hexagon:2611.01029v1

ORBIT CLOSURES OF GENERIC RATIONAL QUADRILATERALS

MANVIR JASWAL

Abstract. Delecroix and Rüth conjectured that their lists of admissible and reduced rational triangle and quadrilateral types whose unfoldings do not have dense orbits in their ambient strata are complete, and they proved this for quadrilaterals with angles $2\pi a_i/N$ when $N \leq 32$. We prove the conjecture for quadrilaterals, for the generic quadrilateral with given angles. For every admissible and reduced type outside their seven quadrilateral exceptions, the orbit closure of the unfolding of the generic member is a connected component of a stratum of Abelian differentials if N is odd or $N \equiv 2 \pmod 4$, and the locus of holonomy double covers of a component of a stratum of quadratic differentials if 4 divides N. The proof reads the rank of the orbit closure off a set of characters of the deck rotation of the unfolding, a cyclic cover of the sphere. Mirzakhani and Wright's rules, as extended for triangles in the two earlier papers of this series, constrain this set, and two further inputs bring in the orbit closures of triangles: the collision of two vertices, through Mirzakhani and Wright's boundary theorem, and symmetric members of the family of cyclic covers, which cover unfoldings of triangles. Parts of the proof, including a census up to $N = 1000$, are computer-assisted.

Contents

1. Introduction 1
2. Unfoldings, characters and the ambient locus 5
3. Colliding two vertices 11
4. Closure rules 16
5. From the characters to the orbit closure 23
6. Odd degree 25
7. Even degree 32
8. Two angles summing to π 43
9. Types with $N \leq 1000$ 45
10. Proof of Theorem 1.1 49
Declarations 49
References 50

1. Introduction

A billiard trajectory in a polygon whose angles are rational multiples of π unfolds to a straight line on a translation surface (X, ω), the *unfolding* of the polygon [31]. The

Date: November 24, 2026.

2020 *Mathematics Subject Classification.* Primary 37C83; Secondary 32G15, 30F30, 11A07.

Key words and phrases. Rational billiards, translation surfaces, orbit closures, affine invariant submanifolds, quadrilaterals.

This paper is Part III of a series on orbit closures of unfoldings of rational polygons: Parts I [19] and II [18] treat triangles, Part III quadrilaterals and Part IV [17] polygons with at least five sides.

group $\mathrm{GL}^+(2,\mathbb{R})$ acts on translation surfaces, and by the theorems of Eskin, Mirzakhani and Mohammadi the closure of the orbit of (X,ω) in its stratum is an affine invariant submanifold [13, 14]. To a tuple θ of angles of a rational polygon Mirzakhani and Wright associate an affine invariant submanifold $\mathcal{M}(\theta)$, which, in their words, "can be defined as the orbit closure of the unfolding of the generic polygon with these angles. One long term goal is to compute these $\mathcal{M}(\theta)$ for all θ" [26, §1]. For a triangle, $\mathcal{M}(\theta)$ is the orbit closure of its unfolding, and Part II shows that it is the ambient locus of the triangle for every scalene triangle outside a list of exceptions [18, Theorem 1.1]. This paper proves the corresponding statement for quadrilaterals.

A quadrilateral has *type* $\theta=(a_1,a_2,a_3,a_4)$, where $a_1\le a_2\le a_3\le a_4$ are positive integers with greatest common divisor 1 and sum N, if its interior angles are $2\pi a_i/N$ in some cyclic order. Following Delecroix and Rüth [11, §4], θ is *admissible* if no angle is π, and an admissible type is *reduced* unless $a_1=a_2<a_3=a_4$. To the six types of quadrilaterals with orbit closures of rank two exhibited by Eskin, McMullen, Mukamel and Wright [12, Theorem 1.1 and §5], Delecroix and Rüth add one, conjecturally also of rank two, which gives their list of exceptional types [11, §4],

$$(1,1,1,7),\ (1,1,1,9),\ (1,1,2,8),\ (1,1,2,12),\ (1,2,2,11),\ (1,2,2,15),\ (2,2,3,13). \tag{1.1}$$

With the flatsurf software they prove that the admissible and reduced types (a,b,c,d) with $a+b+c+d\le 32$ outside (1.1) correspond to quadrilaterals whose unfoldings have dense orbits in their ambient strata [11, Theorem 4.2]. Their Theorem 4.1 is the analogous statement for triangles, and they add: "We conjecture that the finite list of exceptions in Theorem 4.1 and Theorem 4.2 is complete" [11, §4], a conjecture stated for quadrilaterals alone as [7, Conjecture 19]. For reduced triangles with distinct angles this is the theorem of [18] stated above; the equilateral triangle unfolds to a flat torus. We prove it for quadrilaterals, for the generic quadrilateral of each type.

To state the result, let k be the least common denominator of the angles divided by π; so $k=N$ if N is odd and $k=N/2$ if N is even. The unfolding (X,ω) of a quadrilateral of admissible type θ is a cyclic cover of degree k of the sphere, branched over four points [26, §6], and we mark only the zeros of ω on it. If k is odd, the *ambient locus* $\mathcal{L}(\theta)$ is the connected component of the stratum of Abelian differentials that contains (X,ω). If k is even, the deck transformation ι of order two has $\iota^*\omega=-\omega$, so that (X,ω) is the holonomy double cover of the quadratic differential $(X/\iota,\omega^2)$, and $\mathcal{L}(\theta)$ is the locus of the holonomy double covers of all surfaces in the component of the stratum of quadratic differentials that contains $(X/\iota,\omega^2)$. In both cases $\mathcal{M}(\theta)\subset\mathcal{L}(\theta)$ (Section 2.4).

Theorem 1.1. *Let θ be an admissible and reduced type of quadrilaterals that is not one of the seven types (1.1). Then $\mathcal{M}(\theta)=\mathcal{L}(\theta)$: the orbit closure of the unfolding of the generic quadrilateral of type θ is the connected component of its stratum if N is odd or $N\equiv 2 \pmod 4$, and the locus of holonomy double covers of the component of the stratum of quadratic differentials of its quotient by ι if $4\mid N$.*

In terms of k, the two cases are k odd and k even; for the triangles of Parts I and II [19, 18], k is the N used there.

Mirzakhani and Wright ask: "If $n=4$, are there infinitely many $\mathcal{M}(\theta)$ such that θ has more than two distinct angles and $\mathcal{M}(\theta)$ is neither a connected component of a stratum nor a locus $\widetilde{\mathcal{Q}}$ of double covers of all quadratic differentials in a given stratum?" [26, Question 1.6]. Theorem 1.1 answers the case $n=4$ of their question in the negative.

Corollary 1.2. *Let θ be an admissible type of quadrilaterals with more than two distinct angles, other than $(1,1,2,8)$, $(1,1,2,12)$, $(1,2,2,11)$, $(1,2,2,15)$ and $(2,2,3,13)$. Then $\mathcal{M}(\theta)$ is a connected component of a stratum of Abelian differentials if N is odd or $N \equiv 2 \pmod 4$, and the locus $\widetilde{\mathcal{Q}}$ of holonomy double covers of all surfaces in a connected component $\mathcal{Q}$ of a stratum of quadratic differentials if $4 \mid N$. In particular, only finitely many $\mathcal{M}(\theta)$ are as in the case $n = 4$ of Question 1.6 of* [26].

Proof. A type with $a_1 = a_2 < a_3 = a_4$ has two distinct angles, as do $(1,1,1,7)$ and $(1,1,1,9)$. So θ is reduced and not in (1.1), and Theorem 1.1 gives the first statement. For the second, Mirzakhani and Wright admit no angle equal to π [26, §6] and form $\widetilde{\mathcal{Q}}$ from a connected component of a stratum of quadratic differentials [26, §1], as for $\mathcal{L}(\theta)$; so a θ as in the case $n = 4$ of their question is one of the five types excluded above. □

By a lemma of Mirzakhani and Wright, $\mathcal{M}(\theta)$ is the orbit closure of the unfolding of almost every quadrilateral of type θ [26, Lemma 6.3]; Theorem 1.1 says nothing about an individual quadrilateral. The type $(1,1,1,1)$ of the rectangles, which unfold to flat tori, is included.

In outline, the rank of $\mathcal{M}(\theta)$ is read off the set of characters of the deck rotation of the unfolding whose eigenspaces are tangent to $\mathcal{M}(\theta)$. When two vertices whose angles sum to more than π collide, the quadrilateral degenerates to a triangle (Figure 1), and when a condition on greatest common divisors holds the unfolding degenerates to the connected unfolding of that triangle. If the orbit closure of the triangle's unfolding is its ambient locus, Mirzakhani and Wright's boundary theorem puts the characters of the triangle into this set. For a type with equal angles, the unfoldings of kites and the three-fold symmetric members of the cyclic-cover family are covers of unfoldings of triangles, which bounds the rank from below. For the types of Theorem 1.1, rules on the set then force $\mathcal{M}(\theta)$ to have the rank of $\mathcal{L}(\theta)$, except for 35 types settled by cylinder deformations (Section 1.2). When no two angles sum to π, this is arithmetic with bounded computations for $N > 1000$ and a census for $N \le 1000$; the other types have a separate argument (Section 8).

1.1. **Earlier results.** For $N \le 32$, Theorem 1.1 is the theorem of Delecroix and Rüth stated above, with the ambient strata read as the ambient loci defined here; our proof does not

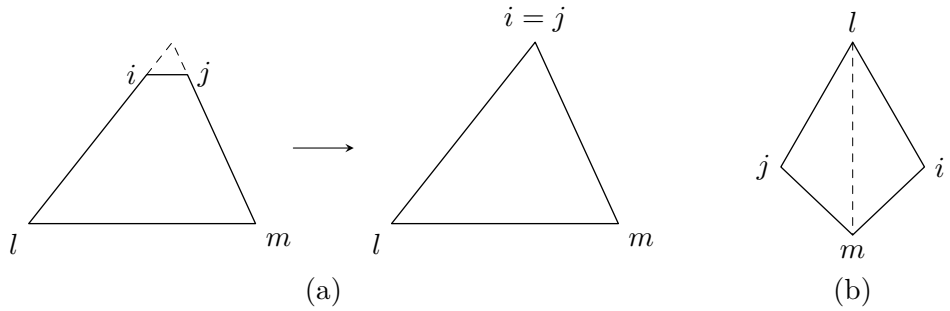


FIGURE 1. (a) When two vertices i and j of a quadrilateral collide, and their angles sum to more than π, the quadrilateral degenerates to a triangle; its angle at the merged vertex is the sum of the two angles minus π (Section 3.2). (b) A kite with equal angles at i and j is made of two copies of a triangle, glued along the side joining l and m, and its unfolding covers that of the triangle (Proposition 4.9).

use it. For the types with two right angles it is a theorem of Apisa, who determines $\mathcal{M}(\theta)$ for the angles $(\frac{a}{n}, \frac{b}{n}, \frac{1}{2}, \frac{1}{2})\pi$ [2, Theorem 1.10]. For the types with two angles summing to π, Apisa and Aygun prove, by other means, that the generic member has an orbit closure as big as possible [3, Corollary 1.3], which for the reduced types of genus at least 2 is $\mathcal{L}(\theta)$; our proof for these types is in Section 8.

The types that are not reduced are those of the parallelograms and isosceles trapezoids other than rectangles, and for them Apisa determined $\mathcal{M}(\theta)$ [2, Theorem 1.9]. On the seven types (1.1) the theorem is silent. For the first six, Eskin, McMullen, Mukamel and Wright prove that the saturation of the cyclic forms, which is $\mathcal{M}(\theta)$, is a primitive, irreducible invariant subvariety of dimension 4 [12, Theorem 1.1], while $\mathcal{L}(\theta)$ has dimension 8, 6, 5, 8, 8, 8 for them (Section 9.3). For $(2, 2, 3, 13)$, where $\mathcal{L}(\theta)$ has dimension 8, Delecroix and Rüth prove that the unfoldings lie in the orbit closure of the unfolding of the triangle with angles $(\frac{3}{20}, \frac{4}{20}, \frac{13}{20})\pi$ and conjecture that this orbit closure has dimension 4 [11, Theorem 1.2 and Conjecture 1.1]; Delecroix announces a positive answer to this conjecture, from joint work with Kai Fu [7, pp. 48–49]; his slides of June 2025, on joint work with K. Fu and B. Zykoski, list this orbit closure with dimension 4 and rank 2 [8, slide 12].

1.2. **The proof.** The deck rotation T of the cyclic cover acts on the cohomology of X, and the absolute projection of the tangent space of $\mathcal{M}(\theta)$ at a generic point of the family $\mathcal{M}_{\mathrm{cyc}}(\theta)$ of cyclic covers is a sum of eigenspaces of T (Proposition 2.5). So the rank of $\mathcal{M}(\theta)$ is read off the set A of the characters $x \in \mathbb{Z}/k$ whose eigenspace is tangent to $\mathcal{M}(\theta)$ (the set called S in Parts I and II), and the theorem requires that A be the set of all characters with a non-zero eigenspace in the absolute tangent space of $\mathcal{L}(\theta)$, together with a statement on relative cohomology when k is even. This is Mirzakhani and Wright's frame [26, §7], used also in [18, 19]. Three of the rules that constrain A are the analogues of rules for triangles, with proofs that follow those of [18, 19]: A is symmetric and is permuted as a block by the units of $\mathbb{Z}/k$; a rule that comes from differentiating a holomorphic eigenform along $\mathcal{M}(\theta)$ adds characters to A; and a rule of the same kind puts relative classes into the tangent space (Section 4). The first two are Mirzakhani and Wright's [26, Lemmas 7.6 and 7.10 and Proposition 7.3], the second in a more general form.

Mirzakhani and Wright write their analysis for polygons with n vertices and assume $n = 3$ at key places [26, §7]. Two of these places change for $n = 4$. The eigenspace decomposition of Proposition 2.5 rests on the irreducibility of the monodromy on the eigenspaces of generic characters (Lemma 2.4), which they observe would extend their Corollary 7.9 beyond triangles. And for $n = 4$ we do not use their Proposition 7.3(2), whose extension to $n > 3$ they pose as Problem 7.4: the differential rule needs only that a certain quotient of eigenforms is not exact (Theorem 4.6 and Lemma 4.5).

Two inputs have no counterpart in the triangle papers. Both use the full ambient rank of the good triangles in Definition 3.3, supplied by [18, Theorem 1.1]. The first is the collision of two vertices, a device of Apisa [2, Sublemma 11.3] and of Apisa and Aygun [3, Lemma 12.3]. By Mirzakhani and Wright's description of the boundary of an affine invariant submanifold [25], the tangent space of a part of the boundary of $\mathcal{M}(\theta)$ that contains the limit, the unfolding of the collision triangle, is cut out of the tangent space of $\mathcal{M}(\theta)$, and we apply this eigenspace by eigenspace (Theorem 3.2). The second comes from the symmetric members, which bound the rank of $\mathcal{M}(\theta)$ from below (Proposition 4.9). Both steps mark regular points on the triangle's unfolding, and use that this does not change the rank of an orbit closure (Lemma 3.1, whose first part, when the triangle's unfolding has a zero, is a lemma of Apisa and Wright [4, Lemma 2.1]).

The rules are formal statements about subsets of $\mathbb{Z}/k$, and the main part of the paper shows that they force A to be the set of all characters of the absolute tangent space of $\mathcal{L}(\theta)$ in the cases they reach. For the types with $N > 1000$ in which no two angles sum to π this is arithmetic, with bounded computations. Once A contains the units, the differential rule gives the rest, by an averaging argument over residue classes (Lemma 6.4); and the units are in A as soon as one of the three triangles obtained by collisions is connected and good, in the sense of Definition 3.3, which excludes the isosceles triangles and the exceptions of [18] (Corollary 6.5). The types for which no collision gives such a triangle are classified, and treated one family at a time (Sections 6 and 7); for even k the classification is longer, and it ends with a table of 46 tuples. The types with two angles summing to π behave differently and are treated in Section 8. For $N \leq 1000$, when no two angles sum to π, the rules are applied by computation (Proposition 9.1). They do not reach 32 of these types and three types with two angles summing to π, all with $N \leq 32$; for each of these 35 types we exhibit one quadrilateral whose unfolding has an orbit closure of the dimension of $\mathcal{L}(\theta)$, by Wright's cylinder deformation theorem [30] and exact computations of cylinders (Proposition 9.4).

To pass from A to $\mathcal{M}(\theta)$ we use, for odd k, Mirzakhani and Wright's theorem that an affine invariant submanifold of full rank is a component of a stratum or the locus of hyperelliptic surfaces in one [26]; for even k we count dimensions, the relative rule supplying the relative part (Section 5). When k is an odd prime, one collision triangle of full rank suffices (Corollary 5.4).

1.3. **Computations.** Finite computations enter the proofs: the census of the types with $N \leq 1000$, the cylinder decompositions of 35 quadrilaterals, and finite ranges inside some arguments of Sections 6, 7 and 8. Each fact is stated where it is used. The computations decide exactly, with integers, rational numbers and arithmetic in number fields; where floating point enters a decision, its result is checked exactly or its error is bounded. For the cylinders we follow Delecroix and Rüth's procedure for bounding orbit closures from below [11, §3], with the flatsurf software [9, 10]. The programs, their output and instructions for rerunning them are deposited [20].

2. Unfoldings, characters and the ambient locus

This section sets up the objects of Theorem 1.1. The unfolding of a quadrilateral of a given admissible type is a cyclic cover of the sphere branched over four points, and the tangent space of the orbit closure of the generic member splits into eigenspaces of the deck rotation; the main result of the section, Proposition 2.5, says that its absolute part is a sum of whole eigenspaces. Along the way we compute the genus and show that for reduced types of genus at least two the automorphisms of the generic member preserving the line of its form are the deck rotations (Lemma 2.2).

2.1. **Types and the cyclic cover.** A *type* is a quadruple $\theta = (a_1, a_2, a_3, a_4)$ of positive integers with $a_1 \leq a_2 \leq a_3 \leq a_4$ and $\gcd(a_1, a_2, a_3, a_4) = 1$; we put $N = a_1 + a_2 + a_3 + a_4$. A quadrilateral *of type* θ is a simple quadrilateral whose interior angles, in some cyclic order, are $2\pi a_i/N$; they sum to 2π, and one of them may exceed π. Following Delecroix and Rüth [11, §4], θ is *admissible* if $2a_i \neq N$ for every i (no angle is π), and an admissible type is *reduced* unless $a_1 = a_2 < a_3 = a_4$. Reducedness is the condition used in Lemma 2.2; with $\gcd(a_1, \dots, a_4) = 1$ alone that lemma fails for $(1, 1, 2, 2)$.

Put

$$k = \begin{cases} N, & N \text{ odd}, \\ N/2, & N \text{ even}, \end{cases} \qquad q_i = \begin{cases} 2a_i, & N \text{ odd}, \\ a_i, & N \text{ even}. \end{cases}$$

Then the angles are $\pi q_i/k$, $q_1+q_2+q_3+q_4 = 2k$ and $\gcd(k, q_1, q_2, q_3, q_4) = 1$, so k is the least common denominator of the angles divided by π and the q_i are the integers that Mirzakhani and Wright attach to the angles [26, §6]. Admissibility says $q_i \neq k$, so $0 < q_i < 2k$ and $q_i \not\equiv 0 \pmod k$. Let $g_i = \gcd(k, q_i)$.

For distinct $z_1, \dots, z_4 \in \mathbb{C}$ let $X = X_z$ be the normalization of the curve $y^k = \prod_i (z - z_i)^{q_i}$, with

$$\omega = \omega_z = \prod_{i=1}^{4} (z - z_i)^{q_i/k - 1}\, dz = y \prod_{i=1}^{4} (z - z_i)^{-1}\, dz, \qquad T(y, z) = (\zeta y, z), \quad \zeta = e^{2\pi i/k}.$$

The automorphism T generates the deck group of $(y, z) \mapsto z$ and $T^*\omega = \zeta\omega$. Over z_i there are g_i points, permuted transitively by T, each with ramification index k/g_i and cone angle $2\pi q_i/g_i$, where ω has order $q_i/g_i - 1$; as $\sum_i q_i \equiv 0 \pmod k$, the cover is unramified over ∞, and ω has neither zeros nor poles there. The points of cone angle 2π, those over the z_i with $q_i \mid k$, are regular points of ω, and they are not marked: $\Sigma = \Sigma_z$ is the set of zeros of ω, the points over the z_i with $q_i \nmid k$, and (X, ω) lies in the stratum of Abelian differentials given by the orders of these zeros. Let $\mathcal{M}_{\mathrm{cyc}}(\theta)$ be the set of all (X_z, ω_z), and $\mathcal{M}(\theta)$ the smallest affine invariant submanifold of that stratum containing it [26, §6]. Both depend only on the multiset of angles, not on their order.

By Schwarz–Christoffel, for real z_i in a suitable order (X_z, ω_z) is the unfolding of a quadrilateral of type θ, and every unfolding of such a quadrilateral arises in this way up to rotation and scaling, as $(X_z, c\,\omega_z)$ with $c \in \mathbb{C}^\times$, which lies in the $\mathrm{GL}^+(2, \mathbb{R})$-orbit of (X_z, ω_z) [26, proof of Lemma 6.3]. Throughout this paper, orbits, orbit closures and invariance are for the group $\mathrm{GL}^+(2, \mathbb{R})$ of real matrices of positive determinant. For the unfoldings of quadrilaterals, and of triangles [19, §2.3], these are the orbits of $\mathrm{GL}(2, \mathbb{R})$ used by Mirzakhani and Wright and in [18, 19], since complex conjugation $(y, z) \mapsto (\bar{y}, \bar{z})$ maps X_z to $X_{\bar{z}}$ and pulls $\omega_{\bar{z}}$ back to $\bar{\omega}_z$, so that $\mathrm{diag}(1, -1)$ preserves $\mathcal{M}_{\mathrm{cyc}}(\theta)$ and, for real z_i, maps (X_z, ω_z) to an isomorphic surface. Mirzakhani and Wright prove that for almost every polygon with given interior angles the unfolding has orbit closure $\mathcal{M}(\theta)$ [26, Lemma 6.3], and their proof treats each connected component of the space of polygons with the angles in a given cyclic order. So $\mathcal{M}(\theta)$ is the orbit closure of the unfolding of almost every quadrilateral of type θ, in every cyclic order of its angles; and since $\mathcal{M}(\theta)$ is closed and $\mathrm{GL}^+(2, \mathbb{R})$-invariant and contains every unfolding of a quadrilateral of type θ, it contains the orbit closure of each of them. We call $\mathcal{M}(\theta)$ the orbit closure of the *generic member* of θ.

Let g be the genus of X. By Riemann–Hurwitz for the cover of degree k branched over $z_1, \dots, z_4$,

$$2g - 2 = -2k + \sum_{i=1}^{4} (k - g_i) = 2k - \sum_{i=1}^{4} g_i. \tag{2.1}$$

Lemma 2.1. *Let θ be admissible. Then $g \geq 1$, and $g = 1$ only for $\theta = (1, 1, 1, 1)$.*

Proof. As $g_i \leq q_i$, (2.1) gives $2g - 2 \geq 2k - \sum_i q_i = 0$. Equality forces $g_i = q_i$, that is $q_i \mid k$, for every i; as $q_i \neq k$, each $q_i \leq k/2$, and $\sum_i q_i = 2k$ forces $q_i = k/2$ for all i. Then $\gcd(k, q_1, \dots, q_4) = k/2$ is 1, so $k = 2$, $N = 4$ and $\theta = (1, 1, 1, 1)$. □

At $\theta = (1, 1, 1, 1)$ the quadrilaterals are the rectangles, X is a flat torus and $\Sigma = \emptyset$. From now on, unless said otherwise, θ is admissible and reduced with $g \geq 2$.

2.2. **Characters.** For $x \in \mathbb{Z}/k$ let H^1_x, $H^1(X,\Sigma)_x$ and $H^{1,0}_x$ be the ζ^x-eigenspaces of T^* on $H^1(X;\mathbb{C})$, on $H^1(X,\Sigma;\mathbb{C})$ and on the space $H^{1,0}(X)$ of holomorphic 1-forms. For a real number t, $\{t\}$ is its fractional part. Let

$$n(x) = \#\{i :\ xq_i \not\equiv 0 \pmod k\}, \qquad h(x) = \sum_{i=1}^{4} \Big\{\frac{-xq_i}{k}\Big\} - 1.$$

As Mirzakhani and Wright recall [26, Lemma 6.1 and Corollary 6.2] (for four points, [28, Lemma 2.6 and Proposition 2.7]), for $x \not\equiv 0$, $H^{1,0}_x$ is spanned by the forms

$$P(z)\prod_{i=1}^{4}(z-z_i)^{-\{-xq_i/k\}}\,dz, \qquad \deg P \le h(x)-1, \tag{2.2}$$

so that $\dim H^{1,0}_x = h(x)$, and $\dim H^1_x = \sum_i\{xq_i/k\}+\sum_i\{-xq_i/k\}-2$; here $\prod_i(z-z_i)^{-\{-xq_i/k\}}$ stands for $y^x\prod_i(z-z_i)^{\lfloor -xq_i/k\rfloor}$. As $\{t\}+\{-t\}$ is 1 for $t\notin\mathbb{Z}$ and 0 for $t\in\mathbb{Z}$, this says $\dim H^1_x = n(x)-2$. Also $H^1_0 = 0$, since $X/\langle T\rangle$ has genus zero.

For $x \not\equiv 0$ we have $n(x) \ge 2$: if $xq_i \equiv 0$ for three indices, then also for the fourth, as $\sum_i xq_i \equiv 0$, and $xq_i \equiv 0$ for all i would give $x\gcd(k,q_1,\dots,q_4) = x \equiv 0$. So H^1_x is a plane, a line or zero according as $n(x)$ is 4, 3 or 2; on a plane $h(x)+h(-x)=2$, and on a line $h(x)+h(-x)=1$. We call x *generic* if $n(x)=4$. Let

$$F = \{x :\ n(x)=4\}, \qquad F_1 = \{x :\ n(x)=3\},$$

where, when k is even, only odd x are admitted in F and F_1. Every unit of $\mathbb{Z}/k$ is generic, since $0<q_i<2k$ and $q_i \ne k$; in particular $1\in F$. Let $U=(\mathbb{Z}/k)^\times$ be the group of units.

Let $p\colon H^1(X,\Sigma;\mathbb{C}) \to H^1(X;\mathbb{C})$ be the restriction to absolute cohomology and $Z = \ker p$, the image of the coboundary from functions on Σ, and let Z_c be the ζ^c-eigenspace of T^* on Z. Since p is surjective and T-equivariant, for every c the sequence

$$0 \to Z_c \to H^1(X,\Sigma)_c \xrightarrow{\ p\ } H^1_c \to 0 \tag{2.3}$$

is exact. For $c \not\equiv 0$ the constant functions contribute nothing to Z_c, and the functions of character c on the orbit of g_i points over z_i form a line if $(k/g_i) \mid c$, that is if $cq_i \equiv 0$, and are zero otherwise. So Z_c, for $c\not\equiv 0$, has dimension equal to the number of vertices i with $q_i \nmid k$ and $cq_i \equiv 0 \pmod k$; in particular $Z_x = 0$ and $H^1(X,\Sigma)_x \cong H^1_x$ for x generic. We call c a *relative character* if $c\not\equiv 0$ and $Z_c \ne 0$.

2.3. **Automorphisms.** The following lemma is the source of the parity statements at even k and of the exclusion of hyperelliptic loci at odd k; for more than two distinct angles this exclusion is [26, Lemma 6.7].

Lemma 2.2. *Let θ be admissible and reduced, with $g \ge 2$. For all z outside a countable union of proper analytic subsets of the configuration space, every automorphism τ of X_z with $\tau^*\omega \in \mathbb{C}\omega$ is a power of T.*

Proof. Let G be the group of automorphisms τ of $X = X_z$ with $\tau^*\omega \in \mathbb{C}\omega$; it is finite, as $g\ge 2$, and contains $\langle T\rangle$. Suppose $\langle T\rangle \ne G$, and let $d = [G:\langle T\rangle] \ge 2$ and $f\colon X/\langle T\rangle \to X/G$ the induced map of degree d. As $X/\langle T\rangle = \mathbb{P}^1$, also $X/G \cong \mathbb{P}^1$.

Suppose first that $X \to X/G$ has at most three branch values. Normalized to lie in $\{0,1,\infty\}$, they leave countably many possibilities for the cover $X\to X/G$, by its permutation monodromy; each X so obtained has finitely many automorphisms, hence finitely many cyclic subgroups whose quotient map to $\mathbb{P}^1$ has exactly four branch values, and each of these gives finitely many cross-ratios of its branch values, one for each order of them. So this case

occurs only for z in a countable union of proper analytic subsets, and we exclude it. Hence $X \to X/G$ has $B \geq 4$ branch values.

Let w be a branch value of $X \to X/G$ and $\xi \in f^{-1}(w)$ a point at which f is unramified. All points of X over w have stabilizers in G of the same order $e_w > 1$, and this order is the product of the ramification indices of $X \to X/\langle T \rangle$ and of f; so $X \to X/\langle T \rangle$ is ramified over ξ, and ξ is one of $z_1, \ldots, z_4$. Let B_1 be the number of such unramified preimages over all B branch values; then $B_1 \leq 4$. Over each branch value the ramification of f is $\sum(e_\xi - 1) \geq \sum_{e_\xi \geq 2} e_\xi/2$, the sums over the preimages ξ, where e_ξ is the ramification index of f at ξ; summing over the B branch values, the total ramification of f is at least $(Bd - B_1)/2$. Riemann–Hurwitz for $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ gives total ramification $2d-2$, so $4d-4 \geq Bd-B_1 \geq 4d-4$. Hence $B = 4$, $B_1 = 4$, every ramified preimage of a branch value has $e_\xi = 2$, and f is unramified elsewhere; in particular $z_1, \ldots, z_4$ are unramified points of f over the four branch values, and the ramified points of f are not among them.

Since $\tau^*\omega = \mu\omega$ with μ a root of unity, the flat metric $|\omega|$ is G-invariant and descends to cone metrics on $X/\langle T \rangle$ and X/G, and f is a local isometry off its ramification points. On $X/\langle T \rangle$ the metric has cone angle $2\pi q_i/k$ at z_i and is smooth elsewhere. Let $2\pi\nu(w)$ be the cone angle of X/G at a branch value w. A ramified point ξ of f over w has $e_\xi = 2$ and is a smooth point of $X/\langle T \rangle$, so $\nu(w) = 1/2$; an unramified point z_i over w has $q_i/k = \nu(w)$. Consequently a vertex z_i with $q_i/k \neq 1/2$ lies over a branch value with no ramified preimage: its fiber consists of d vertices z_j, all with $q_j/k = q_i/k$. If all four q_i/k were $1/2$, then $k = 2$ and $\theta = (1,1,1,1)$, of genus one. Otherwise some $q_i/k = \nu \neq 1/2$, and not all of the other vertices have $q_j/k = 1/2$: else $d\nu + (4-d)/2 = 2$ would give $\nu = 1/2$. So a second fiber of d vertices with a common value $\neq 1/2$ exists, and $2d \leq 4$. Thus $d = 2$, and $\{q_1, \ldots, q_4\}$ consists of two equal pairs; the two values differ, since four equal q_i would all equal $k/2$. So $a_1 = a_2 < a_3 = a_4$, against reducedness. □

We call a point $X = X_z$ of $\mathcal{M}_{\mathrm{cyc}}(\theta)$ *generic* if every automorphism τ of X with $\tau^*\omega \in \mathbb{C}\omega$ is a power of T and X is not in the self-crossing locus of $\mathcal{M}(\theta)$. As in [26, proof of Lemma 7.7], this locus does not contain $\mathcal{M}_{\mathrm{cyc}}(\theta)$: it is a finite union of affine invariant submanifolds of smaller dimension [14, p. 674, after Definition 1.2], a closed invariant set, so if it contained $\mathcal{M}_{\mathrm{cyc}}(\theta)$ it would contain the orbit closure $\mathcal{M}(\theta)$ of the generic member, which has larger dimension and so is not covered by finitely many submanifolds of smaller dimension. With Lemma 2.2 and the proof of Proposition 2.5 below, X_z is generic for all z outside a countable union of proper analytic subsets of the configuration space. At a generic point there are a neighborhood W of $[\omega]$ in period coordinates $H^1(X, \Sigma; \mathbb{C})$ and a single complex linear subspace

$$V = T_X\mathcal{M}(\theta) \subset H^1(X, \Sigma; \mathbb{C})$$

with $\mathcal{M}(\theta) \cap W = V \cap W$ [13, 14]. Mirzakhani and Wright prove that the tangent space of $\mathcal{M}(\theta)$ at a point of $\mathcal{M}_{\mathrm{cyc}}(\theta)$ contains $H^1(X,\Sigma)_1 + H^1(X,\Sigma)_{-1}$ [26, Corollary 6.5]. The rank of $\mathcal{M}(\theta)$ is $\frac{1}{2}\dim_{\mathbb{C}} p(V)$.

