A planar graph without four- or nine-cycles that is not three-colorable

Peter Graham
https://orcid.org/0009-0007-8493-1505

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Abstract

We give an 82-vertex planar graph without cycles of length four or nine that is not three-colorable. Steinberg conjectured in 1976 that planar graphs without cycles of length four or five are three-colorable; Cohen-Addad, Hebdige, Král', Li and Salgado disproved this, and their graph contains a nine-cycle. Lu, Wang, Wang, Bu, Montassier and Raspaud asked, as recorded by Kang, Jin and Wang, for which lengths from five to nine every planar graph without cycles of that length or of length four is three-colorable. Our graph answers the length-nine case negatively. It is built from 27 complete graphs on four vertices by Ore compositions, the edge-deletion and vertex-splitting operation of Dirac, Hajós, Gallai and Ore. It is four-critical: every proper subgraph is three-colorable. Its 136 edges attain Kostochka and Yancey's lower bound, so no four-critical graph on 82 vertices has fewer edges. Lean checks planarity, the exclusion of four- and nine-cycles, and non-three-colorability; criticality and the edge minimum rest on the written proof.

This work has not yet been reviewed by experts. The proofs, the Lean formalization, the search code, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

1 Introduction

All graphs in this paper are finite and simple. A proper three-coloring is a map $f : V(G) \to \{0, 1, 2\}$ with $f(u) \neq f(v)$ for every edge uv. A cycle is simple; it need not be induced. Thus forbidding a cycle of length k, denoted C_k, forbids cycles with chords as well.

A graph is *four-critical* if it has chromatic number four and every proper subgraph is three-colorable. An *Ore composition* glues two graphs by deleting an edge xy of one and splitting a vertex z of the other into two vertices identified with x and y; Section 2 gives the precise definition. A graph obtained from copies of K_4 by Ore compositions is called a *four-Ore graph* [14, Section 1]. For a graph G, write $\chi(G)$ for its chromatic number and $G - e$ for the graph obtained by deleting edge e.

Steinberg conjectured in 1976 that every planar graph without C_4 or C_5 is three-colorable; see Steinberg's account [19] and the disproof [4, opening paragraph]. Grötzsch's theorem [26] gives three-colorability of planar graphs without triangles; see the short proof in [21]. For the relaxation attributed to Erdős in [2], Borodin, Glebov, Raspaud and Salavatipour prove three-colorability

when cycles of lengths four through seven are excluded [30]; earlier bounds by Abbott and Zhou, by Borodin, and by Sanders and Zhao are recorded in [2, Section 1].

Theorem 1. *There exists an 82-vertex planar four-Ore graph with 136 edges, without cycles of length four or nine. Consequently it is four-critical and has the fewest edges among all 82-vertex four-critical graphs.*

The theorem answers the $i = 9$ case of the question of Lu, Wang, Wang, Bu, Montassier and Raspaud [1], as attributed and reproduced by Kang, Jin and Wang [2, Section 1, Problem 1.2]: which $i \in \{5, 6, 7, 8, 9\}$ make every planar graph without C_4 or C_i properly three-colorable? Cohen-Addad, Hebdige, Král', Li and Salgado's counterexample to Steinberg's conjecture answers $i = 5$ negatively [4, Theorem 3]. We observe that the G_1 component in their Figure 1 contains the nine-cycle $a, f, k, n, m, j, i, h, g, a$ in that paper's own labels. This component is retained in the published construction [4, Lemma 2 and proof of Theorem 3]; hence that graph does not settle the case $i = 9$. Kang, Jin, Liu and Wang described the cases $i \in \{6, 7, 8, 9\}$ as "still unsettled" [3, paragraph following Problem 1.2]. Theorem 1 answers the case $i = 9$ negatively; the cases $i = 6, 7, 8$ are not addressed here. On the positive side, Kang, Jin and Wang prove three-colorability when C_6 is also excluded [2, Theorem 1.4], and the $\{C_4, C_7, C_9\}$-free class is three-colorable by Lu and coauthors [1], as also recorded in [2, Section 1, paragraph following Problem 1.3]. Appendix H collects list, DP and defective coloring results for these classes.

