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Smooth convergents and the size of the next denominator

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Abstract

Suppose an irrational ξ has infinitely many continued-fraction convergents P/q for which $|P|$, q, and the next denominator Q have all their prime factors in a fixed finite set S. Marques proved that $\log Q \gg q^c$ at those indices for some $c > 0$. When S consists of two primes, the bound holds with $c = 1$, which is sharp. When S consists of three primes, the supremal exponent is at most $1/2$. In every fixed open subinterval of $(1, 2)$, the minimum of $\log Q$ over ξ and P/q in the interval has order $\sqrt{q}$ at infinitely many powers q of each prime in S. Let N_q be the least S-smooth integer greater than 1 congruent to $\pm 1 \pmod q$. For each S with at least three primes, a single irrational realizes $PQ = N_q$ infinitely often.

1 The size of the next denominator

Let p_n/q_n be the regular continued-fraction convergents of a real irrational ξ, and fix a finite set S of primes. In 1939 Erdős and Mahler proved that if q_{n-1}, q_n, q_{n+1} all have their prime factors in S for infinitely many n, then ξ is a Liouville number (its irrationality exponent is infinite). They also considered the primes dividing $|p_n|q_n$. If these all lie in S for infinitely many n, they observed that ξ must be transcendental and asked whether it must be Liouville [4, pp. 17–18].

In 2017 Lelis and Marques proved the conjecture under a spacing hypothesis on the selected indices [5, Theorem 1.1]. Bugeaud and Nguyen still listed the general case as open in 2023 [2, Question 4.1]. In a November 2026 preprint Marques disproved the conjecture by constructing a number of irrationality exponent 4 with infinitely many convergents of the form $2^a/3^b$ [7].

In an August 2026 paper Marques also imposed the prime restriction on the immediate next denominator. Write $P/q = p_n/q_n$ and $P'/Q = p_{n+1}/q_{n+1}$. Call the transition *S-smooth* if all prime factors of $|P|, q, Q$ belong to S. Marques proved that infinitely many such transitions force $\log Q \gg_{\xi,S} q^{c(S)}$ at those indices for an effective $c(S) > 0$ [6, Theorem 1.1]. Since the error in P/q lies between $1/[q(Q+q)]$ and $1/(qQ)$, this bound forces ξ to be Liouville.

How large can the exponent in Marques's bound be? Let $c_*(S)$ be the supremum of the $c \geq 0$ for which every such ξ satisfies $\log Q \gg_{\xi,S,c} q^c$ along its S-smooth transitions. We prove $c_*(S) = 1$ when S consists of two primes and $c_*(S) \leq \frac{1}{2}$ when S consists of three. In the two-prime case, q and $|P|Q$ are coprime, so each uses a different prime. The linear bound holds, and one irrational attains $\log Q \asymp q$ infinitely often. With three primes, a denominator using two primes still gives a linear bound. With ξ and P/q in any fixed open subinterval of $(1, 2)$, the minimum of $\log Q$ has order $\sqrt{q}$ at infinitely many powers $q = \ell^v$ of each denominator prime ℓ. Minima below $q^{1/2-\epsilon}$ at infinitely many other powers, for some fixed $\epsilon > 0$, would lower the exponent further. Corollary 4.3 identifies $c_*(S)$ with the least of the three lower-limit exponents.

Among continued fractions having $P/q = 65{,}536/59{,}049 = 2^{16}/3^{10}$ as a convergent, how small can the next denominator be? If Q is a power of 2, its least possible value has 5,921 decimal digits. Allow 5 as well and the minimum has just 19 digits. Section 5 computes both minima.

To study one denominator prime, fix ℓ and a nonempty finite set R of other primes. Write $\mathcal{M}(R)$ for the positive integers supported on R, including 1. Take $q = \ell^v > 2$ and $P, Q \in \mathcal{M}(R)$. The

determinant $|P'q - PQ| = 1$ gives $PQ \geq N_q$, where N_q is the least integer > 1 in $\mathcal{M}(R)$ congruent to ± 1 modulo q. For each nonempty open interval $I \subset (1,2)$, let $T_v(I)$ be the minimum of $\log Q$ as the irrational ξ and its convergent P/q both vary in I. An empty minimum has value $+\infty$. All logarithms are natural.

