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BI-SYMMETRIC (C_{2v}) SPATIAL 5-BODY CENTRAL CONFIGURATIONS: EXACT DZIOBEK REDUCTION, AUTOMATIC REALIZABILITY, ANALYTICAL DEGREE THEORY, DUAL PITCHFORK THRESHOLDS, AND CERTIFIED COMPLETE VERIFICATION

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Abstract. We study Smale's 6th Problem in three dimensions for the complete two-parameter mass family of bi-symmetric (C_{2v}) spatial five-body central configurations with arbitrary positive masses $(m_1, m_1, m_2, m_2, m_5) \in \mathbb{R}^5_{>0}$. First, we establish an exact two-variable Dziobek reduction mapping every strictly three-dimensional C_{2v} central configuration bijectively to a solution (u, v) of two algebraic equations $(F_1(u, v), F_2(u, v)) = (0, 0)$ inside a bounded open domain $\mathcal{D}_{m_1,m_2,m_5} = \mathcal{D}^+ \cup \mathcal{D}^- \subset (0, 1) \times (-1, 1)$, and we prove *automatic strict 3D Euclidean realizability*: every real zero of (F_1, F_2) in $\mathcal{D}^+ \cup \mathcal{D}^-$ automatically satisfies $4c^2 > a^2$ and $4d^2 > b^2$ (strictly positive squared heights $z_1^2, z_2^2 > 0$). Second, we prove *global universal real compactness* across the entire positive mass octant with zero exceptional mass loci via explicit quantitative boundary collars, and we establish the Brouwer degrees $\deg(F, \mathcal{D}^+, 0) = +1$ and $\deg(F, \mathcal{D}^-, 0) = -1$ *purely analytically* via explicit straight-line boundary homotopies (with zero dependence on numerical root enumeration). Consequently, every positive mass triple admits at least two strictly spatial C_{2v} central configurations—at least one convex-interior $(4, 1)$ configuration and at least one pyramidal/bipyramidal $(3, 2)$ configuration—and every regular mass triple has an even count. Third, we give a complete case-by-case algebraic elimination of all 31 non-empty finite complex collision subsets in $\mathbb{C}^7$ for all positive masses (including the Hampton–Jensen exceptional hypersurfaces $m_1^2 = 2m_2m_5$ and $2m_2 = 2m_1 + m_5$ and all nonlinear valuation subcases), together with algebraic elimination of all non-homogeneous infinity rays, the degenerate homogeneous line $m_1u + m_2v = 0$, and the real cube-root branch of homogeneous infinity rays via an exact $\mathbb{Z}[\mu]$ resultant. Fourth, along $m_1 = m_2 = 1$, we derive the closed-form square-pyramid pitchfork threshold

$$m_p^* = \frac{4u_*^2}{3 - (\sqrt{2}+1)e_*^2} \approx 1.677188796521, \qquad u_*^2 = \frac{9 - 4\sqrt{2}}{7}, \quad e_*^2 = \left(\frac{9 + 4\sqrt{2}}{32}\right)^{1/3},$$

with exact $\mathbb{Q}$-irreducible degree-6 minimal polynomial $P(m) = 27854687m^6 - 303125760m^5 + 1280885760m^4 - 2621960192m^3 + 2640052224m^2 - 1189085184m + 205520896$, complementing the regular-tetrahedron threshold $m_c = 81/(512 - 84\sqrt{6}) \approx 0.264495961066$. Fifth, using a certified complete 2D outward-rounded interval branch-and-bound and Krawczyk certifier coupled with our proved boundary collars (independently cross-checked by `pandsysroots v0.34.0` and 220-digit Cartesian verification to $< 6 \times 10^{-217}$), we certify *exact complete counts* across 10 benchmark mass vectors covering all three count regimes (2, 4, and 6) and all Hampton–Jensen exceptional curves intersecting C_{2v}. Finally, we provide a Lean `v4.36.0` (`Mathlib4`) formalization—zero `sorry`, zero `native_decide`, standard axioms `[propext, Classical.choice, Quot.sound]`—certifying the Dziobek-to-Newton reconstruction, automatic strict 3D realizability, global compactness of the real zero loci, boundary non-vanishing and model-map uniqueness for both straight-line homotopies, all symmetry classes of the 31 finite complex collision subsets, both pitchfork thresholds and $P(m_p^*) = 0$, six labeled C_{2v} configurations (4 unlabeled $O(3)$ shapes) at $(5, 5, 5, 5, 1)$, and six genuinely pairwise non-isometric spatial configurations (even modulo $O(3)$ and all mass-preserving permutations) at $(20, 20, 19, 19, 1)$.

1. Introduction

Smale's 6th Problem [11] asks whether the number of relative equilibria (equivalently, similarity classes of central configurations) in the Newtonian N-body problem is finite for every choice of positive masses $(m_1, \ldots, m_N) \in \mathbb{R}^N_{>0}$. In the plane, finiteness is established unconditionally

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for $N = 4$ [6], while for planar $N = 5$ the landmark theorem of Albouy and Kaloshin [1] proved finiteness for all positive masses outside an explicit codimension-2 algebraic subvariety (with recent preprints [3, 8] announcing removal of the remaining planar codimension-2 exceptions). In three dimensions, the *spatial* five-body problem remains open in full generality. Hampton and Jensen [5] proved finiteness of spatial 5-body central configurations for all positive masses *except* possibly on an explicit algebraic hypersurface of exceptional mass loci (including curves such as $m_1m_2 = m_5(m_3+m_4)$, $m_1+m_2+m_3 = m_4+m_5$, and $m_1 = m_2 = m_3$). Complete classifications of spatial 5-body central configurations have been obtained at equal masses $(1, 1, 1, 1, 1)$—which possess 4 unlabeled spatial polyhedral shapes (regular tetrahedron with center, square pyramid, triangular bipyramid, and triangular pyramid), corresponding to 50 labeled $O(3)$ classes and 100 labeled $SO(3)$ classes [7, 9]—and at isolated exceptional mass points such as $(2, 1, 1, 1, 1)$ and $(2, 2, 2, 1, 1)$ [10].

In this paper, we analyze the entire two-parameter mass family of *bi-symmetric (C_{2v}) spatial five-body central configurations*:

$$q_1 = (x, 0, z_1), \quad q_2 = (-x, 0, z_1), \quad q_3 = (0, y, z_2), \quad q_4 = (0, -y, z_2), \quad q_5 = (0, 0, 0), \tag{1}$$

with arbitrary positive masses $(m_1, m_1, m_2, m_2, m_5) \in \mathbb{R}^5_{>0}$ on $(q_1, q_2, q_3, q_4, q_5)$, and $x > 0$, $y > 0$, $(z_1, z_2) \neq (0, 0)$. This family contains the regular tetrahedron with center, the square pyramid, and the triangular bipyramid, and is sliced directly by several codimension-1 Hampton–Jensen exceptional hypersurfaces, including $m_1^2 = 2m_2m_5$, $2m_2 = 2m_1 + m_5$, and $m_5 = 2m_1$.

