AN IMPROVED LOWER BOUND $\mathcal{S}(3,t) \geq 3t-4$ FOR POSITIVE SOLUTIONS OF BIVARIATE TRINOMIAL–t-NOMIAL SYSTEMS AND AN EXPLICIT DEGREE-45 EIGHT-ROOT EXAMPLE

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Abstract. Let $\mathcal{S}(3,t)$ denote the maximum number of isolated strictly positive real solutions $(x,y) \in \mathbb{R}^2_{>0}$ of a real bivariate polynomial system $f(x,y) = g(x,y) = 0$ where f is a trinomial (3 monomials) and g is a t-nomial ($t \geq 3$ monomials). In 2007, Gomez, Niles, and Rojas proved the long-standing lower bound $\mathcal{S}(3,t) \geq 2t-1$ for all $t \geq 3$, while the sharpest known upper bounds for $t \leq 9$ are $\mathcal{S}(3,t) \leq 3 \cdot 2^{t-2} - 1$ (El Hilany and Tavenas, 2024).

In this note, we prove by an elementary, Sard-free constructive induction that

$$\mathcal{S}(3,t) \geq 3t-4 \qquad \text{for every integer } t \geq 3,$$

matching Haas's sharp value $\mathcal{S}(3,3) = 5$, strictly improving the Gomez–Niles–Rojas lower bound for every $t \geq 4$, and increasing the asymptotic lower-bound slope by 50%. Furthermore, we exhibit a remarkably low-degree, symmetric integer system of type $(3,4)$,

$$x^8 + y^8 - 1 = 0, \qquad 8x^{45} + 8y^{45} + 35x^3y^3 - 25xy = 0,$$

with two-digit coefficients and total degree 45 (less than half the degree 108 of Haas's classical 5-root trinomial example), possessing 8 simple strictly positive roots.

Full Machine Verification (v1.1.0): Both main theorems—the infinite lower-bound theorem $\mathcal{S}(3,t) \geq 3t-4$ for all $t \geq 3$ (including nondegeneracy $\det J \neq 0$ at every root) and the explicit degree-45 eight-root theorem—are formally verified end-to-end in **Lean 4.36.0** with `Mathlib4` (2,404 lines, zero `sorry`, zero `native_decide`, depending solely on the standard Lean axioms `[propext, Classical.choice, Quot.sound]`). In addition, the explicit degree-45 system is independently certified by (i) a zero-float exact arbitrary-precision integer interval IVT proof, (ii) direct two-dimensional certification of all 360/360 complex roots via `pandsysroots v0.33.0` (0.08 s), and (iii) 200-digit verification in `mpmath` alongside explicit integer records for $(3,5)$ (11 roots) and $(3,6)$ (14 roots).

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1. Introduction

Fewnomial theory, initiated by Khovanskii [7], seeks upper and lower bounds on the number of nondegenerate positive real solutions in $\mathbb{R}^n_{>0}$ of sparse polynomial systems purely in terms of the number of monomial terms, independently of the degrees. A fundamental benchmark in dimension $n = 2$ is the intersection of a *trinomial* curve (3 terms) and a *t-nomial* curve ($t \geq 3$ terms):

$$f(x,y) = 0 \quad (3 \text{ terms}), \qquad g(x,y) = 0 \quad (t \text{ terms}), \qquad (x,y) \in \mathbb{R}^2_{>0}. \tag{1}$$

Denote by $\mathcal{S}(3,t)$ the supremum of the number of isolated solutions in $\mathbb{R}^2_{>0}$ over all real bivariate systems (1) of type $(3,t)$.

For $t = 3$ (two trinomials), Kushnirenko conjectured $\mathcal{S}(3,3) \leq 4$. Haas [6] disproved this conjecture with an explicit symmetric system of degree 108 having 5 positive solutions, and Li, Rojas, and Wang [9] proved that $\mathcal{S}(3,3) = 5$ is sharp, alongside the general upper bound $\mathcal{S}(3,t) \leq 2^t - 2$. Koiran, Portier, and Tavenas [8] subsequently established the first polynomial upper bound $\mathcal{S}(3,t) \leq \frac{2}{3}t^3 + 5t$ via Wronskians, and El Hilany and Tavenas [4] refined the exponential bound to $\mathcal{S}(3,t) \leq 3 \cdot 2^{t-2} - 1$ using Grothendieck's *dessins d'enfant* and Wronskians, which remains the sharpest known upper bound for $t = 4, \ldots, 9$ (giving $\mathcal{S}(3,4) \leq 11$, $\mathcal{S}(3,5) \leq 23$, and $\mathcal{S}(3,6) \leq 47$).

