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PRESCRIBED OSCILLATING SPECTRA AND ISOMORPHISM TYPES IN $C[0,1]$

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Abstract. The oscillating spectrum of a closed subspace of $C[0,1]$ is the set of points at which its unit ball fails to be equicontinuous. Given nonempty F_σ sets $A, B \subseteq [0,1]$, we prove that every closed subspace with spectrum A is nonisomorphic to every closed subspace with spectrum B if and only if one set is countable and the other is nonmeager. We show that a countable spectrum forces the dual to be separable, while a nonmeager spectrum forces the dual to be nonseparable. For every uncountable meager F_σ set M and every separable infinite-dimensional real Banach space E, we construct an isomorphic copy of E in $C[0,1]$ with spectrum M. We also show that every nonempty F_σ set admits an uncomplemented realization.

For a prescribed nonempty finite spectrum, we show that an infinite-dimensional Banach space admits a realization precisely when it embeds isomorphically into c_0. Using the Schreier family, we construct a norm-one complemented subspace of $C[0,1]$ with compact countably infinite spectrum that does not embed isomorphically into c_0. This example and the classification of pairs answer Questions 4.3 and 4.9, respectively, of Enflo, Gurariy, and Seoane-Sepúlveda.

1. Introduction

All Banach spaces are real. For a closed subspace $X \subset C([0,1])$, write B_X for its closed unit ball in the supremum norm. Given $t \in [0,1]$ and $d > 0$, let $I(t,d) = [0,1] \cap [t-d, t+d]$, and write $\operatorname{osc}_A f = \sup_{s,u \in A} |f(s) - f(u)|$. The oscillating spectrum $\Omega(X)$ consists of the points where B_X fails to be equicontinuous. In terms of the oscillation function Γ_X, it is given by

$$\Gamma_X(t) = \inf_{d>0} \sup_{f \in B_X} \operatorname{osc}_{I(t,d)} f, \qquad \Omega(X) = \{t \in [0,1] : \Gamma_X(t) > 0\}. \tag{1.1}$$

Enflo, Gurariy, and Seoane-Sepúlveda introduced the oscillating spectrum and asked when the spectra of two subspaces force the spaces to be nonisomorphic [2, Question 4.9]. Xie showed that a subset of $[0,1]$ occurs as a spectrum if and only if it is F_σ, and that each such set admits a complemented realization [6, Theorems 1.1 and 1.2]. For two nonempty spectra we prove that nonisomorphism is forced exactly when one is countable and the other nonmeager (Theorem 5.2).

A second question concerns the passage from finite to countable spectra. Enflo, Gurariy, and Seoane-Sepúlveda proved that a subspace with finite spectrum embeds isomorphically into c_0 and asked whether countability suffices [2, Theorem 4.2 and Question 4.3]. In Section 2 we prove the converse for prescribed nonempty finite spectra. If $\Omega(X)$ is countable, every infinite-dimensional closed subspace of X contains a copy of c_0. The Schreier example in Section 3 shows that countability alone does not imply embeddability into c_0, even for the range of a norm-one projection.

Section 4 gives the prescribed isomorphic copies and an uncomplemented realization of each nonempty F_σ set, answering Xie's follow-up [6, Question 6.1] to Enflo, Gurariy, and Seoane-Sepúlveda's Question 4.8 [2]. Section 5 completes the classification of pairs.

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2. Finite and countable spectra

We begin with the finite case, where embeddability into c_0 characterizes the spaces that admit a realization with a given spectrum.

Theorem 2.1. *Let X be an infinite-dimensional Banach space and let M be a nonempty finite subset of $[0,1]$. Then X embeds isomorphically into c_0 if and only if there is an isomorphic embedding $T : X \to C([0,1])$ such that $\Omega(T(X)) = M$. If X is a closed subspace of c_0 with the inherited norm, T can be chosen to be an isometry.*

Proof. Suppose first that $T : X \to C([0,1])$ is an isomorphic embedding with $\Omega(T(X)) = M$. Since M is finite, the theorem of Enflo, Gurariy, and Seoane-Sepúlveda [2, Theorem 4.2, p. 620] implies that $T(X)$ embeds isomorphically into c_0. Composing with T gives the required embedding of X.

