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ORBIT CLOSURES OF GENERIC RATIONAL POLYGONS WITH AT LEAST FIVE SIDES

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Abstract. Mirzakhani and Wright attach to the angles θ of a rational polygon the orbit closure $\mathcal{M}(\theta)$ of the unfolding of almost every polygon with these angles. They ask whether, for more than four sides, $\mathcal{M}(\theta)$ can be anything other than a connected component of a stratum of Abelian differentials or a locus of double covers of a component of a stratum of quadratic differentials. It cannot. For every choice of five or more angles, $\mathcal{M}(\theta)$ is the component of its stratum if the least common denominator k of the angles divided by π is odd, and the locus of holonomy double covers of the component of its quadratic stratum if k is even. Polygons in which some s of the angles sum to $(s-1)\pi$ are also treated by Apisa and Aygun, by a different method. We read the rank of $\mathcal{M}(\theta)$ off characters of the deck rotation of the unfolding and induct on the number of sides by colliding two vertices; where no collision applies, arithmetic modulo k takes its place. We also extend Mirzakhani and Wright's nonvanishing of the second fundamental form from triangles to every number of sides.

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1. Introduction

A billiard trajectory in a polygon whose angles are rational multiples of π unfolds to a straight line on a translation surface, the unfolding of the polygon [35]. By the theorems of Eskin, Mirzakhani and Mohammadi, the closure of its $\mathrm{GL}^+(2,\mathbb{R})$-orbit is an affine invariant submanifold [14, 15]. Mirzakhani and Wright attach to the angles θ of a rational polygon an

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This paper is Part IV of a series on orbit closures of unfoldings of rational polygons: Parts I [20] and II [19] treat triangles, Part III [18] quadrilaterals and Part IV polygons with at least five sides.

affine invariant submanifold $\mathcal{M}(\theta)$, the orbit closure of the unfolding of almost every polygon with these angles [30, Lemma 6.3]. Computing $\mathcal{M}(\theta)$ for all θ is one of their long-term goals [30, §1]. They ask whether, for polygons with more than four vertices, $\mathcal{M}(\theta)$ is ever anything other than a connected component of a stratum of Abelian differentials or a locus $\widetilde{\mathcal{Q}}$ of double covers of the quadratic differentials in a component $\mathcal{Q}$ of a stratum [30, Question 1.6]. The answer is no (Corollary 1.3). Parts I [20] and II [19] of this series treat triangles, Part III quadrilaterals [18], and this paper polygons with five or more vertices (Section 1.1).

A polygon with n vertices and angles $\pi q_1/k, \dots, \pi q_n/k$ in some cyclic order, where k is the least common denominator of the angles divided by π, has *type* $\theta = (q_1, \dots, q_n)/k$, and the q_i are its *weights*. So the weights are integers with $1 \leq q_i \leq 2k-1$, $q_i \neq k$, $\sum_i q_i = (n-2)k$ and $\gcd(k, q_1, \dots, q_n) = 1$. For distinct $z_1, \dots, z_n \in \mathbb{C}$ let X_z be the smooth compact curve and ω_z the form given by

$$y^k = \prod_{i=1}^{n} (w - z_i)^{q_i}, \qquad \omega_z = \frac{y\,dw}{\prod_i (w - z_i)}, \qquad T(y, w) = (\zeta y, w), \quad \zeta = e^{2\pi i/k}, \tag{1.1}$$

a cyclic cover of degree k of the sphere with deck rotation T. For z in a suitable open set of real configurations, (X_z, ω_z) is the unfolding of a polygon of type θ, and $\mathcal{M}(\theta)$ is the smallest affine invariant submanifold that contains every (X_z, ω_z) [30, §6]. We mark only the zeros of ω_z, not the regular points over vertices. If k is odd, the *ambient locus* $\mathcal{L}(\theta)$ is the connected component of the stratum of Abelian differentials that contains the (X_z, ω_z). If k is even, $\iota = T^{k/2}$ satisfies $\iota^* \omega_z = -\omega_z$, so (X_z, ω_z) is the holonomy double cover of $(X_z/\iota, \omega_z^2)$, and $\mathcal{L}(\theta)$ is the locus $\widetilde{\mathcal{Q}}$ for the component $\mathcal{Q}$ of the stratum of quadratic differentials containing these quotients. In both cases $\mathcal{M}(\theta) \subset \mathcal{L}(\theta)$ (Section 2).

We call θ *open* if no set S of vertices has $\sum_{i \in S} q_i = (|S| - 1)k$, that is, if the exterior angles $\pi - \pi q_i/k$ at no set of vertices sum to π. Equivalently, no surface in the stratum $\Omega^k \mathcal{M}_0(q_1 - k, \dots, q_n - k)$ of k-differentials on the sphere, whose holonomy covers are the X_z, has a Euclidean (isometrically embedded) cylinder [3, Lemma 7.5]. A type of pentagons is open exactly when no two of its angles sum to π. For quadrilaterals this is the condition that Part III calls "not supplementary". Part III calls a type of quadrilaterals open when it is moreover reduced and not one of Delecroix and Rüth's seven exceptional types [18, §4.6], so these seven types are open in our sense but not in Part III's.

Theorem 1.1. *Let $n \geq 5$ and let θ be an open type of n-gons. Then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$, so the unfolding of almost every polygon of type θ has orbit closure $\mathcal{L}(\theta)$.*

Theorem 1.2 (Apisa and Aygun [3, Theorem 1.2 and Corollary 1.3])**.** *Let $n \geq 5$ and let θ be a type of n-gons that is not open. Then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$.*

Corollary 1.3. *Let $n \geq 5$ and let θ be a type of n-gons. Then $\mathcal{M}(\theta)$ is a connected component of a stratum of Abelian differentials if k is odd, and the locus $\widetilde{\mathcal{Q}}$ for a connected component $\mathcal{Q}$ of a stratum of quadratic differentials if k is even. In particular, there is no θ as in the second part of Question 1.6 of* [30].

Both theorems concern the generic member of a type. All angles of the regular pentagon are $3\pi/5$, and its unfolding has a closed orbit, its Veech group being a lattice [31, proof of Theorem 1.5]. By Theorem 1.1, almost every pentagon with these angles has an unfolding whose orbit is dense in the component of $\mathcal{H}(2, 2, 2, 2, 2)$ that contains it. In fact, as $k = 5$, the pentagons with these angles whose unfoldings have this component as orbit closure form an open dense set of full measure (Corollary 8.3).

In Corollary 1.3, $\widetilde{\mathcal{Q}}$ is formed from a connected component, as in Mirzakhani and Wright's definition [30, §1], and the component matters, since some of these strata are not connected (Section 7.3).

Apisa and Aygun's hypothesis, that some of the orders $q_i - k$ sum to $-k$, says that θ is not open. Before them, Apisa proved this for the types with at most two angles that are not multiples of $\pi/2$ [2, Theorem 1.11], which are never open. For $k = 2$, where every angle is $\pi/2$ or $3\pi/2$, it is due to Athreya, Eskin and Zorich with Chaika [5, Theorem C.1]. For $k = 3$, Theorems 1.1 and 1.2 are Mirzakhani and Wright's Theorem 1.3 [30]. Our method also proves Theorem 1.2, independently of Apisa and Aygun's argument (Section 7).

The two methods divide the work differently. Apisa and Aygun degenerate along a Euclidean cylinder, which exists exactly when θ is not open; they merge two singularities so that such a cylinder remains, and use it at four singularities and where a merge leaves a disconnected cover [3, Lemma 12.6 and Sublemmas 12.8 and 12.9]. We read the rank of $\mathcal{M}(\theta)$ off the set of characters of T whose eigenspaces are tangent to $\mathcal{M}(\theta)$ [30, §7], and induct on n by colliding two vertices, as Apisa [2, Sublemma 11.2] and Apisa and Aygun [3, §12] do. When two suitable vertices collide, the polygon degenerates to one with $n - 1$ vertices (Figure 1), and if $\mathcal{M} = \mathcal{L}$ for the limit type, Mirzakhani and Wright's boundary theorem puts into this set every character that is non-zero at all the weights. For an open type a collision may leave a disconnected cover, but only because a prime p dividing k divides every weight except the two colliding ones. With two such primes, arithmetic modulo k takes the place of the collision and puts every unit into the set.

Section 8 draws the consequences for billiards in almost every polygon P of a type with $n \geq 5$. The number of bands of periodic billiard trajectories in P of length at most L has a weak quadratic asymptotic in the sense of Eskin, Mirzakhani and Mohammadi [15, Theorem 2.12]. We identify its constant in terms of Siegel–Veech constants and the number of regular points over the vertices (Corollary 8.2). The polygons of the type whose unfoldings have smaller orbit closures form a finite union of closed real-analytic subsets of measure zero if $k \notin \{2, 3, 4, 6\}$, and a dense set if $k \in \{2, 3, 4, 6\}$ (Corollary 8.3). Any two distinct points of P, vertices included, are joined by a billiard path that avoids any given finite set of other points (Corollary 8.4).

The angles of a type lie in $(0, \pi) \cup (\pi, 2\pi)$, as in Mirzakhani and Wright's setting [30, §6]. An angle 2π changes the answer. The cyclic forms of the pentagon type $(1, 1, 2, 2, 12)/6$, whose last angle is 2π, generate a primitive invariant subvariety of dimension 6 [13, Theorem 1.6]. We do not know which types with an angle 2π have $\mathcal{M}(\theta) \neq \mathcal{L}(\theta)$.

Mirzakhani and Wright improve their lower bound for the rank of $\mathcal{M}(\theta)$ using the form

$$B_\omega(\alpha, \beta) = \frac{i}{2} \int_X \alpha\beta \frac{\bar{\omega}}{\omega}$$

on holomorphic forms, which computes the second fundamental form of the Hodge bundle [30, §7.1]. For $x \in \mathbb{Z}/k$ let $H^{1,0}_x(X_z)$ be the space of holomorphic forms on which T^* acts by ζ^x, and put $t(x) = \sum_i \{xq_i/k\}$, where $\{\cdot\}$ is the fractional part; then $H^{1,0}_x(X_z) \neq 0$ exactly when $t(-x) > 1$ [30, Lemma 6.1]. For triangles they show that $B_{\omega_z}(\alpha, \beta) \neq 0$ for some z, $\alpha \in H^{1,0}_a(X_z)$ and $\beta \in H^{1,0}_b(X_z)$ whenever $a + b \equiv 2 \pmod{k}$, $t(-a) > 1$ and $t(-b) > 1$, and they ask when this holds for $n > 3$ [30, Proposition 7.3 and Problem 7.4]. For $k > 2$ it always does.

Theorem 1.4. *Let $n \geq 3$, let θ be a type of n-gons with $k > 2$, and let $0 < a, b < k$ with $a + b \equiv 2 \pmod{k}$, $t(-a) > 1$ and $t(-b) > 1$. There are forms $\alpha_z \in H^{1,0}_a(X_z)$ and*

$\beta_z \in H_b^{1,0}(X_z)$, depending holomorphically on z, such that $B_{\omega_z}(\alpha_z, \beta_z) \neq 0$ for every z outside a proper real-analytic subset of the space of configurations.

For a type of triangles, Mirzakhani and Wright prove that $\mathcal{M}(\theta)$ has rank greater than $n-2$ when some unit a modulo k has $2a \not\equiv 2 \pmod{k}$, $t(-a) > 1$ and $t(-(2-a)) > 1$. They remark that this holds for every n once the second statement of their Proposition 7.3 does [30, Theorem 7.5], so by Theorem 1.4 it holds for every $n \geq 3$.

TABLE 1. What each part of the series proves. For a triangle, $\mathcal{M}(\theta)$ is the orbit closure of its unfolding.

Part	Vertices	Result
I [20]	3	Classifies the scalene triangles of rank one; for prime k, every scalene triangle has orbit closure the component of its stratum [20, Theorems 1.1 and 1.2].
II [19]	3	Every scalene triangle that is not on Delecroix and Rüth's list has orbit closure its ambient locus; with Part I, no triangle on the list other than $(3,4,13)/20$ does [19, Theorem 1.1 and Corollary 1.3].
III [18]	4	$\mathcal{M}(\theta) = \mathcal{L}(\theta)$ for every reduced type that is not one of Delecroix and Rüth's seven [18, Theorem 1.1]. For the types with two angles summing to π, Apisa and Aygun prove that the generic orbit closure is as big as possible [3, Corollary 1.3].
IV	≥ 5	$\mathcal{M}(\theta) = \mathcal{L}(\theta)$ for every type: Theorem 1.1, and Theorem 1.2 (also [3, Theorem 1.2]).

1.1. **The series.** Table 1 lists what each part of the series proves. Parts I and II write N for k, and Part III writes the angles of its quadrilaterals as $2\pi a_i/N$, so that N is k or $2k$. Delecroix and Rüth conjecture that their lists of exceptional triangles and quadrilaterals are complete [8, §4]. Part II proves this for triangles and Part III for the generic quadrilateral of each type, and Part III answers the first part of Mirzakhani and Wright's Question 1.6 [18, §1]. The right and isosceles triangles, the quadrilaterals with two right angles, and the quadrilaterals that are not reduced, the parallelograms and isosceles trapezoids other than rectangles, are Apisa's [2, Theorems 7.6, 7.7, 1.10 and 1.9]. For six of Delecroix and Rüth's seven quadrilateral types, $\mathcal{M}(\theta)$ is a primitive invariant subvariety of rank two and dimension 4 identified by Eskin, McMullen, Mukamel and Wright [13, Theorem 1.1 and §5]. With these results (see [19, §1] and [18, §1.1]), Theorems 1.1 and 1.2 leave two types whose $\mathcal{M}(\theta)$ is not known, the triangle $(3,4,13)/20$ and the quadrilateral $(2,2,3,13)/10$. Delecroix and Rüth show that the unfoldings of the second lie in the orbit closure of the unfolding of the first, and conjecture that this orbit closure has dimension 4 [8, Theorem 1.2 and Conjecture 1.1]. Delecroix announces a positive answer, from joint work with Kai Fu [6, pp. 48–49]; his slides of June 2025, on joint work with K. Fu and B. Zykoski, list this orbit closure with dimension 4 and rank 2 [7, slide 12].

Theorem 1.1 is false for $n = 4$: Delecroix and Rüth's seven quadrilateral types are open, and for the six of Eskin, McMullen, Mukamel and Wright $\mathcal{M}(\theta) \neq \mathcal{L}(\theta)$ [13, Theorem 1.1 and §5].

1.2. **The proof.** Fix a type θ of n-gons, $n \geq 5$, and a point $X = X_z$ of the family (1.1) outside a proper closed analytic subset of the configuration space (Section 2.2). The deck rotation T acts on the cohomology of X, and the monodromy over the configuration space acts irreducibly on each non-zero eigenspace H^1_x, $x \in \mathbb{Z}/k$ (Lemma 2.3). So the absolute part $p(V)$ of the tangent space V of $\mathcal{M}(\theta)$ at X is a sum of these eigenspaces, and the rank of $\mathcal{M}(\theta)$ is read off the set A of the characters x with $0 \neq H^1_x \subset p(V)$ (Proposition 2.4). The proof shows that $A = E$, where $p(T_X\mathcal{L}(\theta)) = \bigoplus_{x\in E} H^1_x$ (Lemma 2.2). This is Mirzakhani and Wright's frame [30, §7], used also in Parts I to III, and $\pm 1 \in A$ [30, Corollary 6.5]. For odd k, $A = E$ gives full rank, and Mirzakhani and Wright's theorem on orbit closures of full rank, with their Lemma 6.7, gives $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ [30, Theorem 1.1 and Lemma 6.7]. For even k a statement on relative cohomology is needed as well (Lemma 6.1).

Three rules constrain A (Section 3). Rule (i) says that $A = -A$, and that $uA = A$ for every unit u with $uA \cap A \neq \emptyset$ (Proposition 3.2); it is proved as Mirzakhani and Wright's Lemmas 7.6 and 7.10 [30]. Rules (ii) and (iii) come from differentiating a holomorphic eigenform along $\mathcal{M}(\theta)$ [22], as in Part III [18, Theorems 4.6 and 4.7]. Rule (ii) adds characters to A (Theorem 3.6), and rule (iii) says that if $k > 2$ is even and $A = E$, then V is the (-1)-eigenspace of ι^* on $H^1(X, \Sigma; \mathbb{C})$ (Proposition 3.8). Both rest on a count of exact forms that also proves Theorem 1.4.