Our contribution is an explicit graph for the C_4, C_9 question, built by Ore composition and attaining Kostochka and Yancey's edge bound [13, Theorem 3]; see also [21]. An Ore composition of two non-three-colorable graphs is never three-colorable (Lemma 1), but it can create four- or nine-cycles and destroy planarity. The difficulty is choosing the 26 identifications between planar pieces so that neither happens; the earlier counterexample does not meet these restrictions. The proof checks the resulting graph directly: Ore composition gives the short proof of non-three-colorability, and a rotation system and exhaustive cycle enumeration establish planarity and the cycle exclusions. Section 2 illustrates the first composition and gives the proof; the complete construction is printed in Appendix G. The proof also depends on the finite computations described in Section 3.

2 Construction and proof

The graph G has vertex set $\{0, \ldots, 81\}$. This section builds G by Ore composition, which proves non-three-colorability; Sections 2.1 and 2.2 prove planarity and the cycle exclusions, and Section 2.3 completes the proof of Theorem 1.

For a graph B, $N_B(z)$ denotes the set of neighbors of vertex z. We use the name Ore composition for the operation defined below. Kostochka and Yancey trace it to Dirac [23], Gallai [22], Hajós [24] and Ore [25] [14, Section 1]. Gallai presents the corresponding criticality statement (2.7) as a theorem of Dirac, and its special case (2.9) as Hajós's construction [22, Section 2, (2.7)–(2.9)].

Take vertex-disjoint graphs A and B, an edge xy of A, and a vertex z of B. Partition $N_B(z)$ into two nonempty disjoint sets N_x, N_y. Delete xy from A, replace z in B by two nonadjacent vertices, and identify those vertices with x, y, respectively, assigning neighbors N_x to x and N_y to y. The resulting graph is an *Ore composition*, also called a Dirac–Hajós–Gallai–Ore (DHGO) composition [14, Section 1].

The key point is that a three-coloring of an Ore composition of two non-three-colorable graphs would force the ends of the deleted edge to have the same color: different colors would give a three-coloring of the first graph. Their common color could then be assigned to the split vertex of the second graph, giving another contradiction. Lemma 1 makes this argument precise, so induction

from K_4 proves non-three-colorability. The cycle restrictions are established separately for the resulting graph by the two complete enumerations in Appendix B.

The construction is a binary tree with 27 leaves, each a copy of K_4, and 26 internal nodes, each an Ore composition. The nodes are numbered 0 to 52, with root 52; write J_t for the graph at node t. Node numbers are distinct from vertex labels. The first composition uses copies of K_4 on $\{12, 60, 62, 84\}$ and $\{13, 61, 63, 85\}$. It deletes the edge 62 84, splits vertex 85, and assigns neighbor 63 to 62 and neighbors 13, 61 to 84 (Figure 1). The last composition deletes edge 0 7 from J_{28} and splits vertex 82 of J_{51}: neighbors $2, 3, 4, 5$ are attached to 0 and neighbors $8, 9, 10, 11$ to 7. Its two operands are obtained from J_4, J_{27} and J_{47}, J_{50}, respectively (Figure 2). Vertex labels 82 and above are temporary: each such vertex is split at a later node, so none survives at the root. Appendix G lists every node, and Figure 3 in Appendix A shows the full graph.

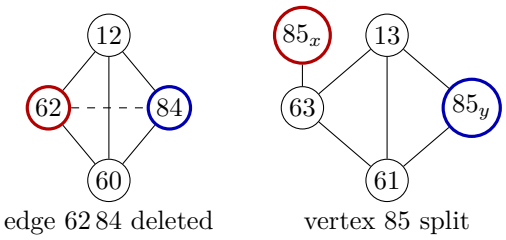


Figure 1: The first composition. Left: edge 62 84 is deleted (dashed). Right: vertex 85 is split. Matching outline styles (red, blue) mark the identifications $85_x = 62$ and $85_y = 84$.

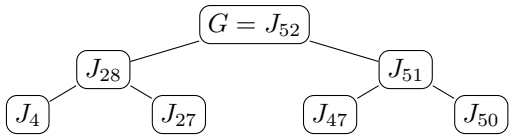


Figure 2: The upper part of the Ore composition tree. At each fork the left graph supplies the deleted edge and the right graph the split vertex. The complete table in Appendix G specifies all 26 compositions.

The following lemma gives the non-colorability property of Ore composition in the form used here.

