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# NON-EFFECTIVE FINITE EQUIVALENCE RELATIONS ON FINITE SCHEMES OVER A FIELD: VENKEN'S EXAMPLE CHECKED, AND A SECOND ONE

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Abstract. In 1961 Grothendieck asked whether a closed equivalence relation on a finite scheme over a field, with finite but not necessarily flat projections, is always effective. In a lecture at Buffalo in 1973 he still called the question open. It had been answered in the negative in 1971 by Venken, in a two-page note about the descent of flat modules whose Corollary 3.1 is a non-effective equivalence relation on the spectrum of a local Artin algebra of dimension 33 in six variables, over any field. We give a second example, in two variables, of length 15 (16 in characteristic 2), obtained by restricting Raicu's non-effective relation on the affine plane to a finite subscheme cut out by squares of invariant functions. It is proved by hand, in every characteristic at once: a general lemma reduces the comparison of the two ideals in degree 5 to the subalgebra $C$ generated by three invariant functions, and a linear form on the degree-5 part of $A \otimes_C A$ separates the obstruction. The example is shorter than Venken's; no minimality is claimed. We then recheck Venken's example by linear algebra, where the ideal of his relation exceeds, by one dimension in degree 4, the ideal of $X \times_Y X$, $Y$ the quotient; this check is formalised in Lean, over any nonzero commutative ring, except for the dimension counts. The computer checks of both examples, over $\mathbb{Q}$ and over the finite prime fields that the pivots require, are reported separately from the proofs.

## 1. The question

Let $k$ be a field, $X = \operatorname{Spec} A$ a finite $k$-scheme and $R \subset X \times_k X$ a closed subscheme whose points with values in every $k$-algebra form an equivalence relation. Its projections $R \rightrightarrows X$ are finite, but they need not be flat. The ring of invariants

$$B = A^R = \{\, a \in A : a \otimes 1 - 1 \otimes a \in I_R \,\},$$

with $I_R \subset A \otimes_k A$ the ideal of $R$, is a finite $k$-algebra, and $Y = \operatorname{Spec} B$ is the quotient of $X$ by $R$ in the category of schemes. The relation is *effective* when the morphism $R \to X \times_Y X$ is an isomorphism.

In the third of his talks on the construction of schemes, after the flat case, Grothendieck wrote that without flatness he did not know whether a finite equivalence relation is effective. He reduced the question to the spectrum of a local Artin ring with algebraically closed residue field, and said that it was open even over a field [3, §2, p. 212-05]. The flat case is SGA 3, exp. V [9].

The answer is in print. Venken's note of 1971 [10], presented to the Académie on 5 April 1971, is titled after the descent of flat modules, and most of it is about modules: it constructs a finite injective morphism $f$ of local Artin rings which is a descent morphism for flat modules, and a descent datum on a free module which is not effective. On the way, its Corollary 3.1 states that a certain ideal of $B \otimes_A B$ is an equivalence ideal and that the equivalence relation it defines is not effective (his $B$ is our $A$, and his $A$ a subring of it). The ring is a local Artin algebra over an arbitrary field $K$, with residue field $K$, so this is a negative answer to the question over every field. Venken takes his definitions from Grothendieck's first talk on descent [2].

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Two years later, at the end of a lecture of his algebraic geometry course at the State University of New York at Buffalo, on 11 July 1973, a listener asked for open problems and he gave this one: he said it was not known, that he expected the answer to be negative, and that it was very elementary [1]. In the appendix to Kollár's paper on quotients by finite equivalence relations, Raicu gave finite equivalence relations on $\mathbb{A}^2$ which are not effective [4, §50], and affine toric equivalence relations are effective [7]. Philippe studied descent along unramified morphisms [6].

**Theorem 1** (Venken, 1971)**.** *Let $K$ be any field. There is a local Artin $K$-algebra of dimension* 33*, with residue field $K$, and a closed equivalence relation on its spectrum, with finite projections, which is not effective.*

**Theorem 2.** *Let $k$ be any field. There is a graded local Artin $k$-algebra of dimension* 15 *(*16 *if $k$ has characteristic* 2*) with two generators, with residue field $k$, and a closed equivalence relation on its spectrum, with finite projections, which is not effective.*

Both examples also answer the more general question, for $X$ finite over a noetherian base. The second is shorter than Venken's example as it is studied here: length 15, or 16 in characteristic 2, against 33, and two generators against six. No minimality is claimed; we do not know whether a shorter example exists.

Section 2 recalls that effectivity is the equality of two ideals and proves a lemma that compares them in the first degree where the relation has an extra equation. Section 3 gives the second example and proves Theorem 2 by hand. Section 4 checks Venken's Corollary 3.1 by machine, since his note gives it in one line after a cohomological computation, and describes its formalisation in Lean. Section 5 reports the computations that found the second example and certify it over every field independently of the proof by hand, and Section 6 separates what is proved by hand, checked by scripts, formalised in Lean, and taken from the historical sources.