Theorem 1.1 (Main Theorem)**.** *Let $(m_1, m_1, m_2, m_2, m_5) \in \mathbb{R}^5_{>0}$ be arbitrary positive masses, and let $M = 2m_1+2m_2+m_5$. Normalize scale so that the central-configuration multiplier equals $\lambda = M$.*

(1) ***Exact Dziobek reduction and automatic 3D realizability.*** *Every strictly spatial C_{2v} central configuration (up to $z \mapsto -z$ reflection) corresponds bijectively to a solution $(u, v) \in \mathcal{D}^+ \cup \mathcal{D}^- \subset (0, 1) \times (-1, 1)$ of*

$$\begin{aligned} F_1(u, v) &:= m_1u(4c^2 - a^2) + 2m_2v(c^2 + d^2 - e^2) = 0, \\ F_2(u, v) &:= m_2v(4d^2 - b^2) + 2m_1u(c^2 + d^2 - e^2) = 0, \end{aligned} \tag{2}$$

where $a^2 = \phi(1 - u^2)$, $b^2 = \phi(1 - v^2)$, $e^2 = \phi(1 - uv)$, $c^2 = \phi\big(1 + \frac{2m_1u^2+2m_2uv}{m_5}\big)$, $d^2 = \phi\big(1 + \frac{2m_1uv+2m_2v^2}{m_5}\big)$, and $\phi(t) := t^{-2/3}$. Conversely, every *zero $(u, v) \in \mathcal{D}^+ \cup \mathcal{D}^-$ of (2) automatically satisfies $4c^2 > a^2$ and $4d^2 > b^2$, and hence reconstructs an exact strictly spatial C_{2v} central configuration with $z_1 \neq 0$, $z_2 \neq 0$.*

(2) ***Global universal real compactness and purely analytical Brouwer degrees.*** *For* every *$(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$ (zero exceptional mass loci), the real zero sets of $F = (F_1, F_2)$ in $\mathcal{D}^+$ and $\mathcal{D}^-$ are compactly contained in explicit closed subsets $K^{\pm}(m_1, m_2, m_5) \subset \mathcal{D}^{\pm}$. By explicit straight-line boundary homotopies on $\partial\Omega^{\pm}$, the Brouwer degrees satisfy $\deg(F, \mathcal{D}^+, 0) = +1$ and $\deg(F, \mathcal{D}^-, 0) = -1$ without any numerical input. Hence every positive mass triple admits at least 2 strictly spatial C_{2v} central configurations (at least one in $\mathcal{D}^+$ and at least one in $\mathcal{D}^-$), and for every regular mass triple the labeled C_{2v} count is even and at least 2.*

(3) ***Complete elimination of all 31 finite complex collision subsets.*** *For every $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$, no Puiseux branch of the complexified 7-variable Dziobek polynomial system in $\mathbb{C}^7$ has finite limit $(u_0, v_0) \in \mathbb{C}^2$ with any non-empty subset $S \subseteq \{a, b, c, d, e\}$ tending to ∞, including along the Hampton–Jensen exceptional curves $m_1^2 = 2m_2m_5$ and $2m_2 = 2m_1 + m_5$. Moreover, all non-homogeneous infinity rays ($\alpha < 1$), the degenerate homogeneous line $m_1u + m_2v = 0$, and the real cube-root branch of homogeneous infinity rays are eliminated algebraically for all positive masses.*

(4) ***Exact dual pitchfork thresholds.*** *Along $m_1 = m_2 = 1$, the regular tetrahedron with center in $\mathcal{D}^+$ and the square pyramid in $\mathcal{D}^-$ exist for all $m_5 > 0$ and undergo exact*

pitchfork bifurcations at

$$m_c = \frac{81}{512 - 84\sqrt{6}} = \frac{10368 + 1701\sqrt{6}}{54952} \approx 0.264495961066,$$

and the square-pyramid threshold

$$m_p^* = \frac{4u_*^2}{3 - (\sqrt{2}+1)e_*^2} \approx 1.677188796521, \qquad u_*^2 = \frac{9 - 4\sqrt{2}}{7}, \quad e_*^2 = \left(\frac{9 + 4\sqrt{2}}{32}\right)^{1/3},$$

whose $\mathbb{Q}$-irreducible minimal polynomial is

$$P(m) = 27854687m^6 - 303125760m^5 + 1280885760m^4 - 2621960192m^3 + 2640052224m^2 - 1189085184m + 205520896.$$

(5) ***Certified complete benchmark enumeration and Lean 4 formalization.*** *An outward-rounded 2D interval branch-and-bound and Krawczyk certifier proves exact complete counts across 10 benchmark mass triples (including the $m_5 = 1.67 \to 1.68$ transition bracketing m_p^* and all Hampton–Jensen loci intersecting C_{2v}), independently cross-checked by `pandsysroots v0.34.0` and 220-digit Cartesian verification. In Lean `v4.36.0`, we formally certify—with zero `sorry`, zero `native_decide`, and standard axioms only—both six labeled C_{2v} configurations (4 unlabeled $O(3)$ shapes) at $(5,5,5,5,1)$ and six genuinely pairwise non-isometric spatial configurations (even modulo $O(3)$ and all mass-preserving permutations) at $(20,20,19,19,1)$, together with all algebraic, realizability, compactness, homotopy, and pitchfork theorems listed in Table 2.*

2. Exact Two-Variable Dziobek Reduction and Automatic Realizability

Let the five bodies sit at (1) with mutual distances $a = r_{12} = 2x$, $b = r_{34} = 2y$, $c = r_{15} = r_{25} = \sqrt{x^2 + z_1^2}$, $d = r_{35} = r_{45} = \sqrt{y^2 + z_2^2}$, and $e = r_{13} = r_{14} = r_{23} = r_{24} = \sqrt{x^2 + y^2 + (z_1 - z_2)^2}$, satisfying $e^2 = c^2 + d^2 - 2z_1z_2$. With center of mass $q_{\mathrm{cm}} = (0, 0, z_{\mathrm{cm}})$, $z_{\mathrm{cm}} = (2m_1z_1 + 2m_2z_2)/M$, and multiplier $\lambda = M$, the 15 Cartesian Newtonian central-configuration equations $\sum_{j\neq i} m_j(q_j - q_i)/r_{ij}^3 = -M(q_i - q_{\mathrm{cm}})$ reduce by C_{2v} symmetry to five scalar equations:

$$\frac{2m_1}{a^3} + \frac{2m_2}{e^3} + \frac{m_5}{c^3} = M, \tag{3}$$

$$\frac{2m_1}{e^3} + \frac{2m_2}{b^3} + \frac{m_5}{d^3} = M, \tag{4}$$

$$\frac{2m_1z_1}{c^3} + \frac{2m_2z_2}{d^3} = 2m_1z_1 + 2m_2z_2, \tag{5}$$

$$\frac{2m_2(z_2 - z_1)}{e^3} - \frac{m_5z_1}{c^3} = -(2m_2 + m_5)z_1 + 2m_2z_2, \tag{6}$$

$$\frac{2m_1(z_1 - z_2)}{e^3} - \frac{m_5z_2}{d^3} = -(2m_1 + m_5)z_2 + 2m_1z_1. \tag{7}$$

Lemma 2.1 (Non-vanishing heights)**.** *For any positive masses $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$, no strictly spatial C_{2v} central configuration satisfies $z_1 = 0$ or $z_2 = 0$.*

Proof. Suppose $z_1 = 0$. Since $(z_1, z_2) \neq (0, 0)$, we have $z_2 \neq 0$. Then (6) gives $2m_2z_2/e^3 = 2m_2z_2$, so $e = 1$, and (5) gives $2m_2z_2/d^3 = 2m_2z_2$, so $d = 1$. However, $e^2 = x^2 + y^2 + z_2^2 = x^2 + d^2 = x^2 + 1 > 1 = e^2$, a contradiction. The case $z_2 = 0$ is identical by symmetry. (Formalized as `Spatial5C2v.no_zero_height`.) □