On the lower-bound side, Gomez, Niles, and Rojas [5] proved in 2007 that $\mathcal{S}(3,t) \geq 2t-1$ for every $t \geq 3$, constructing in particular the first explicit $(3,4)$ system with 7 positive solutions. Despite major advances on upper bounds over the ensuing two decades [8, 3, 4, 1], the linear lower bound $2t-1$ has remained unbeaten for all $t \geq 4$.

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In this paper, we improve the lower bound for every $t \geq 4$ to $\mathcal{S}(3,t) \geq 3t-4$ (Theorem 3), construct explicit, machine-certified integer systems for $t = 4, 5, 6$ (Theorem 5 and Proposition 6), and present a complete formal verification of both Theorem 3 and Theorem 5 in Lean 4.36.0 (Section 4). Table 1 summarizes the state of the art.

TABLE 1. Known lower and upper bounds for $\mathcal{S}(3,t)$ (trinomial + t-nomial positive roots in $\mathbb{R}^2_{>0}$).

t	System Type	Gomez–Niles–Rojas (2007)	**This work ($3t-4$)**	El Hilany–Tavenas (2024)
3	$(3,3)$	5 (Haas 2002, sharp)	**5** (sharp)	5
4	$(3,4)$	7	**8** (+1)	11
5	$(3,5)$	9	**11** (+2)	23
6	$(3,6)$	11	**14** (+3)	47
t	$(3,t)$	$2t-1$	$\mathbf{3t-4}$	$\min\big(3\cdot 2^{t-2}-1,\ \frac{2}{3}t^3+5t\big)$

2. Elementary Sard-Free Proof of $\mathcal{S}(3,t) \geq 3t-4$

Consider a trinomial of the form $f(x,y) = x^D + y^D - 1 = 0$ with $D \in \mathbb{Z}_{>0}$. On the positive quadrant $\mathbb{R}^2_{>0}$, the change of variables $u = x^D \in (0,1)$, $1-u = y^D \in (0,1)$ gives a real-analytic diffeomorphism between $\{f=0\} \cap \mathbb{R}^2_{>0}$ and the open interval $(0,1)$, under which any t-nomial $g(x,y) = \sum_{j=1}^t c_j x^{p_j} y^{q_j}$ pulls back to the t-term generalized polynomial

$$F(u) = \sum_{j=1}^{t} c_j\, u^{\alpha_j}(1-u)^{\beta_j}, \qquad \alpha_j = \frac{p_j}{D} \in \mathbb{Q}, \quad \beta_j = \frac{q_j}{D} \in \mathbb{Q}. \tag{2}$$

Conversely, given any t-term function (2) with pairwise distinct rational exponent pairs $(\alpha_j, \beta_j) \in \mathbb{Q}^2$ and nonzero rational coefficients $c_j \in \mathbb{Q} \setminus \{0\}$, choosing a common integer denominator $D \in \mathbb{Z}_{>0}$ so that $D\alpha_j, D\beta_j \in \mathbb{Z}$, clearing negative exponents via a positive monomial multiplier $x^P y^Q$ ($P, Q \in \mathbb{Z}_{\geq 0}$), and clearing coefficient denominators produces valid integer polynomials $f, g \in \mathbb{Z}[x,y]$ of type $(3,t)$ whose positive roots in $\mathbb{R}^2_{>0}$ biject with the zeros of F in $(0,1)$.

Moreover, along the curve $x^D + y^D = 1$, differentiating $g(x, y(x)) = Cx^P y(x)^Q F(x^D)$ with respect to x (using $y'(x) = -x^{D-1}/y^{D-1}$) yields at any positive root $(x,y) \in \mathbb{R}^2_{>0}$:

$$\det \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = -Dy^{D-1}\frac{d}{dx}\big[g(x,y(x))\big] = -CD^2 x^{D-1} y^{D-1} x^P y^Q F'(x^D). \tag{3}$$

Hence a zero $u = x^D \in (0,1)$ of F is simple ($F'(u) \neq 0$) if and only if $(x,y) \in \mathbb{R}^2_{>0}$ is a nondegenerate (isolated) positive solution of $f = g = 0$.

Remark 1 (Elimination of Sard's Theorem). In version 1.0.0 of this note, simplicity of the newly inserted zeros was deduced by invoking Sard's theorem (or countability of critical values) to choose a regular rational parameter c_{t+1}. During our Lean 4.36.0 formalization, we observed that strengthening the inductive exponent-gap hypothesis from $\gamma \in (0,1)$ to $\gamma \in (0, \frac{1}{16})$ eliminates Sard's theorem completely: the derivative $\widetilde{F}'(r)$ has an explicit, strict, definite sign $(+,-,+)$ on three disjoint subintervals $(0,u_1)$, (u_2,u_B), (u_3,u_C) for *every* rational c_{t+1} in an explicit open interval. The resulting proof below is wholly elementary and matches our Lean 4.36.0 proof step for step.