## 2. Effectivity as linear algebra

With the notation of the question, $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$ and $A \otimes_B A = (A \otimes_k A)/J_B$, where $J_B$ is the ideal generated by the $\delta(b) = b \otimes 1 - 1 \otimes b$, $b \in B$. By definition of $B$, $J_B \subseteq I_R$, and $R \to X \times_Y X$ is the closed immersion defined by $I_R/J_B$. So $R$ is effective if and only if $I_R = J_B$, a comparison of two subspaces of the finite-dimensional space $A \otimes_k A$. For any subalgebra $C \subseteq A$, the ideal $J_C$ generated by the $\delta(c)$, $c \in C$, is the kernel of $A \otimes_k A \to A \otimes_C A$; if $C \subseteq B$, then $J_C \subseteq J_B \subseteq I_R$. Since $\delta(uv) = (u \otimes 1)\,\delta(v) + \delta(u)\,(1 \otimes v)$, $J_C$ is generated by the $\delta(c)$ for $c$ in any set of generators of the algebra $C$.

Suppose now that $A = \bigoplus_{e \geq 0} A_e$ is graded with $A_0 = k$, that $A \otimes_k A$ carries the total grading, and that $I_R$ is homogeneous. Then $B$ is a graded subalgebra: if $a = \sum_e a_e$ is invariant, each $\delta(a_e)$ is the component of degree $e$ of $\delta(a) \in I_R$, so it lies in $I_R$. Hence $J_B$ is generated by the $\delta(b)$ with $b$ homogeneous of positive degree ($\delta$ vanishes on $A_0 = k$), and it is homogeneous; the comparison can be made degree by degree. A homogeneous invariant of degree $e$ contributes to $J_B$ only in degrees $\geq e$, so to compare the two ideals in degree $d$ it suffices to know the invariants of degree at most $d$. The following lemma says that in the first degree where $I_R$ has an equation not coming from a known subalgebra of invariants, the invariants of that degree are the only ones that matter.

**Lemma 3.** *In this graded setting, let $C \subseteq B$ be a graded subalgebra and $d \geq 1$ an integer such that $(I_R)_e = (J_C)_e$ for every $e < d$; this holds, for instance, when $I_R$ is generated by $J_C$ and by homogeneous elements of degree at least $d$. Then:*

1. *$(J_B)_d = (J_C)_d + \delta(B_d)$: the invariants of degree less than $d$ give nothing in degree $d$ beyond $J_C$;*
2. *if $B_d = C_d$, in particular if $A_d = C_d$, then $(J_B)_d = (J_C)_d$, so that $R$ is not effective as soon as $(I_R)_d \neq (J_C)_d$, that is, as soon as some element of $(I_R)_d$ has a nonzero image in $(A \otimes_C A)_d$.*

*Proof.* Since $C \subseteq B$, $J_C \subseteq J_B$, and $\delta(B_d) \subseteq (J_B)_d$; so the right-hand side of (i) lies in the left-hand side. Conversely, $(J_B)_d$ is spanned by the products $\delta(b)\,t$ with $b \in B_e$, $1 \le e \le d$, and $t \in (A \otimes_k A)_{d-e}$. If $e < d$, then $\delta(b) \in (I_R)_e = (J_C)_e$, so $\delta(b)\,t \in J_C$. If $e = d$, then $t \in (A \otimes_k A)_0 = k$, and $\delta(b)\,t \in \delta(B_d)$. This proves (i). If $B_d = C_d$, then $\delta(B_d) = \delta(C_d) \subseteq (J_C)_d$; and $C_d \subseteq B_d \subseteq A_d$, so $A_d = C_d$ forces $B_d = C_d$. Then $(J_B)_d = (J_C)_d$, and if $(I_R)_d \neq (J_C)_d$, then $J_B \neq I_R$. Finally $(A \otimes_C A)_d = (A \otimes_k A)_d/(J_C)_d$. For the instance mentioned in the statement, the ideal generated by homogeneous elements of degree at least $d$ has no nonzero component in degrees $e < d$. □

The lemma does not use that $R$ is an equivalence relation, only that $I_R$ is a homogeneous ideal containing $J_C$, with $B$ defined as above. It reduces a proof of non-effectivity to two facts that do not involve $B$: an equality $A_d = C_d$, and a nonzero class in $(A \otimes_C A)_d$.

## 3. The second example

3.1. **Raicu's relation and its restriction.** Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for the coordinates of $\mathbb{A}^2 \times \mathbb{A}^2$. Raicu's relation [4, §50] is the closed subscheme $R \subset \mathbb{A}^2 \times \mathbb{A}^2$ with ideal

$$I = (x_1^2 - y_1^2,\ x_1x_2 - x_2^2 - y_1y_2 + y_2^2,\ x_2^3 - y_2^3,\ (x_1y_2 - x_2y_1)\,y_2^3) \subset k[x_1, x_2, y_1, y_2].$$

It is a finite equivalence relation on $\mathbb{A}^2$, defined over $\mathbb{Z}$, and it is not effective after base change to any field; the class of $(x_1y_2 - x_2y_1)\,y_2^3$ is the obstruction. These facts are Raicu's; the proof below uses only the first of them. The functions

$$f_1 = x_1^2, \qquad f_2 = x_1x_2 - x_2^2, \qquad f_3 = x_2^3$$

are invariant, since $f_i(x) - f_i(y)$ is one of the first three generators of $I$. Let

$$Z = V(f_1^2, f_2^2, f_3^2) \subset \mathbb{A}^2, \qquad A = k[x_1, x_2]/(f_1^2, f_2^2, f_3^2).$$

