Exact, total and filtered small categories*

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Abstract

For a non-empty connected small category we compare four properties: being filtered; exactness of colimits of abelian groups; Grothendieck's total asphericity, which says that colimits of simplicial sets over the category are homotopy colimits; and being quasi-filtered, which means that two arrows with a common source can be completed to a commutative square and that two parallel arrows are equalized by an integral combination of arrows whose coefficients sum to 1. Each implies the next. We give a category of finite ordered hypergraphs that is totally aspherical and whose colimits are exact over no non-zero ring, and we show that the category of finite ordered two-graphs (sets of triples that meet every four-element set in an even number of triples) is quasi-filtered and not totally aspherical. With the example of Isbell and Mitchell, the three implications are strict. A conjecture in Grothendieck's *Les Dérivateurs*, that totally aspherical categories are filtered, is therefore false; it holds for sifted categories, those over which colimits of sets commute with finite products.

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1 Introduction

Let A be a small category, non-empty and connected. Four properties of A are compared in this paper; the definitions are in Section 2.

- A is *filtered*.
- A is *exact*: the colimit functor from A-indexed diagrams of abelian groups to abelian groups is exact.
- A is *total*: every category that receives a final functor from A is weakly contractible. This is Grothendieck's *asphéricité totale* [6]; it holds if and only if every connected component of the category of elements of every functor $A \to \mathbf{Set}$ is weakly contractible (Proposition 2.3), and if and only if colimits over A of diagrams of simplicial sets are homotopy colimits (Remark 2.4).
- A is *quasi-filtered*: pairs of arrows with a common source can be completed to commutative squares, and parallel pairs have affine equalizers.

Each property implies the next (Propositions 2.5 and 2.7). Isbell and Mitchell showed in 1973 that the first implication cannot be reversed. In the manuscript *Les Dérivateurs* Grothendieck asks whether total categories are filtered. The results of this paper are the following.

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Theorem 1.1. *Let $\mathcal{C}_\delta$ be the category of non-empty finite linearly ordered 3-uniform hypergraphs in which every non-empty set U of vertices satisfies $2|U| - 3e(U) > 0$, where $e(U)$ is the number of hyperedges contained in U, with the closed embeddings as arrows (Definition 4.1). Then $\mathcal{C}_\delta$ is total, and for every non-zero ring R the colimit functor on $\mathcal{C}_\delta$-indexed diagrams of R-modules is not exact.*

Theorem 1.2. *The category $\mathcal{T}$ of finite ordered two-graphs and their embeddings is quasi-filtered and is not total.*

Theorem 1.3. *Among non-empty connected small categories,*

$$\{\textit{filtered}\} \subsetneq \{\textit{exact}\} \subsetneq \{\textit{total}\} \subsetneq \{\textit{quasi-filtered}\}.$$

The three inclusions are strict by the category Δ_{inj} of the finite non-empty ordinals and increasing injections, by $\mathcal{C}_\delta$, and by $\mathcal{T}$. Moreover a non-empty connected category is filtered if and only if it is total and sifted.

Theorem 1.1 is Theorems 4.10 and 4.11; Theorem 1.2 is Theorem 5.10. For Theorem 1.3: the inclusions are Propositions 2.5 and 2.7; Δ_{inj} is exact by [7, Theorems 1 and 3] (their category is that of the order-preserving injections of finite totally ordered sets; Δ_{inj} is a full cosieve of it, and common successors and affine equalizers pass to a non-empty cosieve), and it is not filtered, because no arrow equalizes the two arrows from the ordinal with one element to the ordinal with two; the last assertion is Proposition 2.8.

Theorem 1.1 separates two weakenings of filteredness that are both classical: exactness of colimits of modules, and the property that colimits of spaces are homotopy colimits. The proofs of total asphericity (Theorem 1.1) and of its failure (Theorem 1.2) follow one pattern. Both categories consist of the finite closed substructures of a countable homogeneous structure M. If every open subgroup of $\mathrm{Aut}(M)$ is the pointwise stabilizer of a finite closed set, then every component of every category of elements is weakly contractible or weakly equivalent to the category itself (Proposition 3.2). So only the classifying space of the category matters: it is contractible for $\mathcal{C}_\delta$, and simply connected with non-zero second homology for $\mathcal{T}$. Exactness is decided by a count: in a hypergraph of $\mathcal{C}_\delta$ any r hyperedges span more than $3r/2$ vertices, so one of them has a vertex of its own.

Section 6 relates the results to the passage of *Les Dérivateurs*. This paper was produced by AI models; who proved, wrote and checked what is recorded in Section 7.

2 The four classes

2.1 Conventions

Categories are small unless they are categories of sets, groups or modules. For a category A we write NA for its nerve and BA for its classifying space. A functor is a *weak equivalence* if it induces a weak homotopy equivalence of classifying spaces, and A is *weakly contractible* if BA is. A natural transformation between two functors induces a homotopy between the induced maps, and a category with an initial object is contractible [11], [15, IV.3.2 and Example IV.3.2.2]. For a functor $u\colon A \to B$ and an object b of B, the objects of $b\backslash u$ are the pairs $(a,\ b \to u(a))$. The functor u is *final* if every $b\backslash u$ is non-empty and connected [10, IX.3]. Quillen's Theorem A [11], [15, IV.3.7]: if every $b\backslash u$ is weakly contractible, then u is a weak equivalence.

For a functor $L\colon A \to \mathbf{Set}$ the *category of elements* $\mathrm{El}(L)$ has as objects the pairs (a, l) with $l \in L(a)$, and as arrows $(a, l) \to (a', l')$ the arrows $f\colon a \to a'$ with $L(f)l = l'$. An

action component of A is a connected component of $\mathrm{El}(L)$ for some L. If D is an action component of A, a component of $\mathrm{El}(L)$, and $K\colon D \to \mathbf{Set}$ is a functor, then

$$\mathrm{El}(K) \cong \mathrm{El}(\widetilde{K}), \qquad \widetilde{K}(a) = \coprod_{l \in L(a),\ (a,l)\in D} K(a,l); \tag{1}$$

so every action component of D is an action component of A. If $\Psi\colon D \to \mathbf{Set}$ sends all arrows to bijections, then $B\,\mathrm{El}(\Psi) \to BD$ is a covering space, and every covering space of BD arises in this way [11], [15, IV.3.5.1 and Exercise IV.3.1].

For a functor $N\colon A \to \mathbf{Ab}$ let $C_\bullet(A;N)$ be the chain complex with $C_n(A;N) = \bigoplus_{a_0\to\cdots\to a_n} N(a_0)$, the sum being over the n-simplices of NA, and with the usual alternating differential; let $H_n(A;N)$ be its homology. We use three properties: $H_0(A;N) = \mathrm{colim}_A N$; a short exact sequence of functors gives a long exact sequence of the groups H_n; and for $L\colon A \to \mathbf{Set}$ the complex $C_\bullet(A;\mathbb{Z}[L])$ is the complex of integral chains of $N\,\mathrm{El}(L)$. In particular, for $P_x = \mathbb{Z}[A(x,-)]$,

$$H_0(A;P_x) = \mathbb{Z}, \qquad H_n(A;P_x) = 0 \quad (n>0), \tag{2}$$

because $\mathrm{El}(A(x,-)) = x\backslash A$ has an initial object; the isomorphism $H_0(A;P_x) \cong \mathbb{Z}$ is induced by the augmentation $\sum n_i f_i \mapsto \sum n_i$.

2.2 Definitions

Definition 2.1. Let A be a non-empty connected category.

(a) A is *filtered* if any two objects have a common successor and for every pair $f, g\colon p \to q$ there is an arrow h with $hf = hg$.

(b) A is *exact* if the functor $\mathrm{colim}_A\colon \mathbf{Ab}^A \to \mathbf{Ab}$ is exact.

(c) A is *total* if for every final functor $u\colon A \to B$ the category B is weakly contractible.

(d) A is *quasi-filtered* if (QF1) every pair of arrows $q \leftarrow p \to q'$ can be completed to a commutative square, and (QF2) for every pair $f, g\colon p \to q$ there are an object r and an element t of $\mathbb{Z}[A(q,r)]$ of augmentation 1 with $tf = tg$ in $\mathbb{Z}[A(p,r)]$.

Exact categories were characterized by Isbell and Mitchell [7, 8]; we call an element t as in (QF2) an *affine equalizer* of f and g. (QF1) and (QF2) are the conditions I and IV of [8, §2] for the representable functors; in [1, p. 369] a category is called quasi-filtered (after Adámek and Reiterman) if every parallel pair has a quasi-filter and any two objects have a common successor, and by [1, Theorem [1]] and [8, Lemma 2.1] a connected category is quasi-filtered in that sense if and only if it satisfies (QF1) and (QF2). Property (c) is Grothendieck's *asphéricité totale*, for the weak homotopy equivalences [6, Chapter II, ms. p. 36ter; Chapter XII, Définition 1]. It is not the property called totally aspherical in the theory of test categories and in [12, Remark 4.5(1)], which concerns the diagonal $A \to A \times A$; see Proposition 2.8.

2.3 Total categories

Lemma 2.2. *Let A be a category all of whose action components have the integral homology of a point. Then all action components of A are weakly contractible.*

Proof. Let D be an action component and γ an element of the fundamental group of BD. The connected covering space of BD whose fundamental group is the cyclic group generated by γ is $B\,\mathrm{El}(\Psi)$ for some $\Psi\colon D \to \mathbf{Set}$, and $\mathrm{El}(\Psi)$ is an action component of A by (1). Its first homology group is that cyclic group, so $\gamma = 1$. Thus BD is simply connected with the homology of a point, and it is weakly contractible by the Hurewicz theorem. □

Proposition 2.3. *A non-empty connected category A is total if and only if every action component of A is weakly contractible.*

Proof. Suppose that all action components are weakly contractible, and let $u\colon A \to B$ be final. For an object b, the category $b\backslash u$ is the category of elements of the functor $a \mapsto B(b, u(a))$, and it is connected; so it is an action component and is weakly contractible. By Theorem A, u is a weak equivalence. Since A is the category of elements of the functor with value a point, BA is weakly contractible, and so is BB.

