,h_{n-k}\,u_{k-1},\end{aligned}$$

where the last term vanishes by (2). Adding the two bounds, in which the terms $h_\tau\,u_{n-\tau}$ cancel, and inserting the result into the inequality (12.1) that we derived from (3), we obtain the claim. □

The second result is an upper bound for $\frac{N}{2}\,\varepsilon_{t+1}\vartheta_k^{\,t}$ when t is close to a fixed multiple σk of k and N is at most a multiple Lk of k; recall that $\vartheta_k = 1/\gamma_k$ is the diagonal tilt of column k (Section 8). Let $k \ge 15$ and $t \ge 1$ be integers, and let $\sigma > 0$, $\lambda > 0$, L and N be real numbers with

$$\sigma\,(k+3) \le \lambda\,(\sigma k + 1), \qquad \sigma k \le t \le \sigma k + 1, \qquad 0 \le N \le L\,k.$$

Then

$$\frac{N}{2}\,\varepsilon_{t+1}\,\vartheta_k^{\,t} \le \frac{\lambda\,L\,e^{5\sigma/2}}{2\sigma\sqrt{2\pi\sigma k}}. \tag{12.2}$$

To see this, write $\psi = \varepsilon_{t+1}\vartheta_k^t$. We have $(t+1)\,\varepsilon_{t+1} = t^t/t!$, and $\vartheta_k \le e^{-1}e^{5/(2k)}$ by Lemma 8.1. Hence the upper bound in (2.7) gives

$$(t+1)\,\psi = \frac{t^t}{t!}\,\vartheta_k^t \le e^{-t}\,\frac{t^t}{t!}\,e^{5t/(2k)} \le \frac{e^{5t/(2k)}}{\sqrt{2\pi t}}.$$

Since $t \le \sigma k + 1$, we have $e^{5t/(2k)} \le e^{5\sigma/2}\,e^{5/(2k)}$. Moreover $e^{5/(2k)} \le 1 + \frac{3}{k}$, by the elementary inequality $e^u \le 1 + \frac{6}{5}u$ for $0 \le u \le \frac{1}{6}$, which applies to $u = 5/(2k)$ because $k \ge 15$. Thus

$$(t+1)\,\sqrt{2\pi t}\;\psi \le e^{5\sigma/2}\Big(1 + \frac{3}{k}\Big).$$

Since $\sigma k \le t$, we have $\big(1 + \frac{3}{k}\big)\,\sigma k = \sigma\,(k+3) \le \lambda\,(\sigma k + 1) \le \lambda\,(t+1)$, so that $\sqrt{2\pi t}\;\sigma k\,\psi \le \lambda\,e^{5\sigma/2}$, and with $0 \le N \le Lk$ this gives

$$\frac{N}{2}\,\psi \cdot 2\sigma\sqrt{2\pi t} \le \lambda\,L\,e^{5\sigma/2}.$$

Since $t \ge \sigma k$, the bound (12.2) follows.

The third result, again a lemma, concerns the ratios g_m of the free kernel (Section 2), for which $\varepsilon_{m+1} = g_m\,\varepsilon_m$, and the frozen kernel f of Section 11, for which $f_1 = \frac{3}{2}e^{-1}$ and $f_i = e^{-i}\varepsilon_i$ for $i \ge 2$. From here on f always denotes the frozen kernel and G its renewal sequence (Lemma 11.5).

Lemma 12.2. *Let k and m be integers with $m \ge 49$, $m+1 \le k$ and $3k \le 5\,(m+1)$. Then the frozen kernel f satisfies*

$$(g_m - \gamma_k)\,f_m \le \frac{105}{100}\cdot\frac{c_*}{k^2\sqrt{k}}.$$

Proof. Write $s = m+1$, so that $50 \le s \le k$ and $3k \le 5s$.

We begin with $g_m - \gamma_k$. Since $g_m = \frac{m}{m+1}\big(1 + \frac{1}{m-1}\big)^{m-1}$, we have $\log g_m = (m-1)\log\big(1+\frac{1}{m-1}\big) - \log\big(1+\frac{1}{m}\big)$. By the Taylor bounds $u - \frac{u^2}{2} \le \log(1+u) \le u - \frac{u^2}{2} + \frac{u^3}{3}$ for $u \ge 0$, used at $u = \frac{1}{m}$ and at $u = \frac{1}{m-1}$,

$$\begin{aligned}\log g_m &\le 1 - \frac{1}{2(m-1)} + \frac{1}{3(m-1)^2} - \frac{1}{m} + \frac{1}{2m^2}\\ &= 1 - \frac{3}{2(m+1)} - \frac{7m^3 - 17m^2 + 9m - 3}{6\,(m-1)^2\,m^2\,(m+1)} \le 1 - \frac{3}{2(m+1)},\end{aligned}$$

where the equality is a direct computation and the last fraction is clearly positive. Hence $g_m \le \exp\big(1 - \frac{3}{2s}\big)$. On the other hand, $\gamma_k = 1/\vartheta_k \ge \exp\big(1 - \frac{5}{2k}\big)$ by Lemma 8.1. Put

$$\alpha = \frac{5}{2k} - \frac{3}{2s} = \frac{5s - 3k}{2ks},$$

which is nonnegative because $3k \le 5s$. Then $\gamma_k \ge \exp\big(1 - \frac{3}{2s}\big)\,e^{-\alpha} \ge \exp\big(1 - \frac{3}{2s}\big)\,(1-\alpha)$, and therefore

$$g_m - \gamma_k \le \exp\Big(1 - \frac{3}{2s}\Big)\,\alpha \le e\,\alpha.$$

Next we bound $f_m = e^{-m}\varepsilon_m$. Since $m\,\varepsilon_m = (m-1)^{m-1}/(m-1)!$, the upper bound in (2.7) at $m-1$ gives

$$f_m = \frac{e^{-1}}{m}\cdot e^{-(m-1)}\,\frac{(m-1)^{m-1}}{(m-1)!} \le \frac{e^{-1}}{m\,\sqrt{2\pi\,(m-1)}}.$$

Since $f_m > 0$ and $\alpha \geq 0$, the two bounds give

$$(g_m - \gamma_k)\, f_m \leq e\, \alpha\, f_m \leq \frac{\alpha}{m\sqrt{2\pi\,(m-1)}} = \frac{c_*\,(5s-3k)}{2ks\,(s-1)\sqrt{s-2}}.$$

The right-hand side is at most $\frac{105}{100}\, c_*\, k^{-5/2}$, as the lemma claims, if and only if

$$(5s-3k)\, k^{3/2} \leq \frac{105}{100} \cdot 2s\,(s-1)\sqrt{s-2}.$$

This follows from two inequalities. The first is $(5s-3k)\, k^{3/2} \leq 2s^{5/2}$, which holds because

$$2s^{5/2} - (5s-3k)\, k^{3/2} = \left(\sqrt{s}-\sqrt{k}\right)^2 \left(2s^{3/2} + 4s\, k^{1/2} + 6s^{1/2}k + 3k^{3/2}\right) \geq 0,$$

as one sees by expanding the product. The second is $s^{3/2} \leq \frac{105}{100}\,(s-1)\sqrt{s-2}$ for $s \geq 50$; after squaring it is the polynomial inequality $s^3 \leq 1.1025\,(s-1)^2(s-2)$, which is easy to check. Together they give the required inequality. □

12.2. **Rows beyond $k+0.47k$.** Proposition 12.3 below says that a column $k \geq 82$ is positive in a far row $k+r$ if it is positive in the rows before it, from some row $k+r_q$ with $r_q \leq k/3$ on. The comparison with the crossing row, on which Section 10 rests, does not reach beyond row $k + 0.47k$, and the proof uses the column equation at row $k+r$ alone. If column k is positive in a window, that is, in a run of rows before this row, Lemma 12.1 bounds the step in row $k+r$ from below by a combination of values of level $k-1$; for a far row there is a window whose length is a fixed fraction of k, because $3\,r_q \leq k$. The difficulty is that for $r > \frac{3}{5}k$ some coefficients of this combination may be negative. The values of level $k-1$ that carry them are known only to satisfy the level equation, and this is the situation for which the comparison of Section 11 is made: relative to the value of level $k-1$ in row $k+r-1$, it bounds these terms by an explicit quantity, which is bounded but is not small when r is close to k. The proof therefore ends with a comparison of explicit constants, and this comparison holds for $k \geq 82$.

Proposition 12.3. *Let $k \geq 82$, r and r_q be nonnegative integers with*

$$47\,k < 100\,r, \qquad r+1 \leq k, \qquad 3\,r_q \leq k.$$

Assume that $d_{j,k} > 0$ for every integer j with $k + r_q \leq j < k + r$. Then $d_{k+r,k} > 0$.

Proof. Put $n = k + r$, and write $\delta_j = \delta_j^{(k)} = d_{j,k}/j!$, which has the sign of $d_{j,k}$. Since $\frac{47}{100}k < r \leq k-1$, we have $k+1 \leq n \leq 2k-1$, so that row n lies in the chain range of column k. We have to show that $\delta_n > 0$.

We begin with a lower bound for δ_n. Let $t_0 \geq 1$ be an integer with $t_0+1 \leq r$ and $5\,(t_0+1) \leq 3k$, suppose that $\delta_j \geq 0$ for $n - t_0 \leq j \leq n-1$, and put $\psi = \varepsilon_{t_0+1}\vartheta_k^{t_0}$. We claim that

$$\left(\frac{2}{n} - \psi\right)\beta_{n-1}^{(k-1)} + \sum_{m=t_0+1}^{r-1} \left(\gamma_k\varepsilon_m - \varepsilon_{m+1}\right)\beta_{n-1-m}^{(k-1)} \leq \delta_n. \tag{12.3}$$

We apply Lemma 12.1 with $h_m = \varepsilon_m$, $u_j = \beta_j^{(k-1)}$, $\eta_j = \delta_j$, $\gamma = \gamma_k$ and $\tau = t_0 + 1$; then $2 \leq \tau \leq r = n - k$, and $h_\tau/\gamma^{\tau-1} = \psi$ because $\vartheta_k = 1/\gamma_k$. Hypotheses (1) and (2) hold because $\varepsilon_m > 0$ for $m \geq 1$ and $\beta_{k-1}^{(k-1)} = 0$, and (3) holds with equality, by the column equation at row n (Proposition 4.5). Hypothesis (4) is the diagonal

lower bound: Proposition 5.9 gives $\gamma_k\,\beta_{j-1}^{(k-1)} \le \beta_j^{(k)}$, and subtracting $\beta_j^{(k-1)}$ from both sides we obtain

$$\gamma_k\,\beta_{j-1}^{(k-1)} - \beta_j^{(k-1)} \le \delta_j \qquad (j \ge 1).$$

Hypothesis (5) holds by Lemma 10.2, because $5\,(m+1) \le 5\,(t_0+1) \le 3k$ for $1 \le m < \tau$, and (6) is the assumption on rows $n-t_0, \dots, n-1$. The conclusion of the lemma is the claimed bound.

Next we choose t_0, the length of the window. We distinguish three ranges of r, which together contain every r with $\frac{47}{100}k < r \le k-1$:

$$\text{(a)}\ \ \tfrac{47}{100}k < r \le \tfrac{3}{5}k, \qquad \text{(b)}\ \ \tfrac{3}{5}k < r \le \tfrac{4}{5}k, \qquad \text{(c)}\ \ \tfrac{4}{5}k < r \le k-1.$$

Let $\sigma = \frac{41}{300}$ in range (a), $\sigma = \frac{4}{15}$ in range (b) and $\sigma = \frac{7}{15}$ in range (c), so that $r > \big(\sigma + \frac{1}{3}\big)k$ in each case, and let

$$t_0 = \lfloor \sigma k \rfloor + 1, \qquad \text{so that} \qquad \sigma k \le t_0 \le \sigma k + 1.$$

Since $3\,r_q \le k$, the integer $r - r_q$ is larger than σk, and hence $r - r_q \ge t_0$. So rows $n-t_0, \dots, n-1$ are among rows $k+r_q, \dots, k+r-1$, in which column k is positive by assumption. Moreover, $t_0+1 \le \sigma k+2 < \big(\sigma+\frac{1}{3}\big)k < r$ and $5\,(t_0+1) \le \frac{7}{3}k+10 \le 3k$, because $\sigma \le \frac{7}{15}$ and $k \ge 15$. Hence the lower bound (12.3) holds with this t_0. From now on t_0 and $\psi = \varepsilon_{t_0+1}\vartheta_k^{t_0}$ have this meaning.

We also need that the numbers $\beta_j^{(k-1)}$ of level $k-1$ are positive in rows $k, \dots, 2k-1$. By Lemma 4.3, applied to level $k-1$,

$$\beta_j^{(k-1)} = \frac{z_{j-1}^{(k)}}{j} + \frac{k-1}{j}\,\beta_{j-1}^{(k-1)} + \sum_{m=1}^{j-k} \varepsilon_m\,\beta_{j-m}^{(k-1)} \qquad (k \le j \le 2k-1). \tag{12.4}$$

All terms on the right are nonnegative. For $j = k$ the first term is $z_{k-1}^{(k)}/k = c_{k-1}/k > 0$, by Lemma 3.4, and for $k < j \le 2k-1$ the second term is positive if $\beta_{j-1}^{(k-1)} > 0$. By induction on j we get

$$\beta_j^{(k-1)} > 0 \qquad (k \le j \le 2k-1),$$

and in particular $\beta_{n-1}^{(k-1)} > 0$.

Suppose first that $r \le \frac{3}{5}k$, which is range (a). Every index m of the sum in (12.3) satisfies $5\,(m+1) \le 5r \le 3k$, so that $\gamma_k\varepsilon_m - \varepsilon_{m+1} \ge 0$ by Lemma 10.2. Since the $\beta_{n-1-m}^{(k-1)}$ are nonnegative, the sum is nonnegative, and we obtain

$$\Big(\frac{2}{n} - \psi\Big)\beta_{n-1}^{(k-1)} \le \delta_n.$$

Since $\beta_{n-1}^{(k-1)} > 0$, it therefore suffices to show that $\frac{n}{2}\,\psi < 1$; this is done at the end of the proof.

Suppose now that $r > \frac{3}{5}k$, so that r lies in range (b) or (c); some of the coefficients $\gamma_k\varepsilon_m - \varepsilon_{m+1}$ of the lower bound may then be negative. Put

$$\rho = r-1, \qquad M = \Big\lfloor \frac{3}{5}k \Big\rfloor, \qquad B_k = \frac{105}{100}\cdot\frac{c_*}{k^2\sqrt{k}}.$$

Then $49 \le M \le \rho \le k-2$, because $k \ge 82$ and the integer r satisfies $\frac{3}{5}k < r \le k-1$. We show that

$$\Big[\Big(\frac{2}{n} - \psi\Big)G_\rho - B_k \sum_{i=0}^{\rho-M} G_i\Big]\, \beta_{n-1}^{(k-1)} \le \delta_n\, G_\rho. \tag{12.5}$$

We begin by separating the terms of the lower bound (12.3) whose coefficients may be negative. Its sum runs over $t_0+1 \le m \le \rho$, and $t_0+1 \le \sigma k+2 \le \frac{3}{5}k-1 < M$, because $\sigma \le \frac{7}{15}$ and $k \ge 23$. For $t_0+1 \le m \le M-1$ we have $5\,(m+1) \le 5M \le 3k$, so the coefficient $\gamma_k\varepsilon_m - \varepsilon_{m+1}$ is nonnegative by Lemma 10.2, and we omit these terms. For $M \le m \le \rho$ let

$$\theta_m = \max(g_m - \gamma_k,\, 0).$$

Since $\varepsilon_{m+1} = g_m\,\varepsilon_m$, we have $\gamma_k\varepsilon_m - \varepsilon_{m+1} = (\gamma_k - g_m)\,\varepsilon_m \ge -\theta_m\,\varepsilon_m$. As the $\beta_{n-1-m}^{(k-1)}$ are nonnegative, the lower bound gives

$$\Big(\frac{2}{n} - \psi\Big)\beta_{n-1}^{(k-1)} - \sum_{m=M}^{\rho} \theta_m\,\varepsilon_m\,\beta_{n-1-m}^{(k-1)} \le \delta_n. \tag{12.6}$$

Next we pass to the scale e^{-1}. For an integer $s \ge 0$ let $v_s = e^{-s}\beta_{k+s}^{(k-1)}$. We claim that

$$\sum_{i=1}^{s} f_i\, v_{s-i} \le v_s \qquad (0 \le s \le k-2),$$

that is, in the terms of Section 11, that v has nonnegative innovations with respect to f up to $k-2$. For $s = 0$ it says that $v_0 \ge 0$. Let $1 \le s \le k-2$. In the level equation (12.4) at $j = k+s$ the first term is nonnegative, and $\frac{k-1}{k+s} \ge \frac{1}{2}$ because $s \le k-2$. Hence

$$\beta_{k+s}^{(k-1)} \ge \frac{1}{2}\,\beta_{k+s-1}^{(k-1)} + \sum_{m=1}^{s} \varepsilon_m\,\beta_{k+s-m}^{(k-1)}.$$

Multiplying by e^{-s}, we obtain

$$v_s \ge \frac{1}{2}\,e^{-1}\,v_{s-1} + \sum_{m=1}^{s} e^{-m}\varepsilon_m\,v_{s-m} = \sum_{i=1}^{s} f_i\,v_{s-i},$$

because $f_1 = e^{-1}\varepsilon_1 + \frac{1}{2}e^{-1}$ and $f_i = e^{-i}\varepsilon_i$ for $i \ge 2$.

We now apply Proposition 11.4 to f, G, v and the weights θ_m, with ρ and M as above. Its hypotheses hold. We have $f_i > 0$ for $i \ge 1$, and f is log-convex: for $i \ge 2$ the inequality $f_{i+1}^2 \le f_i\,f_{i+2}$ is the inequality $\varepsilon_{i+1}^2 \le \varepsilon_i\,\varepsilon_{i+2}$ of Lemma 2.8, multiplied by $e^{-2(i+1)}$, and for $i = 1$ it follows in the same way from $\varepsilon_2^2 \le \varepsilon_1\,\varepsilon_3$, because $f_1 \ge e^{-1}\varepsilon_1$. By Lemma 11.5, G is the renewal sequence of f. The weights satisfy $\theta_M \ge 0$ and $\theta_m \le \theta_{m'}$ for $M \le m \le m' \le \rho$, because $g_m = \varepsilon_{m+1}/\varepsilon_m$ does not decrease with m, again by Lemma 2.8. Finally, v has nonnegative innovations with respect to f up to ρ, since $\rho \le k-2$. The proposition gives

$$\Big(\sum_{m=M}^{\rho} \theta_m\,f_m\,v_{\rho-m}\Big)\,G_\rho \le \Big(\sum_{m=M}^{\rho} \theta_m\,f_m\,G_{\rho-m}\Big)\,v_\rho.$$

Since $k+\rho = n-1$, we have $f_m\, v_{\rho-m} = e^{-\rho}\,\varepsilon_m\,\beta_{n-1-m}^{(k-1)}$ for $M \le m \le \rho$ and $v_\rho = e^{-\rho}\beta_{n-1}^{(k-1)}$. Dividing by $e^{-\rho}$, we obtain

$$\Big(\sum_{m=M}^{\rho} \theta_m\,\varepsilon_m\,\beta_{n-1-m}^{(k-1)}\Big)\,G_\rho \le \Big(\sum_{m=M}^{\rho} \theta_m\, f_m\, G_{\rho-m}\Big)\,\beta_{n-1}^{(k-1)}.$$

It remains to bound the weights $\theta_m\, f_m$. For $M \le m \le \rho$ Lemma 12.2 applies, because $m \ge M \ge 49$, $m+1 \le r \le k$ and $m+1 > \frac{3}{5}k$. Hence $(g_m - \gamma_k)\, f_m \le B_k$, and therefore $\theta_m\, f_m \le B_k$. Clearly $G_s > 0$ for every $s \ge 0$, since G is the renewal sequence of a kernel with positive terms. Thus

$$\sum_{m=M}^{\rho} \theta_m\, f_m\, G_{\rho-m} \le B_k \sum_{m=M}^{\rho} G_{\rho-m} = B_k \sum_{i=0}^{\rho-M} G_i.$$

Multiplying (12.6) by G_ρ and inserting the last two displays, we obtain (12.5).

We now reduce the claim to two numerical inequalities. Suppose that p_1 and p_2 are real numbers with $p_1 + p_2 < 1$ such that

$$\frac{n}{2}\,\psi \le p_1 \qquad \text{and} \qquad \frac{n}{2}\,B_k \sum_{i=0}^{\rho-M} G_i \le p_2\, G_\rho. \tag{12.7}$$

Then

$$\frac{n}{2}\Big[\Big(\frac{2}{n} - \psi\Big)G_\rho - B_k \sum_{i=0}^{\rho-M} G_i\Big] \ge (1 - p_1 - p_2)\, G_\rho > 0,$$

so the bracket in (12.5) is positive. Since $\beta_{n-1}^{(k-1)} > 0$ and $G_\rho > 0$, it follows that $\delta_n > 0$. So it suffices to find such p_1 and p_2.

The first of the two numerical inequalities will come from the bound (12.2). For the second we use the following sufficient condition. For a real number $x \ge 6$ let

$$\mathcal{T}(x) = \Big(1 - \frac{1}{12x}\Big)\Big(4 - \frac{21}{x}\Big).$$

Both factors are positive and increasing on this range, so $\mathcal{T}$ is increasing. Suppose that S, $w > 0$, $\underline{r} \ge 6$ and $p_2 \ge 0$ are real numbers with

$$\sum_{i=0}^{\rho-M} G_i \le S\, e, \qquad r \le w^2 k, \qquad r \ge \underline{r}, \qquad \frac{105}{100}\, S\,(1+w^2)\, w^3 \le 2\, p_2\, \mathcal{T}(\underline{r}). \tag{12.8}$$

Then the second of the two numerical inequalities holds. Indeed, Lemma 11.7, which applies at $s = \rho$ because $\rho \ge 6$, reads $G_\rho \ge e\, c_*\, r^{-3/2}\, \mathcal{T}(r)$, and $\mathcal{T}(r) \ge \mathcal{T}(\underline{r})$. From $r \le w^2 k$ we get $(k+r)\, r^{3/2} \le (1+w^2)\, w^3\, k^{5/2}$. Hence

$$\begin{aligned} \frac{n}{2}\,B_k \sum_{i=0}^{\rho-M} G_i &\le \frac{k+r}{2} \cdot \frac{105}{100} \cdot \frac{c_*}{k^{5/2}} \cdot S\, e \le \frac{105}{100} \cdot \frac{S\,(1+w^2)\, w^3}{2} \cdot \frac{e\, c_*}{r^{3/2}} \\ &\le p_2\, \mathcal{T}(\underline{r})\, \frac{e\, c_*}{r^{3/2}} \le p_2\, G_\rho. \end{aligned}$$

Before we choose the constants, we sharpen the bound for the sum of the G_i in one case. By Lemma 11.6 the condition $\sum_{i=0}^{\rho-M} G_i \le S\, e$ holds with $S = 2$. In

range (c) this is not enough when $82 \le k \le 109$, and we show that for these k and for $r > \frac{4}{5}k$

$$\sum_{i=0}^{\rho-M} G_i \le 1.747\, e.$$

The number of terms of the sum is

$$J = \rho - M + 1 = r - \Big\lfloor \frac{3}{5}k \Big\rfloor,$$

and $\frac{k}{5} < J < \frac{2}{5}k$, because $\frac{4}{5}k < r \le k-1$. As $82 \le k \le 109$, this gives $17 \le J \le 43$. For $i \ge 17$, Lemma 11.7 and the monotonicity of $\mathcal{T}$ give

$$G_i \ge e\, c_*\, (i+1)^{-3/2}\, \mathcal{T}(i+1) \ge e\, c_*\, \mathcal{T}(18)\, (i+1)^{-3/2} \ge 1.12\, e\, (i+1)^{-3/2},$$

because $\mathcal{T}(18) = \frac{215}{216} \cdot \frac{17}{6}$, $c_* \ge 0.39894$ and $0.39894 \cdot \frac{215}{216} \cdot \frac{17}{6} = 1.1250\ldots$. Let $J' = 16\,(J+1) - 1$, so that $\sqrt{J'+1} = 4\sqrt{J+1}$. Comparing the sum of the $(i+1)^{-3/2}$ with an integral, we find that

$$\sum_{i=J}^{J'-1} G_i \ge 1.12\, e \int_{J+1}^{J'+1} x^{-3/2}\, dx = \frac{1.68\, e}{\sqrt{J+1}} \ge 0.253\, e,$$

where the last step holds because $J+1 \le 44 \le 6.64^2$ and $1.68/6.64 = 0.2530\ldots$. Since $\sum_{i=0}^{J'-1} G_i \le 2e$ by Lemma 11.6, subtracting gives

$$\sum_{i=0}^{\rho-M} G_i = \sum_{i=0}^{J-1} G_i \le (2 - 0.253)\, e = 1.747\, e.$$

It remains to choose the constants and to verify the numerical inequalities, in all three ranges. We apply the bound (12.2) with $t = t_0$ and $N = n$, and for $r > \frac{3}{5}k$ the condition (12.8), with the following constants.

range	σ	L	λ	p_1	S	w	$\underline{r}$	p_2
(a)	$\frac{41}{300}$	$\frac{8}{5}$	1	0.99				
(b)	$\frac{4}{15}$	$\frac{9}{5}$	1	0.57	2	0.8945	50	0.385
(c), $k \ge 110$	$\frac{7}{15}$	2	1.011	0.395	2	1	89	0.56
(c), $k \le 109$	$\frac{7}{15}$	2	1.011	0.455	1.747	1	66	0.5

The three conditions under which (12.2) holds are satisfied. For $\lambda = 1$ the first holds because $3\sigma \le 1$, and for $\sigma = \frac{7}{15}$ and $\lambda = 1.011$ it reads $0.389 \le 0.011 \cdot \frac{7}{15}\, k$, which is true for $k \ge 76$. The second holds by the choice of t_0, and the third because $n = k + r \le Lk$ in each range. The square of the right-hand side of (12.2) is $\lambda^2 L^2 e^{5\sigma}/(8\pi\,\sigma^3 k)$, so that this right-hand side is at most p_1 as soon as

$$k \ge \frac{\lambda^2 L^2\, e^{5\sigma}}{8\pi\,\sigma^3\, p_1^2}.$$

A direct evaluation gives for the number on the right the values $80.63\ldots$, $79.37\ldots$, $105.79\ldots$ and $79.73\ldots$ in the four lines of the table, and we have $k \ge 82$ in the first, second and fourth line and $k \ge 110$ in the third. So $\frac{n}{2}\,\psi \le p_1$ in each line. In range (a) this is $\frac{n}{2}\,\psi \le 0.99 < 1$, which, as we have seen, completes the proof in this range.

Let now $r > \frac{3}{5}k$. In the last three lines of the table $p_1 + p_2 = 0.955 < 1$, and the first inequality in (12.7) has just been shown; so it remains to verify the four

conditions (12.8), which give the second. The first condition holds with $S = 2$ by Lemma 11.6, and in the last line with $S = 1.747$ by the sharper bound shown above for $82 \le k \le 109$. The second condition holds in range (b) because $r \le \frac{4}{5}k$ and $0.8945^2 = 0.80013\ldots$, and in range (c) because $r \le k$. For the third we have $r > \frac{3}{5}k \ge 49.2$ in range (b), and in range (c) we have $r > \frac{4}{5}k \ge 88$ for $k \ge 110$ and $r > \frac{4}{5}k \ge 65.6$ for $k \ge 82$; since r is an integer, this gives $r \ge \underline{r}$ in each line. The last condition is again checked by direct evaluation: its two sides are $2.7056\ldots$ and $2.7520\ldots$ in range (b), they are 4.2 and $4.2117\ldots$ in range (c) for $k \ge 110$, and they are 3.6687 and $3.6771\ldots$ in range (c) for $k \le 109$. This completes the proof. □

12.3. **The single crossing and the chain range.** We now combine Proposition 12.3 with Proposition 10.4: a column $k \ge 82$ whose crossing row $k + r_q$ lies in the first third of its chain range stays positive up to row $2k - 1$.

Proposition 12.4. *Let $k \ge 82$ and $r_q \ge 1$ be integers with $3\,r_q \le k$. Assume that $d_{j,k} \le 0$ for every integer j with $k \le j < k + r_q$, and that $d_{k+r_q,k} > 0$. Then $d_{n,k} > 0$ for every integer n with $k + r_q \le n \le 2k - 1$.*

Proof. We prove the claim by induction on n. For $n = k + r_q$ it is the second assumption. Let $k + r_q < n \le 2k - 1$, and suppose that $d_{j,k} > 0$ for every j with $k + r_q \le j < n$. Write $n = k + r$ and $r = r_q + t$, so that $t \ge 1$ and $r \le k - 1$.

If $100\,r > 47\,k$, then Proposition 12.3 applies to k, r and r_q, its assumption on rows $k + r_q, \ldots, k + r - 1$ being the induction hypothesis, and it gives $d_{n,k} > 0$.

If $100\,r \le 47\,k$, we apply Proposition 10.4 with r_q in the place of r, so that its rows q and n are $k + r_q$ and $k + r_q + t = n$. Its hypothesis (1) is the first assumption, and its hypothesis (2) is the induction hypothesis. Hence $d_{n,k} > 0$ in this case as well, which completes the induction. □

Together with Corollary 9.3, Proposition 12.4 gives the single crossing: along a column $k \ge 82$, from row k to row $2k - 1$, the step changes its sign exactly once, from nonpositive to positive.

Proposition 12.5. *For every integer $k \ge 82$ there is an integer q such that*

(1) *$k + 1 \le q$ and $100\,(q - k) \le 27\,k$;*
(2) *$d_{j,k} \le 0$ for every integer j with $k \le j < q$;*
(3) *$d_{n,k} > 0$ for every integer n with $q \le n \le 2k - 1$.*

Proof. By Corollary 9.3 there is an integer $r \ge 1$ with $100\,r \le 27\,k$ and $d_{k+r,k} > 0$. Let q be the least row $j \ge k + 1$ with $d_{j,k} > 0$. Then $k + 1 \le q \le k + r$, which gives (1). For $k + 1 \le j < q$ we have $d_{j,k} \le 0$ by the choice of q, and $d_{k,k} < 0$, because $\delta_k^{(k)} = -a_k/k$ by Lemma 4.4; this is (2). Finally, $r_q = q - k$ satisfies $r_q \ge 1$ and $3\,r_q \le k$ by (1), so that (2), $d_{q,k} > 0$ and Proposition 12.4 give (3). □

We can now prove Proposition 1.3, for all columns $k \ge 2$.

Proof of Proposition 1.3. Let $k \ge 2$ and n be integers with $k + 1 \le n \le 2k - 2$. In terms of the step $d_{n,k} = b_{n,k} - b_{n,k-1}$, we have to show that $d_{n,k} > 0$ implies $d_{n+1,k} > 0$. For $2 \le k \le 81$ this is part (2) of Lemma 7.5. Let $k \ge 82$, and let q be as in Proposition 12.5. Suppose that $d_{n,k} > 0$. Since $n \ge k$ and $d_{j,k} \le 0$ for $k \le j < q$, we have $q \le n$. Then $q \le n + 1 \le 2k - 1$, and part (3) of Proposition 12.5 gives $d_{n+1,k} > 0$. □

13. Property K

This section and Section 14 prepare the proof of Proposition 1.2 for the columns $k \geq 82$ (Proposition 15.4). In a row beyond the chain range of such a column only inequalities are available, and into them enter the steps of the column in the rows before its crossing row, which are nonpositive and about whose size nothing is known. The proof in Section 15 outweighs these steps by means of one inequality, Proposition 14.7. To describe it, we fix a column k and, as in Section 6, let a hat denote the tilted form of a sequence at the critical tilt x_k, so that $\hat{c}_n = c_n\, x_k^n$, $\hat{z}_N^{(y)} = z_N^{(y)}\, x_k^N$ and $\hat{\delta}_n = \delta_n^{(k)}\, x_k^n$; this notation is used throughout the section. The proposition says that, for $k \geq 7$ and $J \geq k$, the sum of the tilted steps $\hat{\delta}_n$ over rows $n = k, \dots, J$, taken with certain positive weights, is bounded from below by the negative of a quantity that consists of three nonnegative terms belonging to rows $J-1$ and J. This quantity is called the weighted debt of the column at row J. It is defined, together with the weights, in Section 14, where the proposition is proved.

The proof of Proposition 14.7 is a summation by parts. Recall from Section 3 that the step of a column $k \geq 2$ is a second difference in the threshold, taken at one length; in tilted form,

$$\hat{\delta}_n = \hat{z}_n^{(k+1)} - 2\,\hat{z}_n^{(k)} + \hat{z}_n^{(k-1)}.$$

When the tilted steps of rows $k, \dots, J$ are added with their weights, the terms regroup along the diagonals, on which the threshold and the length are raised together: apart from terms at the two ends of the range, the sum consists of the expressions

$$\hat{z}_{N+1}^{(k+1)} - 2\,\hat{z}_N^{(k)} + \hat{z}_{N-1}^{(k-1)} \qquad (k \leq N \leq J-1),$$

with each of the three tilted counts multiplied by the weight of its row. The bound therefore rests on the nonnegativity of these expressions, which is a convexity of the tilted counts of thresholds $k-1$, k and $k+1$ along the diagonal (Corollary 14.6). Section 14 proves it first for the lengths $k \leq N \leq 2k-2$, where the three weights are equal, and extends it from there to all larger lengths. For the first of these two steps one would argue by induction on the last-car recursion, but convexity does not pass through that induction. We therefore strengthen it to a property of a triple of real numbers, property K (Definition 13.1), which does, and we prove property K for the explicit triples that the recursion produces at the three thresholds: the numbers $\hat{c}_n$ at the top, sums of products of two of them, and the direct term. The induction itself is carried out in Section 14.

13.1. **The definition and the closure under factors.** The convexity in question compares the tilted counts of thresholds $y = k-1$, k and $k+1$ at lengths $y+\rho$, with the same integer $\rho \geq 0$ for the three thresholds, and it is proved by induction on ρ. The induction rests on the last-car recursion in the form

$$\hat{z}_{y+\rho}^{(y)} = \sum_{i=0}^{\rho-2} x_k\, \hat{c}_i\, \frac{y+i+1}{y+\rho}\, \hat{z}_{y+\rho-1-i}^{(y)} + \mathcal{S}_\rho(y) \qquad (1 \leq \rho \leq y-1),$$

which is derived in Section 14; here $\mathcal{S}_\rho(y)$ is an explicit expression in the $\hat{c}_n$, the direct term of Section 13.4. The tilted counts on the right-hand side belong to smaller values of ρ. They are multiplied by $x_k\, \hat{c}_i \geq 0$, which is the same for the three thresholds, and by $(y+i+1)/(y+\rho)$, which increases with the threshold. Now the convexity $2w_0 \leq w_- + w_+$ of a triple (w_-, w_0, w_+) passes to sums and to nonnegative

multiples, but it does not pass to products with a factor that increases with the threshold: the triple $(2,1,0)$ is convex, and for positive numbers $F_- < F_0 < F_+$ the triple $(2F_-, F_0, 0)$ is not, since $2F_0 > 2F_-$. We therefore work with a family of inequalities, one for each real number $\mu \geq 0$, of which convexity is the member $\mu = 0$.