2.4. **The ambient locus.** If k is odd, the *ambient locus* $\mathcal{L}(\theta)$ is the connected component of the stratum of (X, ω) that contains it; it contains $\mathcal{M}_{\mathrm{cyc}}(\theta)$, which is connected, so $\mathcal{M}(\theta) \subset \mathcal{L}(\theta)$.

If k is even, let $\iota = T^{k/2}$. Then $\iota^*\omega = -\omega$ on every surface of $\mathcal{M}_{\mathrm{cyc}}(\theta)$, so ω^2 descends to a quadratic differential q on X/ι; it is not the square of an Abelian differential, for then ω would be ι-invariant. So (X, ω) is the holonomy double cover of $(X/\iota, q)$. The orders of the zeros and poles of q depend only on θ, and $\mathcal{M}_{\mathrm{cyc}}(\theta)$ is connected, so all $(X_z/\iota, q_z)$ lie in one

component $\mathcal{Q}$ of a stratum of quadratic differentials. Let $\widetilde{\mathcal{Q}}$ be the locus of holonomy double covers of the surfaces of $\mathcal{Q}$, in the stratum of (X,ω); it is an affine invariant submanifold [4, §1], and the ambient locus is $\mathcal{L}(\theta) = \widetilde{\mathcal{Q}}$. The double cover of the image of $(X/\iota, q)$ under $g \in \mathrm{GL}^+(2,\mathbb{R})$ is $g(X,\omega)$, so $\mathcal{L}(\theta)$ contains the orbits of the surfaces of $\mathcal{M}_{\mathrm{cyc}}(\theta)$, and $\mathcal{M}(\theta) \subset \mathcal{L}(\theta)$. The regular points of X fixed by ι lie over poles of q and are not marked.

Let k be even and X generic. Then ι is the only automorphism of X with $\iota^*\omega = -\omega$: by Lemma 2.2 such an automorphism is a power T^j with $\zeta^j = -1$. Let

$$E^- = \{v \in H^1(X,\Sigma;\mathbb{C}) :\ \iota^* v = -v\} = \bigoplus_{c \text{ odd}} H^1(X,\Sigma)_c,$$

as ι^* acts on $H^1(X,\Sigma)_c$ by $\zeta^{ck/2} = (-1)^c$. As in [18, §2.2], Lanneau's chart [22, §1.1.1] identifies a neighborhood of $(X/\iota, q)$ in $\mathcal{Q}$, by the double-cover map, with an open subset of E^-: the preimage of a pole of q is one point fixed by ι, and forgetting it does not change the (-1)-eigenspace. As ι is the only involution of its kind on X, $\widetilde{\mathcal{Q}}$ is this one sheet near X, and $T_X\widetilde{\mathcal{Q}} = E^-$. Since $\mathcal{M}(\theta) \subset \widetilde{\mathcal{Q}}$, we get $V \subset E^-$.

Lemma 2.3. *At a generic point X,*

$$p(T_X\mathcal{L}(\theta)) = \bigoplus_{x \in F \cup F_1} H^1_x,$$

so $\operatorname{rank}\mathcal{L}(\theta) = \frac{1}{2}D$ *with* $D = \sum_{x\in F\cup F_1} \dim H^1_x = 2|F| + |F_1|$.

Proof. For k odd, $T_X\mathcal{L}(\theta) = H^1(X,\Sigma;\mathbb{C})$, and $H^1(X;\mathbb{C}) = \bigoplus_{x\neq 0} H^1_x$ is the sum over $F \cup F_1$, as $H^1_x = 0$ for $n(x) = 2$ and $H^1_0 = 0$. For k even, $p(E^-) = \bigoplus_{c \text{ odd}} H^1_c$ by (2.3), and again only c with $n(c) \geq 3$, that is $c \in F \cup F_1$, contribute. □

2.5. **The absolute tangent space is a sum of eigenspaces.** The rotation of (X,ω) by $2\pi/k$ is a loop in $\mathcal{M}(\theta)$ whose monodromy is T; this makes V invariant under T, and the question is how $p(V)$ meets the planes H^1_x, $x \in F$. The answer comes from the monodromy of the family X_z over the configuration space. Mirzakhani and Wright observe that irreducibility of this monodromy would extend their Corollary 7.9 beyond triangles [26, remark after Corollary 7.9]; the proof below is McMullen's reflection argument [23, Proposition 5.1], which Menet and Nguyen use for unequal weights [24, proof of Lemma 7.10]. Compare the irreducibility of hypergeometric groups [5, Proposition 3.3].

Lemma 2.4. *Let $x \in F$, and let* Conf *be the space of ordered quadruples of distinct points of $\mathbb{C}$. The local system over* Conf *with fiber H^1_x at z (for the surface X_z) is irreducible.*

Proof. Let $X^\circ \subset X$ be the complement of the points over $z_1,\dots,z_4$, and $Y = \mathbb{P}^1 \setminus \{z_1,\dots,z_4\}$. As the cover is unramified over ∞, $X^\circ \to Y$ is an unramified cyclic covering with group $\langle T\rangle$, so $H^1(X^\circ;\mathbb{C})_x \cong H^1(Y;\mathbb{L})$, where $\mathbb{L}$ is a rank-one local system on Y whose monodromy around z_i is $\lambda_i = \zeta^{\epsilon x q_i}$, with a sign $\epsilon = \pm 1$ fixed by the conventions. As x is generic, $\lambda_i \neq 1$ for all i, so $H^0(Y;\mathbb{L}) = 0$; $H^2(Y;\mathbb{L}) = 0$ as Y is not compact; and the Euler characteristic $\chi(Y) = -2$ gives $\dim H^1(Y;\mathbb{L}) = 2$. Restriction $H^1(X;\mathbb{C}) \to H^1(X^\circ;\mathbb{C})$ is injective, as $H^1(X,X^\circ;\mathbb{C}) = 0$ by excision, and it is T-equivariant; since $\dim H^1_x = 2$, it identifies H^1_x with $H^1(Y;\mathbb{L})$, compatibly with the Gauss–Manin connections over Conf.

Choose a base point $\star \in Y$ and loops $\gamma_1,\dots,\gamma_4$ around $z_1,\dots,z_4$ generating $\pi_1(Y,\star)$ with $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$, and let $\lambda\colon \pi_1(Y,\star) \to \mathbb{C}^\times$ be the monodromy character of $\mathbb{L}$, so $\lambda(\gamma_i) = \lambda_i$. Then $H^1(Y;\mathbb{L})$ is the space of crossed homomorphisms v, that is, maps

with $v(\gamma\gamma') = v(\gamma) + \lambda(\gamma)v(\gamma')$, modulo the coboundaries $v(\gamma) = (\lambda(\gamma) - 1)c$. A crossed homomorphism is determined by its values $v_i = v(\gamma_i)$, subject only to

$$v_1 + \lambda_1 v_2 + \lambda_1\lambda_2 v_3 + \lambda_1\lambda_2\lambda_3 v_4 = 0,$$

and the coboundaries are the vectors $c(\lambda_i - 1)_i$. For $\gamma, \gamma' \in \pi_1(Y, \star)$ the rule gives

$$v(\gamma\gamma'\gamma^{-1}) = (1 - \lambda(\gamma'))\, v(\gamma) + \lambda(\gamma)\, v(\gamma').$$

Let $1 \le i \le 3$. The full twist of z_i and z_{i+1} is a loop in Conf; its action on $\pi_1(Y, \star)$ sends γ_i and γ_{i+1} to $\gamma\gamma_i\gamma^{-1}$ and $\gamma\gamma_{i+1}\gamma^{-1}$ with $\gamma = \gamma_i\gamma_{i+1}$, and fixes the other two loops. Pulling v back by this action, with $v(\gamma) = v_i + \lambda_i v_{i+1}$ and $\lambda(\gamma) = \lambda_i\lambda_{i+1}$, gives a cocycle v' with

$$v'_i - v_i = (1 - \lambda_i)(v_i + \lambda_i v_{i+1}) + \lambda_i\lambda_{i+1}v_i - v_i = \lambda_i\, \varphi_i(v),$$
$$v'_{i+1} - v_{i+1} = (1 - \lambda_{i+1})(v_i + \lambda_i v_{i+1}) + \lambda_i\lambda_{i+1}v_{i+1} - v_{i+1} = -\varphi_i(v),$$

and $v'_l = v_l$ for $l \ne i, i+1$, where $\varphi_i(v) = (\lambda_{i+1} - 1)v_i + (1 - \lambda_i)v_{i+1}$. So $v' - v = \varphi_i(v)D_i$, where D_i has entries λ_i and -1 in the positions i and $i+1$ and zeros elsewhere; D_i is a cocycle, and φ_i vanishes on coboundaries. The monodromy of the local system along the twist is this map or its inverse, up to a non-zero scalar coming from the choice of a lift of the base point to the cover, and these have the same invariant subspaces.

The classes of $D_1 = (\lambda_1, -1, 0, 0)$ and $D_2 = (0, \lambda_2, -1, 0)$ are independent: a combination equal to a coboundary $c(\lambda_i - 1)_i$ has fourth entry $0 = c(\lambda_4 - 1)$, so $c = 0$, and D_1, D_2 are independent vectors. Moreover $\varphi_1(D_2) = (1 - \lambda_1)\lambda_2 \ne 0$ and $\varphi_2(D_1) = 1 - \lambda_3 \ne 0$. Let $\Lambda \ne 0$ be an invariant subspace and $v \in \Lambda$ a non-zero class. If $\varphi_1(v) \ne 0$, then $D_1 \in \Lambda$, then $\varphi_2(D_1)D_2 \in \Lambda$, and Λ is everything; likewise if $\varphi_2(v) \ne 0$. If $\varphi_1(v) = \varphi_2(v) = 0$, then $v_1/(\lambda_1 - 1) = v_2/(\lambda_2 - 1) = v_3/(\lambda_3 - 1)$, and the cocycle relation gives the same value for $v_4/(\lambda_4 - 1)$, so v is a coboundary, a contradiction. No condition on the products $\lambda_i\lambda_{i+1}$ enters. □

Proposition 2.5. *Let X be a generic point of $\mathcal{M}_{\mathrm{cyc}}(\theta)$. Then $T^*V = V$, so $V = \bigoplus_{c \in \mathbb{Z}/k} V_c$ with $V_c = V \cap H^1(X, \Sigma)_c$. Let*

$$A = \{x \in F \cup F_1 :\ H^1_x \subset p(V)\}.$$

Then

$$p(V) = \bigoplus_{x \in A} H^1_x, \qquad \operatorname{rank} \mathcal{M}(\theta) = \tfrac{1}{2} \sum_{x \in A} \dim H^1_x, \qquad \pm 1 \in A. \tag{2.4}$$

Proof. We follow Mirzakhani and Wright [26, proof of Lemma 7.7]. Near the generic point X, $\mathcal{M}(\theta)$ is the single subspace V in period coordinates. Rotation by $e^{i\phi}$, $0 \le \phi \le 2\pi/k$, is a path in $\mathcal{M}(\theta)$ from (X, ω) to $(X, \zeta\omega)$; as T is an isomorphism from $(X, \zeta\omega)$ to (X, ω), this path is a loop with monodromy T^* on $H^1(X, \Sigma; \mathbb{C})$, and the tangent space of the one branch of $\mathcal{M}(\theta)$ through X is carried to itself. So $T^*V = V$, and V is the sum of its intersections with the eigenspaces of the finite-order map T^*. As p is T-equivariant, $p(V) = \bigoplus_c p(V_c)$ with $p(V_c) \subset H^1_c$.

Let $\mathrm{Conf}^\circ \subset \mathrm{Conf}$ be the set of z for which X_z is not in the self-crossing locus of $\mathcal{M}(\theta)$. The self-crossing locus is a finite union of affine invariant submanifolds of smaller dimension (Section 2.3), so in period coordinates it is locally a finite union of linear subspaces; the periods of ω_z are holomorphic functions of z [26, proof of Lemma 6.3], so the complement of Conf° is a closed analytic subset of Conf, and it is proper, as $\mathcal{M}_{\mathrm{cyc}}(\theta)$ is not contained in the self-crossing locus. Hence Conf° is connected and $\pi_1(\mathrm{Conf}^\circ) \to \pi_1(\mathrm{Conf})$ is surjective, loops being movable off a subset of real codimension at least two. Over Conf°, the tangent

spaces $T_{X_z}\mathcal{M}(\theta)$ form a flat subbundle of the local system with fibers $H^1(X_z, \Sigma_z; \mathbb{C})$: near $z \in \mathrm{Conf}^\circ$, $\mathcal{M}(\theta)$ is one linear subspace in period coordinates and contains the nearby $X_{z'}$. Hence V and $p(V) \cap H^1_c$ are invariant under the monodromy of $\pi_1(\mathrm{Conf})$. For $c \in F$, Lemma 2.4 gives $p(V) \cap H^1_c \in \{0, H^1_c\}$; for $n(c) = 3$ the space H^1_c is a line, and for $n(c) = 2$ it is zero. So $p(V)$ is the sum of the spaces H^1_c it contains. The same holds at every $z \in \mathrm{Conf}^\circ$, and as the subbundle is flat and T-equivariant over the connected Conf°, the set A in the statement does not depend on the generic point.

It remains to see that these c lie in $F \cup F_1$, that is, are odd when k is even; this holds as $V \subset E^-$ (Section 2.4). The rank formula is the first formula. Finally $H^1(X, \Sigma)_{\pm 1} \subset V$ by Mirzakhani and Wright's Corollary 6.5, and $\pm 1 \in F$, as units are generic, and they are odd when k is even. □

By Lemma 2.3 and Proposition 2.5, $\operatorname{rank} \mathcal{M}(\theta) = \operatorname{rank} \mathcal{L}(\theta)$ if and only if $A = F \cup F_1$. In the parity statement at even k the uniqueness of ι, that is Lemma 2.2, is what makes $\widetilde{\mathcal{Q}}$ a single sheet E^- at X.

3. Colliding two vertices

When two branch points z_i, z_j with $q_i + q_j > k$ collide and $\gcd(k, q_l, q_m) = 1$ for the other two indices l, m, the surfaces X_z converge to the unfolding of a triangle, and by Mirzakhani and Wright's theorem on the boundary of an affine invariant submanifold the tangent space of $\mathcal{M}(\theta)$ contains a copy of the tangent space of a boundary component through that unfolding. This section turns this into a rule for the set A of Proposition 2.5: if the triangle's orbit closure has the largest possible rank, the characters that are generic for the triangle lie in A (Theorem 3.2). The limit surface may have marked points of cone angle 2π, and we first show that marking such points does not lower ranks.

3.1. **Marked points.** Following Mirzakhani and Wright, a stratum may have marked points, and a marked point where ω does not vanish is a zero of ω of order zero; an affine invariant submanifold of such a stratum is a properly immersed connected submanifold defined locally in period coordinates by real linear equations with zero constant term [25, §§2.1 and 2.2]. So near each of its points an affine invariant submanifold is a finite union of linear subspaces of the period chart, its *tangent spaces* at the point; for a surface with marked set Σ' the chart is $H^1(\,\cdot\,, \Sigma'; \mathbb{C})$, p denotes the map to absolute cohomology, and the rank is $\frac{1}{2} \dim p(L)$ for a tangent space L. The rank does not depend on L or on the point, the submanifold being connected. Eskin and Mirzakhani call an ergodic $\mathrm{SL}(2, \mathbb{R})$-invariant probability measure ν on the surfaces of area one of a stratum *affine* if its support is the locus of area one $\mathcal{M}_1$ of an affine invariant submanifold $\mathcal{M}$ and, near each point of $\mathcal{M}$, the measure $t^{2m-1}\, dt\, d\nu$ on $\mathcal{M}$, in the coordinates tY with $t > 0$ and $Y \in \mathcal{M}_1$, where $m = \dim_{\mathbb{C}} \mathcal{M}$, is a constant multiple of Lebesgue measure on the tangent spaces [13, Definition 1.1]; in particular $\nu(E) = 0$ for a Borel set $E \subset \mathcal{M}_1$ whose cone $\{tY : t > 0,\ Y \in E\}$ is null for Lebesgue measure on each tangent space. They prove that every ergodic probability measure on the surfaces of area one that is invariant under the upper triangular subgroup of $\mathrm{SL}(2, \mathbb{R})$ is $\mathrm{SL}(2, \mathbb{R})$-invariant and affine [13, Theorem 1.4], and with Mohammadi that the locus of area one of every orbit closure is the support of such a measure [14, Theorem 2.1]. These results hold in strata with marked points [13, Introduction]. We use the marked case for one orbit closure, $\mathcal{N}^P$ below.

Lemma 3.1. *Let (Y, η) be a translation surface with zero set Σ_Y, in the stratum $\mathcal{H}$ whose marked points are the zeros, and let $P \subset Y \setminus \Sigma_Y$ be a non-empty finite set. Let $\mathcal{H}^P$ be the*

stratum in which the points of P are marked as well, and $\mathcal{N} \subset \mathcal{H}$ and $\mathcal{N}^P \subset \mathcal{H}^P$ the orbit closures of (Y, η) and $(Y, \eta; P)$.

(a) *At every point of $\mathcal{N}^P$, the restriction $H^1(\,\cdot\,, \Sigma_Y \cup P; \mathbb{C}) \to H^1(\,\cdot\,, \Sigma_Y; \mathbb{C})$ maps every tangent space L of $\mathcal{N}^P$ onto a tangent space of $\mathcal{N}$. In particular $\dim p(L) = 2 \operatorname{rank} \mathcal{N}$, and $\operatorname{rank} \mathcal{N}^P = \operatorname{rank} \mathcal{N}$.*

(b) *Every affine invariant submanifold of $\mathcal{H}^P$ that contains $(Y, \eta; P)$ has rank at least $\operatorname{rank} \mathcal{N}$.*

When $\Sigma_Y \neq \emptyset$, part (a) is a lemma of Apisa and Wright; the proof below does not use a zero of η. For an affine invariant submanifold $\mathcal{M}$ of a stratum $\mathcal{H}$, with $\mathcal{H}^{*n}$ the surfaces of $\mathcal{H}$ with n distinct marked points that are not zeros and $\pi\colon \mathcal{H}^{*n} \to \mathcal{H}$ the forgetful map, they call an affine invariant submanifold $\mathcal{N}$ of $\mathcal{H}^{*n}$ an *n-point marking* over $\mathcal{M}$ if $\pi(\mathcal{N})$ is a dense subset of $\mathcal{M}$ [4, §1]. They prove that the linear equations defining the fibers of $\mathcal{N}$ over $\mathcal{M}$ can be chosen to hold locally on $\mathcal{N}$, and that each fiber is empty or has dimension $\dim \mathcal{N} - \dim \mathcal{M}$ [4, Lemma 2.1]. By Step 1 below, $\mathcal{N}^P$ is a point marking over $\mathcal{N}$ in this sense. Their proof uses that the image of the marking contains an open subset of the base (compare [1, Lemmas 2.14 and 2.16]), for which we include an argument (Steps 3 and 4).

Proof. Let $\mathfrak{f}\colon \mathcal{H}^P \to \mathcal{H}$ be the map that forgets the points of P. It is continuous, commutes with $\mathrm{GL}^+(2, \mathbb{R})$ and preserves the area. In period coordinates it is the restriction $\operatorname{res}\colon H^1(Y, \Sigma_Y \cup P; \mathbb{C}) \to H^1(Y, \Sigma_Y; \mathbb{C})$, a surjective linear map with $p \circ \operatorname{res} = p$. Its kernel is the annihilator of the image of $H_1(Y, \Sigma_Y)$ in $H_1(Y, \Sigma_Y \cup P)$, so the kernels form a flat subbundle of the bundle of relative cohomology over $\mathcal{H}^P$. At a surface with automorphisms the period chart is the quotient of an open subset of relative cohomology by a finite group, and the statements below are made on that open subset, for the lifts, which the group permutes.

Step 1: the image. $\mathfrak{f}$ maps the orbit of $(Y, \eta; P)$ onto that of (Y, η), and $\mathcal{N}$ is closed; so $\mathfrak{f}(\mathcal{N}^P) \subset \mathcal{N}$, and $\mathfrak{f}(\mathcal{N}^P)$ contains the orbit of (Y, η), which is dense in $\mathcal{N}$. The map $\mathfrak{f}$ need not be proper, so $\mathfrak{f}(\mathcal{N}^P)$ need not be closed, and a dense invariant subset can be small; the next two steps deal with this.

Step 2: the measure. Let $\mathcal{N}_1^P$ and $\mathcal{N}_1$ be the loci of area one. $\mathcal{N}_1^P$ is the support of an ergodic $\mathrm{SL}(2, \mathbb{R})$-invariant probability measure ν that is affine. Then $\mathfrak{f}_*\nu$ is a probability measure on the surfaces of area one of $\mathcal{H}$, invariant and ergodic under $\mathrm{SL}(2, \mathbb{R})$, hence ergodic under the diagonal subgroup by Moore's ergodicity theorem [27] (as in [14, proof of Lemma 11.1]), and so under the upper triangular subgroup, which contains it. By Eskin and Mirzakhani's theorem, in the stratum $\mathcal{H}$, $\mathfrak{f}_*\nu$ is affine: its support is the locus of area one of an affine invariant submanifold $\mathcal{N}'$, on which it is locally Lebesgue measure in the sense above. As $\mathfrak{f}$ is continuous, the support of $\mathfrak{f}_*\nu$ is the closure of $\mathfrak{f}(\mathcal{N}_1^P)$, which by Step 1 lies in $\mathcal{N}_1$ and contains a dense subset of it. So $\mathcal{N}' = \mathcal{N}$.

Step 3: one tangent space with full image. Let $d = \dim \mathcal{N}$. The manifold underlying $\mathcal{N}^P$ is second countable, so $\mathcal{N}_1^P$ is the union of countably many compact sets $K_1, K_2, \ldots$, each contained in one period chart of $\mathcal{H}^P$ and in one tangent space L_i of $\mathcal{N}^P$ there, and so small that $\mathfrak{f}(K_i)$ lies in one period chart of $\mathcal{H}$, in which $\mathcal{N}$ is a finite union of linear subspaces $E_1, \ldots, E_s$ of dimension d, and in finitely many of the open sets on which the measure of Step 2 is Lebesgue measure on such a subspace. A neighborhood of a point of K_i in L_i is mapped into $\mathcal{N}$, so a non-empty open subset of the linear space $\operatorname{res}(L_i)$ lies in $E_1 \cup \cdots \cup E_s$; a non-empty open subset of a real vector space is not contained in finitely many proper subspaces, so $\operatorname{res}(L_i) \subset E_j$ for some j. Suppose that $\dim \operatorname{res}(L_i) < d$ for every

i. Then $\operatorname{res}(L_i) \cap E_j$ is a proper subspace of E_j for every j, so it is null for Lebesgue measure on E_j, and so is the cone over $\mathfrak{f}(K_i)$; hence $\mathfrak{f}_*\nu(\mathfrak{f}(K_i)) = 0$ for every i, by Step 2. But $\mathfrak{f}_*\nu(\bigcup_i \mathfrak{f}(K_i)) \geq \nu(\mathcal{N}_1^P) = 1$. So $\dim \operatorname{res}(L_i) \geq d$ for some i, and then $\operatorname{res}(L_i)$ is one of the E_j.

Step 4: every tangent space. The tangent spaces of $\mathcal{N}^P$ form a flat subbundle of relative cohomology over the connected manifold underlying $\mathcal{N}^P$. Its intersection with the flat subbundle of the kernels of res has locally constant, hence constant, dimension. So $\dim \operatorname{res}(L)$ is the same for all tangent spaces L at all points, and it is d by Step 3. As in Step 3, a neighborhood of the point in L is mapped into $\mathcal{N}$, so a non-empty open subset of $\operatorname{res}(L)$ lies in the union of the tangent spaces of $\mathcal{N}$ at the image point, and $\operatorname{res}(L)$ lies in one of them; they all have dimension d, so it is one of them, L', and $p(L) = p(\operatorname{res}(L)) = p(L')$ has dimension $2 \operatorname{rank} \mathcal{N}$. This proves (a).

Step 5. Let $\mathcal{B}$ be an affine invariant submanifold of $\mathcal{H}^P$ containing $(Y, \eta; P)$. It is closed, as the image of a proper immersion, and $\mathrm{GL}^+(2, \mathbb{R})$-invariant [30, §2], so $\mathcal{N}^P \subset \mathcal{B}$. At $(Y, \eta; P)$ a tangent space L of $\mathcal{N}^P$ lies in the union of the finitely many tangent spaces of $\mathcal{B}$, hence in one of them, $L_{\mathcal{B}}$. Then $p(L_{\mathcal{B}}) \supset p(L)$, of dimension $2 \operatorname{rank} \mathcal{N}$ by (a). This proves (b). □

3.2. **The collision.** For a rational triangle Δ with angles $p_1\pi/d, p_2\pi/d, p_3\pi/d$, where $p_1 + p_2 + p_3 = d$, we write $\Delta = (p_1, p_2, p_3)/d$, and call the expression *reduced* if $\gcd(p_1, p_2, p_3) = 1$; every rational triangle has a unique reduced expression. For a reduced expression the unfolding (Y, η) of Δ [31], the connected cover, is a cyclic cover of degree d of the double S of Δ, and $|\eta|$ is the pull-back of the flat metric of S; with the third branch point at ∞, the unfolding is the cover $y^d = x^{p_1}(x-1)^{p_2}$ of [26, §6] for three branch points. The holonomy of S around its cone points generates the group of d-th roots of unity, and the image of the fundamental group of Y minus the points over the vertices lies in its kernel, of index d, so off the vertices $Y \to S$ is the covering defined by the kernel of the holonomy. Let $F(\Delta)$ be the set of $x \in \mathbb{Z}/d$ with $xp_1, xp_2, xp_3 \not\equiv 0 \pmod d$, odd ones only when d is even, for the reduced expression.

Let θ be admissible and reduced with $g \geq 2$. We call a pair $\{l, m\}$ of indices *short* if $q_l + q_m < k$. For a short pair, the *collision triangle* is

$$\Delta_{lm} = (k - q_l - q_m, q_l, q_m)/k,$$

and we call it *connected* if $\gcd(k, q_l, q_m) = 1$; then this expression is reduced. If i, j are the other two indices, then $q_i + q_j > k$ and $k - q_l - q_m = q_i + q_j - k$: when the vertices i and j of a quadrilateral of type θ collide, their angles $\pi q_i/k$ and $\pi q_j/k$ are replaced by one angle $\pi(q_i + q_j - k)/k$. If no two angles of θ sum to π, each of the three splittings of the four indices into two pairs contains exactly one short pair.

Theorem 3.2 (Collision criterion)**.** *Let $\{l, m\}$ be a short pair with $\gcd(k, q_l, q_m) = 1$, and $\Delta = \Delta_{lm}$. Suppose that the orbit closure of the unfolding of Δ has rank $|F(\Delta)|/2$. Then every $x \in F \cup F_1$ with xq_l, xq_m, $x(q_l + q_m) \not\equiv 0 \pmod k$ lies in A.*

Colliding two vertices and passing to the boundary is a device of Apisa [2, Sublemma 11.3] and of Apisa and Aygun [3, Lemma 12.3]; here the boundary theorem is applied eigenspace by eigenspace. The rest of this subsection is the proof. Let i, j be the other two indices, $\mu = q_i + q_j - k = k - q_l - q_m$, with $0 < \mu < k$, and $\alpha = \mu/k$. Fix distinct points $z_0, z_l, z_m \in \mathbb{C}$.

The limit surface. Let Y be the normalization of the curve $y^k = (z-z_0)^\mu (z-z_l)^{q_l} (z-z_m)^{q_m}$, with

$$\eta = \frac{y\,dz}{(z-z_0)(z-z_l)(z-z_m)}, \qquad T_Y(y, z) = (\zeta y, z).$$

As $\gcd(k,\mu,q_l,q_m)=1$, Y is connected; as $\mu+q_l+q_m=k$, the map z is unramified over ∞ and η has neither zeros nor poles there. The metric $|\eta|$ descends to the flat metric on $\mathbb{P}^1$ with cone angles $2\pi\mu/k$, $2\pi q_l/k$, $2\pi q_m/k$ at z_0, z_l, z_m, the double of a triangle with the angles of Δ, and Y is the covering defined by the kernel of its holonomy; so (Y,η) is the unfolding of Δ, up to a constant factor. Over z_0 there are $\gcd(k,\mu)$ points, of cone angle $2\pi\mu/\gcd(k,\mu)$, and over z_l there are g_l points, of cone angle $2\pi q_l/g_l$ as on X; likewise over z_m. Let Σ_Y be the zero set of η, and let P be the set of points over z_0 if $\mu \mid k$, when they are regular points, and $P=\emptyset$ otherwise. The marked points of the limit are $\Sigma_Y \cup P$: its zeros and, when μ divides k, the μ points over the collided corner z_0, which are the punctures filled with marked points after the collapsed part is discarded, in the convention of Mirzakhani and Wright [25, Definition 2.2 and Remark 2.3]. The points over z_l or z_m that are regular are not marked. Let $\mathcal{N}$ be the orbit closure of (Y,η) in its stratum, with marked set Σ_Y; by hypothesis $\operatorname{rank}\mathcal{N} = |F(\Delta)|/2$. By Riemann–Hurwitz, $2(g-g_Y) = k-g_i-g_j+\gcd(k,\mu)>0$ for the genus g_Y of Y, as $g_i, g_j \le k/2$.

For $c \not\equiv 0$, the ζ^c-eigenspace $H^1(Y;\mathbb{C})_c$ of T_Y^* has dimension $\nu-2$ if the number ν of non-integers among $c\mu/k$, cq_l/k, cq_m/k is at least 2 [26, Corollary 6.2], and $H^1(Y;\mathbb{C})_0 = 0$ as $Y/\langle T_Y\rangle$ has genus zero. So $H^1(Y;\mathbb{C})_c$ is a line if $c\mu, cq_l, cq_m \not\equiv 0 \pmod k$ and is zero otherwise, and

$$\dim \bigoplus_{c\in F(\Delta)} H^1(Y;\mathbb{C})_c = |F(\Delta)|, \tag{3.1}$$

where for even k the sum is over the odd c with a non-zero eigenspace.

The sequence. A point X_z of $\mathcal{M}_{\mathrm{cyc}}(\theta)$ is generic if and only if the conclusion of Lemma 2.2 holds for it and it is not in the self-crossing locus of $\mathcal{M}(\theta)$ (Section 2.3). For $a\in\mathbb{C}^\times$ and $b\in\mathbb{C}$, the substitution $z\mapsto az+b$ identifies (X_{az+b},ω_{az+b}) with $(X_z, a^{-1}\omega_z)$, and the two conditions are invariant under multiplication of the form by a constant; so the set of z with X_z generic is invariant under the affine maps of $\mathbb{C}$, and it is dense in the configuration space. Hence there are $\delta_n\to 0$ and configurations $z^{(n)}$ with $X_n = X_{z^{(n)}}$ generic,

$$z_l^{(n)} = z_l, \qquad z_m^{(n)} = z_m, \qquad |z_i^{(n)} - z_0| < \delta_n, \qquad |z_j^{(n)} - z_0| < \delta_n.$$

Let $\omega_n = \omega_{z^{(n)}}$, Σ_n its zero set, $V^{(n)} = T_{X_n}\mathcal{M}(\theta) \subset H^1(X_n,\Sigma_n;\mathbb{C})$, and choose $\varepsilon_n\to 0$, decreasing, with $\delta_n/\varepsilon_n\to 0$. Let D_n be the union of the disc $|z-z_0|<\varepsilon_n$ and of the discs $|z-z_s|<\varepsilon_n$ for those $s\in\{l,m\}$ with $q_s \nmid k$, and $\mathcal{U}_n \subset Y$ the preimage of D_n. The $\mathcal{U}_n$ are decreasing neighborhoods of $\Sigma_Y\cup P$ with intersection $\Sigma_Y\cup P$.