Since $f_1^2 = x_1^4$ and $f_3^2 = x_2^6$, $Z$ is supported at the origin, and $A$ is a local Artin algebra with residue field $k$; a count in each degree shows that the Hilbert function of $A$ is $1, 2, 3, 4, 3, 2$ (in degree 6 the three relations coming from $f_2^2$ have a $3 \times 3$ matrix of determinant $-4$, invertible when $2 \neq 0$), so that $\dim_k A = 15$. In characteristic 2, $f_2^2 = x_1^2x_2^2 + x_2^4$ loses its middle term, the Hilbert function is $1, 2, 3, 4, 3, 2, 1$ and $\dim_k A = 16$; the proof below uses only degrees at most 5, where nothing changes. Let $R_Z = R \cap (Z \times Z)$, the closed subscheme of $Z \times Z$ with ideal $I_{R_Z}$, the image of $I$ in $A \otimes_k A$.

**Lemma 4.** *$R_Z$ is a closed equivalence relation on $Z$ with finite projections.*

*Proof.* For every $k$-algebra $T$, $R_Z(T) = R(T) \cap (Z(T) \times Z(T))$ is the restriction of an equivalence relation on $\mathbb{A}^2(T)$ to the subset $Z(T)$, hence an equivalence relation. $R_Z$ is closed in $Z \times Z$, which is finite over $Z$ by either projection. □

The restriction to $Z$ is what lets a question about $\mathbb{A}^2$ be asked about a finite scheme. It is not formal that non-effectivity survives it: the ring of invariants of $R_Z$ is computed on $A$, not on $k[x_1, x_2]$, and it can be larger than the image of the invariants on $\mathbb{A}^2$, so that $J_B$ can grow. We now prove that it does not grow enough.

All the ideals above are homogeneous for the standard grading. In $k[x, y] = k[x_1, x_2, y_1, y_2]$ put $K = (f_1^2, f_2^2, f_3^2)(x) + (f_1^2, f_2^2, f_3^2)(y)$, so that $A \otimes_k A = k[x, y]/K$. The ideal of $R_Z$ lifts to $I + K$, and $J_B$ lifts to $J + K$, where $J$ is generated by the $b(x) - b(y)$ for $b$ running over homogeneous lifts of a basis of $B$. Let $C \subset A$ be the subalgebra generated by the classes of $f_1, f_2, f_3$; it lies in $B$. Put $g_i = f_i(x) - f_i(y)$, the first three generators of $I$, and

$$L = (g_1, g_2, g_3) + K \subset k[x, y],$$

the preimage of $J_C$ (Section 2), so that $k[x, y]/L = A \otimes_C A$ and $L \subseteq J + K \subseteq I + K$. The witness is the last generator of $I$,

$$w = (x_1y_2 - x_2y_1)\,y_2^3,$$

Raicu's obstruction, of degree 5.

3.2. **Some degrees of $A$.** In degrees at most 3, $A$ coincides with $k[x_1, x_2]$. In degree 4 the relations are $x_1^4 = 0$ and $f_2^2 = x_1^2x_2^2 - 2x_1x_2^3 + x_2^4 = 0$, so $A_4$ has dimension 3, with basis $x_1^3x_2$, $x_1^2x_2^2$, $x_1x_2^3$, and

$$x_2^4 = 2x_1x_2^3 - x_1^2x_2^2. \tag{1}$$

In degree 5, $x_1^5 = x_1^4x_2 = 0$, and $x_1f_2^2 = x_2f_2^2 = 0$ give $x_1x_2^4 = 2x_1^2x_2^3 - x_1^3x_2^2$ and $x_2^5 = 3x_1^2x_2^3 - 2x_1^3x_2^2$. So $A_5$ is spanned by $x_1^3x_2^2$ and $x_1^2x_2^3$, and since $f_1f_3 = x_1^2x_2^3$ and $f_2f_3 = x_1x_2^4 - x_2^5 = x_1^3x_2^2 - x_1^2x_2^3$,

$$C_5 = A_5. \tag{2}$$

These formulas have integer coefficients and hold in every characteristic. The graded pieces of $C$ in low degree are $C_1 = 0$, $C_2 = kf_1 + kf_2$, $C_3 = kf_3$, $C_4 = kf_1f_2$ (because $f_1^2 = f_2^2 = 0$ in $A$) and $C_5 = A_5$.