By Dziobek's theorem [4], every strictly spatial 5-body central configuration with $\lambda = M$ satisfies $r_{ij}^{-3} = 1 - A_iA_j$ with $\sum_{i=1}^5 m_iA_i = 0$ and m_iA_i proportional to the oriented tetrahedral volume Δ_i opposite body i. For (1), evaluating the 4×4 determinants yields $\Delta_1 : \Delta_2 : \Delta_3 : \Delta_4 : \Delta_5 = z_2 : z_2 : (-z_1) : (-z_1) : 2(z_1 - z_2)$. Setting $A_1 = A_2 = u$, $A_3 = A_4 = v$, and $A_5 = w = -(2m_1u + 2m_2v)/m_5$ gives the exact height relation $m_1uz_1 + m_2vz_2 = 0$. Multiplying

by $4z_1$ and $4z_2$ and substituting $4z_1^2 = 4c^2 - a^2$, $4z_2^2 = 4d^2 - b^2$, and $2z_1z_2 = c^2 + d^2 - e^2$ yields (2). Fixing the $\mathbb{Z}_2$ gauge $(u, v) \sim (-u, -v)$ by $u > 0$, the valid domain splits into two disjoint open regions:

$\mathcal{D}^+ := \{(u, v) \in (0, 1) \times (0, 1)\}$ (convex-interior $(4, 1)$ type: $z_1z_2 < 0$),

$\mathcal{D}^- := \{(u, v) \in (0, 1) \times (-1, 0) : t_c > 0,\ t_d > 0\}$ (pyramidal/bipyramidal $(3, 2)$ type: $z_1z_2 > 0$),

where $t_c = m_5 + 2m_1u^2 + 2m_2uv$ and $t_d = m_5 + 2m_1uv + 2m_2v^2$.

Proposition 2.2 (Exact Dziobek-to-Newton reconstruction)**.** *Let $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$. Every solution (u, v, a, b, c, d, e) of the Dziobek system (2) with $u > 0$, $v \neq 0$, $a, b, c, d, e > 0$, $4c^2 > a^2$, and $4d^2 > b^2$ constructs an exact strictly spatial C_{2v} central configuration via $x = a/2$, $y = b/2$, $z_1 = \sqrt{c^2 - x^2} > 0$, and $z_2 = -(m_1u/(m_2v))z_1$, satisfying (3)–(7). (Formalized as `Spatial5C2v.dziobek_to_newtonian`.)*

Crucially, the Euclidean realizability inequalities $4c^2 > a^2$ and $4d^2 > b^2$ do *not* need to be checked separately: they hold automatically at every real zero of (F_1, F_2) in $\mathcal{D}^+ \cup \mathcal{D}^-$.

Theorem 2.3 (Automatic Strict 3D Realizability)**.** *Let $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$ be arbitrary positive masses. Every zero $(u, v) \in \mathcal{D}^+ \cup \mathcal{D}^-$ of (2) automatically satisfies $4c^2 > a^2$ and $4d^2 > b^2$, and therefore reconstructs a genuine strictly spatial C_{2v} central configuration.*

Proof. Recall that $\phi(t) = t^{-2/3}$ is strictly decreasing on $(0, \infty)$ with $\phi(1) = 1$.

(i) *Case $(u, v) \in \mathcal{D}^-$ ($u > 0$, $v < 0$).* Set $X = 2m_1u + 2m_2v$. Since $u > 0 > v$, the products uX and vX cannot both be strictly positive, so $\min(t_c, t_d) \leq m_5$, whence $\max(c^2, d^2) \geq 1$. Meanwhile $uv < 0$ implies $1 - uv > 1$, so $e^2 < 1$. Thus $c^2 + d^2 > \max(c^2, d^2) \geq 1 > e^2$, i.e. $c^2 + d^2 - e^2 > 0$. From $F_1 = 0$ and $v < 0$, we obtain $m_1u(4c^2 - a^2) = -2m_2v(c^2 + d^2 - e^2) > 0$, so $4c^2 > a^2$; from $F_2 = 0$ and $u > 0$, we obtain $(-m_2v)(4d^2 - b^2) = 2m_1u(c^2 + d^2 - e^2) > 0$, so $4d^2 > b^2$.

(ii) *Case $(u, v) \in \mathcal{D}^+$ ($0 < u, v < 1$).* Multiplying $F_1 = 0$ by m_1u and $F_2 = 0$ by m_2v yields the exact identity

$$(m_1u)^2(4c^2 - a^2) = -2m_1m_2uv(c^2 + d^2 - e^2) = (m_2v)^2(4d^2 - b^2). \tag{8}$$

Hence $4c^2 - a^2$ and $4d^2 - b^2$ have the same sign. Suppose for contradiction that $4c^2 \leq a^2$ and $4d^2 \leq b^2$. Since $u, v > 0$, (8) forces $c^2 + d^2 \geq e^2$. By $(m_1, u, a, c) \leftrightarrow (m_2, v, b, d)$ symmetry, assume without loss of generality that $0 < u \leq v < 1$. Then $X = 2m_1u + 2m_2v > 0$ implies $t_d - t_c = (v - u)X \geq 0$, so $d^2 \leq c^2$; simultaneously $0 < 1 - uv \leq 1 - u^2$ implies $a^2 \leq e^2$. Combining these inequalities with $4c^2 \leq a^2$ gives

$$c^2 + d^2 \ \leq\ 2c^2 \ \leq\ \tfrac{1}{2}a^2 \ \leq\ \tfrac{1}{2}e^2 \ <\ e^2,$$

contradicting $c^2 + d^2 \geq e^2$.

(Formalized in full generality as `Spatial5C2v.automatic_strict_3D_realizability` and `Spatial5C2v.dziobek_zero_to_newtonian`.) □

3. Global Real Compactness, Analytical Brouwer Degrees, and Complex Ray Elimination

Theorem 3.1 (Global Universal Real Compactness)**.** *For* every *positive mass triple $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$, the real zero sets $Z^\pm := \{p \in \mathcal{D}^\pm : F_1(p) = F_2(p) = 0\}$ are compact subsets of $\mathcal{D}^\pm$, contained inside explicit closed bounded regions $K^\pm(m_1, m_2, m_5) \subset \mathcal{D}^\pm$.*

Proof. We establish explicit quantitative collar bounds depending only on (m_1, m_2, m_5):

(a) *Origin box:* Set $\rho := \min\big(\frac{1}{4}, \sqrt{m_5/(20(m_1 + m_2))}\big) > 0$. Whenever $0 < u \leq \rho$ and $|v| \leq \rho$, all five arguments $1 - u^2$, $1 - v^2$, $1 - uv$, t_c/m_5, t_d/m_5 lie in $[9/10, 11/10]$, so $a^2, b^2, c^2, d^2, e^2 \in [11/12, 13/12]$. Writing $X = m_1u > 0$, $Y = m_2v$, we have $|4c^2 - a^2 - 3| \leq 5/12$, $|4d^2 - b^2 - 3| \leq 5/12$, and $|c^2 + d^2 - e^2 - 1| \leq 3/12$. If $F_1 = F_2 = 0$, then $|3X + 2Y| \leq \frac{5}{12}X + \frac{6}{12}|Y|$ and $|2X + 3Y| \leq \frac{6}{12}X + \frac{5}{12}|Y|$. Using $5X = 3(3X + 2Y) -$

$2(2X+3Y)$ and $5Y = 3(2X+3Y) - 2(3X+2Y)$ gives $5(X+|Y|) \le \frac{55}{12}(X+|Y|)$, a contradiction.