Conversely, let $J : X \to c_0$ be an isomorphic embedding and put $Z = J(X)$. Write $M = \{t_1, \dots, t_k\}$ and choose pairwise disjoint relative neighborhoods U_j of the t_j in $[0,1]$. In each $U_j \setminus \{t_j\}$ choose pairwise disjoint nondegenerate closed intervals $I_{j,n}$, $n \geq 1$, such that $\sup_{s \in I_{j,n}} |s - t_j| \to 0$ as $n \to \infty$. Let $h_{j,n}$ be the tent function supported on $I_{j,n}$, equal to zero at its endpoints and to one at its midpoint $q_{j,n}$, and affine on each half.

For $z = (z_n) \in c_0$, define

$$Hz = \sum_{n=1}^{\infty} z_n \sum_{j=1}^{k} h_{j,n}.$$

At most one tent is nonzero at any point, so the tail after the Nth term has supremum norm at most $\sup_{n>N} |z_n|$. Thus the series converges uniformly. Since $(Hz)(q_{j,n}) = z_n$, we have $\|Hz\|_\infty = \|z\|_\infty$. In particular, $H : c_0 \to C([0,1])$ is a linear isometry.

Let B_Z be the closed unit ball of Z. Every point outside M has a neighborhood meeting only finitely many of the intervals $I_{j,n}$. On this neighborhood the members of $H(B_Z)$ are linear combinations of a fixed finite family of tents with coefficients bounded by one. They are therefore equicontinuous there.

Fix j and a neighborhood V of t_j. Choose N so that $q_{j,n} \in V$ for all $n > N$. The coordinate map $z \mapsto (z_1, \dots, z_N)$ has finite rank on the infinite-dimensional space Z. Its kernel therefore contains a unit vector z. As $z \in c_0$, it attains its supremum norm at some coordinate $n > N$. Hence

$$|(Hz)(q_{j,n}) - (Hz)(t_j)| = |z_n| = 1.$$

Thus $H(B_Z)$ fails to be equicontinuous at every t_j. Since H is an isometry, $H(B_Z)$ is the unit ball of $H(Z)$. We obtain $\Omega(H(Z)) = M$. Taking $T = H \circ J$ finishes the proof. When $X \subseteq c_0$ has the inherited norm, we may take J to be the inclusion, so that T is an isometry. □

For countable spectra, the separable-dual assertion below also follows from Godefroy's result on norm-separable James boundaries [3, Lemma 4]. Indeed, the restricted point evaluations have norm-separable image, and their signed norm-one members form a James boundary. The direct proof also constructs a countable compact set on which restriction is an isomorphic embedding. This gives the assertion about copies of c_0.

Proposition 2.2. *If $X \subset C([0,1])$ is a closed subspace and $\Omega(X)$ is countable, then X^* is separable. In particular, X contains no subspace isomorphic to ℓ^1. Moreover, every infinite-dimensional closed subspace of X contains a subspace isomorphic to c_0.*

Proof. Write $e(t) = \delta_t|_X \in X^*$. Since

$$\|e(s) - e(t)\| = \sup_{f \in B_X} |f(s) - f(t)|,$$

the map e is norm-continuous at every point outside $\Omega(X)$. Its restriction to $[0,1] \setminus \Omega(X)$ therefore has separable image. Adding the countable image of $\Omega(X)$ shows that $e([0,1])$ is norm-separable. Choose a norm-dense sequence $(e(t_j))$ in this image and set $H = \overline{\operatorname{span}}\{e(t_j) : j \geq 1\}$.

Fix $x^* \in X^*$. By the Hahn–Banach theorem and the Riesz representation theorem [1, Section 4.1, p. 78], there exists a finite signed Borel measure μ on $[0,1]$ such that $x^*(f) = \int f\, d\mu$ for every $f \in X$. For $\varepsilon > 0$, let

$$U_j = \{t \in [0,1] : \|e(t) - e(t_j)\| < \varepsilon\}.$$

These sets are Borel, since $t \mapsto \|e(t) - e(t_j)\|$ is a supremum of continuous functions and hence lower semicontinuous. They cover $[0,1]$. Set $A_j = U_j \setminus \bigcup_{i<j} U_i$. The series

$$y^*_\varepsilon = \sum_{j\geq 1} \mu(A_j) e(t_j)$$

converges absolutely in H, since $\|e(t_j)\| \leq 1$ and $\sum_j |\mu(A_j)| \leq |\mu|([0,1])$. Moreover,

$$\|x^* - y^*_\varepsilon\| \leq \sup_{f\in B_X} \sum_{j\geq 1} \int_{A_j} |f(t) - f(t_j)|\, d|\mu|(t) \leq \varepsilon |\mu|([0,1]).$$

It follows that $x^* \in H$, and hence $X^* = H$ is separable. The assertion about ℓ^1 follows from the standard quotient description of subspace duals given by the Hahn–Banach theorem and the identity $(\ell^1)^* = \ell^\infty$.