Lemma 1. *If neither A nor B is three-colorable, their Ore composition is not three-colorable.*

Proof. Suppose the composite has a proper three-coloring f. Restrict f to $V(A)$. If $f(x) \neq f(y)$, restoring the deleted edge xy gives a proper three-coloring of A, a contradiction. Hence $f(x) = f(y)$. Restrict f to $V(B) \setminus \{z\}$ and give z this common color. Every neighbor of z belonged to N_x or N_y, and therefore has a color different from that common color. Edges not incident with z are unchanged. This is a proper three-coloring of B, again a contradiction. □

The graph K_4 is not three-colorable: four pairwise adjacent vertices need four distinct colors. Inducting over this explicit tree with Lemma 1 proves that G is not three-colorable. Starting with 27 copies of K_4, each composition reduces the sum of orders and the sum of sizes by one, so the root has $27 \cdot 4 - 26 = 82$ vertices and $27 \cdot 6 - 26 = 136$ edges. The archived checker verifies that the root listed in Appendix C is exactly G; Appendices F and G give its edges, rotation and complete construction tree.

2.1 Planarity

A rotation system specifies the cyclic order of the neighbors at each vertex; the one used here is printed in Appendix F. It yields planarity by an orientable-surface argument. Let $D = \{(u, v) : uv \in E(G)\}$ be the darts. Write $\rho_v(u)$ for the neighbor following u in the cyclic order at v. The face permutation is

$$\phi(u, v) = (v, \rho_v(u)),$$

and F denotes the set of its orbits. The graph is connected, and the face permutation has 56 orbits. To construct the embedding, place an oriented disk at every vertex, attach an untwisted band for each edge according to the cyclic orders, and cap each boundary walk with a disk. The result is a connected closed orientable surface with a cellular embedding of G. The boundary walks are precisely the orbits of ϕ, so its Euler characteristic is

$$|V| - |E| + |F| = 82 - 136 + 56 = 2.$$

By the classification of closed orientable surfaces this surface is the sphere. Deleting a point outside the embedded graph gives a plane embedding. This proves planarity.

2.2 Cycle exclusion

Two independent complete enumerations of simple cycles give the counts in Appendix B. Both find no cycle of length four or nine. The first enumeration grows every simple path from its smallest vertex s, using only other vertices greater than s. It records a closure to s when the path has at least three vertices and its second vertex is smaller than its last. Every undirected simple cycle has a unique smallest vertex and exactly two orientations, exactly one of which meets this last test. Thus each cycle is counted once. Appendix B gives the second enumeration and the counts.

2.3 Criticality and the edge bound

Kostochka and Yancey record the classical observation that an Ore composition of two k-critical graphs is k-critical when the second graph is not k-critical after z has been split [14, Section 1]; Gallai's (2.7) b) requires instead that the split graph be $(k-1)$-colorable [22, Section 2]. Kostochka and Yancey verify this hypothesis for compositions of k-extremal graphs (k-critical graphs attaining their edge bound) using that bound [14, Theorem 4], and conclude that every k-Ore graph is k-extremal, hence k-critical [14, Section 1, paragraphs before Theorem 6]. The proof below establishes the criticality of G directly from the deletion colorings in Appendix C. The edge bound is Kostochka and Yancey's [13, Theorem 3]: every four-critical graph H satisfies $|E(H)| \geq \lceil (5|V(H)| - 2)/3 \rceil$. We now prove Theorem 1.

Proof of Theorem 1. The verified construction tree shows that G is a four-Ore graph. Induction from K_4 using Lemma 1 proves that G is not three-colorable. The rotation system has 56 faces, so $82-136+56 = 2$; the orientable-surface argument in Section 2.1 proves planarity. The enumerations in Section 2.2 and Appendix B find no simple four- or nine-cycle. For an edge e, use the coloring of $G-e$ specified in Appendix C. The two endpoints of e necessarily have equal colors, since otherwise this would color G. Recoloring one endpoint with a fourth color yields a proper four-coloring of G. Together with the preceding induction, this proves $\chi(G) = 4$. The graph has no isolated vertices. Any proper subgraph omits some edge of G, even if it omits a vertex, and is therefore a subgraph of some $G - e$. Restricting that deletion coloring proves its three-colorability. Consequently G is four-critical. Finally $136 = (5 \cdot 82 - 2)/3$, so Kostochka and Yancey's bound proves the edge minimum at order 82. □

3 Computational verification

The computations used above establish the face orbits of the rotation, the cycle counts in Appendix B, the compositions in Appendix G, and a proper three-coloring of $G - e$ for every edge e. Appendix D gives two further non-colorability arguments: exact elimination of coloring constraints and a clausal proof of unsatisfiability.