The proof is carried by the collision of two vertices (Section 4). When two vertices i and j whose angles sum to more than π collide, the polygon degenerates to one with $n-1$ vertices, whose angle at the merged vertex is the sum of the two angles minus π (Figure 1). When this angle lies in $(0, \pi) \cup (\pi, 2\pi)$ and a condition on greatest common divisors holds, $\{i, j\}$ is a collision pair (Definition 4.1), and the unfolding degenerates to the connected unfolding of a type θ_{ij} with $n-1$ vertices. If $\mathcal{M}(\theta_{ij}) = \mathcal{L}(\theta_{ij})$, Mirzakhani and Wright's boundary theorem [29, Theorem 2.9] puts into A every character of E that is non-zero at three or more weights of θ_{ij} (Proposition 4.3). As $n - 2 \geq 3$ weights of θ_{ij} are weights of θ, this includes the set F of the characters of E that are non-zero at all the weights of θ. If θ is open, so is θ_{ij}, and for open types this hypothesis comes by induction on n from Part III. For $n = 5$, θ_{ij} is a quadrilateral, and Part III gives $\mathcal{M}(\theta_{ij}) = \mathcal{L}(\theta_{ij})$ when it is reduced and not one of the seven types [18, Theorem 1.1]; for open θ with $k \geq 11$ both conditions hold. Once $F \subset A$, rule (ii) and an average of fractional parts over the units (Lemma 5.3) give the rest of E, except for some pentagon types with $k \mid 30$. The pentagon types with $k \leq 10$ or $k \in \{15, 30\}$ we settle by computation (Section 1.3).

An open type may have no collision pair (Remark 5.7). Then each pair $\{i, j\}$ that satisfies the first two conditions of Definition 4.1 fails the third at a prime p that divides k and every weight other than q_i and q_j, and different pairs have different primes. Changing a character x by a multiple of k/p changes only the two terms $\{-xq_i/k\}$ and $\{-xq_j/k\}$ of $t(-x)$, which decides whether x has holomorphic eigenforms. We choose the digits of units x and b at two such primes so that rule (ii) puts the quotient b/x into A. These quotients give every unit, by the digit method of Part III [18, Lemma 6.6 and Proposition 6.7]; when $12 \mid k$ this needs a character that is not a unit (Proposition 5.5). From the units, rule (ii) and the average give the rest of E as before.

For the types that are not open, our proof of Theorem 1.2 follows the same plan (Section 7.2). The vertices split into two sets, each with exterior angles summing to π. For a unit x the fractional parts $\{-xq_i/k\}$ over each set sum to a positive integer; so every unit has holomorphic eigenforms, and the average has no exceptions. A collision pair to which the induction applies exists when $k \geq 3$ and $n \geq 6$, and when $k \geq 11$ (Proposition 7.2). The

pentagon types with $3 \le k \le 10$ are settled by computation, and for $k = 2$ the family (1.1) is already all of $\mathcal{L}(\theta)$.

Part III also provides the proof of the collision that we follow [18, Theorem 3.2], the means over fibers behind the average [18, Lemmas 6.2 and 6.3], and, for even k, the passage from ratios to units [18, Lemma 7.1] and the use of a character that is not a unit [18, proof of Proposition 7.4].

1.3. **Computations.** For the 700 pentagon types with $k \le 10$, open or not, and the 936 and 16,711 open ones with $k = 15$ and $k = 30$, we compute the smallest set of characters that contains ± 1 and the characters given by collisions, and is closed under the rules (i) and (ii), as in Part III [18, §4.6] after Mirzakhani and Wright [30, Theorem 7.14 and Appendix A]. For these types we find that it is E. A search of the pentagon types with $k \le 10$ finds the open ones that have no collision pair to which (d) of Definition 5.1 applies (Remark 5.7). The programs are deposited with their output and instructions for rerunning them [21].

2. Cyclic covers and their characters

Throughout, $\theta = (q_1, \dots, q_n)/k$ is a type of n-gons, $\mathrm{Conf} \subset \mathbb{C}^n$ is the space of n-tuples of distinct points, and $X = X_z$, $\omega = \omega_z$ are as in (1.1) for $z \in \mathrm{Conf}$. Let $\mathcal{M}_{\mathrm{cyc}}(\theta)$ be the set of all (X_z, ω_z); it is connected, and $\mathcal{M}(\theta)$ is the smallest affine invariant submanifold that contains it [30, §6]. Orbits and invariance are for $\mathrm{GL}^+(2,\mathbb{R})$. The set-up of this section is that of Part III for four points [18, §§2.1–2.2].

2.1. **Eigenspaces.** Let $g_i = \gcd(k, q_i)$ and $e_i = k/g_i$, so $e_i \ge 2$. Over z_i the curve X has g_i points, permuted cyclically by T, each with ramification index e_i, and ω has order $q_i/g_i - 1$ at each of them. As $\sum_i q_i \equiv 0 \pmod k$, the cover is unramified over ∞, where ω has neither zeros nor poles. So the zero set Σ of ω consists of the points over the z_i with $q_i \nmid k$; the points over the other z_i are regular and are not marked. By Riemann–Hurwitz the genus g of X satisfies

$$2g - 2 = (n-2)k - \sum_i g_i. \tag{2.1}$$

For $n \ge 5$ this gives $g \ge 2$, since $g_i \le k/2$ and so $2g - 2 \ge (n-4)k/2 > 0$.

For real numbers $\epsilon_1, \dots, \epsilon_n$ and an integer j with $\epsilon_i \equiv jq_i/k \pmod 1$ for every i, we write $\prod_i (w - z_i)^{\epsilon_i}$ for the meromorphic function $y^j \prod_i (w - z_i)^{\epsilon_i - jq_i/k}$ on X, on which T^* acts by ζ^j; this is Mirzakhani and Wright's shorthand [30, §6]. Every meromorphic function on X on which T^* acts by ζ^j is $S(w) \prod_i (w - z_i)^{\epsilon_i}$ with S rational, since its quotient by $\prod_i (w - z_i)^{\epsilon_i}$ is T-invariant. In this notation $\omega = \prod_i (w - z_i)^{q_i/k - 1} dw$.

For $x \in \mathbb{Z}/k$ let H^1_x, $H^1(X, \Sigma)_x$ and $H^{1,0}_x$ be the ζ^x-eigenspaces of T^* on $H^1(X; \mathbb{C})$, on $H^1(X, \Sigma; \mathbb{C})$ and on the space of holomorphic 1-forms, so that $\omega \in H^{1,0}_1$. Let $t_i(x) = \{xq_i/k\}$, so that $t(x) = \sum_i t_i(x)$, and let $n(x)$ be the number of i with $xq_i \not\equiv 0 \pmod k$. As $\sum_i xq_i \equiv 0 \pmod k$, $t(x)$ is an integer, and $t(x) + t(-x) = n(x)$, since $\{s\} + \{-s\}$ is 1 for $s \notin \mathbb{Z}$ and 0 for $s \in \mathbb{Z}$. We call $h(x) = t(-x) - 1$ the *height* of x. For $x \not\equiv 0$, $H^{1,0}_x$ is spanned by the forms

$$P(w) \prod_i (w - z_i)^{-t_i(-x)}\, dw, \qquad \deg P \le h(x) - 1, \tag{2.2}$$

so $\dim H^{1,0}_x = h(x)$ and $\dim H^1_x = h(x) + h(-x) = n(x) - 2$ [30, Lemma 6.1 and Corollary 6.2]. Also $H^1_0 = 0$, as $X/\langle T \rangle$ has genus zero. For $x \not\equiv 0$ we have $n(x) \ge 2$: $n(x) = 1$ is impossible as $t(x)$ is an integer, and $n(x) = 0$ would give $x \gcd(k, q_1, \dots, q_n) \equiv 0$. Let E be the set of x with $n(x) \ge 3$, odd if k is even, let F be the set of $x \in E$ with $n(x) = n$, and let U be the

set of units modulo k. Every unit x has $n(x) = n$, and is odd if k is even; so $U \subset F$, and in particular $\pm 1 \in F$.

Let $p : H^1(X, \Sigma; \mathbb{C}) \to H^1(X; \mathbb{C})$ be the restriction and $Z = \ker p$, the image of the coboundary δ from functions on Σ, and let Z_c be the ζ^c-eigenspace of T^* on Z. As p is onto and T-equivariant, for every c the sequence

$$0 \to Z_c \to H^1(X, \Sigma)_c \xrightarrow{p} H^1_c \to 0 \tag{2.3}$$

is exact. For $c \not\equiv 0$ the constant functions contribute nothing to Z_c, and the functions of character c on the g_i points over z_i form a line if $e_i \mid c$, that is if $cq_i \equiv 0$, and are zero otherwise. So for $c \not\equiv 0$, $\dim Z_c = |I(c)|$, where $I(c)$ is the set of i with $q_i \nmid k$ and $cq_i \equiv 0 \pmod k$; in particular $Z_x = 0$ for $x \in F$.

2.2. **Generic points and the ambient locus.** From now on $n \geq 5$. Part III proves the next lemma at four points for the automorphisms that multiply ω by a constant, where it needs the angles [18, Lemma 2.2]; at five or more points a count of dimensions suffices.

Lemma 2.1. *Let $n \geq 5$. For z outside a proper algebraic subset of* Conf*, every automorphism of X_z is a power of T.*

Proof. Let G be the automorphism group of $X = X_z$. It is finite, as $g \geq 2$, and it contains $\langle T \rangle$. Suppose that $G \neq \langle T \rangle$, and let $d = [G : \langle T \rangle] \geq 2$, $\psi : X \to X/G$ the quotient map and $f : X/\langle T \rangle \to X/G$ the induced map of degree d. As $X/\langle T \rangle$ is the w-sphere, X/G is a sphere. Let B be the number of branch values of ψ. Over a branch value s all points of X have stabilizers in G of the same order, greater than one, and this order is the product of the ramification indices of w and of f. So each point of $f^{-1}(s)$ is one of $z_1, \dots, z_n$ or a ramification point of f, and the critical values of f are among the branch values of ψ. Each z_i lies over a branch value, as the points of X over it are fixed by a non-trivial power of T. If r_s is the total ramification of f over s, then $f^{-1}(s)$ has $d - r_s$ points, at most r_s of which are ramification points, so at least $d - 2r_s$ of the z_i lie over s. Summing over the branch values and using $\sum_s r_s = 2d - 2$, we get $Bd - 4d + 4 \leq n$, so

$$B \leq 4 + \frac{n-4}{d} \leq \frac{n+4}{2} < n.$$

Fix d and B, and let R be the algebraic set of triples (f, s, z), where f is a rational function of degree d, $s = (s_1, \dots, s_B)$ is a tuple of distinct points of $\mathbf{P}^1$ that contains the critical values of f, and $z \in$ Conf has $f(z_i) \in \{s_1, \dots, s_B\}$ for every i. For fixed s, the functions f with critical values among the s_j form finitely many orbits of the Möbius group acting on the source, by Riemann's existence theorem, each of dimension 3, and given f and s there are finitely many z; so $\dim R \leq B + 3$. The Möbius group acting on the target acts on R by $\gamma \cdot (f, s, z) = (\gamma \circ f, \gamma(s), z)$, with trivial stabilizers and without changing z. So the image of R in Conf has dimension at most $B < n$, and its Zariski closure is a proper algebraic subset. By the first paragraph z lies in this image for some d and B, and there are finitely many pairs, as $|G| \leq 84(g-1)$ and g depends only on θ. □

Fix a proper algebraic subset $\mathcal{Z}$ of Conf as in Lemma 2.1. We call z, and X_z, *generic* if $z \notin \mathcal{Z}$ and (X_z, ω_z) is not in the self-crossing locus of $\mathcal{M}(\theta)$; then every automorphism of X_z is a power of T. As in Mirzakhani and Wright's proof of their Lemma 7.7 [30], the self-crossing locus does not contain $\mathcal{M}_{\mathrm{cyc}}(\theta)$: it is a finite union of affine invariant submanifolds of smaller dimension [15, p. 674, after Definition 1.2], closed and invariant, so if it contained $\mathcal{M}_{\mathrm{cyc}}(\theta)$ it would contain $\mathcal{M}(\theta)$. The periods of ω_z are holomorphic functions

of z in a local marking [30, proof of Lemma 6.3], and near each of its points the self-crossing locus is a finite union of linear subspaces in period coordinates; so the z with X_z in it form a proper closed analytic subset of Conf. So the set Conf$^\circ$ of generic z is the complement of a proper closed analytic subset of Conf, the union of $\mathcal{Z}$ and this one; it is open, dense and connected. At a generic point X there are a neighborhood W of $[\omega]$ in period coordinates and a single complex linear subspace

$$V = T_X\mathcal{M}(\theta) \subset H^1(X, \Sigma; \mathbb{C}) \qquad \text{with} \qquad \mathcal{M}(\theta) \cap W = V \cap W$$

[14, 15]. In the rest of the paper X is a generic point and V is this subspace, so that $\operatorname{rank}\mathcal{M}(\theta) = \frac{1}{2}\dim p(V)$.

If k is odd, $\mathcal{L}(\theta)$ is the component of the stratum that contains the connected set $\mathcal{M}_{\mathrm{cyc}}(\theta)$, so $\mathcal{M}(\theta) \subset \mathcal{L}(\theta)$. Let k be even. Then $\iota^*\omega = -\omega$, so ω^2 descends to a quadratic differential on X/ι, which is not a square, as ω is not ι-invariant. The orders of its zeros and poles depend only on θ, and $\mathcal{M}_{\mathrm{cyc}}(\theta)$ is connected, so all these quotients lie in one component $\mathcal{Q}$ of a stratum. The locus $\widetilde{\mathcal{Q}}$ of the holonomy double covers of the surfaces of $\mathcal{Q}$ is an affine invariant submanifold [4, §1], and $\mathcal{L}(\theta) = \widetilde{\mathcal{Q}}$. The double cover of the image of $(X/\iota, \omega^2)$ under $\gamma \in \mathrm{GL}^+(2,\mathbb{R})$ is $\gamma(X, \omega)$, so $\mathcal{L}(\theta)$ contains the orbits of the surfaces of $\mathcal{M}_{\mathrm{cyc}}(\theta)$, and $\mathcal{M}(\theta) \subset \mathcal{L}(\theta)$. At a generic X, ι is the only automorphism with $\iota^*\omega = -\omega$, by Lemma 2.1, as it is the only element of order two in $\langle T\rangle$. Let

$$H^1(X, \Sigma; \mathbb{C})^- = \bigoplus_{c \text{ odd}} H^1(X, \Sigma)_c$$

be the (-1)-eigenspace of ι^*, which acts on $H^1(X, \Sigma)_c$ by $(-1)^c$. Lanneau's chart [24, §1.1.1] identifies a neighborhood of $(X/\iota, \omega^2)$ in $\mathcal{Q}$ with an open subset of $H^1(X, \Sigma; \mathbb{C})^-$: a simple pole of the quotient lifts to a single regular point fixed by ι, which is not marked on X, and leaving it unmarked does not change the (-1)-eigenspace. As ι is the only automorphism of X that negates ω, near X the locus $\widetilde{\mathcal{Q}}$ is this one sheet, as in Part III [18, §2.4]. So $T_X\widetilde{\mathcal{Q}} = H^1(X, \Sigma; \mathbb{C})^-$, and $V \subset H^1(X, \Sigma; \mathbb{C})^-$.

Lemma 2.2. *At a generic point X, $p(T_X\mathcal{L}(\theta)) = \bigoplus_{x\in E} H^1_x$. So* $\operatorname{rank}\mathcal{L}(\theta) = \frac{1}{2}\sum_{x\in E}(n(x) - 2)$.

Proof. For odd k, $T_X\mathcal{L}(\theta) = H^1(X, \Sigma; \mathbb{C})$ and $H^1(X; \mathbb{C}) = \bigoplus_{x\not\equiv 0} H^1_x$, where $H^1_x = 0$ unless $n(x) \geq 3$. For even k, $p(H^1(X, \Sigma; \mathbb{C})^-) = \bigoplus_{c \text{ odd}} H^1_c$ by (2.3), and again only the c with $n(c) \geq 3$ contribute. □

2.3. **The absolute tangent space is a sum of eigenspaces.** Mirzakhani and Wright observe that irreducibility of the monodromy of the eigenspaces H^1_x over the configuration space would extend their Corollary 7.9 beyond triangles [30, remark after Corollary 7.9]. The proof of the next lemma follows McMullen [27, Proposition 5.1] and Menet and Nguyen's proof of their Lemma 7.10 [28], as Part III does for four points [18, Lemma 2.4]. Some of the points may have trivial local monodromy. Athira E V and Haridas allow such points too, and prove that the eigenspace of each non-trivial character is irreducible under the larger mixed braid group, whose braids may also exchange points with weights congruent modulo k [9, Theorem A]. When no two weights are congruent modulo k, their theorem is Lemma 2.3.

Lemma 2.3. *Let $n \geq 3$ and $x \in \mathbb{Z}/k$ with $n(x) \geq 3$. The local system over* Conf *with fiber H^1_x at z (for the surface X_z) is irreducible.*

Proof. Let $X^\circ \subset X$ be the complement of the points over $z_1, \dots, z_n$ and $Y = \mathbf{P}^1 \setminus \{z_1, \dots, z_n\}$. As the cover is unramified over ∞, $X^\circ \to Y$ is an unramified cyclic covering with group $\langle T \rangle$, so $H^1(X^\circ; \mathbb{C})_x \cong H^1(Y; L)$, where L is a local system of rank one on Y whose monodromy around z_i is $\lambda_i = \zeta^{\pm x q_i}$, with a sign fixed by the conventions. Then $\lambda_i \neq 1$ exactly when $x q_i \not\equiv 0$; we number the points so that this happens for $i \le r$, where $r = n(x) \ge 3$. Choose a base point $\star \in Y$ and loops $\gamma_1, \dots, \gamma_n$ around $z_1, \dots, z_n$ that generate $\pi_1(Y, \star)$ with $\gamma_1 \cdots \gamma_n = 1$, and let $\lambda : \pi_1(Y, \star) \to \mathbb{C}^\times$ be the monodromy character of L. Then $H^1(Y; L)$ is the space of crossed homomorphisms v, maps with $v(\gamma\gamma') = v(\gamma) + \lambda(\gamma) v(\gamma')$, modulo the coboundaries $v(\gamma) = (\lambda(\gamma) - 1)c$. A crossed homomorphism is determined by $v_i = v(\gamma_i)$, subject only to $\sum_i \lambda_1 \cdots \lambda_{i-1} v_i = 0$, and the coboundaries are the vectors $c\,(\lambda_i - 1)_i$.