To prove the assertion about c_0, we construct a countable compact set $K \subset [0,1]$ such that

$$\|f|_K\|_\infty \geq \tfrac{1}{2}\|f\|_\infty \qquad (f \in X).$$

Begin with the single interval $[0,1]$ at level zero and let S be initially empty. At each level, consider the intervals from left to right. For each interval $J = [a,b]$, first put its midpoint $m = (a+b)/2$ in S. If $\operatorname{diam} e(J) \geq 1/2$, put both halves $[a,m]$ and $[m,b]$ on the list for the next level. If $\operatorname{diam} e(J) < 1/2$, declare J terminal and end the subdivision of J.

For example, if $[0,1]$ is bisected, then at level one we record the midpoints $1/4$ and $3/4$ and test the two halves separately. If only $[0,1/2]$ is bisected, the intervals at level two are $[0,1/4]$ and $[1/4,1/2]$. Continue in this way for as long as there are intervals to consider, and let $K = \overline{S}$ in $[0,1]$.

Every accumulation point of S belongs to $\Omega(X)$. Indeed, if distinct midpoints converge to t, the lengths of their intervals tend to zero, since there are only finitely many dyadic intervals at each level. Their parent intervals also shrink to t, and each parent was bisected because its image under e had diameter at least $1/2$. This rules out norm continuity of e at t. Hence $K \subset S \cup \Omega(X)$ is countable.

For $f \in X$, choose t with $|f(t)| = \|f\|_\infty$. If t lies in a terminal interval J with midpoint s, then

$$|f(t) - f(s)| \leq \|e(t) - e(s)\|\ \|f\|_\infty \leq \tfrac{1}{2}\,\|f\|_\infty .$$

Otherwise, t belongs to a nested sequence of bisected intervals whose midpoints converge to t, so $t \in K$. In either case, $\|f|_K\|_\infty \geq \frac{1}{2}\,\|f\|_\infty$. Thus the restriction map $R : X \to C(K)$ is an isomorphic embedding.

Now let Y be an infinite-dimensional closed subspace of X. Then $R(Y)$ is an infinite-dimensional closed subspace of $C(K)$, and in particular K is infinite. By the theorem of Bessaga and Pełczyński [5, Proposition 3.6], every infinite-dimensional closed subspace of $C(K)$, for countable compact metric K, contains an isomorphic copy of c_0. Pulling this copy back through $R|_Y$ proves the claim. □

3. A countable compact spectrum from the Schreier family

We first compute the oscillation function for the affine extension of $C(L)$ from a compact set $L \subset [0,1]$. Here L' denotes the set of accumulation points of L. We then choose L so that $C(L)$ fails the weak Banach–Saks property.

Let $L \subset [0,1]$ be a nonempty compact set, and write $\alpha = \min L$ and $\beta = \max L$. For $g \in C(L)$, define

$$(E_L g)(t) = \begin{cases} g(t), & t \in L, \\ \dfrac{b-t}{b-a} g(a) + \dfrac{t-a}{b-a} g(b), & t \in (a,b),\ (a,b) \text{ a bounded gap of } L, \\ g(\alpha), & t < \alpha, \\ g(\beta), & t > \beta. \end{cases} \tag{3.1}$$

Here a bounded gap is a connected component (a,b) of $(\alpha,\beta) \setminus L$; both endpoints belong to L. The standard affine extension construction appears in [1, proof of Theorem 1.4.4]. The associated projection is discussed in [1, Section 4.4, p. 96]. We record the resulting properties for later use.

Proposition 3.1 (Affine extension)**.** *The operator $E_L : C(L) \to C([0,1])$ is a linear isometry, and its range $X_L = E_L(C(L))$ is closed. If $R_L : C([0,1]) \to C(L)$ is the restriction map $R_L f = f|_L$, then $P_L = E_L R_L$ is a projection of norm one onto X_L.*

