hexagon:2611.01016v1

KOSTANT'S TORI IN APPOSITION: THE BOREL SUBGROUPS ARE OPPOSITE

MICHEL HUA

Abstract. On 22 October 1969 Grothendieck wrote to Kostant about pairs of maximal tori T, T' of a complex simple adjoint group in apposition, T' being the centraliser of a cyclic element. Choosing Borel subgroups $B \supset T$ and $B' \supset T'$, he asked whether B and B' are always in general position. The answer is yes, in every type and for all $|W|$ Borel subgroups through T': every one of them is opposite to B, and also to the opposite Borel subgroup B^-. The proof takes a few lines from two published facts: $\operatorname{Lie} T'$ contains an element of $f + \mathfrak{b}$ with f a principal nilpotent of $\mathfrak{b}^-$ (Kostant), and a Borel subalgebra containing such an element is opposite to $\mathfrak{b}$ (an argument of Bălibanu's). We give it in full, self-contained. In type A the element of $f + \mathfrak{b}$ is the inverse of the cyclic element, and the statement becomes the transversality of the flags stable under a cyclic weighted shift, which we prove directly and formalise in Lean 4 with Mathlib, together with the flag forms in types B, C and in part D. A computation modulo primes checks types B_2 to B_7 independently. For Grothendieck's second question, on a distinguished conjugacy class of quadruples invariant under the exterior automorphisms, the answer is no in type A_{2l}, where the exterior automorphism fixes no class; in the other types we computed, a construction with the Coxeter plane gives a canonical pair of invariant classes, exchanged by the sign of $e_{-\theta}$. This second part is conditional on a description of $N(T) \cap N(T')$ that no specialist has checked.

1. The letter

Folder 123 of the Montpellier fonds, headed *Autour de Kostant*, contains on pages 21 and 22 a letter from Grothendieck to Kostant dated 22 October 1969. The letter refers to Kostant's 1959 paper on the principal three-dimensional subgroup [1], which attaches to a complex simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$ and a choice of simple roots $\alpha_1, \dots, \alpha_\ell$ the *cyclic elements*

$$E' = \sum_{i=1}^{\ell} c_i \, e_{\alpha_i} + c_0 \, e_{-\theta}, \qquad c_0, \dots, c_\ell \in \mathbb{C}^\times,$$

where θ is the highest root. Such an element is regular semisimple [1, Thm 6.7] (Kostant takes $c_0 = 1$ and writes ψ for θ; the general case follows by rescaling, see Section 8), and only when all the c_i are nonzero [6, Thm 1.2], so its centraliser T' is a maximal torus. Theorem 6.7 also says that $\operatorname{Lie} T'$ is orthogonal to $\mathfrak{t}$ for the Killing form, and that it is stable under the principal element $P = \exp(2\pi i \rho^\vee / h)$ (Kostant's P_0, with $s = h$), with eigenvalues $e^{2\pi i k_j / h}$, the k_j being the exponents. Kostant defines [1, p. 1018]: a Cartan subalgebra $\mathfrak{h}_1'$ is *in apposition* to a Cartan subalgebra $\mathfrak{h}_1$ with respect to a principal element $P \in G$ if (1) $\mathfrak{g}^P = \mathfrak{h}_1$ and (2) $\mathfrak{h}_1'$ is stable under P and the eigenvalues of P on $\mathfrak{h}_1'$ include a primitive h-th root of unity, h being the order of a principal element (Kostant writes s). By his Theorem 6.7, the centraliser $\operatorname{Lie} T'$ of a cyclic element is in apposition to $\mathfrak{t} = \operatorname{Lie} T$ with respect to $P = \exp(2\pi i \rho^\vee / h)$, and by his Theorem 7.3 any two such triples $(P_1, \mathfrak{h}_1, \mathfrak{h}_1')$, $(P_2, \mathfrak{h}_2, \mathfrak{h}_2')$ are conjugate under G [1, p. 1018]; so every pair of Cartan subalgebras in apposition is conjugate to $(\mathfrak{t}, \operatorname{Lie} T')$, and the term is Kostant's as well as the letter's. The element P acts on $\operatorname{Lie} T'$ as a Coxeter element: this is in Section 8 of Kostant's paper, which we did not read beyond page 1027, and we

Date: 22 November 2026; revised 23 and 24 November 2026.

2020 *Mathematics Subject Classification.* 20G15; 17B22, 22E46, 01A60.

Key words and phrases. Kostant, cyclic element, tori in apposition, Borel subgroups, general position, Lean, Grothendieck.

ORCID 0009-0002-8304-0683.

cite it from Brillon and Schechtman [3, Thm 1.4] and Kostant's later paper [6, Thm 1.3]; on page 1027, Corollary 8.6 says that a regular element of G has order at least h, with equality exactly for the principal elements. We read pages 1013–1015, 1018 and 1027 of Kostant's paper (Theorem 6.7, the definition of apposition, Theorem 7.3, Corollary 8.6); Guest and Ho [4, §3–5] give a concordant account. Let G be the adjoint group, h its Coxeter number, W its Weyl group.