Convergence. We show that (X_n,ω_n,Σ_n) converges to $(Y,\eta,\Sigma_Y\cup P)$ in the sense of Mirzakhani and Wright [25, Definition 2.2]: there are maps g_n from $Y\setminus\mathcal{U}_n$ to X_n, diffeomorphisms onto their images, such that $g_n^*\omega_n \to \eta$ uniformly on compact subsets of $Y\setminus(\Sigma_Y\cup P)$ and the injectivity radius at the points of X_n outside the image of g_n (the largest radius of an embedded flat disc about the point that contains no zero of ω_n) tends to 0 uniformly. On $|z-z_0|\ge\varepsilon_n$, including $z=\infty$, the function $\upsilon_n(z) = \big((z-z_i^{(n)})/(z-z_0)\big)^{q_i}\big((z-z_j^{(n)})/(z-z_0)\big)^{q_j}$ tends to 1 uniformly, as $\delta_n/\varepsilon_n\to 0$, so for large n it has a holomorphic k-th root $\upsilon_n^{1/k}$ there that tends to 1 uniformly. As $q_i+q_j = k+\mu$, the equation of X_n is $y^k = (z-z_0)^k\,\upsilon_n(z)\,(z-z_0)^\mu(z-z_l)^{q_l}(z-z_m)^{q_m}$, and

$$g_n(y,z) = \big(y\,(z-z_0)\,\upsilon_n(z)^{1/k},\, z\big)$$

is a biholomorphic map from $Y\setminus\mathcal{U}_n$ onto the part of X_n over $\mathbb{P}^1\setminus D_n$, with $g_n\circ T_Y = T\circ g_n$ and $g_n^*\omega_n = (z-z_0)^2\,\upsilon_n(z)^{1/k}(z-z_i^{(n)})^{-1}(z-z_j^{(n)})^{-1}\eta$, whose factor tends to 1 uniformly

there; every compact subset of $Y \setminus (\Sigma_Y \cup P)$ lies in $Y \setminus \mathcal{U}_n$ for large n, which gives the first condition. The rest of X_n lies over D_n. One of q_i, q_j, say q_i, does not divide k, since two proper divisors of k have sum at most k; so the points over $z_i^{(n)}$ are zeros of ω_n. After the substitution $z = z_0 + \varepsilon_n w$, the density of $|\omega_n|$ over $|w| < 1$ is at most a constant times $\varepsilon_n^\alpha |w - w_i|^{q_i/k-1} |w - w_j|^{q_j/k-1}$, where $z_i^{(n)} = z_0 + \varepsilon_n w_i$ and $z_j^{(n)} = z_0 + \varepsilon_n w_j$, and its integral without the factor ε_n^α along the segment from w_i to any point of $|w| < 1$ is bounded uniformly in w_i and w_j: in the arc length $t \le 2$ from w_i it is at most $\int_0^2 t^{q_i/k-1} |t - t_j|^{q_j/k-1} dt$, t_j the coordinate of the projection of w_j, if $q_j < k$, which by the rearrangement inequality is at most $\int_{-1}^1 2^{q_i/k-1} |t|^{\alpha-1} dt$ when also $q_i < k$, finite as $q_i/k + q_j/k = 1 + \alpha > 1$; and the factor $|w - w_j|^{q_j/k-1}$ is at most $2^{q_j/k-1}$ if $q_j > k$, and $t^{q_i/k-1} \le 2^{q_i/k-1}$ if $q_i > k$. So every point of X_n over $|z - z_0| < \varepsilon_n$ is at flat distance at most a constant times ε_n^α from a zero over $z_i^{(n)}$; likewise, for $s \in \{l, m\}$ with $q_s \nmid k$, a point over $|z - z_s| < \varepsilon_n$ is at distance at most a constant times $\varepsilon_n^{q_s/k}$ from a zero over z_s. The injectivity radius is at most the distance to a zero, and the second condition follows. As $g_Y < g$, the limit is in the boundary of $\mathcal{M}(\theta)$.

The boundary component. Mirzakhani and Wright associate with a convergent sequence collapse maps f_n from X_n to Y, with some points of $\Sigma_Y \cup P$ identified, that map marked points to marked points and, for every $\varepsilon > 0$ and n large, are the inverse of g_n where the injectivity radius is at least ε [25, Proposition 2.4]. In their construction [25, §9.1, proof of Propositions 2.4 and 2.5], f_n is the inverse of g_n on the part of X_n over $\mathbb{P}^1 \setminus D_{n-1}$, maps each component of the part over D_n to a point, and on the annuli over $D_{n-1} \setminus D_n$ interpolates; we take the interpolation with values in the image of $\overline{\mathcal{U}_{n-1}}$. Let $B_n \subset X_n$ be the part over $\overline{D_n}$. It contains Σ_n, and each of its components contains a zero of ω_n: a component over the closed disc around z_0 maps onto that disc, so it contains a point over $z_i^{(n)}$; a component over the closed disc around z_s contains one of the points over z_s, a zero of ω_n as $q_s \nmid k$. Each component of $\overline{\mathcal{U}_n}$ is a closed disc that contains exactly one point of $\Sigma_Y \cup P$, and g_n maps $Y \setminus \mathcal{U}_n$ homeomorphically onto the closure of $X_n \setminus B_n$ and the boundary of $\mathcal{U}_n$ onto that of B_n. Identifying points of $\Sigma_Y \cup P$ does not change relative homology. As f_n is g_n^{-1} on the part of X_n over $\mathbb{P}^1 \setminus D_{n-1}$ and maps the part over $\overline{D_{n-1}}$ into the image of $\overline{\mathcal{U}_{n-1}}$, excision shows that the map induced by f_n on relative homology is the composite

$$H_1(X_n, \Sigma_n; \mathbb{C}) \to H_1(X_n, B_n; \mathbb{C}) \xrightarrow{\cong} H_1(Y, \overline{\mathcal{U}_n}; \mathbb{C}) \xleftarrow{\cong} H_1(Y, \Sigma_Y \cup P; \mathbb{C}) \tag{3.2}$$

with $D_{n-1}, \mathcal{U}_{n-1}$ and the part B'_{n-1} of X_n over $\overline{D_{n-1}}$ in place of $D_n, \mathcal{U}_n$ and B_n; by naturality of excision it is also the composite as displayed, where the middle isomorphism is induced by g_n^{-1} on the closure of $X_n \setminus B_n$ and the last one by inclusion. We write $(f_n)_*$ for it. Its first map is onto, as $H_0(B_n, \Sigma_n; \mathbb{C}) = 0$. The space of vanishing cycles $\mathcal{V}_n = \ker(f_n)_*$ is the image of $H_1(B_n, \Sigma_n; \mathbb{C})$, and the induced map f_n^* on relative cohomology is an isomorphism from $H^1(Y, \Sigma_Y \cup P; \mathbb{C})$ onto $\operatorname{Ann}(\mathcal{V}_n)$ [25, proof of Lemma 9.2]. They prove that if surfaces (X_n, ω_n) of an affine invariant submanifold $\mathcal{M}$ converge to a connected translation surface in the boundary, then the limit lies in a component $\mathcal{M}'$ of the boundary of $\mathcal{M}$, an affine invariant submanifold, whose tangent space is identified by f_n^* with $T_{(X_n, \omega_n)}\mathcal{M} \cap \operatorname{Ann}(\mathcal{V}_n)$ for infinitely many n [25, Theorem 2.9 (1)]. So there are an affine invariant submanifold $\mathcal{M}'$ of the stratum of $(Y, \eta; \Sigma_Y \cup P)$ containing this surface, a tangent space L' of $\mathcal{M}'$ at it, and infinitely many n with

$$f_n^*(L') = V^{(n)} \cap \operatorname{Ann}(\mathcal{V}_n). \tag{3.3}$$

By Lemma 3.1(b) if $P \neq \emptyset$, and because $\mathcal{N} \subset \mathcal{M}'$ if $P = \emptyset$ (as in Step 5 of the proof of Lemma 3.1), $\operatorname{rank} \mathcal{M}' \geq \operatorname{rank} \mathcal{N}$; so

$$\dim p(L') \geq 2 \operatorname{rank} \mathcal{N} = |F(\Delta)|. \tag{3.4}$$

Equivariance. The sets Σ_n, B_n, $\overline{\mathcal{U}_n}$ and $\Sigma_Y \cup P$ are invariant under the deck transformations, and $g_n \circ T_Y = T \circ g_n$; so each map in (3.2) commutes with them, and $(f_n)_* T_* = (T_Y)_* (f_n)_*$, the space $\operatorname{Ann}(\mathcal{V}_n)$ is T^*-invariant, and $f_n^* \circ T_Y^* = T^* \circ f_n^*$. As X_n is generic, $T^* V^{(n)} = V^{(n)}$, and $V^{(n)}$ lies in the sum of the eigenspaces of odd character if k is even (Proposition 2.5 and Section 2.4). By (3.3), L' is T_Y^*-invariant, and for even k it lies in the sum of the eigenspaces of odd character. So L' is the sum of its eigenspaces L'_c, and $p(L') = \bigoplus_c p(L'_c)$ with $p(L'_c) \subset H^1(Y; \mathbb{C})_c$, over odd c if k is even. By (3.1) and (3.4), $p(L'_c) = H^1(Y; \mathbb{C})_c$ for every $c \in F(\Delta)$.

Transfer. Let $x \in F \cup F_1$ with $xq_l, xq_m, x(q_l + q_m) \not\equiv 0$. As $\mu \equiv -(q_l + q_m)$, $x \in F(\Delta)$, so there is $\xi \in L'_x$ with $p(\xi) \neq 0$. Let γ be a closed curve in $Y \setminus (\Sigma_Y \cup P)$ with $\int_\gamma p(\xi) \neq 0$. For large n, γ lies in $Y \setminus \overline{\mathcal{U}_n}$, and by (3.2) $(f_n)_*[g_n(\gamma)] = [\gamma]$. For n as in (3.3), $f_n^* \xi \in V^{(n)}$ has character x, and its value on the absolute cycle $g_n(\gamma)$ is $\xi((f_n)_*[g_n(\gamma)]) = \int_\gamma p(\xi) \neq 0$. So $p(f_n^* \xi)$ is a non-zero element of $p(V^{(n)}) \cap H^1(X_n; \mathbb{C})_x$, and $x \in A$ by Proposition 2.5, at the generic point X_n. This proves Theorem 3.2.

3.3. **Good triangles.**

Definition 3.3. A reduced expression $(p_1, p_2, p_3)/d$ of a rational triangle is *good* if p_1, p_2, p_3 are distinct and, up to order, it is not $(1, 2, c)$ with c odd, $(1, c, c+1)$ with $c \geq 2$, $(2, c, c+2)$ with c odd, or one of

$$(1,4,7), \quad (2,3,4), \quad (3,4,5), \quad (3,5,7), \quad (1,3,6),$$
$$(1,3,8), \quad (1,4,11), \quad (1,4,15), \quad (3,4,13),$$

where we omit the denominator $d = p_1 + p_2 + p_3$.

Among triangles with distinct numerators, those excluded here are the triangles of Delecroix and Rüth's list [11, Theorem 4.1] and $(1, 2, 3)$; their denominators are even or equal to 9 or 15, and those of the nine listed triangles are at most 20. For a good triangle Δ, the orbit closure of its unfolding is its ambient locus, which has rank $|F(\Delta)|/2$ [18, Theorem 1.1 and §1]. Isosceles triangles are left out: most connected ones also have rank $|F(\Delta)|/2$ [2, Theorem 7.7], but the families of Lemmas 6.8 and 7.8 contain members whose isosceles collision triangles have rank one, and Sections 6.5 and 7.6 are needed for them.

Corollary 3.4. *Let $\{l, m\}$ be a short pair whose collision triangle Δ_{lm} is connected and good. Then every $x \in F \cup F_1$ with xq_l, xq_m, $x(q_l + q_m) \not\equiv 0 \pmod{k}$ lies in A.*

We write $\operatorname{Deg}(\theta)$ for the set of all x obtained in this way from the short pairs of θ with connected and good collision triangles.

4. Closure rules

This section proves the rules that constrain the set A of Proposition 2.5: it is symmetric and permuted as a block by the units (Proposition 4.2); a differential rule adds characters (Theorem 4.6); a relative rule puts the relative eigenspaces Z_c into V (Theorem 4.7); and symmetric members of the type bound the rank from below by the rank of a triangle (Proposition 4.9). Throughout, θ is admissible and reduced with $g \geq 2$, X is a generic point of $\mathcal{M}_{\mathrm{cyc}}(\theta)$, and V, p, A are as in Section 2.5. The first two rules are Mirzakhani

and Wright's, who write their analysis for polygons with n vertices and assume $n = 3$ at key places [26, §7]: Proposition 4.2 is their Lemma 7.6, with the extension in their Lemma 7.10, for four points, and Theorem 4.6 with $x' = -1$ is the rule they draw from their Proposition 7.3. The proofs below follow those of the triangle analogues in Part I [19, §2.4 and §3] and, for the relative rule, in Part II [18, Theorem 1.6]. For $n = 4$ we do not use their Proposition 7.3(2), whose extension to $n > 3$ they pose as Problem 7.4: the proof of Theorem 4.6 pairs $\alpha\beta/\omega$ with all of $V_{x'}$, not only with $\bar{\omega}$, and needs only that $\alpha\beta/\omega$ is not exact (Lemma 4.5).

4.1. **Galois conjugates.** Let H^1 be the flat bundle over $\mathcal{M}(\theta)$ with fiber $H^1(X';\mathbb{C})$ at (X',ω'), and κ the field of definition of $\mathcal{M}(\theta)$. Wright proves that $H^1 = \bigoplus_\rho V_\rho \oplus W'$, where ρ runs over the field embeddings $\kappa \to \mathbb{C}$, V_{Id} has fiber $p(V)$ at (X,ω), each V_ρ is a simple flat bundle Galois conjugate to V_{Id}, and W' is a semisimple flat bundle with no subbundle isomorphic to any V_ρ [29, Theorem 1.5]. Here $V_\rho = \sigma(V_{\mathrm{Id}})$ for any automorphism σ of $\mathbb{C}$ extending ρ, acting on $H^1(X';\mathbb{Q}) \otimes \mathbb{C}$ through the coefficients: Wright defines V_ρ in this way [29, proof of Lemma 6.5], and the choice of σ does not matter, as V_{Id} is defined over κ [29, Theorem 1.5]. The V_ρ are pairwise non-isomorphic [29, Lemma 6.5]. Filip proves that each such summand underlies a variation of Hodge structure [15, proof of Theorem 7.3, with Theorem 7.2]; in the form used by Mirzakhani and Wright [26, §7.1], at each point the fiber of each summand is the sum of its intersections with $H^{1,0}$ and $H^{0,1}$.

For a unit u modulo k, let σ_u be an automorphism of $\mathbb{C}$ with $\sigma_u(\zeta) = \zeta^u$. Let W be a neighborhood of $[\omega]$ in period coordinates as in Section 2.3, so that $\mathcal{M}(\theta) \cap W = V \cap W$; over $\mathcal{M}(\theta) \cap W$ the bundle H^1 is the constant bundle $H^1(X;\mathbb{C})$.

Lemma 4.1. *Let u be a unit modulo k. Then $\sigma_u(p(V)) = \bigoplus_{x \in uA} H^1_x$, and this is the fiber at (X,ω) of one of Wright's summands V_ρ. At every point of $\mathcal{M}(\theta) \cap W$ the fiber of V_ρ is the sum of its intersections with $H^{1,0}$ and $H^{0,1}$. Two such conjugates of $p(V)$ are equal or meet only in 0.*

Proof. The proof of [19, Lemma 2.5] applies, with Proposition 2.5 in place of its Lemma 2.4. □

Proposition 4.2. *$A = -A$. If u is a unit modulo k and $A \cap uA \neq \emptyset$, then $uA = A$.*

Proof. The proof of the first two statements of [19, Lemma 3.4] applies, with Lemma 4.1 and Proposition 2.5 in place of its Lemmas 2.5 and 2.4; here $uA \subset F \cup F_1$, as $n(ux) = n(x)$ and u is odd when k is even. □

4.2. **Derivatives of eigenforms.** Classes in $H^1(X,\Sigma;\mathbb{C}) = H^1_c(X \setminus \Sigma;\mathbb{C})$ are represented by closed smooth 1-forms with compact support in $X \setminus \Sigma$, and classes in $H^1(X \setminus \Sigma;\mathbb{C})$ by closed 1-forms on $X \setminus \Sigma$. By Poincaré–Lefschetz duality the pairing

$$\langle [\nu], [\vartheta] \rangle = \int_X \nu \wedge \vartheta$$

between the two spaces is perfect. It is T-invariant, so it pairs $H^1(X,\Sigma)_c$ with $H^1(X \setminus \Sigma;\mathbb{C})_{c'}$ to zero unless $c + c' \equiv 0$, and it restricts to a perfect pairing between $H^1(X,\Sigma)_c$ and $H^1(X \setminus \Sigma;\mathbb{C})_{-c}$. If ϑ is closed on all of X, then $\langle w, [\vartheta] \rangle = \int_X p(w) \wedge \vartheta$. If ϑ is a meromorphic 1-form on X, holomorphic on $X \setminus \Sigma$, and $f \in \mathbb{C}^\Sigma$, then

$$\langle \delta f, [\vartheta] \rangle = -2\pi i \sum_{s \in \Sigma} f(s) \operatorname{Res}_s \vartheta : \tag{4.1}$$

δf is represented by $d(\sum_s f(s)\chi_s)$, where χ_s equals 1 near s and is supported in a small disc D_s, and Stokes' theorem on D_s minus a smaller disc about s gives the residue.

Let $(X_\varpi, \omega_\varpi)$, $\varpi \in W$, be the marked surfaces with period coordinates ϖ, and $[\alpha]_{\mathrm{rel}} \in H^1(X, \Sigma; \mathbb{C})$ the relative class of a holomorphic form α on one of them. For $v \in H^1(X, \Sigma; \mathbb{C})$ we use Kahn and Wright's path $[\omega] + tv$ [21, Lemma 2.2]. Let ν be a closed form with compact support in $X \setminus \Sigma$ that represents v; for small t there is a unique complex structure on the surface, equal to that of X near Σ, for which $\omega + t\nu$ is holomorphic, and with the identity marking this surface has period coordinates $[\omega] + tv$. For holomorphic ϑ, the next lemma is Kahn and Wright's formula for the derivative of the projection from a stratum to moduli space [21, Theorem 1.1, Corollary 2.3 and Remark 2.4], combined with the variational formula for holomorphic forms; it is [19, Lemma 3.1], whose statement and proof use only that ω is a holomorphic 1-form with zero set Σ.

Lemma 4.3. *Let $v \in H^1(X, \Sigma; \mathbb{C})$, and let $\varpi \mapsto \alpha_\varpi$ be a family of holomorphic 1-forms on the surfaces X_ϖ, of class C^1 in t along the path $[\omega] + tv$, as forms on the fixed surface, with $\alpha_{[\omega]} = \alpha$. Let ϑ be a meromorphic 1-form on X, holomorphic on $X \setminus \Sigma$, with at most simple poles. Then*

$$\langle \partial_v [\alpha_\varpi]_{\mathrm{rel}}, [\vartheta] \rangle = \langle v, [\alpha\vartheta/\omega] \rangle.$$

The form $\alpha\vartheta/\omega$ on the right is meromorphic on X and holomorphic on $X \setminus \Sigma$.

The relative class of a holomorphic form pairs to zero with a form of the third kind. Let α' be holomorphic on X and ϑ as in Lemma 4.3. Then $[\alpha']_{\mathrm{rel}}$ is represented by $\alpha' - d(\chi F')$, where χ is a cut-off function equal to 1 near Σ and F' is the primitive of α' near each $s \in \Sigma$ with $F'(s) = 0$. As $\alpha' \wedge \vartheta = 0$ and ϑ has at most simple poles, with w_s a local coordinate at s,

$$\langle [\alpha']_{\mathrm{rel}}, [\vartheta] \rangle = \sum_{s \in \Sigma} \lim_{\varepsilon \to 0} \oint_{|w_s| = \varepsilon} F' \, \vartheta = 0. \tag{4.2}$$

The derivative of an eigenform vanishes against the eigenforms outside its Galois conjugate, because that conjugate is a variation of Hodge structure.

Lemma 4.4. *Let u be a unit modulo k, $x \in uA$ with $h(x) \geq 1$, and $\alpha \in H^{1,0}_x$. There is a holomorphic family α_ϖ of holomorphic 1-forms on the surfaces of $\mathcal{M}(\theta) \cap W$ with $\alpha_{[\omega]} = \alpha$, whose absolute classes lie in the fiber of the summand V_ρ of Lemma 4.1, such that*

$$\langle \partial_v [\alpha_\varpi]_{\mathrm{rel}}, [\beta] \rangle = 0 \qquad \text{for every } v \in V \text{ and every } \beta \in H^{1,0}_b,\ b \notin uA.$$

Proof. This is the proof of [19, Lemma 3.5], with α and β in place of ω_a and ω_b, Lemma 4.1, Proposition 4.2, and Lemma 4.3 in place of its Lemmas 2.5, 3.4 and 3.1, and $L_{[\omega]} = \bigoplus_{c \in uA} H^{1,0}_c$, which contains α; the cup product pairs H^1_c with H^1_b only if $c \equiv -b$, and $-b \notin uA$. □

For real numbers $e_1, \dots, e_4$ and an integer j with $e_i \equiv jq_i/k \pmod 1$ for every i, we write $\prod_i (z - z_i)^{e_i}$ for the meromorphic function $y^j \prod_i (z - z_i)^{e_i - jq_i/k}$ on X; it does not depend on the choice of j modulo k, as $y^k = \prod_i (z - z_i)^{q_i}$, and T^* multiplies it by ζ^j. This is the convention of (2.2). Conversely, every meromorphic function on X on which T^* acts by ζ^j is $S(z) \prod_i (z - z_i)^{e_i}$ for a rational function S, since its quotient by $\prod_i (z - z_i)^{e_i}$ is T-invariant.

Lemma 4.5. *Let $x' \in F$, let $M \geq 0$ be an integer, and let $e_1, \dots, e_4$ be real numbers with $e_i \equiv -x'q_i/k \pmod 1$ and $e_1 + e_2 + e_3 + e_4 = -M - 2$. Let $\mathcal{R}$ be a space of polynomials of degree at most M such that, for $R \in \mathcal{R}$, the form $\psi_R = R(z) \prod_i (z - z_i)^{e_i} \, dz$ is holomorphic*

on $X \setminus \Sigma$. Then the $R \in \mathcal{R}$ for which ψ_R is exact on $X \setminus \Sigma$ form a subspace of dimension at most $\max(0, M-1)$.

Proof. As x' is generic, no e_i is an integer. Let $R \in \mathcal{R}$ with $\psi_R = d\Phi$, Φ holomorphic on $X \setminus \Sigma$. Near a point of Σ, ψ_R is meromorphic with zero residue, so Φ is meromorphic on X. Since $T^*\psi_R = \zeta^{-x'}\psi_R$, the function $\Phi' = \frac{1}{k}\sum_{m=0}^{k-1} \zeta^{mx'}(T^m)^*\Phi$ also has $d\Phi' = \psi_R$, and $T^*\Phi' = \zeta^{-x'}\Phi'$; so $\Phi' = S(z)\prod_i (z-z_i)^{e_i+1}$ with S rational. At a point of $\mathbb{C}$ other than $z_1, \dots, z_4$, Φ' is holomorphic, and so is S. If S had a pole of order $m \geq 1$ at z_i, then near the points over z_i, Φ' would be a non-zero multiple of $(z-z_i)^{e_i+1-m}$ times a unit, and $d\Phi'$ would have the leading term $(e_i+1-m)(z-z_i)^{e_i-m}$ times a non-zero constant, which is non-zero as $e_i \notin \mathbb{Z}$; but ψ_R is $(z-z_i)^{e_i}$ times a holomorphic function of z near z_i, of order at least $e_i > e_i - m$ in $z - z_i$. So S is a polynomial. The cover is unramified over ∞ and ψ_R is holomorphic there, so Φ' is holomorphic at the points over ∞, where it behaves like $z^{\deg S + \sum_i (e_i+1)} = z^{\deg S - M + 2}$; so $\deg S \leq M-2$. Hence ψ_R lies in the image of the linear map $S \mapsto d\big(S(z)\prod_i (z-z_i)^{e_i+1}\big)$ on the polynomials of degree at most $M-2$, a space of dimension $\max(0, M-1)$. □

4.3. **The differential rule.**

Theorem 4.6. *Let u be a unit modulo k, $x \in uA$ with $h(x) \geq 1$, $x' \in A \cap F$, and $b \equiv 1 - x - x'$ with $b \in F \cup F_1$ and $h(b) \geq 1$. Then $b \in uA$.*

Proof. Suppose that $b \notin uA$. Let $\alpha \in H^{1,0}_x$ and $\beta \in H^{1,0}_b$, and take the family α_ϖ of Lemma 4.4. By Lemma 4.4 and Lemma 4.3 with $\vartheta = \beta$,

$$0 = \langle \partial_v [\alpha_\varpi]_{\mathrm{rel}}, [\beta] \rangle = \langle v, [\alpha\beta/\omega] \rangle \qquad \text{for every } v \in V. \tag{4.3}$$

By (2.2), $\alpha = P_\alpha(z)\prod_i (z-z_i)^{-\{-xq_i/k\}}dz$ and $\beta = P_\beta(z)\prod_i (z-z_i)^{-\{-bq_i/k\}}dz$ with polynomials of degrees at most $h(x)-1$ and $h(b)-1$, and $\omega = \prod_i (z-z_i)^{q_i/k-1}dz$. So

$$\alpha\beta/\omega = P_\alpha P_\beta \prod_i (z-z_i)^{e_i}\, dz, \qquad e_i = 1 - \frac{q_i}{k} - \left\{\frac{-xq_i}{k}\right\} - \left\{\frac{-bq_i}{k}\right\}.$$

Here $e_i \equiv (x+b-1)q_i/k \equiv -x'q_i/k \pmod 1$, and $\sum_i e_i = 4 - 2 - (h(x)+1) - (h(b)+1) = -M-2$ with $M = h(x)+h(b)-2 \geq 0$. The form $\alpha\beta/\omega$ is holomorphic off the zeros Σ of ω. The products $P_\alpha P_\beta$ span the space of all polynomials of degree at most M, of dimension $M+1$, so by Lemma 4.5 some $\alpha\beta/\omega$ is not exact on $X \setminus \Sigma$: its class in $H^1(X \setminus \Sigma; \mathbb{C})_{-x'}$ is not zero. As $x' \in A \cap F$, $Z_{x'} = 0$ and $p(V) \supset H^1_{x'}$, so $V_{x'} = H^1(X, \Sigma)_{x'}$ by (2.3). The pairing between $H^1(X,\Sigma)_{x'}$ and $H^1(X \setminus \Sigma; \mathbb{C})_{-x'}$ is perfect, so some $v \in V_{x'}$ has $\langle v, [\alpha\beta/\omega] \rangle \neq 0$, against (4.3). Hence $b \in uA$. □

4.4. **The relative rule.** The rules so far concern $p(V)$. When k is even, the ambient locus has the relative directions Z_c, c odd, and the next rule puts them into V. It uses Filip's theorem that the relative class of a holomorphic form whose absolute class lies in $p(V)$ lies in V: a class in his $H^{1,0}_{\iota_l}$ lifts to relative cohomology through the isomorphism that splits his exact sequence and as the relative class of a holomorphic 1-form, and he proves that the two lifts agree [16, Theorem 4.2 and the paragraph after it]. For the identity embedding ι_0, his $H^1_{\iota_0}$ is $p(V)$ and that isomorphism identifies it with $V/(V \cap Z)$; so for a holomorphic form α' with $p[\alpha'] \in p(V)$, the class $[\alpha']_{\mathrm{rel}}$ lies in $V + (V \cap Z) = V$.

Theorem 4.7. *Let $c \not\equiv 0$ with $Z_c \neq 0$, let $x \in A$ with $h(x) \geq 1$, and let $x' \equiv 1 + c - x$. If $x' \in A \cap F$, then $Z_c \subset V$.*

Proof. Suppose that $V \cap Z_c \neq Z_c$. Let I be the set of vertices i with $q_i \nmid k$ and $cq_i \equiv 0$; by Section 2.2, Z_c is spanned by the classes δf_i, $i \in I$, where $f_i \in \mathbb{C}^\Sigma$ is the function of character c on the g_i points over z_i that is 1 at a chosen point and zero on the other orbits. Choose $r \in \mathbb{C}^\Sigma$ with T^* acting on it by ζ^{-c}, supported on the points over the vertices in I, such that the linear form $\delta f \mapsto \sum_s f(s)r(s)$ on Z_c vanishes on $V \cap Z_c$ and not on Z_c; such r exist, as these linear forms, for r of character $-c$, are all the linear forms on Z_c. The sum of the values of r over each orbit is zero, as $c \not\equiv 0$; so by the residue theorem and the Riemann–Roch theorem there is a meromorphic 1-form on X with at most simple poles on Σ, residues r, and no other poles, and averaging it as in the proof of Lemma 4.5 we obtain such a form φ with $T^*\varphi = \zeta^{-c}\varphi$.

Step 1: φ can be chosen with $\langle w, [\varphi]\rangle = 0$ for all $w \in V$. By T-invariance of the pairing only $w \in V_c$ matter. On $V \cap Z_c$ the pairing is $-2\pi i \sum_s f(s)r(s) = 0$ by (4.1), so $w \mapsto \langle w, [\varphi]\rangle$ factors through $p\colon V_c \to p(V)_c$. By Lemma 4.1 with $u = 1$, $p(V)_c$ is the sum of its intersections with $H_c^{1,0}$ and with $H_c^{0,1}$, the space of classes in $H^{0,1}$ of character c, which is the complex conjugate of $H_{-c}^{1,0}$. If $\alpha' \in p(V)_c \cap H_c^{1,0}$, then $[\alpha']_{\mathrm{rel}} \in V_c$ by Filip's theorem, and $\langle [\alpha']_{\mathrm{rel}}, [\varphi]\rangle = 0$ by (4.2). So the linear form is induced by a linear form on $p(V)_c \cap H_c^{0,1}$; extend it to $H_c^{0,1}$. The pairing $(\xi, \beta) \mapsto \int_X \xi \wedge \beta$ between $H_c^{0,1}$ and $H_{-c}^{1,0}$ is perfect, so there is $\beta \in H_{-c}^{1,0}$ with $\int_X \xi \wedge \beta$ equal to that form. Replace φ by $\varphi - \beta$: the residues are unchanged, $\langle \delta f, [\beta]\rangle = \int_X p(\delta f) \wedge \beta = 0$, and for $w \in V_c$ with $p(w) = \alpha' + \xi$ we get $\langle w, [\beta]\rangle = \int_X \xi \wedge \beta = \langle w, [\varphi]\rangle$, as $\alpha' \wedge \beta = 0$. So the new φ pairs to zero with V.