Restriction $H^1(X; \mathbb{C}) \to H^1(X^\circ; \mathbb{C})$ is injective, as $H^1(X, X^\circ; \mathbb{C}) = 0$ by excision, and T-equivariant; its image consists of the classes that vanish on small loops around the punctures. Such a loop around a point over z_i maps to a conjugate of $\gamma_i^{e_i}$, on which a crossed homomorphism takes the value $(1 + \lambda_i + \cdots + \lambda_i^{e_i - 1}) v_i$ up to a non-zero factor. As $\lambda_i^{e_i} = 1$, this vanishes for $i \le r$, and it is $e_i v_i$ for $i > r$. So H^1_x is identified, compatibly with the Gauss–Manin connection over Conf, with W_x / B_x, where W_x is the space of crossed homomorphisms with $v_i = 0$ for $i > r$, of dimension $r - 1$, and $B_x \subset W_x$ is the line of coboundaries.

Let $1 \le i < r$. The full twist of z_i and z_{i+1} is a loop in Conf; it acts on $\pi_1(Y, \star)$ by sending γ_i and γ_{i+1} to $\gamma\gamma_i\gamma^{-1}$ and $\gamma\gamma_{i+1}\gamma^{-1}$ with $\gamma = \gamma_i\gamma_{i+1}$, and fixes the other loops. The rule gives $v(\gamma\gamma'\gamma^{-1}) = (1 - \lambda(\gamma'))v(\gamma) + \lambda(\gamma)v(\gamma')$, and pulling v back by this action, with $v(\gamma) = v_i + \lambda_i v_{i+1}$ and $\lambda(\gamma) = \lambda_i\lambda_{i+1}$, gives the crossed homomorphism

$$v' = v + \varphi_i(v) D_i, \qquad \varphi_i(v) = (\lambda_{i+1} - 1)v_i + (1 - \lambda_i)v_{i+1},$$

where D_i has entries λ_i and -1 in the positions i and $i + 1$ and zeros elsewhere. Here $D_i \in W_x$, and φ_i vanishes on B_x. The monodromy of the local system along the twist is this map or its inverse, up to a non-zero scalar coming from the choice of a lift of the base point to the cover, and these have the same invariant subspaces.

The vectors $D_1, \dots, D_{r-1}$ are linearly independent, so they span W_x. Moreover $\varphi_{i+1}(D_i) = 1 - \lambda_{i+2} \neq 0$ for $i + 2 \le r$, and $\varphi_{i-1}(D_i) = (1 - \lambda_{i-1})\lambda_i \neq 0$ for $i \ge 2$. Let $\Lambda \neq 0$ be an invariant subspace of W_x / B_x, and $v \in W_x$ with non-zero class in Λ. If $\varphi_i(v) \neq 0$ for some $i < r$, the class of D_i lies in Λ; applying the twists of the neighboring pairs in turn, so do the classes of all D_j, and Λ is everything. Otherwise $v_i / (\lambda_i - 1)$ is the same for all $i \le r$, and $v_i = 0$ for $i > r$, so $v \in B_x$, a contradiction. □

Proposition 2.4. *Let X be a generic point. Then $T^*V = V$, so $V = \bigoplus_c V_c$ with $V_c = V \cap H^1(X, \Sigma)_c$. Let*

$$A = \{x \in E : H^1_x \subset p(V)\}.$$

Then

$$p(V) = \bigoplus_{x \in A} H^1_x, \qquad \operatorname{rank} \mathcal{M}(\theta) = \frac{1}{2} \sum_{x \in A} (n(x) - 2), \qquad \pm 1 \in A, \tag{2.4}$$

and A is the same at every generic point. So $\operatorname{rank} \mathcal{M}(\theta) = \operatorname{rank} \mathcal{L}(\theta)$ if and only if $A = E$.

Proof. We follow Mirzakhani and Wright's proof of their Lemma 7.7 [30], as in Part III [18, Proposition 2.5]. Rotation by $e^{i\phi}$, $0 \le \phi \le 2\pi/k$, is a path in $\mathcal{M}(\theta)$ from (X, ω) to $(X, \zeta\omega)$, and T is an isomorphism from $(X, \zeta\omega)$ to (X, ω); so the path is a loop with monodromy T^* on $H^1(X, \Sigma; \mathbb{C})$, and as $\mathcal{M}(\theta)$ is the single subspace V near X, $T^*V = V$. So V is the sum of its intersections with the eigenspaces of the finite-order map T^*, and $p(V) = \bigoplus_c p(V_c)$ with $p(V_c) \subset H^1_c$. For even k, $V \subset H^1(X, \Sigma; \mathbb{C})^-$, so $p(V_c) = 0$ for even c.

Over Conf$^\circ$ the tangent spaces $T_{X_z}\mathcal{M}(\theta)$ form a flat subbundle of the local system with fibers $H^1(X_z, \Sigma_z; \mathbb{C})$, as near each generic point $\mathcal{M}(\theta)$ is one linear subspace and contains the nearby $X_{z'}$. As Conf$^\circ$ is the complement of a proper analytic subset of Conf, it is connected and $\pi_1(\text{Conf}^\circ) \to \pi_1(\text{Conf})$ is onto. Hence $p(V) \cap H^1_c$ is invariant under the monodromy of $\pi_1(\text{Conf})$, and it is 0 or H^1_c by Lemma 2.3 if $n(c) \geq 3$; if $n(c) \leq 2$, then $H^1_c = 0$. This gives the first formula, the second follows, and A does not depend on the generic point, as $p(T_{X_z}\mathcal{M}(\theta)) \cap H^1_c(X_z)$ is a flat subbundle over the connected Conf$^\circ$. Finally $H^1(X, \Sigma)_{\pm 1} \subset V$ [30, Corollary 6.5], and $\pm 1 \in F \subset E$. The last statement follows by Lemma 2.2. □

3. Rules from differentiating eigenforms

In this section, except in Section 3.4, $n \geq 5$, X is a generic point, and V, p and A are as in Proposition 2.4. The rules are those of Part III at four points [18, §4], which follow Mirzakhani and Wright [30, §7] and Part I [20, §§2–3]; what changes at n points is a count of degrees (Lemma 3.5).

3.1. **Galois conjugates.** Let κ be the field of affine definition of $\mathcal{M}(\theta)$, and $\mathcal{H}^1$ the flat bundle over $\mathcal{M}(\theta)$ with fiber $H^1(X'; \mathbb{C})$ at (X', ω'). Wright proves that $\mathcal{H}^1 = \bigoplus_\rho \mathcal{V}_\rho \oplus \mathcal{W}'$, where ρ runs over the field embeddings $\kappa \to \mathbb{C}$, $\mathcal{V}_{\text{Id}}$ has fiber $p(V)$ at X, $\mathcal{V}_\rho = \sigma(\mathcal{V}_{\text{Id}})$ for any automorphism σ of $\mathbb{C}$ extending ρ, acting on $H^1(X'; \mathbb{Q}) \otimes \mathbb{C}$ through the coefficients, and $\mathcal{W}'$ has no subbundle isomorphic to any $\mathcal{V}_\rho$; the $\mathcal{V}_\rho$ are simple and pairwise non-isomorphic [33, Theorem 1.5, Lemma 6.5 and its proof]. Filip proves that each $\mathcal{V}_\rho$ underlies a variation of Hodge structure [16, Theorem 7.2 and proof of Theorem 7.3]; so, as Mirzakhani and Wright use it [30, §7.1], at each point the fiber of $\mathcal{V}_\rho$ is the sum of its intersections with $H^{1,0}$ and $H^{0,1}$. For a unit u modulo k let σ_u be an automorphism of $\mathbb{C}$ with $\sigma_u(\zeta) = \zeta^u$.

Lemma 3.1. *Let u be a unit modulo k. Then $\sigma_u(p(V)) = \bigoplus_{x \in uA} H^1_x$, and this is the fiber at X of one of the summands $\mathcal{V}_\rho$. Two such conjugates of $p(V)$ are equal or meet only in 0.*

Proof. As T^* is defined over $\mathbb{Q}$, σ_u maps H^1_x onto H^1_{ux}, and the first statement follows from Proposition 2.4. As in Part I's proof of its Lemma 2.5 [20], used again for Part III's Lemma 4.1 [18], the restriction of σ_u to κ is a field embedding ρ, and $\sigma_u(p(V))$ is the fiber at X of $\mathcal{V}_\rho = \sigma_u(\mathcal{V}_{\text{Id}})$. If the fibers at X of two of the summands $\mathcal{V}_\rho$ meet in a non-zero vector, the intersection of the two summands is a non-zero flat subbundle of each, and as both are simple, both are equal to it. So two such conjugates of $p(V)$ are equal or meet only in 0. □

Proposition 3.2 (Rule (i))**.** *$A = -A$. If u is a unit modulo k and $A \cap uA \neq \emptyset$, then $uA = A$.*

Proof. $\mathcal{M}(\theta)$ is defined by real linear equations in period coordinates, so $p(V)$ is invariant under complex conjugation, which maps H^1_x onto H^1_{-x}; so $A = -A$. Also $uA \subset E$, as $n(ux) = n(x)$ and u is odd when k is even. If $x \in A \cap uA$, then H^1_x lies in $p(V)$ and in $\sigma_u(p(V))$, so these are equal by Lemma 3.1, and $uA = A$ by Proposition 2.4. These are Mirzakhani and Wright's Lemmas 7.6 and 7.10 [30], at n points. □

3.2. **Derivatives of eigenforms.** Classes in $H^1(X, \Sigma; \mathbb{C}) = H^1_c(X \setminus \Sigma; \mathbb{C})$ are represented by closed forms with compact support in $X \setminus \Sigma$, and classes in $H^1(X \setminus \Sigma; \mathbb{C})$ by closed forms on $X \setminus \Sigma$. By Poincaré–Lefschetz duality the pairing $\langle [\nu], [\vartheta] \rangle = \int_X \nu \wedge \vartheta$ between the two spaces is perfect. It is T-invariant, so it restricts to a perfect pairing between $H^1(X, \Sigma)_c$

and $H^1(X\setminus\Sigma;\mathbb{C})_{-c}$. If ϑ is closed on X, then $\langle v,[\vartheta]\rangle=\int_X p(v)\wedge\vartheta$. If ϑ is a meromorphic form on X, holomorphic on $X\setminus\Sigma$, and $f\in\mathbb{C}^\Sigma$, then, by Stokes' theorem,

$$\langle\delta f,[\vartheta]\rangle=-2\pi i\sum_{s\in\Sigma}f(s)\operatorname{Res}_s\vartheta. \tag{3.1}$$

Let (X_ϖ,ω_ϖ), $\varpi\in W$, be the marked surfaces with period coordinates ϖ, and $[\alpha]_{\mathrm{rel}}\in H^1(X,\Sigma;\mathbb{C})$ the relative class of a holomorphic form α on one of them. For $v\in H^1(X,\Sigma;\mathbb{C})$ we use Kahn and Wright's path $[\omega]+tv$ [22, Lemma 2.2]. Part I derives the next lemma from their formula for the derivative of the projection from a stratum to moduli space [22, Theorem 1.1, Corollary 2.3 and Remark 2.4]; its statement and proof use only that ω is holomorphic with zero set Σ.

Lemma 3.3 ([20, Lemma 3.1])**.** *Let $v\in H^1(X,\Sigma;\mathbb{C})$, and let $\varpi\mapsto\alpha_\varpi$ be a family of holomorphic 1-forms on the surfaces X_ϖ, of class C^1 along the path $[\omega]+tv$, with $\alpha_{[\omega]}=\alpha$. Let ϑ be a meromorphic 1-form on X, holomorphic on $X\setminus\Sigma$, with at most simple poles. Then $\langle\partial_v[\alpha_\varpi]_{\mathrm{rel}},[\vartheta]\rangle=\langle v,[\alpha\vartheta/\omega]\rangle$.*

The relative class of a holomorphic form α' on X pairs to zero with every such ϑ: it is represented by $\alpha'-d(\chi F')$, where χ is a cut-off function equal to 1 near Σ and F' is the primitive of α' near each $s\in\Sigma$ with $F'(s)=0$, and as $\alpha'\wedge\vartheta=0$,

$$\langle[\alpha']_{\mathrm{rel}},[\vartheta]\rangle=\sum_{s\in\Sigma}\lim_{\varepsilon\to0}\oint_{|w_s|=\varepsilon}F'\vartheta=0, \tag{3.2}$$

w_s being a local coordinate at s. We also use Filip's theorem that the relative class of a holomorphic form whose absolute class lies in the absolute part of a tangent space of $\mathcal{M}(\theta)$ lies in that tangent space [17, Theorem 4.2 and the paragraph after it], in the form recalled in Part III [18, §4.4]: if α' is holomorphic on X_ϖ, $\varpi\in\mathcal{M}(\theta)\cap W$, and $p[\alpha']\in p(V)$, then $[\alpha']_{\mathrm{rel}}\in V$.

Lemma 3.4. *Let u be a unit modulo k, $x\in uA$ with $h(x)\ge1$, and $\alpha\in H^{1,0}_x$. There is a holomorphic family α_ϖ of holomorphic 1-forms on the surfaces of $\mathcal{M}(\theta)\cap W$ with $\alpha_{[\omega]}=\alpha$, whose absolute classes lie in the fiber of the summand $\mathcal{V}_\rho$ of Lemma 3.1, such that $\langle\partial_v[\alpha_\varpi]_{\mathrm{rel}},[\beta]\rangle=0$ for every $v\in V$ and every $\beta\in H^{1,0}_b$ with $b\notin uA$.*

Proof. We follow Part I [20, Lemma 3.5]. Over $\mathcal{M}(\theta)\cap W=V\cap W$ the bundle $\mathcal{H}^1$ is the constant bundle $H^1(X;\mathbb{C})$, and the fiber of $\mathcal{V}_\rho$ is the constant space $\sigma_u(p(V))=\bigoplus_{c\in uA}H^1_c$. As $\mathcal{V}_\rho$ underlies a variation of Hodge structure, its intersections with the spaces of holomorphic forms of the surfaces X_ϖ form a holomorphic subbundle, whose fiber at X is $\bigoplus_{c\in uA}H^{1,0}_c\ni\alpha$; let α_ϖ be a holomorphic section of it through α. For $\varpi\in V\cap W$ we have $\langle[\alpha_\varpi]_{\mathrm{rel}},[\beta]\rangle=\int_X p[\alpha_\varpi]\wedge\beta$, and the cup product pairs H^1_c with H^1_b to zero unless $c\equiv-b$. As $-b\notin uA$ by Proposition 3.2, this vanishes on $V\cap W$, and differentiating along $v\in V$ gives the claim. □

The next lemma is the count of exact forms; for $n=4$ it is Part III's Lemma 4.5 [18], whose proof we follow.