Proposition 3.2. *For every nonempty compact set $L \subset [0,1]$, the space $X_L = E_L(C(L))$ satisfies*

$$\Gamma_{X_L} = 2\mathbf{1}_{L'}, \qquad \Omega(X_L) = L'.$$

Proof. First let $t \notin L'$. On a bounded gap (a,b), every $f = E_L g \in B_{X_L}$ is affine with slope of absolute value at most $2/(b-a)$, since $\|g\|_\infty \le 1$. On each outer interval, all extensions are constant. If t is an isolated point of L, there are at most two adjacent gaps. Their fixed lengths yield finite slope bounds on the corresponding sides of t, while an outer side is constant. Thus, in each case, there exist a relative neighborhood V of t and a finite constant M_t, independent of $f \in B_{X_L}$, such that

$$|f(s) - f(u)| \le M_t |s - u| \qquad (s, u \in V).$$

For sufficiently small $d > 0$, it follows that $\sup_{f \in B_{X_L}} \operatorname{osc}_{I(t,d)} f \le 2M_t d$, and hence $\Gamma_{X_L}(t) = 0$.

Now let $t \in L'$ and fix $d > 0$. Choose $s \in L \setminus \{t\}$ with $|s - t| < d$. The function

$$g(x) = \frac{|x-t| - |x-s|}{|s-t|} \qquad (x \in L)$$

belongs to $C(L)$, has norm one, and satisfies $g(t) = -1$ and $g(s) = 1$. By Proposition 3.1, $E_L g \in B_{X_L}$ and preserves these values. The supremum in (1.1) is therefore at least two for every $d > 0$. It is also at most two, so $\Gamma_{X_L}(t) = 2$. □

A Banach space has the *weak Banach–Saks property* if every weakly null sequence admits a subsequence whose arithmetic means converge to zero in norm.

Proposition 3.3. *Every Banach space isomorphic to a subspace of c_0 has the weak Banach–Saks property.*

This follows from [4, Corollary 2.8], since the weak Banach–Saks property passes to closed subspaces and is invariant under isomorphisms.

Identify subsets of $\mathbb{N} = \{1, 2, \ldots\}$ with their characteristic functions in $\{0,1\}^{\mathbb{N}}$, and define

$$\mathcal{S} = \{\varnothing\} \cup \{F \subset \mathbb{N} : 0 < |F| < \infty,\ |F| \le \min F\}. \tag{3.2}$$

We use the classical Schreier example in the compact-family form described in [5, Propositions 4.20–4.21 and their proofs].

Proposition 3.4 (Schreier)**.** *The family $\mathcal{S}$, endowed with the topology inherited from $\{0,1\}^{\mathbb{N}}$, is countable, compact, and metrizable. The coordinate functions $u_n(F) = \mathbf{1}_F(n)$ form a normalized weakly null sequence in $C(\mathcal{S})$, no subsequence of which has arithmetic means converging to zero in norm. Consequently, $C(\mathcal{S})$ fails the weak Banach–Saks property and, by Proposition 3.3, is not isomorphic to any subspace of c_0.*

Using the standard ternary embedding of the Cantor cube [1, Section 1.4], set

$$\phi(F) = \sum_{n\in F} 2\,3^{-n}, \qquad K = \phi(\mathcal{S}). \tag{3.3}$$

Then ϕ is a homeomorphism from $\mathcal{S}$ onto the countable compact set $K \subset [0,1]$, and the pullback $g \mapsto g \circ \phi$ identifies $C(K)$ linearly and isometrically with $C(\mathcal{S})$.

Theorem 3.5. *There exist a countable compact set $K \subset [0,1]$ and a closed subspace $X \subset C([0,1])$ such that X is linearly isometric to $C(K)$, is the range of a projection of norm one on $C([0,1])$, and is not isomorphic to any subspace of c_0. Moreover,*

$$\Gamma_X(t) = 2\mathbf{1}_{K'}(t) \quad (t \in [0,1]), \qquad \Omega(X) = K',$$

and K' is countably infinite.

Proof of Theorem 3.5. Take K as in (3.3) and set $X = E_K(C(K))$. Proposition 3.1 provides the isometry and the norm-one projection. Since $C(K)$ is linearly isometric to $C(\mathcal{S})$, Proposition 3.4 rules out an isomorphic embedding of X into c_0. Proposition 3.2 gives $\Omega(X) = K'$ and the asserted formula for Γ_X.