The implied constant in Marques's bound can depend on ξ, so sharpness requires infinitely many small successors in one continued fraction. A finite prefix specifies an interval of possible irrational continuations. Constructing the next transition inside that interval preserves the prefix and every transition already chosen.

Theorem 1.1. *The following hold, with positive constants c, C depending only on ℓ and R.*

(i) If $R = \{r\}$, every admissible pair with $P/q \in (1,2)$ satisfies $\log Q \geq cq$ for all large v. In every I, infinitely many v also satisfy $T_v(I) \leq Cq$.

(ii) If $R = \{r, s\}$, there is an infinite set V of levels depending only on the primes such that for every I and all sufficiently large $v \in V$,

$$c\sqrt{q} \;\leq\; T_v(I) \;\leq\; C\sqrt{q}.$$

The lower bound applies to every admissible pair at these levels with $P/q \in (1,2)$.

For each choice of primes there is a dense G_δ subset of the irrationals in $(1,2)$ whose continued fractions attain the indicated two-sided bounds at infinitely many indices.

For $|R| \geq 2$, Theorem 4.1 gives $T_v(I) \asymp_I \log N_q$ at every sufficiently large power, with the primes fixed. To attain $PQ = N_q$ repeatedly in one continued fraction, the minimizing product at each chosen power must have a divisor P for which P/q extends the prefix already chosen. Theorem 5.1 finds such divisors along powers where $\log\log N_q / \log q$ tends to its lower limit. For $q = 3^v$ and $R = \{2, 5\}$, it remains open whether one irrational can attain $\log Q \leq q^{1/2-\epsilon}$ at infinitely many powers outside V for some fixed $\epsilon > 0$.

Precedents. Erdős and Mahler also observed that all the convergent denominators of one irrational can be supported on two prescribed primes [4, p. 18]. Adiceam gives a recursion making successive denominators alternate between prime powers [1, proof of Theorem 2, preprint version]. The numerator at the selected index is unrestricted in that construction. Bugeaud and Nguyen prescribe prime supports at selected rational approximants [2, §4]. If $n_1 < n_2 < \cdots$ are the selected indices, Lelis and Marques bound the denominator $q_{n_{j+1}}$ at the next selected index [5, (2.1)]. Here $Q = q_{n_j+1}$ follows q_{n_j} immediately. For the $2^a/3^b$ convergents in Marques's counterexample, the largest prime factor of Q must tend to infinity by his neighboring-denominator theorem [6, Theorem 1.1]. Marques already constructs infinitely many pairs with $(P, q, Q) = (2^u, 3^v, 2^w)$ [6, p. 2]. The rank-one estimate below quantifies this construction. For two auxiliary primes the main additional ingredient is the geometry of planar ℓ-adic approximation lattices, as in de Weger [3].

2 From a congruence to consecutive convergents

Lemma 2.1. *Let $P/q \in (1,2)$ be reduced, with $q > 2$, and let $Q > q$ be an integer. There exists an irrational whose continued fraction has P/q as a convergent and Q as its next denominator if and only if $PQ \equiv \pm 1 \pmod{q}$. In that case the irrational numbers with a suitable finite prefix form the irrational points of an open interval within distance $1/(qQ)$ of P/q.*

Proof. Necessity follows from $|P'q - PQ| = 1$. Take the finite expansion of P/q whose last partial quotient is at least 2, and also the expansion obtained by splitting that quotient as $a = (a-1) + 1/1$. If the preceding denominator for the first expansion is h, that for the second is $q - h$. They give the two residues $\pm P^{-1} \pmod{q}$ by the determinant identity. Choose the expansion with preceding

denominator h' congruent to Q, and append the positive partial quotient $(Q-h')/q$. The recurrence gives a next convergent P'/Q. As the remaining complete quotient ranges over the irrationals greater than 1, the endpoints of the resulting cylinder are P'/Q and $(P'+P)/(Q+q)$. Their distances from P/q are $1/(qQ)$ and $1/(q(Q+q))$, respectively. □

For example, $4 \cdot 5 \equiv -1 \pmod 3$. The convergent $4/3 = [1;2,1]$ has preceding denominator 2. Appending $(5-2)/3 = 1$ gives $7/5$ as the immediate successor. Section 5 exhibits a second smooth pair inside this same cylinder.