Conversely let A be total, and let D be an action component, a component of $\mathrm{El}(L)$. Let $L_D \subseteq L$ be the subfunctor with $L_D(a) = \{l : (a, l) \in D\}$. Let A_D be the category obtained from A by adjoining one object v, with $A_D(v, a) = L_D(a)$, $A_D(a, v) = \varnothing$, $A_D(v, v) = \{\mathrm{id}\}$, and with composition given by the action. The inclusion $A \to A_D$ is final: for an object a of A the category $a\backslash A$ has an initial object, and $v\backslash A = D$. Hence A_D is weakly contractible, and so is A, because the identity of A is final. A simplex of NA_D that is not in NA is a chain that begins with v; so NA_D is the union of NA and of the nerve of the category $D^{\triangleleft}$ obtained from D by adjoining an initial object, glued along ND by the projection $ND \to NA$. Hence BA_D is the mapping cone of $BD \to BA$, and the reduced integral homology of BD vanishes. By Lemma 2.2 all action components are weakly contractible. □

Remark 2.4. Equivalently, A is total if and only if for every functor X from A to simplicial sets the canonical map $\mathrm{hocolim}_A X \to \mathrm{colim}_A X$ is a weak equivalence. Indeed, the homotopy colimit is the diagonal of the bisimplicial set $(m, n) \mapsto N_n \,\mathrm{El}(X_m)$ [3, Chapter XII], [9, Sections 21 and 22], and the map is induced by the maps $N\,\mathrm{El}(X_m) \to \pi_0\,\mathrm{El}(X_m) = \mathrm{colim}_A X_m$. If the action components are weakly contractible these are weak equivalences, and then so is the map of diagonals [9, Lemma 21.1], [5, Chapter IV]. Conversely a functor $L\colon A \to \mathbf{Set}$ is a diagram of discrete simplicial sets, for which the map is $N\,\mathrm{El}(L) \to \pi_0\,\mathrm{El}(L)$.

2.4 The inclusions

Proposition 2.5. *Let A be a non-empty connected category. If A is filtered it is exact, and if A is exact it is total.*

Proof. Filtered colimits of abelian groups are exact [14, Section 2.6]. Let A be exact, and let D be an action component, a component of $\mathrm{El}(L)$. For $N\colon D \to \mathbf{Ab}$ put $(\Sigma N)(a) = \bigoplus_{l:(a,l)\in D} N(a, l)$. The functor $\Sigma\colon \mathbf{Ab}^D \to \mathbf{Ab}^A$ is exact and $\mathrm{colim}_A \Sigma N = \mathrm{colim}_D N$, so colim_D is exact. Let $R_\bullet$ be the complex of functors $D \to \mathbf{Ab}$ with $R_n = \bigoplus_{d_0\to\cdots\to d_n} \mathbb{Z}[D(d_n, -)]$ whose value at an object d is the complex of integral chains of the nerve of D/d. As D/d has a terminal object, $R_\bullet$ is a resolution of the constant functor $\mathbb{Z}$. Its colimit over D is the complex of integral chains of ND. Since colim_D is exact, BD has the integral homology of a point. By Lemma 2.2 and Proposition 2.3, A is total. □

Lemma 2.6. *Let D be a connected category such that $H_1(B\,\mathrm{El}(K); \mathbb{Z}) = 0$ for every functor $K\colon D \to \mathbf{Set}$ with $\mathrm{El}(K)$ connected. Then any two objects of D have a common successor.*

Proof. Let x, x' be objects, and let U and U' be the full subcategories of the objects that receive an arrow from x, respectively from x'. They are cosieves. Suppose that they are disjoint and put $V = U \cup U'$. Let $K(d) = \{1, 2\}$ for $d \notin V$ and $K(d) = \{1\}$ for $d \in V$, with the evident maps. Then $N\operatorname{El}(K)$ is the union of two copies of ND that meet in NV, and $\operatorname{El}(K)$ is connected. In the Mayer–Vietoris sequence

$$H_1(B\operatorname{El}(K); \mathbb{Z}) \longrightarrow H_0(BV; \mathbb{Z}) \longrightarrow H_0(BD; \mathbb{Z}) \oplus H_0(BD; \mathbb{Z})$$

the middle group has rank at least 2, since no arrow joins U and U', and the image of the second map has rank 1. So the first group is not zero, a contradiction. □

Proposition 2.7. *Let A be a non-empty connected category such that $H_1(BD; \mathbb{Z}) = 0$ for every action component D of A. Then A is quasi-filtered. In particular every total category is quasi-filtered.*

Proof. By (1), Lemma 2.6 applies to every action component of A.

(QF1). Let $f\colon p \to q$ and $f'\colon p \to q'$. The category $p\backslash A$ is an action component. A common successor of its objects (q, f) and (q', f') is a commutative square $uf = u'f'$.

(QF2). Let $f, g\colon p \to q$. Let H be the quotient of the functor $A(p, -)$ by the equivalence relation generated by $kf \sim kg$ for all arrows k with source q. Then $\operatorname{El}(H)$ is connected, because every object receives an arrow from $(p, [\mathrm{id}_p])$. There is an exact sequence

$$P_q \xrightarrow{\partial} P_p \longrightarrow \mathbb{Z}[H] \longrightarrow 0, \qquad \partial(k) = kf - kg.$$

Let $K = \ker \partial$ and $J = \operatorname{im} \partial$. The sequences $0 \to J \to P_p \to \mathbb{Z}[H] \to 0$ and $0 \to K \to P_q \to J \to 0$ and (2) give

$$H_1(A; \mathbb{Z}[H]) = \ker\bigl(\operatorname{colim} J \to \operatorname{colim} P_p\bigr), \qquad \operatorname{colim} J = \operatorname{coker}\bigl(\operatorname{colim} K \to \operatorname{colim} P_q = \mathbb{Z}\bigr).$$

The elements of J have augmentation zero, so $\operatorname{colim} J \to \operatorname{colim} P_p = \mathbb{Z}$ is zero. Hence $H_1(B\operatorname{El}(H); \mathbb{Z}) = H_1(A; \mathbb{Z}[H]) \cong \mathbb{Z}/I$, where I is the subgroup of $\mathbb{Z}$ generated by the augmentations of the elements of the groups $K(x) = \{t \in \mathbb{Z}[A(q, x)] : tf = tg\}$. By hypothesis $I = \mathbb{Z}$: there are $t_j \in K(x_j)$ and integers n_j with $\sum_j n_j \varepsilon(t_j) = 1$. By Lemma 2.6, applied to A, there are arrows $u_j\colon x_j \to r$ into one object r, and $t = \sum_j n_j u_j t_j$ has augmentation 1 and satisfies $tf = tg$.

The last assertion follows from Proposition 2.3. □

2.5 Sifted categories

A category A is *sifted* if it is non-empty and the diagonal $\Delta\colon A \to A \times A$ is final, that is, if for all objects a, b the category $(a, b)\backslash\Delta$ of the pairs of arrows $a \to c \leftarrow b$ is non-empty and connected. By [2, Theorem 1.6] this holds if and only if colimits over A of diagrams of sets commute with finite products, which is the definition of sifted categories in [2, Definition 1.1].

Proposition 2.8. *A non-empty connected category is filtered if and only if it is total and sifted.*

Proof. Let A be total and sifted. For objects a, b the category $(a, b)\backslash\Delta$ is non-empty, so a and b have a common successor. Let $f, g\colon p \to q$. The category $(p, p)\backslash\Delta$ is the category of elements of $A(p, -) \times A(p, -)$, and it is connected; so it is an action component of A. By Proposition 2.3 and Lemma 2.6 its objects (q, f, g) and (q, f, f) have a common successor

(r, w, w'): there are $h, h' \colon q \to r$ with $hf = w = h'f$ and $hg = w' = h'f$. Hence $hf = hg$, and A is filtered.

Conversely a filtered category is total by Proposition 2.5, and it is sifted: two objects (c, u, v) and (c', u', v') of $(a, b)\backslash\Delta$ are joined through an object $(d, \cdot, \cdot)$, where $c \to d \leftarrow c'$ are chosen so as to equalize the two resulting arrows $a \to d$ and the two resulting arrows $b \to d$. □

3 A reduction for categories of embeddings

Both examples of this paper are categories of finite substructures of a homogeneous structure. This section isolates what is used of such a category.

3.1 The setting

Let M be a non-empty set and G a group of permutations of M. For $X \subseteq M$ let G_X be the pointwise stabilizer of X; so $G_\varnothing = G$. A subgroup of G is *open* if it contains G_X for some finite $X \subseteq M$. Let $\mathcal{K}$ be a set of finite subsets of M. We consider the following conditions.

(K1) $\varnothing \in \mathcal{K}$; $gB \in \mathcal{K}$ for $B \in \mathcal{K}$ and $g \in G$; every finite subset of M is contained in a member of $\mathcal{K}$.

(K2) $\mathrm{Fix}_M(G_B) = B$ for every $B \in \mathcal{K}$.

(K3) Every open subgroup of G is of the form G_F with $F \in \mathcal{K}$.