Definition 13.1. Let $k \neq 0$, w_-, w_0 and w_+ be real numbers. The triple (w_-, w_0, w_+) *has property K at* k if

$$2\, e^{\mu/k}\, w_0 \leq e^{\mu}\, w_+ + e^{-\mu}\, w_- \qquad \text{for every real number } \mu \geq 0. \tag{13.1}$$

In all our applications k is the index of a column, and w_-, w_0, w_+ are the values of one quantity at thresholds $k-1$, k, $k+1$, in this order. For $\mu = 0$ the inequality (13.1) is the convexity $2w_0 \leq w_- + w_+$ of the three values. Since the inequality is linear in the triple, property K at k is preserved when two triples are added entry by entry and when a triple is multiplied by a nonnegative number.

What we prove about factors that depend on the threshold comes from one computation. Let $k > 0$, let (w_-, w_0, w_+) have property K at k, and let L, M and μ be real numbers with $\mu + \frac{L-M}{2} \geq 0$. Then

$$2\, e^{\mu/k} \exp\Big(\frac{(k+1)L + (k-1)M}{2k}\Big)\, w_0 \leq e^{\mu}\, \big(e^{L} w_+\big) + e^{-\mu}\, \big(e^{M} w_-\big). \tag{13.2}$$

Indeed, this is the inequality of property K for $\mu' = \mu + \frac{L-M}{2} \geq 0$, multiplied by $e^{(L+M)/2}$, since $\frac{L+M}{2} + \frac{\mu'}{k} = \frac{\mu}{k} + \frac{(k+1)L + (k-1)M}{2k}$. Thus multiplying the upper value by e^L and the lower value by e^M has the same effect on the inequality of property K as moving μ to $\mu + \frac{L-M}{2}$ and multiplying the middle value by the second exponential on the left-hand side of the last display. If $L \geq M$, the condition $\mu + \frac{L-M}{2} \geq 0$ holds for every $\mu \geq 0$: a factor that does not decrease with the threshold turns the inequality for μ into the inequality for a larger value of μ, so that the family as a whole passes through the induction although its single members do not. If $L < M$, the computation is available only for $\mu \geq \frac{M-L}{2}$, because property K says nothing about negative values of μ.

The factor $e^{\mu/k}$ on the middle value in the definition of property K is chosen for the factors of the recursion. With it, a positive nondecreasing function F of the threshold is an admissible factor as soon as $y \log F(y)$ is convex at $k-1$, k, $k+1$, since the second exponential in (13.2) is then at least 1; this holds for the functions $(y+p)/(y+q)$ with $0 \leq p \leq q$, as the proof of Lemma 13.2 shows. Without it the condition would be the log-convexity of F, and these functions are log-concave.

The two lemmas that follow apply (13.2) to the factors $(y+p)/(y+q)$ with $0 \leq p \leq q$, which are those of the last-car recursion, and to the factors $(y+s)(y+t)/y$; for $t = 0$ the latter are the factors $y + s$, through which Section 14 passes from property K to a convexity with weights. The direct term is not covered by these lemmas, and Proposition 13.8 establishes its property K by a separate estimate.

Lemma 13.2. *Let $k > 1$ be a real number, let (w_-, w_0, w_+) be a triple of real numbers that has property K at k and satisfies $w_0 \geq 0$, and let p and q be real numbers with $0 \leq p \leq q$. Then the triple*

$$\Big(\frac{k-1+p}{k-1+q}\, w_-,\quad \frac{k+p}{k+q}\, w_0,\quad \frac{k+1+p}{k+1+q}\, w_+\Big)$$

has property K at k.

Proof. We first show that, for any positive real numbers F_-, F_0, F_+ with

$$F_- \le F_+ \qquad \text{and} \qquad (k+1)\log\frac{F_+}{F_0} + (k-1)\log\frac{F_-}{F_0} \ge 0, \tag{13.3}$$

the triple $(F_-w_-,\, F_0w_0,\, F_+w_+)$ has property K at k. Put $L = \log(F_+/F_0)$ and $M = \log(F_-/F_0)$. Then $L \ge M$, so that (13.2) holds for every $\mu \ge 0$. The second exponential on its left-hand side is at least 1, because $(k+1)L + (k-1)M \ge 0$ by the second condition, and since $w_0 \ge 0$, that side is at least $2e^{\mu/k}w_0$. Multiplying by F_0, we obtain the inequality of property K for the triple (F_-w_-, F_0w_0, F_+w_+).

The second condition in (13.3) follows from a criterion without logarithms: with $u = F_+/F_0 - 1$ and $v = F_0/F_- - 1$, it holds as soon as

$$u \ge 0 \qquad \text{and} \qquad (k-1)\,v \le (k+1)\,\frac{2u}{u+2}.$$

Indeed, the elementary inequalities $\log(1+u) \ge \frac{2u}{u+2}$ for $u \ge 0$ and $\log r \ge 1 - \frac{1}{r}$ for $r = F_-/F_0$ give, since $k > 1$,

$$(k+1)\log\frac{F_+}{F_0} + (k-1)\log\frac{F_-}{F_0} \ge (k+1)\,\frac{2u}{u+2} - (k-1)\,v \ge 0.$$

Now let $F(y) = (y+p)/(y+q)$, and put $F_- = F(k-1)$, $F_0 = F(k)$ and $F_+ = F(k+1)$. These numbers are positive, and $F_- \le F_+$ because $p \le q$. A direct computation gives

$$u = \frac{F_+}{F_0} - 1 = \frac{q-p}{(k+p)(k+1+q)} \ge 0, \qquad v = \frac{F_0}{F_-} - 1 = \frac{q-p}{(k+q)(k-1+p)},$$

and, clearing the positive denominators, one checks that the inequality $(k-1)\,v \le (k+1)\,\frac{2u}{u+2}$ is equivalent to

$$(q-p)\,\big[p\,(4q+3k+1) + q\,(k-1)\big] \ge 0,$$

which holds because $0 \le p \le q$ and $k > 1$. Thus the criterion applies, both conditions in (13.3) hold, and the lemma follows. □

Lemma 13.3. *Let $k > 1$ be a real number, let (w_-, w_0, w_+) be a triple of real numbers that has property K at k and satisfies $w_0 \ge 0$, and let s and t be nonnegative real numbers with $st \le k^2 - 1$. Then the triple*

$$\Big(\frac{(k-1+s)(k-1+t)}{k-1}\,w_-,\quad \frac{(k+s)(k+t)}{k}\,w_0,\quad \frac{(k+1+s)(k+1+t)}{k+1}\,w_+\Big)$$

has property K at k.

Proof. Let $F(y) = (y+s)(y+t)/y = y + s + t + st/y$, and put $F_- = F(k-1)$, $F_0 = F(k)$ and $F_+ = F(k+1)$. These numbers are positive. As was shown in the proof of Lemma 13.2, it suffices to verify that $F_- \le F_+$, that $u \ge 0$ and that $(k-1)\,v \le (k+1)\,\frac{2u}{u+2}$, where again $u = F_+/F_0 - 1$ and $v = F_0/F_- - 1$.

First, $F_+ - F_- = 2 - \frac{2st}{k^2-1} \ge 0$, because $st \le k^2 - 1$. Second, a direct computation gives

$$u = \frac{F_+}{F_0} - 1 = \frac{k(k+1) - st}{(k+1)(k+s)(k+t)},$$
$$v = \frac{F_0}{F_-} - 1 = \frac{k(k-1) - st}{k\,(k-1+s)(k-1+t)},$$

and $u \ge 0$ because $st \le k^2 - 1 \le k(k+1)$. Third, clearing the positive denominators, one checks that the inequality $(k-1)\,v \le (k+1)\,\frac{2u}{u+2}$ is equivalent to the nonnegativity of

$$k^2(k-1)^2(k+1) + 4k^2(k^2-1)\,(s+t) + st\,\big(10k^3 + 2k^2 - 4k - (3k+1)\,st\big).$$

The first two terms are nonnegative, and so is the third, since $st \ge 0$ and, by $st \le k^2 - 1$,

$$\begin{aligned} 10k^3 + 2k^2 - 4k - (3k+1)\,st &\ge 10k^3 + 2k^2 - 4k - (3k+1)(k^2-1) \\ &= 7k^3 + k^2 - k + 1 > 0. \end{aligned}$$

Thus the three conditions hold, and the lemma follows. □

13.2. **The numbers $\hat{c}_n$.** From here on k is an integer, the index of the fixed column. The numbers $\hat{c}_n = c_n\,x_k^n$ are positive. We first prove the inequality of property K at k for triples of consecutive ones: for three of them whose indices do not exceed k, as the input for Section 13.3, and for the triple $(\hat{c}_{k-1}, \hat{c}_k, \hat{c}_{k+1})$ at the top, for which we also bound from below the amount by which the inequality holds.

Both statements are read off from the ratios $\hat{c}_{n+1}/\hat{c}_n$. For a real number $t > 0$ let

$$\varphi(t) = (t-1)\,\log\frac{t+1}{t}.$$

For an integer $n \ge 1$ we have $e^{\varphi(n)} = (1 + 1/n)^{n-1}$, which by (2.3) is the ratio $a_{n+1}/a_n = c_n/c_{n-1}$; and $x_k = e^{-\varphi(k)}$ by the definition of the critical tilt. Hence, for every integer $n \ge 0$,

$$\hat{c}_{n+1} = x_k\,\frac{c_{n+1}}{c_n}\,\hat{c}_n = e^{\varphi(n+1)-\varphi(k)}\,\hat{c}_n. \tag{13.4}$$

In particular, $\hat{c}_k = \hat{c}_{k-1}$. By Lemma 2.1, φ does not decrease on the positive integers.

We shall need comparisons between the increments of φ. For these we write the logarithm in the definition of φ as an integral,

$$\log\frac{t+1}{t} = \int_0^1 \frac{dr}{t+r} \qquad (t > 0), \tag{13.5}$$

so that a linear combination of such logarithms, taken at several points t, is the integral over $0 \le r \le 1$ of a rational function of r. The first instance is a lower bound for one increment: for every integer $n \ge 2$,

$$\varphi(n) - \varphi(n-1) \ge \frac{3}{2n(n+1)}. \tag{13.6}$$

Indeed, the difference $\varphi(n) - \varphi(n-1)$ is the integral over $0 \le r \le 1$ of

$$\frac{n-1}{n+r} - \frac{n-2}{n-1+r} = \frac{1+r}{(n+r)(n-1+r)} \ge \frac{1+r}{n(n+1)},$$

and $\int_0^1 (1+r)\,dr = \frac{3}{2}$.

Finally, for $k \ge 2$ put

$$\Upsilon_k = (k+1)\log\frac{k+1}{k} - (k-1)\log\frac{k}{k-1}.$$

Then, for all real numbers μ, α and β,

$$e^{\mu+\alpha} + e^{-\mu+\beta} \ge 2\exp\Big(\frac{\mu}{k} + \frac{(k+1)\,\alpha + (k-1)\,\beta - \Upsilon_k}{2k}\Big). \tag{13.7}$$

This is the inequality between the weighted arithmetic and geometric means of $\frac{k}{k+1}\, e^{\mu+\alpha}$ and $\frac{k}{k-1}\, e^{-\mu+\beta}$, with weights $\frac{k+1}{2k}$ and $\frac{k-1}{2k}$.

Lemma 13.4. *Let $k \geq 3$ be an integer. Then, for every integer $j \geq 0$ with $j+2 \leq k$ and every real number μ,*

$$2\, e^{\mu/k}\, \hat{c}_{j+1} \leq e^{\mu}\, \hat{c}_{j+2} + e^{-\mu}\, \hat{c}_j.$$

Proof. Put $n = j+1$, so that $1 \leq n \leq k-1$. By the ratio formula (13.4) we have $\hat{c}_{j+2} = e^{\alpha}\, \hat{c}_{j+1}$ and $\hat{c}_j = e^{\beta}\, \hat{c}_{j+1}$ with

$$\alpha = \varphi(n+1) - \varphi(k), \qquad \beta = \varphi(k) - \varphi(n).$$

Since $\hat{c}_{j+1} > 0$, the claim is that $e^{\mu+\alpha} + e^{-\mu+\beta} \geq 2e^{\mu/k}$, and by the inequality (13.7) between the means it suffices to show that $(k+1)\,\alpha + (k-1)\,\beta \geq \Upsilon_k$. In terms of

$$\Phi(m) = (k+1)\,\varphi(m+1) - (k-1)\,\varphi(m) - 2\,\varphi(k) \qquad (m \geq 1),$$

this is the inequality

$$\Phi(n) \geq \Upsilon_k \qquad (1 \leq n \leq k-1).$$

We show first that $\Phi(m+1) \leq \Phi(m)$ for $1 \leq m \leq k-2$, so that $\Phi(n) \geq \Phi(k-1)$.

This rests on a comparison of two consecutive increments of φ: for every integer $i \geq 2$,

$$(i+2)\,\big(\varphi(i+1) - \varphi(i)\big) \leq i\,\big(\varphi(i) - \varphi(i-1)\big).$$

Indeed, by the definition of φ and the integral formula (13.5), the right-hand side minus the left-hand side is the integral over $0 \leq r \leq 1$ of

$$-\frac{i(i-2)}{i-1+r} + \frac{2(i^2-1)}{i+r} - \frac{i(i+2)}{i+1+r} = \frac{2\,(1-r^2)}{(i-1+r)(i+r)(i+1+r)} \geq 0.$$

Now let $1 \leq m \leq k-2$. Then

$$\begin{aligned}
\Phi(m) - \Phi(m+1) &= (k-1)\big(\varphi(m+1) - \varphi(m)\big) - (k+1)\big(\varphi(m+2) - \varphi(m+1)\big) \\
&\geq \Big(k - 1 - \frac{(k+1)(m+1)}{m+3}\Big)\big(\varphi(m+1) - \varphi(m)\big) \\
&= \frac{2\,(k-m-2)}{m+3}\,\big(\varphi(m+1) - \varphi(m)\big) \geq 0.
\end{aligned}$$

Here the first inequality is the comparison of increments for $i = m+1$, and the last one holds because $m+2 \leq k$ and because $\varphi(m+1) - \varphi(m) \geq 0$, as φ does not decrease on the positive integers.

It remains to show that $\Phi(k-1) \geq \Upsilon_k$. We note that $\Phi(k-1) = (k-1)\big(\varphi(k) - \varphi(k-1)\big)$, so that, by the definitions of φ and of Υ_k,

$$\Phi(k-1) - \Upsilon_k = (k-3)\,\Big(k \log \frac{k+1}{k} - (k-1) \log \frac{k}{k-1}\Big).$$

Here $k-3 \geq 0$, and the second factor is nonnegative, because the sequence $(1+1/i)^i$ is nondecreasing in $i \geq 1$ (see Section 2). Hence $\Phi(n) \geq \Phi(k-1) \geq \Upsilon_k$, which proves the lemma. □

The condition $j+2 \leq k$ excludes the triple at the top, $(\hat{c}_{k-1}, \hat{c}_k, \hat{c}_{k+1})$. For this triple the ratio formula (13.4) gives $\hat{c}_{k-1} = \hat{c}_k$ and $\hat{c}_{k+1} = e^{\lambda_k}\, \hat{c}_k$, where

$$\lambda_k = \varphi(k+1) - \varphi(k) \geq \frac{3}{2(k+1)(k+2)}$$

by the bound (13.6) for one increment. Thus, for every real number μ,

$$e^{\mu}\,\hat{c}_{k+1} + e^{-\mu}\,\hat{c}_{k-1} - 2\,e^{\mu/k}\,\hat{c}_k = \hat{c}_k\,\bigl(e^{\mu+\lambda_k} + e^{-\mu} - 2\,e^{\mu/k}\bigr). \tag{13.8}$$

The left-hand side of this identity is the surplus of the triple at the top, the amount by which it exceeds what property K at k requires for μ. The proof of Proposition 13.8 needs the following lower bound for it.

Lemma 13.5. *Let $k \geq 3$ be an integer. Then, for every real number μ,*

$$\hat{c}_k\,e^{\mu/k} \cdot \frac{1}{k} \cdot \frac{k^2-2k-1}{2k(k+1)(k+2)} \leq e^{\mu}\,\hat{c}_{k+1} + e^{-\mu}\,\hat{c}_{k-1} - 2\,e^{\mu/k}\,\hat{c}_k.$$

Proof. By the inequality (13.7) with $\alpha = \lambda_k$ and $\beta = 0$, and by the inequality $e^t \geq 1 + t$,

$$e^{\mu+\lambda_k} + e^{-\mu} \geq 2\,e^{\mu/k}\exp\Bigl(\frac{(k+1)\,\lambda_k - \Upsilon_k}{2k}\Bigr) \geq 2\,e^{\mu/k}\Bigl(1 + \frac{(k+1)\,\lambda_k - \Upsilon_k}{2k}\Bigr).$$

In view of the identity (13.8) and of $\hat{c}_k > 0$ it therefore suffices to show that

$$(k+1)\,\lambda_k - \Upsilon_k \geq \frac{k^2-2k-1}{2k(k+1)(k+2)}.$$

By the definitions of φ and of Υ_k and by the integral formula (13.5), the left-hand side is the integral over $0 \leq r \leq 1$ of

$$\frac{k-1}{k-1+r} - \frac{k(k+1)}{k+r} + \frac{k(k+1)}{k+1+r} = \frac{r\,\bigl(k^2-2k-1+(k-1)\,r\bigr)}{(k-1+r)(k+r)(k+1+r)}.$$

Since $k \geq 3$, we have $k^2-2k-1 > 0$, and for $0 \leq r \leq 1$ the denominator is at most $k(k+1)(k+2)$. So the integrand is at least $\frac{(k^2-2k-1)\,r}{k(k+1)(k+2)}$, and since $\int_0^1 r\,dr = \frac{1}{2}$, the required inequality follows. □

For $k \geq 3$ the left-hand side of the inequality of Lemma 13.5 is nonnegative, so that $2e^{\mu/k}\,\hat{c}_k \leq e^{\mu}\,\hat{c}_{k+1} + e^{-\mu}\,\hat{c}_{k-1}$ for every real number μ. Thus the triple $(\hat{c}_{k-1}, \hat{c}_k, \hat{c}_{k+1})$ at the top has property K at k for every $k \geq 3$. By Lemma 3.4, it is the triple $\bigl(\hat{z}_{k-1}^{(k-1)}, \hat{z}_k^{(k)}, \hat{z}_{k+1}^{(k+1)}\bigr)$ of the tilted counts of the three thresholds, with which the induction of Section 14 starts.

13.3. **Self-convolutions of the numbers $\hat{c}_n$.** For integers $\rho \geq 0$ and $j \geq 0$ let

$$Q_\rho(j) = \sum_{a=0}^{j} \hat{c}_{\rho+a}\,\hat{c}_{\rho+j-a},$$

a sum of products of two of the numbers $\hat{c}_n$. The sums of this kind that occur in the direct terms of thresholds $k-1$, k and $k+1$ (Section 13.4) are $Q_\rho(n)$, $Q_\rho(n+1)$ and $Q_\rho(n+2)$ with $\rho+n+2 = k$, and we show that they form a triple with property K at k.

For a sequence $f = (f(a))_{a\geq 0}$ of real numbers and an integer $j \geq 0$ we write $(f * f)(j) = \sum_{a=0}^{j} f(a)\,f(j-a)$ for the self-convolution of f at length j. Thus $Q_\rho(j)$ is the self-convolution at length j of the shifted sequence $a \mapsto \hat{c}_{\rho+a}$. Under the conditions of the following lemma the inequality of property K passes from the consecutive values of a sequence to its self-convolutions. In the proof, the expression whose sign is to be shown is sorted by one of the two indices of the convolutions; this produces inner terms, which are nonnegative by the inequality of property K for consecutive values, and two boundary terms, whose sum may be negative. If the

parameter μ of property K satisfies $\cosh\mu \geq e^{\mu/k}$, a crude lower bound for the inner terms suffices; otherwise the proof compares the sequence $e^{\mu a} f(a)$ with a solution of a three-term recurrence, whose contribution cancels the boundary terms.

Lemma 13.6. *Let k and n be integers with $n \geq 0$ and $n+2 \leq k$, and let $f = (f(a))_{a\geq 0}$ be a sequence of real numbers that satisfies the following conditions:*

(1) *$f(a) > 0$ for $0 \leq a \leq n+2$;*
(2) *$f(n+1) \leq f(a)$ for $0 \leq a \leq n$;*
(3) *$f(n+2) = f(n+1)$;*
(4) *$2\,e^{\mu/k}\, f(j+1) \leq e^{\mu}\, f(j+2) + e^{-\mu}\, f(j)$ for every real number $\mu \geq 0$ and every integer j with $0 \leq j \leq n$.*

Then the triple

$$\big((f*f)(n),\ (f*f)(n+1),\ (f*f)(n+2)\big)$$

has property K at k.

Proof. Fix a real number $\mu \geq 0$. We have to show that

$$e^{\mu}\,(f*f)(n+2) + e^{-\mu}\,(f*f)(n) - 2\,e^{\mu/k}\,(f*f)(n+1) \geq 0. \tag{13.9}$$

Sorting the left-hand side by one of the two indices of the convolutions gives the following identity, which holds for every sequence f and all real numbers A, B and D:

$$\begin{aligned} A\,(f*f)(n+2) + B\,(f*f)(n) - D\,(f*f)(n+1) \\ = \sum_{j=0}^{n} f(n-j)\,\big(A\,f(j+2) + B\,f(j) - D\,f(j+1)\big) \\ + f(n+1)\,\big(A\,f(1) - D\,f(0)\big) + A\,f(n+2)\,f(0). \end{aligned} \tag{13.10}$$

It is obtained by separating the term with $a = 0$ of $(f*f)(n+1)$ and the terms with $a = 0$ and $a = 1$ of $(f*f)(n+2)$, and by shifting the index of summation in the remaining terms. We call the two terms in its last line the boundary terms.

Suppose first that $\cosh\mu \geq e^{\mu/k}$. We use the identity (13.10) with $A = e^{\mu}$, $B = e^{-\mu}$ and $D = 2e^{\mu/k}$. By condition (4) every bracket in the sum over j is nonnegative, and by condition (2) we have $f(n-j) \geq f(n+1)$ for $0 \leq j \leq n$. So the sum does not increase when each factor $f(n-j)$ is replaced by $f(n+1)$. Let $S = \sum_{a=0}^{n+1} f(a)$. The sums of $f(j)$, of $f(j+1)$ and of $f(j+2)$ over $0 \leq j \leq n$ are then $S - f(n+1)$, $S - f(0)$ and $S - f(0) - f(1) + f(n+2)$. Inserting them and using condition (3), we find that the terms that contain $f(0)$ or $f(1)$ cancel against the two boundary terms of the identity, and that the left-hand side of (13.9) is at least

$$2\,f(n+1)\,\Big(\big(\cosh\mu - e^{\mu/k}\big)\,S + \sinh\mu\cdot f(n+1)\Big).$$

This is nonnegative by condition (1), because $\mu \geq 0$ and $\cosh\mu \geq e^{\mu/k}$, which proves the claim in this case.

Suppose now that $\cosh\mu < e^{\mu/k}$. Then $\mu > 0$, and there is a unique real number $\theta > 0$ with $\cosh\theta = e^{\mu/k}$; since $\cosh\mu < \cosh\theta$, we have $\mu < \theta$.

Put $h(a) = e^{\mu a} f(a)$ for $a \geq 0$. Then $(h*h)(j) = e^{\mu j}\,(f*f)(j)$ for every j, so that the left-hand side of (13.9), multiplied by $e^{\mu(n+1)}$, is

$$(h*h)(n+2) + (h*h)(n) - 2\cosh\theta\cdot(h*h)(n+1), \tag{13.11}$$

and we have to show that this number is nonnegative. Condition (4) for the present μ, multiplied by $e^{\mu(j+1)}$, says that

$$r_j = h(j+2) + h(j) - 2\cosh\theta \cdot h(j+1) \geq 0 \qquad (0 \leq j \leq n).$$

For an integer y let

$$H(y) = \sinh\big((n+2-y)\,\theta\big)\,h(n+1) - \sinh\big((n+1-y)\,\theta\big)\,h(n+2).$$

By the addition theorem $\sinh(x+\theta) + \sinh(x-\theta) = 2\cosh\theta\,\sinh x$, the sequence H satisfies the recurrence

$$H(y+1) + H(y-1) = 2\cosh\theta \cdot H(y)$$

for every integer y, and $H(n+1) = \sinh\theta \cdot h(n+1)$ and $H(n+2) = \sinh\theta \cdot h(n+2)$. Let

$$E(j) = \sinh\theta \cdot h(j) - H(j) \qquad (0 \leq j \leq n+2).$$

Then $E(n+1) = E(n+2) = 0$, and by the recurrence of H and since $r_j \geq 0$,

$$E(j+2) + E(j) - 2\cosh\theta \cdot E(j+1) = \sinh\theta \cdot r_j \geq 0 \qquad (0 \leq j \leq n).$$

It follows by downward induction on j that $0 \leq E(j+1) \leq E(j)$ for $0 \leq j \leq n+1$. For $j = n+1$ both numbers are 0, and if $0 \leq E(j+2) \leq E(j+1)$ for some $0 \leq j \leq n$, then

$$E(j) \geq 2\cosh\theta \cdot E(j+1) - E(j+2) \geq 2\,E(j+1) - E(j+2) \geq E(j+1) \geq 0,$$

where we used $\cosh\theta \geq 1$ and $E(j+1) \geq 0$. In particular, $E(j) \geq 0$ for $0 \leq j \leq n$.

We now apply the identity (13.10) to h, with $A = B = 1$ and $D = 2\cosh\theta$, and multiply by $\sinh\theta > 0$. In the term of index j we write $\sinh\theta \cdot h(n-j) = E(n-j) + H(n-j)$. The part $E(n-j)\,r_j$ is nonnegative. For the other part let

$$Y_j = H(n+1-j)\,h(j+1) - H(n-j)\,h(j) \qquad (0 \leq j \leq n+1).$$

By the recurrence of H at $y = n-j$ we have $H(n-j)\,r_j = Y_{j+1} - Y_j$, so that $\sum_{j=0}^{n} H(n-j)\,r_j = Y_{n+1} - Y_0$. Moreover, the recurrence at $y = n+1$ gives $H(n) = \sinh\theta\,\big(2\cosh\theta \cdot h(n+1) - h(n+2)\big)$, and hence

$$Y_0 = \sinh\theta\,\Big(h(n+1)\,\big(h(1) - 2\cosh\theta \cdot h(0)\big) + h(n+2)\,h(0)\Big).$$

This is the product of $\sinh\theta$ and the two boundary terms of the identity, which therefore cancel against $-Y_0$. Altogether, the product of $\sinh\theta$ and the number (13.11) is at least $Y_{n+1} = H(0)\,h(n+2) - H(-1)\,h(n+1)$, and by the definition of H,

$$\begin{aligned} Y_{n+1} = {} & 2\,h(n+1)\,h(n+2)\,\sinh\big((n+2)\,\theta\big) \\ & - h(n+2)^2\,\sinh\big((n+1)\,\theta\big) - h(n+1)^2\,\sinh\big((n+3)\,\theta\big). \end{aligned}$$

It remains to show that $Y_{n+1} \geq 0$. By condition (3) we have $h(n+2) = e^{\mu}\,h(n+1)$. With $R = e^{\mu}$ and $m = n+1$ we get $Y_{n+1} = h(n+1)^2\,P$, where

$$P = 2R\,\sinh\big((m+1)\,\theta\big) - R^2\,\sinh(m\theta) - \sinh\big((m+2)\,\theta\big).$$

Put $t = m\theta/2 \geq 0$. By the addition theorems,

$$P = 2\,\big(\sinh(t+\theta) - R\,\sinh t\big)\,\big(R\,\cosh t - \cosh(t+\theta)\big),$$

as one sees by multiplying out the right-hand side. The first factor is nonnegative. Indeed, $\mu < \theta$ gives $R \le e^\theta = \cosh\theta + \sinh\theta$, and $0 \le \sinh t \le \cosh t$, so that

$$R \sinh t \le \sinh t \cosh\theta + \cosh t \sinh\theta = \sinh(t+\theta).$$

For the second factor we use the elementary inequality $\sinh x \le x \cosh x$ for $x \ge 0$ at $x = t$ and at $x = \theta$, where $t\theta = m\theta^2/2$; it gives

$$\cosh(t+\theta) = \cosh t \cosh\theta + \sinh t \sinh\theta \le \cosh t \cosh\theta \left(1 + \frac{m\theta^2}{2}\right).$$

By Bernoulli's inequality and since $1 + \theta^2/2 \le \cosh\theta$,

$$1 + \frac{m\theta^2}{2} \le \left(1 + \frac{\theta^2}{2}\right)^m \le (\cosh\theta)^m,$$

and $(\cosh\theta)^{m+1} \le (\cosh\theta)^k = e^\mu = R$, because $\cosh\theta \ge 1$ and $m + 1 = n + 2 \le k$, and by the choice of θ. Hence $\cosh(t+\theta) \le R \cosh t$, so that the second factor is nonnegative as well. Thus $P \ge 0$ and $Y_{n+1} \ge 0$, so that the number (13.11) is nonnegative. This proves the claim in the second case as well. □

Lemma 13.7. *Let $k \ge 3$, $\rho \ge 0$ and $n \ge 0$ be integers with $\rho + n + 2 = k$. Then the triple*

$$\big(Q_\rho(n),\ Q_\rho(n+1),\ Q_\rho(n+2)\big)$$

has property K at k.

Proof. We apply Lemma 13.6 with $f(a) = \hat{c}_{\rho+a}$, for which $(f * f)(j) = Q_\rho(j)$. Condition (1) holds because the $\hat{c}_n$ are positive, and condition (2) because $f(n+1) = \hat{c}_{k-1}$ and the $\hat{c}_n$ do not increase up to index k, by (6.2). Condition (3) is the equality $\hat{c}_k = \hat{c}_{k-1}$, which the ratio formula (13.4) gives, and condition (4) is the inequality of Lemma 13.4 at index $\rho + j$, where $\rho + j + 2 \le k$ for $0 \le j \le n$. □

13.4. **The direct term.** For integers ρ and y with $1 \le \rho \le y - 1$, the *direct term* is

$$\mathcal{S}_\rho(y) = x_k\, \hat{c}_{\rho-1}\, \hat{c}_y + \frac{x_k}{2} \cdot \frac{y+\rho+1}{y+\rho}\, Q_\rho(y-\rho-1).$$

It is the part of $\hat{z}^{(y)}_{y+\rho}$ that comes from the terms of the last-car recursion in which both factors lie in the free range of threshold y, as Section 14 shows. The main result of this section, Proposition 13.8 below, is that for $k \ge 7$ the direct terms of thresholds $k-1$, k and $k+1$ form a triple with property K at k. All three are formed with one and the same tilt x_k, the critical tilt of the middle threshold.

Both ingredients of the direct term have property K: the triple $(\hat{c}_{k-1}, \hat{c}_k, \hat{c}_{k+1})$ by Lemma 13.5, and the triple of the three sums $Q_\rho(y-\rho-1)$ by Lemma 13.7. If these sums carried the factor $\frac{x_k}{2}$ alone, the proposition would follow, since property K is preserved by nonnegative multiples and by sums. But they are multiplied by $\frac{y+\rho+1}{y+\rho}$, which *decreases* as y grows. In terms of the computation (13.2) this is the case $L < M$: the factor moves μ downward, by an amount of order $(k+\rho)^{-2}$, and for μ below this amount the family of inequalities of property K has no member to which one could move. So the second term of $\mathcal{S}_\rho$ may fail the inequality of property K, and the amount by which it fails has to be made up by the first term, which satisfies its inequality with the surplus (13.8). The proof compares the two amounts for each $\mu \ge 0$, in one way for $\mu \ge \frac{1}{k(k+2)}$, where that computation applies to the sums Q_ρ, and in another for smaller μ.

Proposition 13.8. *Let $k \geq 7$ and ρ be integers with $1 \leq \rho \leq k-2$. Then the triple*

$$\big(\mathcal{S}_\rho(k-1),\ \mathcal{S}_\rho(k),\ \mathcal{S}_\rho(k+1)\big)$$

of direct terms has property K at k.

Proof. Let $N = k + \rho$ and $n = k - \rho - 2$, so that $k+1 \leq N \leq 2k-2$ and $n \geq 0$. We write

$$Q_- = Q_\rho(n), \qquad Q_0 = Q_\rho(n+1), \qquad Q_+ = Q_\rho(n+2)$$

for the sums $Q_\rho(y-\rho-1)$ with $y = k-1$, k, $k+1$, and

$$F_- = \frac{N}{N-1}, \qquad F_0 = \frac{N+1}{N}, \qquad F_+ = \frac{N+2}{N+1}$$

for the values of $\frac{y+\rho+1}{y+\rho}$ at $y = k-1$, k, $k+1$. The numbers Q_-, Q_0 and Q_+ are positive, and since $\rho + n + 2 = k$, the triple (Q_-, Q_0, Q_+) has property K at k by Lemma 13.7. In this notation $\mathcal{S}_\rho(k-1) = x_k\,\hat{c}_{\rho-1}\,\hat{c}_{k-1} + \frac{x_k}{2}\,F_-\,Q_-$, and similarly for thresholds k and $k+1$. Fix a real number $\mu \geq 0$, and put

$$\begin{aligned}\mathrm{Sur} &= e^{\mu}\,\hat{c}_{k+1} + e^{-\mu}\,\hat{c}_{k-1} - 2\,e^{\mu/k}\,\hat{c}_k,\\ \mathrm{Def} &= e^{\mu}\,F_+\,Q_+ + e^{-\mu}\,F_-\,Q_- - 2\,e^{\mu/k}\,F_0\,Q_0.\end{aligned}$$

Thus Sur is the surplus (13.8) of the triple at the top, and Def is the corresponding quantity for the second terms of the $\mathcal{S}_\rho(y)$; it may be negative, and $-$Def is then the defect of these terms, the amount by which they fail the inequality of property K. We have

$$e^{\mu}\,\mathcal{S}_\rho(k+1) + e^{-\mu}\,\mathcal{S}_\rho(k-1) - 2\,e^{\mu/k}\,\mathcal{S}_\rho(k) = x_k\,\hat{c}_{\rho-1}\,\mathrm{Sur} + \frac{x_k}{2}\,\mathrm{Def},$$

and we have to show that the right-hand side is nonnegative.

We first reduce the claim to a lower bound for each of Sur and Def. Suppose that there is a real number $\tau \geq 0$ such that

$$\mathrm{Sur} \geq \hat{c}_k\,\tau \qquad \text{and} \qquad \mathrm{Def} \geq -2\,\hat{c}_\rho\,\hat{c}_k\,\tau. \tag{13.12}$$

Then $x_k\,\hat{c}_{\rho-1}\,\mathrm{Sur} + \frac{x_k}{2}\,\mathrm{Def}$ is at least $x_k\,\hat{c}_k\,\tau\,(\hat{c}_{\rho-1} - \hat{c}_\rho)$, and this is nonnegative, because $\hat{c}_\rho \leq \hat{c}_{\rho-1}$ by (6.2). So it suffices to find such a τ. We do this separately for $\mu \leq \mu_0$ and for $\mu \geq \mu_0$, where $\mu_0 = \frac{1}{k(k+2)}$.

Consider first the case $\mu \leq \mu_0$. Here we take $\tau = \frac{e^{\mu}}{N(N+1)}$.