The set K' is compact and countable. It is infinite because, for each $n \geq 2$, the distinct points

$$\phi(\{n,q\}) = 2\,3^{-n} + 2\,3^{-q} \longrightarrow 2\,3^{-n} = \phi(\{n\}) \qquad (q \to \infty,\ q > n)$$

belong to K. Thus $2\,3^{-n} \in K'$ for every $n \geq 2$, which completes the proof. □

4. Realizations with prescribed spectra

4.1. **Prescribing the isomorphism type.**

Theorem 4.1. *Let $M \subseteq [0,1]$ be an uncountable meager F_σ set. For every separable infinite-dimensional Banach space E there exists a linear isomorphism T from E onto a closed subspace $X \subseteq C([0,1])$ such that $\Omega(X) = M$.*

Proof. Write $I = [0,1]$. Since M is an uncountable F_σ set, it contains an uncountable closed subset. The perfect set theorem yields a Cantor set $K \subseteq M$. Choose increasing closed nowhere dense sets F_j, $j \geq 1$, with $K \subseteq F_j$ and $\bigcup_{j\geq 1} F_j = M$. This is possible by taking finite unions of the original closed sets together with K. Put $F_0 = K$.

Equip B_{E^*} with the weak-star topology. Since this ball is compact and metrizable, the Hausdorff–Alexandroff theorem gives a continuous surjection from K onto it. Extend this map to $b : I \to B_{E^*}$ by affine interpolation on each bounded gap between points of K and by constant extension outside $[\min K, \max K]$. Fix $x \in E$ and define $g_x : K \to \mathbb{R}$ by $g_x(t) = b(t)(x)$. Since $b|_K$ is weak-star continuous, g_x is continuous on K. Evaluating the affine interpolation formula for b at x gives

$$b(t)(x) = (E_K g_x)(t) \qquad (t \in I),$$

where E_K is the extension operator from Proposition 3.1. Hence $t \mapsto b(t)(x)$ is continuous on I. Since this holds for every $x \in E$, the map b is weak-star continuous on I.

On each bounded gap (a,c) of K, the map b is affine with values in the line segment joining $b(a)$ and $b(c)$ in E^*. It is therefore norm continuous on (a,c). On the portions of I outside $[\min K, \max K]$, it is constant. Thus b is norm continuous at every point of $I \setminus K$.

Finally, since $b|_K$ maps K onto B_{E^*}, the Hahn–Banach theorem gives

$$\sup_{t\in K} |b(t)(x)| = \sup_{x^*\in B_{E^*}} |x^*(x)| = \|x\| \qquad (x \in E).$$

Choose a dense sequence (u_k) in E. By Hahn–Banach there are norm-one functionals $v_k \in E^*$ vanishing on the span of $u_1, \ldots, u_k$. Then $v_k \to 0$ weak-star. Set $v(0) = 0$ and $v(2^{-k}) = v_k$ for $k \geq 1$. Interpolate affinely between successive dyadic points and put $v(r) = v_1$ for $r \in [1/2, 1]$. Then

$\|v(r)\| \le 1$ everywhere and $\|v(2^{-k})\| = 1$ for every k. The map v is norm continuous on $(0,1]$ and weak-star continuous at zero, since each scalar interpolation tends to zero there.

Write $d_j(t) = \operatorname{dist}(t, F_j)$, put $a_0 = 4$ and $a_j = 4^{-j}$ for $j \ge 1$, and define

$$\Phi(t) = b(t) + \sum_{j\ge 0} a_j\, v(d_j(t)), \qquad (Tx)(t) = \Phi(t)(x).$$

The series converges uniformly in the dual norm, so Φ is bounded and weak-star continuous. Thus $T : E \to C(I)$ is bounded and linear. All added terms vanish on K, so $\|Tx\|_\infty \ge \|x\|$. In particular T is an isomorphism onto its closed range $X = T(E)$. Recall that $\Omega(X)$ consists of the points where the unit ball B_X fails to be equicontinuous. The identity

$$\|\Phi(s) - \Phi(t)\| = \sup_{x\in B_E} |(Tx)(s) - (Tx)(t)|$$

together with $B_{T(E)} \subseteq T(B_E) \subseteq \|T\| B_{T(E)}$ shows that $\Omega(T(E))$ is precisely the set of points where Φ fails to be norm continuous.

Fix $t \in I \setminus M$. Since $K \subseteq M$, the map b is norm continuous at t. Moreover, $d_m(t) > 0$ for every $m \ge 0$, because F_m is closed and $t \notin F_m$. Since v is norm continuous on $(0,1]$, each map $s \mapsto v(d_m(s))$ is norm continuous at t. The series defining $\Phi - b$ converges uniformly in norm, so Φ is norm continuous at t.