Lemma 3.5. *Let $n\ge3$, $x'\in F$, $D\ge0$ an integer, and $\epsilon_1,\dots,\epsilon_n$ real numbers with $\epsilon_i\equiv-x'q_i/k\pmod1$ and $\sum_i\epsilon_i=-D-2$. Let $\mathcal{R}$ be a space of polynomials of degree at most D such that, for $R\in\mathcal{R}$, the form $\psi_R=R(w)\prod_i(w-z_i)^{\epsilon_i}dw$ is holomorphic on $X\setminus\Sigma$. Then the $R\in\mathcal{R}$ for which ψ_R is exact on $X\setminus\Sigma$ form a subspace of dimension at most $\max(0,D+3-n)$.*

Proof. As $x' \in F$, no ϵ_i is an integer. Let $R \in \mathcal{R}$ with $\psi_R = d\Phi$, Φ holomorphic on $X \setminus \Sigma$. Near a point of Σ, ψ_R is meromorphic with zero residue, so Φ is meromorphic on X. As $T^*\psi_R = \zeta^{-x'}\psi_R$, the function $\Phi' = \frac{1}{k}\sum_{m=0}^{k-1}\zeta^{mx'}(T^m)^*\Phi$ also has $d\Phi' = \psi_R$, and $T^*\Phi' = \zeta^{-x'}\Phi'$; so $\Phi' = S(w)\prod_i(w-z_i)^{\epsilon_i+1}$ with S rational, and S is holomorphic on $\mathbb{C}\setminus\{z_1,\dots,z_n\}$. If S had a pole of order $m \geq 1$ at z_i, then near the points over z_i, $d\Phi'$ would have the leading term $(\epsilon_i+1-m)(w-z_i)^{\epsilon_i-m}dw$ times a non-zero constant, non-zero as $\epsilon_i \notin \mathbb{Z}$, whereas ψ_R is $(w-z_i)^{\epsilon_i}dw$ times a holomorphic function of w. So S is a polynomial. The cover is unramified over ∞ and the points over ∞ are not in Σ, so Φ' is holomorphic there, where it behaves like $w^{\deg S+\sum_i(\epsilon_i+1)} = w^{\deg S+n-D-2}$; so $\deg S \leq D+2-n$. Hence ψ_R lies in the image of the map $S \mapsto d\big(S(w)\prod_i(w-z_i)^{\epsilon_i+1}\big)$ on the polynomials of degree at most $D+2-n$, a space of dimension $\max(0, D+3-n)$. □

Theorem 3.6 (Rule (ii))**.** *Let u be a unit modulo k, $x \in uA$ with $h(x) \geq 1$, $x' \in A \cap F$, and $b \equiv 1-x-x'$ with $h(b) \geq 1$. Then $b \in uA$.*

Proof. We follow the proof of Part III's Theorem 4.6 [18]. As $h(b) \geq 1$, $n(b) \geq 3$, and b is odd when k is even, as x and x' are; so $b \in E$. Suppose that $b \notin uA$. Let $\alpha \in H_x^{1,0}$ and $\beta \in H_b^{1,0}$, and take the family α_ϖ of Lemma 3.4. By Lemma 3.4 and Lemma 3.3 with $\vartheta = \beta$,

$$0 = \langle \partial_v[\alpha_\varpi]_{\mathrm{rel}}, [\beta]\rangle = \langle v, [\alpha\beta/\omega]\rangle \qquad \text{for every } v \in V. \tag{3.3}$$

By (2.2), $\alpha = P_\alpha\prod_i(w-z_i)^{-t_i(-x)}dw$ and $\beta = P_\beta\prod_i(w-z_i)^{-t_i(-b)}dw$ with $\deg P_\alpha \leq h(x)-1$ and $\deg P_\beta \leq h(b)-1$, so

$$\alpha\beta/\omega = P_\alpha P_\beta \prod_i (w-z_i)^{\epsilon_i}dw, \qquad \epsilon_i = 1 - \frac{q_i}{k} - t_i(-x) - t_i(-b).$$

Here $\epsilon_i \equiv (x+b-1)q_i/k \equiv -x'q_i/k \pmod 1$, and $\sum_i \epsilon_i = 2 - t(-x) - t(-b) = -D-2$ with $D = h(x)+h(b)-2 \geq 0$. The form $\alpha\beta/\omega$ is holomorphic off Σ, and the products $P_\alpha P_\beta$ span the polynomials of degree at most D, a space of dimension $D+1 > \max(0, D+3-n)$. By Lemma 3.5 some $\alpha\beta/\omega$ is not exact on $X\setminus\Sigma$: its class in $H^1(X\setminus\Sigma;\mathbb{C})_{-x'}$ is not zero. As $x' \in A\cap F$, $Z_{x'} = 0$ and $H^1_{x'} \subset p(V)$, so $V_{x'} = H^1(X,\Sigma)_{x'}$ by (2.3), and this space pairs perfectly with $H^1(X\setminus\Sigma;\mathbb{C})_{-x'}$. So some $v \in V$ has $\langle v, [\alpha\beta/\omega]\rangle \neq 0$, against (3.3). □

3.3. **The relative rule.** When k is even, $\mathcal{L}(\theta)$ has the relative directions Z_c, c odd, and the next theorem puts them into V.

Theorem 3.7. *Let $c \not\equiv 0$ with $Z_c \neq 0$, let $x \in A$ with $h(x) \geq 1$, and let $x' \equiv 1+c-x$. If $x' \in A\cap F$, then $Z_c \subset V$.*

Proof. We follow the proof of Part III's Theorem 4.7 [18], in which only the count of degrees in Steps 3 and 4 changes at n points. Suppose that $V \cap Z_c \neq Z_c$. The first paragraph of that proof and its Steps 1 and 2 hold at n points as written, with Section 2.1, Proposition 2.4, Lemmas 3.3, 3.4 and 3.5, (3.1) and (3.2) for their counterparts at four points. They give a meromorphic 1-form φ on X with $T^*\varphi = \zeta^{-c}\varphi$, at most simple poles, all in Σ, and residues $r \neq 0$, such that

$$\langle v, [\alpha\varphi/\omega]\rangle = 0 \qquad \text{for every } v \in V \text{ and every } \alpha \in H_x^{1,0}. \tag{3.4}$$

Step 3: the shape of φ. At a point over z_i, which is fixed by T^{g_i}, the residue of φ equals that of $(T^{g_i})^*\varphi = \zeta^{-cg_i}\varphi$, so it vanishes unless $cq_i \equiv 0$. Hence φ is holomorphic off the points over the vertices in $I(c)$, where it has at most simple poles. Write $\varphi = P_\varphi(w)\prod_i(w-z_i)^{-t_i(c)-[i\in I(c)]}dw$, with $[i \in I(c)]$ equal to 1 for $i \in I(c)$ and 0 otherwise, and P_φ rational. At a point over z_i, with local coordinate s, $s^{e_i} = w - z_i$, the form $(w-z_i)^\tau dw$

has order $e_i(\tau+1)-1$ when $e_i\tau\in\mathbb{Z}$; so the product has order at least 0 over z_i for $i\notin I(c)$, and -1 for $i\in I(c)$, and a pole of P_φ at z_i would give φ a pole of order at least 2, or a pole over a vertex outside $I(c)$. As φ is holomorphic over the other points of $\mathbb{C}$, P_φ is a polynomial, non-zero as $r\neq 0$, and holomorphy over ∞ gives $\deg P_\varphi\le L$ with

$$L=t(c)+|I(c)|-2.$$

As $t(c)$ is an integer sum of $n(c)$ numbers in $(0,1)$, $t(c)\le n(c)-1$, and $|I(c)|\le n-n(c)$; so $L\le n-3$.

Step 4: a non-zero product. With $\alpha=P_\alpha\prod_i(w-z_i)^{-t_i(-x)}dw$, $\deg P_\alpha\le h(x)-1$,

$$\alpha\varphi/\omega=P_\alpha P_\varphi\prod_i(w-z_i)^{\epsilon_i}dw,\qquad \epsilon_i=1-\frac{q_i}{k}-t_i(-x)-t_i(c)-[i\in I(c)],$$

with $\epsilon_i\equiv(x-c-1)q_i/k\equiv -x'q_i/k \pmod 1$ and $\sum_i\epsilon_i=2-(h(x)+1)-(L+2)=-D-2$, $D=h(x)-1+L$. The form is holomorphic on $X\setminus\Sigma$, and $\deg P_\alpha P_\varphi\le D$. As P_α ranges over the polynomials of degree at most $h(x)-1$, the products $P_\alpha P_\varphi$ form a space of dimension $h(x)$, and $h(x)>\max(0,D+3-n)=\max(0,h(x)+L+2-n)$, as $h(x)\ge 1$ and $L\le n-3$. By Lemma 3.5 some $\alpha\varphi/\omega$ has non-zero class in $H^1(X\setminus\Sigma;\mathbb{C})_{-x'}$. As in the proof of Theorem 3.6, $V_{x'}=H^1(X,\Sigma)_{x'}$ pairs perfectly with this space, against (3.4). Hence $Z_c\subset V$. □

Proposition 3.8 (Rule (iii))**.** *Let $k>2$ be even, and suppose that $A=E$. Then $V=H^1(X,\Sigma;\mathbb{C})^-$.*

Proof. We have $V\subset H^1(X,\Sigma;\mathbb{C})^-$ and $V=\bigoplus_{c\text{ odd}}V_c$ (Section 2.2 and Proposition 2.4). We show below that $Z_c\subset V$ for every odd c with $Z_c\neq 0$. Given this, let c be odd. If $n(c)\ge 3$, then $c\in E=A$, so $p(V_c)=H^1_c$, and with $Z_c\subset V_c$, (2.3) gives $V_c=H^1(X,\Sigma)_c$. If $n(c)=2$, then $H^1_c=0$ and $H^1(X,\Sigma)_c=Z_c\subset V$. So $V=H^1(X,\Sigma;\mathbb{C})^-$.

Let c be odd with $Z_c\neq 0$. Complex conjugation preserves V and maps Z_c onto Z_{-c}, so it suffices to show that Z_c or Z_{-c} lies in V. We apply Theorem 3.7, with $x'\in F\subset A$ and $x\equiv 1+c-x'$ in $E=A$ with $h(x)\ge 1$. If $n(c)\ge 3$, one of $h(c)$, $h(-c)$ is positive, as their sum is $n(c)-2$; replacing c by $-c$ we may assume $h(c)\ge 1$, and we take $x=c$ and $x'=1$.

Let $n(c)=2$, and number the vertices so that $cq_i\not\equiv 0$ exactly for $i=1,2$. Let $o=\gcd(c,k)$, which is odd. For $i\ge 3$, e_i divides c and k, hence o, so e_i is odd and at least 3, and q_i is divisible by the 2-part of k. As the e_i have least common multiple k, one of e_1,e_2 is even; as $q_1+q_2\equiv 0$ modulo the 2-part of k, both are. For $\tau\in\mathbb{Z}$ put $s(\tau)=\sum_{i\ge 3}\{\tau q_i/k\}$, which depends only on τ modulo o. Suppose that

$$\text{(a) } \tau\not\equiv -1 \pmod{e_i} \text{ for every } i\ge 3,\qquad \text{(b) } s(\tau)\ge 1.$$

Choose x' odd with $x'\equiv 1+\tau \pmod o$, and put $x=1+c-x'$. Then $x'\in F$: e_1,e_2 are even and do not divide x', and $e_i\nmid x'$ for $i\ge 3$ by (a). Also x is odd and $x\equiv-\tau \pmod o$, so $t(-x)=s(\tau)+\{-xq_1/k\}+\{-xq_2/k\}>s(\tau)\ge 1$, the last two terms being positive as e_1,e_2 are even; as $t(-x)$ is an integer, $h(x)\ge 1$, so $n(x)\ge 3$ and $x\in E$.

It remains to find τ. The orders e_i, $i\ge 3$, are $n-2\ge 3$ odd numbers. If none of them is 3, then (a) holds for $\tau=\pm 2$, as no e_i divides 3 or 1, and (b) for one of the two, as no e_i divides 2 and so $s(2)+s(-2)=n-2\ge 3$. If some e_i is 3 and at least two are not, then (a) holds for $\tau=\pm 3$, as no e_i divides 4 or 2, and (b) for one of the two, as $s(3)+s(-3)$ is the number of $e_i\neq 3$. If all are 3, take $\tau=1$, with $s(1)\ge(n-2)/3\ge 1$. Suppose finally that exactly one of them, $e=e_{i_0}$, is not 3. Then $e\ge 5$, at least two of the others are 3, and $q'=q_{i_0}/g_{i_0}$ is a unit modulo e. If $3\nmid e$, take $\rho\in\{e-1,e-2\}$ with $\rho\not\equiv -q' \pmod e$, and τ with $\tau\equiv 1 \pmod 3$ and $\tau q'\equiv\rho \pmod e$. If $3\mid e$, take $\rho\in\{e-1,e-2\}$ with $\rho\equiv q'$

(mod 3), and τ with $\tau q' \equiv \rho \pmod e$, so that $\tau \equiv 1 \pmod 3$. In both cases $\tau \not\equiv -1$ modulo 3 and modulo e, and $s(\tau) \geq \frac{2}{3} + \frac{e-2}{e} > 1$. □

3.4. **Proof of Theorem 1.4.** Here $n \geq 3$, $z \in \mathrm{Conf}$ is arbitrary, and the notation of Section 2.1 is used for X_z. Let a, b be as in Theorem 1.4, so $h(a), h(b) \geq 1$, and put $D = h(a) + h(b) - 2 \geq 0$. For $\alpha \in H_a^{1,0}$ and $\beta \in H_b^{1,0}$ let $\Phi = \alpha\beta/\omega$. It is meromorphic on X, holomorphic on $X \setminus \Sigma$, and $T^*\Phi = \zeta^{a+b-1}\Phi = \zeta\Phi$. Let $[\bar\omega]_{\mathrm{rel}} \in H^1(X, \Sigma)_{-1}$ be the relative class of $\bar\omega$, the complex conjugate of $[\omega]_{\mathrm{rel}}$. Then

$$B_\omega(\alpha, \beta) = \frac{i}{2}\int_X \Phi \wedge \bar\omega = -\frac{i}{2}\langle [\bar\omega]_{\mathrm{rel}}, [\Phi]\rangle. \tag{3.5}$$

Indeed, $[\bar\omega]_{\mathrm{rel}}$ is represented by $\bar\omega - d(\chi\bar F)$, with χ as in (3.2) and F the primitive of ω near each $s \in \Sigma$ with $F(s) = 0$; near a zero of ω of order m, $\bar F = O(|w_s|^{m+1})$ and $\Phi = O(|w_s|^{-m})$, so $\int_X d(\chi\bar F) \wedge \Phi = 0$ by Stokes' theorem, and $\int_X \bar\omega \wedge \Phi = -\int_X \Phi \wedge \bar\omega$.

Let $\alpha = w^{j_1}\prod_i (w - z_i)^{-t_i(-a)}dw$ and $\beta = w^{j_2}\prod_i (w - z_i)^{-t_i(-b)}dw$ with integers $0 \leq j_1 < h(a)$ and $0 \leq j_2 < h(b)$, which lie in $H_a^{1,0}$ and $H_b^{1,0}$ by (2.2). As in the proof of Theorem 3.6, $\Phi = w^{j_1+j_2}\prod_i (w - z_i)^{\epsilon_i}dw$ with $\epsilon_i \equiv q_i/k \pmod 1$ and $\sum_i \epsilon_i = -D - 2$, and every integer $0 \leq j \leq D$ is a sum $j_1 + j_2$ of this kind. Fix $z^0 \in \mathrm{Conf}$. By Lemma 3.5 with $x' = -1 \in F$, the polynomials R of degree at most D with $R\prod_i (w - z_i)^{\epsilon_i}dw$ exact on $X_{z^0} \setminus \Sigma$ form a subspace of dimension at most $\max(0, D + 3 - n) < D + 1$, so for some $j = j_1 + j_2 \leq D$ the form Φ is not exact on $X_{z^0} \setminus \Sigma$. With these j_1, j_2 let α_z, β_z be the forms above on X_z; they depend holomorphically on z. Let $\mathcal{B}(z) = B_{\omega_z}(\alpha_z, \beta_z)$ and $\Phi_z = \alpha_z\beta_z/\omega_z$.

On a ball $\mathcal{U} \ni z^0$ in Conf trivialize the cohomology by the Gauss–Manin connection. The classes $\Pi(z) = [\omega_z]_{\mathrm{rel}} \in H^1(X, \Sigma)_1$ and $\Psi(z) = [\Phi_z] \in H^1(X \setminus \Sigma; \mathbb{C})_1$ depend holomorphically on z, as the periods of ω_z and Φ_z do, and $\mathcal{B}(z) = -\frac{i}{2}\langle\overline{\Pi(z)}, \Psi(z)\rangle$ by (3.5). So $\mathcal{B}$ is real-analytic on the connected manifold Conf, and its zero set is a proper real-analytic subset unless $\mathcal{B} \equiv 0$. Suppose that $\mathcal{B} \equiv 0$. The function $(z, z') \mapsto \langle\overline{\Pi(z')}, \Psi(z)\rangle$ on $\mathcal{U} \times \mathcal{U}$ is holomorphic in z and in $\bar z'$ and vanishes on the diagonal $z = z'$, so it vanishes identically; in particular $\langle\overline{\Pi(z')}, \Psi(z^0)\rangle = 0$ for all $z' \in \mathcal{U}$.

Mirzakhani and Wright show that near each of its points $\mathcal{M}_{\mathrm{cyc}}(\theta)$ is an open subset of $H^1(X, \Sigma)_1$ in period coordinates [30, Lemma 6.4]. So the rank of Π equals $\dim H^1(X, \Sigma)_1$ at some point of Conf: otherwise, by Sard's theorem, the images of countably many balls covering Conf would have measure zero, and so would $\mathcal{M}_{\mathrm{cyc}}(\theta)$ near each of its points. The set where the rank is smaller is a proper analytic subset of Conf, so the rank is full at some point of $\mathcal{U}$, and $\Pi(\mathcal{U})$ contains a non-empty open subset of $H^1(X, \Sigma)_1$. Its complex conjugate spans $H^1(X, \Sigma)_{-1}$. So $\Psi(z^0)$ pairs to zero with $H^1(X, \Sigma)_{-1}$, and as the pairing with $H^1(X \setminus \Sigma; \mathbb{C})_1$ is perfect, Φ_{z^0} is exact on $X_{z^0} \setminus \Sigma$, against the choice of j. This proves Theorem 1.4.

4. Colliding two vertices

Definition 4.1. Two vertices $i \neq j$ of θ form a *collision pair* if $k < q_i + q_j < 3k$, $q_i + q_j \neq 2k$ and $\gcd(k, q_l : l \neq i, j) = 1$.