The formal graph G has vertex set $\{0, \ldots, 81\}$ and adjacency given by exactly the 136 unordered edges in Appendix F. The Lean theorem `Steinberg.planar_counterexample` states the following conjunction for this graph:

$$G \text{ has a crossing-free straight-line drawing with rational coordinates;}$$
$$\text{every simple cycle of } G \text{ has length different from four and nine;}$$
$$\text{there is no proper coloring } V(G) \longrightarrow \{0, 1, 2\}.$$

The formal planarity predicate, `RationalPlanar`, requires an injective assignment of rational coordinate pairs to the vertices, with a vertex on an edge segment only at an endpoint and edge segments meeting only at a shared endpoint. Segments are closed real affine segments between those rational points. The proof checks the existing drawing's six-decimal coordinates as exact rationals. Mathlib's simple closed walks and proper colorings supply the other two predicates; their conventions agree with the introduction, including cycles with chords. This proves the planar counterexample to the length-nine question directly from these definitions.

The formal theorem has no hypotheses and is checked by Lean's kernel using only the standard axioms `propext`, `Classical.choice` and `Quot.sound`, without compiled-code evaluation. A second theorem, also with no hypotheses, `Steinberg.counterexample`, proves the same cycle and coloring assertions with the spherical rotation predicate in place of `RationalPlanar`. Build commands and versions are in the accompanying checking notes. Appendix E describes the formal non-colorability argument. The remaining assertions of Theorem 1—Ore membership, criticality and the edge minimum—use the written arguments in Section 2; the last uses Kostochka and Yancey's theorem. The Ore construction, deletion colorings and other cycle counts also have independent executable checks. The mathematical data and checking programs are archived at `https://doi.org/10.5281/zenodo.23491270`, the archive's concept record, which resolves to its latest version.

Appendix A The embedding certificate

Appendix F gives the edge list and cyclic neighbor orders. A checker verifies that the graph is simple and connected and that each vertex's cyclic order lists exactly its neighbors, then traces the face permutation of Section 2.1 and finds 56 orbits.