Now fix $t \in M$, and let j be the least index such that $t \in F_j$. We first choose points approaching t at which the jth summand has norm exactly a_j. For each $i \ge 1$, choose $r_i \in I \setminus F_j$ with $|r_i - t| < 1/i$. Such a point exists because F_j is nowhere dense. Since $d_j(r_i) > 0$, we may choose an integer $k_i \ge i$ such that $2^{-k_i} < d_j(r_i)$. The function d_j is continuous and $d_j(t) = 0$. Hence the intermediate value theorem gives a point s_i between t and r_i such that

$$d_j(s_i) = 2^{-k_i}.$$

Thus $s_i \to t$ and $\|v(d_j(s_i))\| = \|v(2^{-k_i})\| = 1$.

To estimate the contribution of the other terms, set

$$P_j(s) = b(s) + \sum_{m=0}^{j-1} a_m v(d_m(s)),$$

where the sum is empty if $j = 0$. Since the sets F_m are increasing, $t \in F_m$ for every $m \ge j$. Thus $v(d_m(t)) = v(0) = 0$ for all $m \ge j$. Applying the triangle inequality and using $\|v(r)\| \le 1$, we obtain

$$\begin{aligned}\|\Phi(s_i) - \Phi(t)\| &\ge a_j \|v(d_j(s_i))\| - \|P_j(s_i) - P_j(t)\| - \sum_{m>j} a_m \|v(d_m(s_i))\| \\ &\ge a_j - \|P_j(s_i) - P_j(t)\| - \sum_{m>j} a_m.\end{aligned}$$

If $j \ge 1$, the minimality of j gives $t \notin F_m$ for every $m < j$, in particular $t \notin F_0 = K$. Thus b is norm continuous at t, and $d_m(t) > 0$ for every $m < j$. It follows that P_j is norm continuous at t. Since $\sum_{m>j} a_m = a_j/3$, we conclude that

$$\liminf_{i\to\infty} \|\Phi(s_i) - \Phi(t)\| \ge a_j - \frac{a_j}{3} = \frac{2a_j}{3} > 0.$$

If $j = 0$, then $P_0 = b$. Both $b(s_i)$ and $b(t)$ belong to B_{E^*}, so $\|b(s_i) - b(t)\| \le 2$. Using $a_0 = 4$ and $\sum_{m\ge 1} a_m = 1/3$ in the same estimate gives

$$\|\Phi(s_i) - \Phi(t)\| \ge 4 - 2 - \frac{1}{3} = \frac{5}{3} \qquad (i \ge 1).$$

In both cases Φ fails to be norm continuous at t. Its set of norm discontinuities is therefore exactly M, and hence $\Omega(T(E)) = M$. □

4.2. **Uncomplemented realizations.** Xie's complemented realization theorem [6, Theorem 1.2] applies to every F_σ set. We now construct an uncomplemented realization for each nonempty such set.

Theorem 4.2. *Let M be a nonempty F_σ subset of $[0,1]$. There exists a closed linear subspace W of $C[0,1]$ such that W is uncomplemented in $C[0,1]$ and $\Omega(W) = M$.*

Proof. Fix $t \in M$. For $k = 0, 1, \ldots$, set

$$x_k = t - 2^{-k}t, \qquad y_k = t + 2^{-k}(1-t).$$

If $t > 0$, the points x_k increase from 0 to t, and the intervals $[x_k, x_{k+1}]$ cover $[0, t)$. If $t < 1$, the points y_k decrease from 1 to t, and the intervals $[y_{k+1}, y_k]$ cover $(t, 1]$. Enumerate the resulting intervals as $I_n = [a_n, b_n]$ and let $q_n = (a_n + b_n)/2$. Their interiors are pairwise disjoint. Since $x_k \to t$ and $y_k \to t$, every neighborhood of t contains all but finitely many of the I_n.

For each n, the set $M \cap I_n$ is an F_σ subset of I_n. If $M \cap I_n \neq \emptyset$, Xie's realization theorem [6, Theorem 1.1], applied after an affine identification of I_n with $[0,1]$, yields a closed subspace $E_n \subseteq C(I_n)$ with $\Omega(E_n) = M \cap I_n$. If $M \cap I_n = \emptyset$, take $E_n = \{0\}$. In either case, set

$$F_n = \{f \in E_n : f(a_n) = f(q_n) = f(b_n) = 0\}.$$

Then F_n is closed and has codimension at most three in E_n. In particular, $\Omega(F_n) = \Omega(E_n)$.