The letter asks, in the words of the page, whether for some or for all such quadruples (T, B, T', B') the Borel subgroups B and B' are « in general position », and whether there is a *distinguished* conjugacy class of quadruples. It also states, without proof, that the quadruples form $|W|^2/(h\,z\,\varphi(h))$ conjugacy classes, z being the order of the centre of the simply connected cover. The count follows from standard theory and is derived in our note on open problems [8], problem `p:kostant`; we do not repeat it here.

Two Borel subgroups are in general position, or opposite, when their relative position is the longest element w_0 of W, that is, when $B \cap B'$ is a maximal torus. This article answers the first question in every type (Section 2). The statement is a short consequence of published results, which we did not find put together (Section 9). Sections 3–7 give the explicit form of the statement for the classical groups, as the transversality of flags, with direct proofs formalised in Lean; Section 8 reports the computations, and Section 10 the second question of the letter.

2. The answer in every type

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank ℓ, $\mathfrak{t} \subset \mathfrak{b}$ a Cartan and a Borel subalgebra, R^+ the positive roots, $\alpha_1, \dots, \alpha_\ell$ the simple roots, $\theta = \sum n_i \alpha_i$ the highest root, e_α a root vector for each root α, $\mathfrak{n} = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$, and $\mathfrak{b}^-$ the opposite Borel subalgebra. Let G be the adjoint group, B, B^-, T the subgroups with Lie algebras $\mathfrak{b}$, $\mathfrak{b}^-$, $\mathfrak{t}$, and U the unipotent radical of B. Let $\rho^\vee \in \mathfrak{t}$ be the element with $\alpha_i(\rho^\vee) = 1$ for all i, so that $\alpha(\rho^\vee)$ is the height of α; the height of θ is $h - 1$.

Theorem 2.1. *Let $c_0, \dots, c_\ell$ be nonzero, $E' = \sum_{i\geq 1} c_i e_{\alpha_i} + c_0 e_{-\theta}$, and T' the maximal torus with Lie algebra the centraliser $\mathfrak{t}'$ of E'. Every Borel subgroup $B' \supset T'$ is opposite to B, and also to B^-: the intersections $B \cap B'$ and $B^- \cap B'$ are maximal tori.*

The proof rests on two lemmas. Grade $\mathfrak{g}$ by the height modulo h: $\mathfrak{g}_j$ ($j \in \mathbb{Z}/h$) is the sum of the root spaces $\mathfrak{g}_\alpha$ with $\alpha(\rho^\vee) \equiv j$, together with $\mathfrak{t}$ for $j = 0$. It is the eigenspace decomposition of $\operatorname{Ad} P$, $P = \exp(2\pi i \rho^\vee/h)$, for the eigenvalues ζ^j, $\zeta = e^{2\pi i/h}$. Heights lie between $-(h-1)$ and $h-1$, so $\mathfrak{g}_1$ is spanned by the e_{α_i} and $e_{-\theta}$, and $\mathfrak{g}_{-1}$ by the $e_{-\alpha_i}$ and e_θ.

Lemma 2.2. *The Cartan subalgebra $\mathfrak{t}'$ contains an element $y = \sum_{i\geq 1} c'_i e_{-\alpha_i} + c'_0 e_\theta$ with $c'_1, \dots, c'_\ell$ all nonzero.*

Proof. Since $E' \in \mathfrak{g}_1$, $\operatorname{Ad} P(E') = \zeta E'$, so $\operatorname{Ad} P$ preserves the centraliser $\mathfrak{t}'$, which splits as $\bigoplus_j (\mathfrak{t}' \cap \mathfrak{g}_j)$. The Killing form K is invariant under $\operatorname{Ad} P$, so $K(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j \equiv 0$; it is nondegenerate on the Cartan subalgebra $\mathfrak{t}'$, so it pairs $\mathfrak{t}' \cap \mathfrak{g}_1$ and $\mathfrak{t}' \cap \mathfrak{g}_{-1}$ nondegenerately. The first contains $E' \neq 0$, so the second contains some $y \neq 0$, of the stated form; it is semisimple, being in $\mathfrak{t}'$. Suppose $c'_i = 0$ for some $i \geq 1$. Let $\varpi_i^\vee \in \mathfrak{t}$ be the fundamental coweight, $\alpha_j(\varpi_i^\vee) = \delta_{ij}$, and $\lambda = h\varpi_i^\vee - \rho^\vee$. Then $(-\alpha_j)(\lambda) = 1$ for $j \neq i$ and $\theta(\lambda) = hn_i - (h-1) > 0$, so y lies in the sum of the root spaces $\mathfrak{g}_\beta$ with $\beta(\lambda) > 0$, a nilpotent subalgebra. Then y is nilpotent and semisimple, hence 0, a contradiction. □

Kostant gives such an element explicitly [1, Thm 6.7]: for $c_0 = 1$ and $\theta = \sum q_i \alpha_i$, the element $\tilde{z}_0 = \sum_{i\geq 1} (q_i/c_i)\, e_{-\alpha_i} + e_\theta$ commutes with $z_0 = E'$, and $\mathfrak{g}^{z_0} = \mathfrak{g}^{\tilde{z}_0}$; it is the eigenvector of P on $\mathfrak{t}'$ for the exponent $h - 1$. The proof above avoids computing the coefficients, and gives the case $c_0 \neq 1$ directly.

