A sign in the three-loop effective action of pure Yang–Mills theory on S^3, and the resolution of a recent discrepancy

Kevin McMorrow*
Artificial Intelligence Engineering LLC

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Abstract

Aharony, Marsano, Minwalla, Papadodimas and Van Raamsdonk (AMMPV) computed the quartic coefficient b of the Polyakov-loop effective action of large-N pure Yang–Mills theory on a small S^3 at three loops, and found $b = -5.7 \times 10^{-4}\lambda_A^2$. A recent Hamiltonian recomputation agrees with their fourteen three-loop diagrams in six of seven classes, and disagrees in the class of diagrams 3c, 3f and 3n: -2.095 against -3.388, in units of 10^{-4}. We trace the difference to one sign. An identity in Appendix B of AMMPV, for a sum of four spherical-harmonic coefficients $D\,D\,C\,C$ with crossed index structure, is printed with prefactor $-(-1)^{\sum j}$. The correct prefactor is $+(-1)^{\sum j}$. We establish this by direct summation over magnetic quantum numbers and, separately, with explicit harmonics on S^3 that involve no phase conventions. The identity enters only diagram 3n, through a term that is ultraviolet finite and independent of the regulator, so neither of the consistency checks made by AMMPV is sensitive to it. Every entry printed by AMMPV for this class is reproduced by their printed formulas together with the printed identity. With the correct sign, the finite part of 3n changes from -0.92 to $+0.37$, and the class sums to -2.094, in agreement with the Hamiltonian result. The corrected value from the AMMPV diagrams is $b = -4.0(4) \times 10^{-4}\lambda_A^2$. The sign of b is unchanged, and the deconfinement transition remains first order.

1 Introduction

Large-N gauge theories on a small three-sphere are weakly coupled, and they have a deconfinement transition that can be studied in perturbation theory [1, 2]. Its order is decided by the sign of the quartic coefficient b in the effective action of the Polyakov loop, which first appears at three loops. AMMPV computed it for pure Yang–Mills theory [3]. They evaluated fourteen three-loop vacuum diagrams on $S^3 \times S^1$, labelled 3a to 3n, in Coulomb gauge, and found $b = -5.7 \times 10^{-4}\lambda_A^2$. The transition is therefore first order. This result has been the reference point for the subject for two decades.

A recent preprint [4] computed the same coefficient for $\mathcal{N} = 4$ super Yang–Mills theory by Hamiltonian methods. As a check, its codes were restricted to gluons and compared with AMMPV. The two-loop results agree to six digits. At three loops the diagrams fall into seven classes by vertex content. Six agree. The seventh, which contains two Coulomb interactions together with the sextic vertex (diagrams 3c, 3f and 3n), does not: the preprint finds -2.095 and the entries of AMMPV sum to -3.388, in units of 10^{-4} for the finite part of $C_{1,1,-1,-1}$. The preprint states that "one of the two calculations has an error in this class", and lists its resolution as an open problem [4, secs. 8.2 and 10.3].

*ORCID: 0000-0001-6849-7795.

This note resolves it.

Result 1. *The third identity in the second group of four-coefficient sums in Appendix B of [3], which evaluates* $\sum_m D^{\alpha\gamma\rho} D^{\beta\bar{\gamma}\sigma} C^{\bar{\rho}\bar{\beta}\tau} C^{\bar{\tau}\bar{\alpha}\bar{\sigma}}$*, has the wrong overall sign as printed. With the correct sign, the finite part of diagram 3n is* $+0.370 \times 10^{-4}$ *instead of* -0.92×10^{-4}*, and the class 3c + 3f + 3n sums to* -2.094×10^{-4}*. This agrees with the value* -2.095×10^{-4} *of [4].*

All other parts of the AMMPV calculation that we examined are confirmed. The physical conclusion of [3] is unchanged.

2 The identity

We use the notation of [3]. The coefficients $C^{\alpha\beta\gamma}$ and $D^{\alpha\beta\gamma}$ are integrals over S^3 of products of harmonics. Greek indices stand for the full set of quantum numbers (j, m, m', ϵ), and a bar denotes the conjugate harmonic. R_2 and R_3 are the reduced matrix elements defined there. The identity in question is printed as