Step 2: the vanishing. Let $\alpha \in H_x^{1,0}$, and let α_ϖ be the family of Lemma 4.4 with $u = 1$. At each $\varpi \in \mathcal{M}(\theta) \cap W$, $p[\alpha_\varpi]$ lies in the fiber of V_{Id}, which is the absolute part of $T_\varpi \mathcal{M}(\theta) = V$, so $[\alpha_\varpi]_{\mathrm{rel}} \in V$ by Filip's theorem, and $\langle [\alpha_\varpi]_{\mathrm{rel}}, [\varphi]\rangle = 0$ by Step 1. Differentiating along $v \in V$ and applying Lemma 4.3 with $\vartheta = \varphi$, which has at most simple poles,

$$0 = \langle \partial_v [\alpha_\varpi]_{\mathrm{rel}}, [\varphi]\rangle = \langle v, [\alpha\varphi/\omega]\rangle \qquad \text{for every } v \in V. \tag{4.4}$$

Step 3: the shape of φ. At a point over z_i, which is fixed by T^{g_i}, the residue of φ equals that of $(T^{g_i})^*\varphi = \zeta^{-cg_i}\varphi$; so it vanishes unless $\zeta^{cg_i} = 1$, that is, unless $cq_i \equiv 0$. Hence φ is holomorphic off the points over the vertices in I, where it has at most simple poles. Write $\varphi = P_\varphi(z) \prod_i (z - z_i)^{-\{cq_i/k\} - [i \in I]} dz$, with $[i \in I]$ equal to 1 for $i \in I$ and 0 otherwise, and P_φ rational. At a point over z_i, with local coordinate w, $w^{k/g_i} = z - z_i$, the form $(z - z_i)^t dz$ has order $(k/g_i)(t+1) - 1$, where $(k/g_i)t$ is an integer; so a pole of P_φ at z_i would give φ a pole at $i \notin I$ or a pole of order at least 2 at $i \in I$. As φ is holomorphic over the other points of $\mathbb{C}$, P_φ is a polynomial, and holomorphy over ∞ gives $\deg P_\varphi \leq d_\varphi$ with

$$d_\varphi = \sum_i \{cq_i/k\} + |I| - 2.$$

$P_\varphi \neq 0$, as $r \neq 0$. The number of i with $cq_i \equiv 0$ is 1 or 2: it is at least 1 as $Z_c \neq 0$, and three such i would give the fourth, as $\sum_i cq_i \equiv 0$, and then $c \equiv 0$ as $\gcd(k, q_1, \ldots, q_4) = 1$. If it is 1, the three other fractions $\{cq_i/k\}$ have an integral sum in $(0, 3)$, so $d_\varphi \leq 2 + 1 - 2 = 1$; if it is 2, the two others sum to 1 and $d_\varphi \leq 1 + 2 - 2 = 1$. So $0 \leq d_\varphi \leq 1$.

Step 4: a non-zero product. With $\alpha = P_\alpha(z) \prod_i (z - z_i)^{-\{-xq_i/k\}} dz$, $\deg P_\alpha \leq h(x) - 1$,

$$\alpha\varphi/\omega = P_\alpha P_\varphi \prod_i (z - z_i)^{d_i}\, dz, \qquad d_i = 1 - \frac{q_i}{k} - \left\{\frac{-xq_i}{k}\right\} - \left\{\frac{cq_i}{k}\right\} - [i \in I],$$

with $d_i \equiv (x - c - 1)q_i/k \equiv -x'q_i/k \pmod 1$ and $\sum_i d_i = 2 - (h(x) + 1) - (d_\varphi + 2) = -M - 2$, $M = h(x) - 1 + d_\varphi$. The form is holomorphic on $X \setminus \Sigma$, and $\deg P_\alpha P_\varphi \leq M$. As P_α ranges over the polynomials of degree at most $h(x) - 1$, the products $P_\alpha P_\varphi$ form a space of dimension

$h(x)$, and $h(x) > \max(0, M-1)$ since $h(x) \geq 1$ and $M - 1 = h(x) - 2 + d_\varphi \leq h(x) - 1$. By Lemma 4.5 some $\alpha\varphi/\omega$ has non-zero class in $H^1(X \setminus \Sigma; \mathbb{C})_{-x'}$. As in the proof of Theorem 4.6, $V_{x'} = H^1(X, \Sigma)_{x'}$ pairs perfectly with this space, against (4.4). Hence $Z_c \subset V$. □

4.5. **Symmetric members.** A member of the type with a symmetry covers the unfolding of a triangle, and the rank of the orbit closure of a covering surface is at least that of the base. The proof marks the branch values of the covering on the base and uses Lemma 3.1.

Lemma 4.8. *Let (X', ω') and (Y, η) be translation surfaces and $\pi\colon X' \to Y$ a holomorphic map of degree $d \geq 1$ with $\pi^*\eta = \omega'$. Let $\mathcal{M}'$ and $\mathcal{N}$ be the orbit closures of (X', ω') and (Y, η) in their strata, with the zeros marked and no other point. Then* $\operatorname{rank} \mathcal{M}' \geq \operatorname{rank} \mathcal{N}$.

Proof. Let Σ_Y be the zero set of η, $B \subset Y$ the union of Σ_Y and the branch values of π, and $P = B \setminus \Sigma_Y$, a finite set of regular points. The zeros Σ' of ω' lie in $\pi^{-1}(B)$.

Let $\mathcal{N}^P$ be the orbit closure of $(Y, \eta; P)$ in the stratum with the points of P marked as well. Every tangent space L of $\mathcal{N}^P$, at every point, has $\dim p(L) = 2 \operatorname{rank} \mathcal{N}$: if $P = \emptyset$, then $\mathcal{N}^P = \mathcal{N}$ and there is nothing to prove, and if $P \neq \emptyset$ this is Lemma 3.1(a).

Fix a point s_1 of $\mathcal{N}^P$ and a tangent space L of $\mathcal{N}^P$ at s_1, and a simply connected neighborhood W_1 of s_1 in period coordinates, so that the surfaces $(Y_s, \eta_s; P_s)$, $s \in W_1$, are identified topologically with $(Y, \eta; P)$, with $B_s = \Sigma_{Y_s} \cup P_s$. A connected cover of Y_s of degree d branched only over B_s is given by a transitive homomorphism ϱ from $\pi_1(Y \setminus B)$ to the symmetric group, up to conjugation; there are finitely many, $\varrho_1, \dots, \varrho_r$, whose pulled-back forms have the zero orders of ω'. Each gives a continuous map Λ_m from W_1 to the stratum of (X', ω'), $s \mapsto (X'_{s,m}, \pi^*_{s,m}\eta_s)$, which in period coordinates is the linear map

$$H^1(Y, B; \mathbb{C}) \xrightarrow{\pi_m^*} H^1(X', \pi_m^{-1}(B); \mathbb{C}) \longrightarrow H^1(X', \Sigma'; \mathbb{C}),$$

the second arrow being restriction. For $g \in \mathrm{GL}^+(2, \mathbb{R})$ the map π is also a translation covering $g(X', \omega') \to g(Y, \eta)$, branched over $g(B)$. So if $s = g(Y, \eta; P)$ lies in W_1, then $g(X', \omega') = \Lambda_m(s)$ for some m. Points of $L \cap W_1$ off the other branches of $\mathcal{N}^P$ form a dense open subset of $L \cap W_1$, near which $\mathcal{N}^P$ is L; the orbit of $(Y, \eta; P)$ is dense in $\mathcal{N}^P$, hence in $L \cap W_1$. So $L \cap W_1$ is the union of the closures of the sets S_m of orbit points s with $\Lambda_m(s)$ in the orbit of (X', ω'), and by Baire's theorem one closure, $\overline{S_m}$, contains a non-empty open subset O of $L \cap W_1$. By continuity $\Lambda_m(O) \subset \mathcal{M}'$. As Λ_m is linear, $\Lambda_m(O)$ is an open subset of the linear space $\Lambda_m(L)$; near a point of it $\mathcal{M}'$ is a finite union of linear subspaces, one of which contains an open subset of $\Lambda_m(L)$, by Baire's theorem, and hence $\Lambda_m(L)$. So a tangent space T' of $\mathcal{M}'$ contains $\Lambda_m(L)$, and

$$p(T') \supset p(\Lambda_m(L)) = \pi_m^*\, p(L),$$

since pull-back and restriction commute with passing to absolute cohomology. Pull-back $H^1(Y; \mathbb{C}) \to H^1(X'; \mathbb{C})$ by a map of degree d is injective, as push-forward after it is multiplication by d. So $\dim p(T') \geq \dim p(L) = 2 \operatorname{rank} \mathcal{N}$, and since $\mathcal{M}'$ is connected, $\operatorname{rank} \mathcal{M}' \geq \operatorname{rank} \mathcal{N}$. □

Rational triangles $\Delta = (p_1, p_2, p_3)/d$, their reduced expressions, their unfoldings (Y, η) and the sets $F(\Delta)$ are as in Section 3.2.

Proposition 4.9.

(a) *Let $i \neq j$ with $a_i = a_j$, and let l, m be the other two indices. For z such that a Möbius involution J exchanges z_i and z_j and fixes z_l and z_m, (X_z, ω_z) is a translation cover of the unfolding of $\Delta = (2a_i, a_l, a_m)/N$, of degree $2k/k'$, where k' is the denominator of*

the reduced expression of Δ*: the degree is* 1 *when* N *is even and* a_l, a_m *are odd, and* 2 *otherwise.*

(b) *Let* $a_i = a_j = a_l$ *for three distinct indices, and let* m *be the fourth. For* z_i, z_j, z_l *the three cube roots of unity and* $z_m = 0$*,* (X_z, ω_z) *is a translation cover of the unfolding of* $\Delta = (6a_i, 2a_m, N)/(3N)$*, of degree* $\gcd(3, a_m)$*.*

In both cases $\operatorname{rank} \mathcal{M}(\theta)$ *is at least the rank of the orbit closure of the unfolding of* Δ*.*

The two constructions recover the examples of Eskin, McMullen, Mukamel and Wright for $(1, 2, 2, 7)$ and $(1, 1, 1, 7)$, respectively [12, §8, Remarks, (5), and §5, Particular examples, (5)].

For real z the member in (a) is the unfolding of a kite made of two copies of a triangle with the angles of Δ, glued along the side joining the vertices of angles $a_l\pi/N$ and $a_m\pi/N$; when l is one of the three indices of equal entries, $\Delta = (2a_i, a_i, a_m)/N$.

Proof. (a) The metric $|\omega_z|$ on $\mathbb{P}^1 = X_z/\langle T\rangle$ is a flat metric with cone angles $2\pi q_s/k$ at the z_s and no other singularity; such a metric is unique up to a constant factor. $J^*|\omega_z|$ is a flat metric with the same cone points and angles, as $q_i = q_j$, so it is $c|\omega_z|$ with $c > 0$, and $c^2 = 1$ as $J^2 = 1$. So J is an isometry, and the quotient $S = \mathbb{P}^1/J$ is a flat sphere with three cone points: the image of z_i and z_j, of angle $2\pi q_i/k = 2\pi \cdot 2a_i/N$, and the images of the fixed points z_l, z_m, of angles $\pi q_l/k = 2\pi a_l/N$ and $2\pi a_m/N$. It is the double of the triangle with angles $2a_i\pi/N$, $a_l\pi/N$, $a_m\pi/N$. Let (Y, η) be the unfolding of Δ, with $Y \to S$ the covering of degree k' defined by the kernel of the holonomy. Off the cone points, the composite $X_z \to \mathbb{P}^1 \to S$ is a covering whose image in the fundamental group lies in the kernel of the holonomy, since the metric pulls back to $|\omega_z|$, the metric of a global 1-form; so it factors through Y, the factorization extends to a holomorphic map $\pi\colon X_z \to Y$, and $\pi^*\eta$ and ω_z define the same metric, so $\pi^*\eta = \mu\omega_z$ with $|\mu| = 1$; replace η by $\mu^{-1}\eta$. The degree is $\deg(X_z \to S)/\deg(Y \to S) = 2k/k'$. As $\gcd(a_i, a_l, a_m) = 1$, the reduced expression of Δ divides by $\gcd(2a_i, a_l, a_m) \in \{1, 2\}$, which is 2 exactly when a_l, a_m are even. If N is odd, $k = N$ and a_l, a_m have opposite parities, so $k' = N$ and the degree is 2. If N is even, $k = N/2$ and $a_l \equiv a_m \pmod 2$: when they are odd, $k' = N$ and the degree is 1; when they are even, $k' = N/2$ and the degree is 2.

(b) The rotation $R(z) = e^{2\pi i/3}z$ permutes z_i, z_j, z_l, fixes 0 and ∞, and preserves $|\omega_z| = |z^{q_m/k-1}(z^3 - 1)^{q_i/k-1}dz|$, as $q_i = q_j = q_l$. The quotient $\mathbb{P}^1/R$ has cone angles $2\pi q_i/k = 2\pi \cdot 6a_i/(3N)$ at the image of the three points, $2\pi q_m/(3k) = 2\pi \cdot 2a_m/(3N)$ at the image of 0, and $2\pi/3 = 2\pi \cdot N/(3N)$ at the image of ∞, a regular point of $|\omega_z|$; it is the double of the triangle Δ. As in (a), (X_z, ω_z) is a translation cover of the unfolding of Δ, of degree $3k/k''$ with k'' the denominator of the reduced expression of Δ. Here $N = 3a_i + a_m$ and $\gcd(a_i, a_m) = 1$. If N is odd, $\gcd(6a_i, 2a_m, N)$ is odd, hence equal to $\gcd(3a_i, a_m) = \gcd(3, a_m)$, so $k'' = 3N/\gcd(3, a_m)$ and the degree is $\gcd(3, a_m)$. If N is even, a_i and a_m are odd, $\gcd(6a_i, 2a_m, N) = 2\gcd(3a_i, a_m, N/2) = 2\gcd(3, a_m)$, as $3 \mid a_m$ implies $3 \mid N/2$; so $k'' = 3N/(2\gcd(3, a_m))$ and the degree is $3(N/2)/k'' = \gcd(3, a_m)$.

In both cases $(X_z, \omega_z) \in \mathcal{M}_{\mathrm{cyc}}(\theta)$, so its orbit closure $\mathcal{M}'$ lies in $\mathcal{M}(\theta)$, and $\operatorname{rank} \mathcal{M}(\theta) \geq \operatorname{rank} \mathcal{M}'$: at a point of $\mathcal{M}'$, each tangent space of $\mathcal{M}'$ lies in a tangent space of $\mathcal{M}(\theta)$, and the rank of $\mathcal{M}(\theta)$ is the same on all of its branches. By Lemma 4.8, $\operatorname{rank} \mathcal{M}'$ is at least the rank of the orbit closure of the unfolding of Δ. □

4.6. **Closed sets.** The rules of this section and Section 3 are formal statements about subsets of $F \cup F_1$, and they can be applied by computation, as Mirzakhani and Wright do for triangles [26, Theorem 7.14 and Appendix A].

Definition 4.10. A subset $A' \subset F \cup F_1$ is *closed* if $1 \in A'$, $A' = -A'$, $uA' = A'$ for every unit u with $A' \cap uA' \neq \emptyset$, and $b \in uA'$ whenever u is a unit, $x \in uA'$ with $h(x) \geq 1$, $x' \in A' \cap F$, and $b \equiv 1 - x - x'$ lies in $F \cup F_1$ with $h(b) \geq 1$. Let Δ range over the triangles of Proposition 4.9, for the pairs and triples of equal entries of θ, whose reduced expressions are good (Definition 3.3); A' *meets the symmetric bounds* if $\sum_{x \in A'} \dim H^1_x \geq |F(\Delta)|$ for each such Δ. Let $\mathrm{Deg}(\theta) \subset F \cup F_1$ be the set of characters given by Corollary 3.4. The rules *reach* θ if every closed set that contains $\mathrm{Deg}(\theta)$ and meets the symmetric bounds is $F \cup F_1$.

Proposition 4.11. *If the rules reach θ, then $A = F \cup F_1$.*

Proof. A is closed: $1 \in A$ by Proposition 2.5, and the other conditions are Proposition 4.2 and Theorem 4.6. It contains $\mathrm{Deg}(\theta)$ by Corollary 3.4. For a triangle Δ of Proposition 4.9 with good reduced expression, the orbit closure of its unfolding is its ambient locus, of rank $|F(\Delta)|/2$ [18, Theorem 1.1 and §1]; so $\sum_{x \in A} \dim H^1_x = 2 \operatorname{rank} \mathcal{M}(\theta) \geq |F(\Delta)|$ by Propositions 2.5 and 4.9. □

We call θ *supplementary* if two of its angles sum to π, that is, if $q_i + q_j = k$ for some $i \neq j$, and *open* if it is admissible and reduced, is not supplementary, and is not one of the seven types (1.1). An open type has $g \geq 2$ by Lemma 2.1.

5. From the characters to the orbit closure

Once $A = F \cup F_1$, the orbit closure $\mathcal{M}(\theta)$ has the rank of the ambient locus, and this section shows when it is the ambient locus. For odd k the rank suffices, by the classification of Mirzakhani and Wright; for even k the relative characters must be accounted for. Throughout, θ is admissible and reduced with $g \geq 2$, X is a generic point of $\mathcal{M}_{\mathrm{cyc}}(\theta)$, and V, A, D are as in Section 2.

5.1. **Full rank.** Mirzakhani and Wright prove that an affine invariant submanifold of full rank, that is, of rank g, is a connected component of a stratum or the locus of hyperelliptic translation surfaces in one [26, Theorem 1.1]; their surfaces have no marked points, as here.

Proposition 5.1. *Let k be odd, and suppose that* $\operatorname{rank} \mathcal{M}(\theta) = g$. *Then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. This holds in particular when $A = F \cup F_1$.*

Proof. By Mirzakhani and Wright's theorem, $\mathcal{M}(\theta)$ is a connected component of a stratum or the locus of hyperelliptic translation surfaces in one. In the second case the generic point X of $\mathcal{M}_{\mathrm{cyc}}(\theta) \subset \mathcal{M}(\theta)$ has an involution τ with $\tau^*\omega = -\omega$, and by Lemma 2.2 $\tau = T^j$ with $\zeta^j = -1$, which is impossible as k is odd. So $\mathcal{M}(\theta)$ is a connected component of the stratum, the one containing X, which is $\mathcal{L}(\theta)$. If $A = F \cup F_1$, then $\operatorname{rank} \mathcal{M}(\theta) = D/2 = g$ by Proposition 2.5 and Lemma 2.3, since $p(T_X \mathcal{L}(\theta)) = H^1(X; \mathbb{C})$ for odd k. □

5.2. **Even degree: counting dimensions.** Let k be even. The tangent space of $\mathcal{L}(\theta) = \widetilde{\mathcal{Q}}$ at X is $E^- = \bigoplus_{c \text{ odd}} H^1(X, \Sigma)_c$ (Section 2.4). For odd c with $n(c) = 2$ we have $H^1_c = 0$ and $H^1(X, \Sigma)_c = Z_c$ by (2.3); for $c \in F \cup F_1$ the space $H^1(X, \Sigma)_c$ is an extension of H^1_c by Z_c, and $Z_c = 0$ if $c \in F$. So E^- is spanned by lifts of the H^1_x, $x \in F \cup F_1$, and by the Z_c for the odd relative characters c. An odd relative character exists exactly when k/g_i is odd for some vertex i with $q_i \nmid k$, by the description of Z_c in Section 2.2.

Proposition 5.2. *Let k be even. Suppose that $A = F \cup F_1$ and that $Z_c \subset V$ for every odd relative character c. Then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. In particular $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ if $A = F \cup F_1$ and k/g_i is even for every vertex i with $q_i \nmid k$.*

Proof. We have $V \subset E^-$ and $V = \bigoplus_{c \text{ odd}} V_c$ with $V_c = V \cap H^1(X, \Sigma)_c$ (Section 2.4 and Proposition 2.5), and $p(V_c) = p(V) \cap H^1_c$, as p is T-equivariant. Let c be odd. If $n(c) \geq 3$, then $c \in F \cup F_1 = A$, so $p(V_c) = H^1_c$; and $Z_c \subset V_c$, by hypothesis if $Z_c \neq 0$. By (2.3), $\dim V_c = \dim Z_c + \dim H^1_c = \dim H^1(X, \Sigma)_c$, so $V_c = H^1(X, \Sigma)_c$. If $n(c) = 2$, then $H^1(X, \Sigma)_c = Z_c \subset V$. Hence $V = E^-$.

It remains to pass from the equality of tangent spaces at X to $\mathcal{M}(\theta) = \widetilde{\mathcal{Q}}$; we argue as in [18, Lemma 2.2]. Let $f \colon \mathcal{Q} \to \widetilde{\mathcal{Q}}$ be the double-cover map and O the set of $y \in \mathcal{Q}$ such that f maps a neighborhood of y into $\mathcal{M}(\theta)$. It is open, and it contains $(X/\iota, q)$, since near X both $\mathcal{M}(\theta)$ and the image of Lanneau's chart are open subsets of E^- (Section 2.4). It is closed: let $y_n \in O$ tend to y. As $\mathcal{M}(\theta)$ is closed, $f(y) \in \mathcal{M}(\theta)$, and near $f(y)$ the set $\mathcal{M}(\theta)$ is a finite union of linear subspaces through $f(y)$ in period coordinates, read at a lift if $f(y)$ is an orbifold point. Take a connected neighborhood W' of y on which f is a linear chart [22, §1.1.1] and whose image lies in the domain of that description. For large n, $f(W') \cap \mathcal{M}(\theta)$ contains a non-empty open subset of $f(W')$; by Baire's theorem one of the subspaces contains a non-empty open subset of the affine image $f(W')$, hence all of $f(W')$, and $y \in O$. As $\mathcal{Q}$ is connected, $O = \mathcal{Q}$, so $\widetilde{\mathcal{Q}} = f(\mathcal{Q}) \subset \mathcal{M}(\theta) \subset \widetilde{\mathcal{Q}}$. □

Together with Proposition 4.11 this gives the form in which the rules are used.

Corollary 5.3. *Suppose that the rules reach θ. If k is odd, then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. If k is even and $Z_c \subset V$ for every odd relative character c, then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$.*

Proof. $A = F \cup F_1$ by Proposition 4.11; apply Proposition 5.1 for odd k and Proposition 5.2 for even k. □

At prime degree Theorem 3.2 and Proposition 5.1 combine into a criterion that needs one triangle.

Corollary 5.4. *Let θ be a type with k an odd prime, no angle equal to π, and at least three distinct angles. Let $\{l, m\}$ be a short pair, and suppose that the orbit closure of the unfolding of its collision triangle Δ_{lm} has rank $(k-1)/2$. Then $\mathcal{M}(\theta)$ is the connected component of the stratum that contains X.*

Proof. Here $N = k$ and $a_i = q_i/2$ if every q_i is even, and $N = 2k$ and $a_i = q_i$ otherwise (Section 2.1). The type θ is admissible, as no angle is π, that is, no q_i equals k; and it is reduced, as three distinct entries exclude $a_1 = a_2 < a_3 = a_4$. No q_i is divisible by k, so by the definitions of Section 2.2 F is the set of all $x \not\equiv 0$ and $F_1 = \emptyset$, and $g = k - 1 \geq 2$ by (2.1). The numerators of Δ_{lm} are less than k, so Δ_{lm} is connected, $F(\Delta_{lm})$ is the set of all $x \not\equiv 0$, and the hypothesis is that of Theorem 3.2, which gives $A = F$. So $\operatorname{rank} \mathcal{M}(\theta) = g$, and Proposition 5.1 applies. □

For example, for $\theta = (1, 1, 2, 3)$, where $k = 7$ and $(q_i) = (2, 2, 4, 6)$, the short pair of the entries 2 and 4 has the collision triangle $(1, 2, 4)/7$, which is good; so $\mathcal{M}(\theta)$ is the component of the stratum $\mathcal{H}(5, 3, 1, 1)$ containing X.

Remark 5.5. Reducedness cannot be dropped in Proposition 5.1: for the angles $(2, 3, 2, 3)\pi/5$, where $k = 5$ and $g = 4$, Apisa shows that the orbit closure of the generic member is the quadratic double of $\mathcal{Q}(2, 4, -1^{10})$ [2, Theorem 1.9], which has rank 4 and dimension 10 in a stratum of dimension 11.

Nor can the condition $Z_c \subset V$ be dropped in Proposition 5.2. Let $\theta = (1, 1, 2, 8)$, so that $k = 6$ and $(q_i) = (1, 1, 2, 8)$. Then $n(3) = 2$, so $F \cup F_1 = \{1, 5\}$ and $A = F \cup F_1$ by Proposition 2.5, and $\operatorname{rank} \mathcal{M}(\theta) = \operatorname{rank} \mathcal{L}(\theta) = 2$ by Lemma 2.3 and (2.4). The vertex

with $q_i = 8$ makes 3 an odd relative character, with Z_3 a line, so $\mathcal{L}(\theta)$ has dimension 5 (Section 9.3). But $\mathcal{M}(\theta)$ has dimension 4 [12, Theorem 1.1], which is $\dim p(V)$ by (2.4); so p is injective on V, and $Z_3 \not\subset V$. Correspondingly Theorem 4.7 does not apply at $c = 3$, as $x = 1$ gives $x' = 3 \notin F$ and $h(5) = 0$ (see the proof of Proposition 7.15). Mirzakhani and Wright ask whether an affine invariant submanifold of $\widetilde{\mathcal{Q}}$ with the rank of $\widetilde{\mathcal{Q}}$ must be $\widetilde{\mathcal{Q}}$, and attribute the negative answer to Eskin, McMullen, Mukamel and Wright [26, §1]; $(1, 1, 2, 8)$ is an instance.

6. Odd degree

We show that the rules reach every open type with k odd and $N > 1000$ (Theorem 6.12). A good connected collision triangle gives every unit, and a closed set containing the units is everything (Lemma 6.4); so the work is on the types without such a triangle. Either all three collision triangles are disconnected, and the differential rule alone gives the units (Proposition 6.7), or the type lies in one of two families with repeated angles. There a symmetric member gives the full bound or a bound to which the index-two argument applies, except for $(q_i) = (1, 1, 1, 2k - 3)$ with $3 \mid k$, treated directly in Lemma 6.11. Sections 6.1 and 6.2 hold for both parities and are used again in Section 7.

Throughout, θ is an open type, $U = (\mathbb{Z}/k)^\times$, and closed sets are those of Definition 4.10. For a divisor n of k we write $U_n = (\mathbb{Z}/n)^\times$. Primes are denoted by ℓ. For $A' \subset F \cup F_1$ let $D(A') = \sum_{x \in A'} \dim H^1_x$, so that $D = D(F \cup F_1)$. For a divisor d of k let $O_d = \{x \in \mathbb{Z}/k : \gcd(x, k) = d\}$; these are the orbits of U on $\mathbb{Z}/k$, and $x \mapsto n(x)$ is constant on each of them, as $xq_i \equiv 0$ if and only if k/g_i divides $\gcd(x, k)$. So $F \cup F_1$ is a union of orbits O_d, and $\dim H^1_{ux} = \dim H^1_x$ for $u \in U$. Recall that $h(x) + h(-x) = \dim H^1_x$ for $x \neq 0$, so for every $x \in F \cup F_1$ at least one of $h(x)$, $h(-x)$ is positive, and that for a unit x

$$h(x) = 3 - \sum_{i=1}^{4} \Big\{ \frac{xq_i}{k} \Big\}, \tag{6.1}$$

where the sum is an integer because $\sum_i q_i = 2k$.

6.1. Closed sets and units.

Lemma 6.1. *Let A' be a closed set and $H = A' \cap U$.*

(a) *H is a subgroup of U containing -1, and $H = \{u \in U : uA' = A'\}$. Two translates uA', $u'A'$ by units are equal or disjoint, there are $[U : H]$ of them, and $D(uA') = D(A')$.*

(b) *If x and b are units with $h(x) \geq 1$, $h(b) \geq 1$ and $1 - x - b \in A' \cap F$, then $b/x \in H$.*

(c) *If $x \in uA'$ for a unit u, and $v \in U$ satisfies $vx = x$, then $v \in H$.*

(d) *If $U \subset A'$, then A' is a union of orbits O_d.*

Proof. (a) If $uA' \cap u'A' \neq \emptyset$, then $A' \cap u^{-1}u'A' \neq \emptyset$, so $u^{-1}u'A' = A'$. If $uA' = A'$, then $u = u \cdot 1 \in A'$; if $u \in A' \cap U$, then $u \in A' \cap uA'$, so $uA' = A'$. So H is the stabilizer of A', a group, and $-1 \in H$ as $A' = -A'$. (b) This is the argument of Mirzakhani and Wright's Lemma 7.13, where $1 - x - b = -1$ [26]. As $1 \in A'$, $x \in xA'$; the last condition of Definition 4.10 with $u = x$ and $x' = 1 - x - b$ gives $b \in xA'$, so $b/x \in A' \cap U$. (c) $x \in uA' \cap vuA'$, so $vuA' = uA'$ and $vA' = A'$. (d) Then $H = U$ and $uA' = A'$ for every unit u. □

For an integer $n \geq 2$, a *fiber* of n is a non-empty set of the form

$$\{t : 0 < t < n,\ \gcd(t, n) = 1,\ t \equiv t_3 \pmod 3 \text{ if } 3 \mid n,\ t \not\equiv t_\ell \pmod \ell \text{ for } \ell \in \Lambda\},$$

where $t_3 \in \{1,2\}$, Λ is a set of primes $\ell \geq 5$ dividing n, and $t_\ell \not\equiv 0 \pmod{\ell}$. Let $\varkappa(n)$ be the largest mean of $2t/n - 1$ over a fiber of n.

Lemma 6.2. *$\varkappa(3) = \frac{1}{3}$, $\varkappa(6) = \frac{2}{3}$, $\varkappa(12) = \frac{1}{3}$, $\varkappa(15) = \frac{7}{15}$, $\varkappa(30) = \frac{8}{15}$, and $\varkappa(n) < \frac{1}{3}$ for every other $n \geq 2$.*

Proof. Let ρ be the product of the primes dividing n. A fiber of n is the set of $t = t_0 + \rho j$ with t_0 in a fiber of ρ and $0 \leq j < n/\rho$, and the mean of $2(t_0+\rho j)/n - 1$ over j is $(\rho/n)(2t_0/\rho - 1)$; so $\varkappa(n) = (\rho/n)\varkappa(\rho)$. Let S be a fiber of ρ. Its indicator function is a product over the primes $\ell \mid \rho$ of the indicator of one residue class if $\ell \leq 3$, of $1 - \mathbf{1}_{t\equiv 0}$ if $\ell \geq 5$ and $\ell \notin \Lambda$, and of $1 - \mathbf{1}_{t\equiv 0} - \mathbf{1}_{t\equiv t_\ell}$ if $\ell \in \Lambda$. Expanding the product writes it as a signed sum of at most $2^a 3^b$ indicators of single residue classes $t \equiv t_0 \pmod{\rho'}$, $0 \leq t_0 < \rho'$, with $\rho' \mid \rho$, where a and b are the numbers of primes $\ell \geq 5$ of the two kinds. Over such a class, $\sum_{0\leq t<\rho}(t/\rho - \frac{1}{2}) = t_0/\rho' - \frac{1}{2}$, of absolute value at most $\frac{1}{2}$. As $|S| = \prod_{\ell\notin\Lambda}(\ell-1)\prod_{\ell\in\Lambda}(\ell-2)$, the products over the primes $\ell \geq 5$ dividing ρ, and $2/(\ell-1) \leq 3/(\ell-2)$,

$$\varkappa(n) \leq \frac{\rho}{n} \prod_{\ell\mid n,\ \ell\geq 5} \frac{3}{\ell-2}.$$

The right-hand side is at least $\frac{1}{3}$ only if the primes $\ell \geq 5$ dividing n form one of the sets $\emptyset$, $\{5\}$, $\{7\}$, $\{11\}$, $\{5,7\}$, $\{5,11\}$ and $n/\rho \leq 3$. On these finitely many n we evaluate $\varkappa$ directly, and find the five values of the statement and $\varkappa(n) < \frac{1}{3}$ otherwise. □

Lemma 6.3. *Let $3 \mid k$ and $k \notin \{6, 12, 30\}$. Let $t_3 \in \{1,2\}$, let Λ be a set of primes $\ell \geq 5$ dividing k, and let $t_\ell \not\equiv 0 \pmod{\ell}$ for $\ell \in \Lambda$. Then the set C of units x with $x \equiv t_3 \pmod 3$ and $x \not\equiv t_\ell \pmod{\ell}$ for every $\ell \in \Lambda$ contains some x with $h(x) \geq 1$.*

Proof. Suppose that h vanishes on C. By (6.1), $\sum_i \{xq_i/k\} = 3$ for every $x \in C$, so the means over C of the four functions $2\{xq_i/k\} - 1$ sum to 2. Let $n_i = k/g_i \geq 2$. Then $\{xq_i/k\} = t/n_i$, where $0 < t < n_i$ and $t \equiv x\,(q_i/g_i) \pmod{n_i}$, and q_i/g_i is a unit modulo n_i. By the Chinese remainder theorem C is a product of subsets of the groups U_{ℓ^ν}, for the prime powers ℓ^ν exactly dividing k, each of which is a union of classes modulo ℓ; so $x \mapsto x\,(q_i/g_i) \bmod n_i$ maps C onto a fiber of n_i, every point of which has the same number of preimages. Hence the mean of $2\{xq_i/k\} - 1$ over C is at most $\varkappa(n_i)$, and $\sum_i \varkappa(n_i) \geq 2$. Let $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3 \geq \varkappa_4$ be the four values $\varkappa(n_i)$. By Lemma 6.2, $2 \leq \varkappa_1 + \varkappa_2 + 2\varkappa_3 \leq \frac{4}{3} + 2\varkappa_3$, so $\varkappa_3 \geq \frac{1}{3}$, and the three corresponding n_i lie in $\{3, 6, 12, 15, 30\}$. With the values of Lemma 6.2, $\varkappa_1 + \varkappa_2 + 2\varkappa_3 \geq 2$ holds only if two of the three are 6, or all three lie in $\{6, 15, 30\}$ and are not all 15; so their least common multiple is 6, 12 or 30. But the least common multiple of any three of the n_i is k: if a prime power ℓ^ν exactly divides k and divides none of three of the n_i, then ℓ divides three of the q_i, hence the fourth, as $\sum_i q_i = 2k$, against $\gcd(k, q_1, \ldots, q_4) = 1$. So $k \in \{6, 12, 30\}$. □

Lemma 6.4. *Let $k > 30$. Every closed set that contains U is $F \cup F_1$.*

Proof. Let A' be closed with $U \subset A'$. By Lemma 6.1(d) it suffices to show that every orbit $O_d \subset F \cup F_1$ meets A'. We claim that there are $x \in U$ and $b \in O_d$ with $h(x) \geq 1$, $h(b) \geq 1$ and $1 - x - b \in U$. Then the last condition of Definition 4.10, with $u = 1$ and $x' = 1 - x - b \in U \subset A' \cap F$, gives $b \in A'$. Note that $-O_d = O_d$, that d is odd if k is even, and that for a prime ℓ dividing k but not d the residues modulo ℓ of the elements of O_d are all the non-zero ones.