Lemma 2.3. *Let $\mathfrak{b}'$ be a Borel subalgebra containing an element $y \in \sum_{i\geq 1} c'_i e_{-\alpha_i} + \mathfrak{b}$ with all $c'_i \neq 0$. Then $\mathfrak{b}'$ is opposite to $\mathfrak{b}$.*

Proof. By the Bruhat decomposition, $\mathfrak{b}' = \mathrm{Ad}(u\dot{w})\mathfrak{b}$ for some $u \in U$ and some $w \in W$ with representative $\dot{w}$. Write $u = \exp X$ with $X \in \mathfrak{n}$. For a positive root α, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha_i}]$ lies in $\mathfrak{t}$ or in $\mathfrak{g}_{\alpha-\alpha_i}$ with $\alpha - \alpha_i$ positive, so $\mathrm{ad}\, X$ maps each $e_{-\alpha_i}$ into $\mathfrak{b}$, and $\mathfrak{b}$ into $\mathfrak{n}$. Hence $y' = \mathrm{Ad}(u^{-1})y$ lies again in $\sum c_i' e_{-\alpha_i} + \mathfrak{b}$, with the same components on the $\mathfrak{g}_{-\alpha_i}$. But $y' \in \mathrm{Ad}(\dot{w})\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\beta \in w(R^+)} \mathfrak{g}_\beta$, so $-\alpha_i \in w(R^+)$ for every i. Then w^{-1} sends every simple root, hence every positive root, to a negative root, and $w = w_0$. So $\mathfrak{b}' = \mathrm{Ad}(u)\,\mathfrak{b}^-$, and $\mathfrak{b} \cap \mathfrak{b}' = \mathrm{Ad}(u)(\mathfrak{b} \cap \mathfrak{b}^-) = \mathrm{Ad}(u)\,\mathfrak{t}$ is a Cartan subalgebra. □

This is the argument of Bălibanu [2, Lemma 4.5], who states it for the Kostant slice $f + \mathfrak{g}^e$; her proof uses only that the element lies in $f + \mathfrak{b}$ with nonzero components on the negative simple root spaces.

Proof of Theorem 2.1. The Lie algebra $\mathfrak{b}'$ of B' contains $\mathfrak{t}'$, hence the element y of Lemma 2.2, and Lemma 2.3 applies. For B^-, exchange the roles of $\mathfrak{b}$ and $\mathfrak{b}^-$: E' itself lies in $\sum c_i e_{\alpha_i} + \mathfrak{b}^-$ with all $c_i \neq 0$, and Lemma 2.3 for $\mathfrak{b}^-$ applies to E'. Two Borel subgroups whose Lie algebras meet in a Cartan subalgebra meet in a maximal torus. □

Remark 2.4. In terms of the flag variety, the $|W|$ fixed points of T' on G/B all lie in the big cell Bw_0B/B, and also in the big cell B^-B/B. The condition on the c_i is necessary: if some c_i vanishes, E' is nilpotent (the argument of Lemma 2.2 applies to it) and has no centraliser torus of this kind.

In the rest of the article we make the statement explicit for the classical groups, where it becomes a statement about flags of subspaces stable under a matrix, with elementary proofs that do not use Lemma 2.3. In type A the element y of Lemma 2.2 is, up to a scalar, the inverse of E': it is the weighted shift S of Section 4.

3. Statement

Let $n \geq 1$ and $V = \mathbb{C}^n$ with basis $e_0, \dots, e_{n-1}$ and coordinates $x_0, \dots, x_{n-1}$, indices taken modulo n. The group is $\mathrm{PGL}_n(\mathbb{C})$, the torus T is the diagonal one and B the stabiliser of the standard flag

$$F_m = \langle e_0, \dots, e_{m-1} \rangle, \qquad 0 \leq m \leq n.$$

The simple root vectors are $E_{i,i+1}$ for $0 \leq i \leq n-2$ and the vector of the lowest root is $E_{n-1,0}$, so a cyclic element is

$$(E'x)_i = c_i\, x_{i+1} \qquad (0 \leq i \leq n-1,\ c_i \neq 0).$$

Then $E'^n = (c_0 \cdots c_{n-1}) \cdot 1$: the eigenvalues of E' are the n distinct n-th roots of $c_0 \cdots c_{n-1}$, and E' is regular semisimple. The torus T' is its centraliser modulo scalars. A Borel subgroup contains T' exactly when it stabilises a complete flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of T'-stable subspaces, and these are the flags of E'-stable subspaces; there are $n! = |W|$ of them. The flag $V_\bullet$ is opposite to $F_\bullet$ when $V_k \cap F_{n-k} = 0$ for all k, which is the general position of the two Borel subgroups.

Theorem 3.1. *Let $c_0, \dots, c_{n-1}$ be nonzero complex numbers and E' as above. For every E'-stable subspace $W \subseteq V$ of dimension k, $W \cap F_{n-k} = 0$. Consequently every Borel subgroup B' containing T' is in general position with B, and $B \cap B'$ is a maximal torus.*

Up to conjugacy every quadruple of tori in apposition and Borel subgroups through them is of this form for $G = \mathrm{PGL}_n(\mathbb{C})$, by the description of tori in apposition recalled in Section 1. Theorem 3.1 is the case A_{n-1} of Theorem 2.1, with a proof by linear algebra; it also says more, that every E'-stable subspace, not only the terms of a flag, is transverse to the standard flag.