$$\sum_m D^{\alpha\gamma\rho} D^{\beta\bar{\gamma}\sigma} C^{\bar{\rho}\bar{\beta}\tau} C^{\bar{\tau}\bar{\alpha}\bar{\sigma}} = s\,(-1)^{\sum j} \begin{Bmatrix} \frac{j_\rho}{2} & \frac{j_\tau}{2} & \frac{j_\beta+\epsilon_\beta}{2} \\ \frac{j_\sigma}{2} & \frac{j_\gamma+\epsilon_\gamma}{2} & \frac{j_\alpha+\epsilon_\alpha}{2} \end{Bmatrix} \begin{Bmatrix} \frac{j_\rho}{2} & \frac{j_\tau}{2} & \frac{j_\beta-\epsilon_\beta}{2} \\ \frac{j_\sigma}{2} & \frac{j_\gamma-\epsilon_\gamma}{2} & \frac{j_\alpha-\epsilon_\alpha}{2} \end{Bmatrix} \times R_2(j_\rho, j_\beta, j_\tau)\, R_2(j_\tau, j_\alpha, j_\sigma)\, R_{3\epsilon_\alpha\epsilon_\gamma}(j_\alpha, j_\rho, j_\gamma)\, R_{3\epsilon_\beta\epsilon_\gamma}(j_\beta, j_\sigma, j_\gamma), \tag{1}$$

with $s = -1$. We find $s = +1$. The evidence is as follows.

Direct summation. We built C and D from the explicit $3j$ expressions printed in [3] and summed over all magnetic quantum numbers. In all 600 non-vanishing cases with $j_\alpha, j_\beta, j_\gamma \le 3$ and all choices of ϵ, the left side of eq. (1) equals the right side with $s = +1$. As a control, the same code confirms the other identities of that appendix that enter this class as printed: the crossed $D\,D\,D\,D$ sum (96 of 96 cases) and the planar $C\,C$ sum.

Explicit harmonics. The overall sign of such an identity could in principle depend on phase conventions for the vector harmonics. It does not here. Each index appears once plain and once barred, so the sum is a product of projectors and is independent of the basis. To confirm this, we evaluated the left side with explicit real harmonics on S^3 (harmonic polynomials for the scalars, and curl eigenvectors for the vectors), using exact quadrature and no $3j$ or $6j$ symbols. In 72 non-vanishing cases the result agrees with the direct summation to 10^{-14}, and so again gives $s = +1$. Reversing the orientation of the sphere does not change it.

Three separate implementations, written without reference to one another, give the same sign.

3 Diagram 3n

The identity is used only in diagram 3n, the self-contraction of the sextic vertex B_6 that arises when A_0 is integrated out in Coulomb gauge. The integrand printed in [3] contains, in its first bracket,

$$3\,\frac{C^{\bar{\rho}\bar{\alpha}\lambda} C^{\bar{\lambda}\bar{\beta}\bar{\sigma}}}{j_\lambda(j_\lambda+2)} + 3\,\frac{C^{\bar{\rho}\bar{\beta}\lambda} C^{\bar{\lambda}\bar{\alpha}\bar{\sigma}}}{j_\lambda(j_\lambda+2)} + D^{\bar{\alpha}\bar{\lambda}\bar{\rho}} D^{\bar{\beta}\lambda\bar{\sigma}} + D^{\bar{\alpha}\bar{\lambda}\bar{\sigma}} D^{\bar{\beta}\lambda\bar{\rho}}, \tag{2}$$

multiplied by $D^{\alpha\gamma\rho} D^{\beta\bar{\gamma}\sigma} / [j_\rho(j_\rho+2) j_\sigma(j_\sigma+2)]$. The second term of eq. (2) is the crossed $C\,C$ term, and its angular sum is eq. (1).

The integrand is correct. An error in an identity could be compensated by an opposite error in the integrand. It is not. Integrating out A_0 gives $-\frac{1}{2}JM^{-1}J$ with $M = -D_iD_i = G^{-1} + gX_1 + g^2X_2$, where $X_1 = 2i\,\mathrm{ad}(A)\cdot\nabla$ and $X_2 = \mathrm{ad}(A)\cdot\mathrm{ad}(A)$. At second order, GX_1GX_1G gives $+4\,C\,C$, the longitudinal part of X_2 gives $-C\,C$, and its transverse part gives $+D\,D$. This is the combination $3\,C\,C + D\,D$ of eq. (2), with the coefficient 3 that also appears in the Hamiltonian form of the vertex, $C_4 = \frac{3}{2}\rho\Delta^{-1}a\Delta^{-1}a\Delta^{-1}\rho$ [4]. We also contracted B_6 by machine: all 16 words of the trace and all 5 planar pairings, with winding sums. The printed integrand agrees with the contraction, term by term, to 15 digits, including the crossed $C\,C$ term. With the sign of that term reversed, the agreement fails.

Why the checks of AMMPV did not detect it. The crossed $C\,C$ sum converges: its terms fall as J^{-3}. It therefore contributes nothing to the coefficient of the logarithmic divergence, and it is the same for any regulating function. AMMPV checked that the logarithmic coefficients cancel between diagrams, and they recomputed 3n with a second regulator. Both checks are blind to this term.

4 Numerical results

We evaluated the expressions printed in [3] for 3c, 3f and 3n with two codes written separately. The evaluators were validated on the two-loop quantities and on diagram 3a.