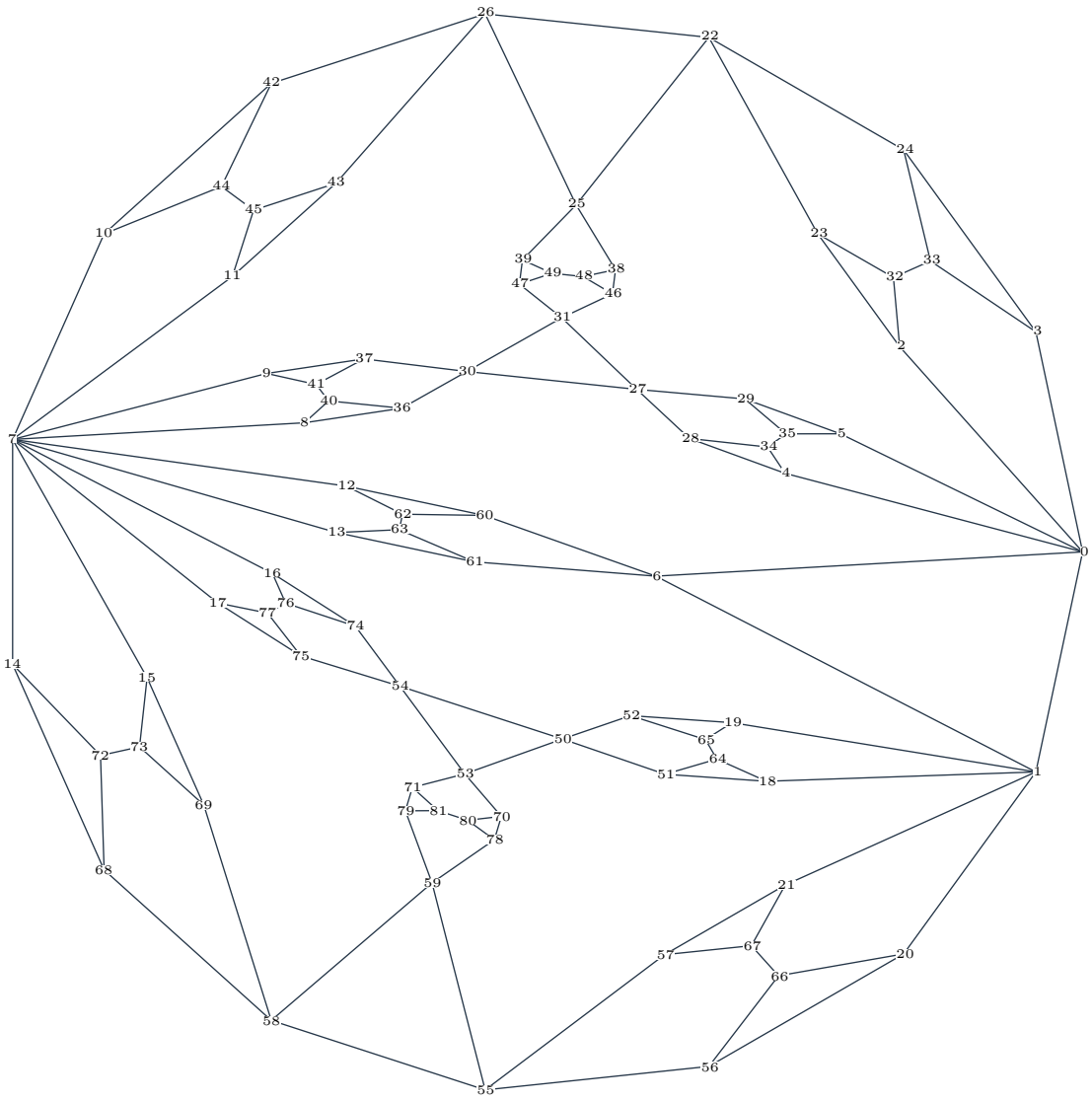


Figure 3: The counterexample, with vertex numbers from the accompanying SVG drawing. The drawing was generated from, and checked against, the edge and rotation data in Appendix F; its rational coordinates are also checked in Lean.

Drawing. The figure is obtained by applying Tutte's barycentric method to the triangulation obtained by adding a new vertex inside each bounded face that is not a triangle, joined to the face's boundary vertices, and then omitting the added vertices and edges. For background see Tutte's "How to Draw a Graph" [18]. The finite-decimal coordinates are exact rationals, checked

in Lean as described in Section 3.

Appendix B Excluding cycles

An independent enumeration checks the cycle counts used in Section 2.2, counting undirected simple cycles without multiplicity. The second enumeration instead chooses an edge (s, t) with $s < t$, and enumerates simple s–t paths avoiding that edge, with all other vertices greater than s. Closing by ts gives a simple cycle. A cycle whose smallest vertex is s is found twice, once for each of its two edges at s; division by two gives the count. Each raw count is even. Both enumerations run through length 14 and return the same table:

length	3	4	5	6	7	8	9	10	11	12	13	14
count	27	0	22	55	44	11	0	0	64	320	832	1824

In particular there is no C_4, C_9 or C_{10}.

Appendix C The construction and deletion colorings

The construction table identifies the graph used in the Ore induction, and the deletion colorings certify its criticality. The tree printed in Appendix G has 27 leaves and 26 internal nodes. Each leaf gives four distinct vertices and all six edges of a K_4. Every internal node gives two earlier, disjoint operands, an edge xy of the first, a vertex z of the second, and two nonempty sets N_x, N_y partitioning $N_B(z)$. Applying the operation in Section 2 at each node gives the root graph. Every non-root node occurs as an operand exactly once. At the root the vertex labels are $0, \dots, 81$ and the edges are exactly those in Appendix F, with the identity correspondence on vertices.

For each of the 136 edges e, the accompanying deletion-coloring data gives a vector $(c_0, \dots, c_{81}) \in \{0, 1, 2\}^{82}$ whose coordinates differ across every edge other than e. These are the deletion colorings used in the proof of Theorem 1; the checking notes specify their location and verification.

C.1 Edge count and other cycle lengths

The tree proves that G is a four-Ore graph. The edge bound of Section 2.3 is also stated as Theorem 4 in [14], and the edge count of four-Ore graphs is discussed in Kostochka and Yancey's Section 1 [14]. Kostochka and Yancey's equality characterization identifies the extremal graphs as the four-Ore graphs [14, Theorem 6]. The edge minimum here follows from the composition count and their lower bound, rather than being a separate new bound. The cycle table also gives the exclusion of C_{10} and the presence of C_5, C_6, C_7, C_8.