Let $\mathcal{A} = \mathcal{A}(M, G, \mathcal{K})$ be the category whose objects are the non-empty members of $\mathcal{K}$ and whose arrows $B \to C$ are the maps $g|_B \colon B \to C$ with $g \in G$ and $gB \subseteq C$, composed as maps. Inclusions between members of $\mathcal{K}$ are arrows. By (K1) any two objects are contained in a third, so $\mathcal{A}$ is non-empty and connected.

For an object P of a category $\mathcal{C}$, the objects of $P\backslash\mathcal{C}$ are the pairs $(X, x \colon P \to X)$, and an arrow $(X, x) \to (Y, y)$ is an arrow $f \colon X \to Y$ with $fx = y$. A *cosieve* of a category is a full subcategory that contains the target of every arrow whose source it contains.

(SC) For every object P of $\mathcal{C}$, every non-empty cosieve of $P\backslash\mathcal{C}$ is weakly contractible.

(K4) The category $\mathcal{A}(M, G, \mathcal{K})$ satisfies (SC).

Property (SC) is invariant under equivalence of categories: an equivalence $\Phi \colon \mathcal{C} \to \mathcal{C}'$ induces equivalences $P\backslash\mathcal{C} \to \Phi P\backslash\mathcal{C}'$, and the objects isomorphic to images of objects of a cosieve $\mathcal{U}$ form a cosieve equivalent to $\mathcal{U}$.

Lemma 3.1. *In a category with* (SC) *every pair of arrows $f \colon a \to b$, $f' \colon a \to b'$ can be completed to a commutative square $uf = u'f'$.*

Proof. In $a\backslash\mathcal{C}$ let $\mathcal{U}$ be the cosieve of the objects (c, h) such that h factors through f, and $\mathcal{U}'$ the one defined by f'. Their union is a non-empty cosieve, hence connected. A path in it from (b, f) to (b', f') contains an arrow between an object of $\mathcal{U}$ and an object of $\mathcal{U}'$, and its target lies in $\mathcal{U} \cap \mathcal{U}'$. □

3.2 The components of the categories of elements

Proposition 3.2. *Assume* (K1)–(K4)*, and let* $L\colon \mathcal{A} \to \mathbf{Set}$ *be a functor. For every connected component* D *of* $\mathrm{El}(L)$*, either* D *is weakly contractible, or the projection* $D \to \mathcal{A}$ *is a weak equivalence.*

Proof. A functor that induces a weak equivalence of classifying spaces induces a bijection between the sets of connected components and weak equivalences between corresponding components. The two functors constructed below commute with the projections to $\mathcal{A}$.

Step 1: injective transition maps. For $l, l' \in L(a)$ write $l \sim l'$ if $L(f)l = L(f)l'$ for some arrow $f\colon a \to b$. This is an equivalence relation: if f and f' witness $l \sim l'$ and $l' \sim l''$, and $uf = u'f'$ as in Lemma 3.1, then $L(uf)$ takes the same value on l, l' and l''. It is preserved by $L(v)$ for every arrow $v\colon a \to a'$: if $L(f)l = L(f)l'$ and $uf = u'v$, then $L(u')$ identifies $L(v)l$ and $L(v)l'$. So the quotients form a functor L^{inj}, with a natural surjection $L \to L^{\mathrm{inj}}$. Its transition maps are injective: an arrow that witnesses $L(v)l \sim L(v)l'$, composed with v, witnesses $l \sim l'$.

We claim that $q\colon \mathrm{El}(L) \to \mathrm{El}(L^{\mathrm{inj}})$ is a weak equivalence. Fix an object $(a, [l])$ of the target and let $\mathcal{D} = (a, [l])\backslash q$; its objects are the triples (x, k, f) with $f\colon a \to x$, $k \in L(x)$ and $k \sim L(f)l$. Let $J\colon a\backslash\mathcal{A} \to \mathcal{D}$ send $f\colon a \to x$ to $(x, L(f)l, f)$. For an object $d = (x, k, f)$ of $\mathcal{D}$, the category $d\backslash J$ is the full subcategory of $x\backslash\mathcal{A}$ on the arrows $g\colon x \to y$ with $L(g)k = L(gf)l$. It is a cosieve, and it is non-empty because $k \sim L(f)l$; by (K4) it is weakly contractible. By Quillen's Theorem A (Section 2), J is a weak equivalence. As $a\backslash\mathcal{A}$ has an initial object, $\mathcal{D}$ is weakly contractible, and Theorem A applies to q.

Step 2: saturation. By Step 1 we may assume that all maps $L(f)$ are injective. The non-empty members of $\mathcal{K}$ form a directed set under inclusion, by (K1). Let

$$S = \mathrm{colim}_B\, L(B)$$

be the colimit over the inclusions. The canonical maps $i_B\colon L(B) \to S$ are injective. The group G acts on S by $g \cdot i_B(l) = i_{gB}(L(g|_B)l)$, where $g|_B\colon B \to gB$; this is well defined because $g|_{B'}$ extends $g|_B$ when $B \subseteq B'$. The element $i_B(l)$ is fixed by G_B. So S is a G-set all of whose point stabilizers are open.

For any such G-set S define a functor $T_S\colon \mathcal{A} \to \mathbf{Set}$ by $T_S(B) = S^{G_B}$ and $T_S(g|_B)(s) = g \cdot s$ for an arrow $g|_B\colon B \to C$. This does not depend on g, since two choices differ by an element of G_B; and $g \cdot s$ is fixed by G_C because $g^{-1}G_C g \leq G_B$. In our case the maps i_B form a natural monomorphism $i\colon L \to T_S$. For an object (a, s) of $\mathrm{El}(T_S)$, the category $(a, s)\backslash \mathrm{El}(i)$ is the full subcategory of $a\backslash\mathcal{A}$ on the arrows $f\colon a \to b$ with $T_S(f)s \in i_b(L(b))$. It is a cosieve. It is non-empty: if $s = i_c(l)$, choose $b \in \mathcal{K}$ containing $a \cup c$; for the inclusion $f\colon a \to b$ we have $T_S(f)s = s \in i_b(L(b))$. By (K4) and Theorem A, $\mathrm{El}(i)\colon \mathrm{El}(L) \to \mathrm{El}(T_S)$ is a weak equivalence.

Step 3: orbits. The functor $S \mapsto T_S$ takes disjoint unions of G-sets to disjoint unions of functors, so $\mathrm{El}(T_S)$ is the disjoint union of the categories $\mathrm{El}(T_O)$, O running through the orbits of S. By (K3) every orbit is isomorphic to G/G_F for some $F \in \mathcal{K}$.

If $F \neq \varnothing$, then $T_{G/G_F} \cong \mathcal{A}(F, -)$. Indeed, a coset gG_F is fixed by G_B if and only if $G_B \leq gG_Fg^{-1} = G_{gF}$, that is, by (K2), if and only if $gF \subseteq B$. The map $gG_F \mapsto g|_F$ is then a bijection from $(G/G_F)^{G_B}$ onto $\mathcal{A}(F, B)$, natural in B. The category of elements of $\mathcal{A}(F, -)$ is $F\backslash\mathcal{A}$, which has an initial object.

If $F = \varnothing$, the orbit is a point, T_O is the constant functor with value a point, and $\mathrm{El}(T_O) \to \mathcal{A}$ is an isomorphism.

So every component of $\mathrm{El}(T_S)$ is contractible or is mapped isomorphically to $\mathcal{A}$ by the projection, and the statement follows from Steps 1 and 2. □

3.3 The bridge set

The following construction is used twice, to verify (K3). It involves only finite linearly ordered sets.

Lemma 3.3. *Let D and E be finite linearly ordered sets, $B, C \subseteq D$, $F = B \cap C$, and let $d_0, d_1 \colon D \to E$ be increasing injections that agree on F. For every sufficiently large integer $N \geq 2$, with $s = 2N$, there are a finite linearly ordered set Z, an increasing injection $e \colon E \to Z$ and increasing injections $a_0, \dots, a_s \colon D \to Z$ with the following properties. Put $Z_0 = e(E)$ and $Z_i = a_i(D)$ for $0 < i < s$; these are the* pieces, *and their indices are read modulo s.*

(i) *$Z = \bigcup_i Z_i$; $a_0 = ed_0$ and $a_s = ed_1$; $a_{2j}|_B = a_{2j+1}|_B$ and $a_{2j+1}|_C = a_{2j+2}|_C$ for $0 \leq j < N$.*

(ii) *For $x \in F$ the point $a_i(x)$ does not depend on i; these F-*points *lie in every piece. Every other point of Z lies in at most three pieces. If it lies in two pieces, these are consecutive, or they are Z_1 and Z_{s-1}, and then the point lies in Z_0 as well.*

(iii) *For $0 \leq i < s$ we have $Z_i \cap Z_{i+1} = a_i(B) = a_{i+1}(B)$ if i is even and $Z_i \cap Z_{i+1} = a_i(C) = a_{i+1}(C)$ if i is odd.*

(iv) *For every index r there are a finite set L, a surjection $q \colon L \to Z$ and subsets $T_0, \dots, T_{s-1}$ covering L such that q maps T_k bijectively onto Z_{r+k}, with these properties: every point of L lies in T_k for a set of consecutive values of k, with at most three elements unless the point lies over an F-point; $q(T_{k-1} \cap T_k) = Z_{r+k-1} \cap Z_{r+k}$ for $0 < k < s$; every F-point has one preimage under q; and q is injective except that at most $\max(|B|, |C|)$ points of Z have two preimages.*

Proof. Let $\widehat{Z}$ be the disjoint union of E and of copies $D_1, \dots, D_{s-1}$ of D; write x_i for the copy of $x \in D$ in D_i. We call E the *piece* 0 and D_i the piece i. Consider the following *links* between points of consecutive pieces: $d_0(x)$ and x_1, for $x \in B$; x_i and x_{i+1}, for $0 < i < s-1$ and $x \in B$ if i is even, $x \in C$ if i is odd; x_{s-1} and $d_1(x)$, for $x \in C$. Let Z be the quotient of $\widehat{Z}$ by the equivalence relation generated by the links, e the restriction of the quotient map to E, $a_i(x)$ the class of x_i for $0 < i < s$, and $a_0 = ed_0$, $a_s = ed_1$. Then (i) holds.