3.3. **Reduction to $A \otimes_C A$.**

**Lemma 5.** *$(J + K)_5 = L_5$, and $(I + K)_5 = L_5 + kw$.*

*Proof.* In $A \otimes_k A$, $J_C$ is generated by the images of $g_1, g_2, g_3$, since $f_1, f_2, f_3$ generate $C$ (Section 2), and $I_{R_Z}$ is generated by $J_C$ and the image of $w$, which has degree 5. Lemma 3 applies with $d = 5$: $(I_{R_Z})_e = (J_C)_e$ for $e < 5$, and $A_5 = C_5$ by (2), so $(J_B)_5 = (J_C)_5$. Taking preimages in $k[x, y]$, all of which contain $K$, gives $(J + K)_5 = L_5$. The second equality holds because $I + K = L + (w)$ and the degree-5 part of the ideal $(w)$ is $kw$. □

So $R_Z$ is not effective as soon as $w \notin L_5$, that is, as soon as the class of $w$ in $(A \otimes_C A)_5$ is not zero. The ring $B$ plays no further role.

3.4. **A presentation of $A \otimes_C A$.** Let $C_+ = \bigoplus_{e>0} C_e$ be the augmentation ideal of $C$. In $A/C_+A = k[x_1, x_2]/(f_1, f_2, f_3)$ one has $x_1^2 = 0$ and $x_1x_2 = x_2^2$, hence $x_1^2x_2 = 0$ and $x_1x_2^2 = x_2^3 = 0$; this ring has basis $1, x_1, x_2, x_2^2$. By the graded Nakayama lemma,

$$e_0 = 1, \quad e_1 = x_1, \quad e_2 = x_2, \quad e_3 = x_2^2$$

generate $A$ as a $C$-module. For a graded module $M$ and an integer $j$, write $M(-j)$ for $M$ with degrees shifted, $M(-j)_d = M_{d-j}$. Let

$$F = C \oplus C(-1) \oplus C(-1) \oplus C(-2)$$

be the graded free $C$-module on four basis elements of degrees $0, 1, 1, 2$, mapped to $e_0, e_1, e_2, e_3$, and $E \subset F$ the graded module of relations: the $(c_0, c_1, c_2, c_3)$ with $\sum c_je_j = 0$ in $A$, an element of $E_d$ having $c_0 \in C_d$, $c_1, c_2 \in C_{d-1}$ and $c_3 \in C_{d-2}$. Tensoring the exact sequence $0 \to E \to F \to A \to 0$ with $A$ over $C$ gives, by right exactness,

$$A \otimes_C A = (A \otimes_C F)/AE, \qquad A \otimes_C F = A \oplus A(-1) \oplus A(-1) \oplus A(-2),$$

where $(a_0, a_1, a_2, a_3)$ stands for $\sum a_j \otimes e_j$ and $AE$ is the $A$-submodule generated by the image of $E$. In degree 5, $(A \otimes_C F)_5 = A_5 \oplus A_4 \oplus A_4 \oplus A_3$, and $(AE)_5$ is spanned by the products $ar$ with $r \in E_d$, $d \leq 5$, and $a \in A_{5-d}$. The class of $w = x_1 \otimes x_2^4 - x_2 \otimes x_1x_2^3 = x_1 \otimes f_3e_2 - x_2 \otimes f_3e_1$ is

$$W = (0,\ -x_2^4,\ x_1x_2^3,\ 0).$$

3.5. **The relations up to degree 5.**

- $E_d = 0$ for $d \leq 2$: in degree 2 a relation reads $c_0 + \gamma x_2^2 = 0$ with $c_0 \in kf_1 + kf_2$ and $\gamma \in k$, and $x_1^2$, $x_1x_2 - x_2^2$, $x_2^2$ are independent; degrees 0 and 1 are clear.
- $E_3 = k\rho$ with $\rho = (f_3,\ f_2,\ f_2 - f_1,\ 0)$. The identity $f_3 + x_1f_2 + x_2(f_2 - f_1) = 0$ holds in $k[x_1, x_2]$. The five products $f_3$, $x_1f_1$, $x_2f_1$, $x_1f_2$, $x_2f_2$ span $A_3$ (they give $x_2^3$, $x_1^3$, $x_1^2x_2$, then $x_1x_2^2$), which has dimension 4, so the relations of degree 3 form a line.

- $E_4 = k\sigma_1 + k\sigma_2$ with
$$\sigma_1 = (0,\ -f_3,\ f_3,\ f_2), \qquad \sigma_2 = (0,\ -2f_3,\ f_3,\ f_1).$$
Indeed $-x_1f_3 + x_2f_3 + x_2^2f_2 = 0$ in $k[x_1, x_2]$, and $-2x_1f_3 + x_2f_3 + x_2^2f_1 = f_2^2$, which is 0 in $A$. The space $F_4 = C_4 \oplus C_3 \oplus C_3 \oplus C_2$ has dimension at most 5, and its image contains $x_1f_3 = x_1x_2^3$, $x_2^2f_1 = x_1^2x_2^2$ and $f_1f_2 + x_2^2f_1 = x_1^3x_2$, which span $A_4$. So $E_4$ has dimension at most 2, and $\sigma_1, \sigma_2$ are independent (look at the last entries).
- An element of $E_5$ has $c_1 \in C_4 = kf_1f_2$ and $c_3 \in C_3 = kf_3$. Nothing more is needed.