When z_i and z_j collide, X_z degenerates to the cover (1.1) of the type θ_{ij} with $n - 1$ vertices in which q_i, q_j are replaced by $q_i + q_j - k$. In the polygon, the two vertices merge into one whose angle is the sum of their angles minus π (Figure 1). The first two conditions put this angle in $(0, \pi) \cup (\pi, 2\pi)$, and the third keeps k as the least common denominator

and makes the limit connected. This section shows that if $\mathcal{M}(\theta_{ij}) = \mathcal{L}(\theta_{ij})$, every character of E that is non-zero at three or more weights of θ_{ij} lies in A (Proposition 4.3).

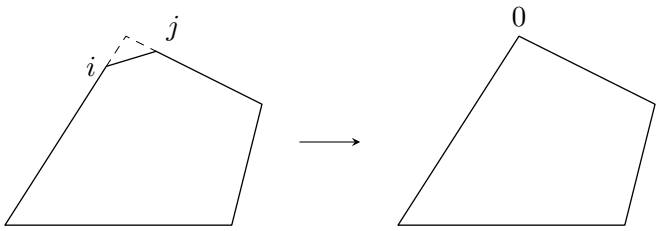

FIGURE 1. When two vertices i and j of a pentagon collide, and their angles sum to more than π, the pentagon degenerates to a quadrilateral; its angle at the merged vertex 0 is the sum of the two angles minus π. The figure shows adjacent vertices; the proof of Proposition 4.3 collides z_i and z_j in $\mathbb{C}$, with the other z_l fixed, for any collision pair.

Let $\{i, j\}$ be a collision pair of θ, $m = q_i + q_j - k$, and J the set of the other $n-2$ vertices. The type θ_{ij} has the weight m at a new vertex 0 and the weights q_l, $l \in J$. For $x \in \mathbb{Z}/k$ let $n_{ij}(x)$ be the number of weights of θ_{ij} at which x is non-zero, $\#\{l \in J : xq_l \not\equiv 0\} + [xm \not\equiv 0]$. A type of quadrilaterals is *reduced*, in Delecroix and Rüth's sense, unless its weights are a, a, b, b with $a < b$ [8, §4], [18, §2.1]; then $a + b = k$, so a type of quadrilaterals that is open is reduced. For the type $(3, 3, 3, 3, 3)/5$ of the regular pentagon, every pair is a collision pair, the merged vertex has weight $3 + 3 - 5 = 1$, and $\theta_{ij} = (1, 3, 3, 3)/5$ is reduced, open and not one of Delecroix and Rüth's seven types.

Lemma 4.2. *Let $\{i, j\}$ be a collision pair of θ.*

(a) *θ_{ij} is a type of $(n-1)$-gons with the same k.*
(b) *If θ is open, so is θ_{ij}.*
(c) *If $n = 5$, θ is open and $k \geq 11$, then θ_{ij} is reduced and is not one of Delecroix and Rüth's seven types.*

Proof. (a) We have $0 < m < 2k$ and $m \neq k$, the weights of θ_{ij} sum to $(n-3)k$, and $\gcd(k, m, q_l : l \in J) = 1$. (b) Let S be a set of vertices of θ_{ij} whose weights sum to $(|S| - 1)k$. If $0 \notin S$, then S is such a set for θ. If $0 \in S$, then $S \setminus \{0\} \cup \{i, j\}$ is one, its sum being larger by k. (c) By (b), θ_{ij} is open, so it is reduced. Part III writes a type of quadrilaterals as $(a_1, \dots, a_4)$, with angles $2\pi a_i/N$; so k is N or $N/2$, and q_i is $2a_i$ or a_i, as N is odd or even [18, §2.1]. Its Theorem 1.1 therefore applies to every type of quadrilaterals here that is reduced and not one of the seven types, and its two cases, N odd or $N \equiv 2 \pmod 4$, and $4 \mid N$, are k odd and k even. In our notation Delecroix and Rüth's seven types are

$$\tfrac{(1,1,1,7)}{5},\ \tfrac{(1,1,1,9)}{6},\ \tfrac{(1,1,2,8)}{6},\ \tfrac{(1,1,2,12)}{8},\ \tfrac{(1,2,2,11)}{8},\ \tfrac{(1,2,2,15)}{10},\ \tfrac{(2,2,3,13)}{10},$$

with $k \leq 10$. □

No two weights of any of the seven types sum to k, so all seven are open.

Proposition 4.3. *Let $n \geq 5$, let $\{i, j\}$ be a collision pair of θ, and suppose that $\mathcal{M}(\theta_{ij}) = \mathcal{L}(\theta_{ij})$. Then every $x \in E$ with $n_{ij}(x) \geq 3$ lies in A. In particular $F \subset A$.*

Colliding two vertices and passing to the boundary is a device of Apisa [2, Sublemma 11.3] and of Apisa and Aygun [3, Lemma 12.3 and §12]. Part III applies Mirzakhani and Wright's

boundary theorem eigenspace by eigenspace when the limit is the unfolding of a triangle [18, Theorem 3.2], and the proof below follows that one.

Proof. Write $\theta' = \theta_{ij}$ and E' for its set E.

The limit surface. The configurations that are not generic for θ form a null set, and so do the configurations z' of $n-1$ points for which the orbit closure of the surface (1.1) of θ' is smaller than $\mathcal{M}(\theta')$: they lie in the preimages of the countably many proper affine invariant submanifolds of $\mathcal{M}(\theta')$, which are proper analytic subsets [30, proof of Lemma 6.3]. By Fubini's theorem we can fix distinct z_0 and z_l, $l \in J$, such that the surface (Y,η) of θ' at $z' = (z_0, (z_l)_{l\in J})$ has orbit closure $\mathcal{N} = \mathcal{M}(\theta')$, and such that the configurations z that are generic for θ and have these z_l are dense in the plane of (z_i, z_j). So Y is the normalization of $y^k = (w-z_0)^m \prod_{l\in J}(w-z_l)^{q_l}$, with $\eta = y\,dw/((w-z_0)\prod_{l\in J}(w-z_l))$ and $T_Y(y,w) = (\zeta y, w)$; it is connected, as $\gcd(k, m, q_l : l \in J) = 1$. Let Σ_Y be the zero set of η, and let P be the set of points over z_0 if $m \mid k$, when they are regular, and $P = \emptyset$ otherwise. We mark $\Sigma_Y \cup P$ as an unordered set, as in [29, Definition 2.1]. By hypothesis $\mathcal{N} = \mathcal{L}(\theta')$, so, by the computation in the proof of Lemma 2.2 (for even k, along the sheet of $\mathcal{L}(\theta')$ at (Y,η) given by $T_Y^{k/2}$; for a reduced type of quadrilaterals of genus at least 2 it is Part III's Lemma 2.3 [18]),

$$2\operatorname{rank}\mathcal{N} = \sum_{c\in E'} \dim H^1(Y;\mathbb{C})_c. \tag{4.1}$$

By Riemann–Hurwitz, $2(g-g_Y) = k - g_i - g_j + \gcd(k,m) > 0$ for the genus g_Y of Y, as $g_i, g_j \le k/2$.

The boundary component. We follow the next three steps of the proof of Part III's Theorem 3.2 [18], up to its (3.4), with the $n-2$ points z_l, $l \in J$, in place of its two and m in place of its μ; its estimate near z_0 needs only $m > 0$, not $m < k$. By the choice of the z_l there are configurations $z^{(\nu)}$, generic for θ, with $z_l^{(\nu)} = z_l$ for $l \in J$ and $z_i^{(\nu)}, z_j^{(\nu)} \to z_0$. Let $X_\nu = X_{z^{(\nu)}}$ and $\omega_\nu = \omega_{z^{(\nu)}}$, with zero set Σ_ν, and $V^{(\nu)} = T_{X_\nu}\mathcal{M}(\theta)$. With radii $\varepsilon_\nu \to 0$ chosen as there, let D_ν be the union of the disc $|w-z_0| < \varepsilon_\nu$ and the discs $|w-z_l| < \varepsilon_\nu$ for the $l \in J$ with $q_l \nmid k$, let $U_\nu \subset Y$ be its preimage, and let $B_\nu \subset X_\nu$ be the part over the closure of D_ν. There are biholomorphic maps g_ν from $Y \setminus U_\nu$ onto the part of X_ν over $\mathbf{P}^1 \setminus D_\nu$, with $g_\nu \circ T_Y = T \circ g_\nu$, and $(X_\nu, \omega_\nu, \Sigma_\nu)$ converges to $(Y, \eta, \Sigma_Y \cup P)$ in the sense of Mirzakhani and Wright [29, Definition 2.2]. As $g_Y < g$, the limit lies in the boundary of $\mathcal{M}(\theta)$. Mirzakhani and Wright's collapse maps $f_\nu : X_\nu \to Y$ [29, Proposition 2.4 and §9.1] induce on relative homology the composite

$$H_1(X_\nu, \Sigma_\nu;\mathbb{C}) \to H_1(X_\nu, B_\nu;\mathbb{C}) \xrightarrow{\cong} H_1(Y, \overline{U_\nu};\mathbb{C}) \xleftarrow{\cong} H_1(Y, \Sigma_Y \cup P;\mathbb{C}), \tag{4.2}$$

where the middle map is induced by g_ν^{-1} and the last by inclusion. The first map is onto, as each component of B_ν contains a zero of ω_ν. Its kernel $\mathcal{V}_\nu$ is the space of vanishing cycles, and f_ν^* is an isomorphism from $H^1(Y, \Sigma_Y \cup P;\mathbb{C})$ onto the annihilator $\operatorname{Ann}(\mathcal{V}_\nu)$ [29, proof of Lemma 9.2]. By Mirzakhani and Wright's boundary theorem [29, Theorem 2.9(1)], there are an affine invariant submanifold $\mathcal{M}'$ of the stratum of $(Y, \eta; \Sigma_Y \cup P)$ that contains this surface, a tangent space L' of $\mathcal{M}'$ there, and infinitely many ν with

$$f_\nu^*(L') = V^{(\nu)} \cap \operatorname{Ann}(\mathcal{V}_\nu). \tag{4.3}$$

By Part III's Lemma 3.1(b) [18] if $P \ne \emptyset$, and because $\mathcal{N} \subset \mathcal{M}'$ if $P = \emptyset$, $\operatorname{rank}\mathcal{M}' \ge \operatorname{rank}\mathcal{N}$; so $\dim p(L') \ge 2\operatorname{rank}\mathcal{N}$.

Equivariance. The sets Σ_ν, B_ν, U_ν and $\Sigma_Y \cup P$ are invariant under the deck transformations, and $g_\nu \circ T_Y = T \circ g_\nu$; so each map in (4.2) commutes with them, $\operatorname{Ann}(\mathcal{V}_\nu)$ is T^*-invariant,

and $f_\nu^* \circ T_Y^* = T^* \circ f_\nu^*$. As X_ν is generic, $T^* V^{(\nu)} = V^{(\nu)}$, and $V^{(\nu)}$ lies in the sum of the eigenspaces of odd character if k is even (Section 2.2 and Proposition 2.4). By (4.3), L' is T_Y^*-invariant, and for even k it lies in the sum of the eigenspaces of odd character. So $p(L') = \bigoplus_c p(L'_c)$ with $p(L'_c) \subset H^1(Y;\mathbb{C})_c$, where $H^1(Y;\mathbb{C})_c = 0$ unless $c \in E'$ or, for even k, c is even; by (4.1), $p(L'_c) = H^1(Y;\mathbb{C})_c$ for every $c \in E'$.

Transfer. Let $x \in E$ with $n_{ij}(x) \geq 3$. Then $x \in E'$, so there is $\xi \in L'_x$ with $p(\xi) \neq 0$. Let γ be a closed curve in $Y \setminus (\Sigma_Y \cup P)$ with $\int_\gamma p(\xi) \neq 0$. For large ν, γ lies in $Y \setminus U_\nu$, and $(f_\nu)_*[g_\nu(\gamma)] = [\gamma]$ by (4.2). For ν as in (4.3), $f_\nu^* \xi \in V^{(\nu)}$ has character x, and its value on the absolute cycle $g_\nu(\gamma)$ is $\int_\gamma p(\xi) \neq 0$. So $p(V^{(\nu)}) \cap H^1(X_\nu;\mathbb{C})_x \neq 0$, and $x \in A$ by Proposition 2.4 at the generic point X_ν. If $x \in F$, then $n_{ij}(x) \geq n-2 \geq 3$. □

5. Units and characters

The rules of Section 3 and the collisions of Section 4 are statements about subsets of $\mathbb{Z}/k$, and this section works with them as such. For $M \mid k$ let $U_M = (\mathbb{Z}/M)^\times$, so $U = U_k$.

Definition 5.1. We call a set $A' \subset E$ *closed* if

(a) $1 \in A'$ and $A' = -A'$,
(b) $uA' = A'$ for every unit u with $A' \cap uA' \neq \emptyset$,
(c) $b \in uA'$ whenever u is a unit, $x \in uA'$ with $h(x) \geq 1$, $x' \in A' \cap F$, and $b \equiv 1 - x - x'$ has $h(b) \geq 1$, and
(d) for every collision pair $\{i,j\}$, with θ_{ij} reduced and not one of the seven types of Lemma 4.2 if $n = 5$, A' contains every $x \in E$ with $n_{ij}(x) \geq 3$.

In (c), $b \in E$, as in the proof of Theorem 3.6. The next lemma is Part III's Lemma 6.1 [18] at n weights.

Lemma 5.2. *Let A' be closed and $H = A' \cap U$.*

(a) *H is a subgroup of U that contains -1, and $H = \{u \in U : uA' = A'\}$.*
(b) *If x and b are units with $h(x) \geq 1$, $h(b) \geq 1$ and $1 - x - b \in A' \cap F$, then $b/x \in H$.*
(c) *If $x \in uA'$ for a unit u, and $v \in U$ has $vx = x$, then $v \in H$.*
(d) *If $U \subset A'$, then A' is a union of orbits of U.*

Proof. Part III's proof, in which (b) is the argument of Mirzakhani and Wright's Lemma 7.13 [30], uses only (a)–(c) of Definition 5.1 and applies as written. □

The next lemma finds units of positive height by averaging, as in Part III [18, Lemma 6.3], with the means computed there. For $m \geq 2$, Part III calls a *fiber* of m a non-empty set of the τ with $0 < \tau < m$ and $\gcd(\tau, m) = 1$ given by conditions of two kinds. If $3 \mid m$, then $\tau \equiv \tau_3 \pmod 3$ for a fixed $\tau_3 \in \{1,2\}$; and for the primes ℓ in a set of primes $\ell \geq 5$ dividing m, $\tau \not\equiv \tau_\ell \pmod \ell$ for a fixed $\tau_\ell \not\equiv 0$. Part III's $\varkappa(m)$ is the largest mean of $2\tau/m - 1$ over a fiber of m. Then

$$\varkappa(3) = \tfrac{1}{3}, \quad \varkappa(6) = \tfrac{2}{3}, \quad \varkappa(12) = \tfrac{1}{3}, \quad \varkappa(15) = \tfrac{7}{15}, \quad \varkappa(30) = \tfrac{8}{15}, \tag{5.1}$$

and $\varkappa(m) < \frac{1}{3}$ for every other $m \geq 2$ [18, Lemma 6.2].

Lemma 5.3. *Let $n \geq 5$, and let $y \in \mathbb{Z}/k$, even if k is even. The set C of units x with $y - x$ a unit is non-empty. If $h(x) = 0$ for every $x \in C$, then $n = 5$ and $k \mid 30$, or $n = 6$ and $k = e_1 = \cdots = e_6 = 6$.*

Proof. C is the set of x with $x \not\equiv 0$ and $x \not\equiv y$ modulo every prime $p \mid k$; at $p = 2$ the two conditions agree, as y is even, so $C \neq \emptyset$. For a unit x, $n(x) = n$ and $h(x) = t(-x) - 1 = n - 1 - t(x)$. Suppose that h vanishes on C. Then the means over C of the n functions $2t_i(x) - 1$ sum to $2(n-1) - n = n - 2$. Now $t_i(x) = \tau/e_i$ with $0 < \tau < e_i$ and $\tau \equiv x\,(q_i/g_i) \pmod{e_i}$, where q_i/g_i is a unit modulo e_i. By the Chinese remainder theorem C is a product over the primes $p \mid k$ of sets of units modulo the powers of p in k, each given by a condition modulo p. So $x \mapsto x\,(q_i/g_i) \bmod e_i$ maps C onto a fiber of e_i, or, if $3 \mid e_i$ and $3 \mid y$, onto the union of two fibers of the same size, one for each τ_3, and every point of the image has the same number of preimages. So the mean of $2t_i - 1$ over C is at most $\varkappa(e_i)$, and $\sum_i \varkappa(e_i) \geq n - 2$ with every term at most $\frac{2}{3}$ by (5.1). For $n \geq 7$ this is impossible, as $n - 2 > 2n/3$. For $n = 6$ it forces $\varkappa(e_i) = \frac{2}{3}$, that is $e_i = 6$, for every i. For $n = 5$, if the fourth largest of the $\varkappa(e_i)$ were less than $\frac{1}{2}$, the sum would be less than $3 \cdot \frac{2}{3} + 2 \cdot \frac{1}{2} = 3$; so four of the e_i have $\varkappa(e_i) \geq \frac{1}{2}$, and lie in $\{6, 30\}$. Finally, the least common multiple of any $n - 1$ of the e_i is k: if a prime power p^ν exactly dividing k divided none of them, p would divide $n - 1$ of the q_i, hence the last, as $\sum_i q_i = (n-2)k$, against $\gcd(k, q_1, \dots, q_n) = 1$. So $k \mid 30$ for $n = 5$ and $k = 6$ for $n = 6$. □

The next lemma is Part III's Lemma 6.6 [18] at n vertices; only the estimate of the height changes.