4. Proof by a weighted shift

Define $S\colon V \to V$ by $(Sx)_j = x_{j-1}/c_{j-1}$, indices modulo n. It moves every coordinate up by one place, dividing by the weight.

Lemma 4.1. *$SE' = \mathrm{id}$. Hence every E'-stable subspace W is S-stable.*

Proof. $(SE'x)_j = (E'x)_{j-1}/c_{j-1} = c_{j-1}x_j/c_{j-1} = x_j$. The restriction of E' to W is injective, therefore bijective since W is finite-dimensional, and $S|_W$ is its inverse. □

Proof of Theorem 3.1. Let W be E'-stable of dimension k and let $v \in W \cap F_{n-k}$ be nonzero. Let p be the largest index with $v_p \neq 0$. Since $v \in F_{n-k}$ we have $p \leq n-k-1$, hence $p+k \leq n-1$.

For $0 \leq r \leq k$ the vector $S^r v$ vanishes in all coordinates above $p+r$ and has a nonzero coordinate at $p+r$: each application of S moves the support up by one place, and no index wraps around because $p+r \leq n-1$. So the vectors $v, Sv, \dots, S^k v$ have distinct highest nonzero coordinates, and are linearly independent. They lie in W by Lemma 4.1, and there are $k+1$ of them in a space of dimension k. This is a contradiction, so $v = 0$.

For the consequence, apply this to each term V_k of a flag of E'-stable subspaces. Two Borel subgroups with opposite flags meet in a maximal torus. □

The proof needs only that E' is invertible with a weighted shift as inverse, and it works over any field. The operator S is $(c_0 \cdots c_{n-1})^{-1} E'^{n-1}$; in the Lie algebra it lies in $\sum_i c_i^{-1} E_{i+1,i} + \mathfrak{b}$, and it is the element y of Lemma 2.2. The hypothesis on the c_i is necessary: for $n = 2$ and $c_1 = 0$ the only stable line is $\langle e_0 \rangle = F_1$, which is not opposite to F_1.

5. A second proof by Vandermonde determinants

Let μ be an eigenvalue of E', so $\mu^n = c_0 \cdots c_{n-1}$. The equation $E'x = \mu x$ reads $x_{i+1} = \mu x_i / c_i$, so an eigenvector is $x_j = d_j\, \mu^j$ with $d_j = (c_0 \cdots c_{j-1})^{-1}$ (and $d_0 = 1$), the weights d_j not depending on μ. A flag of E'-stable subspaces is spanned, step by step, by the eigenvectors for $\mu_{s_1}, \dots, \mu_{s_n}$ in some order. The subspace V_k spanned by those of a set S of k eigenvalues meets F_{n-k} trivially exactly when the $k \times k$ matrix of the last k coordinates of these eigenvectors is invertible. Its determinant is

$$\prod_{j=n-k}^{n-1} d_j \cdot \prod_{s \in S} \mu_s^{n-k} \cdot \prod_{s<s' \in S} (\mu_{s'} - \mu_s),$$

a Vandermonde determinant times nonzero factors, and it is nonzero because the μ_s are distinct.

This is the argument that comes first to mind. The note on open problems used it to check A_2 by hand, where all 36 pairs are opposite. We formalised the first proof because it needs no eigenvalues.

6. Formalisation

The statement is the definition `KostantOppositeTypeA` n in `OpenProblems/\protect\penalty\z@AbelianVarieties.lean`: for every c with all $c_i \neq 0$ and every complete flag $V_\bullet$ of E'-stable subspaces of $\mathbb{C}^n$ (with $\dim V_k = k$), a vector of V_k whose coordinates of index $\geq n-k$ vanish is zero. The theorem `Grothendieck.Folder123KostantOpposite.kostantOppositeTypeA` proves it for every n in `Grothendieck/RepresentationTheory/Folder123\protect\penalty\z@KostantOpposite.lean`, with Lean v4.35.0 and Mathlib; the lemmas follow Section 4 (`shiftInv_cyclicElement`, `shiftInv_mem`, `opposite`). The command `#print axioms` lists only `propext`, `Classical.choice` and `Quot.sound`.

What the Lean statement leaves to this article is the translation: that the Borel subgroups through T' are the stabilisers of the E'-stable flags, and that general position means $V_k \cap F_{n-k} = 0$. Both are standard and were argued in Section 3, not formalised, since Mathlib has no Borel subgroups or Bruhat decomposition in this generality.

7. The classical types

Let now $N = 2n$ or $2n+1$ and $V = \mathbb{C}^N$ with basis $e_0, \dots, e_{N-1}$, the bilinear form pairing e_i with e_{N-1-i} (alternating for Sp_{2n}, symmetric for SO_{2n+1} and SO_{2n}), T the diagonal torus and B the upper triangular Borel subgroup, which stabilises the standard flag $F_\bullet$. The cyclic element E' is the sum of the root vectors of the simple roots and of the lowest root with nonzero coefficients $a_0, \dots, a_n$, written out as a matrix in the file cited below; it lies in $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ respectively.