Suppose that $3 \nmid k$. Choose $b \in O_d$ with $h(b) \geq 1$. By the Chinese remainder theorem there is x with $x \not\equiv 0, \pm(1-b) \pmod{\ell}$ for every odd prime $\ell \mid k$, as $\ell \geq 5$, and x odd if k is

even. Then x and $1-b\pm x$ are units, the latter being odd when k is even. One of x, $-x$ has $h\geq 1$, and the claim follows.

Suppose that $3\mid k$. Let C be the set of units x with $x\equiv 2 \pmod 3$ if $3\mid d$, $x\equiv 1 \pmod 3$ if $3\nmid d$, and $x\not\equiv 1 \pmod \ell$ for every prime $\ell\geq 5$ dividing d. Suppose that some $x\in C$ has $h(x)\geq 1$. Choose $b\in O_d$ with $b\not\equiv \pm(1-x) \pmod \ell$ for every prime $\ell\geq 5$ that divides k and not d; there are at least four non-zero residues to choose from. Then $1-x-b$ and $1-x+b$ are units: modulo a prime $\ell\geq 5$ dividing d they are $\equiv 1-x\not\equiv 0$; modulo 3 they are $\equiv -1$ if $3\mid d$, and $\equiv \mp b\not\equiv 0$ if $3\nmid d$; they are odd if k is even; and modulo the other primes they are non-zero by the choice of b. Replacing b by $-b$ if $h(b)=0$ proves the claim.

Finally, some $x\in C$ has $h(x)\geq 1$, by Lemma 6.3 with $t_3\in\{1,2\}$ as above, Λ the set of primes $\ell\geq 5$ dividing d, and $t_\ell=1$. □

Corollary 6.5. *Let $k>30$. If some collision triangle of θ is connected and good, then the rules reach θ.*

Proof. Let Δ_{lm} be connected and good. A unit x has $xq_l, xq_m, x(q_l+q_m)\not\equiv 0$, as $0<q_l, q_m, q_l+q_m<k$; so $U\subset \mathrm{Deg}(\theta)$ by Corollary 3.4, and Lemma 6.4 applies. □

6.2. **The last digit.** Recall from Section 3.2 the short pairs $\{l,m\}$, those with $q_l+q_m<k$, and their collision triangles $\Delta_{lm}=(k-q_l-q_m,q_l,q_m)/k$. As θ is open, there are three short pairs, one in each splitting of the four indices into two pairs, and any two of them have exactly one index in common. They are the three pairs inside a set of three indices, or the three pairs containing one index. A prime ℓ dividing k, q_l and q_m divides no other q_i, as it would then divide all four; so it belongs to at most one short pair.

Lemma 6.6. *Let $\{l,m\}$ be a short pair and ℓ an odd prime dividing $\gcd(k,q_l,q_m)$. Let A' be closed and $H=A'\cap U$. Then H contains the kernel of $U_k\to U_{k/\ell}$. If $\ell^2\nmid k$, the element e_ℓ of $\mathbb{Z}/k$ with $e_\ell\equiv 0 \pmod \ell$ and $e_\ell\equiv 1 \pmod{k/\ell}$ lies in A', and $h(e_\ell)\geq 1$.*

Proof. Let $z\equiv 1 \pmod{k/\ell}$. As ℓ divides q_l and q_m, $\{zq_l/k\}=q_l/k$ and $\{zq_m/k\}=q_m/k$, with sum in $(0,1)$. So $zq_l, zq_m\not\equiv 0$; the two other zq_i are not both $\equiv 0$, as $\sum_i\{zq_i/k\}$ is an integer; and this integer is less than 3. Also z is odd if k is even. Hence $z\in F\cup F_1$, and $h(z)\geq 1$: if $n(z)=4$, then $h(z)=3-\sum_i\{zq_i/k\}\geq 1$, and if $n(z)=3$, the three non-zero fractions sum to 1 and $h(z)=3-1-1=1$.

For $j\in\mathbb{Z}/\ell$ let $x=1+jk/\ell$ and $b=2-x=1-jk/\ell$, so that $1-x-b=-1\in A'\cap F$. If $\ell^2\mid k$, then x and b are units, and $b/x\in H$ by Lemma 6.1(b). As $(k/\ell)^2\equiv 0 \pmod k$, $b/x=(1-jk/\ell)^2=1-2jk/\ell$, and these elements are the whole kernel. If $\ell^2\nmid k$, choose j with $b\equiv 0 \pmod\ell$; then $b=e_\ell$, and $x\equiv 2 \pmod\ell$ is a unit. The last condition of Definition 4.10 with $u=x$ gives $e_\ell\in xA'$. Every v in the kernel fixes e_ℓ, so the kernel lies in H by Lemma 6.1(c) (compare the second statement of [26, Lemma 7.12]); and x is in the kernel, so $e_\ell\in A'$. □

6.3. **Three disconnected collision triangles.**

Proposition 6.7. *Let k be odd, and suppose that the three collision triangles of θ are disconnected. Then every closed set contains U.*

Proof. Each short pair $\{l,m\}$ has a prime dividing $\gcd(k,q_l,q_m)$; these three primes are odd and distinct. Let $\ell_1<\ell_2<\ell_3$ be such primes, so $\ell_1\geq 3$, $\ell_2\geq 5$, $\ell_3\geq 7$, and $M=k/(\ell_1\ell_2\ell_3)$. Let A' be closed, $H=A'\cap U$, and K the kernel of $U_k\to U_M$. By the Chinese remainder theorem K is the product of the kernels of $U_k\to U_{k/\ell_j}$, so $K\subset H$ by Lemma 6.6.

Step 1: lifting. Let $x_0, b_0 \in U_M$ with $x_0 + b_0 \equiv 2 \pmod M$. We construct units $x \equiv x_0$, $b \equiv b_0 \pmod M$ of $\mathbb{Z}/k$ with $h(x) \geq 1$, $h(b) \geq 1$ and $1 - x - b \in U$. Then $1 - x - b \in -K \subset A' \cap F$, and $b/x \in H$ by Lemma 6.1(b).

The lifts of x_0 to $\mathbb{Z}/k$ are obtained by choosing, for each j, one of ℓ_j *digits*: with the other choices fixed, the lifts are $x + sk/\ell_j$, $s \in \mathbb{Z}/\ell_j$. If $\ell_j \mid M$, all digits give units at ℓ_j, and $1 - x - b \equiv -1 \pmod{\ell_j}$ for all digits of x and b. If $\ell_j \nmid M$, the digit is the residue modulo ℓ_j; one digit of x is excluded by $x \not\equiv 0$, and, given x, at most two digits of b by $b \not\equiv 0, 1 - x$. At ℓ_1 we fix the digit of x, with $x \equiv 1 \pmod{\ell_1}$ if $\ell_1 \nmid M$; then at most one digit of b is excluded there. In all, for x the digit at ℓ_1 is fixed and at most one digit is excluded at each of ℓ_2, ℓ_3; for b at most one digit is excluded at ℓ_1 and at most two at each of ℓ_2, ℓ_3.

Let z be x or b. Changing the digit of z at ℓ_j does not change $\{zq_i/k\}$ if $\ell_j \mid q_i$; if $\ell_j \nmid q_i$, the ℓ_j digits give the ℓ_j values of $\{zq_i/k\}$ in one class modulo $1/\ell_j$. So if at most j' digits are excluded, an admitted digit gives $\{zq_i/k\} < (j'+1)/\ell_j$. By (6.1) it suffices to make $\sum_i \{zq_i/k\} < 3$.

Suppose first that the short pairs are the pairs inside a set of three indices. For each of these indices i, exactly one of the three primes does not divide q_i, the prime of the opposite pair, and this gives a bijection between the three indices and the three primes. So the three fractions $\{zq_i/k\}$ at these indices are governed by different digits. For x, choosing the digits at ℓ_2 and ℓ_3 gives two fractions less than $2/\ell_2$ and $2/\ell_3$, and the sum of the four fractions is less than $\frac{2}{5} + \frac{2}{7} + 1 + 1 < 3$. For b, the three fractions can be made less than $2/\ell_1$, $3/\ell_2$, $3/\ell_3$, with sum less than $\frac{2}{3} + \frac{3}{5} + \frac{3}{7} < 2$, and the total is less than 3.

Suppose now that the short pairs are the pairs $\{i_0, i_j\}$, $j = 1, 2, 3$, containing one index i_0, with ℓ_j the prime of $\{i_0, i_j\}$. The digit at ℓ_j changes the fractions at the two indices $i_{j'}$, $j' \neq j$, and no other. For x, choose the digit at ℓ_2 to make the fraction at i_3 less than $2/\ell_2$, and then the digit at ℓ_3, which does not change it, to make the fraction at i_2 less than $2/\ell_3$; the total is less than $\frac{2}{5} + \frac{2}{7} + 2 < 3$. For b the same choices give two fractions with sum less than $3/\ell_2 + 3/\ell_3$, which is at most 1 unless $(\ell_2, \ell_3) = (5, 7)$. In that case $\ell_1 = 3$, and the digits at 3 and 5 together give the fifteen values of $\{bq_{i_3}/k\}$ in one class modulo $\frac{1}{15}$, a digit at 3 fixing the class modulo $\frac{1}{5}$ and a digit at 5 the class modulo $\frac{1}{3}$. Among the six smallest of the fifteen values, the excluded digit at 3 removes two and the at most two excluded digits at 5 remove at most three; so an admitted pair of digits gives a fraction less than $\frac{6}{15}$ at i_3. The digit at 7 then makes the fraction at i_2 less than $\frac{3}{7}$, and the total is less than $\frac{2}{5} + \frac{3}{7} + 2 < 3$.

Step 2: the group generated. Let $v \in U_M$ with $1 + v \in U_M$, and put $x_0 = 2/(1+v)$, $b_0 = vx_0$. By Step 1 the image of H in U_M contains $b_0/x_0 = v$. Let ℓ^ν be a prime power exactly dividing M. The v that are $\equiv 1$ modulo M/ℓ^ν and $\not\equiv 0, -1$ modulo ℓ form a subset of the factor U_{ℓ^ν} of U_M with $(\ell - 2)\ell^{\nu-1}$ elements. For $\ell \geq 5$ this is more than half of the group, so the image of H contains U_{ℓ^ν}. For $\ell = 3$ it is the subgroup of the elements $\equiv 1 \pmod 3$; the image of H contains -1 and the factors at the other primes, hence the element that is -1 in U_{3^ν} and 1 in the other factors, and so again all of U_{3^ν}. So H maps onto U_M, and with $K \subset H$ this gives $H = U$. □

6.4. **Types without a good collision triangle.**

Lemma 6.8. *Let k be odd, $k > 15$. Then one of the following holds:*

(a) *the three collision triangles of θ are disconnected;*
(b) *some collision triangle of θ is connected and good;*

(c) $N = 2k$ *and, up to order,* $(q_i) = (r, r, s, 3s)$ *with* $k = r + 2s$, $r < 2s$, r *odd and* $\gcd(r, s) = 1$;
(d) *up to order,* $(q_i) = (e, e, e, 2k - 3e)$ *with* $\gcd(e, k) = 1$ *and* $e < k/2$.

Proof. Suppose that (a) fails. A connected collision triangle with distinct numerators is good: its expression with denominator k is reduced, and every triangle excluded in Definition 3.3 has an even denominator or denominator 9 or 15. So suppose that (b) fails too; then some collision triangle Δ_{lm} is connected and isosceles. Its numerators are $(w, w, k - 2w)$, with $\gcd(w, k) = 1$, so q_l and q_m are units modulo k, as k is odd. Every other short pair contains l or m, so its collision triangle is connected. Hence all three collision triangles are connected and isosceles. For a short pair $\{l, m\}$ with $q_l \neq q_m$ this means

$$k = 2q_l + q_m \quad \text{or} \quad k = q_l + 2q_m. \tag{6.2}$$

Suppose that the q_i are distinct, $q_1 < q_2 < q_3 < q_4$. Then $\{1, 2\}$ and $\{1, 3\}$ are short. By (6.2) for both, and since $q_1 < q_2 < q_3$, we get $k = q_1 + 2q_2 = 2q_1 + q_3$: the other three combinations give $q_2 = q_3$ or $q_2 < q_1$. The third short pair is $\{1, 4\}$ or $\{2, 3\}$. For $\{1, 4\}$, (6.2) gives $q_4 = k - 2q_1 = q_3$ or $q_4 = (k - q_1)/2 = q_2$. For $\{2, 3\}$, $q_2 < q_3$ gives $k > 3q_1$, so $q_2 + 2q_3 > 2q_2 + q_3 = 2k - 3q_1 > k$. So this case does not occur.

Suppose that exactly three values occur: $(q_i) = (r, r, s, t)$ with $s < t$. Then $r + s < r + t$, so the pairs of an r and s are short, and (6.2) gives $k = 2r + s$, whence $t = k$, which is excluded, or $k = r + 2s$, whence $t = 3s$. The third short pair consists of the two r if $r < 2s$, and is $\{s, 3s\}$ if $r > 2s$; $r = 2s$ would give $2r = k$. If $r > 2s$, the numerators $(r - 2s, s, 3s)$ are distinct unless $r = 3s = t$ or $r = 5s$, and $r = 5s$ gives $k = 7s$ and $s = \gcd(k, q_1, \dots, q_4) = 1$, so $k = 7$. So $r < 2s$, and $\gcd(r, s) = \gcd(k, q_1, \dots, q_4) = 1$. If N were odd, all q_i would be even. So $N = 2k$, and r is odd as k is. This is (c).

Otherwise two values occur, and as θ is reduced, $(q_i) = (e, e, e, 2k - 3e)$. A common prime factor of e and k would divide all q_i, so $\gcd(e, k) = 1$. If $e > k/2$, the pairs of an e and $2k - 3e$ are short, with collision triangle of numerators $(2e - k, e, 2k - 3e)$; these are distinct, as $e \neq k$, $2e \neq k$, and $5e = 3k$ only for $k = 5$. So $e < k/2$, which is (d). □

6.5. **The two families.** In this subsection k is odd and θ is as in Lemma 6.8(c) or (d). In the first case put $w = s$, and in the second $w = e$; w is a unit modulo k.

Lemma 6.9.

(a) *If* $3 \nmid k$, *then* $F \cup F_1 = F$ *is the set of all* $x \not\equiv 0$, *and* $D = 2k - 2$. *If* $3 \mid k$, *then* $F_1 = \{\pm k/3\}$, F *is the set of the other* $x \not\equiv 0$, *and* $D = 2k - 4$.
(b) *Let* $x \in F \cup F_1$ *and* $\phi = \{wx/k\}$. *For* $(q_i) = (r, r, s, 3s)$, $h(x) = 0$ *if and only if* $\frac{1}{2} < \phi \leq \frac{2}{3}$. *For* $(q_i) = (e, e, e, 2k - 3e)$, $h(x) = 0$ *if and only if* $\phi \geq \frac{2}{3}$.
(c) *Let* $(q_i) = (e, e, e, 2k - 3e)$ *and* $3 \nmid k$. *The triangle* $(3e, 2k - 3e, k)/(3k)$ *of Proposition 4.9(b) is reduced and good for* $k > 5$, *and* $|F(\Delta)| = 2k - 2 = D$.
(d) *Let* $k > 15$ *and let* Δ *be the triangle of Proposition 4.9(a) for two equal entries:* $(2r, s, 3s)/(2k)$ *in the first family, and* $(2e, e, 2k - 3e)/(2k)$ *in the second. Its reduced expression is good, unless* $N = 2k$ *and* $(q_i) = (1, 1, 1, 2k - 3)$, *and*

$$|F(\Delta)| \geq \tfrac{1}{2}D - 1.$$

Proof. (a) Let $x \not\equiv 0$. As r, s and e are units, $xq_i \equiv 0$ can hold only for the last entry, $3s$ or $2k - 3e \equiv -3e$, and it holds there if and only if $3x \equiv 0$.

(b) In the first family $r \equiv -2s$, so $h(x) = 2\{2\phi\} + \{-\phi\} + \{-3\phi\} - 1$. If $3\phi \notin \mathbb{Z}$ this is $1 + \lfloor 3\phi \rfloor - 2\lfloor 2\phi \rfloor$, which vanishes exactly for $\frac{1}{2} \leq \phi < \frac{2}{3}$, and $\phi \neq \frac{1}{2}$ as k is

odd; for $\phi = \frac{1}{3}$ it is 1, and for $\phi = \frac{2}{3}$ it is 0. In the second family $2k - 3e \equiv -3e$, so $h(x) = 3\{-\phi\} + \{3\phi\} - 1 = 2 - \lfloor 3\phi \rfloor$.

(c) In Proposition 4.9(b) the triangle is $(6a_i, 2a_m, N)/(3N)$, which is $(3e, 2k - 3e, k)/(3k)$ for both parities of N. Here $\gcd(3e, k) = 1$, so it is reduced; its numerators are distinct, as $3e \neq k$; and its denominator $3k$ is odd and larger than 15. The numerator $2k - 3e$ is prime to $3k$, so $F(\Delta)$ is the set of x modulo $3k$ with $k \nmid x$ and $3 \nmid x$, which has $3k - k - 3 + 1$ elements.

(d) In the first family $N = 2k$ and $\gcd(2r, s, 3s) = \gcd(2, s)$. If s is odd, the expression is reduced, with distinct numerators; for odd x modulo $2k$, $2rx \equiv 0$ only for $x = k$, $sx \not\equiv 0$ as s is prime to $2k$, and $3sx \not\equiv 0$, as it would need $2k/\gcd(3, k)$, an even number, to divide x. So $|F(\Delta)| = k - 1$. The triangles excluded in Definition 3.3 with even denominator larger than 20 contain the numerator 1 or 2 and are $(1, 2, c)$, $(1, c, c+1)$ or $(2, c, c+2)$; here 1 occurs only for $s = 1$ and 2 only for $r = 1$, and $s = 1$ forces $r = 1$; $(2, s, 3s)$ is of the third form only for $s = 1$; so Δ is excluded only for $k = 3$. If s is even, the reduced expression is $(r, s/2, 3s/2)/k$, with odd denominator; r and $s/2$ are prime to k and $\gcd(3s/2, k) = \gcd(3, k)$, so $|F(\Delta)|$ is $k - 1$ if $3 \nmid k$ and $k - 3$ if $3 \mid k$. Its numerators are distinct unless $r = s/2$ or $r = 3s/2$, which with $\gcd(r, s) = 1$ give $k \leq 7$, and its denominator is odd and larger than 15, so it is good.

In the second family, if $N = 2k$, then e and $2k - 3e$ are odd and the expression is reduced, with distinct numerators, as $e = 2k - 3e$ would give $2e = k$, and $2e = 2k - 3e$ would give $5e = 2k$ with e odd. As before, for odd x modulo $2k$ only $2ex \equiv 0$ can occur, for $x = k$, so $|F(\Delta)| = k - 1$. The numerator 2 occurs only for $e = 1$, and 1 only for $e = 1$ or $2k - 3e = 1$; in the last case $\Delta = (1, e, 2e)$, which for $e > 1$ is none of the three forms. So Δ is good unless $e = 1$. If $N = k$, then e is even and the reduced expression is $(e, e/2, k - 3e/2)/k$, with odd denominator larger than 15; its numerators are distinct unless $e = k - 3e/2$ or $e/2 = k - 3e/2$, that is $5e = 2k$ or $2e = k$; the second is impossible as k is odd, and the first, with $\gcd(e, k) = 1$, forces $(k, e) = (5, 2)$, excluded as $k > 15$; so the reduced expression is good, and $|F(\Delta)|$ is $k - 1$ or $k - 3$ as before.

In all cases $|F(\Delta)| \geq k - 1$ if $3 \nmid k$ and $|F(\Delta)| \geq k - 3$ if $3 \mid k$, which is at least $\frac{1}{2}D - 1$ by (a). □

Lemma 6.10. *Let $k > 300$, and let A' be a closed set with $D(A') \geq \frac{1}{2}D - 1$. Then $U \subset A'$.*

Proof. Let $H = A' \cap U$, and suppose that $H \neq U$. By Lemma 6.1(a) the $[U : H]$ translates of A' are disjoint subsets of $F \cup F_1$ with the same $D(uA')$. Three of them would give $3(\frac{1}{2}D - 1) \leq D$, that is $D \leq 6$, against Lemma 6.9(a). So $[U : H] = 2$, and H is the kernel of a character $\chi\colon U \to \{\pm 1\}$ with $\chi(-1) = 1$. The two translates A' and uA' cover $F \cup F_1$ up to a set E with $D(E) \leq 2$. An orbit $O_d \subset F \cup F_1$ that does not meet A' does not meet uA' either, so it lies in E. Hence every orbit $O_d \subset F \cup F_1$ with $D(O_d) > 2$ meets A', and the stabilizer in U of each of its points lies in H, by Lemma 6.1(c).

Let $\ell \geq 5$ be a prime dividing k. The orbit $O_{k/\ell}$ lies in F and has $\ell - 1$ elements, so $D(O_{k/\ell}) > 2$; the stabilizer of its points is the kernel of $U \to U_\ell$. So χ factors through U_ℓ. If two primes $\ell, \ell' \geq 5$ divided k, the two kernels would generate U and χ would be trivial. If $9 \mid k$, the orbit $O_{k/9}$ lies in F and has six elements, so χ factors through U_9; the non-trivial character of order two of the cyclic group U_9 takes the value -1 at -1, so χ would be trivial on U_9, and then on U. As $k > 15$, it follows that $k = \ell^\nu$ or $k = 3\ell^\nu$ for a prime $\ell \geq 5$, that χ is induced by the Legendre symbol modulo ℓ, also denoted by χ and extended by $\chi(0) = 0$, and that $\ell \equiv 1 \pmod 4$.

Let C be the set of units x of $\mathbb{Z}/k$, with $x \equiv 1 \pmod 3$ if $3 \mid k$; it maps bijectively to U_{ℓ^ν}. For $x, b \in C$, the residue $1 - x - b$ is $\equiv -1 \pmod 3$ if $3 \mid k$, so it is a unit as soon as it is

not divisible by ℓ. By Lemma 6.9(b) a unit x has $h(x) = 0$ only if $\{wx/k\}$ lies in an interval I of length at most $\frac{1}{3}$.

Suppose that $\nu \geq 2$. The elements of C in one class modulo ℓ are $x_1 + jk/\ell^{\nu-1}$, $j \in \mathbb{Z}/\ell^{\nu-1}$, and their fractions $\{wx/k\}$ are the $\ell^{\nu-1} \geq 5$ numbers of one class modulo $1/\ell^{\nu-1}$; so each non-zero class modulo ℓ contains some x with $h(x) \geq 1$. We show by induction that $\chi(j) = 1$ for every odd integer j with $0 < j < \ell$. This holds for $j = 1$. If it holds for j and $j + 2 < \ell$, choose $x, b \in C$ with $x \equiv -j$, $b \equiv j + 2 \pmod{\ell}$ and $h(x), h(b) \geq 1$. Then $1 - x - b \equiv -1 \pmod{\ell}$ is a unit with $\chi = 1$, so it lies in $H \subset A' \cap F$, and $\chi(b) = \chi(x)$ by Lemma 6.1(b); so $\chi(j + 2) = \chi(-j) = 1$. For even j with $0 < j < \ell$, $\chi(j) = \chi(\ell - j) = 1$. So χ is trivial, a contradiction.

Suppose that $\nu = 1$. Then C maps bijectively to U_ℓ, and the fractions $\{wx/k\}$ of the $x \equiv 1 \pmod 3$ (of all x, if $k = \ell$) are ℓ numbers of one class modulo $1/\ell$, of which at most $\ell/3 + 1$ lie in I. Let C^+ and C^- be the sets of $x \in C$ with $h(x) \geq 1$ and $\chi(x) = 1$, respectively $\chi(x) = -1$; each has at least $(\ell - 1)/2 - \ell/3 - 1$ elements. For $x \in C^+$ and $b \in C^-$, $1 - x - b$ is not a unit in H, by Lemma 6.1(b). So, with $\chi(0) = 0$, each term of

$$\Sigma = \sum_{x \in C^+} \sum_{b \in C^-} \chi(1 - x - b)$$

is -1 or 0, and it is 0 for at most $\min(|C^+|, |C^-|)$ pairs. On the other hand, by the Cauchy–Schwarz inequality and $\sum_{x \in \mathbb{Z}/\ell} \chi(x - b)\chi(x - b') = -1$ for $b \not\equiv b'$, as in [19, Lemma 5.1],

$$\Sigma^2 \leq |C^+| \sum_{x \in \mathbb{Z}/\ell} \Big(\sum_{b \in C^-} \chi(1 - x - b) \Big)^2 \leq |C^+|\,|C^-|\,\ell.$$

With $\mu = \sqrt{|C^+||C^-|}$ this gives $\mu^2 - \mu \leq \mu\sqrt{\ell}$, so $\ell/6 - \frac{3}{2} \leq \mu \leq \sqrt{\ell} + 1$, which fails for $\ell \geq 101$. As $k > 300$ and $k \in \{\ell, 3\ell\}$, this is a contradiction. □

Lemma 6.11. *Let k be odd with $3 \mid k$ and $k > 300$, and $(q_i) = (1, 1, 1, 2k - 3)$. Then every closed set contains U.*

Proof. Let A' be closed and $H = A' \cap U$. Here $w = 1$: by Lemma 6.9, $h(z) \geq 1$ exactly when the representative of $z \not\equiv 0$ in $(0, k)$ is less than $2k/3$, and the z with $0 < z < k/3$ lie in F. Let u be the inverse of 2 modulo k.

We show that every odd integer j with $0 < j < k/3$ lies in A'. Let x_0 and i be odd integers in A' with $0 < x_0, i < k/3$ such that $j = 2 + 2i - x_0$ is an odd integer with $0 < j < k/3$. Put $x = ux_0 \in uA'$, $x' = -i \in A' \cap F$ and $b = 1 - x - x' = uj$. The representatives of x and b are $(k + x_0)/2$ and $(k + j)/2$, in $(k/2, 2k/3)$, so $h(x), h(b) \geq 1$, and the last condition of Definition 4.10 gives $b \in uA'$, that is $j \in A'$. With $x_0 = i = 1$ this gives $3 \in A'$. For odd $j \geq 5$, put $i = (j - 1)/2$ and $x_0 = 1$ if $j \equiv 3 \pmod 4$, and $i = (j + 1)/2$ and $x_0 = 3$ if $j \equiv 1 \pmod 4$; then i is odd with $0 < i < j$, and induction on j proves the claim.

If ℓ^ν is a prime power exactly dividing k with $k/\ell^\nu > 3$, then $\ell^\nu \in A'$, and it is fixed by the units $\equiv 1 \pmod{k/\ell^\nu}$, which form the factor U_{ℓ^ν} of U; so this factor lies in H, by Lemma 6.1(c). This gives $H = U$ unless $k = 3^\nu$ or $k = 3\ell^\nu$ with $\ell \geq 5$ prime. If $k = 3^\nu$, then $3^{\nu-2} \in A'$, so H contains the kernel of $U \to U_9$; and $5 \in H$ generates U_9. If $k = 3\ell^\nu$, then $3 \in A'$ gives the factor U_3 of U in H, and it remains to show that H maps onto U_{ℓ^ν}. Every element of U_{ℓ^ν} is $\pm j$ for an odd integer j with $0 < j < \ell^\nu = k/3$ and $\ell \nmid j$. If $3 \nmid j$, then $j \in A' \cap U = H$. Otherwise $j = 3^{\nu'} j'$ with j' of that kind, and it suffices to see that 3 is in the image of H. Let t be the one of $\ell^\nu - 2$, $\ell^\nu - 4$ that is prime to 3. Then t and $3t - 2\ell^\nu$ are odd integers in $(0, \ell^\nu)$ prime to 3ℓ, so they lie in H, and their quotient is $\equiv 3 \pmod{\ell^\nu}$. □

6.6. **Conclusion.**

Theorem 6.12. *The rules reach every open type with k odd and $N > 1000$.*

Proof. Here $k > 500$. Let A' be a closed set that contains $\mathrm{Deg}(\theta)$ and meets the symmetric bounds. We go through the cases of Lemma 6.8. In case (b), Corollary 6.5 applies. In case (a), $U \subset A'$ by Proposition 6.7, and $A' = F \cup F_1$ by Lemma 6.4. In case (d) with $3 \nmid k$, the symmetric bound of Lemma 6.9(c) is $D(A') \geq D$, so $A' = F \cup F_1$. In case (d) with $3 \mid k$ and $(q_i) = (1, 1, 1, 2k-3)$, where $N = 2k$ if $e = 1$, Lemmas 6.11 and 6.4 give $A' = F \cup F_1$. In the remaining cases, case (c) and case (d) with $3 \mid k$ and $(q_i) \neq (1, 1, 1, 2k-3)$, the triangle of Lemma 6.9(d) is good, so $D(A') \geq |F(\Delta)| \geq \frac{1}{2}D - 1$, and Lemmas 6.10 and 6.4 give $A' = F \cup F_1$. □

7. Even degree

We show that the rules reach every open type with k even and $N > 1000$ (Theorem 7.14), and that for every open type with k even the relative rule gives the relative directions of the ambient locus once $A = F \cup F_1$ (Proposition 7.15). As for odd k, a good connected collision triangle suffices. Without one, the argument depends on the number of short pairs whose two entries have an odd prime factor in common with k: with two such pairs the last digits at the two primes give the units, up to a subgroup of index two modulo 12 that is excluded separately; with one, the type is nearly determined and the same argument applies; with none, the type lies in two families with a symmetric member of full rank, in four families of kites, or in a finite list of one-parameter families, on which the differential rule links the characters in an interval.

Throughout this section θ is an open type with k even, so that $N = 2k$ and $a_i = q_i$, and we use the notation of Section 6: U, U_n, O_d, $D(A')$, the short pairs and their collision triangles, and for a closed set A' the group $H = A' \cap U$ of Lemma 6.1. A prime ℓ is a *shared prime* of a short pair $\{l, m\}$ if it divides $\gcd(k, q_l, q_m)$; a prime is shared by at most one short pair (Section 6.2), and a collision triangle is connected exactly when its short pair has no shared prime. For real y let $\langle y \rangle$ be the representative of y modulo 1 in $(0, 1]$, so that $\{-y\} = 1 - \langle y \rangle$ and, for every $z \in \mathbb{Z}/k$,

$$h(z) = 3 - \sum_{i=1}^{4} \Big\langle \frac{z q_i}{k} \Big\rangle, \tag{7.1}$$

where the sum is an integer. An odd z lies in $F \cup F_1$ if and only if at most one of the four terms is 1, and in F if and only if none is.