Given $f_n \in F_n$ with $\|f_n\|_\infty \to 0$, define f on $[0,1]$ by $f|_{I_n} = f_n$ and $f(t) = 0$. The pieces agree at shared endpoints, and the decay of their norms ensures continuity at t. Since $\|f\|_\infty = \sup_n \|f_n\|_\infty$, the space F of all such functions is isometric to the c_0-sum of the F_n and hence is closed. Each $s \neq t$ has a neighborhood meeting only the one or two intervals that contain s. On each such I_n, the restrictions of the unit ball of F form the entire unit ball of F_n, since each member of F_n extends by zero. Thus the unit ball of F is equicontinuous at s precisely when the unit ball of each of those F_n is equicontinuous at s relative to I_n. Consequently,

$$\Omega(F) \setminus \{t\} = M \setminus \{t\}.$$

Choose a closed subspace Z of c_0 that is uncomplemented in c_0. Such a subspace exists by the theorem of Lindenstrauss and Tzafriri, since c_0 is nonreflexive and hence cannot be isomorphic to a Hilbert space. Let h_n be the tent function that vanishes outside I_n, takes the values $0, 1, 0$ at a_n, q_n, b_n, respectively, and is affine on each half of I_n. Define $H : c_0 \to C[0,1]$ and $S : C[0,1] \to c_0$ by

$$Hx = \sum_{n=1}^{\infty} x_n h_n, \qquad Sf = \big(f(q_n) - f(t)\big)_{n \geq 1}.$$

At most one tent function is nonzero at any point. Since $x_n \to 0$, the series converges uniformly and $\|Hx\|_\infty = \|x\|_\infty$. Since $q_n \to t$, the operator S takes values in c_0, and $\|S\| \leq 2$. By construction, $SH = \mathrm{id}_{c_0}$ and $S|_F = 0$.

Set $W = F + H(Z)$. Suppose $f_k + Hz_k \to w$ with $f_k \in F$ and $z_k \in Z$. Applying S gives $z_k \to Sw \in Z$. It follows that $f_k \to w - HSw \in F$, so W is closed. For $w \in W$, the components $HSw \in H(Z)$ and $w - HSw \in F$ depend continuously on w. Near each $s \neq t$, only finitely many tent functions can be nonzero. The restrictions of the unit ball of $H(Z)$ to that neighborhood therefore form a bounded subset of a finite-dimensional space of continuous functions and are equicontinuous at s. Together with $F \subseteq W$, the bounded component projections give

$$\Omega(W) \setminus \{t\} = \Omega(F) \setminus \{t\}.$$

To see that $t \in \Omega(W)$, fix N. Since Z is infinite-dimensional, it contains a unit vector z whose first N coordinates vanish. A nonzero sequence in c_0 attains its supremum norm, so $|z_n| = 1$ for some $n > N$. Then Hz lies in the unit ball of W and

$$|(Hz)(q_n) - (Hz)(t)| = 1.$$

As N can be taken arbitrarily large and $q_n \to t$, the unit ball of W fails to be equicontinuous at t. Since $t \in M$, this proves $\Omega(W) = M$.

Finally, suppose there were a bounded projection $P : C[0,1] \to W$. Then SPH would be a bounded operator on c_0 with range contained in $S(W) = Z$. For every $z \in Z$ we have $Hz \in W$, so $SPHz = SHz = z$. Thus SPH would be a bounded projection of c_0 onto Z, contradicting the choice of Z. □

5. When spectra force nonisomorphism

Proposition 2.2 gives one condition on the spectrum under which the dual is separable. Baire category gives a condition under which the dual is nonseparable.

Theorem 5.1. *Let X be a closed linear subspace of $C([0,1])$ and let $M = \Omega(X)$ be its oscillating spectrum. If M is nonmeager in $[0,1]$, then X^* is nonseparable in the norm topology.*

Proof. We prove the contrapositive. Suppose that X^* is norm separable and choose a norm-dense sequence $(u_n)_{n\geq 1}$ in X^*. For $t \in [0,1]$, let $e(t) = \delta_t|_X$. Since

$$\|e(s) - e(t)\| = \sup_{f \in B_X} |f(s) - f(t)|,$$

the unit ball B_X is equicontinuous at t exactly when e is norm continuous at t. Thus $\Omega(X)$ is the set of norm discontinuities of e.

For integers $m, n \geq 1$, set

$$F_{m,n} = \{t \in [0,1] : \|e(t) - u_n\| \leq 2^{-m}\}.$$

Each $F_{m,n}$ is closed: it is the intersection, over $f \in B_X$, of the closed sets $\{t : |f(t) - u_n(f)| \leq 2^{-m}\}$. For each fixed m, the sets $F_{m,n}$ cover $[0,1]$ by density of (u_n). Their boundaries, taken relative to $[0,1]$, are nowhere dense because the sets are closed. Hence

$$D = \bigcup_{m,n\geq 1} \partial F_{m,n}$$

is meager in $[0,1]$.