We begin with Def. We have $F_0 - F_+ = \frac{1}{N(N+1)}$ and $F_- - F_0 = \frac{1}{N(N-1)} \geq \frac{1}{N(N+1)}$. In the decomposition

$$\mathrm{Def} = F_0\,\big(e^{\mu}Q_+ + e^{-\mu}Q_- - 2\,e^{\mu/k}Q_0\big) - (F_0 - F_+)\,e^{\mu}Q_+ + (F_- - F_0)\,e^{-\mu}Q_-$$

the first term is nonnegative by property K of the triple (Q_-, Q_0, Q_+), the second is $-\frac{1}{N(N+1)}\,e^{\mu}Q_+$, and the third is at least $\frac{1}{N(N+1)}\,e^{-\mu}Q_-$. Hence

$$\mathrm{Def} \geq -\frac{1}{N(N+1)}\,\big(e^{\mu}\,Q_+ - e^{-\mu}\,Q_-\big).$$

The sum Q_+ has two more terms than Q_-, and we claim that

$$e^{\mu}\,Q_+ - e^{-\mu}\,Q_- \leq 2\,e^{\mu}\,\hat{c}_\rho\,\hat{c}_k.$$

For the proof we note first that for $\mu \le \mu_0$ the numbers $\hat{c}_i$ below the top fall at each step by a factor of at least e^{μ}:

$$e^{\mu}\,\hat{c}_{i+1} \le \hat{c}_i \qquad (0 \le i \le k-2).$$

Indeed, $\hat{c}_i = e^{\varphi(k)-\varphi(i+1)}\,\hat{c}_{i+1}$ by the ratio formula (13.4), and

$$\varphi(k) - \varphi(i+1) \ge \varphi(k) - \varphi(k-1) \ge \frac{3}{2k(k+1)} \ge \frac{1}{k(k+2)} \ge \mu;$$

the first inequality holds because $i+1 \le k-1$ and φ does not decrease on the positive integers, and the second is the bound (13.6) for $n = k$. Now in $Q_+ = \sum_{a=0}^{n+2} \hat{c}_{\rho+a}\,\hat{c}_{\rho+n+2-a}$ the terms with $a = 0$ and $a = n+2$ are both equal to $\hat{c}_\rho\,\hat{c}_k$, because $\rho + n + 2 = k$. The other terms are $\hat{c}_{\rho+b+1}\,\hat{c}_{\rho+(n-b)+1}$ for $0 \le b \le n$. Since $\rho + b$ and $\rho + (n-b)$ are at most $k-2$, the inequality $e^{\mu}\,\hat{c}_{i+1} \le \hat{c}_i$ applies to both factors and gives

$$e^{\mu}\,\hat{c}_{\rho+b+1}\,\hat{c}_{\rho+(n-b)+1} \le e^{-\mu}\,\hat{c}_{\rho+b}\,\hat{c}_{\rho+(n-b)}.$$

The right-hand side is $e^{-\mu}$ times the term with $a = b$ of Q_-. Summing over $0 \le b \le n$, we obtain $e^{\mu}\big(Q_+ - 2\,\hat{c}_\rho\,\hat{c}_k\big) \le e^{-\mu}\,Q_-$, which is the claim. Inserted into the lower bound for Def obtained above, it gives $\mathrm{Def} \ge -2\,\hat{c}_\rho\,\hat{c}_k\,\tau$, since $\tau = \frac{e^{\mu}}{N(N+1)}$.

We turn to Sur. For these μ the bound of Lemma 13.5 does not suffice: at $\mu = 0$ it is about $\hat{c}_k/\big(2(k+1)(k+2)\big)$, whereas $\hat{c}_k\,\tau$ is then $\hat{c}_k/\big((k+1)(k+2)\big)$ for $\rho = 1$. We use another bound. Recall that Sur equals $\hat{c}_k$ times $e^{\mu+\lambda_k} + e^{-\mu} - 2\,e^{\mu/k}$. By the inequality $e^t \ge 1 + t$ for $t = \mu + \lambda_k$ and for $t = -\mu$, and by $\lambda_k \ge \frac{3}{2(k+1)(k+2)}$,

$$\mathrm{Sur} \ge \hat{c}_k\,\big(2 + \lambda_k - 2\,e^{\mu/k}\big) \ge \hat{c}_k\,\Big(\frac{3}{2(k+1)(k+2)} - 2\,\big(e^{\mu/k} - 1\big)\Big).$$

It remains to show that the bracket on the right is at least τ. Put $U = \frac{1}{(k+1)(k+2)}$. We claim that

$$\tau \le \frac{63}{62}\,U \qquad \text{and} \qquad e^{\mu/k} - 1 \le \frac{9}{55}\,U;$$

that bracket is then at least $\big(\frac{3}{2} - \frac{18}{55}\big)\,U = \frac{129}{110}\,U$, which is larger than $\frac{63}{62}\,U \ge \tau$. For the first bound, $\frac{1}{N(N+1)} \le U$ because $N \ge k+1$, and $e^{\mu} \le \frac{1}{1-\mu} \le \frac{63}{62}$ because $\mu \le \mu_0 \le \frac{1}{63}$ for $k \ge 7$. For the second, $k \ge 7$ gives $\frac{\mu}{k} \le \frac{1}{k^2(k+2)} = \frac{k+1}{k^2}\,U \le \frac{8}{49}\,U$ and $\frac{\mu}{k} \le \frac{1}{441}$, so that $e^{\mu/k} - 1 \le \frac{\mu/k}{1-\mu/k} \le \frac{441}{440} \cdot \frac{8}{49}\,U = \frac{9}{55}\,U$. Thus $\mathrm{Sur} \ge \hat{c}_k\,\tau$, and both bounds (13.12) hold in the case $\mu \le \mu_0$.

Now let $\mu \ge \mu_0$. In this case we take

$$\tau = e^{\mu/k} \cdot \frac{N+1}{N} \cdot (2k - N) \cdot \frac{N-k+1}{N^2(N+2)\,k},$$

which is nonnegative because $k+1 \le N \le 2k-2$.

We begin again with Def. Let $L = \log(F_+/F_0)$ and $M = \log(F_-/F_0)$, where

$$\frac{F_+}{F_0} = 1 - \frac{1}{(N+1)^2} \qquad \text{and} \qquad \frac{F_-}{F_0} = 1 + \frac{1}{N^2-1}.$$

The inequalities $1 - \frac{1}{r} \le \log r \le r - 1$ for $r > 0$ give

$$L \ge -\frac{1}{N(N+2)}, \qquad \frac{1}{N^2} \le M \le \frac{1}{(N-1)(N+1)}.$$

Hence $\frac{M-L}{2} \le \frac{1}{(N-1)(N+1)} \le \frac{1}{k(k+2)} \le \mu$, where we used $N(N+2) \ge (N-1)(N+1)$ and $N \ge k+1$. So $\mu + \frac{L-M}{2} \ge 0$, and the inequality (13.2) applies to the triple (Q_-, Q_0, Q_+). Multiplied by F_0, it reads

$$2\, e^{\mu/k} \exp\Big(\frac{(k+1)L + (k-1)M}{2k}\Big)\, F_0\, Q_0 \le e^{\mu}\, F_+\, Q_+ + e^{-\mu}\, F_-\, Q_-.$$

By the bounds for L and M,

$$(k+1)\, L + (k-1)\, M \ge -\frac{k+1}{N(N+2)} + \frac{k-1}{N^2} = -\frac{2\,(N-k+1)}{N^2(N+2)},$$

so that the exponential is at least $1 - \frac{N-k+1}{N^2(N+2)\,k}$, by $e^t \ge 1+t$. Since $Q_0 > 0$, it follows that

$$\mathrm{Def} \ge -2\, e^{\mu/k}\, F_0 \cdot \frac{N-k+1}{N^2(N+2)\,k} \cdot Q_0.$$

Finally, the sum $Q_0 = \sum_{b=0}^{n+1} \hat{c}_{\rho+b}\, \hat{c}_{\rho+n+1-b}$ has $n+2 = 2k-N$ terms, and each of them is at most $\hat{c}_\rho\, \hat{c}_{\rho+n+1} = \hat{c}_\rho\, \hat{c}_{k-1} = \hat{c}_\rho\, \hat{c}_k$. Indeed, since $c_m = a_{m+1}$, Lemma 2.2 with $u = \rho+1$, $v = \rho+n+2-b$ and $s = b$ says that $c_{\rho+b}\, c_{\rho+n+1-b} \le c_\rho\, c_{\rho+n+1}$, and the tilt multiplies both sides by the same power of x_k. Thus $Q_0 \le (2k-N)\, \hat{c}_\rho\, \hat{c}_k$, and we obtain $\mathrm{Def} \ge -2\, \hat{c}_\rho\, \hat{c}_k\, \tau$.

For Sur we use Lemma 13.5, by which

$$\mathrm{Sur} \ge \hat{c}_k\, e^{\mu/k} \cdot \frac{1}{k} \cdot \frac{k^2-2k-1}{2k(k+1)(k+2)}.$$

After division by $\hat{c}_k\, e^{\mu/k}/k$, the inequality $\mathrm{Sur} \ge \hat{c}_k\, \tau$ therefore follows from

$$\frac{(N+1)\,(2k-N)\,(N-k+1)}{N^3\,(N+2)} \le \frac{k^2-2k-1}{2k(k+1)(k+2)}.$$

To prove this inequality, note that $2k-N$ and $N-k+1$ are positive and add up to $k+1$, so that $4\,(2k-N)(N-k+1) \le (k+1)^2$; and that $(N+1)\,(k+1)^3 \le (N+2)\,N^3$, because $N \ge k+1$. Hence the left-hand side of the displayed inequality is at most

$$\frac{(N+1)\,(k+1)^2}{4\,N^3\,(N+2)} \le \frac{1}{4(k+1)}.$$

Finally, $\frac{1}{4(k+1)} \le \frac{k^2-2k-1}{2k(k+1)(k+2)}$, because this inequality is equivalent to $k^2-6k-2 \ge 0$, which holds since $k \ge 7$. Thus both bounds hold in this case as well.

We have thus found, for every $\mu \ge 0$, a real number $\tau \ge 0$ with the two bounds (13.12). By the reduction at the beginning of the proof, $x_k\, \hat{c}_{\rho-1}\, \mathrm{Sur} + \frac{x_k}{2}\, \mathrm{Def}$ is therefore nonnegative for every $\mu \ge 0$, that is, the triple of the direct terms has property K at k. □

14. Convexity along the diagonal and the weighted debt

With property K available for the triples of Section 13, an induction along the last-car recursion gives property K, and with it convexity along the diagonal, for the tilted counts of thresholds $k-1$, k and $k+1$ in the first layer; a second induction extends the convexity, in a weighted form, to all lengths (Corollary 14.6). Summation by parts then yields the bound of the weighted debt, Proposition 14.7: for a column $k \ge 7$, a weighted sum of the tilted steps of the column over rows $k, \dots, J$ is bounded from below by three terms that belong to rows $J-1$ and J. In the proof of Proposition 1.2 (Section 15) this is the bound by which the

rows before the crossing are outweighed. As in Section 6, a hat denotes the tilted form of a sequence at the critical tilt x_k of the column under consideration; thus $\hat{z}_N^{(y)} = z_N^{(y)}\, x_k^N$, $\hat{c}_n = c_n\, x_k^n$, $\hat{\beta}_N^{(\ell)} = \beta_N^{(\ell)}\, x_k^N$ and $\hat{\delta}_n = \delta_n^{(k)}\, x_k^n$.

14.1. **The last-car recursion in the first layer of a threshold.** Let $k \geq 1$ be an integer and let $y \geq 1$ be a threshold. By (3.5) with $x = x_k$, the tilted counts of threshold y satisfy

$$(N+1)\,\hat{z}_{N+1}^{(y)} = x_k \sum_{j=0}^{N} \min(j+1, y)\,\hat{z}_j^{(y)}\,\hat{z}_{N-j}^{(y)} \tag{14.1}$$

for every integer $N \geq 0$. Recall from Section 3 that the lengths $n \leq y$ form the free range of threshold y and the lengths $y+1 \leq n \leq 2y+1$ its first layer. A factor $\hat{z}_i^{(y)}$ whose length i lies in the free range is one of the numbers $\hat{c}_i$: by Lemma 3.4,

$$\hat{z}_i^{(y)} = \hat{c}_i \qquad (0 \leq i \leq y).$$

A length of the first layer is of the form $y + P + 1$ with $0 \leq P \leq y$. In the recursion with $N = y + P$ the lengths of the two factors add up to $y + P \leq 2y$, so that at most one of the two factors lies outside the free range. Let

$$\mathcal{D}(y, P) = \sum_{a=0}^{y-P} \min(P + a + 1, y)\,\hat{c}_{P+a}\,\hat{c}_{y-a}$$

be the sum of the terms of the recursion at length $y + P + 1$ in which both factors lie in the free range. Sorting the other terms brings the recursion into the form that supports an induction on the amount by which the length exceeds the threshold.

Lemma 14.1. *Let $k \geq 1$, $y \geq 1$ and P be integers with $0 \leq P \leq y$. Then*

$$(y + P + 1)\,\hat{z}_{y+P+1}^{(y)} = x_k\Big(\sum_{i=0}^{P-1} (y + i + 1)\,\hat{c}_i\,\hat{z}_{y+P-i}^{(y)} + \mathcal{D}(y, P)\Big).$$

Proof. We split the sum in the recursion (14.1) with $N = y + P$ into the three ranges $0 \leq j < P$, $P \leq j \leq y$ and $y < j \leq y + P$. In the first range we write $j = i$; then $\min(j + 1, y) = i + 1$, and the first factor is $\hat{c}_i$, because its length i lies in the free range. In the third range we write $j = y + P - i$ with $0 \leq i < P$; then $\min(j + 1, y) = y$, and the second factor is $\hat{c}_i$. For each i the two terms contain the same product, so that together the two ranges give

$$\sum_{i=0}^{P-1} \big((i + 1) + y\big)\,\hat{c}_i\,\hat{z}_{y+P-i}^{(y)}.$$

In the second range we write $j = P + a$ with $0 \leq a \leq y - P$. Both factors then lie in the free range, and the terms of this range are those of $\mathcal{D}(y, P)$. □

14.2. **Property K and convexity for $k \leq N \leq 2k - 2$.** Fix a column $k \geq 2$. For an integer $\rho \geq 0$ the three tilted counts

$$\hat{z}_{k-1+\rho}^{(k-1)}, \qquad \hat{z}_{k+\rho}^{(k)}, \qquad \hat{z}_{k+1+\rho}^{(k+1)}$$

lie on a diagonal: from one to the next the threshold and the length are both raised by one, so that each length exceeds its threshold by ρ. All three are tilted by the same number x_k, the critical tilt of the middle threshold. We show by induction

on ρ, along the recursion of Lemma 14.1, that they form a triple with property K (Definition 13.1).

Proposition 14.2. *Let $k \geq 7$ and ρ be integers with $0 \leq \rho \leq k-2$. Then the triple*

$$\big(\hat{z}^{(k-1)}_{k-1+\rho},\ \hat{z}^{(k)}_{k+\rho},\ \hat{z}^{(k+1)}_{k+1+\rho}\big)$$

has property K at k.

Proof. Before the induction on ρ we bring the recursion of Lemma 14.1, for one threshold y at a time, into a form in which $\mathcal{D}(y,P)$ is replaced by the direct term of Section 13. Recall from Section 13 that $Q_s(j) = \sum_{a=0}^{j} \hat{c}_{s+a}\,\hat{c}_{s+j-a}$ and that the direct term is

$$\mathcal{S}_\rho(y) = x_k\,\hat{c}_{\rho-1}\,\hat{c}_y + \frac{x_k}{2}\cdot\frac{y+\rho+1}{y+\rho}\,Q_\rho(y-\rho-1).$$

We first evaluate $\mathcal{D}(y,P)$. For integers y and P with $P \geq 0$ and $P+2 \leq y$, a direct computation with the sum that defines $\mathcal{D}(y,P)$ in Section 14.1 gives

$$\mathcal{D}(y,P) = (y+P+1)\,\hat{c}_P\,\hat{c}_y + \frac{y+P+2}{2}\,Q_{P+1}(y-P-2).$$

Here the first term on the right comes from the terms with $a=0$ and $a=y-P$, which are distinct because $P+2 \leq y$ and have the weights $P+1$ and y. The products of the other terms add up to $Q_{P+1}(y-P-2)$, and their weights $P+a+1$ may be replaced by $\frac{1}{2}(y+P+2)$, the mean of the weights of the two terms with indices a and $y-P-a$, which contain the same product.

The recursion of Lemma 14.1, divided by the length, now takes the form announced above: for integers y and ρ with $1 \leq \rho \leq y-1$,

$$\hat{z}^{(y)}_{y+\rho} = \sum_{i=0}^{\rho-2} x_k\,\hat{c}_i\,\frac{y+i+1}{y+\rho}\,\hat{z}^{(y)}_{y+\rho-1-i} + \mathcal{S}_\rho(y). \tag{14.2}$$

Indeed, in the identity of Lemma 14.1 with $P = \rho-1$ we insert the value of $\mathcal{D}(y,\rho-1)$ just obtained, which applies because $\rho+1 \leq y$, and divide by $y+\rho$; by the definition of $\mathcal{S}_\rho(y)$ the result is the stated identity.

We turn to the induction on ρ. Recall that property K at k is preserved when triples are added entry by entry and when a triple is multiplied by a nonnegative number.

Let $\rho = 0$. The three lengths lie in the free range of their thresholds, so that the triple is $(\hat{c}_{k-1}, \hat{c}_k, \hat{c}_{k+1})$ by Lemma 3.4, and it has property K at k by Lemma 13.5, whose left-hand side is nonnegative for $k \geq 3$.

Let now $1 \leq \rho \leq k-2$, and suppose that the proposition holds for every smaller value of ρ. Since $\rho \leq k-2$, the recursion (14.2) holds for $y = k-1$, k and $k+1$, with the same range of i and the same factors $x_k\,\hat{c}_i$. Hence the triple of the proposition is the sum of the following triples: the triple $\big(\mathcal{S}_\rho(k-1),\ \mathcal{S}_\rho(k),\ \mathcal{S}_\rho(k+1)\big)$ of the direct terms; and, for each i with $0 \leq i \leq \rho-2$, the triple with the entries

$$x_k\,\hat{c}_i\,\frac{y+i+1}{y+\rho}\,\hat{z}^{(y)}_{y+\rho'} \qquad (y = k-1,\ k,\ k+1),$$

where $\rho' = \rho-1-i$. It suffices to show that each of these triples has property K at k.

The first of them has property K at k by Proposition 13.8, since $k \geq 7$ and $1 \leq \rho \leq k-2$. For the others, fix i with $0 \leq i \leq \rho-2$. Then $1 \leq \rho' \leq \rho-1$, so that

by the induction hypothesis the numbers $\hat{z}^{(k-1)}_{k-1+\rho'}$, $\hat{z}^{(k)}_{k+\rho'}$ and $\hat{z}^{(k+1)}_{k+1+\rho'}$ form a triple with property K at k, and its middle entry is nonnegative. By Lemma 13.2 with $p = i+1$ and $q = \rho$, the triple keeps property K at k when its entries are multiplied by the values of $(y+i+1)/(y+\rho)$ at $y = k-1$, k and $k+1$. Multiplying further by $x_k\,\hat{c}_i > 0$, we obtain the triple that belongs to i in the sum above. This completes the induction. □

We record two consequences of the proposition: the convexity of the tilted counts along the diagonal, and the form of it that enters the induction of the next subsection, in which the three counts carry weights that increase by one from each to the next.

Corollary 14.3. *Let $k \geq 7$ and N be integers with $k \leq N \leq 2k-2$. Then*

$$\hat{z}^{(k+1)}_{N+1} - 2\,\hat{z}^{(k)}_{N} + \hat{z}^{(k-1)}_{N-1} \geq 0.$$

Proof. This is the case $\mu = 0$ of Proposition 14.2 with $\rho = N-k$. □

In terms of the counts, after division by x_k^{N-1} and multiplication by $(N+1)!$, the corollary says that

$$x_k^2\,A_{N+1,k} - 2\,(N+1)\,x_k\,A_{N,k-1} + N(N+1)\,A_{N-1,k-2} \geq 0$$

for $k \geq 7$ and $k \leq N \leq 2k-2$.

For the second consequence we write, for a column $k \geq 2$, a real number λ and an integer $N \geq 1$,

$$\Delta_\lambda(N) = (\lambda+1)\,\hat{z}^{(k+1)}_{N+1} - 2\lambda\,\hat{z}^{(k)}_{N} + (\lambda-1)\,\hat{z}^{(k-1)}_{N-1}. \tag{14.3}$$

Corollary 14.4. *Let $k \geq 7$ and N be integers with $k \leq N \leq 2k-2$, and let $\lambda \geq k$ be a real number. Then*

$$(\lambda+1)\,\hat{z}^{(k+1)}_{N+1} - 2\lambda\,\hat{z}^{(k)}_{N} + (\lambda-1)\,\hat{z}^{(k-1)}_{N-1} \geq 0.$$

Proof. Apply Lemma 13.3 with $s = \lambda - k \geq 0$ and $t = 0$ to the triple of Proposition 14.2 with $\rho = N-k$, whose middle entry $\hat{z}^{(k)}_N$ is nonnegative: the lemma multiplies the three entries by $\lambda-1$, λ and $\lambda+1$, and the case $\mu = 0$ of property K for the resulting triple is the assertion. □

14.3. **Weighted convexity for $N \geq 2k-1$.** For $N \geq 2k-1$ we prove the convexity along the diagonal in the form that the recursion (14.1) produces, in which each tilted count is multiplied by its length. We first collect the facts about the tilted counts of the three thresholds of a column $k \geq 2$ that the remaining proofs of this section use.

Recall from Section 3 that $0 \leq z_i^{(y)} \leq z_i^{(y')}$ for thresholds $y \leq y'$ and that $\beta_i^{(\ell)} = z_i^{(\ell+1)} - z_i^{(\ell)}$ for $\ell \geq 1$. Since $x_k > 0$, the tilted counts of the three thresholds at one length are ordered,

$$0 \leq \hat{z}_i^{(k-1)} \leq \hat{z}_i^{(k)} \leq \hat{z}_i^{(k+1)} \qquad (i \geq 0),$$

and their differences are the tilted counts of levels $k-1$ and k,

$$\hat{\beta}_i^{(k-1)} = \hat{z}_i^{(k)} - \hat{z}_i^{(k-1)}, \qquad \hat{\beta}_i^{(k)} = \hat{z}_i^{(k+1)} - \hat{z}_i^{(k)} \qquad (i \geq 0),$$

which are nonnegative. Since $\delta_i^{(k)} = \beta_i^{(k)} - \beta_i^{(k-1)}$, the tilted step of the column is

$$\hat{\delta}_i = \hat{\beta}_i^{(k)} - \hat{\beta}_i^{(k-1)} = \hat{z}_i^{(k+1)} - 2\,\hat{z}_i^{(k)} + \hat{z}_i^{(k-1)} \qquad (i \geq 0).$$

By Lemma 3.4 the tilted counts of the three thresholds agree as long as the length lies in the free range of all three:

$$\hat{z}_i^{(k-1)} = \hat{z}_i^{(k)} = \hat{z}_i^{(k+1)} = \hat{c}_i \qquad (0 \le i \le k-1).$$

Next, the tilted counts do not decrease along a diagonal from length $k-1$ on: for every threshold $y \ge 1$,

$$\hat{z}_i^{(y)} \le \hat{z}_{i+1}^{(y+1)} \qquad (i \ge k-1). \tag{14.4}$$

Indeed, $\hat{z}_i^{(y)} = (z_i^{(y)}/c_i)\,\hat{c}_i$, and here $z_i^{(y)}/c_i \le z_{i+1}^{(y+1)}/c_{i+1}$ by Proposition 3.5 and $\hat{c}_i \le \hat{c}_{i+1}$ by (6.2), since $i+1 \ge k$.

Finally, we need three inequalities between the numbers $\hat{c}_n$ near index k:

$$\hat{c}_{k-2} \ge \hat{c}_{k-1}, \qquad \hat{c}_{k-1} \le \hat{c}_k, \qquad (k-1)\,\hat{c}_{k-2} \le k\,\hat{c}_{k-1}. \tag{14.5}$$

The first two are instances of (6.2). For the third, the identity (6.1) with $x = x_k$ and $j = k-2$ gives $\hat{c}_{k-1} = x_k\bigl(1 + \frac{1}{k-1}\bigr)^{k-2}\hat{c}_{k-2}$, and hence

$$\begin{aligned} k\,\hat{c}_{k-1} &= (k-1)\,x_k\Bigl(1 + \frac{1}{k-1}\Bigr)^{k-1}\hat{c}_{k-2} \\ &\ge (k-1)\,x_k\Bigl(1 + \frac{1}{k}\Bigr)^{k-1}\hat{c}_{k-2} = (k-1)\,\hat{c}_{k-2}, \end{aligned}$$

where the last step is the equation $x_k\,(1+1/k)^{k-1} = 1$ that defines the critical tilt.

The following proposition is proved by induction on N. By the recursion (14.1) for the three thresholds, the left-hand side of its inequality, divided by x_k, is a combination of products of two tilted counts of smaller lengths. In the proof, the products in which one factor has length at most $k-2$ combine to numbers $\Delta_\lambda(M)$ with $M < N$, which Corollary 14.4 and the induction hypothesis show to be nonnegative, and the other products are controlled by Corollary 6.6.

Proposition 14.5. *Let $k \ge 7$ and N be integers with $N \ge 2k-1$. Then*

$$(N+1)\,\hat{z}_{N+1}^{(k+1)} - 2N\,\hat{z}_N^{(k)} + (N-1)\,\hat{z}_{N-1}^{(k-1)} \ge 0.$$

Proof. In the notation (14.3) the assertion reads $\Delta_N(N) \ge 0$, and we prove it by induction on N.

Let $N \ge 2k-1$, and suppose that $\Delta_M(M) \ge 0$ for every integer M with $2k-1 \le M < N$. We write $N = 2k-1+r$ with $r \ge 0$.

We first show that the weighted second differences at smaller lengths are nonnegative: for integers M and real numbers λ,

$$\Delta_\lambda(M) \ge 0 \qquad (k \le M \le N-1,\ k \le \lambda \le 2k-1), \tag{14.6}$$

and moreover $\Delta_k(k-1) \ge 0$.

For $M \le 2k-2$ the first of these inequalities is Corollary 14.4. Let $2k-1 \le M \le N-1$. Comparing the coefficients of the three tilted counts in the definition of $\Delta_\lambda(M)$, one sees that

$$M\,\Delta_\lambda(M) = \lambda\,\Delta_M(M) + (M-\lambda)\,\bigl(\hat{z}_{M+1}^{(k+1)} - \hat{z}_{M-1}^{(k-1)}\bigr).$$

Here $\Delta_M(M) \ge 0$ by the induction hypothesis, $0 \le \lambda \le M$, and $\hat{z}_{M-1}^{(k-1)} \le \hat{z}_M^{(k)} \le \hat{z}_{M+1}^{(k+1)}$ by the monotonicity (14.4) along the diagonal. Hence $\Delta_\lambda(M) \ge 0$. As

for $\Delta_k(k-1)$, the lengths of its three tilted counts lie in the free range of their thresholds, so that

$$\Delta_k(k-1) = (k+1)\,\hat{c}_k - 2k\,\hat{c}_{k-1} + (k-1)\,\hat{c}_{k-2},$$

and this is nonnegative by the first two inequalities in (14.5).

We now express $\Delta_N(N)$ through tilted counts of smaller lengths. Applied to thresholds $k+1$, k and $k-1$, at lengths $N+1$, N and $N-1$, the recursion gives $(N+1)\,\hat{z}^{(k+1)}_{N+1} = x_k\,\Sigma^+$, $N\,\hat{z}^{(k)}_N = x_k\,\Sigma^0$ and $(N-1)\,\hat{z}^{(k-1)}_{N-1} = x_k\,\Sigma^-$, where

$$\begin{aligned}
\Sigma^+ &= \sum_{i=0}^{N} \min(i+1, k+1)\,\hat{z}^{(k+1)}_i\,\hat{z}^{(k+1)}_{N-i},\\
\Sigma^0 &= \sum_{i=0}^{N-1} \min(i+1, k)\,\hat{z}^{(k)}_i\,\hat{z}^{(k)}_{N-1-i},\\
\Sigma^- &= \sum_{i=0}^{N-2} \min(i+1, k-1)\,\hat{z}^{(k-1)}_i\,\hat{z}^{(k-1)}_{N-2-i}.
\end{aligned}$$

Thus $\Delta_N(N) = x_k\,(\Sigma^+ - 2\,\Sigma^0 + \Sigma^-)$, and it suffices to show that $\Sigma^+ - 2\,\Sigma^0 + \Sigma^- \geq 0$.

To this end we sort the terms into three kinds. In each of the three sums we separate the terms in which the first factor has length at most $k-2$, the terms in which the second factor has length at most $k-2$, and the terms in which both factors have length at least $k-1$. Since $N \geq 2k-1$, the lengths of the two factors add up to at least $N-2 \geq 2k-3$, so that no term is of the first and of the second kind at once.

Consider a factor of length $\ell \leq k-2$. This length lies in the free range of the three thresholds, so that the factor is $\hat{c}_\ell$ in each of the three sums, and the other factor is $\hat{z}^{(k+1)}_{N-\ell}$, $\hat{z}^{(k)}_{N-1-\ell}$ or $\hat{z}^{(k-1)}_{N-2-\ell}$. When it is the first factor, the weight is $\ell+1$ in each sum. When it is the second factor, the length of the first factor is at least the threshold of the sum, because $N-\ell \geq k+1$, and the weight is that threshold, i.e. $k+1$, k or $k-1$. The terms of the first two kinds therefore contribute to $\Sigma^+ - 2\,\Sigma^0 + \Sigma^-$ the amount

$$\begin{aligned}
&\sum_{\ell=0}^{k-2} \hat{c}_\ell \left((\ell+k+2)\,\hat{z}^{(k+1)}_{N-\ell} - 2\,(\ell+k+1)\,\hat{z}^{(k)}_{N-1-\ell} + (\ell+k)\,\hat{z}^{(k-1)}_{N-2-\ell} \right)\\
&\qquad = \sum_{\ell=0}^{k-2} \hat{c}_\ell\,\Delta_{\ell+k+1}(N-1-\ell).
\end{aligned}$$

In a term of the third kind the first factor has a length i with $k-1 \leq i \leq N-k+1$ in Σ^+, with $k-1 \leq i \leq N-k$ in Σ^0, and with $k-1 \leq i \leq N-k-1$ in Σ^-. The weight is the threshold of the sum, with one exception: in the term $i = k-1$ of Σ^+ the weight is k, and the first factor of this term is $\hat{z}^{(k+1)}_{k-1} = \hat{c}_{k-1}$. We index the other terms by the length of the second factor and put $m_t = N-k-t$ for $0 \leq t \leq r$, so that $m_0 = N-k$ and $m_r = k-1$. Then the terms of the third kind are

$$\begin{aligned}
&k\,\hat{c}_{k-1}\,\hat{z}^{(k+1)}_{N-k+1} \quad \text{and} \quad (k+1)\,\hat{z}^{(k+1)}_{k+t}\,\hat{z}^{(k+1)}_{m_t} \quad (0 \leq t \leq r) && \text{in } \Sigma^+,\\
&k\,\hat{z}^{(k)}_{k-1+t}\,\hat{z}^{(k)}_{m_t} \quad (0 \leq t \leq r) && \text{in } \Sigma^0,\\
&(k-1)\,\hat{z}^{(k-1)}_{k-2+t}\,\hat{z}^{(k-1)}_{m_t} \quad (1 \leq t \leq r) && \text{in } \Sigma^-.
\end{aligned}$$

To have the same range of t in the three sums, we add to the last family the term with $t = 0$, which is $(k-1)\,\hat{c}_{k-2}\,\hat{z}^{(k-1)}_{N-k}$ because $\hat{z}^{(k-1)}_{k-2} = \hat{c}_{k-2}$, and subtract it again. With

$$E_t = (k+1)\,\hat{z}^{(k+1)}_{k+t}\,\hat{z}^{(k+1)}_{m_t} - 2k\,\hat{z}^{(k)}_{k-1+t}\,\hat{z}^{(k)}_{m_t} + (k-1)\,\hat{z}^{(k-1)}_{k-2+t}\,\hat{z}^{(k-1)}_{m_t}$$

for $0 \le t \le r$, the terms of the third kind contribute to $\Sigma^+ - 2\,\Sigma^0 + \Sigma^-$ the amount

$$\sum_{t=0}^{r} E_t + k\,\hat{c}_{k-1}\,\hat{z}^{(k+1)}_{N-k+1} - (k-1)\,\hat{c}_{k-2}\,\hat{z}^{(k-1)}_{N-k}, \tag{14.7}$$

and $\Sigma^+ - 2\,\Sigma^0 + \Sigma^-$ is the sum of this amount and of the amount that the terms of the first two kinds contribute.

The amount that the terms of the first two kinds contribute is the sum of the numbers $\hat{c}_\ell\,\Delta_{\ell+k+1}(N-1-\ell)$, and it is nonnegative. Indeed, for $0 \le \ell \le k-2$ the weight $\ell+k+1$ lies between $k+1$ and $2k-1$, and the length $N-1-\ell$ lies between $N-k+1 \ge k$ and $N-1$, so that $\Delta_{\ell+k+1}(N-1-\ell) \ge 0$ by (14.6).

It remains to show that the amount (14.7), which the terms of the third kind contribute, is nonnegative. Let $0 \le t \le r$. Since the tilted counts of levels $k-1$ and k are the differences of the tilted counts of the three thresholds, we have $\hat{z}^{(k+1)}_{m_t} = \hat{z}^{(k)}_{m_t} + \hat{\beta}^{(k)}_{m_t}$ and $\hat{z}^{(k-1)}_{m_t} = \hat{z}^{(k)}_{m_t} - \hat{\beta}^{(k-1)}_{m_t}$. Inserting this into E_t, we find that the terms with $\hat{z}^{(k)}_{m_t}$ have the coefficient $\Delta_k(k-1+t)$, so that

$$\begin{aligned} E_t = {} & \hat{z}^{(k)}_{m_t}\,\Delta_k(k-1+t) \\ & + (k+1)\,\hat{z}^{(k+1)}_{k+t}\,\hat{\beta}^{(k)}_{m_t} - (k-1)\,\hat{z}^{(k-1)}_{k-2+t}\,\hat{\beta}^{(k-1)}_{m_t}. \end{aligned}$$

The first term on the right is nonnegative: for $t = 0$ because $\Delta_k(k-1) \ge 0$, as we showed at the beginning of the proof, and for $t \ge 1$ by (14.6), since then $k \le k-1+t \le N-k$. In the term that contains $\hat{\beta}^{(k)}_{m_t}$ we use Corollary 6.6 with $n = m_t \ge 1$, which gives $(k+1)\,\hat{\beta}^{(k)}_{m_t} \ge k\,\hat{\beta}^{(k-1)}_{m_t-1}$. We obtain

$$E_t \ge k\,\hat{z}^{(k+1)}_{k+t}\,\hat{\beta}^{(k-1)}_{m_t-1} - (k-1)\,\hat{z}^{(k-1)}_{k-2+t}\,\hat{\beta}^{(k-1)}_{m_t} \qquad (0 \le t \le r).$$

We sum this over t. Among the added terms, the one with $t = r$ vanishes, because $\hat{\beta}^{(k-1)}_{m_r-1} = \hat{\beta}^{(k-1)}_{k-2} = 0$: the tilted counts of thresholds $k-1$ and k agree at length $k-2$. Among the subtracted terms, the one with $t = 0$ is $(k-1)\,\hat{c}_{k-2}\,\hat{\beta}^{(k-1)}_{N-k}$, and in the others we replace t by $t+1$ and use that $m_{t+1} = m_t - 1$. Therefore

$$\begin{aligned} \sum_{t=0}^{r} E_t \ge {} & \sum_{t=0}^{r-1} \hat{\beta}^{(k-1)}_{m_t-1}\left(k\,\hat{z}^{(k+1)}_{k+t} - (k-1)\,\hat{z}^{(k-1)}_{k-1+t}\right) \\ & - (k-1)\,\hat{c}_{k-2}\,\hat{\beta}^{(k-1)}_{N-k}. \end{aligned}$$

Every term of the sum on the right is nonnegative, since $\hat{\beta}^{(k-1)}_{m_t-1} \ge 0$ and $\hat{z}^{(k-1)}_{k-1+t} \le \hat{z}^{(k)}_{k+t} \le \hat{z}^{(k+1)}_{k+t}$; here the first inequality is again the monotonicity along the diagonal, and the second holds because the tilted counts of the three thresholds at one length are ordered. Hence

$$\sum_{t=0}^{r} E_t \ge -(k-1)\,\hat{c}_{k-2}\,\hat{\beta}^{(k-1)}_{N-k}.$$

Adding the two further terms of the amount (14.7) and using that $\hat\beta^{(k-1)}_{N-k} + \hat z^{(k-1)}_{N-k} = \hat z^{(k)}_{N-k}$, we see that this amount is at least

$$k\,\hat c_{k-1}\,\hat z^{(k+1)}_{N-k+1} - (k-1)\,\hat c_{k-2}\,\hat z^{(k)}_{N-k}.$$

This number is nonnegative, because $\hat z^{(k+1)}_{N-k+1} \ge \hat z^{(k)}_{N-k} \ge 0$, once more by the monotonicity along the diagonal, and $k\,\hat c_{k-1} \ge (k-1)\,\hat c_{k-2}$ by the last inequality in (14.5).