7.1. **From ratios to units.**

Lemma 7.1. *Let M be a divisor of k with k/M odd, A' a closed set and $H = A' \cap U$. Suppose that H contains the kernel of $U_k \to U_M$, and that the image of H in U_M contains b_0/x_0 for all $x_0, b_0 \in U_M$ with $x_0 + b_0 \equiv 2 \pmod{M}$. Then H contains every unit u with $u \equiv \pm 1 \pmod{\gcd(k, 12)}$. In particular $H = U$ if $12 \nmid k$.*

Proof. Write U_M as the product of the groups U_{ℓ^ν} over the prime powers ℓ^ν exactly dividing M; here $2 \mid M$. Let $S_\ell \subset U_{\ell^\nu}$ be the set of v with $v \not\equiv -1 \pmod{\ell}$ if ℓ is odd, and with $v \equiv 1 \pmod 4$ if $\ell = 2$ and $4 \mid M$; $S_2 = \{1\}$ if $4 \nmid M$. Let $v = (v_\ell)$ with $v_\ell \in S_\ell$ for every ℓ. There is $x_0 \in U_M$ with $x_0(1 + v) \equiv 2$: at an odd prime power, $1 + v_\ell$ is a unit and $x_0 = 2/(1 + v_\ell)$; at 2^ν with $\nu = 1$ take $x_0 = 1 \bmod 2$, and with $\nu \geq 2$ write $1 + v_2 = 2w$ with w odd and choose x_0 with $x_0 w \equiv 1 \pmod{2^{\nu-1}}$. With $b_0 = v x_0$ we get $x_0 + b_0 \equiv 2$, so

$v = b_0/x_0$ lies in the image $\overline{H}$ of H. Each S_ℓ contains 1, so $\overline{H}$ contains each S_ℓ, as a subset of the factor U_{ℓ^ν}. For $\ell \geq 5$, S_ℓ has $(\ell-2)\ell^{\nu-1}$ elements, more than half of the group, and generates it. For $\ell = 3$, S_3 is the subgroup of the $v \equiv 1 \pmod 3$, and S_2 is the subgroup of the $v \equiv 1 \pmod 4$ if $4 \mid M$, and U_2 otherwise. So $\overline{H}$ contains every $v \in U_M$ with $v \equiv 1 \pmod 3$ if $3 \mid M$ and $v \equiv 1 \pmod 4$ if $4 \mid M$, and it contains -1; hence it contains every $v \equiv \pm 1 \pmod{\gcd(M,12)}$. As H contains the kernel of $U_k \to U_M$, and $\gcd(M,12)$ divides $\gcd(k,12)$, the lemma follows; for $\gcd(k,12) \in \{2,4,6\}$ every unit is $\equiv \pm 1$. □

The congruence modulo 12 in Lemma 7.1 is where ratios of units stop. Let $12 \mid k$, and let $\chi(u)$ be the product of the signs of a unit u modulo 3 and modulo 4, so that $\ker\chi$ is the group of the units $\equiv \pm 1 \pmod{12}$, of index two in U. If x, x' and $1-x-x'$ are units, exactly one of them is $\equiv -1 \pmod 3$ and an odd number are $\equiv -1 \pmod 4$, so $\chi(x)\chi(x')\chi(1-x-x') = 1$. Hence the ratios b/x that Lemma 6.1(b) gives with a unit $1-x-b \in \ker\chi$ lie in $\ker\chi$, and if every element of $F \cup F_1$ is a unit, $\ker\chi$ is closed, as it is for the three supplementary types of Section 8.3. So Proposition 7.4 uses the characters of O_3, which are not units. For triangles the same character appears in [18, Proposition 2.8].

7.2. **The units congruent to ±1 modulo 12.** For an integer $n \geq 2$, a *fiber of n modulo* 12 is a non-empty set of the form

$$\{t:\ 0 < t < n,\ \gcd(t,n) = 1,\ t \equiv t_0 \pmod{\gcd(n,12)},\ t \not\equiv t_\ell \pmod \ell \text{ for } \ell \in \Lambda\},$$

where Λ is a set of primes $\ell \geq 5$ dividing n and $t_\ell \not\equiv 0 \pmod \ell$; it is *plain* if $\Lambda = \emptyset$. Let $\beta(n)$ be the largest mean of $2t/n - 1$ over a fiber of n modulo 12, and $\beta^0(n)$ the largest mean over a plain one.

Lemma 7.2.

(a) *Let $n_1, \dots, n_4 \geq 2$ be integers such that the least common multiple of any three of them is the same number n, with $12 \mid n$. If $\sum_i \beta(n_i) \geq 2$, then $n \leq 420$. If $\sum_i \beta^0(n_i) \geq 2$, then $n \leq 132$.*

(b) *Let $n_1, n_2, n_3 \geq 2$ be integers such that the least common multiple of any two of them is the same number n, with $12 \mid n$. If $\sum_i \beta^0(n_i) \geq 1$, then $n \leq 132$ or $n \in \{168, 180, 204, 264\}$.*

Proof. Let $c(n)$ be the product of $\gcd(n,12)$ and the primes $\ell \geq 5$ dividing n. As in the proof of Lemma 6.2, $\beta(n) = (c(n)/n)\,\beta(c(n))$, the same holds for β^0, and

$$\beta^0(n) \leq \beta(n) \leq \frac{c(n)}{n} \prod_{\ell \mid n,\ \ell \geq 5} \frac{3}{\ell - 2};$$

here the class modulo $\gcd(n,12)$ is imposed in each of the residue classes of the expansion. So for $\tau > 0$ there are finitely many n with $\beta(n) \geq \tau$, and they are found by evaluating β on the n for which the right-hand side is at least τ. The largest value of β is $\beta(12) = \frac{5}{6}$. In (a), the third largest of the four values is therefore at least $\frac{1}{6}$, and in (b) the second largest of the three is at least $\frac{1}{12}$. Evaluating β and β^0 on the finitely many n with bound at least $\frac{1}{6}$, respectively $\frac{1}{12}$, and listing the least common multiples that are divisible by 12 of the triples, respectively pairs, that satisfy the inequality for the sum, we find the multiples of 12 up to 60 and $84, 120, 132, 168, 180, 420$ in the first case, the multiples of 12 up to 60 and $84, 132$ in the second, and the multiples of 12 up to 132 and $168, 180, 204, 264$ in (b). These lists include every value that occurs. □

Lemma 7.3. *Let $12 \mid k$ and $k > 420$. Let t_0 be a unit modulo 12, Λ a set of primes $\ell \geq 5$ dividing k, and $t_\ell \not\equiv 0 \pmod{\ell}$ for $\ell \in \Lambda$. Then the set of units x with $x \equiv t_0 \pmod{12}$ and $x \not\equiv t_\ell \pmod{\ell}$ for every $\ell \in \Lambda$ contains some x with $h(x) \geq 1$.*

Proof. As in the proof of Lemma 6.3: if h vanishes on this set, the means of the four functions $2\{xq_i/k\} - 1$ over it sum to 2, the map $x \mapsto x(q_i/g_i) \bmod n_i$, with $n_i = k/g_i$, sends it onto a fiber of n_i modulo 12, every point of which has the same number of preimages, and the least common multiple of any three of the n_i is k. So $\sum_i \beta(n_i) \geq 2$, against Lemma 7.2(a). □

Proposition 7.4. *Let $12 \mid k$ and $k > 500$. Let A' be a closed set that contains $\mathrm{Deg}(\theta)$, meets the symmetric bounds, and contains every unit $u \equiv \pm 1 \pmod{12}$. Then $A' = F \cup F_1$.*

Proof. If $H = U$, this is Lemma 6.4. Suppose that $H \neq U$; then H is the group of the units $\equiv \pm 1 \pmod{12}$, of index two. The elements of the orbit O_3 are the $b = 3c$ with c a unit modulo $k/3$; they are odd, as $k/3$ is even, and $3cq_i \equiv 0 \pmod{k}$ if and only if $k/3$ divides q_i.

Case 1: at most one q_i is divisible by $k/3$. Then $O_3 \subset F \cup F_1$. For $b \in O_3$ with $h(b) \geq 1$, let C_b be the set of units x with $x \equiv 2 \pmod 3$, $x \equiv b \pmod 4$, and $x \not\equiv 1 - b \pmod{\ell}$ for every prime $\ell \geq 5$ dividing k with $b \not\equiv 1 \pmod{\ell}$. By Lemma 7.3 there is $x \in C_b$ with $h(x) \geq 1$. Then $x' = 1 - x - b$ is a unit with $x' \equiv -1 \pmod 3$ and $x' \equiv 1 - 2b \equiv -1 \pmod 4$, so $x' \in H \subset A' \cap F$, and the last condition of Definition 4.10 with $u = x$ gives

$$b \in xA' \qquad \text{for some unit } x \equiv 2 \pmod 3,\ x \equiv b \pmod 4. \tag{7.2}$$

If $9 \nmid k$, choose $b \in O_3$ with $h(b) \geq 1$, and let v be the unit with $v \equiv 2 \pmod 3$ and $v \equiv 1 \pmod{k/3}$. Then $vb = b$, so $v \in H$ by (7.2) and Lemma 6.1(c). But $v \equiv 5 \pmod{12}$, as $4 \mid k/3$; a contradiction.

If $9 \mid k$, then 12 divides $k/3$. We claim that there are units c_1, c_2 modulo $k/3$ with $c_1 \equiv c_2 \pmod 4$, $c_1 \not\equiv c_2 \pmod 3$ and $h(3c_1), h(3c_2) \geq 1$. Given these, let $b_1 = 3c_1$, $b_2 = 3c_2$, and take x_1, x_2 as in (7.2). Then $x_1 \equiv x_2 \pmod{12}$, so $x_1 A' = x_2 A'$. Let v be a unit modulo k with $v \equiv c_2/c_1 \pmod{k/3}$. Then $vb_1 = b_2$, so $b_2 \in x_1 A' \cap v x_1 A'$ and $v \in H$. But $v \equiv 1 \pmod 4$ and $v \equiv 2 \pmod 3$, so $v \equiv 5 \pmod{12}$; a contradiction.

To prove the claim, let r_i be the residue of q_i modulo $k/3$. The r_i have sum zero and no prime factor in common with $k/3$, at most one of them is zero, and by (7.1)

$$h(3c) = 3 - \sum_{i=1}^{4} \Big\langle \frac{cr_i}{k/3} \Big\rangle.$$

Suppose that $h(3c) = 0$ for every c in a class of units modulo 12. If no r_i is zero, the argument of Lemma 7.3 at the modulus $k/3$, with plain fibers, gives four denominators with $\sum_i \beta^0 \geq 2$, any three of them with least common multiple $k/3$. If $r_4 = 0$, the three other fractions sum to 2 on the class, so the means of the three functions $2\{cr_i/(k/3)\} - 1$ sum to 1, and the three denominators have $\sum_i \beta^0 \geq 1$; any two of them have least common multiple $k/3$, since a prime dividing $k/3$, r_1 and r_2 would divide r_3 as well. By Lemma 7.2, $k/3 \leq 132$ or $k/3 \in \{168, 180, 204, 264\}$. As $k/3 > 166$, the claim holds, with every class modulo 12 available, unless $k/3$ is one of these four numbers. For those, a computation over all $(r_1, \dots, r_4)$ modulo $k/3$ with sum zero, no prime factor in common with $k/3$ and at most one zero entry shows that the classes 1 and 5 modulo 12, or the classes 7 and 11, both contain a unit c with $\sum_i \langle cr_i/(k/3) \rangle \leq 2$.

Case 2: at least two of the q_i are divisible by $k/3$. Then exactly two are, as three would give the fourth, against $\gcd(k, q_1, \dots, q_4) = 1$. They are $jk/3$ and $j'k/3$ with $j, j' \in \{1, 2, 4, 5\}$, $j + j' \neq 3$ and $j + j' < 6$; so (q_i) is, up to order, $(k/3, k/3, R, S)$, $(2k/3, 2k/3, R, S)$ or

$(k/3, 4k/3, R, S)$. A common prime factor of $k/3$ and R would divide S, so R and S are prime to $k/3$, and they are odd.

Let $(q_i) = (jk/3, jk/3, R, S)$ with $j \in \{1, 2\}$. For odd x modulo k, $xjk/3 \equiv 0$ if and only if $3 \mid x$, and $xR, xS \not\equiv 0$, as they would need k or $k/3$, an even number, to divide x. So $F \cup F_1 = F$ is the set of odd x prime to 3, and $D = 2k/3$. The triangle of Proposition 4.9(a) for the two equal entries is $\Delta = (2jk/3, R, S)/(2k)$. This expression is reduced. Its numerators are distinct: $R \neq S$ because θ is reduced, and the first is even. It is good: its only even numerator is larger than 2, which excludes the first and third families of Definition 3.3; if one of R, S is 1, the two other numerators differ by more than 1; and the denominator is larger than 20. For odd x modulo $2k$, $x \cdot 2jk/3 \equiv 0$ if and only if $3 \mid x$, and $xR, xS \not\equiv 0$ as before. So $|F(\Delta)| = 2k/3 = D$, and $A' = F \cup F_1$ since it meets the symmetric bounds.

Let $(q_i) = (k/3, 4k/3, R, S)$. Then $R + S = k/3$, the pair of R and S is short, and its collision triangle $(2k/3, R, S)/k$ is connected, as a common divisor of R and S divides $k/3$; its numerators are distinct, and it is good: its even numerator is larger than 2, and if one of R, S is 1 the two others differ by more than 1. So $A' = F \cup F_1$ by Corollary 6.5. □

7.3. **Two short pairs with odd shared primes.**

Proposition 7.5. *Let $k > 500$, and suppose that two short pairs have odd shared primes. Then the rules reach θ.*

Proof. Let $\ell_1 < \ell_2$ be odd shared primes of two short pairs; so $\ell_1 \geq 3$ and $\ell_2 \geq 5$. The pairs have one index in common: let them be $\{i_0, i_1\}$ for ℓ_1 and $\{i_0, i_2\}$ for ℓ_2, and let i_3 be the fourth index. Then ℓ_1 divides q_{i_0} and q_{i_1} and not q_{i_2}, q_{i_3}, and ℓ_2 divides q_{i_0} and q_{i_2} and not q_{i_1}, q_{i_3}. Let $M = k/(\ell_1\ell_2)$, and let A' be a closed set that contains $\mathrm{Deg}(\theta)$ and meets the symmetric bounds. By Lemma 6.6, H contains the kernel K of $U_k \to U_M$, and $e_{\ell_j} \in A'$ if $\ell_j^2 \nmid k$. For $z_0 \in U_M$, all units $u \equiv z_0 \pmod M$ give the same translate uA', which we denote by $A'(z_0)$.

We show that the image of H in U_M contains b_0/x_0 for all $x_0, b_0 \in U_M$ with $x_0 + b_0 \equiv 2$. By Lemma 7.1 and Proposition 7.4 this gives $A' = F \cup F_1$ if $12 \mid k$, and $H = U$, hence $A' = F \cup F_1$ by Lemma 6.4, if $12 \nmid k$.

Lifts. Let $z_0 \in U_M$. Its lifts to $\mathbb{Z}/k$ are odd, and they are obtained by choosing a digit at ℓ_1 and a digit at ℓ_2, as in the proof of Proposition 6.7. The digit at ℓ_1 does not change $\langle zq_i/k\rangle$ for $i = i_0, i_1$, and as it varies $\langle zq_{i_2}/k\rangle$ takes the ℓ_1 values in $(0, 1]$ of one class modulo $1/\ell_1$; we number the digits at ℓ_1 by $0, 1, \dots$ in the order of increasing $\langle zq_{i_2}/k\rangle$, so that digit number j has $\langle zq_{i_2}/k\rangle \leq (j+1)/\ell_1$. In the same way the digit at ℓ_2 does not change the terms at i_0, i_2, and we number the digits at ℓ_2 in the order of increasing $\langle zq_{i_1}/k\rangle$. The term at i_2 depends only on the digit at ℓ_1, and the term at i_1 only on the digit at ℓ_2. A *candidate* is a lift with digit numbers $(0, 0)$, $(0, 1)$, $(0, 2)$ or $(1, 0)$ that is not divisible by both ℓ_1 and ℓ_2; at most one of the four lifts is excluded by this condition, as it fixes both digits. For a candidate z,

$$\left\langle \frac{zq_{i_1}}{k} \right\rangle + \left\langle \frac{zq_{i_2}}{k} \right\rangle \leq \max\Big(\frac{1}{\ell_1} + \frac{3}{\ell_2}, \frac{2}{\ell_1} + \frac{1}{\ell_2}\Big) \leq \frac{14}{15},$$

so the sum in (7.1) is less than 3, and the two other terms are not both 1. Hence $z \in F \cup F_1$ and $h(z) \geq 1$. Moreover $z \in A'(z_0)$: if z is a unit this holds by definition; otherwise z is divisible by exactly one of the two primes, say ℓ_j, which then divides k only once, and $z = ue_{\ell_j}$ for a unit $u \equiv z_0 \pmod M$, with $e_{\ell_j} \in A'$.

The third character. Let x and b be candidates for x_0 and b_0, and $x' = 1 - x - b$, so $x' \equiv -1 \pmod M$. We choose them with $x' \in A' \cap F$. Then the last condition of Definition 4.10

gives $b \in A'(x_0)$, so $A'(x_0) \cap A'(b_0) \neq \emptyset$, the two translates are equal, and b_0/x_0 is in the image of H.

If x' is a unit, then $-x' \in K \subset H$, so $x' \in A' \cap F$. The residue of x' modulo ℓ_j is -1 if $\ell_j^2 \mid k$, and otherwise it is determined by the digits of x and b at ℓ_j, different digits of b giving different residues. Among the candidates with digit number 0 at ℓ_1 there are, for each of x_0 and b_0, at least two digits at ℓ_2. So if $x' \not\equiv 0 \pmod{\ell_1}$ for the digits number 0 at ℓ_1, we fix such a candidate x and choose b among two with $x' \not\equiv 0 \pmod{\ell_2}$; then x' is a unit. Suppose that $x' \equiv 0 \pmod{\ell_1}$ for the digits number 0; then $\ell_1^2 \nmid k$. If the lift with digit numbers $(1, 0)$ is a candidate for x_0, take it as x; then $x' \not\equiv 0 \pmod{\ell_1}$, and we choose b with digit number 0 at ℓ_1 and $x' \not\equiv 0 \pmod{\ell_2}$ as before. The same applies with the roles of x_0 and b_0 exchanged.

There remains the case that for both x_0 and b_0 the lift with digit numbers $(1, 0)$ is divisible by ℓ_1 and ℓ_2. Then the digit number 1 at ℓ_1 is the one divisible by ℓ_1. We claim that $e_{\ell_1} \in F$. Otherwise k/ℓ_1 divides some q_i; this q_i is divisible by ℓ_2 and not by ℓ_1, as $q_i \neq k$, so $i = i_2$; but then $\langle z q_{i_2}/k \rangle = 1$ for the lifts z divisible by ℓ_1, which makes their digit at ℓ_1 the last one, number $\ell_1 - 1 \neq 1$. Now take x and b with digit number 0 at ℓ_1, where all three digits at ℓ_2 give candidates, and with $x' \not\equiv 0 \pmod{\ell_2}$. Then $x' \equiv 0 \pmod{\ell_1}$ and $x' = -v e_{\ell_1}$ with $v \in K$, so $x' \in A' \cap F$. □

7.4. **One short pair with an odd shared prime.**

Lemma 7.6. *Let $k > 20$. A connected collision triangle of θ that is not good has, up to order, the numerators $(w, w, k - 2w)$ with w odd and $\gcd(w, k) = 1$, or $(1, 2, k - 3)$, or $(1, \frac{k}{2} - 1, \frac{k}{2})$, or $(2, \frac{k}{2} - 2, \frac{k}{2})$ with $\frac{k}{2}$ odd.*

Proof. The expression with denominator k is reduced, and the sum of the numerators is $k > 20$, so the triangle is not one of the nine listed in Definition 3.3. If two numerators are equal to w, then $\gcd(w, k) = 1$, and w is odd. Otherwise it is $(1, 2, c)$ with $c = k - 3$, or $(1, c, c + 1)$ with $2c + 2 = k$, or $(2, c, c + 2)$ with c odd and $2c + 4 = k$. □

We call the last three triangles of Lemma 7.6 the *exceptional scalene* triangles. In the last two, the numerators other than $\frac{k}{2}$ have no odd prime factor in common with k, as $\gcd(\frac{k}{2} - d, k)$ divides $2d$.

Proposition 7.7. *Let $k > 500$, and suppose that exactly one short pair has an odd shared prime. Then the rules reach θ.*

Proof. Let ℓ be an odd shared prime of that pair. Of the two other short pairs at most one has the shared prime 2, so some collision triangle Δ is connected. If it is good, Corollary 6.5 applies. Otherwise it is as in Lemma 7.6. The short pair of ℓ has an index in common with that of Δ, and the entry at this index is a numerator of Δ divisible by ℓ. This is impossible for the numerators $(w, w, k - 2w)$, where $\gcd(k, w) = 1$ and $\gcd(k, k - 2w) = 2$. For $(1, 2, k - 3)$ it forces $\ell = 3$ and the entry $k - 3$, but then the other entry of the short pair of ℓ is at least 3 and the pair is not short. So the numerators of Δ are $(d, \frac{k}{2} - d, \frac{k}{2})$ with $d \in \{1, 2\}$, and the common index, l say, has $q_l = \frac{k}{2}$. Let m be the other index of the short pair of Δ, with $q_m \in \{d, \frac{k}{2} - d\}$, and i the other index of the short pair of ℓ; so $\ell \mid q_i$ and $q_i < \frac{k}{2}$. The fourth entry is $\frac{3k}{2} - q_m - q_i$. The prime ℓ divides q_l and q_i and not the two other entries.

Let A' be a closed set that contains $\mathrm{Deg}(\theta)$ and meets the symmetric bounds, $M = k/\ell$, and K the kernel of $U_k \to U_M$; by Lemma 6.6, $K \subset H$, and $e_\ell \in A'$ if $\ell^2 \nmid k$. As in the

proof of Proposition 7.5 it suffices to show that the image of H in U_M contains b_0/x_0 for all $x_0, b_0 \in U_M$ with $x_0 + b_0 \equiv 2$, and for this to find lifts x, b of x_0, b_0 in $F \cup F_1$ with $h(x), h(b) \geq 1$ and $x' = 1 - x - b \in A' \cap F$; a lift divisible by ℓ, which occurs only if $\ell^2 \nmid k$, is ue_ℓ for a unit u lifting the same class.

A lift z of $z_0 \in U_M$ is odd, so $\langle zq_l/k\rangle = \frac{1}{2}$. As the digit at ℓ varies, $\langle zq_m/k\rangle$ takes the ℓ values in $(0, 1]$ of one class modulo $1/\ell$, and at least $(\ell - 1)/2$ of them are less than $\frac{1}{2}$. For such a digit the sum in (7.1) is less than 3 and the two other terms are not both 1, so $z \in F \cup F_1$ and $h(z) \geq 1$. If $\ell^2 \mid k$, then x' is a unit and $-x' \in K$. If $\ell \geq 5$, there are two such digits for b, and one of them gives $x' \not\equiv 0 \pmod{\ell}$, so again $-x' \in K$. If $\ell = 3$, $9 \nmid k$ and $x' \equiv 0 \pmod 3$, then $x' = -ve_3$ with $v \in K$, which lies in $A' \cap F$ provided that $e_3 \in F$.

It remains to treat $\ell = 3$, $9 \nmid k$, $e_3 \notin F$. Then $k/3$ divides some entry. It does not divide $\frac{k}{2}$; it does not divide q_i, as $q_i < \frac{k}{2}$ is divisible by 3 and $k/3$ is not; and it does not divide d or $\frac{k}{2} - d$, as $k > 12$. So it divides the fourth entry $\frac{3k}{2} - q_m - q_i$, which is not k. If $q_m = d$, this entry lies between $k - d$ and $\frac{3k}{2}$, so it is $\frac{4k}{3}$, and $q_i = \frac{k}{6} - d$; the pair $\{m, i\}$ is short, with collision triangle $(\frac{5k}{6}, d, \frac{k}{6} - d)/k$. If $q_m = \frac{k}{2} - d$, the entry lies between $\frac{k}{2} + d$ and $k + d$, so it is $\frac{2k}{3}$, and $q_i = \frac{k}{3} + d$; the pair $\{m, i\}$ is short, with collision triangle $(\frac{k}{6}, \frac{k}{2} - d, \frac{k}{3} + d)/k$. In both cases a common divisor of the numerators divides $\frac{k}{6}$ and d, and $\frac{k}{6}$ is odd if $d = 2$; so the triangle is connected. Its numerators are distinct, and it is good: in the first case the numerators other than d differ by more than 2 and are larger than 2, and in the second all numerators are larger than 2. So Corollary 6.5 applies. □

7.5. **No short pair with an odd shared prime.**

Lemma 7.8. *Let $k > 500$. Suppose that no short pair has an odd shared prime and that no collision triangle is connected and good. Then, up to order, one of the following holds:*

(a) $(q_i) = (r, r, s, 3s)$ *with* $k = r + 2s$, r *even,* s *odd and* $\gcd(r, s) = 1$;
(b) $(q_i) = (e, e, e, 2k - 3e)$ *with* e *odd,* $\gcd(e, k) = 1$ *and* $e < \frac{k}{2}$;
(c) (q_i) *is one of the tuples of Table 1 not marked "good".*

Proof. Only one short pair can have the shared prime 2, so at least two collision triangles are connected, and by Lemma 7.6 each of them is isosceles, with numerators $(w, w, k - 2w)$, or exceptional scalene.

Suppose that two of them are isosceles, and let c be the entry at the common index of their short pairs and z, z' the other entries of the two pairs; the triangles have the numerators $(k - c - z, c, z)$ and $(k - c - z', c, z')$. If c is even, it is the numerator $k - 2w$ in both, so $z = z' = (k - c)/2$ and the fourth entry is k, which is excluded. So c is odd, it is one of the equal numerators in both, and $z, z' \in \{c, k - 2c\}$. If $z \neq z'$, the fourth entry is again k. If $z = z' = c$, then $(q_i) = (c, c, c, 2k - 3c)$ with $\gcd(c, k) = 1$, and $2c < k$ as the pair is short; this is (b). If $z = z' = k - 2c$, then $(q_i) = (k - 2c, k - 2c, c, 3c)$, which is (a) with $r = k - 2c$ and $s = c$.

Otherwise some connected collision triangle Δ_1 is exceptional scalene, and a second one, Δ_2, is connected and isosceles or exceptional scalene. Let c be the entry at the common index of their short pairs, d the other entry of the pair of Δ_1, and z that of Δ_2. Then c and d are two of the numerators of one of $(1, 2, k - 3)$, $(1, \frac{k}{2} - 1, \frac{k}{2})$, $(2, \frac{k}{2} - 2, \frac{k}{2})$; and either $z \in \{c, k - 2c, (k - c)/2\}$, if Δ_2 is isosceles, or c and z are two of the numerators of one of these three triangles; the six numerators 1, 2, $k - 3$, $\frac{k}{2} - 2$, $\frac{k}{2} - 1$, $\frac{k}{2}$ are distinct integers. The fourth entry is $2k - c - d - z$. This gives finitely many tuples whose entries have the form $ak + b$ with $4a, 4b \in \mathbb{Z}$ and $|b| < 40$. We discard a tuple if, for every even $k > 500$, one

(q_i)	interval	(q_i)	interval
$(1,\ 1,\ 2,\ 2k-4)$	$(\frac{1}{2},\ \frac{3}{4})$	$(2,\ 2,\ k-3,\ k-1)$	$(\frac{2}{3},\ 1)$
$(1,\ 1,\ \frac{k-2}{2},\ \frac{3k-2}{2})$	kite	$(2,\ 4,\ \frac{k-4}{2},\ \frac{3k-8}{2})$	$(\frac{3}{4},\ \frac{7}{8})$
$(1,\ 1,\ \frac{k}{2},\ \frac{3k-4}{2})$	$(\frac{1}{4},\ \frac{3}{4})$	$(2,\ 4,\ k-3,\ k-3)$	$(\frac{3}{4},\ 1)$
$(1,\ 1,\ k-3,\ k+1)$	$(\frac{1}{3},\ \frac{2}{3})$	$(2,\ 5,\ k-4,\ k-3)$	$(\frac{4}{5},\ 1)$
$(1,\ 2,\ 2,\ 2k-5)$	$(\frac{3}{5},\ \frac{4}{5})$	$(2,\ \frac{k}{4},\ \frac{k}{2},\ \frac{5k-8}{4})$	$(\frac{5}{8},\ \frac{7}{8})$
$(1,\ 2,\ \frac{k-4}{2},\ \frac{3k-2}{2})$	$(\frac{1}{2},\ \frac{3}{4})$	$(2,\ \frac{k+4}{4},\ \frac{k-4}{2},\ \frac{5k-4}{4})$	good
$(1,\ 2,\ \frac{k-2}{2},\ \frac{3k-4}{2})$	$(\frac{1}{2},\ \frac{3}{4})$	$(2,\ \frac{k-4}{2},\ \frac{k-4}{2},\ k+2)$	kite
$(1,\ 2,\ \frac{k}{2},\ \frac{3k-6}{2})$	$(\frac{1}{2},\ \frac{5}{6})$	$(2,\ \frac{k-4}{2},\ \frac{k-2}{2},\ k+1)$	$(\frac{1}{2},\ \frac{3}{4})$
$(1,\ 2,\ k-4,\ k+1)$	$(\frac{1}{2},\ \frac{3}{4})$	$(2,\ \frac{k-4}{2},\ \frac{k+6}{2},\ k-3)$	$(\frac{5}{6},\ 1)$
$(1,\ 4,\ k-3,\ k-2)$	$(\frac{3}{4},\ 1)$	$(2,\ \frac{k-4}{2},\ \frac{k+8}{2},\ k-4)$	$(\frac{7}{8},\ 1)$
$(1,\ 5,\ k-3,\ k-3)$	$(\frac{4}{5},\ 1)$	$(2,\ \frac{k-2}{2},\ \frac{k}{2},\ k-1)$	$(\frac{1}{2},\ 1)$
$(1,\ \frac{k}{4},\ \frac{k}{2},\ \frac{5k-4}{4})$	$(\frac{1}{4},\ \frac{3}{4})$	$(2,\ \frac{k-2}{2},\ \frac{k+4}{2},\ k-3)$	$(\frac{3}{4},\ 1)$
$(1,\ \frac{k+2}{4},\ \frac{k-2}{2},\ \frac{5k-2}{4})$	good	$(2,\ \frac{k}{2},\ \frac{k+2}{2},\ k-3)$	$(\frac{2}{3},\ 1)$
$(1,\ \frac{k-4}{2},\ \frac{k-2}{2},\ k+2)$	$(\frac{1}{2},\ \frac{3}{4})$	$(2,\ \frac{k}{2},\ \frac{k+4}{2},\ k-4)$	$(\frac{3}{4},\ 1)$
$(1,\ \frac{k-4}{2},\ \frac{k}{2},\ k+1)$	$(\frac{1}{4},\ \frac{3}{4})$	$(4,\ \frac{k-4}{2},\ \frac{k}{2},\ k-2)$	$(\frac{3}{4},\ 1)$
$(1,\ \frac{k-4}{2},\ \frac{k+8}{2},\ k-3)$	$(\frac{3}{8},\ \frac{5}{8})$	$(\frac{k}{4},\ \frac{k-4}{2},\ \frac{k}{2},\ \frac{3k+8}{4})$	$(\frac{3}{8},\ \frac{5}{8})$
$(1,\ \frac{k-2}{2},\ \frac{k-2}{2},\ k+1)$	kite	$(\frac{k}{4},\ \frac{k-2}{2},\ \frac{k}{2},\ \frac{3k+4}{4})$	$(\frac{3}{4},\ 1)$
$(1,\ \frac{k-2}{2},\ \frac{k+4}{2},\ k-2)$	$(\frac{3}{4},\ 1)$	$(\frac{k+2}{4},\ \frac{k-2}{2},\ \frac{k}{2},\ \frac{3k+2}{4})$	$(\frac{1}{2},\ 1)$
$(1,\ \frac{k-2}{2},\ \frac{k+6}{2},\ k-3)$	$(\frac{5}{6},\ 1)$	$(\frac{k+4}{4},\ \frac{k-4}{2},\ \frac{k}{2},\ \frac{3k+4}{4})$	$(\frac{1}{4},\ \frac{3}{4})$
$(1,\ \frac{k}{2},\ \frac{k+2}{2},\ k-2)$	$(\frac{1}{2},\ 1)$	$(\frac{k-4}{2},\ \frac{k-4}{2},\ \frac{k}{2},\ \frac{k+8}{2})$	$(\frac{3}{8},\ \frac{5}{8})$
$(1,\ \frac{k}{2},\ \frac{k+4}{2},\ k-3)$	$(\frac{1}{3},\ \frac{2}{3})$	$(\frac{k-4}{2},\ \frac{k-2}{2},\ \frac{k}{2},\ \frac{k+6}{2})$	$(\frac{1}{2},\ \frac{3}{4})$
$(2,\ 2,\ \frac{k-4}{2},\ \frac{3k-4}{2})$	kite	$(\frac{k-2}{2},\ \frac{k-2}{2},\ \frac{k}{2},\ \frac{k+4}{2})$	$(\frac{3}{4},\ 1)$
$(2,\ 2,\ \frac{k}{2},\ \frac{3k-8}{2})$	$(\frac{5}{8},\ \frac{7}{8})$	$(1,\ 1,\ 1,\ 2k-3)$	$(\frac{1}{3},\ \frac{2}{3})$

TABLE 1. A list containing the tuples of Lemma 7.8(c) and the last triple family used in Theorem 7.14, as functions of k, each with its interval (ς, ς') of Lemma 7.10. The four rows marked "kite" are treated in Proposition 7.13, and the two marked "good" have a good connected collision triangle.

of its entries or $k-c-z$ is not positive, an entry is equal to k, or two entries have sum k, since θ is open; and we omit the tuples in which some entry is not an integer for any even k. Every tuple that remains is listed in Table 1.