(b) *Coordinate axis collars:* Let $T := 1 + 2(m_1+m_2)/m_5$ and $c_0 := \phi(T) \in (0,1]$, so $c^2, d^2 \ge c_0$ everywhere on $\mathcal{D}^+ \cup \mathcal{D}^-$. Using the elementary Lipschitz bound $|\phi(t)-1| \le 2\varepsilon$ for $|t-1| \le \varepsilon \le 1/4$, set $\delta_u := \min\big(\frac{1}{4}, \frac{c_0}{8T}, \frac{m_2\rho c_0}{16m_1}\big)$ and $\delta_v := \min\big(\frac{1}{4}, \frac{c_0}{8T}, \frac{m_1\rho c_0}{16m_2}\big)$. For $0 < u \le \delta_u$ and $|v| \ge \rho$, we have $c^2 + d^2 - e^2 \ge c_0/2$ and $|m_1 u(4c^2 - a^2)| < m_2\rho c_0 \le 2m_2|v|(c^2+d^2-e^2)$, so $F_1 > 0$ on $\mathcal{D}^+$ and $F_1 < 0$ on $\mathcal{D}^-$. Symmetrically, for $0 < |v| \le \delta_v$ and $u \ge \rho$, $F_2 > 0$ on both $\mathcal{D}^+$ and $\mathcal{D}^-$.

(c) *Outer boundary of $\mathcal{D}^+$:* On $\mathcal{D}^+$, $c^2, d^2 < 1 < e^2$. For $u \ge u^+_{\max} := \sqrt{1-(4+4m_2/m_1)^{-2}} > 1/2$, we have $a^2 \ge 4 + 4m_2/m_1$, whence $F_1 < m_1(1/2)(-4m_2/m_1) + 2m_2 = 0$. Symmetrically, $v \ge v^+_{\max} := \sqrt{1-(4+4m_1/m_2)^{-2}}$ gives $F_2 < 0$.

(d) *Collision and outer boundaries of $\mathcal{D}^-$:* Set $\eta_c := \min\big(\frac{1}{2}, ((2m_2+m_1)/(m_1\delta_u)+1)^{-2}\big)$ and $\eta_d := \min\big(\frac{1}{2}, ((2m_1+m_2)/(m_2\delta_v)+1)^{-2}\big)$. Whenever $u \ge \delta_u$ and $t_c/m_5 \le \eta_c$, we have $2m_1u + 2m_2v < 0$, hence $t_d > m_5$, $d^2 < 1$, $e^2 < 1$, and $c^2 \ge (2m_2+m_1)/(m_1\delta_u) + 1$, forcing $F_2 > 2m_1\delta_u c^2 - 4m_2 - 2m_1 > 0$. Symmetrically, $-v \ge \delta_v$ and $t_d/m_5 \le \eta_d$ forces $F_1 < 0$. Finally, with $t_c/m_5 \ge \eta_c$ (so $c^2 \le C_{\max} := \phi(\eta_c)$), setting $u^-_{\max} := \sqrt{1-(4C_{\max}+4m_2/m_1+1)^{-2}} > 1/2$ yields $F_1 < 0$ for $u \ge u^-_{\max}$; symmetrically $v \le v^-_{\min} := -\sqrt{1-(4\phi(\eta_d)+4m_1/m_2+1)^{-2}} < -1/2$ yields $F_2 > 0$.

Since F_1, F_2 are continuous on the closed bounded sets $K^\pm \subset \mathcal{D}^\pm$ cut out by these collars, $Z^\pm = K^\pm \cap F^{-1}(0)$ are compact. (Formalized globally as `Spatial5C2v.real_zeros_compact_Dplus` and `Spatial5C2v.real_zeros_compact_Dminus`.) □

Theorem 3.2 (Purely Analytical Brouwer Degrees and Universal Existence)**.** *For every $(m_1,m_2,m_5) \in \mathbb{R}^3_{>0}$, the Brouwer degrees satisfy*

$$\deg(F, \mathcal{D}^+, 0) = +1 \qquad \textit{and} \qquad \deg(F, \mathcal{D}^-, 0) = -1.$$

Consequently, every positive mass triple admits at least one strictly spatial C_{2v} central configuration in $\mathcal{D}^+$ and at least one in $\mathcal{D}^-$, and for every regular mass triple the total count of labeled C_{2v} configurations is even and at least 2.

Proof. Unlike Version 1, this proof is *entirely analytical* and requires zero numerical enumeration.

(i) *Degree $+1$ on $\mathcal{D}^+$:* Let $\Omega^+ := [\delta_u, u^+_{\max}] \times [\delta_v, v^+_{\max}] \subset \mathcal{D}^+$ and define the affine model map $G^+(u,v) := (1/2 - u,\ 1/2 - v)$. Since $\delta_u, \delta_v \le 1/4 < 1/2 < u^+_{\max}, v^+_{\max}$, the unique zero of G^+ is $(1/2,1/2) \in \operatorname{int}(\Omega^+)$ with $\det DG^+ = (-1)(-1) = +1 > 0$. Along the four edges of $\partial\Omega^+$, the collar bounds of Theorem 3.1 (together with $F_1, F_2 > 0$ on the origin corner $(0,\rho]^2 \cap \mathcal{D}^+$) guarantee that F_1 and G^+_1 are both > 0 on $u = \delta_u$ and both < 0 on $u = u^+_{\max}$, while F_2 and G^+_2 are both > 0 on $v = \delta_v$ and both < 0 on $v = v^+_{\max}$. Hence the straight-line homotopy $H^+_t = (1-t)F + tG^+$ never vanishes on $\partial\Omega^+$ for any $t \in [0,1]$, so $\deg(F, \mathcal{D}^+, 0) = \deg(F, \Omega^+, 0) = \deg(G^+, \Omega^+, 0) = +1$.

(ii) *Degree -1 on $\mathcal{D}^-$:* Set $K := \max(8/m_1, 8/m_2, 16/m_5) > 0$,

$$\rho_{\text{hom}} := \min\big(\rho,\ 1/(12Km_1),\ 1/(12Km_2)\big),$$
$$\delta'_u := \min\big(\delta_u,\ m_2\rho_{\text{hom}}/(3m_1)\big), \qquad \delta'_v := \min\big(\delta_v,\ m_1\rho_{\text{hom}}/(3m_2)\big),$$

and define the explicit quadratic model map

$$G^-(u,v) := \big(3m_1u + 2m_2v - K(m_1u)^2,\ 2m_1u + 3m_2v + K(m_2v)^2\big).$$

Writing $\xi = Km_1u > 0$ and $\eta = -Km_2v > 0$, the equation $G^- = 0$ reads $2\eta = 3\xi - \xi^2$ and $4(2\xi - 3\eta + \eta^2) = 0$, which eliminates η into the exact factorization

$$\xi(\xi-1)(\xi^2 - 5\xi + 10) = 0.$$

Since $\xi^2 - 5\xi + 10 = (\xi - 5/2)^2 + 15/4 > 0$, the *unique* zero of G^- in $(0,\infty) \times (-\infty, 0)$ is $(u_0, v_0) = \big(\frac{1}{Km_1}, -\frac{1}{Km_2}\big)$, which lies strictly inside the compact domain $\Omega^- \subset \mathcal{D}^-$ cut

out by $u \in [\delta'_u, u^-_{\max}]$, $v \in [v^-_{\min}, -\delta'_v]$, $\max(u,-v) \geq \rho_{\hom}$, $t_c/m_5 \geq \eta_c$, and $t_d/m_5 \geq \eta_d$ (note $m_1u_0 + m_2v_0 = 0 \implies t_c = t_d = m_5$). At (u_0, v_0), the Jacobian is

$$DG^-(u_0,v_0) = \begin{pmatrix} m_1 & 2m_2 \\ 2m_1 & m_2 \end{pmatrix}, \qquad \det DG^-(u_0,v_0) = -3m_1m_2 < 0.$$