Lemma 5.4. *Let $\{i, j\}$ be a pair of vertices with $q_i + q_j > k$, and p an odd prime dividing k and every q_l with $l \neq i, j$. Let A' be closed and $H = A' \cap U$. Then H contains the kernel K_p of $U \to U_{k/p}$.*

Proof. Let $y \equiv 1 \pmod{k/p}$. For $l \neq i, j$, $p \mid q_l$ gives $t_l(-y) = t_l(-1) = 1 - \{q_l/k\} \geq 1 - q_l/k$, so

$$t(-y) \geq \sum_{l \neq i,j} \Big(1 - \frac{q_l}{k}\Big) = \frac{q_i + q_j}{k} > 1,$$

and as $t(-y)$ is an integer, $h(y) \geq 1$. Also $n(y) \geq n - 2 \geq 3$, and y is odd if k is even, so $y \in E$. With this in place of the first paragraph of Part III's proof, its second paragraph, which applies Lemma 5.2 and (c) of Definition 5.1 to $x = 1 + sk/p$ and $b = 2 - x$, both $\equiv 1 \pmod{k/p}$, gives $K_p \subset H$. □

Write $\Delta_i = 1 - q_i/k$ for the exterior angle at vertex i divided by π, which we call the *deficit* at i; so $0 < |\Delta_i| < 1$, $\sum_i \Delta_i = 2$, and a pair $\{i, j\}$ satisfies the first two conditions of Definition 4.1 exactly when $0 < |\Delta_i + \Delta_j| < 1$.

Proposition 5.5. *Let $n \geq 5$ and let θ be open; if $n = 5$, suppose that $k \geq 11$ and $k \nmid 30$. Then every closed set is E.*

Proof. Let A' be closed and $H = A' \cap U$. We first show that $U \subset A'$. If θ has a collision pair, then (d) of Definition 5.1 applies to it, by Lemma 4.2(c) if $n = 5$, and gives $F \subset A'$, so $U \subset A'$. Suppose that θ has no collision pair. If a pair $\{i, j\}$ satisfies the first two conditions of Definition 4.1, the third fails at a prime p that divides k and every q_l with $l \neq i, j$; p divides neither q_i nor q_j, for it divides $q_i + q_j$, and dividing both it would divide every weight. So p belongs to no other such pair, and different pairs have different primes.

Suppose first that $\Delta_a = \epsilon > 0$ and $\Delta_b = -\epsilon$ for some a, b. If a third deficit $\Delta_c > 0$ had $\Delta_c \neq \epsilon$, the pair $\{b, c\}$ would satisfy the first two conditions, and its prime would divide q_a, hence $q_b = 2k - q_a$; likewise for $\Delta_c < 0$, $\Delta_c \neq -\epsilon$, with $\{a, c\}$. So every deficit is $\pm\epsilon$. Let r be the number of positive deficits minus the number of negative ones; then $r\epsilon = 2$, so $r \geq 3$.

Every weight is $k \mp k\epsilon$, with $k\epsilon = k - q_a$ an integer, so $\gcd(k, k\epsilon) = \gcd(k, q_1, \dots, q_n) = 1$. If $r \geq 5$, two positive deficits sum to $4/r < 1$, and as all weights are prime to k their pair is a collision pair. If $r = 4$, two positive deficits sum to 1, and two weights sum to k, against openness. So $r = 3$, $k\epsilon = 2k/3$, $\gcd(k, k\epsilon) = k/3 = 1$, and θ is $(1, \dots, 1, 5, \dots, 5)/3$ with $v + 3$ weights 1 and v weights 5. Then $E = \{\pm 1\} \subset A'$, and we are done; for $n = 5$ this type has $k = 3$ and is excluded.

Suppose now that no two deficits sum to 0. If both signs occur, each of the at least $n - 1$ pairs of opposite signs satisfies the first two conditions. If all deficits are positive, a pair that does not has $\Delta_i + \Delta_j > 1$, since $\Delta_i + \Delta_j = 1$ would make $q_i + q_j = k$; two disjoint such pairs would have deficits summing to more than 2, so these pairs meet pairwise and lie in a star or a triangle, there are at most $n - 1$ of them, and at least $\binom{n}{2} - (n - 1) \geq n - 1$ pairs satisfy the first two conditions. In either case k has at least $n - 1 \geq 4$ prime factors, one for each such pair, and at least two of them are at least 5. Let $P < Q$ be two of these, with pairs π_P and π_Q, so $P \geq 5$ and $Q \geq 7$, and let $M = k/(PQ)$. The pairs have $q_i + q_j > k$, as $\Delta_i + \Delta_j < 1$, so by Lemma 5.4 and the Chinese remainder theorem H contains the kernel $K = K_P K_Q$ of $U \to U_M$.

Lifting. Let $x_0, b_0 \in \mathbb{Z}/M$ with x_0 a unit, $x_0 + b_0 \equiv 2 \pmod M$, $b_0 \not\equiv 0$ modulo P and Q if they divide M, and b_0 odd if k is even. We construct $x \equiv x_0$ and $b \equiv b_0 \pmod M$ with x a unit, $h(x) \geq 1$, $h(b) \geq 1$, and $x' = 1 - x - b$ a unit. Then $x' \equiv -1 \pmod M$, so $x' \in -K \subset A' \cap F$, and (c) of Definition 5.1 with $u = x$ gives $b \in xA'$, that is $b/x \in A'$. We use the digit method of Part III [18, proof of Proposition 6.7, Step 1]. With the other choices fixed, the lifts of x_0 are $x + sk/P$, $s \in \mathbb{Z}/P$, obtained by choosing a digit at P, and likewise at Q. Changing the digit of $y \in \{x, b\}$ at P changes $t_l(-y)$ only for $l \in \pi_P$, as P divides the other weights; for $l \in \pi_P$, $P \nmid q_l$, and the P digits give the values $\delta + s/P$, $0 \leq s < P$, of $t_l(-y)$ for some $0 \leq \delta < 1/P$. So if at most f digits are excluded, an admitted digit gives $t_l(-y) \geq (P - 1 - f)/P$. Choose a vertex $a \in \pi_P \setminus \pi_Q$ and a vertex $c \in \pi_Q$; the digits at Q do not change t_a. We choose the digit of x at P, then that of b at P, then that of b at Q, then that of x at Q, each time maximizing $t_a(-y)$ at P and $t_c(-y)$ at Q among the admitted digits. If $P \nmid M$, the digit of x at P must avoid $x \equiv 0$, one digit, and then that of b must avoid $b \equiv 0$ and $b \equiv 1 - x$, at most two digits; if $P \mid M$, all digits are admitted, as modulo P the residues of x, b and $1 - x - b$ are those of x_0, b_0 and -1. Likewise at Q, with b first. Hence

$$t(-x) \geq \frac{P-2}{P} + \frac{Q-3}{Q} \geq \frac{3}{5} + \frac{4}{7} > 1, \qquad t(-b) \geq \frac{P-3}{P} + \frac{Q-2}{Q} \geq \frac{2}{5} + \frac{5}{7} > 1,$$

so $h(x), h(b) \geq 1$, and x' is a unit.

The units. Taking b_0 a unit, the lifting puts b/x in H with image b_0/x_0 in U_M. So the image of H in U_M contains b_0/x_0 for all units x_0, b_0 with $x_0 + b_0 \equiv 2 \pmod M$. If k is odd, these ratios, $-1 \in H$ and $K \subset H$ give $H = U$, by Step 2 of the proof of Part III's Proposition 6.7 [18]. If k is even, then $k/M = PQ$ is odd. Part III's Lemma 7.1 [18] uses of A' only that H is a subgroup of U that contains -1, which holds by Lemma 5.2(a). With $K \subset H$ and these ratios it shows that H contains every unit $\equiv \pm 1 \pmod{\gcd(k, 12)}$, so $H = U$ unless $12 \mid k$. Let $12 \mid k$, and let 3^s exactly divide k, so that $3^s \parallel M$ and $4 \mid M$. As in the proof of Part III's Proposition 7.4 [18], where $9 \nmid k$, we use a character that is not a unit. Take $b_0 \equiv 0 \pmod{3^s}$ and $b_0 \equiv 1 \pmod{M/3^s}$, and $x_0 = 2 - b_0$. The lifting gives $\beta = b/x \in A'$ with $\beta \equiv 0 \pmod{3^s}$. The unit v with $v \equiv -1 \pmod{3^s}$ and $v \equiv 1 \pmod{k/3^s}$ has $v\beta = \beta$, so $v \in H$ by Lemma 5.2(c), and $v \equiv 5 \pmod{12}$, as $4 \mid k/3^s$. The units $\equiv \pm 1 \pmod{12}$ have index two in U, so $H = U$.

All of E. Now $U \subset A'$, so A' is a union of orbits of U by Lemma 5.2(d). Let $b \in E$. As $h(b) + h(-b) = n(b) - 2 \geq 1$ and $A' = -A'$, we may assume that $h(b) \geq 1$. If some unit x with $h(x) \geq 1$ has $x' = 1 - b - x$ a unit, then (c) with $u = 1$ gives $b \in A'$, as $x' \in U \subset A' \cap F$. By Lemma 5.3 with $y = 1 - b$, which is even if k is, such an x exists unless $n = 6$ and $k = e_1 = \cdots = e_6 = 6$, the case $n = 5$, $k \mid 30$ being excluded. In that case every q_i is prime to 6, so $E = F$; and θ has a collision pair, as without one $k = 3$ or k has at least five prime factors, by the above. So $F \subset A'$ by (d). □

Remark 5.6. When $n = 6$ and $k = e_1 = \cdots = e_6 = 6$, where the mean $\frac{2}{3}$ of (5.1) is attained at every e_i, the proof above uses (d) of Definition 5.1. For the open type $(1, 1, 1, 7, 7, 7)/6$, $E = \{1, 3, 5\}$ with $h(1) = 4$, $h(3) = 2$ and $h(5) = 0$, and the set $\{1, 5\}$ satisfies (a)–(c) of Definition 5.1: the only element with $h \geq 1$ is 1, and (c) with $x = 1$ and $x' \in \{1, 5\}$ gives $b \in \{5, 1\}$. So (d) is needed to reach the character 3, and it does, as the weights 1 and 7 form a collision pair, with $\theta_{ij} = (1, 1, 2, 7, 7)/6$, and $n_{ij}(3) = 4$.

When $12 \mid k$, let $\chi(u)$ be the product of the signs of a unit u modulo 3 and modulo 4; its kernel is the set of units $\equiv \pm 1 \pmod{12}$, of index two in U. If x, x', b are units with $x + x' + b \equiv 1 \pmod{k}$, exactly one of them is $\equiv -1 \pmod 3$ and an odd number are $\equiv -1 \pmod 4$, so $\chi(x)\chi(x')\chi(b) = 1$. So the units $\equiv \pm 1 \pmod{12}$ satisfy (a), (b), and (c) whenever x, x' and b are all units, and the step of the proof above that reaches every unit without a collision pair needs a character that is not a unit, the character β.

Remark 5.7. Collisions alone do not always reach the theorem for $n - 1$ vertices. At seven vertices the open type $(1, 1, 1, 1, 1, 5, 5)/3$ has the deficits $\pm\frac{2}{3}$ only, so no pair satisfies the first two conditions of Definition 4.1; here $E = \{\pm 1\}$. At six vertices, the proof of Proposition 5.5 shows that an open type with no collision pair has $k \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$, and this occurs. For example, $(770, 924, 990, 1050, 1155, 4351)/2310$ is open, the pairs that satisfy the first two conditions are the five that contain 4351, and the primes that divide k and the four other weights are $3, 5, 7, 11, 2$ for these pairs in the order of the displayed weights. At five vertices, a search of the open types with $k \leq 10$ finds three with no collision pair to which (d) of Definition 5.1 applies, $(1, 1, 1, 1, 5)/3$, $(1, 1, 1, 3, 9)/5$ and $(1, 1, 1, 4, 11)/6$; their collision pairs, if any, lead to Delecroix and Rüth's types. For $k \geq 11$ such a type has $k \geq 210$, by Lemma 4.2(c) and the proof of Proposition 5.5, and $(30, 42, 70, 105, 383)/210$ is one, with the primes $7, 5, 3, 2$ for its four pairs that contain 383. For such types with $k \geq 11$ the units come from rule (ii) at two primes.

6. Proof of Theorem 1.1

We first pass from the characters to the orbit closure. The induction on n below follows Apisa and Aygun's argument for their Theorem 1.2 in genus zero [3, §12] and Apisa's [2, Sublemma 11.2]; Part III gives its base case.

Lemma 6.1. *Let $n \geq 5$ and $k \geq 3$, and suppose that $A = E$. Then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$.*

Proof. Let k be odd. By Proposition 2.4 and Lemma 2.2, $p(V) = \bigoplus_{x \in E} H^1_x = H^1(X; \mathbb{C})$, so $\mathcal{M}(\theta)$ has full rank. By Mirzakhani and Wright's theorem it is a connected component of a stratum or the locus of hyperelliptic surfaces in one [30, Theorem 1.1], and by their Lemma 6.7 it does not lie in the hyperelliptic locus, as k is odd and $n > 4$ [30]. So $\mathcal{M}(\theta)$ is the component that contains X, which is $\mathcal{L}(\theta)$.

Let k be even. By Proposition 3.8, $V = H^1(X, \Sigma; \mathbb{C})^- = T_X\mathcal{L}(\theta)$, and by Section 2.2 Lanneau's chart maps a neighborhood of $(X/\iota, \omega^2)$ in $\mathcal{Q}$ onto an open subset of $H^1(X, \Sigma; \mathbb{C})^-$ that contains $[\omega]$. From here the second paragraph of Part III's proof of its Proposition 5.2

[18] shows that the points of $\mathcal{Q}$ near which the double-cover map takes values in $\mathcal{M}(\theta)$ form a non-empty open and closed subset of the connected $\mathcal{Q}$. So $\mathcal{L}(\theta) \subset \mathcal{M}(\theta)$, and $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. □

Proof of Theorem 1.1. Let θ be an open type of n-gons, $n \geq 5$. There is no type with $k = 1$. If $k = 2$, every weight is 1 or 3, and as the weights sum to $2(n-2)$, $(n+4)/2$ of them are 1; two weights 1 sum to k, so θ is not open. Hence $k \geq 3$. We argue by induction on n, and for $n \geq 6$ we assume Theorem 1.1 for the open types of $(n-1)$-gons.

Let X be a generic point and A as in Proposition 2.4. We show that A is closed. Conditions (a) and (b) of Definition 5.1 hold by Propositions 2.4 and 3.2, and (c) by Theorem 3.6. For (d), let $\{i, j\}$ be a collision pair, with θ_{ij} reduced and not one of the seven types if $n = 5$. By Lemma 4.2, θ_{ij} is an open type of $(n-1)$-gons. If $n = 5$, then $\mathcal{M}(\theta_{ij}) = \mathcal{L}(\theta_{ij})$ by Part III [18, Theorem 1.1] (see the proof of Lemma 4.2), and if $n \geq 6$, by the induction hypothesis. So Proposition 4.3 gives (d).

If $n \geq 6$, or $n = 5$, $k \geq 11$ and $k \nmid 30$, then $A = E$ by Proposition 5.5. The remaining types are pentagons with $k \leq 10$ or $k \in \{15, 30\}$. An intersection of closed sets is closed: for (b), if $\bigcap_\alpha A_\alpha$ meets $u \bigcap_\alpha A_\alpha$, then each A_α meets uA_α, so $uA_\alpha = A_\alpha$ for every α; the other conditions pass to intersections directly. As $uE = E$ for every unit u, E is closed; so there is a least closed set, and it lies in A. For the remaining types we find that the least closed set is E (Section 1.3). So $A = E$ in every case.

By Lemma 6.1, $\mathcal{M}(\theta) = \mathcal{L}(\theta)$, and Mirzakhani and Wright's Lemma 6.3 gives the statement for almost every polygon of type θ [30]. □

7. Types that are not open

Throughout this section θ is a type of n-gons, $n \geq 5$, that is not open, and S is a set of vertices with $\sum_{i \in S} q_i = (|S| - 1)k$. As $\sum_i q_i = (n-2)k$, its complement satisfies $\sum_{i \in S^c} q_i = (|S^c| - 1)k$; and $|S|, |S^c| \geq 2$, as a single vertex would have weight 0. In terms of the exterior angles, $\sum_{i \in S} \Delta_i = \sum_{i \in S^c} \Delta_i = 1$.