The two rows marked "good" are $(d, s, \frac{k}{2}-d, 5s-3d)$ with $d \in \{1,2\}$ and $s=(k+2d)/4$. There the pair of the entries d and s is short, with collision triangle $(3s-3d, d, s)/k$. For $d=2$, s is odd, as otherwise all q_i and k would be even; so the triangle is connected. Its numerators are distinct, s and $3s-3d$ are larger than 2 and differ by more than 2; so it is good, and these rows do not occur. □

7.6. **Two families with a full symmetric bound.**

Lemma 7.9. *Let $k>20$, and let A' be a closed set that meets the symmetric bounds. Then $A'=F\cup F_1$ in each of the following cases:*

(a) $(q_i)=(r,r,s,3s)$ *with* $k=r+2s$, *r even, s odd and* $\gcd(r,s)=1$;

(b) $(q_i) = (e, e, e, 2k - 3e)$ *with* e *odd,* $\gcd(e, k) = 1$, $e < \frac{k}{2}$, *and* $3 \nmid k$;

(c) $(q_i) = (e, e, e, 2k - 3e)$ *with* e *odd,* $\gcd(e, k) = 1$, $3 \le e < \frac{k}{2}$, *and* $3 \mid k$.

Proof. In each case we give a triangle Δ of Proposition 4.9 whose reduced expression is good, with $|F(\Delta)| = D$; then $D(A') \ge D$ and $A' = F \cup F_1$. Recall that $N = 2k$ and $a_i = q_i$; the denominators of the triangles are larger than 20. An odd x is not divisible by an even number; we use this repeatedly.

(a) Here s is prime to $2k$, and $\gcd(k, r) = 2$. For odd x modulo k, $sx \not\equiv 0$, $3sx \not\equiv 0$ as $k/\gcd(3, k)$ is even, and $rx \equiv 0$ if and only if $\frac{k}{2}$ divides x. So $F \cup F_1 = F$ is the set of odd x not divisible by $\frac{k}{2}$, and $D = 2|F|$. Let $\Delta = (2r, s, 3s)/(2k)$, the triangle of Proposition 4.9(a) for the two entries r. It is reduced, with distinct numerators, and good: $2r \ge 4$ is its only even numerator, and if $s = 1$ the two others are $2k - 4$ and 3. For odd x modulo $2k$, in the same way, $sx, 3sx \not\equiv 0$, and $2rx \equiv 0$ if and only if $\frac{k}{2}$ divides x. So $F(\Delta)$ is the set of odd x modulo $2k$ whose residue modulo k lies in F, and $|F(\Delta)| = 2|F|$.

(b) For odd x modulo k, $ex \not\equiv 0$ and $(2k - 3e)x \equiv -3ex \not\equiv 0$; so $F \cup F_1 = F$ is the set of all odd x, and $D = k$. Let $\Delta = (3e, 2k - 3e, k)/(3k)$, the triangle of Proposition 4.9(b). It is reduced, as $\gcd(3e, k) = 1$, with distinct numerators, as $3 \nmid k$. It is good: its numerators are larger than 2, since $3e \ge 3$ and $2k - 3e > k/2 > 10$. For odd x modulo $3k$: $kx \equiv 0$ if and only if $3 \mid x$; $3ex \not\equiv 0$; and $(2k - 3e)x \not\equiv 0$, as $2k - 3e$ is prime to $3k$. So $F(\Delta)$ is the set of odd x modulo $3k$ prime to 3, and $|F(\Delta)| = k$.

(c) As $k/3$ is even, again $F \cup F_1 = F$ is the set of all odd x and $D = k$. Let $\Delta = (2e, e, 2k - 3e)/(2k)$, the triangle of Proposition 4.9(a) for two of the equal entries, the two other entries being e and $2k - 3e$. It is reduced, with distinct numerators. It is good, since $2e \ge 6$, $e \ge 3$, and $2k - 3e > k/2 > 10$. For odd x modulo $2k$, $2ex \not\equiv 0$, $ex \not\equiv 0$, and $(2k - 3e)x \not\equiv 0$ as $2k/3$ is even. So $|F(\Delta)| = k$. □

7.7. **Intervals of characters.** In a row of Table 1 with an interval (ς, ς'), the entries are $q_i = \alpha_i k + b_i$ with $4\alpha_i, 4b_i \in \mathbb{Z}$, $\sum_i \alpha_i = 2$ and $\sum_i b_i = 0$; an *instance* of the row is an open type whose (q_i) is given by the row. For an instance let J be the set of odd integers x with $\varsigma k + 2 < x < \varsigma' k$.

Lemma 7.10. *Let θ be an instance of a row of Table 1 with an interval. For every $x \in J$, both x and $2 - x$ lie in F, and $h(x) \ge 1$, $h(2 - x) \ge 1$. For every closed set A' and every unit u, either $J \subset uA'$ or $J \cap uA' = \emptyset$. Each interval has length at least $\frac{1}{8}$.*

Proof. Let x be an odd integer, $\varrho \in \{1, 3\}$ its residue modulo 4, and $t = x/k$, $t' = (x - 2)/k$. As $4\alpha_i \in \mathbb{Z}$,

$$\frac{xq_i}{k} \equiv \varrho\alpha_i + b_i t, \qquad \frac{(2 - x)q_i}{k} \equiv (2 - \varrho)\alpha_i - b_i t' \pmod 1.$$

For each row, each $\varrho \in \{1, 3\}$ and every real t with $\varsigma < t < \varsigma'$, none of the eight numbers $\varrho\alpha_i + b_i t$, $(2 - \varrho)\alpha_i - b_i t$ is an integer, and $\sum_i \{\varrho\alpha_i + b_i t\} \le 2$, $\sum_i \{(2 - \varrho)\alpha_i - b_i t\} \le 2$. This is checked row by row: the two sums are integers, and they are constant between consecutive points t at which one of the eight numbers is an integer; so it suffices to find these points, none of which lies in the interval, and to evaluate the sums at one point of the interval. For $x \in J$ both t and t' lie in the interval, which gives the first statement, by (7.1).

Let A' be closed, u a unit and $x \in J$. If $x \in uA'$, the last condition of Definition 4.10, with $x' = -1$ and $b = 2 - x$, gives $2 - x \in uA'$, so $x - 2 \in uA'$. If $x - 2 \in uA'$, then $2 - x \in uA'$, and the same condition, applied to $2 - x$ with $x' = -1$ and $b = x$, gives $x \in uA'$. So $x \in uA'$

if and only if $x-2 \in uA'$, for every $x \in J$; as J consists of consecutive odd integers, the second statement follows. □

For example, in the last row, $(1,1,1,2k-3)$, the fractional parts in the two sums are $t,t,t,\{-3t\}$ and $1-t,1-t,1-t,\{3t\}$, and for $\frac{1}{3} < t < \frac{2}{3}$ both sums are 2.

Lemma 7.11. *Let $n \geq 1$ and $n' \geq 1$ be coprime integers and $\omega(n)$ the number of primes dividing n. In every residue class modulo n', an open interval of length λ contains at least $\lambda\varphi(n)/(nn') - 2^{\omega(n)}$ integers prime to n. Moreover $\varphi(n) > 10 \cdot 2^{\omega(n)}$ for every $n > 660$, and for every odd $n > 165$.*

Proof. By inclusion and exclusion the number is $\sum_e \pm N_e$, over the $2^{\omega(n)}$ squarefree divisors e of n, where N_e is the number of integers in the interval that lie in the class and are divisible by e; these form a residue class modulo en', so $|N_e - \lambda/(en')| \leq 1$. For the second statement let $f(n) = \varphi(n)/2^{\omega(n)}$, which is multiplicative, with $f(2^\nu) = 2^\nu/4$, $f(3^\nu) = 3^\nu/3$, and $f(\ell^\nu) \geq \max(2, 2\ell^\nu/5)$ for $\ell \geq 5$. If $f(n) \leq 10$, then n has at most four prime factors $\ell \geq 5$, and $n \leq 4 \cdot 3 \cdot (5/2)^4 f(n) < 5000$. Going through the $n < 5000$, the largest with $f(n) \leq 10$ is 660, and the largest odd one is 165. □

Proposition 7.12. *Let $k > 500$, and let θ be an instance of a row of Table 1 with an interval, with no connected and good collision triangle. Then every closed set contains U.*

Proof. Let A' be closed. The units of $\mathbb{Z}/k$ are odd. By Lemma 7.10, J is the set of odd integers in an open interval of length at least $k/8-2$; and if J contains a unit x_0, then $J \subset x_0 A'$, so that for $x, x'' \in J$ and a unit v with $vx \equiv x'' \pmod k$ we get $x'' \in x_0 A' \cap v x_0 A'$ and $v \in H$. In particular $x/x_0 \in H$ for every unit $x \in J$.

For $k \leq 5280$ a computation over all such instances shows that J contains a unit x_0 and that -1 and the ratios x/x_0 of the units $x \in J$ generate U. Let $k > 5280$. Then $k/8-2 > k/10$, and J contains a unit x_0, by Lemma 7.11 with $n = k$ and $n' = 1$. Write U as the product of the groups U_{ℓ^ν} over the prime powers exactly dividing k; we show that H contains each factor.

Odd primes. Let ℓ^ν exactly divide k, with ℓ odd, and $M = k/\ell^\nu$, which is even. If $M \geq 18$, the interval has length more than $k/9 \geq 2\ell^\nu$, so J contains an odd multiple x of ℓ^ν. Every element of the factor U_{ℓ^ν} fixes x, so the factor lies in H. Let $M \leq 16$; then $\ell^\nu > 330$. The integers $1+Mt$ in the interval are odd, and t runs through at least $\ell^\nu/8-2$ consecutive integers. Such an integer is a unit modulo k unless ℓ divides it, and the ratio of two of them that are units lies in H and in the factor U_{ℓ^ν}.

If $\nu \geq 2$, then $\ell \leq \sqrt{\ell^\nu}$ and $\ell^\nu/8 - 2 > \ell + 3$. So there are t and $t+\ell$ in the range with $1+Mt$ prime to ℓ; the ratio of $1+M(t+\ell)$ and $1+Mt$ is $\equiv 1 \pmod \ell$ and $\not\equiv 1 \pmod{\ell^2}$, so it generates the kernel of $U_{\ell^\nu} \to U_\ell$. The integers $1+Mt$ prime to ℓ in the range represent every non-zero class modulo ℓ. So H contains the factor.

If $\nu = 1$, suppose that the ratios generate a proper subgroup of U_ℓ. Then some character χ of U_ℓ of prime order is trivial on it, so $\chi(1+Mt)$ is the same for all t in the range with $1+Mt \not\equiv 0$. As $1+Mt \equiv M(t+M^{-1})$, χ is constant on the non-zero members of a set I of R consecutive residues modulo ℓ, with $\ell/8-2 \leq R < \ell$; so $|\sum_{y\in I} \chi(y)| \geq R-1$, with $\chi(0) = 0$. For $\ell \leq 16384$ a computation shows that for $\ell > 73$ no character of prime order of U_ℓ is constant on the non-zero members of such an I. For $\ell > 16384$ we use the Pólya–Vinogradov inequality, in the explicit form its standard proof gives:

$$\Big|\sum_{y\in I} \chi(y)\Big| \leq \sqrt{\ell}\,(1+\log \ell).$$

Let $\tau(a) = \sum_y \chi(y) e^{2\pi i a y/\ell}$; then $\tau(0) = 0$, $|\tau(a)| = \sqrt{\ell}$ for $a \not\equiv 0$, and $\chi(y) = \ell^{-1} \sum_a \tau(a) e^{-2\pi i a y/\ell}$. The sum of $e^{-2\pi i a y/\ell}$ over $y \in I$ has absolute value at most $1/|\sin(\pi a/\ell)| \le \ell/(2\min(a, \ell - a))$ for $0 < a < \ell$. So the sum of χ over I is at most $\sqrt{\ell} \sum_{0<a<\ell/2} 1/a \le \sqrt{\ell}(1 + \log \ell)$ in absolute value. As $\ell/8 - 3 > \sqrt{\ell}(1 + \log \ell)$ at $\ell = 16384$, where the two sides are 2045 and less than 1371, and the difference increases with ℓ, this is a contradiction.

The prime 2. Let $k = 2^\nu n$ with n odd. If $2^\nu \ge 64$, the interval has length at least $k/8 - 2 > 4n + 2$, as $k \ge 64n$ and $k > 5280$; so there is $x \in J$ with $x + 2n, x + 4n \in J$. Let v, v' be the units with $v \equiv v' \equiv 1 \pmod n$, $v \equiv (x + 2n)/x$ and $v' \equiv (x + 4n)/x \pmod{2^\nu}$. Then $vx \equiv x + 2n$ and $v'x \equiv x + 4n \pmod k$, so $v, v' \in H$ by the first paragraph. In U_{2^ν}, $v' \equiv 5 \pmod 8$ generates the subgroup of the elements $\equiv 1 \pmod 4$, and $v \equiv 3 \pmod 4$; so H contains the factor U_{2^ν}. If $2^\nu \le 32$, then $n > 165$, and for every odd class modulo 2^ν the interval contains an integer of this class prime to n, by Lemma 7.11 with $n' = 2^\nu$, as $\varphi(n)/10 > 2^{\omega(n)}$. These integers are units in J, so the image of H in U_{2^ν} is everything; as H contains the factors at the odd primes, $H = U$. □

7.8. **Kites.** The four rows of Table 1 marked "kite" are the tuples $(R, R, S, S + k)$ with $R + S = \frac{k}{2}$ and $\min(R, S) \in \{1, 2\}$. For an open type of this form, $\gcd(R, S) = 1$.

Proposition 7.13. *Let $k > 50$ and $(q_i) = (R, R, S, S + k)$ with $R + S = \frac{k}{2}$, $\gcd(R, S) = 1$ and $\min(R, S) \in \{1, 2\}$. Then every closed set that meets the symmetric bounds is $F \cup F_1$.*

Proof. Put $L = \frac{k}{2}$; the denominators of the triangles below are larger than 20. Then R and S are prime to L, and they are both odd if L is even. For odd x modulo k, $Rx \equiv 0$ if and only if R is even and $x = L$, and $Sx \equiv (S + k)x \equiv 0$ if and only if S is even and $x = L$. So $F \cup F_1 = F$; it is the set of all odd x, and $D = 2L$, if L is even, and the set of the odd $x \ne L$, with $D = 2L - 2$, if L is odd.

Let $\Delta = (2R, S, S + k)/(2k)$, the triangle of Proposition 4.9(a) for the two entries R, and let A' be a closed set that meets the symmetric bounds.

S odd. The expression is reduced, with distinct numerators. It is good: the numerator 2 occurs only for $R = 1$, and then the two others, $L - 1$ and $3L - 1$, are larger than 1 and differ by k; the numerator 1 occurs only for $S = 1$, and then the two others are $2L - 2$ and $2L + 1$. For odd x modulo $2k$, $Sx, (S + k)x \not\equiv 0$, as S and $S + k$ are prime to $2k$, and $2Rx \equiv 0$ if and only if R is even and L divides x. So $|F(\Delta)|$ is $2L$ if L is even and $2L - 2$ if L is odd; this is D, and $A' = F \cup F_1$.

S even. Then L and R are odd, and the reduced expression of Δ is $(R, S/2, S/2 + L)/k$. Its numerators are distinct, as $R = S/2$ would give $L = 3$. It is good: the numerators 1 and 2 can only be R and $S/2$, and then $L \le 5$; if $R = 1$ the two others differ by L; if $S/2 = 1$ the two others are $L - 2$ and $L + 1$; and if $S/2 = 2$ they are $L - 4$ and $L + 2$. For odd x modulo k, $Rx \not\equiv 0$; exactly one of $S/2$, $S/2 + L$ is even, with greatest common divisor 2 with k, and the other is prime to k. So $F(\Delta)$ is the set of odd $x \ne L$, and $|F(\Delta)| = L - 1 = \frac{1}{2}D$. As $\dim H^1_x = 2$ for every $x \in F$, we get $|A'| \ge \frac{1}{2}|F|$.

Suppose that $A' \ne F \cup F_1$. Then $H \ne U$ by Lemma 6.4, and by Lemma 6.1(a) there are exactly two translates of A'; they partition F, $|A'| = (L - 1)/2$, and H has index two. Reduction modulo L identifies the odd residues modulo k with $\mathbb{Z}/L$, F with the non-zero classes, and U with U_L; we use this identification. For a prime ℓ dividing L, the classes of additive order ℓ form an orbit of U in F, which meets A' since it meets one of the two translates; by Lemma 6.1(c), H contains the kernel of $U_L \to U_\ell$. So the character of U_L with kernel H factors through U_ℓ for every prime ℓ dividing L. Hence $L = \ell^\nu$ is a prime

power, H is the set of the units that are squares modulo ℓ, and $\ell \equiv 1 \pmod 4$ as $-1 \in H$. Let $n_0 = (L-1)/4$; then $n_0 \geq 3$.

For odd x, the four numbers $\{xq_i/k\}$ are $y, y, \{\frac{1}{2} - y\}, \{\frac{1}{2} - y\}$ with $y = \{x \min(R,S)/k\}$, since $xS \equiv xL - xR$ modulo k and $\{xL/k\} = \frac{1}{2}$. So for $x \in F$, $h(x) = 2$ if $y < \frac{1}{2}$ and $h(x) = 0$ if $y > \frac{1}{2}$. If $\min(R,S) = 1$, the $x \in F$ with $h(x) = 2$ are the odd integers $\xi_j = 1+2j$, $0 \leq j \leq 2n_0 - 1$. If $\min(R,S) = 2$, they are the odd x with $x \equiv \xi_j \pmod L$, where $\xi_j = 1+j$, $0 \leq j \leq 2n_0 - 1$. In both cases exactly one of x, $-x$ has $h = 2$, for every $x \in F$; so the set E of the j with $\xi_j \in A'$ has n_0 elements, and $0 \in E$. For $j \leq j'$ in E, the last condition of Definition 4.10 with $u = 1$, $x = \xi_j$ and $x' = -\xi_{j'}$ gives $b = 1 - \xi_j + \xi_{j'} = \xi_{j'-j} \in A'$. So E is closed under non-negative differences. Let δ be its least positive element. Subtracting δ repeatedly shows that E consists of multiples of δ, and that it contains every multiple of δ up to its largest element; so $E = \{0, \delta, \dots, (n_0 - 1)\delta\}$. As $(n_0 - 1)\delta \leq 2n_0 - 1$ and $n_0 \geq 3$, $\delta \in \{1, 2\}$. Hence, modulo L,

$$A' = \{\pm(1 + tj) : \ 0 \leq j < n_0\}, \qquad t \in \{1, 2, 4\},$$

where $t = 2\delta$ if $\min(R,S) = 1$ and $t = \delta$ if $\min(R,S) = 2$.

The units among these classes are squares modulo ℓ. If $n_0 \geq \ell$, the numbers $1 + tj$, $0 \leq j < \ell$, would represent all classes modulo ℓ, and those prime to ℓ would be units in A', not all of them squares. So $n_0 < \ell$, that is $\ell^\nu < 4\ell + 1$, and $\nu = 1$. Then $L = \ell$, every non-zero class is a unit, and $A' = H$ is the group of the non-zero squares modulo ℓ, of order $2n_0 > 2$. The sum of the squares of the elements of a subgroup of U_ℓ of order larger than 2 is zero modulo ℓ, as it is unchanged by multiplication by $v^2 \neq 1$ for an element v of the subgroup. On the other hand, as $4n_0 \equiv -1 \pmod \ell$,

$$\sum_{x \in H} x^2 = 2 \sum_{j=0}^{n_0 - 1} (1 + tj)^2 = 2n_0 \Big(1 + t(n_0 - 1) + t^2 \frac{(n_0 - 1)(2n_0 - 1)}{6}\Big)$$
$$\equiv \frac{2n_0}{16}(16 - 20t + 5t^2) \pmod \ell,$$

and $16 - 20t + 5t^2$ is 1, -4, 16 for $t = 1, 2, 4$. This is not zero modulo ℓ; a contradiction. □

7.9. **Conclusion.**

Theorem 7.14. *The rules reach every open type with k even and $N > 1000$.*

Proof. Here $k > 500$. Let A' be a closed set that contains $\mathrm{Deg}(\theta)$ and meets the symmetric bounds. If some collision triangle is connected and good, Corollary 6.5 applies; suppose that there is none. If at least two short pairs have odd shared primes, Proposition 7.5 applies, and if exactly one has, Proposition 7.7. Otherwise θ is as in Lemma 7.8. In case (a), and in case (b) with $3 \nmid k$ or $e \geq 3$, $A' = F \cup F_1$ by Lemma 7.9. The remaining tuple of case (b) is $(1, 1, 1, 2k - 3)$, the last row of Table 1, which we treat with case (c). There, if the row is marked "kite", then $A' = F \cup F_1$ by Proposition 7.13; otherwise $U \subset A'$ by Proposition 7.12, and $A' = F \cup F_1$ by Lemma 6.4. □

7.10. **Relative characters.**

Proposition 7.15. *Let θ be an open type with k even, and suppose that $A = F \cup F_1$. Then $Z_c \subset V$ for every odd relative character c.*

Proof. Let c be an odd relative character. By Theorem 4.7 it suffices to find $x \in F \cup F_1$ with $h(x) \geq 1$ and $1 + c - x \in F$. We look for a unit x with $h(x) \geq 1$ such that $1 + c - x$ is a unit; units lie in F. Note that $1 + c - x$ is odd for odd x.

If $3 \nmid k$, there is a unit x with $x \not\equiv \pm(1+c) \pmod{\ell}$ for every odd prime ℓ dividing k, as $\ell \geq 5$. Then $1+c-x$ and $1+c+x$ are units, and one of x, $-x$ has $h \geq 1$.

Let $3 \mid k$. Choose $t_3 \in \{1,2\}$ with $t_3 \not\equiv 1+c \pmod 3$, and let C be the set of units x with $x \equiv t_3 \pmod 3$ and $x \not\equiv 1+c \pmod{\ell}$ for every prime $\ell \geq 5$ dividing k with $1+c \not\equiv 0 \pmod{\ell}$. For $x \in C$, $1+c-x$ is a unit. If $k \notin \{6,12,30\}$, some $x \in C$ has $h(x) \geq 1$, by Lemma 6.3. For $k \in \{6,12,30\}$, an enumeration of the admissible, reduced types with $k \in \{6,12,30\}$ that are not supplementary shows that for every odd relative character c there is $x \in F \cup F_1$ with $h(x) \geq 1$ and $1+c-x \in F$, except for $\theta = (1,1,2,8)$ and $c=3$; and $(1,1,2,8)$ is not open. □

Together with Corollary 5.3, Theorem 7.14 and Proposition 7.15 give $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ for every open type with k even that the rules reach.

8. Two angles summing to π

The types excluded in Sections 6 and 7 are the supplementary ones, in which $q_i + q_j = k$ for some $i \neq j$. They have two short pairs instead of three, but the rules are simpler on them, because every generic character has $h = 1$. In this section θ is an admissible, reduced, supplementary type with $g \geq 2$; the result is Theorem 8.8, and for $t = \frac{k}{2}$ in the normal form below it is [2, Theorem 1.10].

8.1. **Normal form and characters.**

Lemma 8.1. *N is even, $k = N/2$ and $q_i = a_i$, and*

$$(q_1, q_2, q_3, q_4) = (s,\ t,\ k-t,\ k-s) \qquad \text{with } 0 < s < t \leq \tfrac{k}{2},\ \gcd(s,t,k) = 1.$$

The short pairs are the pairs of the entries $\{s,t\}$ and $\{s, k-t\}$. For $x \not\equiv 0$, $n(x) \in \{2,4\}$; so $F_1 = \emptyset$, and F is the set of x, odd if k is even, with $xs, xt \not\equiv 0 \pmod k$. Every unit lies in F, and $h(x) = 1$ for every $x \in F$.

Proof. If N is odd, then k is odd and the $q_i = 2a_i$ are even, so no two of them sum to k. So N is even, and if two entries sum to k, so do the two others. No entry is k, so the entries lie strictly between 0 and k, and the two pairs are $\{s, k-s\}$ and $\{t, k-t\}$ with $s \leq t \leq \frac{k}{2}$; here $\gcd(s,t,k) = \gcd(a_1, \dots, a_4) = 1$. As θ is reduced, $s < t$, or $s = t = \frac{k}{2}$; in the second case $k = 2$ and $g = 1$. The sums of the four other pairs are $s + t < k$, $s + k - t < k$, and two sums larger than k.

As $x(k-s) \equiv -xs$, the entries with $xq_i \equiv 0$ come in pairs, and not all four vanish, as $\gcd(s,t,k) = 1$. For $x \in F$, $\{-xs/k\} + \{-x(k-s)/k\} = 1$, and likewise for t; so $h(x) = 1$. For a unit x, $xs, xt \not\equiv 0$ since $0 < s, t < k$. □

So on θ the last condition of Definition 4.10 reads: if u is a unit, $x \in uA'$, $x' \in A'$, and $b \equiv 1 - x - x'$ lies in F, then $b \in uA'$.

Lemma 8.2. *Every closed set that contains U is F.*

Proof. Let A' be such a set and $b \in F$. There is a unit x with $x \not\equiv 1 - b \pmod{\ell}$ for every odd prime ℓ dividing k. Then $x' = 1 - b - x$ is a unit, being odd if k is even, as b is odd. So $x, x' \in A'$, and $b \equiv 1 - x - x' \in A'$. □

Lemma 8.3. *Let A' be a closed set, u a unit and $x \in uA'$. If $x + 2 \in F$, then $x + 2 \in uA'$.*

Proof. $-x \in uA'$, and the condition above with $-x$ in place of x and $x' = -1$ gives $b = x + 2 \in uA'$. □

8.2. **The collision triangles.** The collision triangles of the two short pairs are

$$\Delta_1 = (k-s-t, s, t)/k, \qquad \Delta_2 = (t-s, s, k-t)/k,$$

and both are connected, as $\gcd(k,s,t) = \gcd(k,s,k-t) = 1$. A unit x satisfies the conditions of Corollary 3.4 for both, as $0 < s+t < k$ and $0 < t-s < k$. So if Δ_1 or Δ_2 is good, then $U \subset \mathrm{Deg}(\theta)$, and every closed set that contains $\mathrm{Deg}(\theta)$ is F, by Lemma 8.2.

Lemma 8.4. *Let $k > 20$, and suppose that neither Δ_1 nor Δ_2 is good. Then k is even, and (s,t) is one of*

$$(1,2), \qquad (1,\tfrac{k}{2}), \qquad (\tfrac{k}{2}-1,\tfrac{k}{2}), \qquad (2,\tfrac{k}{2}) \ \text{ with } \tfrac{k}{2} \text{ odd}, \qquad (\tfrac{k}{2}-2,\tfrac{k}{2}) \ \text{ with } \tfrac{k}{2} \text{ odd}.$$

Proof. By Lemma 7.6, whose proof applies to θ, each of the two triangles has two equal numerators or is exceptional scalene: $(1,2,k-3)$ with k even, $(1,\frac{k}{2}-1,\frac{k}{2})$, or $(2,\frac{k}{2}-2,\frac{k}{2})$ with $\frac{k}{2}$ odd.

The largest numerator of Δ_2 is $k-t$, as s and $t-s$ are less than $t \le k-t$. So for Δ_2 one of the following holds: $t = 2s$; or k is even, $t = 3$ and $s \in \{1,2\}$; or $t = \frac{k}{2}$ and $s \in \{1, \frac{k}{2}-1\}$; or $t = \frac{k}{2}$ is odd and $s \in \{2, \frac{k}{2}-2\}$. The last two give four of the five pairs of the statement.

Let $t = 2s$. Then $\gcd(s,k) = 1$ and Δ_1 has the numerators $s, 2s, k-3s$. If two of them are equal, then $k = 4s$ or $k = 5s$, so $s = 1$ and $k \le 5$. If Δ_1 is $(1,2,k-3)$ with k even, then $s = 1$, as $k-3s \in \{1,2\}$ would give $k \le 8$; this is the pair $(1,2)$. If the largest numerator of Δ_1 is $\frac{k}{2}$, then $2s = \frac{k}{2}$ or $k-3s = \frac{k}{2}$, so s divides k, $s = 1$ and $k \le 6$.

Let k be even, $t = 3$ and $s \in \{1,2\}$. Then Δ_1 has the numerators $s, 3, k-s-3$, which are distinct; the largest is larger than $\frac{k}{2}$, and 1 and 2 are not both among them. So Δ_1 is good. □

Lemma 8.5. *Let k be even and (s,t) one of the five pairs of Lemma 8.4. Then every closed set is F.*

Proof. For odd x, $x\frac{k}{2} \not\equiv 0$. In each of the five cases the entries among s,t other than $\frac{k}{2}$ are 1, 2, $\frac{k}{2}-1$, or $\frac{k}{2}-2$ with $\frac{k}{2}$ odd; their greatest common divisors with k are 1, 2, $\gcd(2,\frac{k}{2}-1)$ and 1. For an entry e prime to k no $x \not\equiv 0$ has $xe \equiv 0$, and for an entry e with $\gcd(k,e) = 2$ this holds exactly when $\frac{k}{2}$ divides x. So F is the set of all odd residues, or the set of all odd residues other than $\frac{k}{2}$, with $\frac{k}{2}$ odd. Let A' be closed. By Lemma 8.3 with $u = 1$, starting from $1 \in A'$, A' contains the odd integers $1, 3, 5, \dots$ up to $k-1$, or up to $\frac{k}{2}-2$ if $\frac{k}{2} \notin F$; in the second case $A' = -A'$ gives the others. □

8.3. **The three exceptions.**

Proposition 8.6. *If θ is not one of $(1,4,8,11)$, $(3,4,8,9)$, $(4,5,7,8)$, then every closed set that contains $\mathrm{Deg}(\theta)$ is F; in particular the rules reach θ.*

Proof. For $k > 20$ this follows from Lemmas 8.4 and 8.5 and the remark before them. For $k \le 20$ we compute the least closed set containing $\mathrm{Deg}(\theta)$ for each of the finitely many types; it is F except for the three types of the statement. □

The three types have $k = 12$ and $(s,t) = (1,4)$, $(3,4)$, $(4,5)$, and $F = \{1,5,7,11\}$. Their collision triangles are $(1,4,7)/12$ and $(1,3,8)/12$, $(3,4,5)/12$ and $(1,3,8)/12$, $(3,4,5)/12$ and $(1,4,7)/12$, none of which is good, so $\mathrm{Deg}(\theta) = \emptyset$; no two entries are equal, so there are no symmetric bounds; and $\{1,11\}$ is closed. Indeed, for $x, x' \in \{\pm 1\}$ the residue $1-x-x'$ is ± 1 or $3 \notin F$; for $x \in \{5,7\}$ and $x' \in \{\pm 1\}$ it is 5, 7 or $9 \notin F$; and $\{1,11\} \cap \{5,7\} = \emptyset$. So the rules do not reach these types, and we treat them by certificates in Section 9.