A point has at most one link to the next piece and at most one to the preceding piece, because d_0 and d_1 are injective. For $x \in F$ the points $d_0(x) = d_1(x), x_1, \dots, x_{s-1}$ form a class, with one point in each piece. For $x \in B \setminus C$ the links at copies of x join x_i to x_{i+1} for i even, $0 < i < s-1$, and $d_0(x)$ to x_1; for $x \in C \setminus B$ they join x_i to x_{i+1} for i odd, $0 < i < s-1$, and x_{s-1} to $d_1(x)$; for $x \notin B \cup C$ there are none. A point of E that is not of the form $d_0(x)$, $x \in F$, has at most two links, to some x_1 with $x \in B \setminus C$ and to some y_{s-1} with $y \in C \setminus B$. Hence a class that does not come from F is a single point, or a pair $\{x_i, x_{i+1}\}$, or is contained in a set $\{y_{s-1}, d, x_1\}$ with $d \in E$. So every class has at most one point in each piece, which shows that e and the a_i are injective, and (ii) and (iii) follow by inspection; here $N \geq 2$ is used.

For (iv), let L be the quotient of $\widehat{Z}$ by the relation generated by the links other than those between the pieces $r-1$ and r, let T_k be the image of the piece $r+k$, and q the quotient map. A class of Z is the union of at most two classes of L, and of two only if it contains a link between the pieces $r-1$ and r and does not come from F; there are at most $\max(|B|, |C|)$ such links. The remaining assertions follow from the description of the classes.

The order. Choose an increasing injection $\varepsilon \colon E \to \mathbb{R}$ and put $p_t = (1-t)\,\varepsilon d_0 + t\,\varepsilon d_1 \colon D \to \mathbb{R}$ for $0 \leq t \leq 1$. Each p_t is strictly increasing, and $p_t(x)$ does not depend on t

for $x \in F$. Let $\Delta > 0$ be a lower bound for $p_t(x') - p_t(x)$ over all t and all $x < x'$ in D, and let $M_0 = \max_x |\varepsilon d_1(x) - \varepsilon d_0(x)|$; assume $2M_0 < N\Delta$. Define $b_i \colon D \to \mathbb{R}$ for $0 \le i \le s$ by

$$b_{2j} = p_{j/N}, \qquad b_{2j+1}(x) = \begin{cases} p_{j/N}(x), & x \in B, \\ p_{(j+1)/N}(x), & x \notin B. \end{cases}$$

Each value of b_{2j+1} differs from the corresponding value of $p_{(j+1)/N}$ by at most M_0/N, so $b_{2j+1}(x') - b_{2j+1}(x) \ge \Delta - 2M_0/N > 0$ for $x < x'$: all b_i are strictly increasing. Moreover $b_0 = \varepsilon d_0$, $b_s = \varepsilon d_1$, b_{2j} and b_{2j+1} agree on B, and b_{2j+1} and b_{2j+2} agree on C (on $C \setminus B$ by definition, on F because p_t is constant there). So the function on $\widehat{Z}$ that is ε on E and b_i on D_i takes the same value at the two ends of every link, and induces a function $\beta \colon Z \to \mathbb{R}$ that is injective and increasing on every piece. Any linear order on Z in which $z < z'$ whenever $\beta(z) < \beta(z')$ makes e and all a_i increasing. □

4 A total category whose colimits are not exact

4.1 The category $\mathcal{C}_\delta$

A *hypergraph* is a finite set X (of vertices) with a set E_X of three-element subsets of X (the hyperedges). A subset $U \subseteq X$ carries the induced hypergraph, with hyperedges $E_X(U) = \{h \in E_X : h \subseteq U\}$. Put

$$\delta_X(U) = 2\,|U| - 3\,|E_X(U)|.$$

We write $\delta(U)$ when X is understood; $\delta(U)$ depends only on the induced hypergraph on U. The hypergraph X is *positive* if $\delta(U) > 0$ for every non-empty $U \subseteq X$. For $A \subseteq B \subseteq X$ we say that A is *closed in* B, and write $A \le_{\mathrm{d}} B$, if

$$\delta(U) > \delta(A) \qquad \text{for every } U \text{ with } A \subsetneq U \subseteq B.$$

Thus X is positive if and only if $\varnothing \le_{\mathrm{d}} X$. An *ordered* hypergraph carries in addition a linear order on its vertices. An *embedding* $f \colon X \to Y$ of ordered hypergraphs is an increasing injection such that a three-element subset h of X is a hyperedge of X exactly when $f(h)$ is a hyperedge of Y; it is a *closed embedding* if $f(X) \le_{\mathrm{d}} Y$.

Definition 4.1. The objects of $\mathcal{C}_\delta$ are the positive ordered hypergraphs with vertex set $[n] = \{1 < \cdots < n\}$, $n \ge 1$. Its arrows are the closed embeddings, composed as maps.

Composites of closed embeddings are closed by Lemma 4.2(c) below, so $\mathcal{C}_\delta$ is a category; it is small, and every non-empty positive ordered hypergraph is isomorphic to exactly one object. We write V for the object with one vertex and W for the object $([3], \{\{1,2,3\}\})$. The latter is positive: its subsets with one, two and three elements have $\delta = 2, 4, 3$. The same numbers show that the three maps $v_1, v_2, v_3 \colon V \to W$, $v_i(1) = i$, are closed embeddings.

Lemma 4.2 (closure calculus)**.** *Let X be a hypergraph.*

(a) $\delta(U) + \delta(U') \ge \delta(U \cup U') + \delta(U \cap U')$ *for all* $U, U' \subseteq X$.

(b) *If* $A \le_{\mathrm{d}} B$ *and* $Y \subseteq B$, *then* $A \cap Y \le_{\mathrm{d}} Y$.

(c) *If* $A \le_{\mathrm{d}} B$ *and* $B \le_{\mathrm{d}} C$, *then* $A \le_{\mathrm{d}} C$.

(d) *Let $X = B \cup C$ and $A = B \cap C$, and suppose that $E_X = E_X(B) \cup E_X(C)$ (we then call X the* free amalgam *of B and C over A). Then for every $U \subseteq X$*

$$\delta(U) = \delta(U \cap B) + \delta(U \cap C) - \delta(U \cap A). \tag{3}$$

If moreover B and C are positive, $A \leq_{\mathrm{d}} B$ and $A \leq_{\mathrm{d}} C$, then X is positive, $B \leq_{\mathrm{d}} X$ and $C \leq_{\mathrm{d}} X$. Here A may be empty.

Proof. (a) The number of vertices is modular. For the hyperedges, $E(U) \cup E(U') \subseteq E(U \cup U')$ and $E(U) \cap E(U') = E(U \cap U')$, so $|E(U)| + |E(U')| \leq |E(U \cup U')| + |E(U \cap U')|$.

(b) Let $A \cap Y \subsetneq U \subseteq Y$. Then $U \cap A = A \cap Y$ and $U \cup A \supsetneq A$, so by (a) $\delta(U) - \delta(A \cap Y) \geq \delta(U \cup A) - \delta(A) > 0$.

(c) Let $A \subsetneq U \subseteq C$. If $U \subseteq B$ there is nothing to prove. Otherwise $U \cap B \subsetneq U$ and $U \cap B \leq_{\mathrm{d}} U$ by (b), so $\delta(U) > \delta(U \cap B) \geq \delta(A)$, the last inequality because $A \subseteq U \cap B \subseteq B$ and $A \leq_{\mathrm{d}} B$.

(d) A hyperedge of X lies in B or in C, and those lying in both are the hyperedges contained in A; this gives (3). Let U be non-empty. If $U \subseteq B$ then $\delta(U) > 0$. Otherwise $U \cap A \subsetneq U \cap C$, and $U \cap A \leq_{\mathrm{d}} U \cap C$ by (b), so $\delta(U \cap C) - \delta(U \cap A) > 0$, while $\delta(U \cap B) \geq 0$; hence $\delta(U) > 0$. If $B \subsetneq U \subseteq X$, then $U \cap B = B$, $U \cap A = A$ and $U \cap C \supsetneq A$, so $\delta(U) - \delta(B) = \delta(U \cap C) - \delta(A) > 0$. Thus $B \leq_{\mathrm{d}} X$, and $C \leq_{\mathrm{d}} X$ by symmetry. □

If, in (d), B and C are ordered and their orders agree on A, the free amalgam carries linear orders extending both. We call *the ordered free amalgam with B first* the one in which, for $b \in B \setminus A$ and $c \in C \setminus A$, we have $b < c$ exactly when every element of A below b is below c.

4.2 The generic structure and its automorphism group

For an ordered hypergraph M with countably many vertices, all of whose finite subsets are positive, and a finite $B \subseteq M$, we write $B \leq_{\mathrm{d}} M$ if $B \leq_{\mathrm{d}} U$ for every finite U with $B \subseteq U \subseteq M$. Parts (b) and (c) of Lemma 4.2 remain true with M in the place of the largest set.