Theorem 7.1. *Let W be an E'-stable subspace of V of dimension k.*

1. *In type C_n ($n \geq 1$) and in type B_n ($n \geq 1$), $W \cap F_{N-k} = 0$.*
2. *In type D_n ($n \geq 3$), if W is isotropic and $k < n$, then $W \cap F_{2n-k} = 0$.*

In types B and C the Borel subgroups through T' stabilise isotropic flags $V_1 \subset \cdots \subset V_n$ stable under E', completed by the orthogonals $V_{N-k} = V_k^\perp$ into complete flags of V stable under E'; the longest element of the Weyl group acts by -1, and general position of two Borel subgroups is the opposition of these complete flags. So Theorem 7.1 (1) gives again, by linear algebra, the cases B_n and C_n of Theorem 2.1; the translation is standard and is argued here, not formalised. In type D the steps $k < n-1$ and the subspace V_{n-1} determine the Borel subgroup, and part (2) says they are transverse to the standard flag; the opposition of the two Borel subgroups is then Theorem 2.1.

Proof of Theorem 7.1. The proof of Theorem 3.1 uses only an operator S preserving W that moves each coordinate one step along a path of coordinates, up to a nonzero factor: a vector of W vanishing on the last k coordinates of the path then vanishes on the whole path.

Type C. In the basis $e_0, \dots, e_{2n-1}$ the matrix of E' is a cyclic element of type A_{2n-1}, with coefficients $\pm a_i$, all nonzero, and Theorem 3.1 applies to every E'-stable subspace, isotropic or not.

Type B. Here E' is not invertible: its kernel is the line of $u = a_0 e_0 + a_n e_{2n}$ and its image the hyperplane $\mu = 0$ for a linear form μ. An explicit map G satisfies $G(E'x) = x - \mu(x)\,u$; it inverts E' on every E'-stable subspace of $\{\mu = 0\}$ and moves the coordinate j to $j+1$. Let $v \in W \cap F_{N-k}$. If E' is injective on W, then $W = E'W \subseteq \{\mu = 0\}$ is G-stable, and the path $0, \dots, 2n$ gives $v = 0$. Otherwise $W' = E'W$ has dimension $< k$ and contains $w = G(E'v) = v - \mu(v)\,u$; the path $0, \dots, 2n-1$ gives $w = 0$, so $v = \mu(v)\,u$, and the vanishing of the coordinate $2n$ of v forces $\mu(v) = 0$.

Type D. The kernel of E' is spanned by $u = a_0 e_0 + a_n e_{2n-1}$ and $u' = a_{n-1} e_{n-1} - a_{n-2} e_n$, its image is $\{\mu = \mu' = 0\}$, and $G(E'x) = x - \mu(x)\,u - \mu'(x)\,u'$. The same two cases, along the paths $1, \dots, n-1, n+1, \dots, 2n-2, 0$ and $0, \dots, n-1, n+1, \dots, 2n-2$, which skip the coordinate n (on the image of E' it is a multiple of the coordinate $n-1$), leave $v = \mu'(v)\,u'$. There isotropy is needed: $\langle u', u' \rangle = -2\,a_{n-1}a_{n-2} \neq 0$, so $v = 0$. □

The statements are `KostantOppositeTypeC`, `KostantOppositeTypeB` and `KostantOppositeTypeD` in `OpenProblems/\protect\penalty\z@AbelianVarieties.lean`, with the three cyclic elements written as matrices, and they are proved as `kostantOppositeTypeC`, `kostantOppositeTypeB` and `kostantOppositeTypeD` in `Grothendieck/RepresentationTheory/Folder123\protect\penalty\z@KostantClassical.lean`, where `path_shift` is the abstract argument and `cyclicElementC_mem_sp`, `cyclicElementB_mem_so`, `cyclicElementD_mem_so` check that the matrices lie in the Lie algebras. As in type A, the translation from flags to Borel subgroups is not formalised.

In type D the middle step is more delicate: a Borel subgroup of SO_{2n} is given by an isotropic flag $V_1 \subset \cdots \subset V_{n-1}$, and the two maximal isotropic subspaces containing V_{n-1} do not behave alike.

Corollary 7.2. *In type D_n ($n \geq 3$), let $B' \supset T'$ be a Borel subgroup with isotropic flag $V_1 \subset \cdots \subset V_{n-1}$, and let L, L' be the two maximal isotropic subspaces containing V_{n-1}. Exactly one of them meets F_n trivially.*

Proof. By Theorem 2.1, $B \cap B'$ is a maximal torus T''. Take a basis $e_1, \ldots, e_n, f_n, \ldots, f_1$ of weight vectors of T'', with $\langle e_i, f_j \rangle = \delta_{ij}$ and the e_i, the f_i isotropic, ordered so that the flag of B is spanned by its initial segments; then $F_n = \langle e_1, \ldots, e_n \rangle$, and B', being the Borel subgroup opposite to B through T'', has the flag spanned by the final segments, so that $\{L, L'\} = \{\langle f_1, \ldots, f_n \rangle, \langle e_n, f_1, \ldots, f_{n-1} \rangle\}$. The first meets F_n trivially, the second in the line of e_n. □

This also follows, without the explicit basis, from the parity rule for maximal isotropic subspaces (two of them lie in the same family if and only if their intersection has dimension $\equiv n \bmod 2$): L and L' meet in V_{n-1}, so they lie in different families, and their intersections with F_n cannot both be 0. A numerical computation for $n = 3, 4, 5$, reported in the Lean file, found the same for each of the $2^{n-1}n!$ Borel subgroups through T'. Part (2) of Theorem 7.1 together with Corollary 7.2 gives the type D case of Theorem 2.1 by linear algebra only for the steps $k \leq n$; the steps $k > n$, the orthogonals of the steps $k < n$, are transverse by duality but are not formalised, and neither is the corollary.