Diagram	Code I	Code II	AMMPV
3c	+0.22245	+0.2225	+0.2224
3f	−2.6864	−2.686(2)	−2.69
3n, $s=-1$ (as printed)	−0.9212	−0.9212	−0.92
3n, $s=+1$ (correct)	+0.3698	+0.3698	—
Class, $s=-1$	−3.3852	−3.385	−3.388
Class, $s=+1$	−2.0941	−2.094(3)	—

Table 1: Finite parts of the three-loop contributions to $C_{1,1,-1,-1}$ at the Hagedorn point $x_c = 2-\sqrt{3}$, in units of 10^{-4}. Every printed entry of AMMPV follows from their printed formulas with the printed identity. The Hamiltonian result of [4] for the class is −2.095. The correct sign shifts the class by +1.291; the discrepancy reported in [4] is +1.293.

A sharper test. Reference [4] quotes the first coefficients of the expansion of this class in $x = e^{-\beta}$, obtained with two engines: 36.9553(5) and 36.957(6) at order x^4, and 65.93(5) and 65.87(6) at order x^5, in units of $1/64\pi^4$. The crossed $C\,C$ term contributes ±30.000 at order x^4, so the two signs differ by 60. The AMMPV expressions give 36.9553 with $s=+1$ and −23.05 with $s=-1$. At order x^5 we find 65.85(7) with $s=+1$. The six-digit agreement at order x^4 cannot be accidental.

Operator ordering plays no role. One might suspect the difference between a Weyl-ordered Hamiltonian and a path integral with contact terms, that is, terms of the Christ–Lee type [5]. These contribute nothing to this class at this order. At order g^4 they are single-trace and quadratic in the gauge field, so they produce at most two traces of the holonomy, where $|\mathrm{tr}\,U|^4$ requires four. Numerically, the $\delta(t)$ contact terms of AMMPV in 3c and 3f (+0.131 and +3.328) cancel the $D\,D\,D\,D$ part of the sextic vertex (−3.459), and the remainder is the Hamiltonian class.

5 Consequences

Quantity	AMMPV, as printed	Corrected
3n, finite part	-0.92×10^{-4}	$+0.370 \times 10^{-4}$
Sum of three-loop finite parts	$-4.5(3) \times 10^{-4}$	$-3.2(3) \times 10^{-4}$
$C_{1,1,-1,-1}$	-4.3×10^{-4}	$-3.0(3) \times 10^{-4}$
b/λ_A^2	-5.7×10^{-4}	$-4.0(4) \times 10^{-4}$

Table 2: Corrected values from the diagrams of AMMPV. The uncertainty is inherited from the entries that AMMPV print to one digit (3i and 3l); we did not recompute those.

The corrected coefficient is $b = -4.0(4) \times 10^{-4}\lambda_A^2$. It is consistent with the pure-glue value $-4.3 \times 10^{-4}\lambda_A^2$ of [4], which is the more precise number. The remaining difference lies within the rounding of AMMPV's entries 3i and 3l. The sign of b is unchanged. The deconfinement transition of pure Yang–Mills theory on a small S^3 is first order, as AMMPV concluded. All seven classes of three-loop diagrams now agree between the Lagrangian and Hamiltonian calculations, which strengthens both.

6 Limitations

- We did not have access to the computer files of AMMPV. That their numerical evaluation used the identity as printed is an inference: the printed identity reproduces their entry for 3n, -0.9212 against -0.92, and the correct sign gives $+0.37$. The precise statement is that the printed entry follows from the printed identity.
- We did not recompute the value -2.095 of [4] in its Hamiltonian formalism. The correct sign is established here without reference to that paper; the agreement is then a check of it.
- The diagrams 3c and 3f, and the remaining parts of 3n, were evaluated by two codes that agree with each other and with AMMPV. The third, adversarial check recomputed only the crossed $C\,C$ term, the identity, and the vertex.
- We did not recompute the other eleven diagrams. The corrected b uses AMMPV's printed entries for them.

Methods and author contributions

The calculations were made by AI agents (Anthropic's Claude) in a project set up and directed by the author. Two agents worked in separate directories and did not see each other's work. Each located the same identity and obtained the same numbers. A third agent, instructed to act as a hostile referee and to assume the claim was wrong, wrote its own code from the definitions in [3]. It confirmed the sign of the identity, rederived the sextic vertex, and checked the integrand of 3n by machine contraction. The agents are instances of the same model, so their errors may be correlated; the checks with explicit harmonics and the comparison with the independent numbers of [4] are the external anchors. A hypothesis proposed at the outset, that operator-ordering terms explain the discrepancy, was tested and found to be wrong. No human physicist has yet checked this work. All scripts and their outputs are in the accompanying repository.

Acknowledgements

The calculation of AMMPV is long and intricate, and its authors printed every expression needed to check it. That made this note possible. We also thank the authors of [4] for isolating the discrepancy to a single class of diagrams.

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