3.6. **The linear form.** Let $\lambda$ be the linear form on the forms of degree 4 with $\lambda(x_1^i x_2^{4-i}) = 1$ for $0 \le i \le 3$ and $\lambda(x_1^4) = 0$. Since $\lambda(f_2^2) = 1 - 2 + 1 = 0$, it is defined on $A_4$. Let $\nu$ be the linear form on $A_3$ with
$$\nu(x_1^3) = \nu(x_1^2x_2) = 2, \qquad \nu(x_1x_2^2) = 1, \qquad \nu(x_2^3) = 0,$$
and put $\varphi(a_0, a_1, a_2, a_3) = \lambda(a_1) + \nu(a_3)$ on $(A \otimes_C F)_5$.

**Lemma 6.** *$\varphi$ vanishes on $(AE)_5$, and $\varphi(W) = -1$.*

*Proof.* By the list above it suffices to evaluate $\varphi$ on $a\rho$ for $a \in A_2$, on $b\sigma_1$ and $b\sigma_2$ for $b \in A_1$, and on $E_5$.

- $\varphi(a\rho) = \lambda(af_2)$, and for $a = x_1^2, x_1x_2, x_2^2$, $af_2$ is $x_1^3x_2 - x_1^2x_2^2$, $x_1^2x_2^2 - x_1x_2^3$, $x_1x_2^3 - x_2^4$: each has $\lambda = 1 - 1 = 0$.
- $\varphi(b\sigma_1) = -\lambda(bf_3) + \nu(bf_2)$. For $b = x_1$: $-\lambda(x_1x_2^3) + \nu(x_1^2x_2) - \nu(x_1x_2^2) = -1 + 2 - 1 = 0$. For $b = x_2$: $-\lambda(x_2^4) + \nu(x_1x_2^2) - \nu(x_2^3) = -1 + 1 - 0 = 0$.
- $\varphi(b\sigma_2) = -2\lambda(bf_3) + \nu(bf_1)$. For $b = x_1$: $-2 + \nu(x_1^3) = 0$. For $b = x_2$: $-2 + \nu(x_1^2x_2) = 0$.
- For $r \in E_5$, $\varphi(r) = \lambda(c_1) + \nu(c_3)$ is a combination of $\lambda(f_1f_2) = \lambda(x_1^3x_2) - \lambda(x_1^2x_2^2) = 0$ and $\nu(f_3) = \nu(x_2^3) = 0$.

Finally $\varphi(W) = -\lambda(x_2^4) = -1$. □

**Proposition 7.** *Over every field, $w \in (I + K)_5$ and $w \notin (J + K)_5$. Hence $R_Z$ is not effective.*

*Proof.* By Lemma 6, $\varphi$ induces a linear form on $(A \otimes_C A)_5 = (A \otimes_C F)_5/(AE)_5$ that does not vanish on the class of $w$, so $w \notin L_5$. By Lemma 5, $(J + K)_5 = L_5$. □

*Proof of Theorem 2.* $A$ is a graded local Artin $k$-algebra with two generators, residue field $k$ and dimension 15 (16 in characteristic 2); $R_Z$ is a closed equivalence relation on $Z = \operatorname{Spec} A$ with finite projections by Lemma 4, and it is not effective by Proposition 7. □

All the numbers in the proof are integers, and the only one that has to be nonzero is $\varphi(W) = -1$; so the argument is the same in every characteristic, 2 and 3 included, although $A$ has length 16 in characteristic 2. The proof does not determine $B$: through Lemma 3, it only uses that invariants of degree at most 4 give differences in $L$, and that $A_5 = C_5$. It does not transport Raicu's argument on $\mathbb{A}^2$ either; it compares $R_Z$ with $Z \times_{\operatorname{Spec} C} Z$ directly. The script `hand.py` of the directory `computations/effective/raicu` of the repository [8] checks each identity used above over $\mathbb{Z}$.

## 4. Venken's example

Let $K$ be a field. In Venken's notation [10], $\mathbf{B} = K[M, N, X, Y, Z, W]/\mathbf{I}$, where $\mathbf{I}$ is generated by
$$M^2,\ MN,\ N^2,\ X^3,\ Y^3,\ Z^3,\ W^3,\ YZ,\ NY,\ MZ,\ NXY,\ MYW$$
and all monomials of degree 3 in $X, Y, Z, W$; lower-case letters denote the classes. Let $\mathbf{A} \subset \mathbf{B}$ be the subalgebra generated by $m$, $n$, $mx$, $nw$ and $my - nz$. Its maximal ideal has square zero, so $\dim_K \mathbf{A} = 6$. Venken proves that $\mathbf{A} \to \mathbf{B}$ is a descent morphism for flat modules, that $H^1(\mathbf{B}/\mathbf{A}, \mathbb{G}_a) \neq 0$, with $mxy \otimes w$ giving a nonzero class, and deduces:

> *Corollary 3.1.* The ideal $\mathbf{J}$ of $\mathbf{B} \otimes_{\mathbf{A}} \mathbf{B}$ generated by $mxy \otimes w$ is an equivalence ideal, and the equivalence relation it defines is not effective.