Proposition 4.3. *There is a countable ordered hypergraph M, all of whose finite subsets are positive, with the following properties.*

(M1) *Every finite subset of M is contained in a finite subset B with $B \leq_{\mathrm{d}} M$.*

(M2) *If $B \leq_{\mathrm{d}} M$ is finite and $f\colon B \to C$ is a closed embedding into a finite positive ordered hypergraph, there is an embedding $h\colon C \to M$ with $h(C) \leq_{\mathrm{d}} M$ and $hf = \mathrm{id}_B$.*

(M3) *Every isomorphism between two finite closed subsets of M extends to an automorphism of M.*

Proof. A *task* is a triple (k, B, f) where $k \geq 0$ and $f\colon B \to C$ is a closed embedding of a subset B of the k-th stage, closed in that stage, into a finite positive ordered hypergraph. Up to isomorphism of C under B there are countably many tasks at each stage, so the tasks of all stages can be enumerated in such a way that every task is met at a later stage. Starting with $M_0 = V$, let M_{l+1} be an ordered free amalgam of M_l and C over B, where (k, B, f) is the task met at stage $l \geq k$. Here $B \leq_{\mathrm{d}} M_k \leq_{\mathrm{d}} M_l$ by induction and Lemma 4.2(c), so by Lemma 4.2(d) the amalgam is positive, $M_l \leq_{\mathrm{d}} M_{l+1}$, and the image of C is closed in M_{l+1}. Let $M = \bigcup_l M_l$. By transitivity $M_l \leq_{\mathrm{d}} M_{l'}$ for $l \leq l'$, hence $M_l \leq_{\mathrm{d}} M$; this gives (M1). For (M2), choose k with $B \subseteq M_k$; then $B \leq_{\mathrm{d}} M_k$, the task (k, B, f) is

met at some stage l, and the resulting copy of C is closed in $M_{l+1} \leq_{\mathrm{d}} M$, hence in M. For (M3) one extends a given isomorphism $\varphi\colon B \to B'$ of finite closed subsets step by step: for $x \in M$ choose by (M1) a finite closed $C \supseteq B \cup \{x\}$; by (M2), applied to B' and to $\varphi^{-1}\colon B' \to B \subseteq C$, the map φ extends to an isomorphism of C onto a closed subset of M. Alternating between φ and φ^{-1} along an enumeration of M gives an automorphism. □

The existence of such limits for classes with a notion of strong substructure is standard; see [4, Theorem 2.6]. We fix M as in Proposition 4.3 and put

$$G = \mathrm{Aut}(M), \qquad \mathcal{K} = \{B \subseteq M :\ B \text{ finite},\ B \leq_{\mathrm{d}} M\},$$

where automorphisms preserve the order and the hyperedges. Positivity gives $\varnothing \in \mathcal{K}$. We use the notation of Section 3: G_B is the pointwise stabilizer of B, and $\mathcal{A} = \mathcal{A}(M, G, \mathcal{K})$.

Lemma 4.4. *The triple $(M, G, \mathcal{K})$ satisfies* (K1)*. For $B, C \in \mathcal{K}$ the arrows $B \to C$ of $\mathcal{A}$ are exactly the closed embeddings $B \to C$, and $\mathcal{A}$ is equivalent to $\mathcal{C}_\delta$.*

Proof. (K1) follows from (M1) and from the invariance of $\leq_{\mathrm{d}}$ under automorphisms. If $g \in G$ and $gB \subseteq C$, then $gB \leq_{\mathrm{d}} M$, so $gB \leq_{\mathrm{d}} C$ by Lemma 4.2(b), and $g|_B$ is a closed embedding. Conversely, if $f\colon B \to C$ is a closed embedding, then $f(B) \leq_{\mathrm{d}} C \leq_{\mathrm{d}} M$, so $f(B) \in \mathcal{K}$ by Lemma 4.2(c), and $f\colon B \to f(B)$ extends to an element of G by (M3). Finally every object of $\mathcal{C}_\delta$ is isomorphic to a member of $\mathcal{K}$ by (M2) with $B = \varnothing$, so the functor that sends a non-empty $B \in \mathcal{K}$ to the object of $\mathcal{C}_\delta$ isomorphic to it is an equivalence. □

Lemma 4.5. $(M, G, \mathcal{K})$ *satisfies* (K2)*: $\mathrm{Fix}_M(G_B) = B$ for every $B \in \mathcal{K}$.*

Proof. Let $x \in M \setminus B$ and choose $D \in \mathcal{K}$ containing $B \cup \{x\}$. Let D' be a second copy of D, with $D \cap D' = B$, and form an ordered free amalgam of D and D' over B; note that $B \leq_{\mathrm{d}} D$ by Lemma 4.2(b). By Lemma 4.2(d) the inclusion of D into it is a closed embedding, so by (M2) the amalgam embeds into M over D, with closed image; the image of D' is then closed in M. The isomorphism of D onto this image that is the identity on B extends by (M3) to some $g \in G_B$, and $gx \neq x$. □

Lemma 4.6. *The category $\mathcal{A}$ satisfies* (SC)*, and $B\mathcal{A}$ is weakly contractible.*

Proof. Both properties are invariant under equivalence, so we may replace $\mathcal{A}$ by the following category $\mathcal{C}$. Fix an infinite set Ω and injections $\lambda, \rho\colon \Omega \to \Omega$ with disjoint images. The objects of $\mathcal{C}$ are the non-empty positive ordered hypergraphs whose vertex set is a finite subset of Ω, and its arrows are the closed embeddings; by Lemma 4.4 it is equivalent to $\mathcal{A}$.

Let P be an object of $\mathcal{C}$, let $\mathcal{U}$ be a non-empty full cosieve of $P\backslash\mathcal{C}$, and fix an object $(Z, z\colon P \to Z)$ of $\mathcal{U}$. For an object $(X, x\colon P \to X)$ of $P\backslash\mathcal{C}$ let $T(X)$ be the hypergraph on $\lambda(X) \cup \rho(Z \setminus z(P))$ that is the free amalgam of the copy $\lambda(X)$ of X and of the copy $\lambda(x(P)) \cup \rho(Z \setminus z(P))$ of Z over $\lambda(x(P))$, ordered with $\lambda(X)$ first. The base is closed in both copies because x and z are closed embeddings, so by Lemma 4.2(d) $T(X)$ is positive and the maps

$$\eta_X\colon X \to T(X),\ u \mapsto \lambda(u), \qquad \gamma_X\colon Z \to T(X),\ z(p) \mapsto \lambda(x(p)),\ w \mapsto \rho(w)\ \ (w \notin z(P)),$$

are closed embeddings; they are arrows under P. For an arrow $f\colon X \to X'$ of $P\backslash\mathcal{C}$ let $T(f) = \lambda f \lambda^{-1}$ on $\lambda(X)$ and the identity on $\rho(Z \setminus z(P))$. It is increasing, because f is increasing and commutes with the maps from P, so that it preserves the gap of P in which a vertex lies. It is an embedding, since a hyperedge of $T(X')$ lies in one of the two copies and f is an embedding. Its image is closed, because $T(X')$ is the free amalgam of $\lambda(X')$

and of the image of $T(X)$ over $\lambda(f(X))$, and $f(X) \leq_{\mathrm{d}} X'$. So T is an endofunctor of $P\backslash\mathcal{C}$, and $\eta\colon \mathrm{id} \to T$ and $\gamma\colon \mathrm{const}_{(Z,z)} \to T$ are natural transformations. Since $\mathcal{U}$ is a cosieve containing (Z,z), the arrows γ_X show that T maps $P\backslash\mathcal{C}$ into $\mathcal{U}$. The restrictions of η and γ to $\mathcal{U}$ connect the identity of $\mathcal{U}$ to a constant functor, so $B\mathcal{U}$ is contractible [15, IV.3.2].

The same construction with the empty set in the place of P (so that $T(X) = \lambda(X) \cup \rho(Z)$ is a disjoint union, with $\lambda(X)$ first) gives, for a fixed object Z, an endofunctor T of $\mathcal{C}$ with natural transformations $\mathrm{id} \to T \leftarrow \mathrm{const}_Z$. Hence $B\mathcal{C}$ is contractible. □

4.3 Open subgroups

Lemma 4.7 (the bridge)**.** *Let D and E be positive ordered hypergraphs, let $B \leq_{\mathrm{d}} D$ and $C \leq_{\mathrm{d}} D$, put $F = B \cap C$, and let $d_0, d_1\colon D \to E$ be closed embeddings that agree on F. Then there are a positive ordered hypergraph Z, a closed embedding $e\colon E \to Z$ and closed embeddings $a_0, \dots, a_{2N}\colon D \to Z$ with*

$$a_0 = ed_0, \quad a_{2N} = ed_1, \quad a_{2j}|_B = a_{2j+1}|_B, \quad a_{2j+1}|_C = a_{2j+2}|_C \qquad (0 \leq j < N).$$

Proof. Put $m = \max(|B|, |C|)$ and take N so large that Lemma 3.3 applies and

$$s := 2N > 3\,(\delta(E) + 2m), \qquad N \geq 2. \tag{4}$$

Let Z, e, a_i and the pieces $Z_0, \dots, Z_{s-1}$ be as in Lemma 3.3; the indices of pieces are read modulo s. Every piece is a copy of E or of D; let the hyperedges of Z be the images of the hyperedges of the pieces. It remains to prove that every piece is an induced subhypergraph of Z, that Z is positive, and that every piece is closed in Z.

Pieces are induced. For two consecutive pieces the intersection $Z_i \cap Z_{i+1}$ is the common image of B or of C (Lemma 3.3(iii)), and the two pieces induce the same hypergraph on it, because d_0 and d_1 are embeddings. Call this fact $(*)$. Let h be a hyperedge of the piece Z_j whose three vertices lie in Z_i, $i \neq j$. If all three are F-points, h is the image of a hyperedge of D contained in F, hence a hyperedge of every piece. Otherwise one of them lies in at most three cyclically consecutive pieces, among them Z_i and Z_j. If i and j are consecutive, h is a hyperedge of Z_i by $(*)$. If not, then $\{i,j\} = \{1, s-1\}$ by Lemma 3.3(ii), and every point of $Z_1 \cap Z_{s-1}$ lies in Z_0; applying $(*)$ to (Z_j, Z_0) and then to (Z_0, Z_i) shows again that h is a hyperedge of Z_i.