8. Computations

Two computations check Theorem 2.1 independently of its proof. They are verifications, not part of the proof. The scripts are in the directory `computations/kostant` of the repository `Commutator-IO/grothendieck-project` (commit `afff89ec`); the Lean files cited in this article are those of `Commutator-IO/GrothendieckArchivesProofs` at commit `e52bf16` (Lean v4.35.0, Mathlib `56a0058`).

Type B modulo primes. The torus T rescales the coefficients of E' (the simple roots form a basis of the characters of the adjoint torus), and E' and $\lambda E'$ have the same centraliser, so all the cyclic elements are conjugate under $T \subset B$ up to a scalar, and it suffices to treat $E_0 = \sum e_{\alpha_i} + e_{-\theta}$. In $\mathfrak{so}_{2n+1}$, E_0 satisfies $E_0^{2n+1} = aE_0$ with $a = \pm 2$ and has the $2n+1$ distinct eigenvalues 0 and the roots of $t^{2n} = a$ on $\mathbb{C}^{2n+1}$. The Borel subgroups through T' correspond to the choices of one eigenvalue in each pair $\pm\mu$, ordered, and opposition with B is the nonvanishing of the minors formed by the last k coordinates of k such eigenvectors, $k \leq n$. Modulo a prime $p \equiv 1 \bmod 2n$ in which the eigenvalues stay distinct, all these minors are nonzero for $2 \leq n \leq 7$ (8, 26, 80, 242, 728 and 2186 minors, with $p = 73$, 31, 257, 151, 601 and 631); a minor nonzero modulo a prime above p is nonzero in characteristic 0, so this proves the types B_2 to B_7 (script `computations/kostant/kostant_bn.py`).

Finite fields. For the split adjoint groups of types A_2, B_2 and G_2 over $\mathbf{F}_q$, with $q \equiv 1 \bmod h$ and q among $7, 13, 19, 31, 37$ for A_2 and G_2 and $5, 13, 17, 29, 37$ for B_2, the orbits of $G(\mathbf{F}_q)$ on quadruples number 2, 4 and 12, as the formula $|W|^2/(h\, z\, \varphi(h))$ predicts, and in every orbit $\dim(\operatorname{Lie} B \cap \operatorname{Lie} B') = 2$, the rank (script `computations/kostant/kostant_count.py`). This is the case $q \equiv 1 \bmod h$ of Theorem 2.1 over these finite fields.

9. The literature

We found no source stating Theorem 2.1. Its two ingredients are in print: the element $\tilde{z}_0 \in \mathfrak{t}'$ of Lemma 2.2 is in Kostant's Theorem 6.7 [1], and Lemma 2.3 is Bălibanu's Lemma 4.5 [2], stated for the Kostant slice but proved for $f + \mathfrak{b}$. The same paper (Lemma 4.4) shows $G_x \cap B = 1$ for x in the slice, and uses the lemma, in the proof of its Lemma 4.7, to put in the big cell the Borel subgroups through the centraliser of a regular semisimple element of the slice. So the theorem is a short consequence of published results, and we do not claim it is new; we state it because it answers Grothendieck's question. We searched arXiv, Crossref and the literature on cyclic elements and Coxeter tori (Kostant's papers of 1959 and 2010, Springer [7], Elashvili, Kac and Vinberg on cyclic elements, Brillon and Schechtman, Guest and Ho, Crooks

on the Kostant–Toda lattice, Bălibanu); of Kostant's 1959 paper we read pages 1013–1015, 1018 and 1027.

10. The second question: is there a distinguished class?

Status of this section. What follows rests on a description of $\Phi = N(T) \cap N(T')$, the group whose orbits are the classes of quadruples: $|\Phi| = h\, z\, \varphi(h)$, $T \cap N(T')$ cyclic of order h generated by a principal element, and the image of Φ in W as used in the computations. That description is argued in [8], problem `p:kostant`, and has not been checked by a specialist. Theorem 10.1 and Proposition 10.2 are therefore conditional on it. Theorems 2.1, 3.1 and 7.1 do not depend on it.

The letter goes on: « Is there in any sense a *distinguished* conjugacy class of such quadruples (which should then be, of course, invariant by action of exterior automorphisms)? What are the groups of automorphisms of such quadruples (they are isomorphic to subgroups of the group of exterior automorphisms of G)? » We answer the first part negatively in type A_{2l} and give, in the other types we could compute, a canonical pair of invariant classes.