As a closed subscheme of $\operatorname{Spec}(\mathbf{B} \otimes_K \mathbf{B})$, this relation $R_V$ has ideal

$$I_V = (a \otimes 1 - 1 \otimes a : a = m, n, mx, nw, my - nz) + (mxy \otimes w),$$

since the first generators cut $\mathbf{B} \otimes_{\mathbf{A}} \mathbf{B}$ out of $\mathbf{B} \otimes_K \mathbf{B}$. Its points are an equivalence relation on $\operatorname{Spec} \mathbf{B}$ over $K$, and its projections are finite because $\mathbf{B}$ is. All the generators are monomials or binomials of one degree, so the graded setting of Section 2 applies with every variable of degree 1.

**Proposition 8.** *Over every field $K$:*

1. $\dim \mathbf{B} = 33$*, and* $\mathbf{B}$ *vanishes in degrees above* 3*;*
2. $I_V$ *satisfies the three axioms of an equivalence relation: it lies in the kernel of the multiplication, it is stable under the exchange of factors, and its image under $p_{13}$ lies in the ideal generated by its images under $p_{12}$ and $p_{23}$ in $\mathbf{B}^{\otimes 3}$;*
3. *the ring of invariants $\mathbf{B}^{R_V}$ has dimension* 10*, so it is strictly larger than* $\mathbf{A}$*;*
4. $I_V$ *and* $J_{\mathbf{B}^{R_V}}$ *differ only in degree* 4*, where their dimensions are* 269 *and* 268*, and $mxy \otimes w$ does not lie in $J_{\mathbf{B}^{R_V}}$.*

So $R_V$ is not effective in the sense of Section 2, as Venken states. Venken notes that the ideal generated by $M^2$, $MN$, $N^2$ and the monomials of degree 3 in $X, Y, Z, W$ would suffice, the other generators only simplifying his computations. The same check on that ideal gives $\dim \mathbf{B} = 45$, invariants of dimension 8, and again a gap of one dimension in degree 4 ($410 > 409$).

The proposition is checked by computer. The script `venken.py` of the directory `computations/effective/venken` of the repository [8] works in the monomial basis of $\mathbf{B}^{\otimes 2}$ and $\mathbf{B}^{\otimes 3}$ and performs the comparisons by Gaussian elimination over $\mathbb{Q}$. Every pivot it meets is $\pm 1$, so every operation of the elimination has integer coefficients, and the same operations, reduced modulo any prime $p$, form an elimination over $\mathbb{F}_p$ with the same pivot positions. The ranks are therefore the same over $\mathbb{Q}$ and over every $\mathbb{F}_p$, and the integer bases of the kernels produced over $\mathbb{Q}$ reduce modulo $p$ to bases of the kernels over $\mathbb{F}_p$; since ranks do not change under extension of scalars, the dimensions hold over every field. The script was also run directly over $\mathbb{F}_p$ for $p = 2, 3, 5, 7, 11, 101$, with the same result.

Most of the proposition is also formalised in Lean 4 with Mathlib, over every nontrivial commutative ring, from integer certificates checked by the Lean kernel, in the repository [5]. Two files and three theorems are concerned:

- `Grothendieck/AlgebraicGeometry/EffectiveVenken.lean`, theorem `Grothendieck.EffectiveVenken.venken_not_effective`: (ii), the equality of $\mathbf{B}^{R_V}$ with the span of $1, m, n, mx, nz - my, nw, mx^2, my^2, nz^2, nw^2$, and $mxy \otimes w \in I_V$, $mxy \otimes w \notin J_{\mathbf{B}^{R_V}}$, the last by an integer linear form on $(\mathbf{B} \otimes \mathbf{B})_4$ that vanishes on $J_{\mathbf{B}^{R_V}}$ and takes the value 1 at $mxy \otimes w$;
- `Grothendieck/AlgebraicGeometry/EffectiveVenkenRelation.lean`, theorems `venken_isEquivalenceRelation` and `venken_existsUnique_groupoidScheme` of the namespace `Grothendieck.EffectiveVenken.Relation`: $R_V$ is an equivalence relation of schemes over $\operatorname{Spec} K$ and extends uniquely to a groupoid scheme, with the vocabulary of AxiomMath's Lean formalisation of the Stacks Project, tags 022P and 0233.

The dimensions 33, 10, 269 and 268 are not stated in Lean, and the identification of $J_{\mathbf{B}^{R_V}} \neq I_V$ with non-effectivity in the language of schemes, the criterion of Section 2, is not formalised.

## 5. Computations

The second example was found by computer before it was proved by hand, and the computation also certifies it over every field independently of Section 3. We report it here with the notation of that section.