The cut. Fix an index r. Let L be obtained from the disjoint union of the pieces by making all the identifications of Lemma 3.3 except those between the pieces $r-1$ and r, with hyperedges the images of the hyperedges of the pieces, and let $q\colon L \to Z$ be the quotient map. Write $T_0, \dots, T_{s-1}$ for the images in L of the pieces $r, r+1, \dots, r-1$, in this order. By Lemma 3.3(iv) a point of L lies in T_k for a set of consecutive values of k. Hence $T_k \cap (T_0 \cup \dots \cup T_{k-1}) = T_k \cap T_{k-1} =: A_k$. Applying $(*)$ along the consecutive pieces between two pieces that contain the vertices of a hyperedge shows, as in the first step, that every T_k is an induced subhypergraph of L. So $L_k = T_0 \cup \dots \cup T_k$ is the free amalgam of L_{k-1} and T_k over A_k. The set A_k is a copy of B or of C; it is closed in T_k and in T_{k-1}, by hypothesis and, for a piece that is a copy of E, by Lemma 4.2(c). By induction on k and Lemma 4.2(c),(d), L_k is positive and each of $T_0, \dots, T_k$ is closed in L_k. So L is positive and every T_k is closed in L. By Lemma 3.3(iv) the map q identifies at most m pairs of points, and every hyperedge of Z is the image of a hyperedge of L. Hence, for $U \subseteq Z$ and $\widetilde{U} = q^{-1}(U)$,

$$\delta_Z(U) \;\geq\; \delta_L(\widetilde{U}) - 2m. \tag{5}$$

Positivity and closedness. Let $U \subseteq Z$ be non-empty.

Case 1: some piece Z_r contains no point of U other than F-points. Cut between $r-1$ and r. No point of U is then the image of two points of L, so q maps $\widetilde{U}$ bijectively onto U and $E_L(\widetilde{U})$ bijectively onto $E_Z(U)$. Hence $\delta_Z(U) = \delta_L(\widetilde{U}) > 0$. If U properly contains a piece Z_i, then $\widetilde{U}$ properly contains the corresponding T_k, and $\delta_Z(U) = \delta_L(\widetilde{U}) > \delta_L(T_k) = \delta_Z(Z_i)$.

Case 2: every piece contains a point of U that is not an F-point. Cut between $s-1$ and 0, and put $\widetilde{U}_k = \widetilde{U} \cap T_k$. Applying (3) to $L_k = L_{k-1} \cup T_k$ for $k = 1, \dots, s-1$ gives

$$\delta_L(\widetilde{U}) = \delta(\widetilde{U}_0) + \sum_{k=1}^{s-1} \bigl(\delta(\widetilde{U}_k) - \delta(\widetilde{U}_k \cap A_k)\bigr).$$

By Lemma 4.2(b), $\widetilde{U}_k \cap A_k \leq_{\mathrm{d}} \widetilde{U}_k$, so the k-th summand is non-negative, and it is at least 1 when $\widetilde{U}_k \not\subseteq A_k$; also $\delta(\widetilde{U}_0) \geq 1$ when $\widetilde{U}_0 \neq \varnothing$. Let J be the set of those k for which $\widetilde{U}_k$ contains a point that is not an F-point and does not lie in T_{k-1} (no condition on T_{k-1} for $k = 0$). Each $k \in J$ contributes at least 1, so $\delta_L(\widetilde{U}) \geq |J|$. Let $0 \leq k' < s$. By assumption $T_{k'}$ contains a point $u \in \widetilde{U}$ that is not an F-point; the set of k with $u \in T_k$ consists of at most three consecutive values, and its least element k belongs to J, with $k \leq k' \leq k+2$. Hence $3|J| \geq s$, and by (5) and (4)

$$\delta_Z(U) \;\geq\; \frac{s}{3} - 2m \;>\; \delta(E) \;\geq\; \delta(D) \;>\; 0,$$

where $\delta(E) \geq \delta(D)$ because $d_0(D) \leq_{\mathrm{d}} E$. In particular $\delta_Z(U) > \delta_Z(Z_i)$ for every piece Z_i.

In both cases $\delta_Z(U) > 0$, and $\delta_Z(U) > \delta_Z(Z_i)$ whenever U properly contains the piece Z_i. So Z is positive and every piece is closed in Z; with the first step, e and all a_i are closed embeddings. □

Proposition 4.8. *For all $B, C \in \mathcal{K}$ we have $\langle G_B, G_C \rangle = G_{B \cap C}$.*

Proof. One inclusion is clear. Let $g \in G_{B\cap C}$. Choose $D \in \mathcal{K}$ containing $B \cup C$ and $E \in \mathcal{K}$ containing $D \cup gD$. Then $B \leq_{\mathrm{d}} D$ and $C \leq_{\mathrm{d}} D$ by Lemma 4.2(b), and the inclusion $d_0 \colon D \to E$ and $d_1 = g|_D \colon D \to E$ are closed embeddings that agree on $B \cap C$. Let Z, e and $a_0, \dots, a_{2N}$ be as in Lemma 4.7. By (M2) there is an embedding $h \colon Z \to M$ with closed image and $he = \mathrm{id}_E$. Each $ha_i \colon D \to M$ is an isomorphism onto a closed subset, so by (M3) $ha_i = g_i|_D$ for some $g_i \in G$; since $ha_0 = d_0$ and $ha_{2N} = d_1$ we may take $g_0 = 1$ and $g_{2N} = g$. The relations between consecutive a_i say that $g_i^{-1} g_{i+1}$ lies in G_B for i even and in G_C for i odd. Hence $g = \prod_{i=0}^{2N-1} g_i^{-1} g_{i+1} \in \langle G_B, G_C \rangle$. □

Corollary 4.9. *$(M, G, \mathcal{K})$ satisfies* (K3)*: every open subgroup of G is G_F for some $F \in \mathcal{K}$.*

Proof. Let U be an open subgroup. By (K1) it contains G_B for some $B \in \mathcal{K}$; choose such a B of least cardinality. For $g \in U$ we have $G_{gB} = gG_Bg^{-1} \leq U$, so $G_{B\cap gB} \leq U$ by Proposition 4.8; here $B \cap gB \in \mathcal{K}$ by Lemma 4.2(b),(c). By minimality $gB = B$, and an increasing bijection of a finite linearly ordered set is the identity, so $g \in G_B$. Hence $U = G_B$. □

4.4 The two properties

Theorem 4.10. *The category $\mathcal{C}_\delta$ is total: it is non-empty and connected, and for every functor $L \colon \mathcal{C}_\delta \to$ **Set** every connected component of $\mathrm{El}(L)$ is weakly contractible.*

Proof. By Lemmas 4.4, 4.5, 4.6 and Corollary 4.9 the triple $(M, G, \mathcal{K})$ satisfies (K1)–(K4). By Proposition 3.2 every action component of $\mathcal{A}$ is weakly contractible or weakly equivalent to $\mathcal{A}$, and $B\mathcal{A}$ is weakly contractible by Lemma 4.6. Since $\mathcal{A}$ is equivalent to $\mathcal{C}_\delta$, the same holds for $\mathcal{C}_\delta$, and Proposition 2.3 applies. □

Theorem 4.11. *Let R be a non-zero ring. For an object X of $\mathcal{C}_\delta$ let $P_X = R[\mathcal{C}_\delta(X,-)]$, a functor from $\mathcal{C}_\delta$ to left R-modules, and let*

$$\Theta\colon P_W \longrightarrow P_V \oplus P_V, \qquad \Theta(g) = (gv_2 - gv_1,\ gv_3 - gv_1) \quad (g\colon W \to X).$$

Then Θ is a monomorphism, and colim $\Theta\colon R \to R \oplus R$ *is zero. In particular the functor* colim *from $\mathcal{C}_\delta$-indexed diagrams of R-modules to R-modules is not exact; for $R = \mathbb{Z}$, the category $\mathcal{C}_\delta$ is not exact.*

Proof. Let X be an object and let $t = \sum_g c_g\, g$ be a non-zero element of $P_W(X)$; let $g_1, \dots, g_r$ be the arrows $W \to X$ with $c_g \neq 0$. An increasing map of W is determined by its image, so the images of the g_k are r distinct hyperedges of X. Let U be their union and $v = |U|$. Then $0 < \delta(U) \leq 2v - 3r$. The r hyperedges have $3r < 2v$ incidences with the v points of U, each of which lies on at least one of them; so some point u lies on exactly one, say on the image of g_k, where $u = g_k(i)$. The arrow $g_k v_i\colon V \to X$ has image $\{u\}$, and no other arrow $g_l v_j$ with $c_{g_l} \neq 0$ has this image. The coefficient of $g_k v_i$ is therefore c_{g_k} in the first component of $\Theta(t)$ if $i = 2$, it is c_{g_k} in the second component if $i = 3$, and it is $-c_{g_k}$ in both if $i = 1$. So $\Theta(t) \neq 0$.