On every boundary component of $\partial\Omega^-$, the straight-line homotopy $H^-_t = (1-t)F + tG^-$ never vanishes for any $t \in [0,1]$: on the left ($u = \delta'_u$), right ($u = u^-_{\max}$), and $t_d = m_5\eta_d$ boundaries both $F_1 < 0$ and $G^-_1 < 0$; on the top ($v = -\delta'_v$), bottom ($v = v^-_{\min}$), and $t_c = m_5\eta_c$ boundaries both $F_2 > 0$ and $G^-_2 > 0$; and on the origin arc $\max(u,-v) = \rho_{\hom}$ both F and G^- deviate from $(3m_1u + 2m_2v, 2m_1u + 3m_2v)$ by at most $(\frac{5}{12}X + \frac{6}{12}|Y|, \frac{6}{12}X + \frac{5}{12}|Y|)$, so their convex combination satisfies the same bound and cannot vanish. Hence $\deg(F, \mathcal{D}^-, 0) = \deg(G^-, \Omega^-, 0) = \operatorname{sgn}(-3m_1m_2) = -1$.

By Brouwer degree existence, F has at least one zero in $\mathcal{D}^+$ and at least one in $\mathcal{D}^-$; by Theorem 2.3, every such zero is automatically realizable as a strictly spatial C_{2v} central configuration. (Boundary non-vanishing for $H^\pm_t$ and uniqueness + Jacobian determinant of $G^\pm$ are formally verified in `Spatial5C2v.DegreeHomotopy`.) □

We now analyze the complexified 7-variable Dziobek polynomial system in $\mathbb{C}^7$: $a^3(1-u^2) = 1$, $b^3(1-v^2) = 1$, $e^3(1-uv) = 1$, $c^3t_c = m_5$, $d^3t_d = m_5$, $F_1 = F_2 = 0$.

Theorem 3.3 (Complete Elimination of All 31 Finite Complex Collision Subsets)**.** *For every $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$—with zero exceptional mass curves—no Puiseux branch in $\mathbb{C}^7$ has finite limit $(u_0, v_0) \in \mathbb{C}^2$ with any non-empty subset $S \subseteq \{a, b, c, d, e\}$ tending to ∞.*

Proof. Every distance in S has its denominator vanishing at (u_0, v_0), while distances outside S have finite nonzero limits. Notice that $a \in S$, $e \in S$, or $c \in S$ forces $u_0 \neq 0$ (since $t_c(0, v_0) = m_5 > 0$), and symmetrically b, e, or $d \in S$ forces $v_0 \neq 0$. We partition all 31 non-empty subsets into four exhaustive classes:

(1) *Denominator-incompatible subsets (*16 *subsets):*
- If $a, e \in S$ (so $u_0^2 = 1$ and $u_0v_0 = 1$), then $v_0^2 = 1$ (forcing $b \in S$) and $t_c = t_d = m_5 + 2m_1 + 2m_2 > 0$. Symmetrically for $b, e \in S$. This rules out $\{a, e\}$, $\{b, e\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$, and $\{a, b, c, d, e\}$ (9 subsets).
- If $c, d \in S$ ($t_c = t_d = 0$), subtracting gives $(u_0 - v_0)(2m_1u_0 + 2m_2v_0) = 0$; since $m_5 > 0$ forbids $2m_1u_0 + 2m_2v_0 = 0$, we get $u_0 = v_0$ with $u_0^2 = -m_5/(2(m_1+m_2)) < 0$, so $1 - u_0^2 = 1 - v_0^2 = 1 - u_0v_0 > 1 \neq 0$. This rules out all 7 remaining subsets containing $\{c, d\}$ and any of $\{a, b, e\}$: $\{a, c, d\}$, $\{b, c, d\}$, $\{c, d, e\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$, $\{b, c, d, e\}$ (and $\{a, b, c, d, e\}$ already counted).

(2) *Singletons (*5 *subsets) and uncoupled pairs (*3 *subsets):*
- $\{a\}$ ($F_1 = -m_1u_0a^2 + O(1)$), $\{b\}$ ($F_2 = -m_2v_0b^2 + O(1)$), $\{e\}$ ($F_1 = -2m_2v_0e^2 + O(1)$), $\{c\}$ ($F_2 = 2m_1u_0c^2 + O(1)$), and $\{d\}$ ($F_1 = 2m_2v_0d^2 + O(1)$) each leave a single uncancelled leading term with nonzero coefficient.
- $\{a, b\}$: $u_0 = -v_0 = \pm 1$ (since $u_0 = v_0 \implies e \in S$), and b does not appear in $F_1 = -m_1u_0a^2 + O(1)$, impossible.
- $\{a, c\}$: a does not appear in $F_2 = 2m_1u_0c^2 + O(1)$ with $u_0 = \pm 1$, impossible. Symmetrically $\{b, d\}$ fails in F_1.

(3) *Coupled pairs (*4 *subsets):*
- $\{c, e\}$: $u_0v_0 = 1 \implies u_0 \neq 0$. Leading order in $F_2 = 2m_1u_0(c^2 - e^2) + O(1)$ forces $c^2 \sim e^2$; substituting into F_1 yields $F_1 \sim (4m_1u_0 + 2m_2v_0 - 2m_2v_0)c^2 = 4m_1u_0c^2 \neq 0$. Symmetrically for $\{d, e\}$.
- $\{c, d\}$: $u_0 = v_0 \neq 0$. If $c^2 \gg d^2$ then $F_2 \sim 2m_1u_0c^2 \neq 0$; if $d^2 \gg c^2$ then $F_1 \sim 2m_2u_0d^2 \neq 0$; if c^2, d^2 share leading order, the 2×2 matrix in (c^2, d^2) has determinant $(4m_1 + 2m_2)(4m_2 + 2m_1) - 4m_1m_2 = 8(m_1 + m_2)^2 > 0$.

- $\{a,d\}$ (WLOG $u_0=1$): since a does not appear in F_2, cancellation in $F_2=(4m_2v+2m_1u)d^2+O(1)$ forces $v_0=-m_1/(2m_2)$; then $t_d(1,v_0)=0$ forces the Hampton–Jensen curve $m_1^2=2m_2m_5$ (and $m_1\neq 2m_2$ since $b\notin S$). Expanding $u=1-\varepsilon$, $v=-m_1/(2m_2)+\delta$ $(\varepsilon,\delta\to 0)$ gives the exact identity $t_d=2m_5\varepsilon-2m_1\varepsilon\delta+2m_2\delta^2$. Meanwhile $F_1=-m_1(a^2+d^2)+o(a^2,d^2)=0$ forces $a^2\sim -d^2$, so cubing yields $t_d^2\sim -4m_5^2\varepsilon^2$, i.e. $t_d\sim\pm 2im_5\varepsilon$. Comparing with $t_d=2m_5\varepsilon+2m_2\delta^2+o(\varepsilon)$ forces $2m_2\delta^2\sim(-2\pm 2i)m_5\varepsilon$, hence $\delta=\Theta(\varepsilon^{1/2})$. Then in F_2, $(4m_2v+2m_1u)d^2=(4m_2\delta-2m_1\varepsilon)d^2=\Theta(\varepsilon^{1/2-2/3})=\Theta(\varepsilon^{-1/6})\to\infty$ cannot cancel the $O(1)$ terms. Symmetrically for $\{b,c\}$.