Lemma 2.2. *For a fixed positive rational ℓ-adic unit $z \neq 1$, if $m \geq 1$ and $z^m \equiv \pm 1 \pmod{\ell^v}$, then $m \gg_{\ell,z} \ell^v$.*

Proof. Put $h = 2(\ell-1)$ and $e = \operatorname{ord}_\ell(z^h - 1)$. Here $e \geq 1$, and $e \geq 3$ when $\ell = 2$ since an odd unit squares to 1 modulo 8. By the binomial expansion, raising $1 + \ell^e u$, with u a unit, to an exponent prime to ℓ preserves the valuation of its difference from 1. Raising it to ℓ increases that valuation by one. Hence $\operatorname{ord}_\ell(z^{hm} - 1) = e + \operatorname{ord}_\ell(m)$. Since h is even, $v \leq e + \operatorname{ord}_\ell(m)$. □

If $R = \{r\}$, write $PQ = r^b$. Lemmas 2.1 and 2.2 give $b \gg q$, and $\log P \leq \log(2q)$ gives the lower bound in Theorem 1.1(i). For the converse let J be an open interval whose closure lies inside I. Infinitely many v admit $u \geq 0$ with $r^u/\ell^v \in J$, by the irrationality of $\log \ell / \log r$ and density of the tails of an irrational rotation. The least $b > 0$ with $r^b \equiv \pm 1 \pmod q$ lies between $c_0 q$ and $\varphi(q)$. For large such v take $P = r^u$, $Q = r^{b-u} > q$. Lemma 2.1 gives an entire cylinder in I with $\log Q \leq q \log r$.

For two distinct primes r, s we will use a stronger elementary observation. For *every* sufficiently large real y, a fixed such interval J contains $r^u s^w / y$ with $u, w \geq 0$ and $w \leq M_J$. To see this, shrink $J = (a, b)$ so that $\log(b/a) < \log r$. The positive multiples of $\log s$ are dense modulo $\log r$, so a fixed finite initial segment meets every translate of the open arc of length $\log(b/a)$. Hence some $w \leq M_J$ and integer u satisfy $\log(ay) < u \log r + w \log s < \log(by)$. Boundedness of w gives $u \geq 0$ for large y.

3 Powers with a square-root minimum

For $R = \{r, s\}$ let

$$\Lambda_v = \{(a, b) \in \mathbb{Z}^2 : r^a s^b \equiv \pm 1 \pmod{\ell^v}\}.$$

Negative powers are defined using modular inverses. The index D_v of Λ_v is the order of the subgroup generated by r, s in $(\mathbb{Z}/q\mathbb{Z})^\times / \{\pm 1\}$. The order of r alone is $\gg q$ by Lemma 2.2, so $D_v \asymp q$. Successive lattices satisfy $\Lambda_v \subseteq \Lambda_{v-1}$ and $\ell \Lambda_{v-1} \subseteq \Lambda_v$ for $v \geq 2$. The second follows by raising the defining congruence to the ℓth power.

Lemma 3.1. *There are infinitely many v at which the length of the shortest nonzero vector of Λ_v is $\gg \sqrt{q}$ and every point of $\mathbb{R}^2$ is within $O(\sqrt{q})$ of a point of Λ_v. The constants depend only on the three primes.*

Proof. Choose a shortest vector z_v. Among the integer points of the square $[0, \lceil \sqrt{D_v} \rceil]^2$, two have the same class modulo Λ_v. Thus $|z_v| \ll \sqrt{q}$. Its direction cannot be eventually constant. On a fixed rational line generated by a primitive (a, b), each nonzero vector of Λ_v is $m(a, b)$ with $|m| \gg q$, by Lemma 2.2 applied to $r^a s^b \neq 1$.