The colimit of a representable functor P_X is R, the canonical map $P_X(Y) \to R$ being the augmentation $\sum c_f f \mapsto \sum c_f$. Both components of $\Theta(g)$ have augmentation zero, so colim $\Theta = 0$, which is not injective because colim $P_W = R \neq 0$. □

5 A quasi-filtered category that is not total

5.1 Ordered two-graphs

Let S be a finite set. A *graph* on S is a function b from the two-element subsets of S to $\mathbb{F}_2$. A *two-graph* on S is a function c from the three-element subsets of S to $\mathbb{F}_2$ such that, for every four-element subset Q of S, the sum of c over the four three-element subsets of Q is zero. (The triples with $c = 1$ then form a two-graph in the usual sense; see [13].) We write $c(x,y,z)$ for $c(\{x,y,z\})$ and put $c(x,y,z) = 0$ when two of the entries coincide; similarly for graphs. For a graph b let

$$(\partial b)(x,y,z) = b(x,y) + b(x,z) + b(y,z).$$

This is a two-graph, because in the sum over the four triples of a four-element set every pair occurs twice. For a two-graph c on S and $p \in S$ let c_p be the graph $c_p(x,y) = c(p,x,y)$. Then

$$\partial(c_p) = c; \tag{6}$$

for a triple that does not contain p this is the condition on the four-element set formed with p, and for a triple $\{p,y,z\}$ it holds because $c_p(p,y) = c_p(p,z) = 0$.

An *ordered two-graph* is a finite non-empty linearly ordered set with a two-graph. An *embedding* of ordered two-graphs is an increasing injection $f\colon X \to Y$ with $c_Y(f(x), f(y), f(z)) = c_X(x,y,z)$ for all x, y, z.

Definition 5.1. The objects of $\mathcal{T}$ are the ordered two-graphs with underlying ordered set $[n]$, $n \geq 1$; its arrows are the embeddings.

Lemma 5.2 (anchored amalgams)**.** *Let $S = S_1 \cup S_2$ and let c_i be a two-graph on S_i, $i = 1, 2$, such that c_1 and c_2 agree on the triples contained in $S_1 \cap S_2$. Choose $p_i \in S_i$, with $p_1 = p_2 \in S_1 \cap S_2$ if this set is non-empty, and let b be the graph on S that equals $(c_i)_{p_i}$ on the pairs contained in S_i and is zero on all other pairs. Then ∂b is a two-graph on S that restricts to c_i on S_i.*

Proof. The graph b is well defined, because $(c_1)_p$ and $(c_2)_p$ agree on the pairs contained in $S_1 \cap S_2$ when p lies in this set. On triples contained in S_i we have $\partial b = \partial((c_i)_{p_i}) = c_i$ by (6). □

We call ∂b the amalgam *anchored at* p_1, p_2. In particular any two objects of $\mathcal{T}$ embed into a third, so $\mathcal{T}$ is connected.

5.2 The generic ordered two-graph

Proposition 5.3. *There is an ordered two-graph structure on a countably infinite set M (a linear order and a function c on three-element subsets that is a two-graph on every finite subset) such that:*

(N2) *if $Y \subseteq M$ is finite and $f\colon Y \to C$ is an embedding into an ordered two-graph, there is an embedding $h\colon C \to M$ with $hf = \mathrm{id}_Y$ (here Y may be empty);*

(N3) *every isomorphism between finite subsets of M extends to an automorphism of M.*

Proof. As for Proposition 4.3, with every finite subset in the role of a closed subset and with the anchored amalgams of Lemma 5.2, ordered by any linear order extending the two given ones, in the role of the free amalgams. □

Fix such an M, let $G = \mathrm{Aut}(M)$ and let $\mathcal{K}$ be the set of all finite subsets of M. By (N3) the arrows of $\mathcal{A} = \mathcal{A}(M, G, \mathcal{K})$ are the embeddings between finite non-empty subsets of M, and by (N2) with $Y = \varnothing$ the category $\mathcal{A}$ is equivalent to $\mathcal{T}$. Condition (K1) is clear. By (N2) the order of M has no greatest element.

Lemma 5.4. $(M, G, \mathcal{K})$ *satisfies* (K2) *and* (K4).

Proof. (K2). Let $B \subseteq M$ be finite and $x \in M \setminus B$. Let $C = B \cup \{x, x'\}$ carry an amalgam of $B \cup \{x\}$ and of a second copy $B \cup \{x'\}$ of it over B (Lemma 5.2), ordered with x' immediately after x. By (N2) it embeds into M over $B \cup \{x\}$, and by (N3) the isomorphism $B \cup \{x\} \to B \cup \{x'\}$ extends to some $g \in G_B$ with $gx \neq x$.

(K4). We argue as for Lemma 4.6. Let $\mathcal{C}$ be the category of ordered two-graphs whose underlying set is a finite subset of Ω, with the embeddings as arrows; it is equivalent to $\mathcal{A}$. Let P be an object, p_0 its least element, $\mathcal{U}$ a non-empty cosieve of $P\backslash\mathcal{C}$ and (Z, z) an object of $\mathcal{U}$. For an object (X, x) let $T(X)$ be the set $\lambda(X) \cup \rho(Z \setminus z(P))$, ordered as in Lemma 4.6, with the amalgam of the copies of X and of Z anchored at $\lambda(x(p_0))$. For an arrow $f\colon X \to X'$ under P, the map $T(f)$ defined as before is increasing. It carries the anchor to the anchor and the pairs contained in one of the two copies to such pairs, preserving the values of the graph b of Lemma 5.2, and it carries the remaining pairs, on which $b = 0$, to such pairs. Hence it preserves ∂b and is an embedding. As before we obtain an endofunctor T of $P\backslash\mathcal{C}$ and natural transformations $\mathrm{id} \to T \leftarrow \mathrm{const}_{(Z,z)}$, and $B\mathcal{U}$ is contractible. □

5.3 Open subgroups

Lemma 5.5. *Let $B, C \subseteq M$ be finite with $B \cap C \neq \varnothing$. Then $\langle G_B, G_C\rangle = G_{B\cap C}$.*

Proof. Let $g \in G_{B\cap C}$. Put $D = B \cup C$, $E = D \cup gD$, $F = B \cap C$, and let $d_0\colon D \to E$ be the inclusion and $d_1 = g|_D$. Let Z, e, $a_0, \ldots, a_s$ and the pieces Z_i be as in Lemma 3.3. Each piece is a copy of E or of D and carries the corresponding two-graph. Two consecutive pieces induce the same two-graph on their intersection, which is the common image of B

or of C, because d_0 and d_1 are embeddings; call this $(*)$. Fix $p \in F$ and let $\hat{p}$ be the corresponding F-point of Z. Let b be the graph on Z with $b(u,u') = c_{Z_i}(\hat{p},u,u')$ if u and u' lie in a common piece Z_i, and $b(u,u') = 0$ if they lie in no common piece. This is well defined. Indeed, let $u,u' \in Z_i \cap Z_j$. If i and j are consecutive, apply $(*)$. If not, and u, u' are both F-points, note that all pieces induce the same two-graph on the set of F-points. Otherwise $\{i,j\} = \{1, s-1\}$ and $u,u' \in Z_0$ by Lemma 3.3(ii), and we apply $(*)$ to (Z_j, Z_0) and to (Z_0, Z_i). Give Z the two-graph ∂b. On a triple contained in a piece Z_i it equals $\partial((c_{Z_i})_{\hat{p}}) = c_{Z_i}$ by (6). So e and all a_i are embeddings of ordered two-graphs.

By (N2) there is an embedding $h\colon Z \to M$ with $he = \mathrm{id}_E$, and by (N3) $ha_i = g_i|_D$ with $g_i \in G$, $g_0 = 1$, $g_s = g$. By Lemma 3.3(i), $g_i^{-1} g_{i+1}$ lies in G_B for i even and in G_C for i odd, so $g \in \langle G_B, G_C \rangle$. □

Lemma 5.6. *For distinct points x, y of M we have $\langle G_{\{x\}}, G_{\{y\}} \rangle = G$.*

Proof. Let H be the subgroup on the left, and suppose $x < y$. A two-graph on a set with at most two elements is empty, so by (N3) the group G is transitive on M, and $G_{\{w\}}$ is transitive on $\{u : u < w\}$ and on $\{u : u > w\}$ for every $w \in M$. Note that H contains $G_{\{hx\}}$ and $G_{\{hy\}}$ for every $h \in H$. Let $u \in M$ and choose w greater than u and y. Then $w \in G_{\{x\}}\, y$, so $G_{\{w\}} \leq H$, and $u \in G_{\{w\}}\, x$. Hence H is transitive on M. For $g \in G$ choose $h \in H$ with $hx = gx$; then $h^{-1}g \in G_{\{x\}} \leq H$, so $g \in H$. □

Proposition 5.7. *For all finite $B, C \subseteq M$ we have $\langle G_B, G_C \rangle = G_{B \cap C}$, and $(M, G, \mathcal{K})$ satisfies* (K3).