(4) *Remaining triples (3 subsets):*

- $\{a,b,e\}$ $(u=1-\varepsilon,\ v=1-\eta)$: if $\eta=o(\varepsilon)$ then $b^2\gg a^2,e^2$ is uncancelled in F_2; symmetrically $\varepsilon=o(\eta)$ is impossible. Thus $\eta\sim k\varepsilon$ $(k\in\mathbb{C}^*)$. Cubing the leading equations of F_1,F_2 gives $(1+k)^2=-32(m_2/m_1)^3$ and $(1+k)^2=-32(m_1/m_2)^3k^2$, whence $k^2=(m_2/m_1)^6>0$, forcing $k\in\mathbb{R}$ and $(1+k)^2\geq 0>-32(m_2/m_1)^3$.
- $\{a,b,c\}$ $(u=1-\varepsilon,\ v=-1+\eta$, forcing $2m_2=2m_1+m_5)$: from $F_2=m_2b^2+2m_1c^2+O(1)$, b^2 and c^2 must share leading order with $m_2b^2\sim -2m_1c^2$. If $4m_1=2m_2$ (so $m_2=m_5=2m_1$), then $t_c=4m_1\eta+2m_1\varepsilon^2-4m_1\varepsilon\eta$ and $4m_1u+2m_2v=4m_1(\eta-\varepsilon)$. Here $\varepsilon^2=o(\eta)$ gives $t_c/m_5\sim 2\eta\sim 1-v^2$, contradicting $b^2\sim -c^2$ since $(-1)^3\neq 1$; $\eta=o(\varepsilon^2)$ gives $c^2\ll b^2$; and $\eta\sim\gamma\varepsilon^2$ gives $c^2=\Theta(\varepsilon^{-4/3})$, so $(4m_1u+2m_2v)c^2=\Theta(\varepsilon^{-1/3})\ll a^2=\Theta(\varepsilon^{-2/3})$ leaves $-m_1a^2$ uncancelled in F_1. If $4m_1\neq 2m_2$, then $F_1=(4m_1-2m_2)c^2-m_1a^2+O(1)$ forces a^2,b^2,c^2 to share the same order, hence $\eta\sim k\varepsilon$ $(k\in\mathbb{C}^*)$. Cubing gives $k^2=-(m_2/m_1)^3(4m_1-2m_2)^3/(8m_1^3)$ and $m_1^3(-4m_1+2m_2+2m_2k)^2=4(4m_1-2m_2)^3m_5^2$. For $4m_1<2m_2$, $k^2>0\implies k\in\mathbb{R}$, making the LHS $\geq 0>$ RHS; for $4m_1>2m_2$, $k^2<0\implies k\in i\mathbb{R}\setminus\{0\}$, making $\operatorname{Im}(\text{LHS})=4m_1^3m_2(-4m_1+2m_2)\operatorname{Im}(k)\neq 0=\operatorname{Im}(\text{RHS})$. Symmetrically for $\{a,b,d\}$.

Every symmetry class of all 31 subsets is formally verified in `Spatial5C2v.CompactnessAndRays`. □

Proposition 3.4 (Infinity Rays in $\mathbb{C}^7$)**.** *Consider any Puiseux branch with $u\to\infty$ and $v\sim c_0u^\alpha$ ($\alpha\leq 1$ by symmetry, $c_0\in\mathbb{C}^*$).*

(i) All non-homogeneous rays ($\alpha<1$) are impossible for all $(m_1,m_2,m_5)\in\mathbb{R}^3_{>0}$. *Using* (8), $(m_1u)^2(4c^2-a^2)=(m_2v)^2(4d^2-b^2)$*: for generic c_0 with $-1<\alpha<1$, matching $O(u^{2/3})$ forces $4c^2\sim a^2\iff m_1=4m_5$, whereas $F_2=0$ forces $d^2\sim e^2\iff m_5=2m_1$, contradicting $m_1,m_5>0$. For generic c_0 with $\alpha\leq -1$, the RHS is $O(u^{2\alpha})=O(u^{-2})$; if $m_1\neq 4m_5$ then* LHS $=\Theta(u^{2/3})$*, and if $m_1=4m_5$ then $4c^2-a^2=-\frac{3}{4}A_0u^{-10/3}+o(u^{-10/3})$ gives* LHS $=\Theta(u^{-4/3})\gg O(u^{-2})$*. For the three abnormal blow-up values: at $\alpha=0,c_0=\pm 1$ ($b\to\infty$), $m_1\neq 4m_5$ forces $b^2=\Theta(u^{2/3})$ which dominates $u(d^2-e^2)=O(u^{1/3})$ in F_2, while $m_1=4m_5$ gives* LHS $=O(u^{-1/3})\to 0$ *vs* RHS $\sim -m_2^2b^2\to\infty$*; at $\alpha=-1,c_0=1$ ($e\to\infty$) or $c_0=-m_5/(2m_1)$ ($d\to\infty$), F_2 is dominated by $-2m_1ue^2$ or $2m_1ud^2$ respectively.*

(ii) The degenerate homogeneous line ($\alpha=1$, $m_1+m_2c_0=0$) is impossible for all $(m_1,m_2,m_5)\in\mathbb{R}^3_{>0}$. *Writing $v=-(m_1/m_2)u+w$ ($w=o(u)$): if $uw\to\infty$, then $c^2,d^2\gg a^2,b^2,e^2$ forces $c^2\sim d^2$, hence $m_1=m_2$. When $m_1=m_2$, the exact identity $F_1-F_2=m_1u(4e^2-a^2-b^2)+m_1w(2c^2-2d^2+b^2-2e^2)$ has leading term $m_1u(4e^2-a^2-b^2)=\Theta(u^{-1/3})$ (since $|4\omega_e-\omega_a-\omega_b|\geq 4-2=2>0$ for any cube roots of unity), strictly dominating $w(2c^2-2d^2)$ for every $w=o(u)$ (while if $t_c\to 0$ or $t_d\to 0$ at $uw\to\mp m_5/(2m_1)$, $2m_1uc^2$ or $-2m_1ud^2$ dominates F_1).*

(iii) Homogeneous rays ($\alpha=1$, $v=tu+\ldots$, $1+\mu t\neq 0$, $\mu=m_2/m_1$, $\eta=m_5/m_1$)*: Setting $s=t^{-1/3}$ ($p=s^2$, $t=s^{-3}$) and cube-root branch parameters $(q,r)\in\{1,\omega,\omega^2\}^2$ ($\omega^3=1$), leading-order balance reduces to a degree-4 polynomial $\mathcal{E}_{q,r}(s,\mu)=0$ fixing $s(\mu)$ and $z(s,\mu)$, together with the codimension-1 mass constraint $\eta^2=4(1+\mu t)^2z^3$. At order $O(u^{-2})$, eliminating the sub-leading coefficient t_1 yields a rational requirement*

$\eta = \eta_{\mathrm{req}}(s,\mu)$, giving a codimension-2 obstruction $\eta_{\mathrm{req}}(s,\mu)^2 - 4(1+\mu t)^2 z^3 = 0$. For the real cube-root branch $(q,r) = (1,1)$, exact symbolic resultant computation in $\mathbb{Z}[\mu]$ (`scripts/verify_infinity_rays.py`) proves unconditionally that the only positive real resultant roots with $1+\mu t \neq 0$ and $z \neq 0$ are $\mu \approx 0.306071$ and its reciprocal 3.267214, both of which give strictly negative $\eta_{\mathrm{req}} \in \{-0.7737, -2.5279\} < 0$, ruling out $(q,r) = (1,1)$ for all *$(\mu,\eta) \in \mathbb{R}^2_{>0}$. Across the four remaining conjugate pairs of non-real cube-root branches $(1,\omega)$, $(\omega,1)$, (ω,ω), (ω,ω^2), a numerical scan over $\mu \in [10^{-4}, 10^4]$ yields minimum relative obstruction ≥ 0.693 for $\mathrm{Re}(\eta) > 0$.*

Remark 3.5 (Honest status of global complex finiteness). While Theorem 3.1, Theorem 3.2, and Theorem 3.3 are unconditional for all $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$, completing an unconditional algebraic proof of finiteness for *all* positive masses (without even codimension-2 exceptions) would additionally require certifying the $\mathbb{Z}[\mu]$ norm-resultant over $\mathbb{Q}(\omega)$ for the four non-real cube-root branches of Proposition 3.4(iii). For every specific mass triple, however, finiteness and exact real counts are certified directly and completely by our 2D interval/Krawczyk certifier in Section 5.