7.1. **From Apisa and Aygun's theorem.** Let $\mathcal{C} = \Omega^k \mathcal{M}_0(\kappa)$, with $\kappa_i = q_i - k$, be the stratum of the k-differentials $\xi = c \prod_i (w - z_i)^{\kappa_i} (dw)^k$, $c \in \mathbb{C}^\times$, on the sphere. The orders exceed $-k$, so these differentials have finite area, and as $\gcd(k, \kappa_1, \ldots, \kappa_n) = 1$, none is a power of a differential of lower order. The stratum is connected, being parametrized by c and the z_i. The k-th roots of ξ are the forms $c^{1/k}\omega_z$ on X_z, so the holonomy cover of ξ is $(X_z, c^{1/k}\omega_z)$. A set S' of the orders sums to $-k$ exactly when $\sum_{i \in S'} q_i = (|S'| - 1)k$, so the hypothesis of the second statement of Apisa and Aygun's Theorem 1.2 is that θ is not open, and they conclude that the orbit closure $\mathcal{M}_\mathcal{C}$ of these covers is as big as possible [3, Theorem 1.2]. For $n \geq 5$, a general configuration of n points of the sphere is preserved by no involution, so $\mathcal{C}$ is not hyperelliptic [3, proof of Lemma 8.6], and by their definition [3, §1.1] $\mathcal{M}_\mathcal{C}$ is the component of the stratum that contains it if k is odd, and the quadratic double that contains it if k is even. Their Corollary 1.3 gives the same for the unfolding of almost every polygon of type θ [3]. Whether or not the points over the vertices are kept as marked points, forgetting those that are not zeros of the form is continuous and commutes with $\mathrm{GL}^+(2, \mathbb{R})$; it maps the orbit closure of a marked surface into that of the unmarked one, and the locus just described onto $\mathcal{L}(\theta)$. So the unfolding of almost every polygon of type θ has orbit closure $\mathcal{L}(\theta)$, and $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ by Mirzakhani and Wright's Lemma 6.3 [30]. This is Theorem 1.2.

Before Apisa and Aygun, Apisa proved that the unfolding of almost every polygon with at least five sides and at most two angles that are not multiples of $\pi/2$ has dense orbit in the hyperelliptic locus that contains it [2, Theorem 1.11]. Such a type is not open, since its other $n-2 \geq 3$ angles are $\pi/2$ or $3\pi/2$ and all the angles sum to $(n-2)\pi$, so that two of them are $\pi/2$; and this locus is $\mathcal{L}(\theta)$, since X/ι is a cyclic cover of the sphere branched only over the at most two vertices whose angles are not multiples of $\pi/2$, so that it has genus zero and ι is the hyperelliptic involution of X.

7.2. **An independent proof.** The partition into S and S^c makes the arithmetic of Section 5 short.

Lemma 7.1. *Every unit x has $h(x) \geq 1$, and every closed set that contains U is E.*

Proof. Let x be a unit. Each $t_i(-x) = \{-xq_i/k\}$ is positive, as $q_i \not\equiv 0 \pmod{k}$, and $\sum_{i\in S} t_i(-x) \equiv -x(|S|-1) \equiv 0 \pmod{1}$; so this sum is a positive integer, and so is the sum over S^c. Hence $t(-x) \geq 2$, and $h(x) \geq 1$.

Let A' be closed with $U \subset A'$, and let $b \in E$. As $A' = -A'$ and $h(b)+h(-b) = n(b)-2 \geq 1$, we may assume that $h(b) \geq 1$. By Lemma 5.3, applied with $y = 1-b$, which is even if k is, there is a unit x such that $x' = 1-b-x$ is a unit. Then $x \in A'$ with $h(x) \geq 1$ and $x' \in U \subset A' \cap F$, and (c) of Definition 5.1 with $u = 1$ gives $b \in A'$. □

Proposition 7.2. *Suppose that $k \geq 3$, and that $n \geq 6$ or $k \geq 11$. Then θ has a collision pair $\{i, j\}$, with θ_{ij} reduced and not one of the seven types of Lemma 4.2 if $n = 5$.*

Proof. A pair $\{i, j\}$ with $i \in S$ and $j \in S^c$ satisfies the third condition of Definition 4.1: a prime that divides k and every q_l with $l \neq i, j$ divides q_i and q_j, by the sums over S and S^c, and so every weight. Recall that the first two conditions of Definition 4.1 say that $0 < |\Delta_i + \Delta_j| < 1$.

Let $n = 5$, so $k \geq 11$. We may take S to have two vertices, with weights p and $k-p$, $p \leq k/2$; let $r \leq s \leq t$ be the other weights, so $r+s+t = 2k$ and $t \geq 2k/3 > p$. A type of quadrilaterals is reduced unless its weights are a, a, b, b with $a < b$, and then $a + b = k$ [8, §4], [18, §2.1]; and as $k \geq 11$, it is not one of the seven types. So we need a collision pair whose θ_{ij} does not have this form.

First let $t \neq k+p$. The pair of weights $k-p$ and t has sum in $(k, 3k)$, not $2k$, so it is a collision pair, and θ_{ij} has the weights $t-p$, p, r, s. Suppose that these are a, a, b, b with $a < b$. Then $p \neq k/2$, the weights are $p, p, k-p, k-p$, and as $t \neq k$, $t = 2p$ and $r = s = k-p$. So $k/3 \leq p < k/2$, and $p = k/3$ would make $\gcd(k, q_1, \ldots, q_5) = k/3 > 1$. Now the pair of the two weights $k-p$ has sum $2k-2p \in (k, 4k/3)$, and the other weights p, $k-p$, $2p$ satisfy $\gcd(k, p, k-p, 2p) = \gcd(k, p) = 1$; so it is a collision pair, and its θ_{ij} has the four distinct weights $k-2p < p < k-p < 2p$.

Now let $t = k+p$, so that $r+s = k-p$. If $p < k/2$, take the pair of weights p and t, with sum $k+2p \in (k, 2k)$; its θ_{ij} has the weights $2p$, $k-p$, r, s. As $r+s < k$, these could be a, a, b, b only with $r = s = a$ and $2p = k-p = b$, so that $p = r = s = k/3$ and $\gcd(k, q_1, \ldots, q_5) = k/3 > 1$. If $p = k/2$, take the pair of weights s and $t = 3k/2$, with sum in $(3k/2, 2k)$ as $0 < s < k/2$. The other weights $k/2$, $k/2$, r have $\gcd(k, k/2, r) = \gcd(k/2, r)$, which divides $s = k/2 - r$ and t, so it is 1. This θ_{ij} has two weights $k/2$, so it is not a, a, b, b with $a < b$.

Let $n \geq 6$. If every Δ_l is positive, let $i \in S$ and $j \in S^c$ have the least deficits in their sets; then $0 < \Delta_i + \Delta_j \leq \frac{1}{|S|} + \frac{1}{|S^c|} \leq \frac{1}{2} + \frac{1}{4}$, as $|S|, |S^c| \geq 2$ and $|S| + |S^c| \geq 6$. Otherwise some Δ_a is negative, say with $a \in S$, and as the deficits over S^c sum to 1, some $b \in S^c$ has

$\Delta_b > 0$. Then $-1 < \Delta_a + \Delta_b < 1$, and $\{a, b\}$ is a collision pair unless $\Delta_a = -\Delta_b$, that is, $q_a + q_b = 2k$. In that case suppose first that some c has $|\Delta_c| \neq \Delta_b$. If $\Delta_c > 0$, the pair $\{a, c\}$ has $0 < |\Delta_a + \Delta_c| < 1$, and a prime that divides k and the weights other than q_a and q_c divides q_b, hence $q_a = 2k - q_b$, hence q_c, as the weights sum to $(n-2)k$; so $\{a, c\}$ is a collision pair. If $\Delta_c < 0$, the pair $\{b, c\}$ is one in the same way. Finally suppose that $|\Delta_l| = \Delta_b$ for every l, and let $\sigma = \sum_{l \in S} \operatorname{sign} \Delta_l$. Then $k\Delta_b = k - q_b$ is a positive integer, and the sum over S gives $k = \sigma \cdot k\Delta_b$; so $k\Delta_b$ divides k and every $q_l = k \mp k\Delta_b$, and $k\Delta_b = 1$. So every weight is $k \pm 1$, prime to k, and as $\sigma = k \geq 3$, at least two deficits equal $1/k$; two of them form a collision pair, with $\Delta_i + \Delta_j = 2/k < 1$. □

Proof of Theorem 1.2. Let $k = 2$. Every weight is 1 or 3; as they sum to $2(n-2)$, $r = (n+4)/2$ of them are 1 and $v = (n-4)/2$ are 3, so n is even. Here $\iota = T$, X_z/ι is the w-sphere, and the quotient of ω_z^2 is $\prod_i (w - z_i)^{q_i - 2} dw^2$, with simple zeros at the z_i with $q_i = 3$, simple poles at those with $q_i = 1$ and a regular point at ∞; it lies in $\mathcal{Q}(1^v, -1^r)$. Conversely, after a Möbius map that takes a regular point to ∞, every surface of this stratum is $c \prod_i (w - z_i)^{q_i - 2} dw^2$ for some $z \in \operatorname{Conf}$ and $c \in \mathbb{C}^\times$, and its holonomy double cover is $(X_z, c^{1/2}\omega_z)$. The stratum is connected, being parametrized by c and the z_i, so $\mathcal{L}(\theta)$ is the locus of these covers. As $\mathcal{M}(\theta)$ is invariant under $\mathrm{GL}^+(2, \mathbb{R}) \supset \mathbb{C}^\times$ and contains every (X_z, ω_z), $\mathcal{L}(\theta) \subset \mathcal{M}(\theta)$, so $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. This case is due to Athreya, Eskin and Zorich, with Chaika [5, Proposition 3.2, Theorem C.1 and §6.1], and is contained in Apisa's Theorem 1.11 [2]; here it is immediate from the cyclic family.

Let $k \geq 3$. We argue by induction on n, and for $n \geq 6$ we assume the theorem for the types of $(n-1)$-gons with this k that are not open; Theorem 1.1 covers those that are open. Let X be a generic point and A as in Proposition 2.4. As in the proof of Theorem 1.1, A satisfies (a)–(c) of Definition 5.1. It satisfies (d): for a collision pair $\{i, j\}$, with θ_{ij} reduced and not one of the seven types if $n = 5$, θ_{ij} is a type of $(n-1)$-gons with the same k by Lemma 4.2(a), and $\mathcal{M}(\theta_{ij}) = \mathcal{L}(\theta_{ij})$, by Part III [18, Theorem 1.1] if $n = 5$ (see the proof of Lemma 4.2), and by Theorem 1.1 or the induction hypothesis if $n \geq 6$; so Proposition 4.3 applies. So A is closed. If $n = 5$ and $k \leq 10$, we find that the least closed set is E (Section 1.3), so $A = E$. Among these types, $(1, 1, 1, 2, 4)/3$, $(1, 2, 2, 2, 2)/3$, $(1, 1, 1, 4, 8)/5$ and $(1, 1, 1, 5, 10)/6$ are the four with no collision pair to which (d) applies. If $n \geq 6$ or $k \geq 11$, Proposition 7.2 gives a collision pair to which (d) applies, (d) puts F, and so U, into A, and $A = E$ by Lemma 7.1. In both cases Lemma 6.1 gives $\mathcal{M}(\theta) = \mathcal{L}(\theta)$.

For every k, Mirzakhani and Wright's Lemma 6.3 then gives the statement for almost every polygon [30]. □

7.3. **Proof of Corollary 1.3.** By Theorems 1.1 and 1.2, $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ for every type with $n \geq 5$, which is the statement. For the open type $(1, 1, 2, 7, 7)/6$, $\iota = T^3$ fixes the single point over each vertex of weight 1, which is regular, and the single point over each vertex of weight 7, a zero of order 6; the quotient differential has the orders -1 and 5 there. It exchanges the two regular points over the vertex of weight 2 and has no other fixed points, as it is a deck transformation. So $(X/\iota, \omega^2)$ lies in $\mathcal{Q}(5, 5, -1, -1)$, a stratum of genus 3 with two components, one of them hyperelliptic [23, Corollary 1], [24, Theorem 1.2], and by Lemma 8.1 it lies in the other one.

8. Billiards in a generic polygon

In this section θ is a type of n-gons with $n \geq 5$, and $\mathcal{P}$ is a connected component of the space of polygons with the angles $\pi q_1/k, \dots, \pi q_n/k$ in a fixed cyclic order, with its

Lebesgue measure class. Mirzakhani and Wright's proof of their Lemma 6.3 identifies, by the Schwarz–Christoffel formula, the unfoldings of the polygons in $\mathcal{P}$ with the surfaces (X_z, ω_z) for z in an open subset of $\mathbb{R}^n$, whose periods are real-analytic in z [30]. Write $M = \mathcal{L}(\theta)$, and let $\mathcal{P}^\circ$ be the set of $P \in \mathcal{P}$ whose unfolding X has orbit closure M and no automorphisms other than the powers of T. By Theorems 1.1 and 1.2, Mirzakhani and Wright's Lemma 6.3, and Lemma 2.1, the complement of $\mathcal{P}^\circ$ has measure zero, as the real points of a proper algebraic subset of $\mathbb{C}^n$ form a null set.

If k is odd, let $(Y, q) = (X, \omega)$ and $\mathcal{S} = M$. If k is even, let $(Y, q) = (X/\iota, \omega^2)$, and let $\mathcal{S}$ be the component $\mathcal{Q}$ of its stratum, so that $M = \widetilde{\mathcal{Q}}$. Let $e = \gcd(2, k)$, $d = k/e$ and $b = 2d$. The quotient $X/\langle T\rangle$ is the double $D(P)$ of P, two copies of P glued along their edges, and $Y \to D(P)$ is cyclic of degree d; so Y is tiled by b copies of P, and has area bA, where A is the area of P. Over the vertex i the surface Y has $s_i = \gcd(d, q_i)$ points, each of cone angle $2\pi q_i/(es_i)$, so

$$m = \sum_{i:\, q_i = es_i} s_i$$

is the number of regular points of Y over the vertices. Let M_* be the orbit closure of X with all the regular points over the vertices marked.

Lemma 8.1. *Let $P \in \mathcal{P}^\circ$. Then $\operatorname{rank} M \geq n - 2$ if $k > 2$, and $\operatorname{rank} M \geq 2$ if $k = 2$. Moreover X is not a translation cover of a torus, and $\mathcal{S}$ is not hyperelliptic. If k is even, the orbit of (Y, q) is dense in $\mathcal{S}$.*

Proof. By Lemma 2.2, $\operatorname{rank} M = \frac{1}{2}\sum_{x \in E}(n(x) - 2)$. If $k > 2$, the distinct characters ± 1 lie in E and have $n(\pm 1) = n$, so $\operatorname{rank} M \geq n - 2 \geq 3$; if $k = 2$, then $E = \{1\}$, $n \geq 6$ (Section 7.2), and $\operatorname{rank} M = (n-2)/2 \geq 2$. As $\operatorname{rank} M \geq 2$, X is not a translation cover of a torus, whose orbit closure would have rank one [34, Proposition 4.18].

If k is odd, the automorphisms of $Y = X$ are the powers of T, of odd order, so Y has no involution and $\mathcal{S}$ is not hyperelliptic. Let k be even, and let φ be an automorphism of Y with $\varphi^* q = q$. As X is the curve of the square roots of q, φ lifts to an automorphism $\tilde{\varphi}$ of X with $\tilde{\varphi}^*\omega = \pm\omega$; so $\tilde{\varphi} = T^j$ with $\zeta^j = \pm 1$, that is, $\tilde{\varphi} \in \{1, \iota\}$, and φ is the identity. In a hyperelliptic component every surface has an involution that preserves its quadratic differential [23, Definition 1 and §4.1], [4, §1 and Lemma 4.5], so $\mathcal{S}$ is not hyperelliptic.

Let k be even, and let $f : \mathcal{S} \to M$ send a half-translation surface to its holonomy double cover. It is continuous and commutes with $\mathrm{GL}^+(2, \mathbb{R})$. For $g \in \mathrm{GL}^+(2, \mathbb{R})$, a homeomorphism of the surface is an automorphism of $g \cdot X$ that negates its form exactly when it is an automorphism of X that negates ω, as both mean that it is affine with derivative $-I$ and $g(-I)g^{-1} = -I$. By Lemma 2.1 the only such automorphism is ι, so the only point of $\mathcal{S}$ over $g \cdot X$ is $g \cdot (Y, q)$. Near each point of $\mathcal{S}$, Lanneau's chart [24, §1.1.1] makes f the inclusion of an open subset of a linear subspace of dimension $\dim \mathcal{S} = \dim M$ into period coordinates, up to the finite groups of the orbifold charts, and near each of its points M is a finite union of linear subspaces of this dimension [4, §1]. So the image under f of each non-empty open subset of $\mathcal{S}$ contains a non-empty open subset of M, hence a point $g \cdot X$, as the orbit of X is dense in M, and the orbit of (Y, q) is dense in $\mathcal{S}$. □

8.1. **Counting periodic trajectories.** A cylinder on a translation or half-translation surface is a maximal family of parallel closed regular geodesics, counted once and without orientation, with its circumference; on a surface with marked points the geodesics must also avoid the marked points. Let $N_X(L)$ be the number of cylinders on X of circumference at

most L, and $B_P(L)$ the number of bands of periodic billiard trajectories in P of length at most L, as in Apisa's [2, Definition 8.5 and Lemma 8.6]. For a counting function N let

$$W(N) = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-2s} N(e^s)\, ds.$$

Let $\mathcal{N}$ be an affine invariant submanifold, possibly with marked points, with its affine invariant probability measure on the surfaces of area one [15, Definition 1.1], or a component of a stratum of quadratic differentials, with its Masur–Veech probability measure. The mean number of cylinders of circumference at most L on the surfaces of $\mathcal{N}$ is $\pi c_{\mathrm{cyl}}(\mathcal{N})L^2$, and the mean total area of these cylinders is $\pi c_{\mathrm{area}}(\mathcal{N})L^2$ [11, §8], [5, §4].