Fix $t \notin D$ and $\varepsilon > 0$. Choose m with $2^{1-m} < \varepsilon$ and then n with $t \in F_{m,n}$. As $t \notin \partial F_{m,n}$, some relative neighborhood U of t is contained in $F_{m,n}$. For every $s \in U$,

$$\|e(s) - e(t)\| \leq \|e(s) - u_n\| + \|u_n - e(t)\| \leq 2^{1-m} < \varepsilon.$$

Thus e is norm continuous at t. It follows that $\Omega(X) \subseteq D$, so $\Omega(X)$ is meager. □

Together with the realization theorems, these two implications give the following answer to [2, Question 4.9] for nonempty spectra.

Theorem 5.2. *Let A and B be nonempty F_σ subsets of $[0,1]$. The following assertions are equivalent.*

(i) *For every pair of closed subspaces $X, Y \subseteq C([0,1])$ with $\Omega(X) = A$ and $\Omega(Y) = B$, the spaces X and Y are not isomorphic.*
(ii) *One of A and B is countable and the other is nonmeager in $[0,1]$.*

Proof. Suppose that (ii) holds and, by symmetry, that A is countable and B is nonmeager. For any X and Y as in (i), Proposition 2.2 gives that X^* is separable, whereas Theorem 5.1 gives that Y^* is nonseparable. An isomorphism between X and Y would induce an isomorphism between their duals. This proves (i).

For the converse, we show that every nonempty meager F_σ set is the spectrum of a copy of c_0, and that every uncountable F_σ set is the spectrum of a copy of $C([0,1])$.

If M is uncountable, meager, and F_σ, both realizations follow from Theorem 4.1, applied first to c_0 and then to $C([0,1])$. If M is nonempty and finite, Theorem 2.1 gives the required copy of c_0.

Now let $M = \{t_j : j \geq 1\}$ be countably infinite, with the points listed without repetition. For each j, use the singleton tent isometry $H_j : c_0 \to C([0,1])$ from the proof of Theorem 2.1. Thus $H_j(B_{c_0})$ is equicontinuous away from t_j. If e_n is the nth unit vector of c_0 and $q_{j,n}$ is the midpoint of its tent, then $q_{j,n} \to t_j$ and

$$(H_j e_n)(q_{j,n}) = 1, \qquad (H_j e_n)(t_j) = 0.$$

Set $a_j = 4^{1-j}$ and $T = \sum_{j\geq 1} a_j H_j$. The series converges in operator norm. Since H_1 is an isometry and $\sum_{j>1} a_j = 1/3$, we have

$$\tfrac{2}{3}\|x\|_\infty \leq \|Tx\|_\infty \leq \tfrac{4}{3}\|x\|_\infty \qquad (x \in c_0).$$

In particular, T is an isomorphic embedding, and $T(B_{c_0})$ and the unit ball of $T(c_0)$ are equicontinuous at the same points.

At each $t \notin M$, the images of B_{c_0} under the partial sums are equicontinuous at t. Operator norm convergence gives the same property for $T(B_{c_0})$. At t_j, the terms with index $i < j$ are equicontinuous on the unit ball at t_j, and each term with $i > j$ contributes at most $2a_i$ to the absolute difference between the values at $q_{j,n}$ and t_j. Testing on e_n therefore gives

$$\liminf_{n\to\infty} |(Te_n)(q_{j,n}) - (Te_n)(t_j)| \geq a_j - 2\sum_{i>j} a_i = \tfrac{1}{3}a_j > 0.$$

Hence $\Omega(T(c_0)) = M$.

Finally, suppose that M is nonmeager and F_σ. By Xie's complemented realization theorem [6, Theorem 1.2], there is a complemented subspace Z of $C([0,1])$ with $\Omega(Z) = M$. Theorem 5.1 yields that Z^* is nonseparable. Rosenthal's theorem [5, Theorem 5.3] then gives $Z \cong C([0,1])$.

If (ii) fails, then either both A and B are meager or both are uncountable, since countable subsets of $[0,1]$ are meager. In the first case choose copies of c_0 with spectra A and B. In the second case choose copies of $C([0,1])$ with those spectra. In either case the two spaces are isomorphic, so (i) fails. □

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