Take the infinitely many v for which z_v and z_{v-1} are not parallel. Both z_v and ℓz_{v-1} belong to Λ_v, and the absolute determinant of these two vectors is a positive multiple of D_v. Their lengths are $O(\sqrt{q})$, so the determinant lower bound forces $|z_v| \gg \sqrt{q}$. Rounding the two real coefficients in this pair of independent vectors puts any point of the plane within $(|z_v| + \ell |z_{v-1}|)/2$ of their integer span. □

The change of shortest direction has a direct precedent in de Weger's treatment of planar approximation lattices [3, Definition 2.1 and proof of Theorem 4.1(ii)]. The covering assertion lets us choose a congruence solution in the positive quadrant *after* specifying the numerator.

Corollary 3.2. *There is an effectively computable $K = K(\ell, r, s) > 1$ such that each interval $(v, Kv]$ of sufficiently large level indices contains a change of shortest direction.*

Proof. Let $u = (a, b)$ be a primitive vector on the shortest direction at v. Then $|u| \ll \ell^{v/2}$. With $h = 2(\ell - 1)$ put $e = \mathrm{ord}_\ell(r^{ha}s^{hb} - 1)$. Marques deduces a fixed-base estimate from Yu's theorem [8, §1.1, p. 190]. Applied to the signed exponents ha, hb, this estimate [6, Lemma 2.1] gives $e \ll \log(2 + |u|) \ll v$, effectively. If the same line contains a shortest vector mu at a later level w, the valuation calculation of Lemma 2.2 gives $\ell^{w-e} \le |m|$. Since $|mu| \ll \ell^{w/2}$ and $|u| \ge 1$, this forces $w \le 2e + O(1)$. Choosing K large enough forces a change by level $\lfloor Kv \rfloor$. □

Proof of Theorem 1.1(ii) and the category assertion. Take V from Lemma 3.1. For any admissible pair write $PQ = r^A s^B$. The nonzero vector (A, B) belongs to Λ_v. Both coordinates are nonnegative, so $\log(PQ) \ge \min(\log r, \log s)|(A, B)| \gg \sqrt{q}$. Subtracting $\log P \le \log(2q)$ proves the lower bound.

For the upper bound choose $P = r^u s^w$ with $P/q \in J$, where the closure of J lies inside I. Let $D\sqrt{q}$ be a uniform covering bound from the lemma. Some $(A, B) \in \Lambda_v$ is within $D\sqrt{q}$ of $(u + 2D\sqrt{q}, w + 2D\sqrt{q})$. Each of $A - u, B - w$ therefore lies in $[D\sqrt{q}, 3D\sqrt{q}]$. The integer $Q = r^{A-u}s^{B-w}$ satisfies $\log Q \asymp \sqrt{q}$ and $Q > q$ for large v. Since $PQ \equiv \pm 1$, Lemma 2.1 places a whole cylinder inside I. Only the threshold on v depends on I.

In either part (i) or part (ii), take the union, over the constructed cylinders at levels $v \ge k$, of their irrational points. This is open and dense in the irrationals of $(1, 2)$ for every k. Their intersection over k is dense by the Baire category theorem. Its elements attain the two bounds infinitely often and are Liouville, since $|\xi - P/q| < 1/(qQ)$ at these indices. One may also intersect over all choices of the primes, which form a countable family. □

Levels realized by any one irrational can be much sparser than the set V. If its next selected denominator after a square-root transition is $\ell^{v'}$, then $\ell^{v'} \ge Q$ and hence $v' \gg \ell^{v/2}$. Corollary 3.2 concerns the supply of available levels across all cylinders.

4 The minimum at an individual prime power

For $R = \{r_1, \dots, r_d\}$ define

$$H_v = \min\{\log N : N \in \mathcal{M}(R),\ N > 1,\ N \equiv \pm 1 \pmod{\ell^v}\}.$$

Thus $H_v = \log N_q$ for $q = \ell^v$. This is an optimization in *nonnegative* exponents. A short vector with coordinates of opposite sign need not give a competitor. The next result compares the positive minimum directly with the original continued-fraction problem, even at powers outside V.

Theorem 4.1. *If $d \ge 2$, then for each open interval $I \subset (1, 2)$ of positive length there exist constants $c, C > 0$ such that at every sufficiently large level v,*

$$cH_v \le T_v(I) \le CH_v.$$

Proof. For any admissible pair $N = PQ$ is a competitor, so $H_v \le \log P + \log Q \le \log 2 + 2\log Q$. Also $H_v \ge \log(q - 1)$. This proves the lower bound.