**Proposition 9.** *Over $\mathbb{Q}$ and over $\mathbb{F}_p$ for every prime $p$:*

(1) *for* $d = 1, \dots, 7$*, the forms* $a \in k[x_1, x_2]_d$ *with* $a(x) - a(y) \in I + K$ *make up a space of dimension* $0, 2, 1, 3, 6, 7, 8$ *respectively,* 27 *in all;*
(2) *in the degrees* $d \leq 14$ *the ideals* $I + K$ *and* $J + K$ *differ only in degree* 5*, where*

$$\dim_k (I + K)_5 = 52 > 51 = \dim_k (J + K)_5,$$

*and* $w \in (I + K)_5$*,* $w \notin (J + K)_5$*.*

The first version of this note stated (i) as 27 *independent differences* $b(x) - b(y)$ *modulo* $I + K$, which was wrong: such a difference lies in $I + K$ by definition of an invariant. The number 27 counts the independent forms $a$ found by the script, that is, the homogeneous lifts of invariants of positive degree at most 7, the forms of $(f_1^2, f_2^2, f_3^2)$, which lift 0, included. As $a \mapsto a(x) - a(y)$ is injective on forms of positive degree, it is also the dimension of the span of these differences in $k[x, y]$, before the quotient by $K$. The forms of $(f_1^2, f_2^2, f_3^2)$ of degree $d = 1, \dots, 7$ have dimension $0, 0, 0, 2, 4, 7, 8$ ($0, 0, 0, 2, 4, 6, 8$ in characteristic 2), by the Hilbert function of Section 3. So $\dim B_d = 0, 2, 1, 1, 2, 0, 0$ for $d = 1, \dots, 7$ ($0, 2, 1, 1, 2, 1, 0$ in characteristic 2). Comparing with the graded pieces of $C$ (Section 3; $C_6 = 0$, every product of degree 6 of the $f_i$ containing a square): $B = C$, of dimension 7, outside characteristic 2, while in characteristic 2, $B = C \oplus A_6$ has dimension 8. As $A$ vanishes in degrees above 6, these are all the invariants. The proof of Section 3 does not depend on any of these counts: it uses neither $B$ nor the number 27, only $C \subseteq B$ and $A_5 = C_5$; and Lemma 5 accounts for the dimensions in (ii), $52 = 51 + 1$.

The proposition is a finite computation: the degree-$d$ parts are spanned by monomial multiples of the generators, the forms of (i) are the kernel of a linear map on the forms of degree $d$ in $x$, and the dimensions are ranks. The script `scan.py` of the directory `computations/effective/raicu` of the repository [8], run with `EXP=2,2,2`, performs it by Gaussian elimination over $\mathbb{Q}$ or over $\mathbb{F}_p$, and prints the dimensions of (i) degree by degree, those of (ii), and the two membership tests for $w$.

*Proof that it holds over every field.* A field $k$ contains $\mathbb{Q}$ or some $\mathbb{F}_p$, and all the data are defined over the prime field: ranks of matrices with entries in the prime field do not change under extension of scalars. It remains to cover every prime field.

Over $\mathbb{Q}$ the elimination divides only by pivots. Each coefficient it produces is an integer combination of the input coefficients, all integers, divided by a product of pivots. If a prime $p$ divides neither the numerator nor the denominator of any pivot, all these coefficients are $p$-integral, and their reductions modulo $p$ form an elimination over $\mathbb{F}_p$ with the same pivot positions. It gives the same ranks, and its kernel bases are the reductions modulo $p$ of the kernel bases found over $\mathbb{Q}$; this is the sense in which the kernels are the same, the spaces themselves living over different fields. The script `certify_scan.py` reruns the computation over $\mathbb{Q}$ and records the prime factors of all pivots; they are $2, 3, 5, 7, 13, 41$ and 61. For every other prime the proposition over $\mathbb{F}_p$ follows from the one over $\mathbb{Q}$. For these seven primes `scan.py` was run directly over $\mathbb{F}_p$, with the same result. □

*Remark* 10. Why this $Z$. Among the subschemes $V(f_1^a, f_2^b, f_3^c)$ with $a, b, c \leq 3$, sixteen were tried over $\mathbb{F}_3$, and the smallest on which the obstruction survives is $a = b = c = 2$; with exponents $(3, 3, 2)$ one gets length 27, with gaps in degrees 5 and 6. This is a search among these sixteen subschemes only, and it does not show that $Z$ is minimal among all examples. We do not know whether a shorter example exists, from other ideals. A random search over 48 small monomial algebras over $\mathbb{F}_2$ and $\mathbb{F}_3$ found no non-effective relation; Raicu's theorem that affine toric equivalence relations are effective [7] suggests why.

*Remark* 11. Grothendieck reduced the question to an algebraically closed residue field; both examples have residue field $k$, which may be chosen algebraically closed. Both are also graded, which the question does not require.

**Reproducibility.** The scripts are those of the repository [8] at commit `0c1e5880`, in the directory `computations/effective`; they use Python 3 and its standard library only, and the argument `P` is 0 for $\mathbb{Q}$ or a prime $p$ for $\mathbb{F}_p$. From `computations/effective/raicu`:

- `python3 hand.py`: the identities of Section 3, over $\mathbb{Z}$;
- `EXP=2,2,2 python3 scan.py P`: Proposition 9, over $\mathbb{Q}$ or $\mathbb{F}_p$;
- `EXP=2,2,2 python3 certify_scan.py`: the primes dividing a pivot over $\mathbb{Q}$;
- `EXP=a,b,c python3 scan.py 3`: the search among the $V(f_1^a, f_2^b, f_3^c)$ of the first remark.