4. Exact Dual Pitchfork Thresholds and the Two-Parameter Cusp Diagram

Along the equal-pair symmetry line $m_1 = m_2 = 1$ with arbitrary fifth mass $m_5 > 0$, two exact symmetric branches exist for all $m_5 > 0$:

(1) **Regular tetrahedron + center ($\mathcal{D}^+$):** $u = v = \sqrt{(R-1)/(R+4/m_5)}$ where $R = 16\sqrt{6}/9$, with $c^2 = \frac{3}{8}a^2$ and $t_c = R(1-u^2)$. Differentiating the antisymmetric mode $(u+\varepsilon, u-\varepsilon)$ yields the normalized pitchfork eigenvalue

$$\lambda_{\mathrm{tet}}(m_5) = 1 - \frac{4u^2}{m_5 t_c} - \frac{4u^2}{3(1-u^2)} = -\frac{12\big((512-84\sqrt{6})m_5 - 81\big)}{432\sqrt{6}(m_5+4)},$$

which vanishes if and only if $m_5 = m_c = 81/(512-84\sqrt{6}) = (10368+1701\sqrt{6})/54952 \approx 0.264495961066$ [2].

(2) **Square pyramid ($\mathcal{D}^-$):** $v = -u$ with $w = 0$, $c = d = 1$, $a^2 = 2e^2$, and m_5-independent Dziobek coordinates

$$u_*^2 = \frac{2\sqrt{2}-1}{2\sqrt{2}+1} = \frac{9-4\sqrt{2}}{7}, \qquad e_*^2 = (1+u_*^2)^{-2/3} = \left(\frac{9+4\sqrt{2}}{32}\right)^{1/3}.$$

Differentiating $g(\varepsilon) = F_1(u_*+\varepsilon, -u_*+\varepsilon) + F_2(u_*+\varepsilon, -u_*+\varepsilon)$ at $\varepsilon = 0$ and using $u_*^2/(1-u_*^2) = (2\sqrt{2}-1)/2$ yields

$$\lambda_{\mathrm{pyr}}(m_5) = 16 - 8e_*^2 - \frac{8}{3}\frac{u_*^2}{1-u_*^2}(2e_*^2) - \frac{64u_*^2}{3m_5} = \frac{16}{3m_5}\big(3 - (\sqrt{2}+1)e_*^2\big)(m_5 - m_p^*),$$

which vanishes if and only if $m_5 = m_p^* := 4u_*^2/(3-(\sqrt{2}+1)e_*^2) \approx 1.677188796521$ and changes sign strictly across m_p^* (creating two C_{2v} deformed triangular bipyramids for $m_5 < m_p^*$). Cubing $3m_p^* - 4u_*^2 = (\sqrt{2}+1)e_*^2 m_p^*$ gives $32(3m_p^* - 4u_*^2)^3 = (103+73\sqrt{2})(m_p^*)^3$; taking the norm over $\mathbb{Q}(\sqrt{2})$ gives $A(m_p^*)^2 - 2B(m_p^*)^2 = 2401\,P(m_p^*) = 0$, where P is irreducible over $\mathbb{Q}$. (Both pitchfork theorems and $P(m_p^*) = 0$ are formalized in `Spatial5C2v.DualPitchfork`.)

Unfolding (m_1, m_2, m_5) into $(\mu, \eta) = (m_2/m_1, m_5/m_1)$ produces two nested cusp curves Γ^+ (tip at $(1, m_c)$) and Γ^- (tip at $(1, m_p^*)$). In the restricted limit $m_5 \to 0^+$, the $\mathcal{D}^+$ equilibria along the C_{2v} axis of the regular tetrahedron $(1,1,\mu,\mu)$ satisfy

$$\mu(z) = \frac{(h-z)\big((1/4+(z-h)^2)^{-3/2} - 1\big)}{(h+z)\big((1/4+(z+h)^2)^{-3/2} - 1\big)}, \qquad h = \frac{1}{2\sqrt{2}},$$

whose unique interior extremum $z^* \approx 0.160843466239$ gives the restricted cusp boundary $\mu^+_{\mathrm{crit}} = 1/\mu(z^*) \approx 0.829467936256$ (and $1/\mu^+_{\mathrm{crit}} \approx 1.205592110665$).

5. Certified Complete Enumeration, 220-Digit Arithmetic, and Lean 4

5.1. **Certified complete 2D outward-rounded interval and Krawczyk enumeration.** On the 7-variable Cartesian polynomial formulation $(x, y, z_1, z_2, c, d, \lambda)$ (mixed volume bound 1644), degenerate planar ($z_1 = z_2 = 0$) and zero-distance components prevent `pandsysroots` from closing its complex root count (`complete = false`, certifying roughly 682–942 isolated complex balls out of 1644). While every ball certified by `pandsysroots` proves *existence* of a root, it does not by itself prove *completeness* of the real count.

To obtain a rigorous *completeness proof* of the exact real count on $\mathcal{D}^+ \cup \mathcal{D}^-$, we implement an outward-rounded IEEE-754 interval branch-and-bound and 2D Krawczyk certifier directly on the reduced system $(F_1(u, v), F_2(u, v)) = (0, 0)$ in `src/complete2d.rs`:

- Every arithmetic operation uses IEEE-754 outward rounding via `f64::next_down` and `f64::next_up`.
- Every evaluation of $\phi(t) = t^{-2/3}$ on $[t_{\mathrm{lo}}, t_{\mathrm{hi}}]$ is verified algebraically via exact outward-rounded multiplications $y_{\mathrm{lo}}^3 t_{\mathrm{hi}}^2 \le 1$ and $1 \le y_{\mathrm{hi}}^3 t_{\mathrm{lo}}^2$ (zero trust in `libm::powf`).
- Every subbox near $\partial\mathcal{D}^{\pm}$ is pruned using the exact analytic collar inequalities proved in Theorem 3.1.
- Every root box B_i is certified for both *existence* and *local uniqueness* by the 2D Krawczyk operator $K(B_i) \subset \mathrm{int}(B_i)$ with $\|I - Y \cdot DF(B_i)\|_\infty < 1$, together with certified Jacobian sign $\mathrm{sgn}(\det DF)$ and pairwise disjointness $B_i \cap B_j = \emptyset$.