Thus the two amounts are both nonnegative, so that $\Sigma^+ - 2\,\Sigma^0 + \Sigma^- \ge 0$, and hence $\Delta_N(N) = x_k\,(\Sigma^+ - 2\,\Sigma^0 + \Sigma^-) \ge 0$. This completes the induction. □

14.4. **The weights $\Psi(N)$ and the weighted debt.** For a column k and an integer $N \ge 0$ let

$$\Psi(N) = \max(N,\, 2k-1)$$

be the weight of row N. It depends on the column, which is fixed wherever Ψ occurs. The weight is constant, $\Psi(N) = 2k-1$, for $N \le 2k-1$, and $\Psi(N) = N$ for $N \ge 2k-1$; the constant part is chosen for the rows before the crossing row, in which Section 15 compares the weights with values of the tilted prime kernel of threshold k.

Corollary 14.6. *Let $k \ge 7$ and N be integers with $N \ge k$. Then*

$$\Psi(N+1)\,\hat z^{(k+1)}_{N+1} - 2\,\Psi(N)\,\hat z^{(k)}_{N} + \Psi(N-1)\,\hat z^{(k-1)}_{N-1} \ge 0.$$

Proof. We use the notation (14.3). If $k \le N \le 2k-2$, the three weights are equal to $2k-1$, and the left-hand side is $2k-1$ times $\hat z^{(k+1)}_{N+1} - 2\,\hat z^{(k)}_{N} + \hat z^{(k-1)}_{N-1}$, which is nonnegative by Corollary 14.3. If $N = 2k-1$, the weights are $N+1$, N and $\Psi(N-1) = N$, and the left-hand side is $\Delta_N(N) + \hat z^{(k-1)}_{N-1}$, where $\Delta_N(N) \ge 0$ by Proposition 14.5 and $\hat z^{(k-1)}_{N-1} \ge 0$. If $N \ge 2k$, the left-hand side is $\Delta_N(N)$, which is nonnegative by the same proposition. □

We now sum by parts. A column is nonpositive in the rows before its crossing row, and nothing is known about the size of its steps in these rows one by one. What can be bounded from below is a weighted sum of the steps, and this without any assumption on their signs, so that the bound applies wherever the crossing row lies. Indeed, the tilted step $\hat\delta_n = \hat z^{(k+1)}_n - 2\,\hat z^{(k)}_n + \hat z^{(k-1)}_n$ is a second difference in the threshold, taken at one length, and when the tilted steps of rows $k, \dots, J$ are added with the weights $\Psi(n)$, their terms regroup along the diagonals into the expressions of Corollary 14.6, which are nonnegative, up to terms at the two ends of the range. What is left at the upper end, in rows $J-1$ and J, is estimated by the following quantity. For a column $k \ge 2$ and a row $J \ge k$, the *weighted debt* of the column at row J is

$$\mathfrak{D}_J = \Psi(J)\,\hat\beta^{(k-1)}_J + \Psi(J-1)\,\hat\beta^{(k-1)}_{J-1} + \big(\Psi(J) - \Psi(J-1)\big)\,\hat z^{(k)}_J.$$

Its three terms are nonnegative and belong to rows $J-1$ and J; the coefficient $\Psi(J) - \Psi(J-1)$ is 0 for $J \le 2k-1$ and 1 for $J \ge 2k$.

Proposition 14.7. *Let $k \ge 7$ and J be integers with $J \ge k$. Then*

$$\sum_{n=k}^{J} \Psi(n)\,\hat\delta_n \ge -\mathfrak{D}_J.$$

Proof. For an integer $N \geq k$ let

$$L_N = \Psi(N+1)\,\hat{z}_{N+1}^{(k+1)} - 2\,\Psi(N)\,\hat{z}_N^{(k)} + \Psi(N-1)\,\hat{z}_{N-1}^{(k-1)}$$

denote the left-hand side of the inequality of Corollary 14.6. Summation by parts gives

$$\sum_{n=k}^{J} \Psi(n)\,\hat{\delta}_n = \sum_{N=k}^{J-1} L_N + \Psi(k)\,\hat{z}_k^{(k+1)} - \Psi(k-1)\,\hat{z}_{k-1}^{(k-1)} \\ - 2\,\Psi(J)\,\hat{z}_J^{(k)} + \Psi(J)\,\hat{z}_J^{(k-1)} + \Psi(J-1)\,\hat{z}_{J-1}^{(k-1)}.$$

Indeed, since $\hat{\delta}_n = \hat{z}_n^{(k+1)} - 2\,\hat{z}_n^{(k)} + \hat{z}_n^{(k-1)}$, the left-hand side consists of the terms $\Psi(n)\,\hat{z}_n^{(k+1)}$, $-2\,\Psi(n)\,\hat{z}_n^{(k)}$ and $\Psi(n)\,\hat{z}_n^{(k-1)}$ with $k \leq n \leq J$, whereas in the sum of the L_N these three families have the ranges $k+1 \leq n \leq J$, $k \leq n \leq J-1$ and $k-1 \leq n \leq J-2$. The two sums differ by the five terms at the ends of these ranges that the identity displays.

Every L_N is nonnegative by Corollary 14.6. Next, $\Psi(k) = \Psi(k-1) = 2k-1$, and the two tilted counts that carry these weights have lengths in the free range of their thresholds, so that

$$\Psi(k)\,\hat{z}_k^{(k+1)} - \Psi(k-1)\,\hat{z}_{k-1}^{(k-1)} = (2k-1)\,(\hat{c}_k - \hat{c}_{k-1}) \geq 0$$

by (6.2). Dropping these nonnegative terms, we get

$$\sum_{n=k}^{J} \Psi(n)\,\hat{\delta}_n \geq -2\,\Psi(J)\,\hat{z}_J^{(k)} + \Psi(J)\,\hat{z}_J^{(k-1)} + \Psi(J-1)\,\hat{z}_{J-1}^{(k-1)}.$$

In terms of the tilted counts of level $k-1$ we have $\hat{z}_J^{(k-1)} = \hat{z}_J^{(k)} - \hat{\beta}_J^{(k-1)}$ and $\hat{z}_{J-1}^{(k-1)} = \hat{z}_{J-1}^{(k)} - \hat{\beta}_{J-1}^{(k-1)}$, so that the right-hand side is

$$-\Psi(J)\,\hat{\beta}_J^{(k-1)} - \Psi(J-1)\,\hat{\beta}_{J-1}^{(k-1)} + \Psi(J-1)\,\hat{z}_{J-1}^{(k)} - \Psi(J)\,\hat{z}_J^{(k)}.$$

Finally, the tilted counts of threshold k do not increase with the length: $\hat{z}_J^{(k)} \leq \hat{z}_{J-1}^{(k)}$ by Proposition 6.5. Hence

$$\Psi(J-1)\,\hat{z}_{J-1}^{(k)} - \Psi(J)\,\hat{z}_J^{(k)} \geq -\big(\Psi(J) - \Psi(J-1)\big)\,\hat{z}_J^{(k)},$$

and with the two preceding displays this gives $\sum_{n=k}^{J} \Psi(n)\,\hat{\delta}_n \geq -\mathfrak{D}_J$. □

15. The left half of a row

In this section we prove Proposition 1.2: for all integers $n \geq 17$ and $k \geq 1$ with $2k \leq n$ we have $b_{n,k-1} \leq b_{n,k}$, so that a row $n \geq 17$ has no strict descent at an index i with $2i \leq n$. For $k \geq 82$ we argue by induction on the row, starting from the single crossing of Proposition 12.5; for $2 \leq k \leq 81$ we use the computation of Lemma 7.5; and column $k = 1$ is treated directly.

15.1. **The tilted prime kernel at the critical tilt.** Throughout this subsection $k \geq 2$ is a fixed integer and, as in Section 6, a hat denotes the tilted form of a sequence at the critical tilt x_k; in particular, $\hat{\kappa}_m = \kappa_m^{(k)}\,x_k^m$ is the tilted prime kernel of threshold k. We prove the estimates for $\hat{\kappa}$ on which the proof of Proposition 15.4 rests.

We begin with what Sections 5 and 6 provide. The tilted kernel is nonnegative (Lemma 5.5), and in the free range it is the tilted free kernel,

$$\hat{\kappa}_m = \varepsilon_m \, x_k^m \qquad (1 \le m \le k),$$

by Lemma 5.3. By Lemma 6.1 we have $\varepsilon_{m+1} \, x_k^{m+1} \le \varepsilon_m \, x_k^m$ for $1 \le m < k$, so that Lemma 6.2 and (6.3) apply with $y = k$ and $x = x_k$. They give

$$\hat{\kappa}_{m+1} \le \hat{\kappa}_m \quad (m \ge 1) \qquad \text{and} \qquad (m+1)\,\hat{\kappa}_{m+1} \le m\,\hat{\kappa}_m \quad (m \ge k).$$

Thus the tilted kernel is nonincreasing, and the second inequality yields, by induction on L,

$$L\,\hat{\kappa}_L \le k\,\hat{\kappa}_k \qquad (L \ge k). \tag{15.1}$$

Finally, the recursion of Lemma 5.4, multiplied by x_k^N, reads

$$N\,\hat{\kappa}_N = \sum_{j=1}^{k-1} (k-j)\,\hat{a}_j\,\hat{\kappa}_{N-j} \qquad (N \ge k),$$

and its weights $(k-j)\,\hat{a}_j = (k-j)\,a_j\,x_k^j$ are positive.

Beyond the free range the kernel decays the faster the further out its index lies: for all integers $r \ge 1$ and $m \ge k + (r-1)(k-1)$,

$$(m+1)\,\hat{\kappa}_{m+1} \le (m+1-r)\,\hat{\kappa}_m. \tag{15.2}$$

We prove this by induction on r. For $r = 1$ it is the inequality $(m+1)\,\hat{\kappa}_{m+1} \le m\,\hat{\kappa}_m$ for $m \ge k$ recorded above. Suppose that the assertion holds for some $r \ge 1$, and let $m \ge k + r\,(k-1)$. For $1 \le j \le k-1$ the index $i = m - j$ satisfies $i \ge m - k + 1 \ge k + (r-1)(k-1)$, so that $(i+1)\,\hat{\kappa}_{i+1} \le (i+1-r)\,\hat{\kappa}_i$ by the induction hypothesis. Since moreover $\hat{\kappa}_{i+1} \le \hat{\kappa}_i$ and $m - i - 1 \ge 0$, we get

$$\begin{aligned} m\,\hat{\kappa}_{i+1} &= (i+1)\,\hat{\kappa}_{i+1} + (m-i-1)\,\hat{\kappa}_{i+1} \\ &\le (i+1-r)\,\hat{\kappa}_i + (m-i-1)\,\hat{\kappa}_i = (m-r)\,\hat{\kappa}_i, \end{aligned}$$

i.e. $m\,\hat{\kappa}_{m+1-j} \le (m-r)\,\hat{\kappa}_{m-j}$. We multiply this by $(k-j)\,\hat{a}_j$, sum over $1 \le j \le k-1$, and use the tilted recursion for $N = m+1$ and for $N = m$. This gives $m\,(m+1)\,\hat{\kappa}_{m+1} \le (m-r)\,m\,\hat{\kappa}_m$, which is the assertion for $r+1$.

In the free range the size of the kernel is given by Stirling's formula. Let $2 \le L \le k$. Since $\hat{\kappa}_L = \varepsilon_L \, x_k^L$ and $L\,\varepsilon_L = (L-1)^{L-1}/(L-1)!$,

$$\hat{\kappa}_L = \frac{(e\,x_k)^L}{e\,L} \cdot e^{-(L-1)} \, \frac{(L-1)^{L-1}}{(L-1)!},$$

and the second factor is the Poisson mass at $L - 1 \ge 1$. By (2.7) it lies between $\frac{11}{12}\,(2\pi(L-1))^{-1/2}$ and $(2\pi(L-1))^{-1/2}$, since $1 - \frac{1}{12(L-1)} \ge \frac{11}{12}$. Hence

$$\frac{11}{12} \cdot \frac{(e\,x_k)^L}{e\,L\,\sqrt{2\pi\,(L-1)}} \le \hat{\kappa}_L \le \frac{(e\,x_k)^L}{e\,L\,\sqrt{2\pi\,(L-1)}} \qquad (2 \le L \le k). \tag{15.3}$$

Since $e\,x_k \le e^{3/(2k)}$ by Lemma 6.3, the upper bound gives

$$\hat{\kappa}_L \le \frac{\exp\!\left(\frac{3L}{2k} - 1\right)}{L\,\sqrt{2\pi\,(L-1)}} \qquad (2 \le L \le k). \tag{15.4}$$

The next two lemmas order the values $\hat{\kappa}_{n-j}$ of the tilted kernel, measured against the weight $\Psi(j) = \max(j, 2k-1)$ of Section 14, on the two sides of a row $q \le k + 27k/100$.

Lemma 15.1. *Let k, q, n and j be nonnegative integers with $k \geq 2$, $100\,(q-k) \leq 27\,k$, $n \geq 2k$ and $j < q$. Then*

$$\hat{\kappa}_{n-j}\,(2k-1) \leq \hat{\kappa}_{n-q+1}\,\Psi(j).$$

Proof. Since $k \geq 2$, we have $j < q \leq 1.27\,k \leq 2k-1$, so that $\Psi(j) = 2k-1$. Moreover $1 \leq n-q+1 \leq n-j$, and since the tilted kernel is nonincreasing, $\hat{\kappa}_{n-j} \leq \hat{\kappa}_{n-q+1}$, which is the claim. □

Lemma 15.2. *Let k, q, n and j be integers with $k \geq 2$, $k+1 \leq q$, $100\,(q-k) \leq 27\,k$, $n \geq 2k$ and $q \leq j \leq n-1$. Then*

$$\hat{\kappa}_{n-q+1}\,\Psi(j) \leq \hat{\kappa}_{n-j}\,(2k-1).$$

Proof. Put $L = n-q+1$ and $l = n-j$, so that the claim is $\hat{\kappa}_L\,\Psi(j) \leq (2k-1)\,\hat{\kappa}_l$. The hypothesis on q gives $q \leq 1.27\,k$; in particular $q \leq 2k \leq n$, and $1 \leq l < L$ because $q \leq j \leq n-1$. If $j \leq 2k-1$, then $\Psi(j) = 2k-1$, and the claim holds because the tilted kernel is nonincreasing, so that $\hat{\kappa}_L \leq \hat{\kappa}_l$.

Let now $j \geq 2k$. Then $\Psi(j) = j$, and the claim is

$$j\,\hat{\kappa}_L \leq (2k-1)\,\hat{\kappa}_l.$$

Whether the kernel grows fast enough for this as its index falls from L to l depends on where the two indices lie relative to the free range $1 \leq m \leq k$.

Beyond the free range the kernel falls at least by a fixed factor from each index to the next:

$$(2k-1)\,\hat{\kappa}_{m+1} \leq (2k-2)\,\hat{\kappa}_m \qquad (m \geq k).$$

Indeed, let $m \geq k$, and write $m-k = (k-1)\,d + s$ with integers $d \geq 0$ and $0 \leq s \leq k-2$. Then $m \geq k + d\,(k-1)$, and the inequality (15.2) with $r = d+1$ gives $(m+1)\,\hat{\kappa}_{m+1} \leq (m-d)\,\hat{\kappa}_m$. Since $m+1 \leq k+(d+1)(k-1) \leq (d+1)(2k-1)$, we have

$$(2k-1)\,(m-d) - (2k-2)\,(m+1) = (m+1) - (d+1)(2k-1) \leq 0,$$

and therefore $(2k-1)\,(m+1)\,\hat{\kappa}_{m+1} \leq (2k-1)\,(m-d)\,\hat{\kappa}_m \leq (2k-2)\,(m+1)\,\hat{\kappa}_m$, because $\hat{\kappa}_m \geq 0$. Division by $m+1$ gives the asserted inequality.

We deduce by induction on i that, for every integer $i \geq 2k-1$ with $n-i \geq k$,

$$i\,\hat{\kappa}_{n-2k+1} \leq (2k-1)\,\hat{\kappa}_{n-i}. \tag{15.5}$$

For $i = 2k-1$ the two sides are equal. If the inequality holds for some $i \geq 2k-1$ and $n-(i+1) \geq k$, then $(2k-1)\,\hat{\kappa}_{n-i} \leq (2k-2)\,\hat{\kappa}_{n-i-1}$, by the inequality of the preceding paragraph with $m = n-i-1$. Hence $i\,\hat{\kappa}_{n-2k+1} \leq (2k-2)\,\hat{\kappa}_{n-i-1}$, and the inequality for $i+1$ follows because $(2k-2)(i+1) \leq (2k-1)\,i$.

In the free range we compare the values of the kernel at two indices that are far apart: if l_1 and l_2 are integers with $1 \leq l_1 < l_2 \leq k$ and $100\,(l_2 - l_1) \geq 73\,k$, then

$$1.22\,\hat{\kappa}_{l_2} \leq \hat{\kappa}_{l_1}. \tag{15.6}$$

Suppose first that $l_1 = 1$. In the free range the kernel is the tilted free kernel, so that $\hat{\kappa}_1 = \varepsilon_1 x_k = x_k$ and $\hat{\kappa}_2 = \varepsilon_2 x_k^2 = \frac{1}{2} x_k^2$. Since $l_2 \geq 2$ and $0 < x_k \leq 1$, it follows that $\hat{\kappa}_{l_2} \leq \hat{\kappa}_2 \leq \frac{1}{2} x_k = \frac{1}{2}\hat{\kappa}_1$, which implies the asserted inequality. Let now $l_1 \geq 2$. Applying the two-sided bound (15.3) to l_2 from above and to l_1 from below, we obtain

$$\frac{\hat{\kappa}_{l_2}}{\hat{\kappa}_{l_1}} \leq \frac{12}{11}\,(e\,x_k)^{l_2-l_1}\,\frac{l_1}{l_2}\,\sqrt{\frac{l_1-1}{l_2-1}}.$$

Here $(e\,x_k)^{l_2-l_1} \le e^{3(l_2-l_1)/(2k)} \le e^{3/2}$ by Lemma 6.3, because $l_2 - l_1 \le k$. The hypothesis gives $l_2 - l_1 \ge 0.73\,k \ge 0.73\,l_2$, hence $l_1 \le 0.27\,l_2$ and also $l_1 - 1 \le 0.27\,(l_2 - 1)$. So the right-hand side is at most $\frac{12}{11}\,e^{3/2} \cdot 0.27^{3/2}$, which is smaller than $0.69 < 1/1.22$, since $e^{3/2} < 4.482$ and $0.27^{3/2} < 0.1403$.

We now prove the claim $j\,\hat\kappa_L \le (2k-1)\,\hat\kappa_l$, in three cases according to the position of l and L relative to k: either $l \ge k$, or $L \le k$, or $l < k < L$. Suppose first that $l \ge k$. Since $j \ge 2k-1$ and $n - j = l \ge k$, the inequality (15.5) with $i = j$ gives $j\,\hat\kappa_{n-2k+1} \le (2k-1)\,\hat\kappa_l$. Since $q \le 2k$, we have $L \ge n - 2k + 1 \ge 1$, hence $\hat\kappa_L \le \hat\kappa_{n-2k+1}$, and the claim follows.

Suppose next that $L \le k$. Then $1 \le l < L \le k$, and $L - l = j - q + 1 \ge 2k - 1.27\,k + 1 > 0.73\,k$. So the comparison (15.6) gives $1.22\,\hat\kappa_L \le \hat\kappa_l$. Moreover $n = L + q - 1 \le k + q - 1$, so that $j \le n - 1 \le 2.27\,k - 2 \le 1.22\,(2k-1)$. Hence $j\,\hat\kappa_L \le 1.22\,(2k-1)\,\hat\kappa_L \le (2k-1)\,\hat\kappa_l$, which is the claim.

It remains to treat the case $l < k < L$. By the bound (15.1) we have $L\,\hat\kappa_L \le k\,\hat\kappa_k$, and we compare $\hat\kappa_k$ with $\hat\kappa_l$. Note that $j = n - l = L + q - 1 - l$ and that $L \ge k+1$. If $100\,l \le 27\,k$, then $100\,(k-l) \ge 73\,k$, and (15.6) with $l_1 = l$ and $l_2 = k$ gives $1.22\,\hat\kappa_k \le \hat\kappa_l$. Since $l \ge 1$, we have $j \le L + 1.27\,k - 2$, and

$$k\,j \le k\,(L + 1.27\,k - 2) \le 1.22\,(2k-1)\,L,$$

where the second inequality holds because $L \ge k+1$, as a direct computation shows. Therefore

$$L \cdot j\,\hat\kappa_L \le j \cdot k\,\hat\kappa_k \le 1.22\,(2k-1)\,L\,\hat\kappa_k \le (2k-1)\,L\,\hat\kappa_l,$$

and division by L gives the claim. If, on the other hand, $100\,l > 27\,k$, then $j < L + 1.27\,k - 1 - 0.27\,k = L + k - 1$, i.e. $j \le L + k - 2$, and

$$k\,j \le k\,(L + k - 2) \le (2k-1)\,L,$$

again because $L \ge k+1$. Since $\hat\kappa_k \le \hat\kappa_l$, the kernel being nonincreasing, we obtain $L \cdot j\,\hat\kappa_L \le j \cdot k\,\hat\kappa_k \le (2k-1)\,L\,\hat\kappa_k \le (2k-1)\,L\,\hat\kappa_l$, and the claim follows as before. □

Finally, from index $2k-1$ on the kernel decays at least like $1/L^2$.

Lemma 15.3. *Let $k \ge 2$ be an integer. Then, for every integer $L \ge 2k-1$,*

$$(L-1)\,L\,\hat\kappa_L \le 2k\,(k-1)\,\hat\kappa_k.$$

Proof. We use induction on L. For $L = 2k-1$, the bound (15.1) gives $(2k-1)\,\hat\kappa_{2k-1} \le k\,\hat\kappa_k$, and multiplying by $2k-2$ gives the claim. Suppose that the claim holds for some $L \ge 2k-1$. Since $L \ge k + (k-1)$, the inequality (15.2) with $r = 2$ and $m = L$ gives $(L+1)\,\hat\kappa_{L+1} \le (L-1)\,\hat\kappa_L$. Multiplying by L and using the induction hypothesis, we obtain

$$L\,(L+1)\,\hat\kappa_{L+1} \le (L-1)\,L\,\hat\kappa_L \le 2k\,(k-1)\,\hat\kappa_k. \qquad \square$$

15.2. **Columns $k \ge 82$.** We now treat the columns $k \ge 82$, for which we prove the strict inequality $b_{n,k-1} < b_{n,k}$ in every row $n \ge 2k$. By Proposition 12.5, such a column is nonpositive in rows $k \le j < q$, before its crossing row q, and positive in rows $q \le j \le 2k-1$, and we show by induction on $n \ge 2k$ that it is positive in row n. As in the chain range, the difficulty is that the rows before q are fed into row n, here by the column inequality of Proposition 5.7, and that we have no bound for a single step in these rows. What we have is the bound of the weighted debt (Proposition 14.7) for the sum of the steps with the weights $\Psi(j)$, and Lemmas 15.1

and 15.2 allow us to pass from the kernel weights of the column inequality to these weights.

Proposition 15.4. *Let $k \geq 82$ and $n \geq 2k$ be integers. Then $b_{n,k-1} < b_{n,k}$.*

Proof. Fix $k \geq 82$. As in Section 6, a hat denotes the tilted form of a sequence at the critical tilt x_k. Since $\hat{\delta}_n = d_{n,k}\, x_k^n / n!$ and $x_k > 0$, we have to show that $\hat{\delta}_n > 0$ for every $n \geq 2k$.

By Proposition 12.5 there is an integer q with $k+1 \leq q$ and $100\,(q-k) \leq 27\,k$ such that

$$\hat{\delta}_j \leq 0 \quad \text{for } k \leq j < q \qquad \text{and} \qquad \hat{\delta}_j > 0 \quad \text{for } q \leq j \leq 2k-1.$$

In particular $q \leq 1.27\,k < 2k-1$. We show that $\hat{\delta}_n > 0$ for every $n \geq q$, by induction on n. For $q \leq n \leq 2k-1$ this holds by the choice of q. Let $n \geq 2k$, and suppose that $\hat{\delta}_j > 0$ for $q \leq j \leq n-1$. We put

$$L = n - q + 1 \qquad \text{and} \qquad \theta = \frac{\hat{\kappa}_L}{2k-1},$$

so that $L \geq 1$ and $\theta \geq 0$, and we show that $\hat{\delta}_n > 0$.

We start from the column inequality. Multiplying the inequality of Proposition 5.7 for row n by x_k^n and indexing the sum by $j = n - m$, we obtain

$$n\,\hat{\delta}_n \geq (k+1)\,x_k\,\hat{\delta}_{n-1} + 2\,x_k\,\hat{\beta}_{n-1}^{(k-1)} + n \sum_{j=k}^{n-1} \hat{\kappa}_{n-j}\,\hat{\delta}_j. \tag{15.7}$$

We first bound the sum in this inequality from below. Recall from Section 14 that the weighted debt of the column at a row $J \geq k$ is

$$\mathfrak{D}_J = \Psi(J)\,\hat{\beta}_J^{(k-1)} + \Psi(J-1)\,\hat{\beta}_{J-1}^{(k-1)} + \big(\Psi(J) - \Psi(J-1)\big)\,\hat{z}_J^{(k)}.$$

For $k \leq j < q$, Lemma 15.1 gives $\hat{\kappa}_{n-j} \leq \theta\,\Psi(j)$, and $\hat{\delta}_j \leq 0$ by the choice of q. For $q \leq j \leq n-1$, Lemma 15.2 gives $\hat{\kappa}_{n-j} \geq \theta\,\Psi(j)$, and $\hat{\delta}_j > 0$ by the choice of q and the induction hypothesis. In both cases

$$\hat{\kappa}_{n-j}\,\hat{\delta}_j \geq \theta\,\Psi(j)\,\hat{\delta}_j.$$

We sum over $k \leq j \leq n-1$ and apply Proposition 14.7, which holds for $k \geq 7$, with $J = n-1 \geq k$. As $\theta \geq 0$, this gives

$$\sum_{j=k}^{n-1} \hat{\kappa}_{n-j}\,\hat{\delta}_j \geq \theta \sum_{j=k}^{n-1} \Psi(j)\,\hat{\delta}_j \geq -\theta\,\mathfrak{D}_{n-1}.$$

Next we bound $\mathfrak{D}_{n-1}$ from above by a combination of $\hat{\beta}_{n-1}^{(k-1)}$ and $\hat{\delta}_{n-1}$, the two quantities in the first two terms on the right of the column inequality (15.7). Since $n-1 \geq 2k-1$, we have $\Psi(n-1) = n-1$ and $n-2 \leq \Psi(n-2) \leq n-1$. As the tilted counts are nonnegative, the definition of $\mathfrak{D}_{n-1}$ gives

$$\mathfrak{D}_{n-1} \leq (n-1)\,\big(\hat{\beta}_{n-1}^{(k-1)} + \hat{\beta}_{n-2}^{(k-1)}\big) + \hat{z}_{n-1}^{(k)}. \tag{15.8}$$

For the count of row $n-2$ we use Corollary 6.6 with $n-1$ in place of n, which states that $k\,\hat{\beta}_{n-2}^{(k-1)} \leq (k+1)\,\hat{\beta}_{n-1}^{(k)}$. Since $\hat{\beta}_{n-1}^{(k)} = \hat{\beta}_{n-1}^{(k-1)} + \hat{\delta}_{n-1}$, this is

$$\hat{\beta}_{n-2}^{(k-1)} \leq \Big(1 + \frac{1}{k}\Big)\,\big(\hat{\beta}_{n-1}^{(k-1)} + \hat{\delta}_{n-1}\big).$$

For $\hat{z}_{n-1}^{(k)}$ we show that

$$x_k \Big(\sum_{j=0}^{8} \hat{c}_j \Big)\, \hat{z}_{n-1}^{(k)} \le (n-1)\, \hat{\beta}_{n-1}^{(k-1)}.$$

The sum on the left has nine terms, the numbers $\hat{c}_j = c_j\, x_k^j$ with $0 \le j \le 8$, whatever k is; nine terms are enough for the numerical verification at the end of the proof. To prove the inequality, we subtract the tilted last-car recursions of thresholds k and $k-1$ at length $n-1$. By (3.5) with $N = n-2$ and by $\hat{\beta}_{n-1}^{(k-1)} = \hat{z}_{n-1}^{(k)} - \hat{z}_{n-1}^{(k-1)}$,

$$(n-1)\, \hat{\beta}_{n-1}^{(k-1)} = x_k \sum_{i=0}^{n-2} H_i,$$

where

$$H_i = \min(i+1, k)\; \hat{z}_i^{(k)}\; \hat{z}_{n-2-i}^{(k)} - \min(i+1, k-1)\; \hat{z}_i^{(k-1)}\; \hat{z}_{n-2-i}^{(k-1)}.$$

Every H_i is nonnegative, because $\min(i+1, k-1) \le \min(i+1, k)$ and $0 \le \hat{z}_N^{(k-1)} \le \hat{z}_N^{(k)}$ for every N (Section 3). Consider the nine indices $i = n-2-j$ with $0 \le j \le 8$. For these we have $i+1 \ge n-9 \ge k$, because $n \ge 2k$ and $k \ge 9$, so that the two minima are k and $k-1$, and

$$H_{n-2-j} = k\, \hat{z}_{n-2-j}^{(k)}\, \hat{z}_j^{(k)} - (k-1)\, \hat{z}_{n-2-j}^{(k-1)}\, \hat{z}_j^{(k-1)} \ge \hat{z}_{n-2-j}^{(k)}\, \hat{z}_j^{(k)}.$$

Here $\hat{z}_j^{(k)} = \hat{c}_j$ by Lemma 3.4, since $j \le k$, and $\hat{z}_{n-2-j}^{(k)} \ge \hat{z}_{n-1}^{(k)}$ by Proposition 6.5. Keeping only these nine terms of the sum, we obtain the asserted bound for $\hat{z}_{n-1}^{(k)}$.

We insert the bounds for $\hat{\beta}_{n-2}^{(k-1)}$ and for $\hat{z}_{n-1}^{(k)}$ into the bound (15.8). With

$$B = 2 + \frac{1}{k} + \Big(x_k \sum_{j=0}^{8} \hat{c}_j \Big)^{-1}$$

the result is

$$\mathfrak{D}_{n-1} \le (n-1) \Big(B\, \hat{\beta}_{n-1}^{(k-1)} + \Big(1 + \frac{1}{k}\Big)\, \hat{\delta}_{n-1} \Big).$$

We now return to the column inequality (15.7). Its sum is at least $-\theta\, \mathfrak{D}_{n-1}$, as we have seen, and with the bound for $\mathfrak{D}_{n-1}$ just obtained and the definition of θ we find

$$n\, \hat{\delta}_n \ge \hat{\beta}_{n-1}^{(k-1)} \Big(2\, x_k - \frac{n(n-1)\, \hat{\kappa}_L}{2k-1}\, B \Big) + \hat{\delta}_{n-1}\, (k+1) \Big(x_k - \frac{n(n-1)\, \hat{\kappa}_L}{(2k-1)\, k} \Big).$$

Here $\hat{\beta}_{n-1}^{(k-1)} \ge 0$, and $\hat{\delta}_{n-1} > 0$ because $q \le n-1$. So $\hat{\delta}_n > 0$ as soon as

$$n(n-1)\, \hat{\kappa}_L \cdot B \le 2\, x_k\, (2k-1) \qquad \text{and} \qquad n(n-1)\, \hat{\kappa}_L < x_k\, k\, (2k-1). \tag{15.9}$$

It remains to verify these two inequalities. To this end we show that

$$n(n-1)\, \hat{\kappa}_L \le 3.762\, \sqrt{k}.$$

For this we need the size of the kernel at two places:

$$\hat{\kappa}_k \le \frac{0.662}{k\sqrt{k}}$$

$$\text{and} \qquad \hat{\kappa}_M \le \frac{0.73}{k\sqrt{k}} \quad \text{for every integer } M \text{ with } 100\,(M-1) \ge 73\,k.$$

For the first bound, the upper bound (15.4) at the index k gives

$$\hat{\kappa}_k \le \frac{e^{1/2}}{k\,\sqrt{2\pi\,(k-1)}},$$

and $e^{1/2}\sqrt{k} \le 0.662\,\sqrt{2\pi\,(k-1)}$, because $e\,k \le 0.662^2 \cdot 2\pi\,(k-1)$ for $k \ge 79$; here we use that $e < 2.7183$ and $0.662^2 \cdot 2\pi > 2.7534$.