For the other direction we first find a competitor $L = \prod_i r_i^{a_i}$ with at least two positive exponents and $H_v \le \log L \ll H_v$. Take a minimizer. If it has a single prime factor r_i, Lemma 2.2 gives $H_v \gg q$.

Multiplying by $r_j^{\varphi(q)}$ for any $j \neq i$ preserves its residue and costs $O(q)$ in the logarithm. There are only finitely many possible i.

Choose a coordinate i with $a_i \log r_i \geq (\log L)/d$ and another positive coordinate j. As in Section 2 there exist $P = r_i^u r_j^w$ with $P/q \in J$, $u \log r_i \leq \log(2q)$, and $w \leq M$. Here J has closure in I, and M can be chosen for all the finitely many ordered pairs i, j. Fix an integer $K \geq \max(M, 4d)$. Then $Ka_i \log r_i \geq 4 \log(q-1) > \log(2q)$ for large q, and $Ka_j \geq w$. Consequently P divides L^K and $Q = L^K/P > q$. We have $PQ \equiv \pm 1 \pmod q$ and $\log Q \leq K \log L \ll_I H_v$ with the primes fixed. Lemma 2.1 puts the required cylinder inside I. □

Write $\alpha = \liminf_{v\to\infty} \log H_v / \log(\ell^v)$. For an irrational $\xi \in (1,2)$ let $\tau(\xi)$ be the lower limit of $\log\log q_{n+1}/\log q_n$ along indices with $q_n = \ell^v$ and $p_n, q_{n+1} \in \mathcal{M}(R)$. Assign value $+\infty$ if there are only finitely many such indices.

Corollary 4.2. *For $d \geq 2$, every ξ satisfies $\tau(\xi) \geq \alpha$, and equality holds on a dense G_δ subset of the irrationals in $(1,2)$. Moreover $0 < \alpha \leq 1/2$.*

Proof. The pointwise lower bound follows from $H_v \leq \log 2 + 2\log q_{n+1}$. The minimum H_v is at most $\varphi(q)\log r_1$, so α is finite. For every $\epsilon > 0$ there are arbitrarily large v with $H_v \leq q^{\alpha+\epsilon/2}$. In any fixed interval the proof of Theorem 4.1 supplies a cylinder at each sufficiently large such v with $\log Q \leq q^{\alpha+\epsilon}$. Take open unions over cylinders at arbitrarily late levels, then intersect as in the proof of Theorem 1.1, using $\epsilon = 1/j$ for $j \geq 1$. This proves equality on the indicated set.

Restricting the upper construction of Theorem 1.1(ii) to two primes in R gives $\alpha \leq 1/2$. For positivity use [6, Lemma 2.2] with $A = N$, $B = q$ and $S = R \cup \{\ell\}$, where N realizes H_v. Since $q \mid N^2 - 1$, that lemma gives $\log q \ll_S \log\log N = \log H_v$ for large v. □

Corollary 4.3. *For every two-prime set S, the optimal exponent in Marques's bound is $c_*(S) = 1$, and $\log Q \gg_{\xi,S} q$ holds along the S-smooth transitions. For every three-prime set S,*

$$c_*(S) = \min_{\ell\in S} \alpha_\ell, \qquad 0 < \alpha_\ell \leq \tfrac{1}{2},$$

where α_ℓ is the exponent of Corollary 4.2 for denominator prime ℓ and auxiliary primes $S \setminus \{\ell\}$.

Proof. Fix a real irrational ξ. At a sufficiently late S-smooth transition, $Q > q > 1$ and $1 \leq |P| \ll_\xi q$. The determinant gives $|P|Q \equiv \pm 1 \pmod q$, so $\gcd(q, |P|Q) = 1$. If $|S| = 2$, it follows that $q = \ell^v$ and $|P|Q = r^m$ for distinct primes $\ell, r \in S$. Lemma 2.2 gives $m \gg_S q$. Subtracting $\log|P|$ from $m \log r$ costs only $O_\xi(\log q)$, which proves the bound at 1. Conversely Theorem 1.1(i) gives one irrational with $\log Q \ll_S q$ infinitely often. Every exponent greater than 1 fails for this irrational.