8.4. **Relative characters and the result.**

Proposition 8.7. *Let k be even, and suppose that $A = F$. Then $Z_c \subset V$ for every odd relative character c.*

Proof. There is a unit x with $x \not\equiv 1 + c \pmod{\ell}$ for every odd prime ℓ dividing k. Then $1+c-x$ is a unit, being odd. So $x \in A$ with $h(x) = 1$, and $1+c-x \in A \cap F$, by Lemma 8.1; Theorem 4.7 gives $Z_c \subset V$. □

Theorem 8.8. *Let θ be an admissible, reduced, supplementary type with $g \geq 2$. Then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$: the connected component of the stratum containing X if $N \equiv 2 \pmod 4$, and the locus $\widetilde{\mathcal{Q}}$ if $4 \mid N$.*

Proof. For the three types $(1, 4, 8, 11)$, $(3, 4, 8, 9)$, $(4, 5, 7, 8)$ this is Proposition 9.4. Otherwise the rules reach θ, by Proposition 8.6. If k is odd, Corollary 5.3 gives $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. If k is even, $A = F$ by Proposition 4.11, so $Z_c \subset V$ for every odd relative character c by Proposition 8.7, and Corollary 5.3 applies. By Lemma 8.1, $k = N/2$, which is odd exactly when $N \equiv 2 \pmod 4$. □

9. Types with $N \leq 1000$

For $N \leq 1000$ we apply the rules by computation. They reach every open type except 32, all with $N \leq 32$ (Proposition 9.1). For these 32 types, and for the three supplementary types of Proposition 8.6, we show $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ by exhibiting one quadrilateral of the type whose unfolding has an orbit closure of the dimension of $\mathcal{L}(\theta)$; the dimension is bounded from below with Wright's cylinder deformations, by Delecroix and Rüth's procedure [11, §3] (Proposition 9.4).

9.1. **The census.**

Proposition 9.1. *The rules reach every open type with $N \leq 1000$, except the following 32 types, which they do not reach:*

$$\begin{gathered}(1,1,2,6);\qquad (1,1,4,10),\ (1,2,3,10),\ (1,2,4,9),\ (2,4,5,5);\\ (1,1,2,16),\ (1,1,4,14),\ (1,1,6,12),\ (1,2,3,14),\ (1,2,4,13),\ (1,2,5,12),\\ (1,2,6,11),\ (1,3,4,12),\ (2,3,3,12),\ (2,3,4,11),\ (2,3,6,9),\ (3,3,6,8);\\ (1,1,1,21),\ (1,1,2,20),\ (1,1,4,18),\ (1,1,6,16),\ (1,1,8,14),\ (1,2,4,17),\\ (1,2,5,16),\ (1,2,8,13),\ (1,3,4,16),\ (1,4,5,14),\ (1,4,6,13),\ (2,4,5,13);\\ (1,1,2,24),\ (1,2,6,19);\qquad (1,2,4,25).\end{gathered} \tag{9.1}$$

The types of (9.1) have $N = 10$, 16, 20, 24, 28 and 32, in this order, so that k is 5, 8, 10, 12, 14 and 16.

Proof. This is a finite computation with the definitions of Definitions 4.10 and 3.3, Corollary 3.4, and Proposition 4.9. An intersection of closed sets is closed, so there is a least closed set A_0 that contains $\mathrm{Deg}(\theta)$; it is obtained from $\mathrm{Deg}(\theta) \cup \{1\}$ by adding the elements that the conditions of Definition 4.10 require until none is left, as each element added in this way lies in every closed set that contains $\mathrm{Deg}(\theta)$. If $A_0 = F \cup F_1$, the rules reach θ. For every admissible type with $N \leq 1000$ that is not supplementary, the computation first tests a sufficient condition for $A_0 = F \cup F_1$: that $U \subset A_0$, because some collision triangle is connected and good (Corollary 3.4) or because the ratios $(2-x)/x$, over the units x such that $2 - x$ is a unit and $h(x), h(2-x) \geq 1$, generate U together with -1 (Lemma 6.1(a)

and (b), the latter with $b = 2 - x$); and that every orbit $O_d \subset F \cup F_1$ with $d \neq 1$ lies in $\mathrm{Deg}(\theta)$ by Corollary 3.4, or contains some b with $h(b) \geq 1$ such that $1 - x - b$ is a unit for some unit x with $h(x) \geq 1$, so that $b \in A_0$ by the last condition of Definition 4.10 with $u = 1$, and $O_d \subset A_0$ by Lemma 6.1(d). This leaves 1402 types with $N \leq 350$ and 42,579 with $351 \leq N \leq 1000$. For each of the first, the computation determines A_0; six of them are types of (1.1), which are not open. On the 32 types of (9.1), $A_0 \neq F \cup F_1$, and no triangle of Proposition 4.9 for θ has a good reduced expression; so A_0 meets the symmetric bounds, and the rules do not reach θ. On every other open one with $A_0 \neq F \cup F_1$, the computation finds that no closed set that contains A_0 and is not $F \cup F_1$ meets the symmetric bounds. None of the 42,579 types with $351 \leq N \leq 1000$ has a connected good collision triangle. For each of them the computation adds to $\{\pm 1\}$ the elements that the conditions of Definition 4.10 require, until the set contains U; then $A_0 = F \cup F_1$ by Lemma 6.4, as $k > 175$. This fails on 488 types, which are the members of the four kite families described next with $351 \leq N \leq 1000$, and Proposition 7.13 applies to them. □

For $33 \leq N \leq 1000$ the open types with $A_0 \neq F \cup F_1$ are the members of the four families $(1, 1, m, 3m + 2)$, $(1, m, m, 2m + 3)$, $(2, 2, m, 3m + 4)$ and $(2, m, m, 2m + 6)$, with m odd in the last two, and the types $(1, 4, 4, 31)$, $(3, 4, 4, 29)$ and $(4, 4, 13, 19)$; the four families are the rows of Table 1 marked "kite". On the seven types of (1.1) the same computation gives $A_0 = \{\pm 1\}$. For six of them this is not $F \cup F_1$. For $(1, 1, 2, 8)$, where $k = 6$, it is $F \cup F_1$: the rules reach this type, so that $A = F \cup F_1$, and what fails is the relative part (Remark 5.5).

9.2. **Cylinders and tangent vectors.** Let (X', ω') be a translation surface with non-empty zero set Σ', and $\mathcal{M}'$ its orbit closure in its stratum. The orbit of (X', ω') is dense in $\mathcal{M}'$, so (X', ω') does not lie in the self-crossing locus of $\mathcal{M}'$, a closed invariant set that is a finite union of affine invariant submanifolds of smaller dimension (Section 2.3). So there is one linear subspace $V' \subset H^1(X', \Sigma'; \mathbb{C})$ with which $\mathcal{M}'$ agrees near (X', ω') in period coordinates, and $\dim \mathcal{M}' = \dim_{\mathbb{C}} V'$. It contains the real classes $\mathrm{Re}[\omega']$ and $\mathrm{Im}[\omega']$, which are tangent to the orbit.

A *cylinder of direction* $\mathbf{v}$, for a non-zero vector $\mathbf{v} \in \mathbb{R}^2$, is a maximal open annulus in X' swept out by closed geodesics of $|\omega'|$ whose holonomy is a positive multiple of $\mathbf{v}$; its boundary consists of saddle connections between points of Σ'. For such a cylinder $\mathcal{C}$ let $|\mathcal{C}|$ be its circumference and $\mathrm{mod}(\mathcal{C})$ its modulus, the height divided by the circumference; and let $I_{\mathcal{C}} \in H^1(X', \Sigma'; \mathbb{Z})$ be the class that vanishes on the relative cycles that can be realized disjointly from $\mathcal{C}$ and takes the value 1 on a path in the closure of $\mathcal{C}$ that goes from a point of Σ' on the boundary component to the right of the core curves, oriented by $\mathbf{v}$, to a point of Σ' on the other boundary component. Equivalently, $I_{\mathcal{C}}$ takes on a relative cycle its algebraic intersection number with a core curve of $\mathcal{C}$ oriented by $\mathbf{v}$, a crossing counting $+1$ when the core curve and the cycle, in this order, form a positive basis there. After the rotation that makes $\mathbf{v}$ horizontal and positive, these are the bottom and the top of $\mathcal{C}$, and $I_{\mathcal{C}}$ is the class that Wright denotes by I_α, for the core curve α of $\mathcal{C}$ [30, Lemma 2.4].

Wright's cylinder deformation theorem says that, for the collection $\mathcal{K}$ of all horizontal cylinders of (X', ω') and all $s, t \in \mathbb{R}$, the surface $a_s^{\mathcal{K}}(u_t^{\mathcal{K}}(X', \omega'))$ lies in $\mathcal{M}'$ [30, Theorem 1.1]. Here $u_t^{\mathcal{K}}$ shears the cylinders of the collection by the matrix $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ and leaves the rest of the surface unchanged, and $a_s^{\mathcal{K}}$ stretches them. For the cylinders $C_1, \ldots, C_r$ of $\mathcal{K}$, with heights h_i and circumferences c_i, Wright writes $u_{t_1, \ldots, t_r}^{\mathcal{K}}$ for the deformation that shears C_i by t_i, and proves that the closure of the orbit of (X', ω') under the $u_t^{\mathcal{K}}$ is the set of the surfaces

$u^{\mathcal{K}}_{t_1/m_1,\dots,t_r/m_r}(X',\omega')$, where $m_i = h_i/c_i$ and the t_i satisfy every homogeneous linear relation with rational coefficients that the m_i satisfy [30, Corollary 3.4].

Lemma 9.2. *Let $\mathcal{C}_1,\dots,\mathcal{C}_r$ be all the cylinders of (X',ω') of a direction $\mathbf{v}$, and let $t_1,\dots,t_r$ be real numbers that satisfy every homogeneous linear relation with rational coefficients that holds among $\operatorname{mod}(\mathcal{C}_1),\dots,\operatorname{mod}(\mathcal{C}_r)$. Then*

$$\sum_{i=1}^{r} t_i\,|\mathcal{C}_i|\,I_{\mathcal{C}_i} \in V'.$$

Proof. A rotation multiplies the period coordinates by a complex scalar and maps $\mathcal{M}'$ to itself; so, with the marking of (X',ω'), the rotated surface has the same space V', and we may assume that $\mathbf{v}$ is horizontal and positive. By Wright's theorem with $s = 0$, the surfaces $u^{\mathcal{K}}_t(X',\omega')$, for the collection $\mathcal{K}$ of all horizontal cylinders, lie in $\mathcal{M}'$, which is closed. By Wright's corollary above, applied with $(\sigma t_i)_i$ in place of $(t_i)_i$, $\mathcal{M}'$ contains the surface obtained from (X',ω') by shearing $\mathcal{C}_i$ by $\sigma t_i/\operatorname{mod}(\mathcal{C}_i)$ for $i = 1,\dots,r$, for every real σ. A shear of $\mathcal{C}_i$ by t moves the period coordinates by t times the height of $\mathcal{C}_i$ times $I_{\mathcal{C}_i}$, for small t [30, Lemmas 2.3 and 2.4], and shears of distinct cylinders commute. So for small σ this surface has the period coordinates $[\omega'] + \sigma\sum_i t_i|\mathcal{C}_i|I_{\mathcal{C}_i}$, as in Wright's proof of [30, Lemma 7.3]. Near $[\omega']$ the set $\mathcal{M}'$ is V', which is a linear subspace containing $[\omega']$; the claim follows. □

9.3. **One member of full dimension.** By Sections 2.4 and 2.2, for odd k the ambient locus has dimension $\dim H^1(X,\Sigma;\mathbb{C}) = 2g + |\Sigma| - 1$, where $|\Sigma|$ is the sum of the g_i over the vertices i with $q_i \nmid k$; for even k it has dimension

$$\dim E^- = D + \sum_{c \text{ odd}} \dim Z_c,$$

where $\dim Z_c$ is the number of vertices i with $q_i \nmid k$ and $cq_i \equiv 0 \pmod{k}$.

Lemma 9.3. *Let θ be admissible and reduced with $g \geq 2$, and let Π be a quadrilateral of type θ. If the orbit closure of the unfolding of Π, in its stratum with the zeros marked and no other point, has dimension at least $\dim \mathcal{L}(\theta)$, then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$.*

Proof. Let (X_Π,ω_Π) be the unfolding, $\mathcal{M}_\Pi$ its orbit closure and V_Π the subspace with which $\mathcal{M}_\Pi$ agrees near $[\omega_\Pi]$ (Section 9.2). The unfolding is a point of $\mathcal{M}_{\mathrm{cyc}}(\theta)$, whatever the cyclic order of the angles of Π (Section 2.1); so $\mathcal{M}_\Pi \subset \mathcal{M}(\theta) \subset \mathcal{L}(\theta)$. Near $[\omega_\Pi]$ the set $\mathcal{M}(\theta)$ is a finite union of linear subspaces, each of dimension $\dim \mathcal{M}(\theta)$, read at a lift if (X_Π,ω_Π) is an orbifold point; this union contains a neighborhood of $[\omega_\Pi]$ in V_Π, so one of the subspaces contains V_Π, as a non-empty open subset of a vector space is not contained in a finite union of proper subspaces. Hence $\dim \mathcal{M}(\theta) \geq \dim V_\Pi \geq \dim \mathcal{L}(\theta)$.

Now let X be a generic point of $\mathcal{M}_{\mathrm{cyc}}(\theta)$ and $V = T_X\mathcal{M}(\theta)$. Then $V \subset T_X\mathcal{L}(\theta)$ and $\dim V = \dim \mathcal{M}(\theta) \geq \dim \mathcal{L}(\theta)$, so $V = T_X\mathcal{L}(\theta)$. If k is odd, this is $H^1(X,\Sigma;\mathbb{C})$; so $p(V) = H^1(X;\mathbb{C})$ and $\operatorname{rank}\mathcal{M}(\theta) = g$, and Proposition 5.1 gives $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. If k is even, $V = E^-$, and the second paragraph of the proof of Proposition 5.2 gives $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. □

9.4. **The certificates.**

Proposition 9.4. *Let θ be one of the 32 types of (9.1) or one of the supplementary types $(1,4,8,11)$, $(3,4,8,9)$, $(4,5,7,8)$. Then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$.*

Each of these types has $N \leq 32$, so the proposition is also contained in Delecroix and Rüth's [11, Theorem 4.2], with the ambient strata read as our ambient loci; we prove it here from certificates in the deposit.

Proof. For each of these 35 types, the deposit [20] gives a quadrilateral Π of type θ (its vertices are listed there; for $(1,1,4,14)$ and $(1,2,3,14)$ the angles in cyclic order are $(1,4,1,14)$ and $(1,3,2,14)$, which Lemma 9.3 allows) whose vertices have coordinates in a real number field $\Bbbk$, and a list of non-zero vectors $\mathbf{v} \in \Bbbk^2$; Table 2 records their number. Let (X_Π, ω_Π) be the unfolding of Π, with zero set Σ_Π; its periods lie in $\Bbbk + i\Bbbk$. We describe what is computed, in exact arithmetic in $\Bbbk$.

For each listed $\mathbf{v}$ the computation finds cylinders $\mathcal{C}_1, \dots, \mathcal{C}_r$ of (X_Π, ω_Π) of direction $\mathbf{v}$, by the flow decomposition of the flatsurf library [9, 10], and it establishes that they are all the cylinders of direction $\mathbf{v}$. In most of the directions the areas of the $\mathcal{C}_i$ add up to the area of X_Π. In the others, whose number is in the last column of Table 2, a second program shows that every closed leaf of direction $\mathbf{v}$ lies in one of the $\mathcal{C}_i$, as follows.

Let J be the union of the edges, not parallel to $\mathbf{v}$, of the triangulation of X_Π that cuts each copy of Π by a diagonal (its vertices are the points over the vertices of Π, zeros and points of angle 2π). A point z of an edge of J has the coordinate $\det(\mathbf{v}, z)$, with z written in the plane of the copy of Π that contains one of the two triangles on the edge, chosen once for each edge; this copy is the image of Π under a product of reflections in lines through 0 parallel to sides of Π, a matrix over $\Bbbk$, so that its vertices lie in $\Bbbk^2$. This coordinate is constant along each leaf of direction $\mathbf{v}$ inside a triangle; the copies of the two triangles on an edge are glued along it by a translation by a vector of $\Bbbk^2$, which changes the coordinate by a constant in $\Bbbk$; and it is strictly monotone along the edge, which is not parallel to $\mathbf{v}$. We orient each edge of J so that the coordinate increases along it and lay the edges end to end, so that J becomes, up to finitely many points, an interval of length $\sum_e |\det(\mathbf{v}, e)| \in \Bbbk$, the sum over the holonomies e of the edges of J. Every leaf of direction $\mathbf{v}$ that meets no vertex crosses J, a closed one in finitely many points, and the map that sends a point of J to the next point where its leaf crosses J is an interval exchange in this coordinate, with the endpoints of its intervals in $\Bbbk$, which translates its j-th interval by $\epsilon_j \in \Bbbk$.

Rauzy induction replaces an interval exchange by its first-return map to a shorter interval that every orbit meets, and an exchange that maps the union of its first m intervals onto itself, for some m less than their number, is split into the two exchanges on that union and on the rest; so a periodic orbit of the first exchange gives a periodic orbit of one of the exchanges the program ends with. These are of two kinds. An exchange of one interval that it maps to itself consists of closed leaves of one cylinder; these are grouped into cylinders by the classes in $H^1(X_\Pi, \Sigma_\Pi; \mathbb{Z})$ given by intersection with their core curves (closed leaves of distinct cylinders have distinct classes, as two disjoint closed leaves homologous in $X_\Pi \smallsetminus \Sigma_\Pi$ bound a flat annulus), and a third program compares the cylinders so found with the flow decomposition: their number, their circumferences and areas, and the area outside them. For each other exchange the program gives a $\mathbb{Q}$-linear map $\Phi\colon \Bbbk \to \mathbb{Q}$ and rational numbers Ψ_j such that $\Phi(\epsilon'_j) + \Psi_{j'} - \Psi_j > 0$, $\epsilon'_j \in \Bbbk$ the translation of its j-th interval, whenever the image of the j-th interval meets the j'-th, which it checks in exact arithmetic; such an exchange has no periodic orbit, as along one the translations add up to zero and the differences $\Psi_{j'} - \Psi_j$ cancel, while each term $\Phi(\epsilon'_j) + \Psi_{j'} - \Psi_j$ is positive. This extends the one-step case of Boshernitzan's criterion [6, Lemma 6.7], stated there for exchanges of rational rank two with Φ the coefficient of the irrational parameter and all $\Psi_j = 0$.

Now a cylinder of direction $\mathbf{v}$ contains a closed leaf that avoids the vertices and the countably many points of J where the exchanges are not defined; that leaf gives a periodic orbit of a final exchange of the first kind, so the cylinder is one of those found. The $\mathcal{C}_i$ are cylinders of direction $\mathbf{v}$, so they are among those found; as the cylinders found are as many as the $\mathcal{C}_i$, they are the $\mathcal{C}_i$.

The holonomy of the core curve of $\mathcal{C}_i$ is $s_i\mathbf{v}$ with $s_i \in \Bbbk$ positive, and $|\mathbf{v}|^2 \bmod(\mathcal{C}_i)$ is the area of $\mathcal{C}_i$ divided by s_i^2, an element μ_i of $\Bbbk$. Writing the μ_i in a basis of $\Bbbk$ over $\mathbb{Q}$ gives a rational matrix with r rows; a rational relation holds among the moduli if and only if it holds among the entries of each column, so the vectors (t_i) of Lemma 9.2 are the real combinations of the columns. By Lemma 9.2, V_Π contains the class $\sum_i t_i s_i I_{\mathcal{C}_i}$ for each column (t_i).

Finally the computation evaluates these classes, for all the listed $\mathbf{v}$, and the classes $\mathrm{Re}[\omega_\Pi]$ and $\mathrm{Im}[\omega_\Pi]$ on a basis of $H_1(X_\Pi, \Sigma_\Pi; \mathbb{Z})$; the values lie in $\Bbbk$, and the rank of the matrix of values is $\dim \mathcal{L}(\theta)$, the number in the third column of Table 2. Real classes that are linearly independent over $\mathbb{R}$ are linearly independent over $\mathbb{C}$, so $\dim \mathcal{M}_\Pi = \dim V_\Pi \geq \dim \mathcal{L}(\theta)$, and Lemma 9.3 applies. □

For the quadrilaterals Π of the deposit the proof gives more: the orbit closure $\mathcal{M}_\Pi$ of the unfolding of Π is $\mathcal{L}(\theta)$ itself. By the proof of Lemma 9.3, $\mathcal{M}_\Pi \subset \mathcal{L}(\theta)$, and V_Π lies in one of the linear subspaces with which $\mathcal{L}(\theta) = \mathcal{M}(\theta)$ agrees near $[\omega_\Pi]$; as $\dim V_\Pi \geq \dim \mathcal{L}(\theta)$, it is that subspace. So near a point x' of V_Π close to $[\omega_\Pi]$ and on none of the other subspaces, both $\mathcal{L}(\theta)$ and $\mathcal{M}_\Pi$ agree with V_Π. The argument of the second paragraph of the proof of Proposition 5.2, with $\mathcal{M}_\Pi$ in place of $\mathcal{M}(\theta)$ and started at a point y of $\mathcal{Q}$ with $f(y) = x'$ in place of $(X/\iota, q)$ (it lies in O, as f maps a neighborhood of y into $\mathcal{L}(\theta)$, hence into V_Π near x'), and for odd k with the component of the stratum, in its own period charts, in place of $\mathcal{Q}$ and x' in place of y, then gives $\mathcal{M}_\Pi = \mathcal{L}(\theta)$.

10. Proof of Theorem 1.1

Let θ be an admissible and reduced type that is not one of the seven types (1.1). We go through the cases.

If $\theta = (1,1,1,1)$, then X is a flat torus and $\Sigma = \emptyset$ (Section 2.1). The stratum of flat tori is a single orbit of $\mathrm{GL}^+(2,\mathbb{R})$, which acts transitively on the lattices in $\mathbb{C}$; so the non-empty invariant subsets $\mathcal{M}(\theta) \subset \mathcal{L}(\theta)$ of this stratum are both the whole stratum.

Otherwise $g \geq 2$, by Lemma 2.1. If θ is supplementary, $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ by Theorem 8.8; this includes the types $(1,4,8,11)$, $(3,4,8,9)$ and $(4,5,7,8)$, which rest on Proposition 9.4. If θ is not supplementary, it is open. If it is one of the 32 types (9.1), then $\mathcal{M}(\theta) = \mathcal{L}(\theta)$ by Proposition 9.4. In every other case the rules reach θ: by Proposition 9.1 if $N \leq 1000$, by Theorem 6.12 if $N > 1000$ and k is odd, and by Theorem 7.14 if $N > 1000$ and k is even. For odd k, Corollary 5.3 then gives $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. For even k, $A = F \cup F_1$ by Proposition 4.11, so $Z_c \subset V$ for every odd relative character c by Proposition 7.15, and Corollary 5.3 gives $\mathcal{M}(\theta) = \mathcal{L}(\theta)$.

In every case $\mathcal{M}(\theta) = \mathcal{L}(\theta)$. Finally k is odd if and only if N is odd or $N \equiv 2 \pmod 4$, and $\mathcal{L}(\theta)$ is then the connected component of the stratum of (X,ω); and k is even if and only if $4 \mid N$, and $\mathcal{L}(\theta)$ is then the locus $\widetilde{\mathcal{Q}}$ of holonomy double covers (Section 2.4). □

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Fable 5.2 in Claude Code and OpenAI's GPT-6.2 Astra in Codex. The author chooses the problems and prioritizes

θ	k	$\dim \mathcal{L}(\theta)$	dir.	n.c.p.	θ	k	$\dim \mathcal{L}(\theta)$	dir.	n.c.p.
$(1,1,2,6)$	5	9	7	2	$(1,1,2,20)$	12	12	10	2
$(1,1,4,10)$	8	8	6	2	$(1,1,4,18)$	12	10	8	2
$(1,2,3,10)$	8	8	6	2	$(1,1,6,16)$	12	12	10	2
$(1,2,4,9)$	8	8	6	2	$(1,1,8,14)$	12	12	10	2
$(2,4,5,5)$	8	8	6	2	$(1,2,4,17)$	12	10	8	2
$(1,1,2,16)$	10	9	7	2	$(1,2,5,16)$	12	12	10	2
$(1,1,4,14)$	10	10	8	2	$(1,2,8,13)$	12	12	10	2
$(1,1,6,12)$	10	10	8	3	$(1,3,4,16)$	12	10	8	2
$(1,2,3,14)$	10	9	7	2	$(1,4,5,14)$	12	10	8	2
$(1,2,4,13)$	10	9	7	2	$(1,4,6,13)$	12	10	8	2
$(1,2,5,12)$	10	9	7	2	$(2,4,5,13)$	12	10	8	2
$(1,2,6,11)$	10	9	7	2	$(1,1,2,24)$	14	13	7	0
$(1,3,4,12)$	10	10	8	2	$(1,2,6,19)$	14	13	7	0
$(2,3,3,12)$	10	9	7	2	$(1,2,4,25)$	16	16	6	0
$(2,3,4,11)$	10	9	7	4	$(1,4,8,11)$	12	10	4	0
$(2,3,6,9)$	10	9	4	0	$(3,4,8,9)$	12	10	3	0
$(3,3,6,8)$	10	10	3	0	$(4,5,7,8)$	12	10	3	0
$(1,1,1,21)$	12	12	10	2					

TABLE 2. The types of Proposition 9.4: the dimension of the ambient locus, the number of directions used (dir.), and the number of them that are not completely periodic (n.c.p.).

the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [20].

REFERENCES

1. Paul Apisa, $\mathrm{GL}_2\mathbb{R}$*-invariant measures in marked strata: generic marked points, Earle–Kra for strata and illumination*, Geom. Topol. **24** (2020), no. 1, 373–408, doi:10.2140/gt.2020.24.373.
2. ———, *Billiards in right triangles and orbit closures in genus zero strata*, 2021, Preprint, arXiv:2110.07540v1.
3. Paul Apisa and Juliet Aygun, *Components of strata of k-differentials and their orbit closures*, 2026, Preprint, arXiv:2604.26177v1.
4. Paul Apisa and Alex Wright, *Marked points on translation surfaces*, Geom. Topol. **25** (2021), no. 6, 2913–2961, doi:10.2140/gt.2021.25.2913; numbering as in arXiv:1708.03411v3.
5. Frits Beukers and Gert Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$, Invent. Math. **95** (1989), no. 2, 325–354, doi:10.1007/BF01393900.
6. Michael D. Boshernitzan, *Rank two interval exchange transformations*, Ergodic Theory Dynam. Systems **8** (1988), no. 3, 379–394, doi:10.1017/S0143385700004521.
7. Vincent Delecroix, *Surfaces: combinatorics, geometry, algorithms and dynamics*, Mémoire d'habilitation, Université de Bordeaux, 2024, HAL tel-05410741v1.
8. ———, *Computing the multiscale boundary of linear subvarieties*, 2025, Slides, joint work with K. Fu and B. Zykoski, JHH80, Toulouse, June 2025, https://indico.math.cnrs.fr/event/12318/.

9. Vincent Delecroix, W. Patrick Hooper, and Julian Rüth, *sage-flatsurf, version 0.8.0*, 2025, Zenodo, doi:10.5281/zenodo.17650831.
10. Vincent Delecroix and Julian Rüth, *flatsurf, version 3.15.2*, 2024, Zenodo, doi:10.5281/zenodo.13838321.
11. ———, *A new orbit closure in genus 8?*, Exp. Math. **34** (2025), no. 2, 254–259, doi:10.1080/10586458.2024.2361608; numbering as in arXiv:2110.05407v2.
12. Alex Eskin, Curtis T. McMullen, Ronen E. Mukamel, and Alex Wright, *Billiards, quadrilaterals and moduli spaces*, J. Amer. Math. Soc. **33** (2020), no. 4, 1039–1086, doi:10.1090/jams/950; numbering as in the authors' accepted manuscript.
13. Alex Eskin and Maryam Mirzakhani, *Invariant and stationary measures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space*, Publ. Math. Inst. Hautes Études Sci. **127** (2018), 95–324, doi:10.1007/s10240-018-0099-2, arXiv:1302.3320v6.
14. Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi, *Isolation, equidistribution, and orbit closures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space*, Ann. of Math. (2) **182** (2015), no. 2, 673–721, doi:10.4007/annals.2015.182.2.7.
15. Simion Filip, *Semisimplicity and rigidity of the Kontsevich–Zorich cocycle*, Invent. Math. **205** (2016), no. 3, 617–670, doi:10.1007/s00222-015-0643-3; numbering as in arXiv:1307.7314v3.
16. ———, *Splitting mixed Hodge structures over affine invariant manifolds*, Ann. of Math. (2) **183** (2016), no. 2, 681–713, doi:10.4007/annals.2016.183.2.5, arXiv:1311.2350v3.
17. Manvir Jaswal, *Orbit closures of generic rational polygons with at least five sides*, 2026, Zenodo, doi:10.5281/zenodo.24452738.
18. ———, *Orbit closures of scalene rational triangles: every denominator*, 2026, Zenodo, doi:10.5281/zenodo.24434358.
19. ———, *Orbit closures of scalene rational triangles: rank one and prime denominators*, 2026, Zenodo, doi:10.5281/zenodo.24434354.
20. ———, *Programs and data for "Orbit closures of generic rational quadrilaterals"*, 2026, Zenodo, doi:10.5281/zenodo.24434360.
21. Jeremy Kahn and Alex Wright, *Hodge and Teichmüller*, J. Mod. Dyn. **18** (2022), 149–160, doi:10.3934/jmd.2022007; numbering as in arXiv:2106.04665v1.
22. Erwan Lanneau, *Connected components of the strata of the moduli spaces of quadratic differentials*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 1, 1–56, doi:10.24033/asens.2062, arXiv:math/0506136v2.
23. Curtis T. McMullen, *Braid groups and Hodge theory*, Math. Ann. **355** (2013), no. 3, 893–946, doi:10.1007/s00208-012-0804-2; numbering as in the author's preprint of April 25, 2009.
24. Gabrielle Menet and Duc-Manh Nguyen, *Representations of braid groups via cyclic covers of the sphere: Zariski closure and arithmeticity*, 2023, Preprint, arXiv:2310.10401v3.
25. Maryam Mirzakhani and Alex Wright, *The boundary of an affine invariant submanifold*, Invent. Math. **209** (2017), 927–984, doi:10.1007/s00222-017-0722-8; numbering as in arXiv:1508.01446v4.
26. ———, *Full-rank affine invariant submanifolds*, Duke Math. J. **167** (2018), no. 1, 1–40, doi:10.1215/00127094-2017-0036; numbering as in arXiv:1608.02147v2.
27. Calvin C. Moore, *Ergodicity of flows on homogeneous spaces*, Amer. J. Math. **88** (1966), 154–178, doi:10.2307/2373052.
28. Alex Wright, *Schwarz triangle mappings and Teichmüller curves: abelian square-tiled surfaces*, J. Mod. Dyn. **6** (2012), no. 3, 405–426, doi:10.3934/jmd.2012.6.405, arXiv:1203.2683v3.
29. ———, *The field of definition of affine invariant submanifolds of the moduli space of abelian differentials*, Geom. Topol. **18** (2014), no. 3, 1323–1341, doi:10.2140/gt.2014.18.1323.
30. ———, *Cylinder deformations in orbit closures of translation surfaces*, Geom. Topol. **19** (2015), no. 1, 413–438, doi:10.2140/gt.2015.19.413; numbering and pages as in the journal.
31. Alexander N. Zemljakov and Anatole B. Katok, *Topological transitivity of billiards in polygons*, Math. Notes **18** (1975), no. 2, 760–764, doi:10.1007/BF01818045; Russian original: Mat. Zametki 18 (1975), no. 2, 291–300.

Toronto, Canada
Email address: manvirjaswal@icloud.com
URL: https://manvirjaswal.github.io