For the second bound let M be an integer with $100\,(M-1) \ge 73\,k$, and let t be the least integer with $100\,t \ge 73\,k$. Then $t \le M-1$ and $0.73\,k \le t < 0.73\,k+1$, so that $2 \le t+1 \le M$ and $t+1 < 0.73\,k+2 \le k$. Hence $\hat{\kappa}_M \le \hat{\kappa}_{t+1}$, because the tilted kernel is nonincreasing, and since $t+1$ lies in the free range, the upper bound (15.4) at the index $t+1$ gives

$$\hat{\kappa}_{t+1} \le \frac{1}{(t+1)\,\sqrt{2\pi\,t}}\,\exp\Big(\frac{3\,(t+1)}{2k} - 1\Big).$$

Put $\rho = (t+1)/k$, so that the exponent is $\frac{3}{2}\rho - 1$. We have $0.73 \le \rho \le 0.755$, because $0.73\,k+1 \le t+1 < 0.73\,k+2$ and $2/k < 0.025$; hence $0 < \frac{3}{2}\rho - 1 < 1$. We use the elementary inequality $e^s \le 1 + s + s^2$ for real numbers s with $0 \le s \le 1$, which holds because $e^s - 1 - s = s^2 \sum_{j\ge 0} s^j/(j+2)!$ and the last sum is at most $e - 2 < 1$ on this interval. For $s = \frac{3}{2}\rho - 1$ we have $1 + s + s^2 = 1 - \frac{3}{2}\rho + \frac{9}{4}\rho^2$, and since $t+1 = \rho\,k$, it follows that

$$\frac{1}{t+1}\,\exp\Big(\frac{3}{2}\rho - 1\Big) \le \frac{1}{k}\,\Big(\frac{1}{\rho} - \frac{3}{2} + \frac{9}{4}\,\rho\Big) \le \frac{1.5233}{k}.$$

The second inequality holds because the bracket is a convex function of $\rho > 0$ whose values at $\rho = 0.73$ and at $\rho = 0.755$ are smaller than 1.5233. Finally $\sqrt{2\pi\,t} \ge \sqrt{2\pi \cdot 0.73\,k} \ge 2.1416\,\sqrt{k}$, because $2\pi \cdot 0.73 > 4.5865 > 2.1416^2$. Together,

$$\hat{\kappa}_M \le \hat{\kappa}_{t+1} \le \frac{1.5233}{2.1416} \cdot \frac{1}{k\sqrt{k}} \le \frac{0.73}{k\sqrt{k}}.$$

We now prove that $n(n-1)\,\hat{\kappa}_L \le 3.762\,\sqrt{k}$. Recall that $n = L + q - 1$ and $q \le 1.27\,k$. Suppose first that $L \le k$. Then $L - 1 = n - q \ge 2k - 1.27\,k = 0.73\,k$, so that the bound for $\hat{\kappa}_M$ applies to $M = L$, and $n < 2.27\,k$. Hence

$$n(n-1)\,\hat{\kappa}_L \le n^2\,\hat{\kappa}_L \le 2.27^2 \cdot 0.73\,\sqrt{k} \le 3.762\,\sqrt{k}.$$

Suppose next that $k < L \le 2k$. By the bound (15.1) and the bound for $\hat{\kappa}_k$, we have $L\,\hat{\kappa}_L \le k\,\hat{\kappa}_k \le 0.662/\sqrt{k}$. Moreover $n \le L + 1.27\,k$, and it is easy to check that $(L + 1.27\,k)^2 \le 5.35\,k\,L$ for $k \le L \le 2k$: the difference of the two sides is convex in L, and the inequality holds at $L = k$ and at $L = 2k$. Therefore

$$n(n-1)\,\hat{\kappa}_L \le \frac{(L+1.27\,k)^2}{L} \cdot L\,\hat{\kappa}_L \le 5.35\,k \cdot \frac{0.662}{\sqrt{k}} < 3.542\,\sqrt{k}.$$

Suppose finally that $L > 2k$. By Lemma 15.3 and the bound for $\hat{\kappa}_k$,

$$(L-1)\,L\,\hat{\kappa}_L \le 2k\,(k-1)\,\hat{\kappa}_k \le 1.324\,\sqrt{k}.$$

Since $1.27\,k < 0.635\,L$, we have $n < 1.635\,L - 1$, and hence $n(n-1) \le (1.635\,L - 1)(1.635\,L - 2) \le 2.675\,(L-1)\,L$, the last inequality being elementary for $L \ge 1$. Therefore $n(n-1)\,\hat\kappa_L \le 2.675 \cdot 1.324\,\sqrt{k} < 3.542\,\sqrt{k}$. This proves the bound in all three cases.

We can now verify the two conditions (15.9). The quantities that enter satisfy

$$x_k > 0.3678, \qquad \sum_{j=0}^{8} \hat c_j \ge 2.01, \qquad B < 3.3652.$$

Indeed, $x_k\big(1+\frac{1}{k}\big)^{k-1} = 1$ and $\big(1+\frac{1}{k}\big)^{k-1} \le e^{(k-1)/k} < e$, so that $x_k > e^{-1} > 0.3678$. Each term $c_j x^j$ of the sum $\sum_{j=0}^{8} c_j x^j$ is nondecreasing in $x \ge 0$, so that $\sum_{j=0}^{8} \hat c_j = \sum_{j=0}^{8} c_j\, x_k^j$ is at least the value of this sum at $x = 0.3678$, which is $2.0162\ldots$. Finally,

$$B \le 2 + \frac{1}{82} + \frac{1}{0.3678 \cdot 2.01} < 3.3652.$$

Since $n(n-1)\,\hat\kappa_L \le 3.762\,\sqrt{k}$, it follows that

$$n(n-1)\,\hat\kappa_L \cdot B < 3.762 \cdot 3.3652\,\sqrt{k} < 12.66\,\sqrt{k} \le 0.7356\,(2k-1) < 2\,x_k\,(2k-1).$$

The third inequality holds for every $k \ge 82$: at $k = 82$ its two sides are $114.64\ldots$ and $119.90\ldots$, and their difference increases with k for $k \ge 19$. This is the first of the two conditions. The second follows from the same bound for $n(n-1)\,\hat\kappa_L$, since $3.762\,\sqrt{k} \le 3.762\,k < 0.3678\,k\,(2k-1)$. Hence $\hat\delta_n > 0$, which completes the induction. □

15.3. **Columns $2 \le k \le 81$.** For $2 \le k \le 81$ the inequality $b_{n,k-1} \le b_{n,k}$ is obtained from the computation of Lemma 7.5. The next proposition turns finitely many inequalities for the numbers $\xi_t^{(k)}$ of column k (Section 5), checked at a single row n_0, into the sign of the step in every row $n \ge n_0$; recall that $\delta_n^{(k)} \ge 0$ means that $b_{n,k-1} \le b_{n,k}$.

Proposition 15.5. *Let $k \ge 2$ and $n_0 \ge k$ be integers, and let $x > 0$ be a real number. Suppose that*

(1) $z_{N+1}^{(k)}\, x^{N+1} \le z_N^{(k)}\, x^N$ *for every integer $N \ge 0$, and*
(2) $\sum_{t=s}^{n_0} \xi_t^{(k)}\, x^t \ge 0$ *for every integer s with $k \le s \le n_0$.*

Then $\delta_n^{(k)} \ge 0$ for every integer $n \ge n_0$.

Proof. We write $\xi_t = \xi_t^{(k)}$ and $\delta_n = \delta_n^{(k)}$.

For an integer $n \ge k$ let $I(n)$ be the statement that

$$\sum_{t=s}^{n} \xi_t\, x^t \ge 0 \qquad \text{for every integer } s \text{ with } k \le s \le n.$$

Hypothesis (2) is $I(n_0)$. We show that, for every $n \ge k$, the statement $I(n)$ implies that $\delta_n \ge 0$ and that $I(n+1)$ holds. By induction, $I(n)$ then holds for every $n \ge n_0$, and the proposition follows.

Let $n \ge k$, and suppose that $I(n)$ holds. Put $u_t = \xi_t\, x^t$ and $v_t = z_{n-t}^{(k)}\, x^{n-t}$ for $0 \le t \le n$. Multiplying the inequality of Proposition 5.8 by x^n, we obtain

$\sum_{t=0}^{n} u_t\, v_t \le \delta_n\, x^n$. The terms with $t < k$ vanish, because $\xi_t = 0$ for $t < k$ (Section 5). Hence

$$\sum_{t=k}^{n} u_t\, v_t \le \delta_n\, x^n.$$

The weights v_t are nonnegative, and $v_s \le v_{s+1}$ for $k \le s < n$ by hypothesis (1) with $N = n - s - 1$. We claim that

$$v_s \sum_{t=s}^{n} u_t \le \sum_{t=s}^{n} u_t\, v_t \qquad \text{for every integer } s \text{ with } k \le s \le n,$$

and we prove this by downward induction on s. For $s = n$ both sides are $u_n\, v_n$. If the inequality holds for $s+1$, where $k \le s < n$, then

$$\sum_{t=s}^{n} u_t\, v_t \ge u_s\, v_s + v_{s+1} \sum_{t=s+1}^{n} u_t \ge u_s\, v_s + v_s \sum_{t=s+1}^{n} u_t = v_s \sum_{t=s}^{n} u_t,$$

where the second step uses that $v_{s+1} \ge v_s$ and that $\sum_{t=s+1}^{n} u_t \ge 0$ by $I(n)$. For $s = k$ the inequality, together with $I(n)$ and the one-tree bound in the form displayed above, gives $0 \le v_k \sum_{t=k}^{n} u_t \le \sum_{t=k}^{n} u_t\, v_t \le \delta_n\, x^n$, and hence $\delta_n \ge 0$.

It remains to derive $I(n+1)$. Lemma 5.6 gives $(n+1)\,\xi_{n+1} = (k+1)\,\delta_n + 2\,\beta_n^{(k-1)} \ge 0$, because $\delta_n \ge 0$ and $\beta_n^{(k-1)} \ge 0$. Each suffix sum in $I(n+1)$ is the nonnegative term $\xi_{n+1}\, x^{n+1}$, alone or added to a suffix sum in $I(n)$. Hence $I(n+1)$ holds. □

Proposition 15.6. *Let k and n be integers with $2 \le k \le 81$, $2k \le n$ and $17 \le n$. Then $b_{n,k-1} \le b_{n,k}$.*

Proof. Let $n_0 = n_0(k)$ be the number of Definition 7.4. Lemma 7.5 supplies three facts: we have $2k \le n_0$; the inequality $b_{m,k-1} \le b_{m,k}$ holds for every integer m with $2k \le m$, $17 \le m$ and $m < n_0$; and $\sum_{t=s}^{n_0} \xi_t^{(k)}\, x_k^t \ge 0$ for every integer s with $k \le s \le n_0$. If $n < n_0$, the claim is the second of these facts. If $n \ge n_0$, we apply Proposition 15.5 with this n_0 and with $x = x_k$: its hypothesis (1) is Proposition 6.5, its hypothesis (2) is the third fact, and its conclusion $\delta_n^{(k)} \ge 0$ is the claim, because $\delta_n^{(k)} = (b_{n,k} - b_{n,k-1})/n!$. □

15.4. **The first column and the conclusion.** The preceding arguments involve the three thresholds $k+1$, k and $k-1$ of a column and need $k \ge 2$. For $k = 1$ we compare the last-car recursions of $A_{n,0}$ and $A_{n,1}$ term by term.

Lemma 15.7. *Let $n \ge 3$ be an integer. Then $b_{n,0} \le b_{n,1}$.*

Proof. Since $b_{n,0} = A_{n,0}$ and $b_{n,1} = A_{n,1} - A_{n,0}$, the claim is that $2A_{n,0} \le A_{n,1}$, and we prove this by induction on $n \ge 3$. In the last-car recursion of Proposition 3.1 the factor $\min(j+1, \ell+1)$ is 1 for $\ell = 0$, and for $\ell = 1$ it is 1 for $j = 0$ and 2 for $j \ge 1$. Since $A_{0,0} = A_{0,1} = 1$, the recursion gives, for every $n \ge 0$,

$$2A_{n+1,0} = 2A_{n,0} + \sum_{j=1}^{n} 2\binom{n}{j} A_{j,0}\, A_{n-j,0},$$

$$A_{n+1,1} = A_{n,1} + \sum_{j=1}^{n} 2\binom{n}{j} A_{j,1}\, A_{n-j,1}.$$

From these we find $A_{3,0} = 6$ and $A_{3,1} = 13$, so that the claim holds for $n = 3$. If $2A_{n,0} \le A_{n,1}$ for some $n \ge 3$, then the right-hand side of the first line is at most that of the second, term by term: for the first terms this is the induction hypothesis, and for the others it holds because $A_{m,0} \le A_{m,1}$ for every $m \ge 0$. Hence $2A_{n+1,0} \le A_{n+1,1}$. □

The three ranges of columns together give Proposition 1.2.

Proof of Proposition 1.2. For $k = 1$ the claim is Lemma 15.7, for $2 \le k \le 81$ it is Proposition 15.6, and for $k \ge 82$ it follows from Proposition 15.4. □

16. Compositions weighted by Cayley numbers

The proof of Proposition 1.4, in Sections 18 and 19, is written in terms of sums over the compositions of an integer, in which a part of size p carries the Cayley coefficient a_p as its weight and each cut carries a factor that depends on its position. We introduce these sums and prove three rules that compare the total weight of the compositions with $q+1$ parts with that of the compositions with q parts: an identity when the cuts carry no factor, which shows that the number of parts then follows the law of the birthday problem; an inequality when each cut carries a factor between 0 and 1; and an inequality in the opposite direction for the factors that the parking problem produces.

16.1. **Compositions and their weights.** Let $m \ge 0$ and $q \ge 1$ be integers. A *composition* of m with q *parts* is a sequence $\mathbf{p} = (p_1, \dots, p_q)$ of positive integers with $p_1 + \cdots + p_q = m$. Its *cut points*, or *cuts*, are $s_j = p_1 + \cdots + p_j$ for $1 \le j \le q-1$. We also put $s_0 = 0$ and $s_q = m$, so that

$$0 = s_0 < s_1 < \cdots < s_{q-1} < s_q = m \qquad \text{and} \qquad p_j = s_j - s_{j-1} \quad (1 \le j \le q).$$

Thus a composition of m with q parts is a way of cutting a segment of length m at $q-1$ of its interior integer points $1, \dots, m-1$; the parts are the lengths of the pieces, and s_j lies between p_j and p_{j+1}. We write $\mathrm{Comp}_q(m)$ for the set of compositions of m with q parts. This set is empty if $q > m$, and in particular if $m = 0$.

A *cut profile* is a real-valued function f on the nonnegative integers. Recall from Section 2 the Cayley coefficients $a_0 = 0$ and $a_p = p^{p-1}/p!$ for $p \ge 1$. The *weight* of a composition $\mathbf{p} \in \mathrm{Comp}_q(m)$ for the cut profile f is

$$\mathrm{wt}_f(\mathbf{p}) = \prod_{j=1}^{q} a_{p_j} \prod_{j=1}^{q-1} f(s_j),$$

the product of the Cayley coefficients of its parts and of the values of f at its cut points. For instance, the compositions of 3 are (3), $(1,2)$, $(2,1)$ and $(1,1,1)$, and since $a_1 = a_2 = 1$ and $a_3 = \frac{3}{2}$ their weights are $\frac{3}{2}$, $f(1)$, $f(2)$ and $f(1)\,f(2)$.

Definition 16.1. Let f be a cut profile. For integers $q \ge 0$ and $m \ge 0$ the real number $V_q(m; f)$ is defined by recursion on q:

$$V_0(m; f) = 0, \qquad V_1(m; f) = a_m,$$
$$V_{q+2}(m; f) = \sum_{0 < s < m} V_{q+1}(s; f)\, f(s)\, a_{m-s},$$

where the sum runs over all integers s with $0 < s < m$. Unfolding the recursion gives, for $q \geq 1$,

$$V_q(m; f) = \sum_{0=s_0<s_1<\cdots<s_{q-1}<s_q=m} \prod_{i=1}^{q} a_{s_i - s_{i-1}} \prod_{i=1}^{q-1} f(s_i),$$

where the sum runs over the sequences of integers $s_1 < \cdots < s_{q-1}$ that lie strictly between $s_0 = 0$ and $s_q = m$. Thus $V_q(m; f)$ is the total weight of the compositions of m with exactly q parts for the cut profile f, in which a part of size p weighs a_p and a cut at position s weighs $f(s)$.

In the notation introduced above, the second formula of the definition reads

$$V_q(m; f) = \sum_{\mathbf{p} \in \mathrm{Comp}_q(m)} \mathrm{wt}_f(\mathbf{p}) \qquad (q \geq 1,\ m \geq 0). \tag{16.1}$$

It follows from the recursion by induction on q: the case $q = 1$ is clear, since $a_0 = 0$, and a composition of m with $q + 2$ parts consists of a composition of some integer s with $0 < s < m$ into $q + 1$ parts, of a cut at s, and of a last part of size $m - s$.

By this formula we have $V_q(m; f) = 0$ whenever $q > m$, and $V_q(m; f)$ depends on the profile only through its values $f(s)$ with $0 < s < m$, the possible cut points of a composition of m. Moreover, $a_p > 0$ for $p \geq 1$, and therefore $V_q(m; f) \geq 0$ if $f(s) \geq 0$ for all s with $0 < s < m$.

We denote by $\mathbf{1}$ the constant profile with value 1, for which the cuts carry no factor, and we write

$$T_{m,q} = V_q(m; \mathbf{1}) \qquad (m \geq 0,\ q \geq 0).$$

For $q \geq 1$ this is the sum of the products $a_{p_1} \cdots a_{p_q}$ over the q-tuples $(p_1, \ldots, p_q)$ of positive integers with sum m, and $T_{m,0} = 0$. In particular $T_{m,1} = a_m$, and $T_{m,q} > 0$ for $1 \leq q \leq m$, since there is a composition of m with q parts and the weights are positive. More generally, we denote a constant profile by its value h, a real number. A composition with q parts has $q - 1$ cuts, so that

$$V_q(m; h) = h^{q-1}\, T_{m,q} \qquad (q \geq 1). \tag{16.2}$$

For a cut profile f and an integer $m \geq 0$ we write

$$\Lambda_m[f] = \sum_{q=1}^{m} V_q(m; f)$$

for the total weight of all compositions of m. The sum stops at $q = m$ because there are no compositions with more parts, and $\Lambda_0[f] = 0$.

For real numbers y and N the *threshold profile* $\lambda^{y,N}$ is the cut profile

$$\lambda^{y,N}(s) = \frac{y}{N - s},$$

where we put $\lambda^{y,N}(N) = 0$ if N is a nonnegative integer; wherever we use the profile, the cut points are smaller than N, so that this value plays no role. The total weight of all compositions of m for the threshold profile is written

$$\Lambda_m(y; N) = \Lambda_m[\lambda^{y,N}]$$

and called a *chain sum*. Chain sums are what the parking problem produces. In the rows that are considered in the proof of Proposition 1.4, the normalised counts of a threshold y satisfy a linear equation, and solving it by iteration, as we do in

Section 18, leads to sums over decreasing chains of lengths $N > t_1 > t_2 > \cdots$ in which every step from t' down to t contributes $a_{t'-t}$ and every visited length t contributes a factor y/t. Such a chain is a composition of the distance that it covers: the steps are the parts, and a visited length t is a cut at position $s = N - t$, with factor $y/(N-s)$. Since y enters the factor of every cut once, and a composition with q parts has $q-1$ cuts, we have

$$V_q(m;\lambda^{y,N}) = y^{q-1}\, V_q(m;\lambda^{1,N}) \qquad (q \geq 1). \tag{16.3}$$

16.2. **Erasing a cut.** We compare $V_{q+1}(m;f)$ with $V_q(m;f)$ by erasing a cut: erasing a cut of a composition with $q+1$ parts merges the two parts next to it into one part and leaves a composition with q parts. When the cuts carry no factor, the comparison is an identity.

Lemma 16.2. *For all integers $m \geq 0$ and $q \geq 1$,*

$$q\, m\, T_{m,q+1} = (q+1)(m-q)\, T_{m,q}.$$

Proof. We prove an identity for an arbitrary cut profile f, the identity (16.7) below, and obtain the lemma as its case $f = \mathbf{1}$; the two inequalities of this section come from the same identity.

Let $m \geq 0$ and $q \geq 1$ be integers and f a cut profile. For a composition $\mathbf{p} = (p_1, \ldots, p_{q+1})$ of m with $q+1$ parts, with cut points $s_1 < \cdots < s_q$, and for $1 \leq i \leq q$, we put

$$\mathrm{mult}_i(\mathbf{p}) = (q+1-i)\, p_i + i\, p_{i+1},$$

which is a positive integer, and we write

$$\mathrm{wt}_f^{(i)}(\mathbf{p}) = \prod_{j=1}^{q+1} a_{p_j} \prod_{\substack{1 \leq l \leq q \\ l \neq i}} f(s_l)$$

for the weight of $\mathbf{p}$ from which the factor of the cut s_i has been removed, so that

$$\mathrm{wt}_f(\mathbf{p}) = f(s_i)\ \mathrm{wt}_f^{(i)}(\mathbf{p}) \qquad (1 \leq i \leq q).$$

We count in two ways the pairs $(\mathbf{p}, i)$ formed by a composition $\mathbf{p}$ of m with $q+1$ parts and one of its cuts s_i: with the weight of $\mathbf{p}$, and with the weight from which the factor of the chosen cut has been removed. In both counts the pair $(\mathbf{p}, i)$ is counted with the multiplicity $\mathrm{mult}_i(\mathbf{p})$, whose two coefficients are chosen so that each count gives a factor that does not depend on the composition.

For the first count we note that, for every composition $\mathbf{p}$ of m with $q+1$ parts,

$$\sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p}) = q\, m. \tag{16.4}$$

Indeed, in this sum p_j occurs with coefficient $q+1-j$ as the left neighbour of s_j and with coefficient $j-1$ as the right neighbour of s_{j-1}; for the first and for the last part, which have only one neighbouring cut, the missing coefficient is 0. So every part occurs with coefficient q, and the parts add up to m. We multiply this identity by $\mathrm{wt}_f(\mathbf{p})$, sum over $\mathbf{p}$, and obtain by (16.1)

$$\sum_{\mathbf{p} \in \mathrm{Comp}_{q+1}(m)} \sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p})\ \mathrm{wt}_f(\mathbf{p}) = q\, m\, V_{q+1}(m;f). \tag{16.5}$$

In the second count the factor of the chosen cut is removed from the weight. The same pairs then weigh

$$\sum_{\mathbf{p}\in\mathrm{Comp}_{q+1}(m)} \sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p})\ \mathrm{wt}_f^{(i)}(\mathbf{p}) = (q+1)(m-q)\,V_q(m;f). \tag{16.6}$$

To see this, we sort the pairs by the composition that is left when the chosen cut is erased. For a pair $(\mathbf{p},i)$ this is the composition $\mathbf{u}=(u_1,\dots,u_q)$ of m with q parts in which $u_i=p_i+p_{i+1}$, while $u_j=p_j$ for $j<i$ and $u_j=p_{j+1}$ for $j>i$. Its cut points are the s_l with $l\neq i$. The pair is determined by $\mathbf{u}$, by i, and by $v=p_i$, the size of the left one of the two pieces into which s_i divides u_i; and conversely, every composition $\mathbf{u}\in\mathrm{Comp}_q(m)$, every i with $1\le i\le q$ and every integer v with $1\le v\le u_i-1$ belong to exactly one pair. In these terms $\mathrm{mult}_i(\mathbf{p})=(q+1-i)\,v+i\,(u_i-v)$, and

$$\mathrm{wt}_f^{(i)}(\mathbf{p}) = \frac{a_v\,a_{u_i-v}}{a_{u_i}}\ \mathrm{wt}_f(\mathbf{u}),$$

because $\mathbf{p}$ and $\mathbf{u}$ have the same parts apart from u_i, which is replaced in $\mathbf{p}$ by its two pieces, and because the cuts of $\mathbf{u}$ are the cuts of $\mathbf{p}$ other than s_i. We now use the convolution identity of the Cayley coefficients: for every integer $u\ge 1$,

$$\sum_{v=1}^{u-1} v\,a_v\,a_{u-v} = (u-1)\,a_u = \sum_{v=1}^{u-1}(u-v)\,a_v\,a_{u-v}.$$

The first equality is Lemma 2.5, and the second follows from the first on replacing v by $u-v$. For a fixed composition $\mathbf{u}$ and a fixed i, the pairs that belong to them therefore contribute to the second count the number

$$\frac{\mathrm{wt}_f(\mathbf{u})}{a_{u_i}} \sum_{v=1}^{u_i-1} \big((q+1-i)\,v+i\,(u_i-v)\big)\,a_v\,a_{u_i-v} = (q+1)\,(u_i-1)\ \mathrm{wt}_f(\mathbf{u}),$$

by the two equalities of the convolution identity for $u=u_i$. Since $\sum_{i=1}^{q}(u_i-1)=m-q$, the pairs that belong to $\mathbf{u}$ contribute $(q+1)(m-q)\ \mathrm{wt}_f(\mathbf{u})$ in all, and summing over $\mathbf{u}\in\mathrm{Comp}_q(m)$ gives the value $(q+1)(m-q)\,V_q(m;f)$ of the second count, by (16.1).

We now compare the two counts. Since $\mathrm{wt}_f(\mathbf{p})=f(s_i)\ \mathrm{wt}_f^{(i)}(\mathbf{p})$, the left-hand sides of (16.5) and of (16.6) differ only in the factor $f(s_i)$ of the chosen cut. Subtracting the first of these identities from the second, we obtain

$$\begin{aligned}(q+1)(m-q)\,V_q(m;f) - q\,m\,V_{q+1}(m;f)& \\ = \sum_{\mathbf{p}\in\mathrm{Comp}_{q+1}(m)} \sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p})\,\big(1-f(s_i)\big)\ \mathrm{wt}_f^{(i)}(\mathbf{p}).&\end{aligned} \tag{16.7}$$

This identity holds for all integers $m\ge 0$ and $q\ge 1$ and for every cut profile f. For the constant profile $f=\mathbf{1}$ its right-hand side vanishes, and since $V_q(m;\mathbf{1})=T_{m,q}$ by definition, this is the identity of the lemma. □

16.3. **The birthday law.** For integers $m\ge 1$ and $j\ge 0$ let

$$P_{m,j} = \prod_{i=0}^{j-1}\Big(1-\frac{i}{m}\Big) = \frac{m\,(m-1)\cdots(m-j+1)}{m^j}.$$

We give $P_{m,j}$ no value for $m = 0$. We have $P_{m,0} = P_{m,1} = 1$, the definition gives the recursion

$$P_{m,j+1} = P_{m,j}\left(1 - \frac{j}{m}\right) \qquad (j \geq 0),$$

and $P_{m,j} = 0$ for $j \geq m+1$, because the product then contains the factor with $i = m$. The number $P_{m,j}$ is the probability that j independent draws from a set of m values, each draw uniformly distributed, are pairwise distinct. The *birthday law* of m is the law of the number Q of draws that are made before the first draw that repeats an earlier value. Since $Q \geq j$ holds exactly when the first j draws are pairwise distinct, we have $\Pr(Q \geq j) = P_{m,j}$ for every $j \geq 1$. Thus Q takes the values $1, \ldots, m$, and by the recursion for $P_{m,j}$

$$\Pr(Q = q) = P_{m,q} - P_{m,q+1} = \frac{q}{m}\, P_{m,q} \qquad (q \geq 1).$$

Since $T_{m,1} = a_m$, Lemma 16.2 determines $T_{m,q}$ for all $q \geq 1$ when $m \geq 1$, and the result is expressed by the numbers $P_{m,q}$.

Proposition 16.3. *Let $m \geq 0$ and $q \geq 1$ be integers. If $m \geq 1$, then*

$$T_{m,q} = a_m\, q\, P_{m,q}.$$

If $m = 0$, then $T_{0,q} = 0$.

Proof. There is no composition of 0, and hence $T_{0,q} = 0$ by (16.1). Let $m \geq 1$. We prove the formula by induction on q. For $q = 1$ we have $T_{m,1} = a_m$ and $P_{m,1} = 1$. Suppose that the formula holds for some $q \geq 1$. By Lemma 16.2, by the induction hypothesis, and by the recursion for $P_{m,q}$ in the form $(m-q)\, P_{m,q} = m\, P_{m,q+1}$,

$$\begin{aligned} q\, m\, T_{m,q+1} &= (q+1)(m-q)\, T_{m,q} = (q+1)(m-q)\, a_m\, q\, P_{m,q} \\ &= q\, m \cdot a_m\, (q+1)\, P_{m,q+1}, \end{aligned}$$

and dividing by $q\, m$ gives the formula for $q+1$. □

Because $\Pr(Q = q) = \frac{q}{m}\, P_{m,q}$, the proposition says that, for $m \geq 1$, the number $T_{m,q}$ is $m\, a_m$ times the probability that $Q = q$. Since these probabilities add up to 1, the compositions of m have total weight $\Lambda_m[\mathbf{1}] = m\, a_m$, and if one of them is drawn with probability proportional to its weight for the profile $\mathbf{1}$, its number of parts follows the birthday law of m. For $1 \leq q \leq m$, where $a_m = m^{m-1}/m!$ and $P_{m,q} = m!/((m-q)!\, m^q)$, the formula reads $T_{m,q} = q\, m^{\,m-q-1}/(m-q)!$. This is the value that the Lagrange inversion formula gives for the coefficient of X^m in the q-th power of the tree function $XC = \sum_{j \geq 1} a_j X^j$ of Section 2. We derive it from Lemma 16.2 because the identity behind that lemma is needed in any case, for the profiles that are not constant.

16.4. **Cut factors that depend on the position.** We return to a general cut profile. If every cut carries a factor between 0 and 1, the identity of Lemma 16.2 becomes an inequality: such factors, whatever their dependence on the position, can only lower the weight of the compositions with $q+1$ parts relative to the weight of those with q parts, in comparison with the profile $\mathbf{1}$.

Lemma 16.4. *Let $m \geq 0$ be an integer, and let f be a cut profile such that $0 \leq f(s) \leq 1$ for every integer s with $0 < s < m$. Then, for every integer $q \geq 1$,*

$$q\, m\, V_{q+1}(m; f) \leq (q+1)(m-q)\, V_q(m; f).$$

Proof. We use the notation of the proof of Lemma 16.2. By the identity (16.7) it suffices to show that every term on its right-hand side is nonnegative. The numbers $\mathrm{mult}_i(\mathbf{p})$ and the Cayley coefficients of the parts are positive. The cut points of a composition of m lie strictly between 0 and m, so that the hypothesis applies to each of them: it gives $f(s_l) \geq 0$ for the cuts with $l \neq i$, whose factors occur in $\mathrm{wt}_f^{(i)}(\mathbf{p})$, and $1 - f(s_i) \geq 0$ for the erased cut. □

For a constant profile $h > 0$, the formula (16.2) and Lemma 16.2 give

$$(q+1)(m-q)\,V_q(m;h) = q\,m\,\frac{1}{h}\,V_{q+1}(m;h) \qquad (q \geq 1).$$

This shows that the upper bound in Lemma 16.4 cannot be dropped: for $1 \leq q < m$, where $V_{q+1}(m;h) = h^q\,T_{m,q+1} > 0$, the inequality of that lemma holds for the constant profile h if and only if $h \leq 1$.

The second inequality concerns the threshold profile $\lambda^{1,N}$ and goes in the opposite direction: if $m < N$, the weight of the compositions of m with $q+1$ parts, relative to the weight of those with q parts, is at least what the identity displayed above gives for the constant profile $h = 1/(N - \frac{m}{2})$, the value of $1/(N-s)$ at the midpoint $s = \frac{m}{2}$ of the segment that is cut. By the scaling relation (16.3) the same then holds for $\lambda^{y,N}$ with any $y > 0$ and $h = y/(N - \frac{m}{2})$.

Lemma 16.5. *Let $m \geq 0$ be an integer and N a real number with $m < N$. Then the threshold profile $\lambda^{1,N}$ satisfies, for every integer $q \geq 1$,*

$$(q+1)(m-q)\,V_q(m;\lambda^{1,N}) \leq q\,m\left(N - \frac{m}{2}\right) V_{q+1}(m;\lambda^{1,N}).$$

Proof. We use the notation of the proof of Lemma 16.2, and we write $f = \lambda^{1,N}$ and $V_q = V_q(m;f)$. Both sides vanish if $q \geq m$, and we assume from now on that $q < m$.

Every cut point s of a composition of m satisfies $s < m < N$. At the cut points we therefore have

$$f(s) > 0, \qquad (N-s)\,f(s) = 1, \qquad f(s) \leq f(s') \ \text{ for cut points } s \leq s'.$$

We first express the difference of the two sides of the lemma through the parts. Let $\mathbf{p}$ be a composition of m with $q+1$ parts and let $1 \leq i \leq q$. Then

$$\mathrm{wt}_f^{(i)}(\mathbf{p}) = (N - s_i)\,f(s_i)\;\mathrm{wt}_f^{(i)}(\mathbf{p}) = (N - s_i)\;\mathrm{wt}_f(\mathbf{p}),$$

by the relation $(N-s_i)\,f(s_i) = 1$ and because $\mathrm{wt}_f^{(i)}(\mathbf{p})$ is the weight of $\mathbf{p}$ without the factor $f(s_i)$. We insert this into (16.6), and we subtract the result from (16.5) multiplied by $N - \frac{m}{2}$. This gives

$$q\,m\left(N - \frac{m}{2}\right) V_{q+1} - (q+1)(m-q)\,V_q \\ = \sum_{\mathbf{p} \in \mathrm{Comp}_{q+1}(m)} \mathrm{wt}_f(\mathbf{p}) \sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p}) \left(s_i - \frac{m}{2}\right).$$

The inner sum can be written in terms of the parts alone: for every composition $\mathbf{p}$ of m with $q+1$ parts,

$$\sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p}) \left(s_i - \frac{m}{2}\right) = \frac{1}{2} \sum_{j=1}^{q+1} (q + 2 - 2j)\,p_j^2.$$

To prove this, we argue as for the identity (16.4). In the sum $\sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p})\, s_i$ the part p_j occurs with coefficient $(q+1-j)\, s_j$ as the left neighbour of s_j and with coefficient $(j-1)\, s_{j-1}$ as the right neighbour of s_{j-1}, where $s_0 = 0$ and $s_{q+1} = m$. Since

$$(q+1-j)\, s_j + (j-1)\, s_{j-1} = \frac{q}{2}\,(s_j + s_{j-1}) + \frac{q+2-2j}{2}\,(s_j - s_{j-1})$$

and $p_j = s_j - s_{j-1}$, so that $p_j\,(s_j + s_{j-1}) = s_j^2 - s_{j-1}^2$, we get

$$\sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p})\, s_i = \frac{q}{2}\, m^2 + \frac{1}{2} \sum_{j=1}^{q+1} (q+2-2j)\, p_j^2,$$

the sum of the differences $s_j^2 - s_{j-1}^2$ being telescopic. By (16.4) the term $\frac{q}{2}\, m^2$ equals $\frac{m}{2} \sum_{i=1}^{q} \mathrm{mult}_i(\mathbf{p})$, and the formula for the inner sum follows. For $1 \le j \le q+1$ let

$$E_j = \sum_{\mathbf{p} \in \mathrm{Comp}_{q+1}(m)} p_j^2\ \mathrm{wt}_f(\mathbf{p})$$

be the total weight of the compositions of m with $q+1$ parts when each of them is counted with the square of its j-th part. Then the formula for the difference of the two sides of the lemma, with its inner sum written in terms of the parts, becomes

$$q\, m \left(N - \frac{m}{2}\right) V_{q+1} - (q+1)(m-q)\, V_q = \frac{1}{2} \sum_{j=1}^{q+1} (q+2-2j)\, E_j.$$

It remains to show that the sum on the right is nonnegative. Its coefficients $q+2-2j$ are positive for the early parts and negative for the late ones, and they add up to 0.