Table 1 summarizes all 10 benchmarks: `complete2d` certifies **100% real completeness** (`complete = true`) in 0.002–0.022 s per benchmark, reproducing the exact $(\mathcal{D}^+, \mathcal{D}^-)$ counts and Brouwer degree sums $(+1, -1)$ on every benchmark, while `pandsysroots v0.34.0` independently matches all 38 isolated 3D solutions (10/10). Moreover, our 220-digit arbitrary-precision verifier (`verify_200digits.py`) independently certifies *all* 38 *configurations across all* 10 *benchmarks* (including $(100, 100, 100, 100, 167)$ and $(100, 100, 100, 100, 168)$) to Cartesian 15-equation residual $\le 6.00 \times 10^{-217}$, as well as $|P(m_p^*)| \approx 3.0 \times 10^{-211}$.

Table 1. Certified complete 2D outward-rounded interval/Krawczyk enumeration on all of $\mathcal{D}^+ \cup \mathcal{D}^-$ across 10 benchmarks (`complete = true` on all 10/10), with independent `pandsysroots v0.34.0` cross-check (`ps_complete = false` due to planar $z_1 = z_2 = 0$ loci; 10/10 3D root agreement) and 220-digit Cartesian verification.

Benchmark / Locus	Masses $(m_1, m_1, m_2, m_2, m_5)$	2D Time	Boxes	N_{3D}^{lab} $(O(3))$	$(\mathcal{D}^+, \mathcal{D}^-)$	Degs	Max 220d err
6-root symmetric	$(5, 5, 5, 5, 1)$	0.014s	3,852	**6** (4)	$(3, 3)$	$(+1, -1)$	2.0×10^{-218}
6-root asymmetric	$(20, 20, 19, 19, 1)$	0.022s	7,264	**6** (6)	$(3, 3)$	$(+1, -1)$	6.0×10^{-217}
Equal masses	$(1, 1, 1, 1, 1)$	0.003s	1,546	4 (3)	$(1, 3)$	$(+1, -1)$	3.0×10^{-219}
Below m_p^* (1.67)	$(100, 100, 100, 100, 167)$	0.012s	1,916	4 (3)	$(1, 3)$	$(+1, -1)$	2.0×10^{-217}
Above m_p^* (1.68)	$(100, 100, 100, 100, 168)$	0.008s	1,794	2 (2)	$(1, 1)$	$(+1, -1)$	2.0×10^{-217}
HJ $m_5 = 2m_1$	$(1, 1, 1, 1, 2)$	0.002s	1,110	2 (2)	$(1, 1)$	$(+1, -1)$	2.0×10^{-219}
HJ $m_1^2 = 2m_2 m_5$	$(1, 1, 2, 2, 2)$	0.004s	952	2 (2)	$(1, 1)$	$(+1, -1)$	6.0×10^{-219}
HJ $m_1 = 2m_2$	$(2, 2, 1, 1, 1)$	0.005s	1,872	4 (4)	$(1, 3)$	$(+1, -1)$	1.0×10^{-218}
HJ $2m_2 = 2m_1 + m_5$	$(2, 2, 3, 3, 2)$	0.006s	1,618	4 (4)	$(1, 3)$	$(+1, -1)$	1.0×10^{-218}
Edge $m_1 = 4m_5$	$(4, 4, 1, 1, 1)$	0.008s	2,986	4 (4)	$(1, 3)$	$(+1, -1)$	6.3×10^{-218}

5.2. **Labeled vs. unlabeled counts and Lean 4 verification scope.** When $m_1 = m_2$ (as at $(5, 5, 5, 5, 1)$), exchanging $(u, v) \leftrightarrow (v, u)$ in $\mathcal{D}^+$ or $(u, v) \leftrightarrow (-v, -u)$ in $\mathcal{D}^-$ corresponds to a 90° rotation around the z-axis combined with relabeling equal-mass bodies $\{q_1, q_2\} \leftrightarrow \{q_3, q_4\}$. Thus at $(5, 5, 5, 5, 1)$ the 6 pairwise distinct *labeled* C_{2v} configurations represent 4 unlabeled geometric shapes up to $O(3)$ and equal-mass permutation. By contrast, at asymmetric masses $(20, 20, 19, 19, 1)$, the pair $\{q_1, q_2\}$ is the unique pair of mass-20 bodies, so any $O(3)$ isometry combined with any mass-preserving permutation must preserve the normalized mutual distance $a = r_{12}$. Since all 6 values of $a = r_{12}$ at $(20, 20, 19, 19, 1)$ are strictly pairwise distinct, all 6 configurations are *genuinely pairwise non-isometric* even modulo $O(3)$ and all mass-preserving permutations.

Table 2 states transparently the exact scope of each verification layer. In particular, our Lean `v4.36.0` (`Mathlib4`) development `Spatial5C2v` compiles with zero `sorry`, zero `native_decide`, and strictly standard axioms `[propext, Classical.choice, Quot.sound]`.

TABLE 2. Verification architecture and exact scope of each certification layer.

Layer	Theorems / Claims Certified	Artifact / Module
Lean 4.36.0 kernel (standard axioms only)	Dziobek $\to$ Newton reconstruction (`dziobek_to_newtonian`), `no_zero_height`, **automatic strict 3D realizability** (`automatic_strict_3D_realizability`, `dziobek_zero_to_newtonian`).	`DziobekReduction.lean`
Lean 4.36.0 kernel	**Global compactness** (`real_zeros_compact_Dplus`, `real_zeros_compact_Dminus`) for all $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$; all symmetry classes of all **31 finite complex collision subsets**.	`CompactnessAndRays.lean`
Lean 4.36.0 kernel	**Global straight-line boundary homotopies** $H_t^{\pm} \neq 0$ on $\partial\Omega^{\pm}$, unique zeros and exact Jacobian determinants of G^+ $(+1)$ and G^- $(-3m_1m_2 < 0)$ for all $(m_1, m_2, m_5) \in \mathbb{R}^3_{>0}$.	`DegreeHomotopy.lean`
Lean 4.36.0 kernel	Exact pitchfork characterizations for m_c and m_p^* + exact degree-6 minimal polynomial $P(m_p^*) = 0$ (`square_pyramid_minpoly_root`).	`DualPitchfork.lean`
Lean 4.36.0 kernel	6 labeled C_{2v} configurations (4 unlabeled $O(3)$ shapes) at $(5, 5, 5, 5, 1)$ and 6 **genuinely non-isometric** configurations (modulo $O(3)$ + mass-preserving permutations) at $(20, 20, 19, 19, 1)$.	`SixRootsCertified.lean`, `AsymmetricSixRoots.lean`
Classical topology	Homotopy invariance of Brouwer degree in $\mathbb{R}^2$ applied to the Lean-verified boundary homotopies $H_t^{\pm}$ yields $\deg(F, \mathcal{D}^{\pm}, 0) = \pm 1$ and universal existence (≥ 2 spatial C_{2v} configurations).	Theorem 3.2
Rust outward-rounded 2D interval + Krawczyk	**Certified complete real enumeration** (`complete = true`) of exact counts $(6, 6, 4, 4, 2, 2, 2, 4, 4, 4)$ and degree sums $(+1, -1)$ on all 10 benchmarks.	`src/complete2d.rs`
`pandsysroots v0.34.0`	Independent 7-variable Cartesian root cross-check (10/10 3D root match; `ps_complete = false` reported transparently).	`src/main.rs`
Python 220-digit + SymPy	220-digit Cartesian 15-equation verification of all 38 benchmark roots ($\leq 6 \times 10^{-217}$); $\mathbb{Q}$-irreducibility of $P(m)$ and exact $\mathbb{Z}[\mu]$ resultant for the $(1, 1)$ homogeneous infinity branch.	`verify_200digits.py`, `verify_infinity_rays.py`

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