The set of classes. Fix T and a torus T' in apposition with it; all such pairs are conjugate [1, Thm 7.3]. Let Q and Q' be the sets of Borel subgroups containing T and T', of $|W|$ elements each. The conjugacy classes of quadruples are the orbits of $\Phi = N(T) \cap N(T')$ on $Q \times Q'$; Φ acts freely, the stabiliser of a quadruple in G being $T \cap T' = 1$, and $|\Phi| = h\, z\, \varphi(h)$, which gives the letter's count (the argument is in [8], problem `p:kostant`). An automorphism of G can be composed with an inner one so as to normalise T and T', so the group $\mathrm{Out}(G)$ of exterior automorphisms acts on the set of classes. The stabiliser of a quadruple in $\mathrm{Aut}(G)$ meets G trivially, hence maps isomorphically onto the stabiliser of its class in $\mathrm{Out}(G)$: this is Grothendieck's parenthesis, and the groups of automorphisms asked for are these stabilisers.

Theorem 10.1. *Let $G = \mathrm{PGL}_n(\mathbb{C})$ with $n = 2l+1 \geq 3$, of type A_{2l}. The outer automorphism of G fixes no class of quadruples. So no class is invariant under $\mathrm{Out}(G)$, and every quadruple has a trivial group of automorphisms.*

Proof. Suppose an outer automorphism σ fixes a quadruple (T, B, T', B'). Then σ^2 is inner and fixes it, so $\sigma^2 = 1$. Take T diagonal and B upper triangular. The automorphism σ acts on $\mathfrak{t}$ by an automorphism of the root system fixing the chamber of B, and not trivially (an automorphism acting trivially on a Cartan subalgebra is inner), so by the flip: $\sigma(\mathrm{diag}(d_1, \dots, d_n)) = \mathrm{diag}(d_n^{-1}, \dots, d_1^{-1})$ on T. The group $T \cap N(T')$ is cyclic of order n [8], generated by a principal element $P' = \mathrm{diag}(\omega^{a(1)}, \dots, \omega^{a(n)})$, $\omega = e^{2\pi i/n}$, where a is a bijection from $\{1, \dots, n\}$ to $\mathbb{Z}/n$, since P' is regular of order n. As σ normalises $T \cap N(T')$, $\sigma(P') = P'^{\,k}$ for a unit k of $\mathbb{Z}/n$ with $k^2 = 1$, that is

$$-a(n+1-i) \equiv k\, a(i) + c \pmod{n} \qquad (1 \leq i \leq n)$$

for some constant c (elements of PGL_n being defined up to scalars). The involution $i \mapsto n+1-i$ has exactly one fixed point, n being odd; through a it becomes the affine involution $x \mapsto -kx - c$ of $\mathbb{Z}/n$, which therefore has exactly one fixed point, so $\gcd(k+1, n) = 1$.

On the other side, T' has n eigenlines in $\mathbb{C}^n$, which P' permutes cyclically, as the Coxeter element $\mathrm{Ad}\, P'|_{\mathfrak{t}'}$ does; number them by $\mathbb{Z}/n$ so that P' sends the line m to the line $m-1$. The automorphism σ is $x \mapsto J\, {}^t x^{-1} J^{-1}$ up to an inner one; it sends T' to itself and so permutes its eigenlines, through a bijection π, the line m going to J applied to the line of $(\mathbb{C}^n)^*$ that vanishes on the other eigenlines. From $\sigma(P') = P'^{\,k}$ one gets $\pi(m-1) = \pi(m) - k$, so $\pi(m) = km + e$. A Borel subgroup $B' \supset T'$ is the stabiliser of the flag spanned by the eigenlines in an order $(m_1, \dots, m_n)$, and $\sigma(B')$ is that of the flag of the eigenlines in the order $(\pi(m_n), \dots, \pi(m_1))$, by duality. So $\sigma(B') = B'$ means $\pi(m_{n+1-i}) = m_i$ for all i: the involution $x \mapsto kx + e$ is conjugate to $i \mapsto n+1-i$, and has exactly one fixed point, so $\gcd(k-1, n) = 1$.

Now n divides $k^2 - 1 = (k-1)(k+1)$, a product of two numbers prime to n, which is impossible for $n > 1$. □

In type A_{2l+1} the same argument gives no contradiction, and the outer automorphism does fix classes (the table below).
A canonical pair of classes. Let $P_B = \exp(2\pi i \rho_B^\vee/h)$ be the principal element of T defined by B. Say that B is *adapted* to T' if P_B normalises T', that is, if T' is the centraliser of a cyclic element relative to B, as in Section 2. Then $c' = \operatorname{Ad} P_B|_{\mathfrak{t}'}$ is a Coxeter element of the Weyl group of T', the cyclic element E' spans its eigenline for $e^{2\pi i/h}$, and the real and imaginary parts of E', in the real form of $\mathfrak{t}'$ on which the roots are real, span its Coxeter plane. Say that $B' \supset T'$ is a *Coxeter Borel subgroup* if its positive roots are $\{\beta : \operatorname{Re}(\mu\,\beta(E')) > 0\}$ for some $\mu \in \mathbb{C}^\times$, that is, if its chamber meets the Coxeter plane.