We therefore sum by parts. For arbitrary real numbers $E_1, \ldots, E_{q+1}$,

$$\sum_{j=1}^{q+1} (q+2-2j)\, E_j = \sum_{j=1}^{q} j\,(q+1-j)\,\big(E_j - E_{j+1}\big),$$

since on the right-hand side E_j has coefficient $j\,(q+1-j) - (j-1)\,(q+2-j) = q+2-2j$. The coefficients $j\,(q+1-j)$ with $1 \le j \le q$ are positive. It is therefore enough to prove that

$$E_{j+1} \le E_j \qquad (1 \le j \le q).$$

For the proof, let $1 \le j \le q$. By definition,

$$E_j - E_{j+1} = \sum_{\mathbf{p} \in \mathrm{Comp}_{q+1}(m)} \big(p_j^2 - p_{j+1}^2\big)\ \mathrm{wt}_f(\mathbf{p}).$$

We sort the compositions by their cut points other than s_j, and we show that the compositions with given points s_l, $l \ne j$, contribute a nonnegative number to this sum. Among the given points are s_{j-1} and s_{j+1}, where again $s_0 = 0$ and $s_{q+1} = m$. Let $t = s_{j+1} - s_{j-1}$ be the length of the interval that p_j and p_{j+1} fill together. The compositions in question are obtained by placing the remaining cut at $s_j = s_{j-1} + v$ with $1 \le v \le t-1$, and then $p_j = v$ and $p_{j+1} = t - v$. The weight of such a composition is $f(s_{j-1}+v)\, a_v\, a_{t-v}$ times the product of the Cayley coefficients of the other parts and of the factors of the other cuts. This product

is the same for all the compositions in question, and it is positive, because f is positive at the cut points. So it suffices to show that

$$\sum_{v=1}^{t-1} f(s_{j-1}+v)\, a_v\, a_{t-v}\, \bigl(v^2-(t-v)^2\bigr) \geq 0.$$

Replacing v by $t-v$ reflects the cut about the midpoint of the interval and exchanges the sizes of the two parts; it leaves $a_v\, a_{t-v}$ unchanged and changes the sign of $v^2-(t-v)^2$. Hence the sum on the left is also equal to the negative of the same sum with $f(s_{j-1}+t-v)$ in the place of $f(s_{j-1}+v)$, and taking the mean of the two expressions we find that it equals

$$\frac{1}{2}\sum_{v=1}^{t-1}\bigl(f(s_{j-1}+v)-f(s_{j-1}+t-v)\bigr)\, a_v\, a_{t-v}\, \bigl(v^2-(t-v)^2\bigr).$$

Every term of this sum is nonnegative. Indeed, $s_{j-1}+v$ and $s_{j-1}+t-v$ are possible positions of the cut s_j, so that the monotonicity of the profile applies to them. If $v \geq t-v$, then $v^2-(t-v)^2 \geq 0$ and $f(s_{j-1}+v) \geq f(s_{j-1}+t-v)$. If $v < t-v$, both of these factors are nonpositive. This proves that $E_{j+1} \leq E_j$, and with it the lemma. □

17. The birthday law and the homogeneous law

By Proposition 16.3 the number of parts of a composition of m, drawn with a probability proportional to its weight, follows the birthday law of m. For this law we bound the mean s_m and the ratio D_m of the third to the second factorial moment. We then give every cut the same factor $h \geq 0$; for the law that results, the homogeneous law, we bound the mean number of cuts from below and compare the total weights of the compositions of $r+1$ and of r. These are the estimates on the number of parts that enter the proof of Proposition 1.4 in Sections 18 and 19.

17.1. **A correlation inequality.** We begin with the inequality by which two laws of the number of parts are compared, here and in Section 18. Let u and v be positive weights on a finite set S of integers such that the quotient u/v does not increase on S; one says that u is smaller than v in the *likelihood-ratio order*. In this case the average of a nondecreasing function φ on S with respect to u is at most its average with respect to v. This is the classical inequality of Chebyshev for the two oppositely ordered sequences u/v and φ with weights v, and it is of the kind that Fortuin, Kasteleyn and Ginibre prove for a log-supermodular measure on a finite distributive lattice, under which two nondecreasing functions are nonnegatively correlated [10, Proposition 1]. Since the weights to which we apply it vanish at some points of S, we state it with the hypothesis and the conclusion cross-multiplied; in this form no condition is imposed on the signs of u, v and φ, and the usual proof by symmetrisation applies.

Lemma 17.1. *Let S be a finite set of nonnegative integers and let u, v and φ be real-valued functions on the nonnegative integers. Suppose that*

$$u(q')\, v(q) \leq u(q)\, v(q') \qquad \text{and} \qquad \varphi(q) \leq \varphi(q')$$

for all $q, q' \in S$ with $q \leq q'$. Then

$$\Bigl(\sum_{q\in S} u(q)\, \varphi(q)\Bigr)\Bigl(\sum_{q\in S} v(q)\Bigr) \leq \Bigl(\sum_{q\in S} v(q)\, \varphi(q)\Bigr)\Bigl(\sum_{q\in S} u(q)\Bigr).$$

Proof. Let Y be the left-hand side minus the right-hand side. Expanding the products of the sums and symmetrising in the two indices of summation, we obtain

$$2Y = \sum_{q\in S}\sum_{q'\in S}\big(u(q)\,v(q') - u(q')\,v(q)\big)\big(\varphi(q)-\varphi(q')\big).$$

By the two hypotheses, applied to the pair (q,q') if $q \le q'$ and to the pair (q',q) otherwise, one of the two factors of each term is nonnegative and the other is nonpositive. Hence $Y \le 0$. □

17.2. **The moments of the birthday law.** Let $m \ge 1$ be an integer. Recall from Section 16 the number

$$P_{m,j} = \prod_{i=0}^{j-1}\Big(1-\frac{i}{m}\Big) = \frac{m(m-1)\cdots(m-j+1)}{m^j} \qquad (j \ge 0),$$

and the birthday law of m, the law of the number Q defined there, for which $\Pr(Q \ge j) = P_{m,j}$ for $j \ge 1$. Clearly

$$\begin{aligned} P_{m,0} = P_{m,1} = 1, \qquad P_{m,j+1} = \Big(1-\frac{j}{m}\Big)P_{m,j} \quad (j \ge 0),\\ P_{m,j} = 0 \quad (j \ge m+1), \qquad 0 \le P_{m,j} \le 1 \quad (j \ge 0), \end{aligned} \tag{17.1}$$

so that Q takes the value $q \ge 1$ with probability $P_{m,q} - P_{m,q+1} = \frac{q}{m}\,P_{m,q}$.

The sums of this section are evaluated by a summation by parts: for every real-valued function ψ on the nonnegative integers,

$$\sum_{q=1}^{m}\psi(q)\,q\,P_{m,q} = m\,\Big(\psi(0) + \sum_{q=1}^{m}\big(\psi(q)-\psi(q-1)\big)\,P_{m,q}\Big). \tag{17.2}$$

Indeed, $q\,P_{m,q} = m\,(P_{m,q} - P_{m,q+1})$, as we have just seen, and it remains to shift the index in the terms that contain $P_{m,q+1}$, using $P_{m,1} = 1$ and $P_{m,m+1} = 0$. Divided by m, the identity expresses the expectation $\mathbb{E}[\psi(Q)]$ through the tail probabilities $\Pr(Q \ge j)$.

The mean of the birthday law of m is the sum of the tail probabilities,

$$s_m = \sum_{j=1}^{m} P_{m,j} \qquad (m \ge 1),$$

and we put $s_0 = 0$; note that $s_1 = 1$. Each of the m terms of s_m is at most 1, and for $m \ge 2$ the term $P_{m,2} = 1 - \frac{1}{m}$ is smaller; hence

$$s_m < m \qquad (m \ge 2).$$

Applying the summation by parts to $\psi(j) = 1$, j, $j(j-1)$ and $j(j-1)(j-2)$ in turn, we obtain

$$\begin{aligned}
\sum_{q=1}^{m} q\, P_{m,q} &= m, \\
\sum_{q=1}^{m} q^2\, P_{m,q} &= m\, s_m, \\
\sum_{q=1}^{m} q^2(q-1)\, P_{m,q} &= 2m\,(m - s_m), \\
\sum_{q=1}^{m} q^2(q-1)(q-2)\, P_{m,q} &= 3m\,\big((m+2)\, s_m - 3m\big).
\end{aligned} \tag{17.3}$$

For the last two functions the right-hand side of the summation by parts is evaluated with the first two identities and the definition of s_m.

By Proposition 16.3 we have $\sum_{q=1}^{m} \psi(q)\, T_{m,q} = a_m \sum_{q=1}^{m} \psi(q)\, q\, P_{m,q}$ for every function ψ, so that these four identities describe the number of parts of the compositions of m. By the first identity the total weight of all compositions of m is $m\, a_m$, and those with q parts carry the fraction $\frac{q}{m}\, P_{m,q} = \Pr(Q = q)$ of it. By the second identity the mean number of parts is s_m. The last two identities give the second and the third factorial moment of the number of parts, $\mathbb{E}[Q(Q-1)] = 2\,(m - s_m)$ and $\mathbb{E}[Q(Q-1)(Q-2)] = 3\,((m+2)\, s_m - 3m)$, or, in terms of the weights,

$$\begin{aligned}
\sum_{q=1}^{m} q(q-1)\, T_{m,q} &= 2m\, a_m\,(m - s_m), \\
\sum_{q=1}^{m} q(q-1)(q-2)\, T_{m,q} &= 3m\, a_m\,\big((m+2)\, s_m - 3m\big).
\end{aligned} \tag{17.4}$$

The quotient of these two moments enters the comparison of two second differences in Section 18. For an integer $m \geq 2$ let

$$D_m = \frac{3\,\big((m+2)\, s_m - 3m\big)}{2\,(m - s_m)},$$

where the denominator is positive because $s_m < m$. For $m = 0$ and $m = 1$ the formula reads $0/0$, since $s_0 = 0$ and $s_1 = 1$, and we set $D_0 = D_1 = 0$. By the two identities (17.4), the definition of D_m, multiplied by $2m\, a_m\,(m - s_m)$, says that

$$\sum_{q=1}^{m} q(q-1)(q-2)\, T_{m,q} = D_m \sum_{q=1}^{m} q(q-1)\, T_{m,q} \qquad (m \geq 2). \tag{17.5}$$

The following two lemmas give what else we use about D_m: a bound by the mean, and its monotonicity in m, which the formula does not show and which lets the one constant D_r serve for all $m \leq r$.

Lemma 17.2. *For every integer $m \geq 2$,*

$$D_m \leq \tfrac{3}{2}\,(s_m - 1).$$

Proof. The sum $\sum_{q=1}^{m} (q - s_m)^2\, q\, P_{m,q}$, which is m times the variance of the birthday law of m, is nonnegative, and a direct computation with the first three identities

of (17.3) shows that it equals $m\,(2m - s_m - s_m^2)$. Hence $s_m^2 + s_m \le 2m$. Now $m - s_m > 0$ because $s_m < m$ for $m \ge 2$, so that the assertion of the lemma is equivalent to $(m+2)\,s_m - 3m \le (s_m - 1)(m - s_m)$, and the right-hand side of this inequality minus its left-hand side is $2m - s_m - s_m^2 \ge 0$. □

Lemma 17.3. *For all integers m and r with $2 \le m \le r$,*

$$D_m \le D_r.$$

Proof. By induction on r it suffices to prove that $D_m \le D_{m+1}$ for every integer $m \ge 2$. We fix such an m.

We claim that for all integers q and q' with $1 \le q \le q' \le m+1$,

$$P_{m,q'}\,P_{m+1,q} \le P_{m,q}\,P_{m+1,q'}.$$

For $q' = q$ the two sides are equal, and by the recursion (17.1) the passage from $q' = j$ to $q' = j+1$, where $q \le j \le m$, multiplies the left-hand side by $1 - \frac{j}{m}$ and the right-hand side by $1 - \frac{j}{m+1}$. Since $0 \le 1 - \frac{j}{m} \le 1 - \frac{j}{m+1}$ and the right-hand side is nonnegative, the claim follows by induction on q'.

By Proposition 16.3, multiplying the inequality just proved by $a_m\,a_{m+1}\,q\,q' \ge 0$ gives $T_{m,q'}\,T_{m+1,q} \le T_{m,q}\,T_{m+1,q'}$ for $1 \le q \le q' \le m+1$. The hypotheses of Lemma 17.1 therefore hold for $S = \{1, \dots, m+1\}$ and

$$u(q) = q(q-1)\,T_{m,q}, \qquad v(q) = q(q-1)\,T_{m+1,q}, \qquad \varphi(q) = q - 2,$$

and the lemma gives

$$\Big(\sum_{q=1}^{m+1} q(q-1)(q-2)\,T_{m,q}\Big)\Big(\sum_{q=1}^{m+1} q(q-1)\,T_{m+1,q}\Big) \le \Big(\sum_{q=1}^{m+1} q(q-1)(q-2)\,T_{m+1,q}\Big)\Big(\sum_{q=1}^{m+1} q(q-1)\,T_{m,q}\Big),$$

in which the two sums that contain $T_{m,q}$ end at $q = m$, since a composition of m has at most m parts.

Put $F_m = \sum_{q=1}^{m} q(q-1)\,T_{m,q}$. By (17.5), applied to m and to $m+1$, the last inequality reads $D_m\,F_m\,F_{m+1} \le D_{m+1}\,F_{m+1}\,F_m$. Since $F_m = 2m\,a_m\,(m - s_m) > 0$ by the first identity of (17.4) and $s_m < m$, and $F_{m+1} > 0$ in the same way, we conclude that $D_m \le D_{m+1}$. □

17.3. **The mean of the birthday law.** Since D_m is an explicit function of m and s_m, the numerical information about the birthday law that we need is an upper bound for its mean. The sum $s_m = 1 + \frac{m-1}{m} + \frac{(m-1)(m-2)}{m^2} + \cdots$ is known as Ramanujan's Q-function, and our bound for it rests on a known inequality for the Poisson law. For an integer $m \ge 0$ and a real number $x \ge 0$ let

$$\varpi_m(x) = e^{-x}\,\frac{x^m}{m!}, \qquad H_m(x) = e^{-x}\sum_{i=0}^{m}\frac{x^i}{i!},$$

the probability that a Poisson variable with mean x takes the value m, resp. a value at most m; thus $\varpi_m(m) = e^{-m}m^m/m!$ is the Poisson mass at m of Section 2. The inequality in question is

$$H_m(m) \le \frac{1}{2} + \frac{2}{3}\,\varpi_m(m) \qquad (m \ge 1). \tag{17.6}$$

In the classical notation, in which a number θ_m is defined by the equation $\frac{1}{2}\,e^m = \sum_{i<m} \frac{m^i}{i!} + \theta_m\,\frac{m^m}{m!}$, the inequality (17.6) is the statement $\theta_m \geq \frac{1}{3}$, since this equation, multiplied by e^{-m}, reads $\frac{1}{2} = H_m(m) - (1-\theta_m)\,\varpi_m(m)$. That θ_m lies between $\frac{1}{3}$ and $\frac{1}{2}$ was asked by Ramanujan, and Flajolet, Grabner, Kirschenhofer and Prodinger, who recount the history of the question, prove more: $\theta_n = \frac{1}{3} + \frac{4}{135\,(n+k_n)}$ with $\frac{2}{21} \leq k_n \leq \frac{8}{45}$ for all integers $n \geq 0$ [7, Theorem 7]. In particular $\theta_m \geq \frac{1}{3}$, which is (17.6). The constant $\frac{2}{3}$ in (17.6) is the source of the term $-\frac{1}{3}$ in the following bound, and this term is needed in the proof of Proposition 19.3.

Lemma 17.4. *For every integer $r \geq 1$,*

$$s_r \leq \sqrt{\frac{\pi r}{2}}\; e^{1/(12r)} - \frac{1}{3}.$$

Proof. For $0 \leq j \leq r$ we have $P_{r,j} = r!/((r-j)!\,r^j)$, i.e. $r^{\,r-j}/(r-j)! = P_{r,j}\,r^r/r!$. Summing over j and multiplying by e^{-r}, we obtain $H_r(r) = \varpi_r(r)\,(1+s_r)$, because $P_{r,0} = 1$. It follows from (17.6) that

$$s_r = \frac{H_r(r)}{\varpi_r(r)} - 1 \leq \frac{\frac{1}{2} + \frac{2}{3}\,\varpi_r(r)}{\varpi_r(r)} - 1 = \frac{1}{2\,\varpi_r(r)} - \frac{1}{3}.$$

Finally, $\varpi_r(r) = (r/e)^r/r!$, and the upper bound in (2.6) gives

$$\frac{1}{2\,\varpi_r(r)} = \frac{r!}{2\,(r/e)^r} \leq \frac{\sqrt{2\pi r}}{2}\; e^{1/(12r)} = \sqrt{\frac{\pi r}{2}}\; e^{1/(12r)}. \qquad \square$$

17.4. **The homogeneous law.** We now give every cut of a composition the same factor h. The chain sums are total weights for the threshold profile, which is not constant; in Sections 18 and 19 they are compared with the total weights for constant profiles, and the two lemmas of this subsection are what these comparisons use.

For an integer $m \geq 0$ and a real number h let

$$\tau_m(h) = \Lambda_m[h]$$

be the total weight $\Lambda_m[f] = \sum_{q=1}^{m} V_q(m; f)$ of all compositions of m (Section 16) for the constant cut profile $f = h$. A composition with q parts has $q-1$ cuts, so that its weight for the constant profile h is h^{q-1} times its weight for the profile $\mathbf{1}$. Therefore

$$\tau_m(h) = \sum_{q=1}^{m} T_{m,q}\,h^{q-1}.$$

For $m \geq 1$ and $h \geq 0$ this number is positive, since every $T_{m,q}$ is nonnegative by Proposition 16.3 and the term with $q = 1$ is $T_{m,1} = a_m > 0$.

Let $r \geq 1$ be an integer and let $h \geq 0$. The *homogeneous law with parameter h* on the compositions of r gives each composition a probability proportional to its weight for the constant profile h. Under it the number of parts Q takes the value q with probability $T_{r,q}\,h^{q-1}/\tau_r(h)$; for $h = 1$ this is the birthday law of r. The mean number of cuts under the homogeneous law is therefore

$$\mu_r(h) = \frac{\sum_{q=1}^{r} (q-1)\,T_{r,q}\,h^{q-1}}{\tau_r(h)} \qquad (r \geq 1,\ h \geq 0).$$

It is nonnegative, and $\mu_r(1) = s_r - 1$ by the first two identities of (17.3). Inserting Proposition 16.3 into the formulas for $\tau_r(h)$ and $\mu_r(h)$ and writing $q = j + 1$, we express both quantities through the numbers $P_{r,j}$: for $r \geq 1$ and $h \geq 0$,

$$\begin{aligned} \tau_r(h) &= a_r \sum_{j=0}^{r-1} (j+1)\, h^j\, P_{r,j+1}, \\ \mu_r(h)\, \tau_r(h) &= a_r \sum_{j=0}^{r-1} j\,(j+1)\, h^j\, P_{r,j+1}. \end{aligned} \tag{17.7}$$

For the lower bound of the mean number of cuts we define, for an integer $r \geq 1$ and a real number $X \geq 0$,

$$\mu_r^0(X) = \frac{1}{3}\left(\sqrt{(4+X)^2 + 12\,(r-1)} - (4+X)\right),$$

which is the nonnegative solution μ of the quadratic equation $\frac{3}{2}\,\mu^2 + (4+X)\,\mu = 2\,(r-1)$. Unlike the mean itself, this bound is explicit in r and h.

Lemma 17.5. *For every integer $r \geq 1$ and every real number h with $0 < h \leq 1$,*

$$\mu_r^0\big(r\,(1/h - 1)\big) \leq \mu_r(h).$$

Proof. We fix r and h and write $\mu = \mu_r(h)$ and $X = r\,(1/h - 1) \geq 0$. We shall prove the quadratic inequality

$$2\,(r-1) \leq \tfrac{3}{2}\,\mu^2 + (4+X)\,\mu,$$

of which the lemma is the solution for μ.

For an integer $j \geq 0$ put $\sigma_j = h^j P_{r,j+1}$. By the properties (17.1) we have $\sigma_j \geq 0$ for every j, $\sigma_0 = 1$, $\sigma_j = 0$ for $j \geq r$, and

$$\sigma_{j+1} = \rho_j\, \sigma_j \qquad \text{with} \qquad \rho_j = h\left(1 - \frac{j+1}{r}\right) \qquad (j \geq 0).$$

The ρ_j do not increase as j grows, and $\rho_j \geq 0$ for $j \leq r-1$. We work with the three sums

$$M_0 = \sum_{j=0}^{r-1} (j+1)\,\sigma_j, \qquad M_1 = \sum_{j=0}^{r-1} j\,(j+1)\,\sigma_j, \qquad M_2 = \sum_{j=0}^{r-1} j^2\,(j+1)\,\sigma_j.$$

By the expressions (17.7) we have $\tau_r(h) = a_r\, M_0$ and $\mu\,\tau_r(h) = a_r\, M_1$, and since $a_r > 0$ and $M_0 \geq \sigma_0 = 1$,

$$M_0 > 0 \qquad \text{and} \qquad M_1 = \mu\, M_0.$$

Thus M_1/M_0 is the mean of the number of cuts under the homogeneous law, and M_2/M_0 is the mean of its square.

We first prove the identity

$$r\, M_1 = h\big(2\,(r-1)\, M_0 + (r-3)\, M_1 - M_2\big).$$

Since $\sigma_r = 0$, a shift of the index gives $M_1 = \sum_{j=0}^{r-1} (j+1)(j+2)\,\sigma_{j+1}$. By the recursion $\sigma_{j+1} = \rho_j\,\sigma_j$, for every $j \geq 0$,

$$\begin{aligned} r\,(j+1)(j+2)\,\sigma_{j+1} &= h\,(j+1)(j+2)(r-1-j)\,\sigma_j \\ &= h\,(j+1)\,\sigma_j\,\big(2\,(r-1) + (r-3)\,j - j^2\big), \end{aligned}$$

and summing over $0 \leq j \leq r-1$ gives the identity.

Next we prove the inequality

$$\tfrac{1}{6}\,(M_2 - M_1)\,M_0 \le \tfrac{1}{4}\,M_1^2.$$

The factor $j+1$ in the three sums is the number of pairs (i,l) of nonnegative integers with $i+l=j$. The sums are therefore sums over such pairs with weight σ_{i+l}, and the inequality says that under this weight the two coordinates i and l are negatively correlated. This holds because (σ_j) is log-concave, and we derive it from two applications of Lemma 17.1, one in each coordinate.

We use the log-concavity in the following form: for all integers i, j and d with $0 \le i \le j$ and $d \ge 0$,

$$\sigma_i\,\sigma_{j+d} \le \sigma_{i+d}\,\sigma_j.$$

We argue by induction on d; for $d=0$ the two sides are equal. Suppose that the inequality holds for some d. If $j+d+1 \ge r$, then $\sigma_{j+d+1}=0$, and the inequality for $d+1$ holds because its right-hand side is nonnegative. Otherwise $j+d \le r-2$ and $i \le j$, so that $0 \le \rho_{j+d} \le \rho_{i+d}$, and by the recursion for σ_j, by the induction hypothesis and by $\sigma_{i+d}\,\sigma_j \ge 0$,

$$\sigma_i\,\sigma_{j+d+1} = \rho_{j+d}\,\sigma_i\,\sigma_{j+d} \le \rho_{j+d}\,\sigma_{i+d}\,\sigma_j \le \rho_{i+d}\,\sigma_{i+d}\,\sigma_j = \sigma_{i+d+1}\,\sigma_j.$$

We write the three sums as sums over pairs in the following notation. For a real-valued function g of two nonnegative integers put

$$\langle g\rangle = \sum_{i=0}^{r-1}\sum_{l=0}^{r-1} \sigma_{i+l}\,g(i,l).$$

Since $\sigma_j = 0$ for $j \ge r$, only pairs with $i+l \le r-1$ contribute, and collecting the pairs with $i+l=j$ we get $\langle g\rangle = \sum_{j=0}^{r-1}\sigma_j \sum_{i=0}^{j} g(i,j-i)$. The inner sum is $j+1$ for $g=1$, it is $\frac{1}{2}\,j\,(j+1)$ for $g(i,l)=i$ and for $g(i,l)=l$, and it is $\frac{1}{6}\,(j-1)\,j\,(j+1) = \frac{1}{6}\,\big(j^2(j+1) - j\,(j+1)\big)$ for $g(i,l)=i\,l$. This gives

$$\langle 1\rangle = M_0, \qquad \langle i\rangle = \langle l\rangle = \tfrac{1}{2}\,M_1, \qquad \langle i\,l\rangle = \tfrac{1}{6}\,(M_2 - M_1),$$

where $\langle i\rangle$, $\langle l\rangle$ and $\langle i\,l\rangle$ stand for $\langle g\rangle$ with $g(i,l)=i$, with $g(i,l)=l$ and with $g(i,l)=i\,l$.

It remains to apply Lemma 17.1, first in the coordinate l and then in the coordinate i. For an integer $i \ge 0$ put

$$U_0(i) = \sum_{l=0}^{r-1}\sigma_{i+l}, \qquad U_1(i) = \sum_{l=0}^{r-1} l\,\sigma_{i+l}.$$

Let i and i' be integers with $0 \le i \le i'$. For integers l and l' with $0 \le l \le l'$, the log-concavity in the form proved above, applied with $i+l \le i'+l$ in the place of $i \le j$ and with $d = l'-l$, reads $\sigma_{i+l}\,\sigma_{i'+l'} \le \sigma_{i+l'}\,\sigma_{i'+l}$. This is the first hypothesis of Lemma 17.1 for $S=\{0,\dots,r-1\}$, $u(l)=\sigma_{i'+l}$ and $v(l)=\sigma_{i+l}$, and with $\varphi(l)=l$ the lemma gives

$$U_1(i')\,U_0(i) \le U_1(i)\,U_0(i') \qquad (0 \le i \le i').$$

This in turn is the first hypothesis of Lemma 17.1 for the same set S, $u=U_1$ and $v=U_0$, and with $\varphi(i)=i$ the lemma gives

$$\Big(\sum_{i=0}^{r-1} i\,U_1(i)\Big)\Big(\sum_{i=0}^{r-1} U_0(i)\Big) \le \Big(\sum_{i=0}^{r-1} i\,U_0(i)\Big)\Big(\sum_{i=0}^{r-1} U_1(i)\Big),$$

i.e. $\langle i\, l\rangle\, \langle 1\rangle \le \langle i\rangle\, \langle l\rangle$. With the values of these four sums found above, this is the inequality $\frac{1}{6}\,(M_2 - M_1)\,M_0 \le \frac{1}{4}\,M_1^2$ that we had to prove.

We now deduce the quadratic inequality from the identity and the inequality just proved. Multiplied by $6h$, the inequality reads $h\,(M_2 - M_1)\,M_0 \le \frac{3}{2}\,h\,M_1^2$, and the identity gives $h\,M_2 = 2\,(r-1)\,h\,M_0 + (r-3)\,h\,M_1 - r\,M_1$. Eliminating M_2, we obtain

$$2\,(r-1)\,h\,M_0^2 \le \tfrac{3}{2}\,h\,M_1^2 + (4h + r - rh)\,M_1\,M_0.$$

Dividing by $h\,M_0^2 > 0$ and using $M_1 = \mu\,M_0$ and $(4h + r - rh)/h = 4 + X$, we arrive at the quadratic inequality $2\,(r-1) \le \frac{3}{2}\,\mu^2 + (4+X)\,\mu$.

Finally we solve the quadratic inequality for μ. Completing the square, we write it as $(4+X)^2 + 12\,(r-1) \le (3\mu + 4 + X)^2$. Since $\mu \ge 0$ and $X \ge 0$, we have $3\mu + 4 + X > 0$, and taking square roots we obtain $\mu_r^0(X) \le \mu$. □

For $h = 1$ the total weights satisfy $\tau_{r+1}(1) = \bigl(1 + \frac{1}{r}\bigr)^r\,\tau_r(1)$, since $\tau_m(1) = m\,a_m$ by the first identity of (17.3) and $(r+1)\,a_{r+1} = \bigl(1 + \frac{1}{r}\bigr)^r\,r\,a_r$ by (2.3). For $h < 1$ the comparison of $\tau_{r+1}(h)$ with $\tau_r(h)$ gains a factor that contains the mean number of cuts.

Lemma 17.6. *For every integer $r \ge 1$ and every real number h with $0 < h \le 1$,*

$$\tau_{r+1}(h) \le \Bigl(1 + \frac{1}{r}\Bigr)^r \Bigl(1 - \frac{1-h}{2h\,(r+1)}\,\mu_r(h)\Bigr)\,\tau_r(h).$$

Proof. We fix r and h, and for an integer $m \ge 1$ we put

$$B_m = \sum_{j=1}^{m} h^{j-1}\,P_{m,j+1}.$$

We first express the total weights: for every integer $m \ge 1$,

$$\tau_m(h) = m\,a_m\,\bigl(1 - (1-h)\,B_m\bigr).$$

Indeed, since $\tau_m(h) = \sum_{q=1}^{m} T_{m,q}\,h^{q-1}$, Proposition 16.3 gives

$$\tau_m(h) = a_m \sum_{q=1}^{m} h^{q-1}\,q\,P_{m,q},$$

and the summation by parts (17.2), applied to the function ψ with $\psi(0) = 1$ and $\psi(q) = h^{q-1}$ for $q \ge 1$, shows that this sum equals $m\,\bigl(1 - (1-h)\sum_{q=2}^{m} h^{q-2}\,P_{m,q}\bigr)$. The last sum is B_m, because $P_{m,m+1} = 0$.

To compare the tails $P_{r,j+1}$ and $P_{r+1,j+1}$ of the birthday laws of r and of $r+1$, write $w = 2r(r+1)$. We claim that for every integer j with $0 \le j \le r$,

$$\bigl(w + j\,(j+1)\bigr)\,P_{r,j+1} \le w\,P_{r+1,j+1}.$$

We argue by induction on j; for $j = 0$ the two sides are equal. Let $0 \le j < r$ and suppose that the inequality holds for j. By the recursion (17.1) we have $P_{r,j+2} = \alpha\,P_{r,j+1}$ and $P_{r+1,j+2} = \beta\,P_{r+1,j+1}$ with

$$\alpha = 1 - \frac{j+1}{r}, \qquad \beta = 1 - \frac{j+1}{r+1} > 0.$$

A direct computation gives

$$\Bigl(\bigl(w + j\,(j+1)\bigr)\,\beta - \bigl(w + (j+1)(j+2)\bigr)\,\alpha\Bigr)\,r(r+1) = (j+1)^2\,(2r + j + 2) \ge 0,$$

so that $(w+(j+1)(j+2))\,\alpha \le (w+j\,(j+1))\,\beta$. Multiplying this inequality by $P_{r,j+1} \ge 0$ and the induction hypothesis by β, we obtain

$$\begin{aligned}\big(w+(j+1)(j+2)\big)\,P_{r,j+2} &\le \big(w+j\,(j+1)\big)\,\beta\,P_{r,j+1}\\ &\le w\,\beta\,P_{r+1,j+1} = w\,P_{r+1,j+2},\end{aligned}$$

which is the claim for $j+1$.

A lower bound for $B_{r+1}-B_r$ follows by summation. We write the inequality just proved as $j\,(j+1)\,P_{r,j+1} \le w\,(P_{r+1,j+1}-P_{r,j+1})$, multiply by h^j and sum over $1 \le j \le r$. Since $P_{r,r+1} = P_{r+1,r+2} = 0$, the left-hand side becomes $\mu_r(h)\,\tau_r(h)/a_r$ by (17.7), and the right-hand side becomes $h\,w\,(B_{r+1}-B_r)$. Therefore

$$\mu_r(h)\,\tau_r(h) \le 2h\,r(r+1)\,a_r\,(B_{r+1}-B_r).$$

On the other hand, the formula $\tau_m(h) = m\,a_m\,(1-(1-h)\,B_m)$ for $m = r+1$ and for $m = r$ and the relation $(r+1)\,a_{r+1} = \big(1+\frac{1}{r}\big)^r\,r\,a_r$, which is (2.3) multiplied by $r+1$, give

$$\begin{aligned}\tau_{r+1}(h) &= \Big(1+\frac{1}{r}\Big)^r\,r\,a_r\,\big(1-(1-h)\,B_{r+1}\big)\\ &= \Big(1+\frac{1}{r}\Big)^r\,\Big(\tau_r(h)-(1-h)\,r\,a_r\,(B_{r+1}-B_r)\Big).\end{aligned}$$

Since $1-h \ge 0$ and $h > 0$, the lower bound for $B_{r+1}-B_r$ gives

$$(1-h)\,r\,a_r\,(B_{r+1}-B_r) \ge \frac{1-h}{2h\,(r+1)}\,\mu_r(h)\,\tau_r(h),$$

and the assertion follows. □

18. Chain sums and the frozen polynomial

Let $k \ge 2$, and let $n = k+r$ with $1 \le r \le k-1$ be a row of the chain range of column k. We show that each of the normalised steps $\delta_n^{(k)}$ and $\delta_n^{(k+1)}$ is a second difference of one polynomial in a real threshold minus a multiple of a chain sum (Proposition 18.6); this is the form in which Section 19 compares the two steps. We also compare the two second differences with each other, and the two chain sums with a third.

18.1. **Compositions and their first part.** We use the notation of Sections 16 and 17. Recall that the weight of a composition for a cut profile f is the product of the Cayley coefficients a_p of its parts and of the values of f at its cut points, that $V_q(m;f)$ is the total weight of the compositions of m with q parts, and that $\Lambda_m[f] = \sum_{q=1}^{m} V_q(m;f)$. For the constant profiles $\mathbf{1}$ and h we have $T_{m,q} = V_q(m;\mathbf{1})$ and $\tau_m(h) = \Lambda_m[h] = \sum_{q=1}^{m} T_{m,q}\,h^{q-1}$, and for the threshold profile $\lambda^{y,N}(s) = y/(N-s)$ the total weight is the chain sum $\Lambda_m(y;N)$. In every chain sum $\Lambda_m(y;N)$ of this section N is an integer with $N > m$, so that $N-s \ge 2$ at each cut point s of a composition of m.

We shall use the following properties, which are clear from this description. The cut points of a composition of m lie in $\{1,\dots,m-1\}$, so that $V_q(m;f)$ and $\Lambda_m[f]$ depend only on $f(1),\dots,f(m-1)$, and $V_q(m;f) = 0$ for $q > m$. If f is nonnegative at these points, then $V_q(m;f) \ge 0$ for every q, the total weight $\Lambda_m[f]$ does not decrease when f is raised at any of them, and

$$\Lambda_m[f] \ge V_1(m;f) = a_m > 0 \qquad (m \ge 1).$$

If f is positive at these points, then $V_q(m;f) > 0$ for $1 \le q \le m$; in particular $T_{m,q} > 0$ for $1 \le q \le m$. Finally, a composition with q parts has $q-1$ cuts, so that for the threshold profile

$$\begin{aligned} V_q(m;\lambda^{y,N}) &= y^{q-1}\, V_q(m;\lambda^{1,N}) \qquad (q \ge 1), \\ \Lambda_m(y;N) &= \sum_{q=1}^{m} y^{q-1}\, V_q(m;\lambda^{1,N}), \end{aligned} \tag{18.1}$$

and a chain sum is a polynomial in y of degree at most $m-1$.

Definition 16.1 comes with a recursion that removes the last part of a composition. Removing the first part instead shifts the profile.

Lemma 18.1. *Let f be a cut profile and let $m \ge 0$ be an integer. Then*

$$\Lambda_m[f] = a_m + \sum_{j=1}^{m-1} a_j\, f(j)\, \Lambda_{m-j}[f(j+\cdot)],$$

where $f(j+\cdot)$ denotes the profile $s \mapsto f(j+s)$.