Now let $|S| = 3$. If $q = p^i t^j$ uses two primes, then $|P|Q = r^m$ for the remaining prime r. For $u = p, t$ set $e_u = \mathrm{ord}_u(r^{2(u-1)} - 1)$. The valuation calculation in Lemma 2.2 gives $\mathrm{ord}_u(q) \leq e_u + \mathrm{ord}_u(m)$ for both primes. Hence $q \mid p^{e_p} t^{e_t} m$, so again $m \gg_S q$ and $\log Q \gg_{\xi,S} q$. A denominator using all three primes would force $Q = 1$.

It remains to consider $q = \ell^v$. Write $H_{v,\ell}$ for H_v with $R = S \setminus \{\ell\}$, and put $\gamma = \min_{\ell\in S} \alpha_\ell$. Given $0 \leq c < c' < \gamma$, the definition of the lower limit gives $H_{v,\ell} \geq q^{c'}$ for all sufficiently large v for each of the three choices of ℓ. Since $|P|Q$ is a competitor, $\log Q \geq H_{v,\ell} - \log|P| \gg_{\xi,S,c} q^c$. Thus every $c < \gamma$ is admissible. For the reverse inequality choose ℓ with $\alpha_\ell = \gamma$. Corollary 4.2 supplies a single irrational with $\liminf \log\log Q/\log q = \gamma$ on its transitions with this denominator prime. For every $c > \gamma$, the ratio $(\log Q)/q^c$ tends to zero along a subsequence. This proves the formula for the supremum. □

For three primes the bound at the supremum remains open in general. Even the numerical value of α_3 for $S = \{2, 3, 5\}$ is open. Take nonnegative integers a, b with $a + b > 0$. Can the least product $2^a 5^b \equiv \pm 1 \pmod{3^v}$ under this restriction have logarithm below $(3^v)^{1/2-\epsilon}$ for infinitely

many v, for some fixed $\epsilon > 0$? An affirmative answer would lower the exponent actually realized along a subsequence of consecutive convergents. A negative answer for every $\epsilon > 0$ would give $\alpha_3 = 1/2$. In de Weger's terminology the relevant 3-adic number is $\theta = -\log_3(25)/\log_3(4)$, where $\log_3$ denotes the 3-adic logarithm. Indeed $2^a 5^b \equiv \pm 1 \bmod 3^v$ is equivalent to $4^a 25^b \equiv 1 \bmod 3^v$. The 3-adic logarithm preserves the valuation of the difference from 1 on $1 + 3\mathbb{Z}_3$, and $\mathrm{ord}_3(\log_3 4) = 1$. Thus the congruence amounts to $\mathrm{ord}_3(a - b\theta) \geq v - 1$, with $a, b \geq 0$ and $a + b > 0$. Nonnegative exponents are necessary for a factorization into positive integers P, Q. Obtaining a divisor P of the right size is the remaining constraint.

5 Realizing the least product itself

Theorem 4.1 used a bounded power of a competitor so its prime factors could supply a numerator near q. Along a sequence of levels the minimizer itself has large exponents at two distinct primes. One exponent locates a numerator on the scale of q, and the other adjusts its position inside the prescribed interval I. Once $PQ = N_q$, we have $\log Q = H_v - \log P$. To minimize Q we therefore seek a divisor P as close as possible to $q \sup I$ from below, with P/q in I.

Theorem 5.1. *Let $d = |R| \geq 2$. There are levels $w_j \to \infty$, depending only on the primes, with $\log H_{w_j}/\log(\ell^{w_j}) \to \alpha$ and the following properties. Write $q = \ell^{w_j}$.*

(i) For every fixed open interval $I \subset (1, 2)$ of positive length,

$$T_{w_j}(I) = H_{w_j} - \log(q \sup I) + o_I(1).$$

(ii) A dense G_δ subset of the irrationals in $(1, 2)$ has infinitely many admissible transitions at these levels satisfying $PQ = \exp H_{w_j}$ exactly.