Corollary 8.2. *For almost every $P \in \mathcal{P}$, of area A,*

$$W(N_X) = \frac{\pi c_{\mathrm{cyl}}(M)}{2kA}, \qquad W(B_P) = \frac{\pi c_{\mathrm{cyl}}(M_*)}{(2k)^2 A} = \frac{\pi\big(c_{\mathrm{cyl}}(\mathcal{S}) + m\, c_{\mathrm{area}}(\mathcal{S})\big)}{b^2 A}.$$

Eskin, Mirzakhani and Mohammadi's Theorem 2.12 gives $W(B_P)$ for every rational polygon [15, Theorem 2.12]. Its constant is a Siegel–Veech constant of the orbit closure of the unfolding with the regular points over the vertices marked, and the corollary identifies it for almost every P of a type with five or more vertices. For $k = 2$, Athreya, Eskin and Zorich give exact asymptotics for almost every right-angled polygon, configuration by configuration [5, Theorem 2.4], and Apisa notes that their work gives such counts for almost every polygon of a type with at most two angles that are not multiples of $\pi/2$ [2, after Theorem 1.11]. The term $m\, c_{\mathrm{area}}(\mathcal{S})$ comes from the m regular points of Y over the vertices, each of which cuts the cylinder of Y that contains it into two of the same circumference. So the regular points over the vertices change the constant whenever $m > 0$, as $c_{\mathrm{area}}(\mathcal{S}) > 0$, every translation or half-translation surface having cylinders [26]. For the type $(1, 2, 2, 2, 2)/3$, which is not open, $m = 1$ and $b = 6$; for the open type $(1, 5, 5, 5, 5)/7$, with $X \in \mathcal{H}(4, 4, 4, 4)$, $m = 1$ and $b = 14$.

Proof. Let $P \in \mathcal{P}^\circ$. Eskin, Mirzakhani and Mohammadi's proof of their Theorem 2.12 shows that a translation surface of area one with orbit closure $\mathcal{N}$ has $W(N) = \pi c_{\mathrm{cyl}}(\mathcal{N})$ for its count N of cylinders [15, Theorem 2.6 and proof of Theorem 2.12], [11, Theorem 8.2]. As X has area $2kA$ and orbit closure M, scaling gives the first formula.

With marked points, Eskin, Mirzakhani and Mohammadi's results still hold [25, §2.2], but two hypotheses of Eskin, Marklof and Morris's Theorem 8.2 [11] are not given by these citations. Let x be a surface of area one and genus at least 2 with r regular marked points, $\mathcal{N}$ its orbit closure and ν the affine measure of $\mathcal{N}$. A cylinder of x with its marks forgotten gives at most $r + 1$ cylinders with the marks, and every cylinder has a saddle connection on its boundary. So, by Eskin and Masur's Theorem 5.1 [12], the bound (B′) of [11, Remark 8.5] holds with the length of the shortest saddle connection of the unmarked surface, to which Eskin and Masur's Theorem 5.2 applies. For condition (D), off the self-crossing set of $\mathcal{N}$, which is closed and ν-null [15, Definition 1.1], $\mathcal{N}$ is locally one linear subspace, and on it the cylinder transform $\hat{h}$ of the indicator of a trapezoid is locally constant off the zero sets of finitely many real-analytic functions, where the holonomy of a cylinder meets the boundary of the trapezoid or the width of a cylinder vanishes. Each of these zero sets is ν-null or the whole subspace, along which $\hat{h}$ does not change, so $\hat{h}$ is continuous off a ν-null set, which gives (D). The second formula uses this for X with the regular points over the vertices marked, with $\mathcal{N} = M_*$.

Let $N_{Y,*}$ count the cylinders of Y with the m regular points over the vertices marked. Here the cylinders of $D(P)$ are as in Apisa's Definition 8.5, maximal families of homotopic

closed geodesics in $D(P)$ minus its cone points, which need not be embedded. Apart from finitely many, those that avoid the vertices come in pairs exchanged by the reflection of $D(P)$, one pair for each band of P, and each lifts to d cylinders of Y of the same circumference; so $W(B_P) = W(N_{Y,*})/b$ [2, Lemma 8.6 and its proof]. If k is odd, then $Y = X$, Y with these marks has orbit closure M_*, and $b = 2k$, which gives $W(B_P) = \pi c_{\mathrm{cyl}}(M_*)/(b \cdot 2kA) = \pi c_{\mathrm{cyl}}(M_*)/((2k)^2 A)$. Let k be even. The core curve of a cylinder of Y has trivial linear holonomy, so each cylinder of Y with these marks lifts to two cylinders of X of the same circumference, which avoid all the regular points over the vertices, the fixed points of ι among them. Every such cylinder of X arises so, as a cylinder that ι maps to itself contains two fixed points of ι. So $N_{X,*} = 2N_{Y,*}$ for the corresponding count on X. As $b = k$, $W(B_P) = W(N_{X,*})/(2b) = \pi c_{\mathrm{cyl}}(M_*)/((2k)^2 A)$.

For the last expression, let $\mathcal{S}_*$ be the orbit closure of (Y, q) with its m regular points over the vertices marked. We show that it is the locus of all surfaces of $\mathcal{S}$ with m distinct regular marked points. If k is odd, X has dense orbit in the component M, which is not hyperelliptic, and genus at least 2, so every finite set of regular points of X is generic in Apisa's sense [1, Theorem 1.1]. Let k be even. The orbit of (Y, q) is dense in $\mathcal{S}$, which has rank at least 2 and is not hyperelliptic (Lemma 8.1). So there are no irreducible markings of two or more points over $\mathcal{S}$ [4, Corollary 5.2], and no regular point of (Y, q) is periodic, as it would lift to a periodic point of the double cover [4, §5] that is not fixed by the involution [4, Theorem 1.4]. In both cases each marked point moves freely, so $\mathcal{S}_*$ has dimension $\dim \mathcal{S} + m$; it is closed and invariant, and the locus of all surfaces of $\mathcal{S}$ with m distinct regular marked points is connected, so it is $\mathcal{S}_*$. The invariant measure on $\mathcal{S}_*$ is the product of that of $\mathcal{S}$ and the area measures of the marked points [15, Definition 1.1(ii)], [5, §4.3]. A cylinder of area a on a surface of area one that contains j marked points at distinct heights is cut into $1 + j$ cylinders of the same circumference, and j has mean ma; so $c_{\mathrm{cyl}}(\mathcal{S}_*) = c_{\mathrm{cyl}}(\mathcal{S}) + m\, c_{\mathrm{area}}(\mathcal{S})$. If k is odd, $\mathcal{S}_* = M_*$ and $b = 2k$. If k is even, $N_{X,*} = 2N_{Y,*}$ and X has twice the area of Y, so $c_{\mathrm{cyl}}(M_*) = 4c_{\mathrm{cyl}}(\mathcal{S}_*)$ at area one, and $(2k)^2 = 4b^2$. □

8.2. **The exceptional polygons.** Let $F(P)$ be the unfolding of $P \in \mathcal{P}$, and $\mathcal{E}$ the set of P for which the orbit closure of $F(P)$ is smaller than M.

Corollary 8.3. *If $k \notin \{2, 3, 4, 6\}$, $\mathcal{E}$ is a finite union of closed real-analytic subsets of measure zero. If $k \in \{2, 3, 4, 6\}$, $\mathcal{E}$ is a countable union of such sets and is dense in $\mathcal{P}$.*

Proof. The set $\mathcal{E}$ is the union of the sets $F^{-1}(\mathcal{D})$ over the countably many affine invariant submanifolds $\mathcal{D} \subsetneq M$, and each of these is a closed real-analytic subset of $\mathcal{P}$ of measure zero. If the orbit closure of $F(P)$ is $\mathcal{D} \neq M$, then $P \in F^{-1}(\mathcal{D})$; conversely, if $F(P) \in \mathcal{D} \subsetneq M$, its orbit closure lies in $\mathcal{D}$, which is closed and invariant. There are countably many affine invariant submanifolds [15, Proposition 2.16], and near each of its points $\mathcal{D}$ is a finite union of linear subspaces in period coordinates [15, Definitions 1.1 and 1.2]; as the periods of $F(P)$ are real-analytic in P, $F^{-1}(\mathcal{D})$ is a closed real-analytic subset, of measure zero as $\mathcal{P}^\circ$ has full measure.

Let $k \notin \{2, 3, 4, 6\}$, so that $k \geq 5$ and $\operatorname{rank} M \geq n - 2 \geq 3$ (Lemma 8.1). We argue as in Apisa's proof of his Proposition 10.3 [2]. By Eskin, Filip and Wright, all but finitely many affine invariant submanifolds of the stratum of M have rank 1 and degree at most 2, the degree being that of the field of affine definition over $\mathbb{Q}$. All but finitely many of those of rank 1 and degree 2 lie in one of finitely many affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_r$ of rank 2 [10, Theorem 1.5], and those of rank 1 and degree 1 consist of covers of tori [10, after Corollary 1.6]. No unfolding of a polygon of type θ covers a torus, as that forces

$k \in \{2,3,4,6\}$ [4, proof of Proposition 3.11]. As $\operatorname{rank} M \geq 3$, each $\mathcal{N}_\alpha \cap M$ is a closed invariant proper subset of M, locally a finite union of linear subspaces, and every point of it has orbit closure smaller than M. So $\mathcal{E}$ is the union of the sets $F^{-1}(\mathcal{D})$ for finitely many $\mathcal{D}$ and the sets $F^{-1}(\mathcal{N}_\alpha \cap M)$, each a closed real-analytic subset of measure zero.

Let $k \in \{2,3,4,6\}$, and let Λ be the lattice $\mathbb{Z}[i]$ if $k \in \{2,4\}$ and $\mathbb{Z}[e^{\pi i/3}]$ if $k \in \{3,6\}$. Rotate the polygons so that one edge is horizontal. The directions of the edges are then multiples of π/k, and each such direction contains a non-zero vector of Λ; so the edge vectors are $t_1 u_1, \dots, t_n u_n$ with fixed $u_l \in \Lambda$ and $t_l > 0$ subject to $\sum_l t_l u_l = 0$, two linear equations with rational coefficients in a basis of Λ. Rational solutions are dense among the real ones, and simplicity is an open condition; so the polygons of $\mathcal{P}$ whose vertices lie in Λ after a translation and a scaling are dense. For such a polygon the reflections in its edges preserve Λ, so the developing map of its unfolding sends every point over a vertex into Λ, and integrating ω modulo Λ makes the unfolding a cover of the torus $\mathbb{C}/\Lambda$ branched over one point. Its orbit is closed, of rank one [34, Remark 3.7 and Proposition 4.18], while $\operatorname{rank} M \geq 2$ (Lemma 8.1); so these polygons lie in $\mathcal{E}$. □

8.3. **Periodic points and illumination.** A regular point of a translation surface is periodic if marking it does not increase the dimension of the orbit closure [1, §1]. Two points are finitely blocked if a finite set of other points meets every straight segment that joins them [4, §3.1]. A billiard path in P is a path that is straight in the interior of P, reflects at the edges, and meets no vertex except possibly at its ends.

Corollary 8.4. *For almost every $P \in \mathcal{P}$ the following hold. If k is odd, X has no periodic points; if k is even, its periodic points are the regular fixed points of ι, which lie over vertices. Any two distinct points of P, vertices included, are joined by a billiard path that avoids any given finite set of other points of P.*

For every rational polygon, Lelièvre, Monteil and Weiss show that each point is joined by a billiard path to all but finitely many points [25, Corollary 3]. Apisa and Wright show that only finitely many pairs are finitely blocked when the polygon is not Gaussian, Eisenstein or rectangle-tiled and some angle is not a multiple of $\pi/2$ [4, Theorem 1.1(3)]. For almost every polygon of a type with five or more vertices, Corollary 8.4 leaves no such pairs of distinct points. For the types with at most two angles that are not multiples of $\pi/2$, Apisa notes that the finite blocking problem for almost every polygon can be solved as in his Theorem 1.3 [2, after Theorem 1.11].

Proof. Let $P \in \mathcal{P}^\circ$. If k is odd, X has dense orbit in the component M, which is not hyperelliptic (Lemma 8.1), and genus at least 2, so it has no periodic points [1, Theorem 1.1]. If k is even, $\mathcal{S}$ has rank at least 2, so it is not $\mathcal{Q}(-1^4)$, $\mathcal{Q}(2,-1^2)$ or $\mathcal{Q}(2^2)$ [4, Corollary 4.3], and it is not hyperelliptic; so the periodic points of X are the regular fixed points of its involution that negates ω [4, Theorem 1.4], which is ι by Lemma 2.1. As $\iota = T^{k/2}$, its fixed points lie over the vertices.

As X is not a translation cover of a torus, it has a translation covering $\pi_{\min}$ onto a half-translation surface which factors through every translation covering of X onto a half-translation surface [4, Lemma 3.3]. We show that $\pi_{\min}$ is an isomorphism if k is odd, and the quotient map $X \to Y$ followed by an isomorphism if k is even. In both cases $\pi_{\min}$ is the covering $X \to Y$ followed by a half-translation covering g of Y. Fix a degree greater than one, a target stratum and branching data. The surfaces of $\mathcal{S}$ that admit a half-translation covering with these data form an invariant set C, the image of an algebraic family of coverings. The generic surface of the non-hyperelliptic component $\mathcal{S}$ admits no non-bijective

half-translation covering [4, Lemma 4.5 and proof of Corollary 5.2], so C lies in a proper algebraic subset of $\mathcal{S}$, and its closure is a proper closed invariant subset. It contains no surface with dense orbit in $\mathcal{S}$, and Y has dense orbit in $\mathcal{S}$, by the choice of P if k is odd and by Lemma 8.1 if k is even; so Y is not in C, and g is bijective.

Let $p \neq p'$ be points of P and $x, x' \in X$ points over them. The map from X to P factors through Y, so $\pi_{\min}(x) \neq \pi_{\min}(x')$, and if x and x' are finitely blocked, each is a periodic point or a zero [4, Theorem 3.6], and so lies over a vertex. So if p or p' is not a vertex, no two points over p and p' are finitely blocked.

Let p and p' be the vertices $i \neq j$. Mark on each X_z, $z \in \mathrm{Conf}$, the points over each z_l with the label l. As in Mirzakhani and Wright's proof of their Lemma 6.3, the periods of these marked surfaces are holomorphic in z in a local marking, and there are countably many affine invariant submanifolds in strata with marked points [25, §2.2]; so for almost every $P \in \mathcal{P}$ the orbit closure of the marked unfolding of P contains every marked X_z, and we take P with this property. Choose $z \in \mathbb{R}^n \cap \mathrm{Conf}$ with no z_l between z_i and z_j. Over this interval $\omega_z = \prod_l (w - z_l)^{q_l/k-1} dw$ has constant argument on each lift and is integrable at the ends, as $q_l/k - 1 > -1$; so a lift of the interval is a straight segment from a point over z_i to a point over z_j whose interior meets no marked point. Having such a segment is an open condition in the stratum of the marked surfaces, invariant under $\mathrm{GL}^+(2,\mathbb{R})$ [4, proof of Lemma 3.7]. As the orbit of the marked unfolding of P comes arbitrarily close to the marked X_z, the unfolding X has a straight segment σ from a point x over p to a point x' over p' whose interior meets no point over a vertex. Suppose that x and x' are blocked by a minimal finite set B. Then x and x' are periodic points or zeros, as above, and the points of B are locally determined by the unmarked surface and x, x' [4, Lemma 3.8]; so marking them does not increase the dimension, and they are periodic points, which lie over vertices. Then σ avoids B, a contradiction.

Finally let $p \neq p'$, let x, x' be points over them that are not finitely blocked, and let $B \subset P$ be finite with $p, p' \notin B$. If p and p' lie on one edge of P, we add to B a point of that edge between them. As x and x' are not finitely blocked, there is a straight segment from x to x' whose interior avoids the zeros of ω and the points other than x and x' that lie over B, over p or p', or over a vertex. Its part from its last visit to x before its first visit to x' then meets no point over a vertex, over p or p', or over B in its interior. It does not run along the lift of an edge, as it would then lie in that lift, whose ends are over vertices, and its image would be the part of an edge between p and p', which contains a point of B. So its image in P is a billiard path from p to p' that avoids B, by Wolecki's Lemma 2.1 and Remark 2.2 [32] on its subsegments whose ends are not over vertices, extended continuously to p and p'. □

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Fable 5.2 in Claude Code and OpenAI's GPT-6.2 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [21].

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