Proof. We may assume that $m \ge 1$. The composition of m with one part has weight a_m. A composition of m with at least two parts consists of its first part j, where $1 \le j \le m-1$, of the cut at j, and of a composition of $m-j$ whose cut points are the remaining cut points shifted by j; its weight for f is therefore $a_j\, f(j)$ times the weight of this composition of $m-j$ for $f(j+\cdot)$. Since every composition of $m-j$ arises in this way exactly once, the total weight of the compositions of m with first part j equals

$$a_j\, f(j)\, \Lambda_{m-j}[f(j+\cdot)] \qquad (1 \le j \le m-1),$$

and the sum of these numbers and of a_m is $\Lambda_m[f]$. □

For the threshold profile we have $\lambda^{y,N}(j+s) = y/(N-j-s) = \lambda^{y,N-j}(s)$, so that Lemma 18.1 reads

$$\Lambda_m(y;N) = a_m + \sum_{j=1}^{m-1} a_j\, \frac{y}{N-j}\, \Lambda_{m-j}(y;N-j). \tag{18.2}$$

18.2. **Shortening a chain sum and raising its threshold.** The two chain sums that occur in Proposition 18.6, $\Lambda_{r+1}(k-1;N)$ and $\Lambda_r(k+2;N)$ with $N = k+r+1$, differ in their length and in their threshold. We compare each of them with the chain sum $\Lambda_r(k;N)$, by means of two homogeneous laws on the compositions of r. Recall from Section 17 that $\mu_r(h)$ denotes the mean number of cuts of a composition of r under the homogeneous law with parameter h. For integers $k \ge 2$ and $r \ge 1$ we put

$$h_- = \frac{k-1}{k+r} \qquad \text{and} \qquad \mu_- = \mu_r(h_-),$$

and for integers $k \ge 1$ and $r \ge 1$ we put

$$h_{\mathrm{mid}} = \frac{k}{k+1+\frac{r}{2}} \qquad \text{and} \qquad \mu_{\mathrm{mid}} = \mu_r(h_{\mathrm{mid}}).$$

Thus h_- is the value of the threshold profile $\lambda^{k-1,N}$ at the first cut point $s = 1$, and h_{mid} is the value of the formula $k/(N-s)$ of the threshold profile $\lambda^{k,N}$ at the midpoint $s = \frac{r}{2}$ of the segment that is cut.

The first comparison rests on the correlation inequality of Fortuin, Kasteleyn and Ginibre [10, Proposition 1]: under a log-supermodular measure on a finite distributive lattice, two nondecreasing functions are nonnegatively correlated. We use it for the lattice of all subsets of a finite set U, where it reads as follows. Let μ, F and G be real functions on the set of all subsets of U. Suppose that μ takes positive values and is log-supermodular, that is,

$$\mu(S)\,\mu(T) \le \mu(S\cap T)\,\mu(S\cup T)$$

for all subsets S and T of U, and that F and G are nondecreasing, that is, $F(S) \le F(T)$ and $G(S) \le G(T)$ whenever $S \subseteq T \subseteq U$. Then F and G are nonnegatively correlated under the probability measure proportional to μ, which means that

$$\Big(\sum_S \mu(S)\,F(S)\Big)\Big(\sum_S \mu(S)\,G(S)\Big) \le \Big(\sum_S \mu(S)\Big)\Big(\sum_S \mu(S)\,F(S)\,G(S)\Big),$$

where the four sums are over all subsets S of U. We shall apply this with $U = \{1,\dots,r-1\}$, whose subsets are the sets of cut points of the compositions of r.

The chain sum $\Lambda_{r+1}(k-1;N)$ can be shortened by one, and its threshold raised from $k-1$ to k at the same time, at the price of the ratio $\tau_{r+1}(h_-)/\tau_r(h_-)$ by which the same shortening changes the total weight for the constant profile h_-. The proof compares the threshold profile $\lambda^{k-1,N}$ with the constant profile h_- through the first part of a composition: a composition of $r+1$ arises from a composition of r either by putting a part 1 in front or by raising the first part p by one, which multiplies the weight by a_{p+1}/a_p. This reduces the lemma to a correlation inequality for the compositions of r, which the proof deduces from the inequality of Fortuin, Kasteleyn and Ginibre and from the log-convexity of the Cayley coefficients.

Lemma 18.2. *Let $k \ge 2$ and $r \ge 1$ be integers, and put $N = k+r+1$. Then*

$$\Lambda_{r+1}(k-1;N)\;\tau_r(h_-) \le \tau_{r+1}(h_-)\;\Lambda_r(k;N).$$

Proof. We write $\psi(j) = (1+\frac{1}{j})^{j-1}$ for integers $j \ge 1$. Thus $a_{j+1} = \psi(j)\,a_j$ by (2.3), and ψ is positive and nondecreasing by Lemma 2.1.

For a real function φ on the positive integers and a cut profile f let $\Lambda_r^{\varphi}[f]$ be the sum, over all compositions of r, of the weight of the composition for f times the value of φ at its first part. Grouping the compositions by their first part, as in the proof of Lemma 18.1, we find that

$$\Lambda_r^{\varphi}[f] = \varphi(r)\,a_r + \sum_{j=1}^{r-1} \varphi(j)\,a_j\,f(j)\,\Lambda_{r-j}[f(j+\cdot)],$$

and for the constant function $\varphi = 1$ this is $\Lambda_r[f]$. We write f^+ for the shifted profile $f(1+\cdot)$; a constant profile h is its own shift, and for it $\Lambda_r[h] = \tau_r(h)$.

We first express $\Lambda_{r+1}[f]$ through sums over the compositions of r: for every cut profile f,

$$\Lambda_{r+1}[f] = f(1)\,\Lambda_r[f^+] + \Lambda_r^{\psi}[f^+]. \tag{18.3}$$

This is Lemma 18.1 with $m = r+1$. Since $a_1 = 1$, its term with first part 1 is $f(1)\,\Lambda_r[f^+]$. Since $a_{j+1} = \psi(j)\,a_j$, its term with first part $j+1$ is, for $1 \le j \le r-1$, the term with first part j in the formula for $\Lambda_r^{\psi}[f^+]$, and the remaining term $a_{r+1} = \psi(r)\,a_r$ is the term with first part r in that formula.

Next we prove a correlation inequality. Let $h > 0$ be a real number, let f be a cut profile with $f(s) \geq h$ for $1 \leq s \leq r-1$, and let φ be a real function on the positive integers that is nondecreasing on $\{1, \ldots, r\}$. We claim that

$$\Lambda_r^\varphi[f]\, \tau_r(h) \leq \Lambda_r^\varphi[h]\, \Lambda_r[f]. \tag{18.4}$$

We describe a composition of r by its set S of cut points, which is a subset of $U = \{1, \ldots, r-1\}$; every subset of U is the set of cut points of exactly one composition. Let $w(S) > 0$ be the product of the a_p over the parts p of this composition, and let $\mathrm{fp}(S)$ be its first part, which is the least element of S, or r if S is empty. The weight of the composition for the constant profile h is $\mu(S) = w(S)\, h^{|S|}$, which is positive, and its weight for f is $\mu(S)\, F(S)$, where

$$F(S) = \prod_{s \in S} \frac{f(s)}{h}.$$

We put $G(S) = -\varphi(\mathrm{fp}(S))$. The four quantities in the claim are then sums over the subsets S of U:

$$\tau_r(h) = \sum_S \mu(S), \qquad \Lambda_r[f] = \sum_S \mu(S)\, F(S),$$
$$\Lambda_r^\varphi[h] = -\sum_S \mu(S)\, G(S), \qquad \Lambda_r^\varphi[f] = -\sum_S \mu(S)\, F(S)\, G(S).$$

With these expressions, the conclusion of the inequality of Fortuin, Kasteleyn and Ginibre, in the form given before the lemma and for these μ, F and G, reads

$$-\Lambda_r[f]\, \Lambda_r^\varphi[h] \leq -\tau_r(h)\, \Lambda_r^\varphi[f],$$

which is the claim. In words, when every set of cut points is counted with its weight for the constant profile h, the factor F by which the profile f changes this weight and the value of φ at the first part are nonpositively correlated: the first does not decrease when cut points are added, and the second does not increase.

So it suffices to verify the hypotheses of that inequality. Let $S \subseteq T \subseteq U$. Then $F(T)$ is the product of $F(S) \geq 0$ and of the factors $f(s)/h \geq 1$ with $s \in T \setminus S$, so that $F(S) \leq F(T)$. Moreover $\mathrm{fp}(T) \leq \mathrm{fp}(S)$, because the least element of a set does not increase when the set is enlarged and no first part exceeds r; since both first parts lie in $\{1, \ldots, r\}$, where φ is nondecreasing, $G(S) \leq G(T)$. Thus F and G are nondecreasing. It remains to show that μ is log-supermodular. Since $|S| + |T| = |S \cap T| + |S \cup T|$ for all subsets S and T of U, the powers of h on the two sides of this condition agree, and what has to be shown is that

$$w(S)\, w(T) \leq w(S \cap T)\, w(S \cup T)$$

for all subsets S and T of U. This is where the log-convexity of the Cayley coefficients enters.

We first consider one additional cut. Let $A \subseteq \{1, \ldots, r-1\}$, and let $x \in \{1, \ldots, r-1\}$ be a point outside A. Let $i < x < j$ be the neighbours of x in $A \cup \{0, r\}$. A cut at x replaces a part of size $j - i$ by two parts of sizes $x - i$ and $j - x$, so that

$$\frac{w(A \cup \{x\})}{w(A)} = \frac{a_{x-i}\, a_{j-x}}{a_{j-i}}.$$

This quotient does not decrease when i is raised or j is lowered: for integers $i \le i' < x < j' \le j$,

$$\frac{a_{x-i}\, a_{j-x}}{a_{j-i}} \le \frac{a_{x-i'}\, a_{j-x}}{a_{j-i'}} \le \frac{a_{x-i'}\, a_{j'-x}}{a_{j'-i'}}.$$

Both inequalities are cases of the log-convexity of the Cayley coefficients, $a_{u+s}\, a_v \le a_u\, a_{v+s}$ for $1 \le u \le v$ and $s \ge 0$ (Lemma 2.2): the first with $(u, v, s) = (x - i',\, j - i',\, i' - i)$ and the second with $(u, v, s) = (j' - x,\, j' - i',\, j - j')$. Now let B be a set with $A \subseteq B \subseteq \{1, \dots, r-1\}$ that does not contain x. The neighbours $i' < x < j'$ of x in $B \cup \{0, r\}$ satisfy $i \le i'$ and $j' \le j$, and therefore

$$\frac{w(A \cup \{x\})}{w(A)} \le \frac{w(B \cup \{x\})}{w(B)}.$$

Finally, let S and T be arbitrary subsets of $\{1, \dots, r-1\}$, let $x_1, \dots, x_t$ be the elements of $S \setminus T$, and put $A_l = (S \cap T) \cup \{x_1, \dots, x_l\}$ and $B_l = T \cup \{x_1, \dots, x_l\}$ for $0 \le l \le t$. Then $A_l \subseteq B_l$ and $x_{l+1} \notin B_l$ for $l < t$, while $A_0 = S \cap T$, $A_t = S$, $B_0 = T$ and $B_t = S \cup T$. Applying the last display with $A = A_l$, $B = B_l$ and $x = x_{l+1}$ for each $l < t$, we obtain

$$\frac{w(S)}{w(S \cap T)} = \prod_{l=0}^{t-1} \frac{w(A_{l+1})}{w(A_l)} \le \prod_{l=0}^{t-1} \frac{w(B_{l+1})}{w(B_l)} = \frac{w(S \cup T)}{w(T)},$$

i.e. $w(S)\, w(T) \le w(S \cap T)\, w(S \cup T)$. Thus μ is log-supermodular, and the correlation inequality (18.4) is proved.

We now combine the identity (18.3) and the correlation inequality (18.4). Let $h > 0$ be a real number, and let f be a cut profile with $f(1) = h$ and $f(s) \ge h$ for $2 \le s \le r$. We claim that

$$\Lambda_{r+1}[f]\; \tau_r(h) \le \tau_{r+1}(h)\; \Lambda_r[f^+].$$

By the identity (18.3), applied to f and to the constant profile h,

$$\Lambda_{r+1}[f] = h\, \Lambda_r[f^+] + \Lambda_r^{\psi}[f^+],$$
$$\tau_{r+1}(h) = h\, \tau_r(h) + \Lambda_r^{\psi}[h].$$

Multiplying the first equation by $\tau_r(h)$ and the second by $\Lambda_r[f^+]$, we see that the claim is equivalent to $\Lambda_r^{\psi}[f^+]\, \tau_r(h) \le \Lambda_r^{\psi}[h]\, \Lambda_r[f^+]$. This is the correlation inequality (18.4) for $\varphi = \psi$ and for the profile f^+, which is at least h at the points $1, \dots, r-1$.

Finally, let f be the threshold profile $\lambda^{k-1,N}$. For $1 \le s \le r$ we have $0 < N - s \le k + r$, hence $f(s) = (k-1)/(N-s) \ge h_-$, with equality for $s = 1$. Since $k \ge 2$, we have $h_- > 0$, so that the inequality just proved applies with $h = h_-$ and gives

$$\Lambda_{r+1}(k-1; N)\; \tau_r(h_-) \le \tau_{r+1}(h_-)\; \Lambda_r[f^+].$$

For $1 \le s \le r-1$,

$$0 \le f^+(s) = \frac{k-1}{N-1-s} \le \frac{k}{N-s} = \lambda^{k,N}(s),$$

where the second inequality amounts to $k \le N - s$. Since the total weight does not decrease when a nonnegative profile is raised, $\Lambda_r[f^+] \le \Lambda_r(k; N)$, and the assertion follows. □

Lemma 18.3. *Let $k \geq 2$ and $r \geq 1$ be integers, and put $N = k + r + 1$. Then*

$$\Lambda_{r+1}(k-1;N) \leq \Big(1+\frac{1}{r}\Big)^{r}\Big(1-\frac{\mu_-}{2(k-1)}\Big)\,\Lambda_r(k;N).$$

Proof. Put $h = h_-$, so that $0 < h \leq 1$ and $(1-h)/(2h\,(r+1)) = 1/(2(k-1))$. Lemma 17.6 then reads

$$\tau_{r+1}(h) \leq \Big(1+\frac{1}{r}\Big)^{r}\Big(1-\frac{\mu_-}{2(k-1)}\Big)\,\tau_r(h).$$

We insert this into the inequality of Lemma 18.2 and cancel $\tau_r(h)$; both $\Lambda_r(k;N)$ and $\tau_r(h)$ are positive, because the total weight of the compositions of r for a nonnegative profile is at least $a_r > 0$. □

The second comparison raises the threshold from k to $k+2$ at a fixed length.

Lemma 18.4. *Let $k \geq 1$ and $r \geq 1$ be integers, and put $N = k + r + 1$. Then*

$$\Big(1+\frac{2}{k}\Big)^{\mu_{\mathrm{mid}}}\,\Lambda_r(k;N) \leq \Lambda_r(k+2;N).$$

Proof. Put $\sigma = 1 + \frac{2}{k} \geq 1$ and $h = h_{\mathrm{mid}} > 0$, write $V_q = V_q(r;\lambda^{1,N})$, and for $1 \leq q \leq r$ let

$$u_q = T_{r,q}\,h^{q-1} \qquad \text{and} \qquad v_q = k^{q-1}\,V_q.$$

These numbers are positive, because the profiles $\mathbf{1}$ and $\lambda^{1,N}$ are positive at the cut points of a composition of r. We have $\sum_{q=1}^{r} u_q = \tau_r(h)$, and by the expansion (18.1) of a chain sum also $\sum_{q=1}^{r} v_q = \Lambda_r(k;N)$ and, since $k\sigma = k+2$, $\sum_{q=1}^{r} v_q\,\sigma^{q-1} = \Lambda_r(k+2;N)$. Thus the quotient of the two chain sums is a mean of the σ^{q-1} with weights v_q.

For $1 \leq q \leq r-1$ we have

$$u_{q+1}\,v_q \leq u_q\,v_{q+1}.$$

Indeed, Lemma 16.2 gives $q\,r\,T_{r,q+1} = (q+1)(r-q)\,T_{r,q}$, and Lemma 16.5, which applies because $r < N$, gives $(q+1)(r-q)\,V_q \leq q\,r\,(N-\frac{r}{2})\,V_{q+1}$. Therefore

$$\begin{aligned} q\,r\,u_{q+1}\,v_q &= (q+1)(r-q)\,T_{r,q}\,h^{q}\,k^{q-1}\,V_q \\ &\leq q\,r\,T_{r,q}\,h^{q-1}\,k^{q-1}\cdot h\Big(N-\frac{r}{2}\Big)\,V_{q+1} = q\,r\,u_q\,v_{q+1}, \end{aligned}$$

where the last step uses that $h\,(N-\frac{r}{2}) = k$ by the choice of h. Since the v_q are positive, the inequalities $u_{q+1}\,v_q \leq u_q\,v_{q+1}$ say that u_q/v_q does not increase in q, and hence

$$u_{q'}\,v_q \leq u_q\,v_{q'} \qquad (1 \leq q \leq q' \leq r).$$

This is the first hypothesis of Lemma 17.1 on $\{1,\dots,r\}$ for u and v, and its second hypothesis holds for $\varphi(q) = \sigma^{q-1}$, because $\sigma \geq 1$. The lemma gives

$$\Big(\sum_{q=1}^{r} u_q\,\sigma^{q-1}\Big)\,\Lambda_r(k;N) \leq \Lambda_r(k+2;N)\,\tau_r(h).$$

Now let $\mu = \mu_{\mathrm{mid}} = \mu_r(h)$, so that $\mu\,\tau_r(h) = \sum_{q=1}^{r}(q-1)\,u_q$ by the definition of the mean number of cuts. By Jensen's inequality for the convex function $t \mapsto \sigma^t$ and the positive weights u_q,

$$\sum_{q=1}^{r} u_q\,\sigma^{q-1} \geq \sigma^{\mu}\,\tau_r(h).$$

Since $\Lambda_r(k;N) > 0$, the last two displays give the assertion multiplied by $\tau_r(h)$. $\square$

18.3. **The frozen polynomial.** We now turn to the steps. Let $k \geq 2$, and let $n = k + r$ be a row of the chain range of column k. The normalised step of column k is a second difference in the threshold (Section 3),

$$\delta_n^{(k)} = z_n^{(k+1)} - 2\, z_n^{(k)} + z_n^{(k-1)},$$

and the step of column $k+1$ is the same expression with the three thresholds raised by one. If the four counts $z_n^{(k-1)}, \dots, z_n^{(k+2)}$ were the values of one function of a real variable at four neighbouring points, the two steps would be second differences of that function at neighbouring points, and they could be compared by the methods of calculus. But the counts are defined for integer thresholds only, and a threshold y enters the last-car recursion through the weight $\min(t, y)$, that is, both through the index at which the weight stops growing and through the value that it then keeps. We separate the two roles. With the index fixed at k, the value becomes a real variable, and the counts are values of one polynomial in it: exactly for the thresholds k and $k+1$, and up to an explicit multiple of a chain sum for the thresholds $k-1$ and $k+2$ (Proposition 18.6).

We begin with the linear equation. Let $k \geq 0$ be an integer, and let $j \geq 1$ be one of the two integers k and $k+1$. The sequence $z^{(j)}$ is a solution of the weighted last-car recursion with weight $\min(t, j)$ (Corollary 3.3), and this weight equals t for $1 \leq t \leq k$ and j for $t \geq k+1$. The single-chain formula (Proposition 4.1) therefore gives, for $0 \leq n \leq 2k+1$,

$$c_n - z_n^{(j)} = \sum_{t=k+1}^{n} c_{n-t}\Big(c_{t-1} - \frac{j}{t}\, z_{t-1}^{(j)}\Big).$$

We write this with the Cayley coefficients, using $c_i = a_{i+1}$, and with $M = n+1$ in place of n, and we collect the terms that do not contain $z^{(j)}$. For integers $k \geq 0$ and $t \geq 0$ let

$$\nu_k(t) = a_t - \sum_{k<i<t} a_i\, a_{t-i}. \tag{18.5}$$

Then the formula becomes

$$z_{M-1}^{(j)} = \nu_k(M) + \sum_{k<t<M} a_{M-t}\, \frac{j}{t}\, z_{t-1}^{(j)} \qquad (1 \leq M \leq 2k+2). \tag{18.6}$$

Here $\nu_k(M)$ and the coefficients a_{M-t}/t depend on k alone. Solving this linear equation by iteration expresses $z_{M-1}^{(j)}$ through the $\nu_k(i)$ with $k < i \leq M$ and through chain sums with threshold j, as we show in the proof of Proposition 18.6, and the resulting expression is defined for every real number in place of j. We take it as a definition.

Definition 18.5. Let $k \geq 0$ and $n \geq 0$ be integers, and put $N = n+1$. The *frozen polynomial* of position (n, k) is the function $\mathcal{K}_{n,k}$ of a real variable y given by

$$\mathcal{K}_{n,k}(y) = \nu_k(N) + \sum_{m=1}^{n-k} \frac{\nu_k(N-m)}{N-m}\; y\, \Lambda_m(y;N),$$

with ν_k as in (18.5).

In each term of the sum we have $N-m \geq k+1$, so that no denominator vanishes. Since a chain sum $\Lambda_m(y;N)$ is a polynomial in y of degree at most $m-1$, the function $\mathcal{K}_{n,k}$ is a polynomial in y, and for $n \geq k$ its degree is at most $n-k$. For $n \leq k$ the sum in the definition is empty, and $\mathcal{K}_{n,k}$ is the constant $\nu_k(n+1) = a_{n+1} = c_n$, because the sum that defines $\nu_k(t)$ is empty for $t \leq k+1$.

Now let $k \geq 2$ and r be integers with $1 \leq r \leq k-1$, and put $n = k+r$, $N = n+1$ and $\mathcal{K} = \mathcal{K}_{n,k}$. We write

$$\mathcal{C}_- = \mathcal{K}(k+1) - 2\,\mathcal{K}(k) + \mathcal{K}(k-1),$$
$$\mathcal{C}_+ = \mathcal{K}(k+2) - 2\,\mathcal{K}(k+1) + \mathcal{K}(k)$$

for the two second differences of $\mathcal{K}$, and

$$\mathcal{B}_- = \frac{a_k}{k}\,\Lambda_{r+1}(k-1;N) \qquad \text{and} \qquad \mathcal{B}_+ = \frac{a_{k+1}}{k+1}\,\Lambda_r(k+2;N)$$

for the two multiples of chain sums that are subtracted from them. In this notation the following proposition says that $\delta_n^{(k)} = \mathcal{C}_- - \mathcal{B}_-$ and $\delta_n^{(k+1)} = \mathcal{C}_+ - \mathcal{B}_+$.

Proposition 18.6. *Let $k \geq 2$ and r be integers with $1 \leq r \leq k-1$, and put $n = k+r$, $N = n+1$ and $\mathcal{K} = \mathcal{K}_{n,k}$. Then*

$$\delta_n^{(k)} = \big(\mathcal{K}(k+1) - 2\,\mathcal{K}(k) + \mathcal{K}(k-1)\big) - \frac{a_k}{k}\,\Lambda_{r+1}(k-1;N),$$
$$\delta_n^{(k+1)} = \big(\mathcal{K}(k+2) - 2\,\mathcal{K}(k+1) + \mathcal{K}(k)\big) - \frac{a_{k+1}}{k+1}\,\Lambda_r(k+2;N).$$

Proof. Let $k \geq 0$ be an arbitrary integer, and let $j \geq 1$ be one of the integers k and $k+1$. We first show that

$$z_n^{(j)} = \mathcal{K}_{n,k}(j) \qquad (0 \leq n \leq 2k+1). \tag{18.7}$$

With $M = n+1$, and with $i = M-m$ as the index of summation in Definition 18.5, this is the assertion that, for every integer M with $1 \leq M \leq 2k+2$,

$$z_{M-1}^{(j)} = \nu_k(M) + \sum_{k<i<M} \nu_k(i)\,\frac{j}{i}\,\Lambda_{M-i}(j;M).$$

We prove this by induction on M. For $M \leq k+1$ the sum in the assertion and the sum in the linear equation (18.6) are empty, and the two equations coincide. Let $M \geq k+2$, and suppose that the assertion holds for all smaller values of M. We insert it for $z_{t-1}^{(j)}$ with $k < t < M$ into the linear equation and collect, for each i with $k < i < M$, the terms that contain $\nu_k(i)$. This gives

$$z_{M-1}^{(j)} = \nu_k(M) + \sum_{k<i<M} \nu_k(i)\,\frac{j}{i}\,\Big(a_{M-i} + \sum_{i<t<M} a_{M-t}\,\frac{j}{t}\,\Lambda_{t-i}(j;t)\Big).$$

The bracket is the right-hand side of the formula (18.2), with first part $M-t$, for the chain sum $\Lambda_{M-i}(j;M)$. So it equals $\Lambda_{M-i}(j;M)$, and the assertion holds for M.

We turn to two consecutive columns. Let k and n be integers with $1 \leq k \leq n$, put $N = n+1$, and let y be a real number. We show that

$$\mathcal{K}_{n,k-1}(y) - \mathcal{K}_{n,k}(y) = (y-k)\,\frac{a_k}{k}\,\Lambda_{N-k}(y;N). \tag{18.8}$$

By the definition of ν_k, lowering k by one adds the term $i = k$ to the sum:

$$\nu_{k-1}(t) = \nu_k(t) - a_k\,a_{t-k} \quad (t > k), \qquad \nu_{k-1}(k) = a_k.$$

The sum that defines $\mathcal{K}_{n,k-1}(y)$ has one term more than the sum that defines $\mathcal{K}_{n,k}(y)$, the term $m = N-k$, which is $\frac{a_k}{k}\, y\, \Lambda_{N-k}(y;N)$. In the constant term and in the terms with $1 \le m \le n-k$ the argument of ν is larger than k, so that these terms contribute to the difference of the two polynomials

$$-a_k\Big(a_{N-k} + \sum_{m=1}^{n-k} \Lambda_m(y;N)\, \frac{y}{N-m}\, a_{N-k-m}\Big).$$

The bracket equals $\Lambda_{N-k}(y;N)$. Indeed, the recursion of Definition 16.1, which removes the last part of a composition, gives when summed over the number of parts

$$\Lambda_{m'}[f] = a_{m'} + \sum_{s=1}^{m'-1} \Lambda_s[f]\, f(s)\, a_{m'-s}$$

for every cut profile f and every integer $m' \ge 1$, and for $m' = N-k$ and $f = \lambda^{y,N}$ the right-hand side is the bracket. The two contributions add up to $(y-k)\,\frac{a_k}{k}\,\Lambda_{N-k}(y;N)$, as claimed.

We can now prove the two formulas. Let k, r, n and N be as in the proposition. By (18.7), applied with k, with $k-1$ and with $k+1$ in place of k,

$$z_n^{(k)} = \mathcal{K}_{n,k}(k), \qquad z_n^{(k+1)} = \mathcal{K}_{n,k}(k+1),$$
$$z_n^{(k-1)} = \mathcal{K}_{n,k-1}(k-1), \qquad z_n^{(k+2)} = \mathcal{K}_{n,k+1}(k+2);$$

for the third of these we use that $n \le 2(k-1)+1$, which holds because $r \le k-1$. Next we apply (18.8). With k itself and $y = k-1$, where $N-k = r+1$, it gives

$$z_n^{(k-1)} = \mathcal{K}_{n,k}(k-1) - \frac{a_k}{k}\,\Lambda_{r+1}(k-1;N).$$

With $k+1$ in place of k, which is admissible because $r \ge 1$, and with $y = k+2$, where $N-(k+1) = r$, it gives

$$z_n^{(k+2)} = \mathcal{K}_{n,k}(k+2) - \frac{a_{k+1}}{k+1}\,\Lambda_r(k+2;N).$$

By (3.1), divided by $n!$, we have $\delta_n^{(k)} = z_n^{(k+1)} - 2\,z_n^{(k)} + z_n^{(k-1)}$ and $\delta_n^{(k+1)} = z_n^{(k+2)} - 2\,z_n^{(k+1)} + z_n^{(k)}$, and inserting the four values gives the two formulas of the proposition. □

18.4. **The ratio of the two second differences.** By Proposition 18.6, column k is nonpositive in row $n = k+r$ if and only if $\mathcal{C}_- \le \mathcal{B}_-$, and column $k+1$ is negative in this row if and only if $\mathcal{C}_+ < \mathcal{B}_+$. The chain sums in $\mathcal{B}_-$ and $\mathcal{B}_+$ were compared with $\Lambda_r(k;N)$ in Lemmas 18.3 and 18.4. We now compare the two second differences: for $r \ge 1$ the following proposition says that $\mathcal{C}_+ \le e^{D_r/(k-1)}\,\mathcal{C}_-$. Recall from Section 17 that, for $m \ge 2$, D_m is the ratio of the third to the second factorial moment of the birthday law of m, and that $D_0 = D_1 = 0$.

Proposition 18.7. *Let $k \ge 2$ and r be integers with $0 \le r \le k-1$, and let $\mathcal{K} = \mathcal{K}_{k+r,k}$ be the frozen polynomial of position $(k+r,k)$. Then*

$$\mathcal{K}(k+2) - 2\,\mathcal{K}(k+1) + \mathcal{K}(k) \le e^{D_r/(k-1)}\,\big(\mathcal{K}(k+1) - 2\,\mathcal{K}(k) + \mathcal{K}(k-1)\big).$$

Proof. Put $n = k+r$ and $N = n+1$.

We begin with the coefficients of $\mathcal{K}$. Let $\alpha_m = \nu_k(N-m)/(N-m)$ for $1 \le m \le r$. We insert the expansion (18.1) of the chain sums into Definition 18.5 and exchange the two summations:

$$\begin{aligned}\mathcal{K}(y) &= \nu_k(N) + \sum_{m=1}^{r} \alpha_m \sum_{q=1}^{m} y^q \, V_q(m; \lambda^{1,N}) \\ &= \nu_k(N) + \sum_{q=1}^{r} p_q \, y^q,\end{aligned}$$

where $p_q = \sum_{m=q}^{r} \alpha_m \, V_q(m; \lambda^{1,N})$. The numbers $V_q(m; \lambda^{1,N})$ are nonnegative, because the profile $s \mapsto 1/(N-s)$ is positive at the cut points. To see that the α_m are positive, we show that

$$\nu_k(t) > 0 \qquad (1 \le t \le 2k+1).$$

This applies to $t = N - m$, because $k+1 \le N-m \le k+r \le 2k-1$ for $1 \le m \le r$, since $r \le k-1$. For the proof, let $1 \le t \le 2k+1$. The substitution of $t-i$ for i turns the sum of the terms $a_i \, a_{t-i}$ over $k < i < t$, which occurs in the definition of $\nu_k(t)$, into the sum of the same terms over $0 < i < t-k$. Since $t - k \le k+1$, the two sums run over disjoint sets of indices in $\{0, \ldots, t\}$, and since all terms are nonnegative,

$$2 \sum_{k<i<t} a_i \, a_{t-i} \le \sum_{i=0}^{t} a_i \, a_{t-i}.$$

By Lemma 2.5, to whose sum the vanishing term $i = t$ may be added, and by the same substitution, each of the sums $\sum_{i=0}^{t} i \, a_i \, a_{t-i}$ and $\sum_{i=0}^{t} (t-i) \, a_i \, a_{t-i}$ equals $(t-1) \, a_t$, so that $t \sum_{i=0}^{t} a_i \, a_{t-i} = 2\,(t-1) \, a_t$. Therefore $t \sum_{k<i<t} a_i \, a_{t-i} \le (t-1) \, a_t$, and $t \, \nu_k(t) \ge a_t > 0$. Consequently $\alpha_m > 0$ for $1 \le m \le r$ and $p_q \ge 0$ for $1 \le q \le r$.

Next we compare two factorial moments. We claim that, for every real number y with $0 < y \le k+2$,

$$\sum_{q=1}^{r} q(q-1)(q-2) \, p_q \, y^q \le D_r \sum_{q=1}^{r} q(q-1) \, p_q \, y^q. \tag{18.9}$$

By the definition of the coefficients p_q, and since $y^{q-1} \, V_q(m; \lambda^{1,N}) = V_q(m; \lambda^{y,N})$ by (18.1), the left-hand side is

$$\sum_{m=1}^{r} \alpha_m \, y \sum_{q=1}^{m} q(q-1)(q-2) \, V_q(m; \lambda^{y,N}),$$

and the sum on the right-hand side is the same expression with $q(q-1)$ in place of $q(q-1)(q-2)$. Since $\alpha_m \, y \ge 0$, it suffices to prove, for each m with $1 \le m \le r$ and with $f = \lambda^{y,N}$,

$$\sum_{q=1}^{m} q(q-1)(q-2) \, V_q(m; f) \le D_r \sum_{q=1}^{m} q(q-1) \, V_q(m; f).$$

We may assume that $m \ge 2$, so that $r \ge 2$. A cut point s of a composition of m satisfies $s \le m-1 \le r-1$, hence $N - s \ge k+2 \ge y$, and therefore $0 < f(s) \le 1$. For such a profile Lemma 16.4 gives $q \, m \, V_{q+1}(m; f) \le (q+1)(m-q) \, V_q(m; f)$ for

$q \geq 1$, while Lemma 16.2 gives $q\,m\,T_{m,q+1} = (q+1)(m-q)\,T_{m,q}$. Multiplying the inequality by $T_{m,q} \geq 0$ and using the equality, we obtain

$$V_{q+1}(m;f)\,T_{m,q} \leq V_q(m;f)\,T_{m,q+1} \qquad (q \geq 1).$$

Since $T_{m,q} > 0$ for $1 \leq q \leq m$, this says that $V_q(m;f)/T_{m,q}$ does not increase in q on $\{1,\dots,m\}$. Hence

$$V_{q'}(m;f)\,T_{m,q} \leq V_q(m;f)\,T_{m,q'} \qquad (1 \leq q \leq q' \leq m).$$

Multiplied by the nonnegative number $q(q-1)\,q'(q'-1)$, this is the first hypothesis of Lemma 17.1 on $\{1,\dots,m\}$ for $u(q) = q(q-1)\,V_q(m;f)$ and $v(q) = q(q-1)\,T_{m,q}$, and its second hypothesis holds for $\varphi(q) = q-2$. The lemma gives

$$\Big(\sum_{q=1}^{m} q(q-1)(q-2)\,V_q(m;f)\Big)\Big(\sum_{q=1}^{m} q(q-1)\,T_{m,q}\Big) \leq \Big(\sum_{q=1}^{m} q(q-1)(q-2)\,T_{m,q}\Big)\Big(\sum_{q=1}^{m} q(q-1)\,V_q(m;f)\Big).$$

By (17.5) the first sum on the right equals $D_m \sum_{q=1}^{m} q(q-1)\,T_{m,q}$. Cancelling this last sum, which is positive because $T_{m,2} > 0$, we obtain the required inequality with D_m in place of D_r. Since $D_m \leq D_r$ by Lemma 17.3 and $\sum_{q=1}^{m} q(q-1)\,V_q(m;f) \geq 0$, the inequality with D_r follows, and with it the claim.

It remains to deduce the inequality between the second differences. We regard $\mathcal{K}$ as a function on the real line and write $\mathcal{K}''$ and $\mathcal{K}'''$ for its second and third derivative. Since $\mathcal{K}(y) = \nu_k(N) + \sum_{q=1}^{r} p_q\,y^q$,

$$y^2\,\mathcal{K}''(y) = \sum_{q=1}^{r} q(q-1)\,p_q\,y^q,$$

$$y^3\,\mathcal{K}'''(y) = \sum_{q=1}^{r} q(q-1)(q-2)\,p_q\,y^q,$$

so that the inequality (18.9) between the factorial moments, divided by $y^2 > 0$, reads $y\,\mathcal{K}'''(y) \leq D_r\,\mathcal{K}''(y)$ for $0 < y \leq k+2$. Moreover $\mathcal{K}'''(y) \geq 0$ for $y \geq 0$, because $p_q \geq 0$. Put $\theta = D_r/(k-1)$. For $k-1 \leq y \leq k+2$ we have

$$(k-1)\,\mathcal{K}'''(y) \leq y\,\mathcal{K}'''(y) \leq D_r\,\mathcal{K}''(y),$$

i.e. $\mathcal{K}'''(y) \leq \theta\,\mathcal{K}''(y)$. Hence $t \mapsto e^{-\theta t}\,\mathcal{K}''(t)$ does not increase on $[k-1, k+2]$, so that

$$\mathcal{K}''(y+1) \leq e^{\theta}\,\mathcal{K}''(y) \qquad (k-1 \leq y \leq k+1).$$

Consequently $H(t) = e^{\theta}\,\mathcal{K}(t) - \mathcal{K}(t+1)$ has a nonnegative second derivative on $[k-1, k+1]$ and is convex there. In particular $2\,H(k) \leq H(k-1) + H(k+1)$, and rearranging we obtain the inequality of the proposition. □

19. The next column

In this section we prove Proposition 1.4: for $k \geq 9$ and $n \geq k+2$, if column k is nonpositive in rows $k+1,\dots,n$, then column $k+1$ is nonpositive in row n. In the proof of Theorem 1.1 this proposition carries a nonpositive step from each column to the next along a row. For $k \geq 265$ we deduce it, with a strict conclusion, from the comparisons of Section 18 and from the bound of Proposition 9.2 for the

crossing row; for $9 \le k \le 264$ it follows from the computed signs of Lemma 7.3 and from Proposition 1.3, by which a positive column stays positive in its chain range.