Appendix D Two additional checks of non-colorability

These checks establish non-colorability independently of the Ore induction.

D.1 Exact constraint elimination

A relation's *scope* is the set of vertices whose colors it specifies. The *interaction graph* joins two remaining vertices when they occur in the scope of a common relation.

For each edge uv start with the relation $R_{uv} = \{(a, b) \in \{0, 1, 2\}^2 : a \neq b\}$. Maintain a conjunction of relations on the remaining vertices. To eliminate v, collect all relations mentioning it, let S be the union of their scopes minus v, and replace them by the relation

$$R_S = \{a \in \{0, 1, 2\}^S : \exists c \in \{0, 1, 2\} \bigwedge_{R \text{ mentioning } v} R(a, v = c)\}.$$

All other relations remain. The new conjunction is exactly the existential projection of the old one. Induction on eliminations shows that satisfiability is preserved. An empty relation proves that no coloring exists; a nonempty relation on the empty scope is a tautology and may be discarded.

Each elimination enumerates all $3^{|S|}$ assignments and all three values of c. A vertex of minimum degree in the current interaction graph is chosen, breaking ties by vertex label. This choice affects cost, not correctness. For this graph an empty relation is reached; the largest remaining scope S has size three.

D.2 Exact graph-to-CNF reduction and symmetry

The coloring constraints form a conjunction of clauses, or conjunctive normal form (CNF). Independently introduce Boolean variables $x_{v,c}$, numbered $3v + c + 1$, for $0 \leq v < 82$ and $c \in \{0, 1, 2\}$. The generator adds the clauses

$x_{v,0} \vee x_{v,1} \vee x_{v,2}$	for each v,
$\neg x_{v,a} \vee \neg x_{v,b}$	for each v and $0 \leq a < b < 3$,
$\neg x_{u,c} \vee \neg x_{v,c}$	for each $uv \in E(G)$ and c.

The first two lines specify exactly one color at each vertex, and the third specifies distinct colors across every edge. Assigning the unique true color is therefore a bijection between satisfying assignments of these clauses and proper three-colorings of G.

The lexicographically first triangle is $(0, 1, 6)$. The generator also adds $x_{0,0}$, $x_{1,1}$ and $x_{6,2}$, that is, unit clauses 1, 5 and 21. Every proper coloring gives distinct colors to this triangle. There is a permutation of the three color names taking them, in this order, to $0, 1, 2$; the other clauses are invariant under this permutation. Adding these units thus preserves satisfiability.

The resulting formula has a clausal proof ending with the empty clause, generated with CaDiCaL [16] and checked by DRAT-trim [15]. Each clause addition satisfies a checked redundancy condition preserving satisfiability; a clause deletion weakens the formula. The final empty clause therefore proves unsatisfiability. Together with the color-permutation argument and the exact reduction above, this proves that G has no proper three-coloring. The formula and proof, their sizes and the checking commands are given in the checking notes.

Appendix E Formal verification

The formal proof certifies non-three-colorability by deducing a contradiction from the coloring clauses. Lean checks its own clausal refutation of the coloring formula, in which every added clause follows by reverse unit propagation (RUP); this certificate is separate from the DRAT proof of Appendix D, which Lean does not check. Its coloring formula has the edge and vertex clauses of Appendix D, without units that prescribe a color for an individual vertex. The formal reduction proves that a proper coloring would satisfy this formula. Equality of the formula produced by the reduction with the certificate's input is proved in Lean.

Unit propagation assigns the only remaining literal of a clause when all its other literals are false. Clause additions are justified by reverse unit propagation: assuming the negation of a proposed clause, unit propagation in previously justified clauses yields a contradiction. The formal proof establishes this implication and applies it to every addition. Hints specify the clauses used in these deductions. Deletion instructions leave earlier clauses available; every such clause remains entailed by the original input. The final empty clause proves unsatisfiability of that input. The checker uses no general RAT steps. It does not check deletions; this is sound because every retained clause is entailed. The executable DRAT-trim check instead reads the symmetry-breaking formula and its text DRAT proof described above.

The use of hints follows Cruz-Filipe, Heule, Hunt, Kaufmann and Schneider-Kamp [17, abstract], whose format extends DRAT, a clausal proof format allowing clause additions and deletions, "with hints that facilitate a simple and fast validation algorithm." The checking notes give the exact certificate formats and formal theorem names.