From `computations/effective`, `python3 search.py 2 400 3 all 6` runs the random search of the same remark over $\mathbb{F}_2$. From `computations/effective/venken`: `python3 venken.py P` for Proposition 8 (with `P` = 0 it also prints the primes dividing a pivot), `python3 venken.py P simple` for the ideal of Venken's footnote, and `python3 lean_cert.py DIR`, which writes the Lean certificates to the directory `DIR`. The Lean files are those of the repository [5] at commit `e52bf16` (Lean v4.35.0, Mathlib pinned in `lakefile.toml`). From a clone, with `elan` installed:

- `lake exe cache get`;
- `lake build Grothendieck.AlgebraicGeometry.EffectiveVenken Grothendieck.AlgebraicGeometry.EffectiveVenkenRelation`.

The build fails on any warning, hence on any `sorry`; the build of that commit passed in the repository's continuous integration.

## 6. What is proved, checked and formalised

| Status | Results |
|---|---|
| Proved by hand | The criterion $I_R = J_B$ and Lemma 3 (Section 2); Lemma 4, Lemmas 5 and 6, Proposition 7, hence Theorem 2 (Section 3). Not checked by a specialist; not formalised. |
| Checked by scripts | Every identity of Section 3 (`hand.py`, over $\mathbb{Z}$); Proposition 8 (`venken.py`); Proposition 9 and the counts that follow it (`scan.py`, `certify_scan.py`); the searches of the first remark of Section 5. |
| Formalised in Lean | Proposition 8 (ii), the ten invariants spanning $\mathbf{B}^{R_V}$, $mxy \otimes w \in I_V \setminus J_{\mathbf{B}^{R_V}}$, over every nontrivial commutative ring; $R_V$ as an equivalence relation of schemes and a groupoid scheme. Not formalised: the dimensions 33, 10, 269 and 268; the criterion of Section 2; Venken's statements on flat descent; the second example. |
| Historical claims | Grothendieck's question and reduction: [3, §2, p. 212-05]. Venken's Corollary 3.1, its setting and dates: his note [10]; the repository records a reading on Gallica on 7 November 2026 (`computations/effective/README.md`). The Buffalo problem: a transcript of the passage made from the automatic captions of the recording [1], not checked against the audio by a person. Raicu's relation and its non-effectivity on $\mathbb{A}^2$: [4, §50]; the toric result: [7]. Philippe's paper is cited by its title only and is not used. |

**Use of language models.** The question was located in the recording of the Buffalo lecture and in [3] with language models (Claude Opus 5.5, Anthropic), which also found Raicu's example, proposed the restriction to $Z$, wrote the scripts and a first draft of this text, under the author's direction. Venken's note was read on Gallica, and its Corollary 3.1 found to answer the question, after the first version of this text had claimed the case of a finite scheme over a field to be new. The proof by hand of Section 3 was found the same way, the scripts serving to search for the linear form $\varphi$; each of its steps is a short computation that a reader can redo with pencil and paper. No specialist has checked it. This revision answers a reading report produced with another language model (Perplexity): it corrects the statement of Proposition 9, whose count was misdescribed in the first version, extracts Lemma 3 from the proof by hand,

and was itself written with Claude Opus 5.5. Propositions 8 and 9 rest on the scripts, which anyone can rerun. The author has checked the criterion of Section 2 and Venken's example (Section 4); Lemma 3 is new in this revision.

## References


[1] A. Grothendieck, course on algebraic groups (with algebraic geometry), State University of New York at Buffalo, lecture of 11 July 1973, recording at 6:59:50–7:00:54 of `https://www.youtube.com/watch?v=3ULFIl_2Wis`.

[2] A. Grothendieck, Technique de descente et théorèmes d'existence en géométrie algébrique I : généralités, descente par morphismes fidèlement plats, Séminaire Bourbaki 5 (1958/60), exp. 190, 299–327, `http://www.numdam.org/item/SB_1958-1960__5__299_0`.

[3] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique III : préschémas quotients, Séminaire Bourbaki 6 (1960/61), exp. 212, 99–118, `http://www.numdam.org/item/SB_1960-1961__6__99_0`.

[4] J. Kollár, Quotients by finite equivalence relations, with an appendix by C. Raicu, in *Current Developments in Algebraic Geometry*, MSRI Publ. 59, Cambridge University Press, 2012, 227–256, arXiv:0812.3608.

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[6] A. Philippe, Morphisme net d'anneaux, et descente, *Bull. Sci. Math. (2)* 97 (1973), 57–64.

[7] C. Raicu, Affine toric equivalence relations are effective, *Proc. Amer. Math. Soc.* 138 (2010), 3835–3847, doi:10.1090/S0002-9939-2010-10416-1.

[8] Commutator-IO, grothendieck-project, `https://github.com/Commutator-IO/grothendieck-project`, directories `computations/effective/raicu` and `computations/effective/venken`.

[9] M. Demazure, A. Grothendieck (eds.), *Schémas en groupes* (SGA 3), Lecture Notes in Math. 151, 152, 153, Springer, 1970.

[10] J. Venken, Non effectivité de la descente de modules plats par un morphisme fini d'anneaux locaux artiniens, *C. R. Acad. Sci. Paris Sér. A* 272 (14 June 1971), A1553–A1554, `https://gallica.bnf.fr/ark:/12148/bpt6k480300n/f1562.item`.



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