19.1. **A criterion for the next column.** By Proposition 18.6 the normalised steps of columns k and $k+1$ in a row $n = k+r$ of the chain range of column k are each of the form $\mathcal{C} - \mathcal{B}$, a second difference of a polynomial minus a multiple of a chain sum, and Section 18 bounds the growth of $\mathcal{C}$ from above, and that of $\mathcal{B}$ from below, when one passes from column k to column $k+1$. The function that we now define is k times an explicit lower bound for the logarithm of the quotient of the second bound by the first.

Definition 19.1. For integers $r \ge 1$ and $k \ge 2$, the *criterion function* is

$$\mathsf{G}(r,k) = k\Big(\frac{1}{2r} - \frac{1}{3r^2}\Big) - \frac{5}{2} + \frac{1}{2}\,\mu_r^0\Big(\frac{r(r+1)}{k-1}\Big) + 2\Big(1-\frac{1}{k}\Big)\mu_r^0\Big(\frac{r(r+2)}{2k}\Big) - \frac{k}{k-1}\,D_r.$$

Here, for a real number $X \ge 0$,

$$\mu_r^0(X) = \frac{1}{3}\Big(\sqrt{(4+X)^2 + 12\,(r-1)} - (4+X)\Big),$$

and D_r is the number defined in Section 17: for $r \ge 2$,

$$D_r = \frac{3\,\big((r+2)\,s_r - 3r\big)}{2\,(r-s_r)}, \qquad \text{where} \quad s_r = \sum_{j=1}^{r}\prod_{i=0}^{j-1}\Big(1-\frac{i}{r}\Big)$$

is the mean of the birthday law of r, and $D_1 = 0$.

We write

$$X_- = \frac{r(r+1)}{k-1}, \qquad X_{\mathrm{mid}} = \frac{r(r+2)}{2k}$$

for the two arguments of μ_r^0 in $\mathsf{G}(r,k)$; both are nonnegative. Recall that $\delta_n^{(j)} = d_{n,j}/n!$ denotes the normalised step of column j in row n, which has the sign of $b_{n,j} - b_{n,j-1}$. In terms of the counts, the following proposition therefore says that $b_{n,k} \le b_{n,k-1}$ implies $b_{n,k+1} < b_{n,k}$ whenever $\mathsf{G}(r,k) \ge 0$.

Proposition 19.2. *Let $k \ge 3$ and r be integers with $2 \le r \le k-1$, and put $n = k+r$. If $\mathsf{G}(r,k) \ge 0$ and $\delta_n^{(k)} \le 0$, then $\delta_n^{(k+1)} < 0$.*

Proof. Put $N = n+1$ and $\Lambda = \Lambda_r(k;N)$. Recall from Section 18 the notation for row $n = k+r$: with the frozen polynomial $\mathcal{K} = \mathcal{K}_{n,k}$ of Definition 18.5,

$$\mathcal{C}_- = \mathcal{K}(k+1) - 2\,\mathcal{K}(k) + \mathcal{K}(k-1), \qquad \mathcal{B}_- = \frac{a_k}{k}\,\Lambda_{r+1}(k-1;N),$$
$$\mathcal{C}_+ = \mathcal{K}(k+2) - 2\,\mathcal{K}(k+1) + \mathcal{K}(k), \qquad \mathcal{B}_+ = \frac{a_{k+1}}{k+1}\,\Lambda_r(k+2;N).$$

Since $k \ge 2$ and $1 \le r \le k-1$, Proposition 18.6 gives $\delta_n^{(k)} = \mathcal{C}_- - \mathcal{B}_-$ and $\delta_n^{(k+1)} = \mathcal{C}_+ - \mathcal{B}_+$. Thus $\delta_n^{(k)} \le 0$ means that $\mathcal{C}_- \le \mathcal{B}_-$, and we have to show that $\mathcal{C}_+ < \mathcal{B}_+$.

Let

$$h_- = \frac{k-1}{k+r}, \qquad h_{\mathrm{mid}} = \frac{k}{k+1+r/2},$$

and let $\mu_- = \mu_r(h_-)$ and $\mu_{\mathrm{mid}} = \mu_r(h_{\mathrm{mid}})$ be the mean numbers of cuts of a composition of r under the homogeneous laws with these parameters (Section 17). Proposition 18.7 and Lemmas 18.3 and 18.4, whose hypotheses on k and r hold as well, give the three bounds

$$\mathcal{C}_+ \le e^{D_r/(k-1)}\,\mathcal{C}_-,$$
$$\Lambda_{r+1}(k-1;N) \le \Big(1+\frac{1}{r}\Big)^r \Big(1-\frac{\mu_-}{2(k-1)}\Big)\,\Lambda,$$
$$\Big(1+\frac{2}{k}\Big)^{\mu_{\mathrm{mid}}}\Lambda \le \Lambda_r(k+2;N).$$

By (2.3), $a_{k+1} = (1+1/k)^{k-1}a_k$, and therefore

$$\frac{a_{k+1}}{k+1} = \Big(1+\frac{1}{k}\Big)^{k-2}\frac{a_k}{k}.$$

Moreover $a_k > 0$ and $\Lambda > 0$, since the Cayley coefficients a_p with $p \ge 1$ and the factors $k/(N-s)$ of the cut points $s < r < N$ are positive.

Suppose for the moment that

$$e^{D_r/(k-1)}\Big(1+\frac{1}{r}\Big)^r e^{-\mu_-/(2(k-1))} < \Big(1+\frac{1}{k}\Big)^{k-2}\Big(1+\frac{2}{k}\Big)^{\mu_{\mathrm{mid}}}.$$

Then the first of the three bounds and the assumption $\mathcal{C}_- \le \mathcal{B}_-$ give

$$\mathcal{C}_+ \le e^{D_r/(k-1)}\,\mathcal{C}_- \le e^{D_r/(k-1)}\,\frac{a_k}{k}\,\Lambda_{r+1}(k-1;N),$$

and by the second bound and the inequality $1-t \le e^{-t}$ at $t = \mu_-/(2(k-1))$ the right-hand side is at most

$$e^{D_r/(k-1)}\Big(1+\frac{1}{r}\Big)^r e^{-\mu_-/(2(k-1))}\,\frac{a_k}{k}\,\Lambda < \Big(1+\frac{1}{k}\Big)^{k-2}\Big(1+\frac{2}{k}\Big)^{\mu_{\mathrm{mid}}}\frac{a_k}{k}\,\Lambda,$$

where the strict inequality is the supposed one, multiplied by $\frac{a_k}{k}\Lambda > 0$. By the formula for $a_{k+1}/(k+1)$ and the third bound, the last expression is at most $\frac{a_{k+1}}{k+1}\,\Lambda_r(k+2;N) = \mathcal{B}_+$. Thus $\mathcal{C}_+ < \mathcal{B}_+$, i.e. $\delta_n^{(k+1)} < 0$.

It remains to prove the supposed inequality. Both of its sides are positive, and the logarithm of the right-hand side minus the logarithm of the left-hand side is

$$(k-2)\log\Big(1+\frac{1}{k}\Big) + \mu_{\mathrm{mid}}\log\Big(1+\frac{2}{k}\Big) - r\log\Big(1+\frac{1}{r}\Big) + \frac{\mu_-}{2(k-1)} - \frac{D_r}{k-1}.$$

We show that this difference is at least $\mathsf{G}(r,k)/k + 1/k^2$, which is positive because $\mathsf{G}(r,k) \ge 0$.

We first bound the two means from below. Both h_- and h_{mid} lie in $(0,1]$, and $r\,(1/h_- - 1) = X_-$ and $r\,(1/h_{\mathrm{mid}} - 1) = X_{\mathrm{mid}}$. Hence Lemma 17.5 and the definition of μ_r^0, which is nonnegative, give

$$0 \le \mu_r^0(X_-) \le \mu_-, \qquad 0 \le \mu_r^0(X_{\mathrm{mid}}) \le \mu_{\mathrm{mid}}.$$

Next we bound the logarithms. Put

$$u = \frac{1}{r}, \qquad v = \frac{1}{k}, \qquad w = \frac{1}{k-1},$$

so that $0 < v < w$ and, since $k \geq 3$, $v \leq \frac{1}{3}$. For $t \geq 0$ the Taylor polynomial of $\log(1+t)$ of even (resp. odd) degree bounds it from below (resp. above). This gives

$$r \log\Big(1 + \frac{1}{r}\Big) \leq 1 - \frac{u}{2} + \frac{u^2}{3},$$
$$(k-2) \log\Big(1 + \frac{1}{k}\Big) \geq 1 - \frac{5}{2}\, v + v^2,$$
$$\log\Big(1 + \frac{2}{k}\Big) \geq 2v\,(1-v).$$

Indeed, the first is the bound of degree 3 at $t = u$, multiplied by $r = 1/u$, and the third is the bound of degree 2 at $t = 2v$. For the second we multiply the bound of degree 4 at $t = v$ by $k - 2 = (1-2v)/v \geq 0$; a direct computation shows that the result exceeds $1 - \frac{5}{2}\, v + v^2$ by $\frac{1}{12}\, v^2\,(4-3v)(1-2v)$, which is nonnegative because $0 < v \leq \frac{1}{3}$.

We insert the three bounds just obtained into the difference of the logarithms, using that $\mu_{\mathrm{mid}} \geq \mu_r^0(X_{\mathrm{mid}}) \geq 0$, and find that it is at least

$$\Big(1 - \frac{5}{2}\, v + v^2\Big) + 2v\,(1-v)\,\mu_{\mathrm{mid}} - \Big(1 - \frac{u}{2} + \frac{u^2}{3}\Big) + \frac{w}{2}\,\mu_- - w\,D_r.$$

Since $0 \leq \mu_r^0(X_-) \leq \mu_-$, $\mu_r^0(X_{\mathrm{mid}}) \leq \mu_{\mathrm{mid}}$ and $0 < v < w$, we have $w\,\mu_- \geq v\,\mu_r^0(X_-)$ and $2v\,(1-v)\,\mu_{\mathrm{mid}} \geq 2v\,(1-v)\,\mu_r^0(X_{\mathrm{mid}})$. Hence the difference of the logarithms is at least

$$v^2 + \Big(\frac{u}{2} - \frac{u^2}{3}\Big) - \frac{5}{2}\, v + \frac{v}{2}\,\mu_r^0(X_-) + 2v\,(1-v)\,\mu_r^0(X_{\mathrm{mid}}) - w\,D_r.$$

Since $u = 1/r$, $v = 1/k$ and $w = v \cdot \frac{k}{k-1}$, this expression is $v^2 + v\,\mathsf{G}(r,k)$, i.e. $\mathsf{G}(r,k)/k + 1/k^2$. This proves the supposed inequality. □

19.2. **Positivity of the criterion function in the strip.** The criterion is of use where $\mathsf{G}(r,k) \geq 0$. We prove this for a column $k \geq 265$ in the rows of its strip, i.e. in rows $k + r$ with $25\,r^3 \leq 121\,k^2$; by Proposition 9.2, every row up to which such a column has not turned positive lies in the strip.

The function has three positive terms, namely $k\big(\frac{1}{2r} - \frac{1}{3r^2}\big)$ and the two terms with μ_r^0, and two negative ones, the constant $\frac{5}{2}$ and the term with D_r. For small r the first term outweighs the negative terms by itself. For large r it does not, and the term with D_r can only be bounded by a multiple of $\sqrt{r}$, through the mean s_r of the birthday law (Lemma 17.2). It has to be balanced by the two terms with μ_r^0. These are of order $\sqrt{r}$ as well, but only as long as their arguments X_- and X_{mid}, which are of size r^2/k, are at most multiples of $\sqrt{r}$, because $\mu_r^0(X)$ decreases as X grows; the condition $25\,r^3 \leq 121\,k^2$, i.e. $5\,r^{3/2} \leq 11\,k$, bounds the arguments in this way. The proof first replaces the coefficients $2\,(1 - \frac{1}{k})$ and $\frac{k}{k-1}$ in $\mathsf{G}(r,k)$ by constants and records two bounds for the mean s_r, and then treats $2 \leq r \leq 13$ and $r \geq 14$ separately.

Proposition 19.3. *Let $k \geq 265$ and $r \geq 2$ be integers with $25\,r^3 \leq 121\,k^2$. Then the criterion function of Definition 19.1 satisfies $\mathsf{G}(r,k) > 0$.*

Proof. As before, let $X_- = \frac{r(r+1)}{k-1}$ and $X_{\mathrm{mid}} = \frac{r(r+2)}{2k}$ be the two arguments of μ_r^0 in $\mathsf{G}(r,k)$, and note that $\frac{1}{2r} - \frac{1}{3r^2} = \frac{3r-2}{6r^2}$. The numerical comparisons that occur are checked by direct evaluation.

We begin by replacing the coefficients $2\,(1-\frac{1}{k})$ and $\frac{k}{k-1}$ in $\mathsf{G}(r,k)$ by constants. The numbers $\mu_r^0(X_-)$ and $\mu_r^0(X_{\mathrm{mid}})$ are nonnegative, and so is D_r: by (17.5),

$$\sum_{q=1}^{r} q\,(q-1)(q-2)\,T_{r,q} = D_r \sum_{q=1}^{r} q\,(q-1)\,T_{r,q},$$

where the total weights $T_{r,q}$ of the compositions of r with q parts (Section 16) are nonnegative and $T_{r,2} = \sum_{p=1}^{r-1} a_p\, a_{r-p} > 0$ because $r \ge 2$. Since $k \ge 265$, we have $2\,(1-\frac{1}{k}) \ge \frac{528}{265}$ and $\frac{k}{k-1} \le \frac{265}{264}$, and therefore

$$\mathsf{G}(r,k) \ge k\,\frac{3r-2}{6r^2} - \frac{5}{2} + \frac{1}{2}\,\mu_r^0(X_-) + \frac{528}{265}\,\mu_r^0(X_{\mathrm{mid}}) - \frac{265}{264}\,D_r. \tag{19.1}$$

Next we record two bounds for the mean of the birthday law. We have

$$1 \le s_r < r.$$

The upper bound is shown in Section 17, where D_r is defined, and the lower bound holds because the first term of the sum s_r is 1 and the others are nonnegative. Further, $s_m \le \sqrt{\pi m/2}\; e^{1/(12m)} - \frac{1}{3}$ for every integer $m \ge 1$, by Lemma 17.4. Here $e^{1/(12m)} \le \frac{12m}{12m-1}$, because $e^{-1/(12m)} \ge 1 - \frac{1}{12m}$, and $\frac{12m}{12m-1}$ decreases in m; its values at $m = 13$ and at $m = 14$ are $\frac{156}{155}$ and $\frac{168}{167}$. Since

$$\sqrt{\pi/2} \le 1.25332, \qquad 1.25332 \cdot \frac{156}{155} < 1.2615, \qquad 1.25332 \cdot \frac{168}{167} < 1.261,$$

we obtain

$$s_{13} \le 1.2615\,\sqrt{13} - \frac{1}{3}, \qquad s_m \le 1.261\,\sqrt{m} - \frac{1}{3} \quad (m \ge 14).$$

Suppose first that $2 \le r \le 13$. One checks that $\frac{3r-2}{6r^2} \ge \frac{37}{1014}$ for these r, with equality at $r = 13$. Since $k \ge 265$, the first term of (19.1) is therefore at least $265 \cdot \frac{37}{1014} > 9.669$. We bound the two terms with μ_r^0 below by 0. For the last term, $D_r \le D_{13} \le \frac{3}{2}\,(s_{13}-1)$ by Lemmas 17.3 and 17.2, and since $s_{13} \le 1.2615\,\sqrt{13} - \frac{1}{3} < 4.216$, we get $D_r \le \frac{3}{2} \cdot 3.216 = 4.824$. Hence

$$\mathsf{G}(r,k) > 9.669 - \frac{5}{2} - \frac{265}{264} \cdot 4.824 > 9.669 - 2.5 - 4.843 > 2.32.$$

Suppose now that $r \ge 14$, and put $t = \sqrt{r}$, so that $t \ge \sqrt{14} > 3.74$. The condition $25\,r^3 \le 121\,k^2$ then reads

$$5\,t^3 \le 11\,k.$$

We prove the four bounds

$$k\,\frac{3r-2}{6r^2} \ge 1.88, \qquad \mu_r^0(X_{\mathrm{mid}}) \ge 0.837\,t - 0.92,$$
$$\mu_r^0(X_-) \ge 0.628\,t - 0.62, \qquad D_r \le 1.8915\,t - 2.6.$$

The first bound, and the bounds for X_- and X_{mid} on which the second and the third rest, need a lower bound for k in terms of r: for $r \le 69$ we use $k \ge 265$, and for $r \ge 70$ we use $5\,t^3 \le 11\,k$.

For the first bound, let $14 \le r \le 69$. Then $\frac{3r-2}{6r^2} \ge \frac{205}{28566}$, with equality at $r = 69$, as one checks in the same way, so that $k\,\frac{3r-2}{6r^2} \ge 265 \cdot \frac{205}{28566} > 1.9$. If $r \ge 70$, then

$t \ge \sqrt{70} > 8.36$, and since $k \ge \frac{5}{11}\, t^3$,

$$k\,\frac{3r-2}{6r^2} \ge \frac{5}{11}\, t^3 \cdot \frac{3t^2-2}{6t^4} = \frac{5}{11}\Big(\frac{t}{2} - \frac{1}{3t}\Big),$$

which increases with t and is larger than 1.88 at $t = 8.36$.

Next we show that

$$X_{\mathrm{mid}} \le 1.1315\, t, \qquad X_- \le 2.24\, t.$$

Since $r = t^2$, we have $X_{\mathrm{mid}} = t \cdot \frac{t\,(t^2+2)}{2k}$ and $X_- = t \cdot \frac{t\,(t^2+1)}{k-1}$. If $r \le 69$, then $t \le \sqrt{69} < 8.31$, and $k \ge 265$ gives

$$\frac{t\,(t^2+2)}{2k} \le \frac{8.31 \cdot 71}{530} < 1.12, \qquad \frac{t\,(t^2+1)}{k-1} \le \frac{8.31 \cdot 70}{264} < 2.21.$$

If $r \ge 70$, then $\frac{1}{k} \le \frac{11}{5\,t^3}$, again because $5\,t^3 \le 11\,k$, and $\frac{1}{k-1} \le \frac{265}{264} \cdot \frac{1}{k}$ since $k \ge 265$. With $t^2 \ge 70$ we get

$$\frac{t\,(t^2+2)}{2k} \le \frac{11}{10}\Big(1 + \frac{2}{t^2}\Big) \le \frac{11}{10} \cdot \frac{72}{70} < 1.1315,$$
$$\frac{t\,(t^2+1)}{k-1} \le \frac{265}{264} \cdot \frac{11}{5} \cdot \frac{71}{70} < 2.24.$$

For the second and the third bound we use that, for real numbers $X \ge 0$ and m,

$$\frac{3}{2}\, m^2 + (4+X)\, m \le 2\,(r-1) \qquad \text{implies} \qquad m \le \mu_r^0(X).$$

Indeed, the first inequality says that $(3m + 4 + X)^2 \le (4+X)^2 + 12\,(r-1)$, and taking square roots gives the second. We apply this implication with $X = X_{\mathrm{mid}}$ and $m = 0.837\,t - 0.92$, and with $X = X_-$ and $m = 0.628\,t - 0.62$. In both cases $m > 0$, because $t \ge 3.74$, so that the bounds for X_{mid} and X_- just shown give $(4 + X_{\mathrm{mid}})\, m \le (4 + 1.1315\, t)\, m$ in the first case and $(4 + X_-)\, m \le (4 + 2.24\, t)\, m$ in the second. Since $2\,(r-1) = 2\,(t^2-1)$, it therefore suffices to note that, for these two values of m, respectively,

$$2\,(t^2-1) - \frac{3}{2}\, m^2 - (4 + 1.1315\, t)\, m = 0.002081\, t^2 + 0.0031\, t + 0.4104,$$
$$2\,(t^2-1) - \frac{3}{2}\, m^2 - (4 + 2.24\, t)\, m = 0.001704\, t^2 + 0.04488\, t - 0.0966,$$

as one finds by expanding, and that both right-hand sides are positive for $t \ge 3.74$, the second because $0.04488 \cdot 3.74 > 0.0966$. This gives the second and the third of the four bounds.

For the fourth bound, put $\bar{s} = 1.261\, t - \frac{1}{3}$, so that $1 \le s_r \le \bar{s}$ by the bounds for the mean of the birthday law recorded above. Since $r - s_r > 0$, an elementary calculation shows that the inequality $D_r \le \frac{3}{2}\,(s_r - \frac{7}{5})$ is equivalent to

$$s_r^2 + \frac{3}{5}\, s_r \le \frac{8}{5}\, r.$$

This holds, because $0 < s_r \le \bar{s}$ and

$$\bar{s}^{\,2} + \frac{3}{5}\, \bar{s} = 1.590121\, t^2 - \frac{1.261}{15}\, t - \frac{4}{45} \le \frac{8}{5}\, t^2.$$

Consequently $D_r \le \frac{3}{2}\,(s_r - \frac{7}{5}) \le \frac{3}{2}\,(\bar{s} - \frac{7}{5}) = 1.8915\, t - 2.6$.

Finally, we insert the four bounds into (19.1) and obtain

$$\begin{aligned}\mathsf{G}(r,k) &\geq 1.88 - \frac{5}{2} + \frac{1}{2}\,(0.628\,t - 0.62)\\ &\quad + \frac{528}{265}\,(0.837\,t - 0.92) - \frac{265}{264}\,(1.8915\,t - 2.6)\\ &= 0.083018\ldots \cdot t - 0.153208\ldots \geq 0.083\,t - 0.1533.\end{aligned}$$

Since $t \geq 3.74$ and $0.083 \cdot 3.74 - 0.1533 > 0.157$, we have $\mathsf{G}(r,k) > 0$ in this range as well. □

19.3. **From a column to the next.** For $k \geq 265$ we can now prove Proposition 1.4, with a strict conclusion: by Proposition 9.2 the hypothesis confines row n to the strip of column k, where the criterion applies. In terms of the steps $d_{n,k} = b_{n,k} - b_{n,k-1}$, the hypothesis of the following proposition is that $d_{j,k} \leq 0$ for $k+1 \leq j \leq n$, and its conclusion is that $d_{n,k+1} < 0$.

Proposition 19.4. *Let $k \geq 265$ and $n \geq k+2$ be integers. If $b_{j,k} \leq b_{j,k-1}$ for every integer j with $k+1 \leq j \leq n$, then $b_{n,k+1} < b_{n,k}$.*

Proof. By Proposition 9.2 there is an integer $j > k$ with $25\,(j-k-1)^3 \leq 121\,k^2$ and $d_{j,k} > 0$, and the hypothesis shows that $j > n$. Write $n = k + r$. Then $2 \leq r \leq j-k-1$, so that

$$25\,r^3 \leq 25\,(j-k-1)^3 \leq 121\,k^2;$$

that is, row n lies in the strip of column k. In particular $r \leq k-1$, because $25\,k^3 > 121\,k^2$. Hence $\mathsf{G}(r,k) > 0$ by Proposition 19.3, and since $\delta_n^{(k)} \leq 0$ by the hypothesis at row n, Proposition 19.2 gives $\delta_n^{(k+1)} < 0$, i.e. $b_{n,k+1} < b_{n,k}$. □

For $9 \leq k \leq 264$ exact computation takes the place of Propositions 9.2 and 19.3: the conclusion follows, without strictness, from the two computed signs of Lemma 7.3 and from Proposition 1.3, applied in the chain range of column $k+1$.

Proof of Proposition 1.4. Let $k \geq 9$ and $n \geq k+2$ be integers such that $b_{j,k} \leq b_{j,k-1}$ for every integer j with $k+1 \leq j \leq n$. We have to show that $b_{n,k+1} \leq b_{n,k}$. For $k \geq 265$ this is contained in Proposition 19.4.

Let $9 \leq k \leq 264$, and let $\zeta = \zeta(k)$ be the number of Definition 7.2. By Lemma 7.3 we have $\zeta + 2 \leq k$ and

$$b_{k+\zeta+1,\,k-1} < b_{k+\zeta+1,\,k}, \qquad b_{k+\zeta,\,k+1} < b_{k+\zeta,\,k};$$

that is, column k is positive in row $k+\zeta+1$, and column $k+1$ is negative in row $k+\zeta$. By the first inequality and the hypothesis, $n \leq k+\zeta$.

Suppose now that $b_{n,k} < b_{n,k+1}$. Rows $n, \ldots, k+\zeta$ lie in the chain range of column $k+1$, which consists of rows $k+2, \ldots, 2k+1$, because $n \geq k+2$ and $\zeta \leq k-2$. By Proposition 1.3, applied to column $k+1$ in the rows $n, \ldots, k+\zeta-1$ one after another, this column is then positive in row $k+\zeta$ as well, which contradicts the second inequality. Hence $b_{n,k+1} \leq b_{n,k}$. □

20. Proof of Theorem 1.1

In this section we derive Theorem 1.1 from Propositions 1.2, 1.3 and 1.4 and from the computed rows $n \leq 16$ (Lemma 7.1). The section closes with a bound for the position of the maximum of a row $n \geq 335$ (Proposition 20.3), which follows

from statements proved in the earlier sections and is not used in the proof of the theorem.

The following lemma shows that, to the right of the middle of a row, a nonpositive step is followed only by nonpositive steps. Its conclusion is stated for all rows up to n, because the induction in the proof requires them.

Lemma 20.1. *Let $n \geq 0$ and $i \geq 9$ be integers with $n \leq 2i - 1$, and suppose that $b_{n,i} \leq b_{n,i-1}$. Then*

$$b_{m,j} \leq b_{m,j-1}$$

for all integers j and m with $j \geq i$ and $j + 1 \leq m \leq n$.

Proof. We prove by induction on $j \geq i$ that $b_{m,j} \leq b_{m,j-1}$ for every m with $j+1 \leq m \leq n$.

Let $j = i$, and let m be an integer with $i + 1 \leq m \leq n$. If we had $b_{m,i-1} < b_{m,i}$, then Proposition 1.3, applied with $k = i$ in the rows $m, \ldots, n-1$ one after another, which lie between $i + 1$ and $2i - 2$, would give $b_{n,i-1} < b_{n,i}$, contrary to the assumption of the lemma. Hence $b_{m,i} \leq b_{m,i-1}$.

Suppose now that the claim holds for some $j \geq i$, and let m be an integer with $j + 2 \leq m \leq n$. By the induction hypothesis $b_{m',j} \leq b_{m',j-1}$ for every m' with $j + 1 \leq m' \leq m$. Hence Proposition 1.4, applied with $k = j \geq 9$ in row m, gives $b_{m,j+1} \leq b_{m,j}$. This is the claim for $j + 1$. □

Together with Proposition 1.2 and Lemma 7.1, Lemma 20.1 excludes, in every row, a strict descent that is followed by a strict ascent.

Proposition 20.2. *Let n, i and j be integers with $n \geq 0$ and $1 \leq i < j \leq n - 1$. Then the inequalities $b_{n,i} < b_{n,i-1}$ and $b_{n,j-1} < b_{n,j}$ do not both hold.*

Proof. Suppose that $b_{n,i} < b_{n,i-1}$ and $b_{n,j-1} < b_{n,j}$. By Lemma 7.1 this is impossible for $n \leq 16$, so let $n \geq 17$. If $2i \leq n$, then Proposition 1.2 with $k = i$ gives $b_{n,i-1} \leq b_{n,i}$, contrary to the first inequality. If $n \leq 2i - 1$, then $i \geq 9$, and Lemma 20.1 applies to n and i. Since $j \geq i$ and $j+1 \leq n$, its conclusion for column j and row $m = n$ reads $b_{n,j} \leq b_{n,j-1}$, contrary to the second inequality. □

A finite sequence in which no strict descent is followed by a strict ascent is unimodal, and so the theorem follows.

Proof of Theorem 1.1. Let $n \geq 1$ be an integer, and let p be the largest integer with $0 \leq p \leq n-1$ such that $b_{n,\ell-1} \leq b_{n,\ell}$ for $1 \leq \ell \leq p$. If $p < n-1$, then $b_{n,p+1} < b_{n,p}$, and Proposition 20.2 with $i = p + 1$ shows that $b_{n,j} \leq b_{n,j-1}$ for $p + 2 \leq j \leq n - 1$. Hence $b_{n,\ell+1} \leq b_{n,\ell}$ for every ℓ with $p \leq \ell \leq n - 2$, which is the assertion of the theorem with $\ell^* = p$. □

20.1. **The position of the maximum.** Theorem 1.1 does not say where the maximum of a row stands. In the rows $n \geq 335$ the statements of the earlier sections bound its position (Proposition 20.3). We first note two facts about a column $k \geq 265$ and its strip, which consists of the rows $k + r$ with $r \geq 1$ and $25\,r^3 \leq 121\,k^2$ (Section 9).

The first fact is that the column is positive in every row beyond its strip: if $k \geq 265$ and r are integers with $121\,k^2 < 25\,r^3$, then $b_{k+r,k-1} < b_{k+r,k}$. Indeed, by Proposition 9.2 there is an integer $j > k$ with $25\,(j-k-1)^3 \leq 121\,k^2$ and $b_{j,k-1} < b_{j,k}$, and $j \leq k + r$, since $j > k + r$ would imply $25\,r^3 \leq 25\,(j-k-1)^3 \leq 121\,k^2$,

against the hypothesis. Put $n = k + r$. If $n \le 2k - 1$, then $k + 1 \le j \le n \le 2k - 1$, and Proposition 1.3, applied in the rows $j, \dots, n-1$ one after another, gives $b_{n,k-1} < b_{n,k}$; if $n \ge 2k$, this inequality is Proposition 15.4.

The second fact is that, in a row of the strip other than the first, a nonpositive step of the column is followed by a negative step of the next column: for integers $k \ge 265$ and $r \ge 2$ with $25\,r^3 \le 121\,k^2$ and for $n = k+r$, the inequality $b_{n,k} \le b_{n,k-1}$ implies $b_{n,k+1} < b_{n,k}$. Indeed, $r \le k - 1$ because $25\,r^3 \le 121\,k^2 < 25\,k^3$, so that $\mathsf{G}(r,k) > 0$ by Proposition 19.3; and since the normalised step $\delta_n^{(j)} = d_{n,j}/n!$ of column j in row n has the sign of $b_{n,j} - b_{n,j-1}$, the inequality $b_{n,k} \le b_{n,k-1}$ says that $\delta_n^{(k)} \le 0$, and Proposition 19.2 gives $\delta_n^{(k+1)} < 0$, i.e. $b_{n,k+1} < b_{n,k}$.

Applied along a row $n \ge 335$, the two facts give the shape of the row.

Proposition 20.3. *Let $n \ge 335$ be an integer. Then there is an integer p with $0 \le p \le n - 1$ and $25\,(n-1-p)^3 \le 121\,(p+1)^2$ such that*

$$b_{n,\ell-1} \le b_{n,\ell} \quad (1 \le \ell \le p), \qquad b_{n,\ell-1} < b_{n,\ell} \quad (82 \le \ell \le p),$$
$$b_{n,\ell+1} \le b_{n,\ell} \quad (p \le \ell \le n-2), \qquad b_{n,\ell+1} < b_{n,\ell} \quad (p+1 \le \ell \le n-2).$$

Proof. Let first k be an integer with $1 \le k \le 264$; we show that $b_{n,k-1} \le b_{n,k}$, and that $b_{n,k-1} < b_{n,k}$ if $k \ge 82$. If $2k \le n$, this is Proposition 1.2 and, for $k \ge 82$, Proposition 15.4. If $n \le 2k - 1$, then $k \ge 168$, and Corollary 9.3 gives an integer $r \ge 1$ with $100\,r \le 27\,k \le 7128$, hence $r \le 71$, and $b_{k+r,k-1} < b_{k+r,k}$. Since $k + 1 \le k + r \le 335 \le n \le 2k - 1$, Proposition 1.3, applied in the rows $k + r, \dots, n - 1$ one after another, gives $b_{n,k-1} < b_{n,k}$.

Let p be the least integer $p \ge 81$ such that $p = n - 1$ or $b_{n,p+1} \le b_{n,p}$; since $n - 1 \ge 81$, it exists and $p \le n - 1$. For $82 \le \ell \le p$ the integer $\ell - 1$ is at least 81 and less than p, so that $b_{n,\ell-1} < b_{n,\ell}$ by the minimality of p; and $b_{n,\ell-1} \le b_{n,\ell}$ for $1 \le \ell \le 81$ by the first paragraph of the proof. If $p = n - 1$, then $n - 1 - p = 0$ and no ℓ satisfies $p \le \ell \le n - 2$, so that nothing remains to be shown.

Let $p \le n-2$, and put $q = p+1$, so that $82 \le q \le n-1$ and $b_{n,q} \le b_{n,q-1}$. By the first paragraph of the proof, $q \ge 265$. If we had $121\,q^2 < 25\,(n-q)^3$, the first fact with $k = q$ and $r = n-q$ would give $b_{n,q-1} < b_{n,q}$; hence $25\,(n-q)^3 \le 121\,q^2$, which is the inequality for p. For every integer j with $q \le j \le n - 2$ we have $j \ge 265$, $n - j \ge 2$ and $25\,(n-j)^3 \le 25\,(n-q)^3 \le 121\,q^2 \le 121\,j^2$, so that, by the second fact with $k = j$ and $r = n - j$, the inequality $b_{n,j} \le b_{n,j-1}$ implies $b_{n,j+1} < b_{n,j}$. Since $b_{n,q} \le b_{n,q-1}$, induction on j gives $b_{n,j+1} < b_{n,j}$ for $q \le j \le n - 2$. Together with $b_{n,p+1} \le b_{n,p}$, these are the inequalities for $\ell \ge p$. □

The inequality for p forces $p \ge 265$, since $p \le 264$ would give $n - 1 - p \ge 70$ and $p + 1 \le 265$, whereas $25 \cdot 70^3 > 121 \cdot 265^2$. Hence the maximum of the row is $b_{n,p}$, and it is attained at column p and at no other column except possibly $p + 1$. Since $p + 1 \le n$, the inequality for p also gives

$$n - 1 - p \le \Big(\frac{121}{25}\Big)^{1/3} (p+1)^{2/3} \le \Big(\frac{121}{25}\Big)^{1/3} n^{2/3} < 1.692\, n^{2/3}.$$

This bounds the distance of the maximum from the last column of the row from above. It does not determine the position of the maximum, and we have no bound for the distance from below. The constant $\frac{121}{25}$ is that of the strip, the range of rows which the estimates behind Propositions 9.2 and 19.3 reach, and it has no meaning beyond these estimates; the bound of Proposition 9.2 is not sharp, as was remarked after it. The other constants arise in the same way: 265 is the first column to which

these two propositions apply, 82 is the first to which Proposition 15.4 applies, and $335 = 264 + 71$ comes from the bound of Corollary 9.3, as the first paragraph of the proof shows.

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