Appendix F Explicit graph and rotation

The graph list gives every edge, and the rotation gives every neighbor in cyclic order. Together they specify the labeled graph and its embedding.

Graph

```
82 136
0 1
0 2
0 3
0 4
0 5
0 6
1 6
1 18
1 19
1 20
1 21
2 23
2 32
3 24
3 33
4 28
4 34
5 29
5 35
6 60
6 61
7 8
7 9
7 10
7 11
7 12
7 13
7 14
7 15
7 16
7 17
8 36
8 40
9 37
9 41
10 42
10 44
11 43
11 45
12 60
12 62
13 61
13 63
14 68
14 72
15 69
15 73
16 74
16 76
17 75
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18 51
18 64
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21 57
21 67
22 23
22 24
22 25
22 26
23 32
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25 26
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25 39
26 42
26 43
27 28
27 29
27 30
27 31
28 34
29 35
30 31
30 36
30 37
31 46
31 47
32 33
34 35
36 40
37 41
38 46
38 48
39 47
39 49
40 41
42 44
43 45
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51 64
52 65
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53 70
53 71
54 74
54 75
55 56
55 57
55 58
55 59
56 66
57 67
58 59
58 68
58 69
59 78
59 79
60 62
61 63
62 63
64 65
66 67
68 72
69 73
70 78
70 80
71 79
71 81
72 73
74 76
75 77
76 77
78 80
79 81
80 81
```

Rotation

```
0: 1 3 2 5 4 6
1: 0 6 19 18 21 20
2: 23 0 32
3: 33 0 24
4: 28 0 34
5: 35 0 29
6: 1 0 60 61
7: 12 8 9 11 10 14 15 17 16 13
8: 7 36 40
9: 37 7 41
10: 42 7 44
11: 45 7 43
12: 60 7 62
13: 7 61 63
14: 7 68 72
15: 69 7 73
16: 74 7 76
17: 77 7 75
18: 51 1 64
19: 65 1 52
20: 56 1 66
21: 67 1 57
22: 25 23 24 26
23: 22 2 32
24: 3 22 33
25: 38 22 26 39
26: 22 42 43 25
27: 30 28 29 31
28: 27 4 34
29: 5 27 35
30: 36 27 31 37
31: 27 46 47 30
32: 2 33 23
33: 32 3 24
34: 4 35 28
35: 34 5 29
36: 8 30 40
37: 30 9 41
38: 46 25 48
39: 25 47 49
40: 41 8 36
41: 9 40 37
42: 26 10 44
43: 11 26 45
44: 10 45 42
45: 44 11 43
46: 31 38 48
47: 39 31 49
48: 49 46 38
49: 47 48 39
50: 53 51 52 54
51: 50 18 64
52: 19 50 65
53: 70 50 54 71
54: 50 74 75 53
55: 58 56 57 59
56: 55 20 66
57: 21 55 67
58: 68 55 59 69
59: 55 78 79 58
60: 6 12 62
61: 13 6 63
62: 63 60 12
63: 61 62 13
64: 18 65 51
65: 64 19 52
66: 20 67 56
67: 66 21 57
68: 14 58 72
69: 58 15 73
70: 78 53 80
71: 53 79 81
72: 73 14 68
73: 15 72 69
74: 54 16 76
75: 17 54 77
76: 16 77 74
77: 76 17 75
78: 59 70 80
79: 71 59 81
80: 81 78 70
81: 79 80 71
```

Appendix G The explicit Ore tree

Each leaf row specifies the four vertices of a complete graph. Each internal row specifies the two operand indices, then x, y, z, N_x, N_y in the notation of Section 2. Operands are disjoint at every composition. The root graph is G, with the identity correspondence on $0, \ldots, 81$.