Proof. Write $q_v = \ell^v$, so $N_{q_v} = \exp H_v$. Unique factorization identifies each distinct minimizing product with a distinct vector of d nonnegative exponents. The costs H_v are nondecreasing and unbounded, and $H_{v+1} \leq \ell H_v$ because $N_{q_v}^{\ell}$ is admissible at the next level. Start at a sequence of v with $\log H_v/\log q_v \to \alpha$. We need enough distinct minimizers to force two exponents above a growing threshold D, but few enough changes that their total increase in the logarithmic cost $\log H_v$ is $o(v)$. Put $D = \lfloor v^{1/(2(d-1))} \rfloor$ and $B = d(D+1)^{d-1} = O_d(\sqrt{v})$. Take the first $B + 1$ *distinct* minimizers appearing from level v onward, including N_{q_v}. Each solves the signed congruence modulo q_v. At each change the cost increases by at most a factor ℓ, so every cost in this list is at most

$$\ell^B H_v = q_v^{\alpha + o(1)} = o(q_v),$$

using $\alpha \leq 1/2$ for the last equality.

If two exponent vectors in the list lay on the same line parallel to a coordinate axis, the quotient of their products would give a pure power $r_i^m \equiv \pm 1 \bmod q_v$ with $0 < |m| \log r_i = o(q_v)$, contradicting Lemma 2.2. All vectors with at most one coordinate greater than D are covered by B such lines. Each line is specified by choosing a free coordinate and fixing each of the other $d - 1$ coordinates in $\{0, \ldots, D\}$. For $d = 2$ these are the first $D + 1$ rows and columns of the quadrant. The $B + 1$ distinct vectors therefore include one with at least two coordinates greater than D. Let $w \geq v$ be a level with this minimizer. Its second-largest exponent tends to infinity along the chosen levels, and

$$\frac{\log H_w}{\log q_w} \leq \frac{\log H_v + B \log \ell}{\log q_v} = \alpha + o(1).$$

The definition of the lower limit gives convergence to α. Passing to a subsequence makes the chosen levels w_j strictly increasing.

Put $q = \ell^{w_j}$. By Corollary 4.2, $\alpha > 0$, so $H_{w_j} \geq q^{\alpha/2}$ for all large j. Choose a main coordinate of the minimizer contributing at least H_{w_j}/d to its logarithm. It can supply any exponent whose logarithmic cost is at most $\log(2q)$. At least one other coordinate exceeds D, which eventually exceeds the finite bound from the numerator construction in Section 2 for every ordered pair of primes in R. Hence for every interval J with closure inside I, all large j admit a *divisor* $P \mid N_q$ with $P/q \in J$. Set $Q = N_q/P$. Since $Q > q$ for large j, Lemma 2.1 gives a whole cylinder inside I. Intersecting the open dense unions of such cylinders at $j \geq k$ proves part (ii).

Finally every admissible pair in I satisfies $\log Q \geq H_{w_j} - \log(q \sup I)$. Fix $\delta > 0$ and take J with closure in $I \cap (e^{-\delta} \sup I, \sup I)$. The divisor just constructed gives an upper bound exceeding this lower bound by at most δ, for every sufficiently large j. This proves part (i). □

Two examples at the same modulus. First fix the convergent $P/q = 2^{16}/3^{10}$ from Section 1. For Q a power of 2, write $PQ = 2^m$. Since q is an odd prime power, $2^m \equiv \pm 1 \pmod{q}$ is equivalent to $4^m \equiv 1 \pmod{q}$. As $\operatorname{ord}_3(4^m - 1) = 1 + \operatorname{ord}_3(m)$, the first positive such m is $3^9 = 19{,}683$. The least Q is therefore $2^{19{,}667}$, with 5,921 decimal digits. With 5 also allowed, the least nontrivial product is $N = 2^{59}5^8$ (Figure 1). The source bundle's `checks/check_examples.py` enumerates all 1,352 products $2^a5^b \leq N$ with $a, b \geq 0$. Only 1 and N are ± 1 modulo q. The same P divides N, giving the least possible $Q = 2^{43}5^8 = 3{,}435{,}973{,}836{,}800{,}000{,}000$, with 19 digits. Both choices satisfy $Q > q$, so Lemma 2.1 realizes each as an immediate successor denominator of the specified convergent.