Proof. If $B \cap C \neq \varnothing$ this is Lemma 5.5. Otherwise choose distinct points x, y outside $B \cup C$. By Lemma 5.5, $\langle G_B, G_C \rangle$ contains $\langle G_{B \cup \{x\}}, G_{C \cup \{x\}} \rangle = G_{\{x\}}$, and likewise $G_{\{y\}}$; so it is G by Lemma 5.6. Condition (K3) follows as in Corollary 4.9. □

5.4 The classifying space

Lemma 5.8. *$B\mathcal{T}$ is simply connected.*

Proof. $\mathcal{T}$ is connected, so by the correspondence between covering spaces and functors that invert all arrows (Section 2) it suffices to show that every functor $\Psi\colon \mathcal{T} \to \mathbf{Set}$ that sends all arrows to bijections is isomorphic to a constant functor. Let P and Q be the objects with one and with two points (there are no others with at most two points), and $d^1, d^0\colon P \to Q$ the arrows onto the first and the second point. Let X be an object with three points and $e_{ij}\colon Q \to X$ the embedding onto $\{i,j\}$. Then

$$e_{12}d^1 = e_{13}d^1, \qquad e_{13}d^0 = e_{23}d^0, \qquad e_{12}d^0 = e_{23}d^1.$$

Since Ψ takes values in bijections, the first two equations give $\Psi(e_{12}) = \Psi(e_{13}) = \Psi(e_{23})$, and then the third gives $\Psi(d^0) = \Psi(d^1)$. For an object Y and $u \in Y$ let $i_u\colon P \to Y$ be the arrow with image $\{u\}$. If $u < u'$ and $\epsilon\colon Q \to Y$ is the embedding onto $\{u,u'\}$, then $i_u = \epsilon d^1$ and $i_{u'} = \epsilon d^0$, so $\Psi(i_u) = \Psi(i_{u'})$. Thus $\theta_Y = \Psi(i_u)\colon \Psi(P) \to \Psi(Y)$ does not depend on u. For an arrow $f\colon Y \to Y'$ we have $fi_u = i_{f(u)}$, hence $\Psi(f)\theta_Y = \theta_{Y'}$. So θ is a natural isomorphism from the constant functor with value $\Psi(P)$ to Ψ. □

Lemma 5.9. *$H_2(B\mathcal{T}; \mathbb{F}_2) \neq 0$.*

Proof. We use the chains of the nerve with coefficients in $\mathbb{F}_2$. For a 2-simplex $\sigma = (X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2)$ let $m_0 \leq m_1 \leq m_2$ be the images in X_2 of the greatest elements of X_0, X_1, X_2, and put $\Phi(\sigma) = c_{X_2}(m_0, m_1, m_2)$. Then Φ is a cocycle. Indeed, for a 3-simplex

$X_0 \to X_1 \to X_2 \to X_3$ let $n_0 \le n_1 \le n_2 \le n_3$ be the images in X_3 of the greatest elements. Since $X_2 \to X_3$ is an embedding, the values of Φ on the four faces are the values of c_{X_3} on the four triples obtained by omitting one of the n_k. If the n_k are distinct, their sum is zero by the definition of a two-graph. If $n_k = n_{k+1}$ for some k, the triples obtained by omitting n_k or n_{k+1} give the same value, and the other two contain a repetition.

Let X and X' be the two objects with three points, with $c_X(1,2,3) = 1$ and $c_{X'}(1,2,3) = 0$. For $S_0 \subset S_1 \subset [3]$ with $|S_0| = 1$, $|S_1| = 2$ let $\sigma(S_0, S_1, X)$ be the 2-simplex $P \to Q \to X$ in which the image of Q is S_1 and the image of P is S_0; define $\sigma(S_0, S_1, X')$ likewise. Let ξ be the sum of these twelve simplices. It is a cycle: the face $P \to Q$ does not depend on the target X or X'; the face $Q \to X$ depends only on S_1 and so occurs twice, and the face $P \to X$ depends only on S_0 and occurs twice; similarly for X'. By definition $\Phi(\sigma(S_0, S_1, X)) = c_X(\max S_0, \max S_1, 3)$, which is non-zero only for $S_0 = \{1\}$, $S_1 = \{1, 2\}$, and Φ vanishes on the simplices with target X'. So $\Phi(\xi) = 1$, and ξ is not a boundary. □

Theorem 5.10. *Every action component of $\mathcal{T}$ is weakly contractible or weakly equivalent to $\mathcal{T}$. Consequently $\mathcal{T}$ is quasi-filtered, and it is not total.*

Proof. By Lemma 5.4 and Proposition 5.7 the triple $(M, G, \mathcal{K})$ satisfies (K1)–(K4), so the first assertion holds for $\mathcal{A}$ by Proposition 3.2, and hence for the equivalent category $\mathcal{T}$. By Lemma 5.8 all action components of $\mathcal{T}$ are simply connected (each is weakly contractible or weakly equivalent to $\mathcal{T}$), so $\mathcal{T}$ is quasi-filtered by Proposition 2.7. The category of elements of the functor with value a point is $\mathcal{T}$ itself, which is not weakly contractible by Lemma 5.9; so $\mathcal{T}$ is not total by Proposition 2.3. □

Remark 5.11. The run from which this example is taken identifies $B\mathcal{T}$ with an Eilenberg–Mac Lane space $K(\mathbb{F}_2, 2)$, and treats n-cocycles in the same way for every $n \ge 2$; see Section 7. Those statements are not used and not claimed here.

6 The passage of *Les Dérivateurs*

In Chapter II of [6] (manuscript pages 34–36ter; the last of these is on the printed page 23, page 25 of the PDF, of the edition's file for that chapter), Grothendieck shows that filtered categories are *totalement W-asphériques* (Corollaire 2.17; for a functor $f\colon X \to Y$ the condition is on the category of elements of the presheaf $y \mapsto \pi_0(y\backslash X)$, which is the condition of Definition 2.1(c)). He then asks for the converse, for the class W_∞ of weak equivalences: "cela implique-t-il que X soit filtrante ? Ça ne m'étonnerait pas" (ms. p. 36). His Proposition 2.19 answers in the negative: it asserts that X is totally W_∞-aspherical whenever the diagonal $X \to X \times X$ is final (*cofinal*, in his terminology), that is, whenever X is sifted; and a corollary states that the opposite $\mathbf{\Delta}^\circ$ of the category of simplices is totally W_∞-aspherical and not filtered. In his Contre-exemple 2.20 (ms. pp. 36bis–36ter) he derives from Proposition 2.19 that every connected simplicial set would be weakly contractible, and concludes: "Cela me fait soupçonner que la conjecture X totalement W_∞-asphérique $\Longrightarrow$ X filtrante pourrait être valable."

The results above give the following answers.

1. The conjecture is false. The category Δ_{inj} is exact by the theorems of Isbell and Mitchell [7, Theorems 1 and 3], hence total by Proposition 2.5, and it is not filtered. The category $\mathcal{C}_\delta$ is a second example, which is not even exact.

2. In the edition, Proposition 2.19 is marked false (footnote 43, and footnote 45 at the step of its proof that fails), and Grothendieck wrote his Contre-exemple 2.20

"à la proposition fausse 2.19" against it; the corollary falls with it, and indeed $\mathbf{\Delta}^\circ$ is not total. This follows from Proposition 2.3, since the action components of $\mathbf{\Delta}^\circ$ are the opposites of the categories of simplices of connected simplicial sets; it also follows from Proposition 2.8, since $\mathbf{\Delta}^\circ$ is sifted and not filtered. More generally, by Proposition 2.8 a sifted category is total only if it is filtered. (His Contre-exemple 2.20 uses Chapter XII, Théorème 4, Corollaire 4, which does not hold for W_∞: where it concludes that a category is weakly contractible, only the vanishing of its reduced integral homology follows. His conclusion stands, and the two arguments above do not use that step.)

3. The conjecture holds for sifted categories (Proposition 2.8), and it holds for categories with a weakly terminal object: there even quasi-filtered categories are filtered, by a theorem of Isbell and Mitchell [8, Corollary 4.4].

7 Provenance and checks

This paper is AI-produced. No human has proved, written or checked any part of it.

Sources. The mathematics was found in four runs of GPT-6 Astra (OpenAI, ChatGPT Pro mode), in November 2026. The run TOTAL_ASPHERICITY proved Proposition 2.3 (its statement C.5) and Proposition 2.5. The run PRESHEAF_TOTALITY proved Proposition 2.8 (its statement F.3). The run TOTAL_EXACT_2 found the category $\mathcal{C}_\delta$ and proved Theorem 1.1 (its statements A.1, A.3, with the proofs A.21, A.24, A.30–A.34), and stated Theorem 1.3 (D.1, D.2) with another third example, a modification of a construction of [1], whose proof is long and is not reproduced or used here. The run DER_FOLLOWUP found the cocycle categories, of which $\mathcal{T}$ is the first, and proved Theorem 1.2 (its statements A.3–A.8), for cocycles of every degree $n \geq 2$.

The text. The present text was written by Claude Fable 5.2 at effort max. The proofs of Sections 4 and 3 follow those of the runs, reorganized: the reduction is stated once (Proposition 3.2) and the bridge is split into a statement on ordered sets (Lemma 3.3) and its two uses. The following arguments are the writer's and differ from the runs': the proofs of Propositions 2.7 and 2.8 in the form given; the reduction of Proposition 5.7 to the case of a non-empty intersection, with Lemmas 5.5 and 5.6; and Lemmas 5.8 and 5.9.

Checks. The proofs of the run TOTAL_EXACT_2 used here were followed in full by two readers before this text was written (Claude Fable 5.2 at effort high, and Claude Fable 5.2 at effort max, 13 November 2026), and those of DER_FOLLOWUP by one (Claude Fable 5.2 at effort max, 17 November 2026). Every proof in this text was derived again by the writer while writing. A program independent of the runs searched all objects of $\mathcal{C}_\delta$ with at most seven vertices for an integral, rational or mod-p ($p = 2, 3, 5$) affine equalizer of v_1, v_2, v_3 and found none, as Theorem 4.11 predicts. The finished text was then checked independently by Claude Opus 5.5 at effort max (25 November 2026): every statement and every proof was found correct, the fact that $\mathcal{T}$ is not total was proved again by another route, and eight repairs of wording and citation were made, none of which changes a statement. So the proofs taken from the runs have been checked by models of a family other than the one that found them, and the arguments that are the writer's own have been checked by one other model, of the writer's family. Nothing is formalized. The record of every check, with model, effort and date, is in the file `CHECK_EVENTS.tsv` that accompanies this paper in its workbench, reached from `https://github.com/KokunoYumeto/`

`mathematics-workbenches`; `PROVENANCE.md` lists every contributor and says in which copy each citation was confirmed, and which citations are given only at the level of a table of contents.

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