node	vertices of K_4
0	$0, 1, 6, 7$
1	$12, 60, 62, 84$
2	$13, 61, 63, 85$
5	$50, 53, 54, 83$
6	$16, 74, 76, 89$
7	$17, 75, 77, 90$
10	$55, 58, 59, 88$
11	$14, 68, 72, 92$
12	$15, 69, 73, 93$
15	$20, 56, 66, 91$
16	$21, 57, 67, 94$
20	$70, 78, 80, 87$
21	$71, 79, 81, 95$
24	$18, 51, 64, 86$
25	$19, 52, 65, 96$
29	$22, 25, 26, 82$
30	$10, 42, 44, 100$
31	$11, 43, 45, 101$
34	$27, 30, 31, 99$
35	$8, 36, 40, 103$
36	$9, 37, 41, 104$
39	$4, 28, 34, 102$
40	$5, 29, 35, 105$
44	$38, 46, 48, 98$
45	$39, 47, 49, 106$
48	$2, 23, 32, 97$
49	$3, 24, 33, 107$

node	A	B	x	y	z	N_x	N_y
3	1	2	62	84	85	63	13, 61
4	0	3	6	7	84	60, 61	12, 13
8	6	7	76	89	90	77	17, 75
9	5	8	54	83	89	74, 75	16, 17
13	11	12	72	92	93	73	15, 69
14	10	13	58	88	92	68, 69	14, 15
17	15	16	66	91	94	67	21, 57
18	14	17	55	88	91	56, 57	20, 21
19	9	18	53	83	88	59	14, 15, 20, 21
22	20	21	80	87	95	81	71, 79
23	19	22	53	59	87	70, 71	78, 79
26	24	25	64	86	96	65	19, 52
27	23	26	50	83	86	51, 52	18, 19
28	4	27	1	7	83	18, 19, 20, 21	14, 15, 16, 17
32	30	31	44	100	101	45	11, 43
33	29	32	26	82	100	42, 43	10, 11
37	35	36	40	103	104	41	9, 37
38	34	37	30	99	103	36, 37	8, 9
41	39	40	34	102	105	35	5, 29
42	38	41	27	99	102	28, 29	4, 5
43	33	42	25	82	99	31	4, 5, 8, 9
46	44	45	48	98	106	49	39, 47
47	43	46	25	31	98	38, 39	46, 47
50	48	49	32	97	107	33	3, 24
51	47	50	22	82	97	23, 24	2, 3
52	28	51	0	7	82	2, 3, 4, 5	8, 9, 10, 11

Appendix H Related coloring results

DP-coloring is correspondence coloring, introduced by Dvořák and Postle [20]; see also [5, Section 1]. Kang, Jin and Zhu prove DP-3-colorability of the $\{C_4, C_7, C_9\}$-free class of planar graphs [5, Theorem 1.1]. Their Corollary 1.2 states that these graphs are "3-choosable (and hence 3-colorable)" [5].

Liu, Loeb, Rolek, Yin and Yu prove that planar graphs without cycles of lengths $4, a, b, 9$, for distinct $a, b \in \{6, 7, 8\}$, are DP-3-colorable [6, Theorem 1.3]. This extends the three-choosability results of Wang, Lu and Chen [27], Shen and Wang [28], and Wang, Wu and Shen [29], which are summarized in [6, Theorem 1.2]. These results impose two additional cycle exclusions beyond the pair considered here.

Defective colorings are a different notion: in a (d_1, d_2, d_3)-coloring, color class j induces a graph of maximum degree at most d_j. Related work includes Kang, Jin, Liu and Wang on $(1, 0, 0)$-colorability [3, Theorem 1.4]. For the four-and-nine pair, Dai, Wang and Xu prove that "Every planar graph without cycles of length 4 or 9 is $(1, 1, 0)$-colorable" [7], while Chen, Dai, Nie, Wang and Yu prove $(2, 0, 0)$-colorability [8, Theorem 1.4]. Xiao and Huang prove weak DP-$(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_0)$-colorability [9, Corollary 1.2], where $\mathcal{F}_d$ is the class of forests of maximum degree at most d. These conclusions allow defects or impose forest conditions, unlike proper three-colorability. For further bibliographic context see Ang and Leo [10], Bu, Wang and Wang [11], and Li, Niu and Tian [12].

Acknowledgments

Kostochka and Yancey describe the construction as "found in Dirac's paper" and "used by Gallai . . . and Hajós", and say that Ore "used it for a composition of complete graphs" [14, Section 1]. The sharp edge bound is Kostochka and Yancey's [13, Theorem 3], also reproduced as Theorem 4 in [14]. We acknowledge the researchers cited above for posing and developing the forbidden-cycle questions, and Cohen-Addad and coauthors for their disproof of Steinberg's conjecture. The formalization uses Lean 4 and its standard library and Mathlib, developed by their contributors [31, 32]. The

SAT certificate was generated with CaDiCaL [16] and checked with Wetzler, Heule and Hunt's DRAT-trim [15].

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