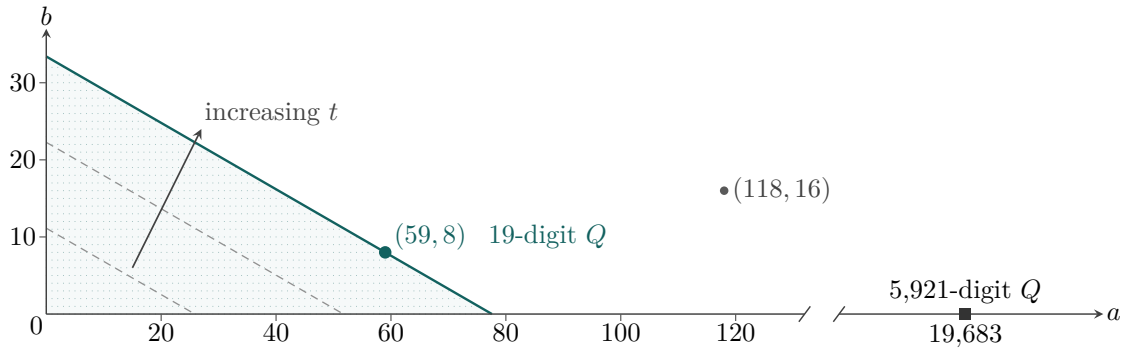


Figure 1. Minimizing $PQ = 2^a5^b$ at the fixed convergent $P/q = 2^{16}/3^{10}$. The fine dots represent the 1,352 pairs with $2^a5^b \leq 2^{59}5^8$. The large markers show nonzero solutions of $2^a5^b \equiv \pm 1 \pmod{q}$. As t increases, the line $a \log 2 + b \log 5 = t$ first reaches such a solution at $(59, 8)$. On $b = 0$ the first positive solution has $a = 19{,}683$, shown off scale.

The same minimum product also exhibits two smooth transitions of one continued fraction. The prefix $[1; 2, 1, 1]$ contains the consecutive convergents $4/3, 7/5$. Its cylinder is $(11/8, 7/5)$. Inside it lies $P/q = 81920/59049 = 2^{14}5/3^{10}$. Taking $Q = N/P = 2^{45}5^7$ and $P' = (N + 1)/q$ gives a second admissible pair in the very same continued fraction. Indeed the finite prefix

$$[1; 2, 1, 1, 2, 1, 1, 4, 31, 1, 3, 2, 1, 46550814906941]$$

has precisely $P/q, P'/Q$ for its last two convergents. Every infinite continuation therefore exhibits both pairs. For this fixed P/q the value of Q is minimal, since any smaller successor would give a smaller admissible product.

At $q = 59{,}049$, the two numerators 65,536 and 81,920 thus attain the same least product N_q. The second numerator is 25% larger, so its next denominator $Q = N_q/P$ is 20% smaller. In Theorem 5.1(i) the construction finds divisors with $P/(q \sup I) \to 1$, giving the endpoint term $-\log(q \sup I)$. Along the exact-product transitions of a fixed ξ in part (ii), $P/q \to \xi$, so $Q \sim N_q/(q\xi)$.

Verification

Two AI reviewers independently checked the proofs of Theorems 1.1, 4.1, and 5.1 in the 24 November 2026 working draft. A separate audit of the six-page version of that date confirmed the proofs, reviewed the exposition, and supplied the argument for Corollary 4.3.

At $q = 3^{10}$ the audit checker independently found $N_q = 2^{59}5^8$ using a table of powers of 2 modulo q and checked all 1,352 products through N_q. It reproduced the 5,921- and 19-digit minima at $P = 65{,}536$ and recovered all fourteen partial quotients of the nested example by applying the Euclidean algorithm to P'/Q. To reproduce the finite checks, run these commands using Python 3 from the project repository root or the source archive's `anc/` directory.

```
python3 checks/check_examples.py
python3 checks/check_parents.py
python3 checks/independent-checks.py
```

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