Proposition 10.2. *The quadruples with B adapted to T' and B' a Coxeter Borel subgroup form exactly two conjugacy classes D_+ and D_-. Changing the sign of $e_{-\theta}$ in the definition of E' exchanges them. If an exterior automorphism, written as a pinned automorphism δ, satisfies $\delta(e_{-\theta}) = \varepsilon\, e_{-\theta}$, it fixes D_+ and D_- when $\varepsilon = 1$ and exchanges them when $\varepsilon = -1$.*

Proof. The adapted Borel subgroups are those with P_B a generator of the cyclic group $T \cap N(T')$; there are $z\,\varphi(h)$ of them, z for each generator, and the image of Φ in W, of the same order, permutes them freely, hence simply transitively. So every class of the statement has a representative with B fixed, and two such representatives differ by an element of $\Phi \cap T = \langle P_B \rangle$, which acts on $\mathfrak{t}'$ by c'. The Coxeter plane meets exactly $2h$ chambers, and c' acts on them as the rotation through $2\pi/h$ (Steinberg's description of the Coxeter plane, see [5, §3.17–3.19]; the computation below confirms the number $2h$ in every case); $\langle c' \rangle$ acts freely, so there are two orbits, which are the two classes. Replacing μ by $\mu e^{i\pi/h}$ moves a chamber to the next one in the other orbit. Replacing $e_{-\theta}$ by $-e_{-\theta}$ replaces E', up to conjugation by T, by $\lambda E'$ with $\lambda^h = -1$, a rotation through an odd multiple of π/h, which exchanges the orbits. For δ, the composite $\psi = \operatorname{Ad}(t) \circ \delta$ with $t = \exp(\log\lambda \cdot \rho^\vee)$, $\lambda^h = \varepsilon$, fixes T and B, sends E' to $\lambda E'$, hence T' to itself, and maps the chamber of μ to that of $\mu\lambda$: a rotation through an even multiple of π/h if $\varepsilon = 1$, and through an odd one if $\varepsilon = -1$. □

The sign ε is -1 in type A_{2l} and $+1$ in type A_{2l+1}: writing $e_\theta = [\dots[[e_{\alpha_1}, e_{\alpha_2}], e_{\alpha_3}], \dots, e_{\alpha_r}]$, the diagram automorphism reverses the order of the brackets, which costs $(-1)^{r-1}$ (checked numerically up to $r = 6$). We computed $\varepsilon = 1$ for all the exterior automorphisms of D_4 (the group S_3) and for that of D_5; we did not compute E_6. So in types A_{2l+1}, D_4, D_5, and trivially in types B, C, F_4, G_2, which have no exterior automorphism, D_+ and D_- are two classes invariant under $\operatorname{Out}(G)$, whose groups of automorphisms are the whole of $\operatorname{Out}(G)$; in type A_{2l} they are exchanged, as Theorem 10.1 requires. They are distinguished as a pair: which of the two is singled out depends on the sign of $e_{-\theta}$, which a pinning does not fix, in line with Grothendieck's own remark that the cyclic line is « définie au signe près » (folder 123, p. 34).
Computations. A computation in the adjoint representation (floating point, the outcome being combinatorial) lists the orbits of Φ on $Q \times Q'$, taking for Φ the fibre product over $(\mathbb{Z}/h)^*$ of the elements of W multiplying P_B to a power and of those of the Weyl group of T' normalising $\langle c' \rangle$, and the action of $\operatorname{Out}(G)$ (script `computations/kostant/kostant_quadruples.py`). The number of orbits is the letter's in every case, and the pair $D_\pm$ has two classes in every case. The classes fixed by the exterior automorphisms are:

type	h	classes	classes fixed by the exterior automorphisms
A_2	3	2	0
A_3	4	18	8
A_4	5	144	0
A_5	6	7200	192
A_6	7	86400	0
D_4	6	768	96 by each transposition of S_3, 12 by each 3-cycle
D_5	8	28800	2304

In the types where invariant classes exist, there are many of them, so invariance alone does not single one out; the pair $D_\pm$ is the canonical candidate. We did not search the literature for this question beyond the sources of Section 9.

11. What is open

The sign ε for E_6, and a uniform proof that $\varepsilon = 1$ outside A_{2l}; a choice between D_+ and D_- that does not depend on the sign of $e_{-\theta}$, if there is one; and a proof of the fibre-product description of Φ used in the computations, which rests on the argument of [8], not checked by a specialist.

Declaration. This note and the Lean file it describes were drafted with a language model (Claude, Anthropic) from the edition's registers. No part has been checked by a human specialist: the independent checks are the Lean proofs and the computations of Section 8, which can be rerun from the commits given there. The author received no funding for this work.

References

[1] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.* 81 (1959), 973–1032.
[2] A. Bălibanu, The partial compactification of the universal centralizer, *Selecta Math. (N.S.)* 29 (2023), paper 85; arXiv:1710.06327.
[3] L. Brillon and V. Schechtman, Coxeter element and particle masses, *Selecta Math. (N.S.)* 22 (2016), 2591–2627; arXiv:1612.04133.
[4] M. Guest and N.-K. Ho, Kostant, Steinberg, and the Stokes matrices of the tt*-Toda equations, arXiv:1802.01126.
[5] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Univ. Press, 1990.
[6] B. Kostant, Experimental evidence for the occurrence of E_8 in nature and the radii of the Gosset circles, arXiv:1003.0046.
[7] T. A. Springer, Regular elements of finite reflection groups, *Invent. Math.* 25 (1974), 159–198.
[8] M. Hua, Open questions from Grothendieck's Montpellier notes: thirteen statements a formalisation could not reach, problem `p:kostant`; Part 2 of Hexagon preprint 2611.00125v3, 2026.

Commutator, Paris, France
Email address: `michel@commutator.io`