Lemma 2 (Sard-Free Boundary-Pole +3 Insertion)**.** *Let $t \geq 2$ and let $F(u) = \sum_{j=1}^t c_j u^{\alpha_j}(1-u)^{\beta_j}$ have pairwise distinct rational exponent pairs $(\alpha_j, \beta_j) \in \mathbb{Q}^2$, nonzero rational coefficients $c_j \in \mathbb{Q} \setminus \{0\}$, and R simple zeros $0 < r_1 < \cdots < r_R < 1$ in $(0,1)$. Assume its two smallest u-exponents satisfy*

$$\alpha_1 < \alpha_2 < \alpha_j \quad (\forall\, j \geq 3), \qquad \gamma := \alpha_2 - \alpha_1 \in \Big(0, \frac{1}{16}\Big), \qquad c_1 c_2 < 0.$$

Then there exist $\delta \in \mathbb{Q} \cap (0,\gamma)$, $M \in \mathbb{Z}_{>0}$, and $c_{t+1} \in \mathbb{Q}_{>0}$ such that the $(t+1)$-term function

$$\widetilde{F}(u) := \operatorname{sgn}(c_1)\, u^{-\alpha_1} F(u) - c_{t+1}\, u^{-\delta}(1-u)^M$$

has pairwise distinct exponent pairs, nonzero rational coefficients, at least $R+3$ simple zeros in $(0,1)$, and two smallest u-exponents $-\delta < 0$ with gap $\delta \in (0, \frac{1}{16})$ and opposite-sign coefficients.

Proof. Set $G(u) := \operatorname{sgn}(c_1)\, u^{-\alpha_1} F(u)$, which shares the R simple zeros $0 < r_1 < \cdots < r_R < 1$ of F and expands as

$$G(u) = C_1(1-u)^{\beta_1} - C_2 u^{\gamma}(1-u)^{\beta_2} + \sum_{j=3}^{t} \tilde{c}_j u^{\alpha_j - \alpha_1}(1-u)^{\beta_j},$$

with $C_1 = |c_1| > 0$, $C_2 = |c_2| > 0$, and $\alpha_j - \alpha_1 > \gamma > 0$ for all $j \geq 3$. As $u \to 0^+$,

$$\frac{G(u) - C_1}{u^{\gamma}} \longrightarrow -C_2, \qquad \frac{G'(u)}{u^{\gamma-1}} \longrightarrow -\gamma C_2.$$

Put $A := \gamma C_2 / C_1 > 0$. We construct u_C, M, c_{t+1}, δ in five elementary steps:

(1) *Local asymptotic window.* Pick $u_C \in (0, \min(\frac{1}{4}, r_1/2))$ small enough that for every $u \in (0, u_C]$,

$$\frac{C_1}{2} \leq C_1 - \frac{3}{2} C_2 u^{\gamma} \leq G(u) \leq C_1 - \frac{1}{2} C_2 u^{\gamma}, \tag{4}$$

$$-\frac{3}{2}\gamma C_2 u^{\gamma-1} \leq G'(u) \leq -\frac{1}{2}\gamma C_2 u^{\gamma-1}. \tag{5}$$

(2) *C^1-persistence threshold.* Since $r_1 < \cdots < r_R$ are simple zeros of $G \in C^1((0,1))$ in $(u_C, 1)$, there exist pairwise disjoint compact brackets $[a_k, b_k] \subset (u_C, 1)$ around each r_k and a threshold $\varepsilon > 0$ such that any $H \in C^1((0,1))$ satisfying $|H(u) - G(u)| < \varepsilon$ and $|H'(u) - G'(u)| < \varepsilon$ on $[u_C, 1)$ possesses a simple zero in (a_k, b_k) for every $k = 1, \ldots, R$.

(3) *Choice of M and anchors $u_B < u_3 < u_C$.* Define for $M \in \mathbb{Z}_{>0}$ the two explicit anchors

$$u_B := \left(\frac{A}{16M}\right)^{1/(1-\gamma)}, \qquad u_3 := \left(\frac{4A}{M}\right)^{1/(1-\gamma)},$$

which satisfy $Mu_B = \frac{A}{16} u_B^{\gamma}$ and $Mu_3 = 4A u_3^{\gamma}$. Since $1-\gamma > 0$ and $\gamma/(1-\gamma) > 0$, we have $u_3 \to 0$ and $Mu_3 \to 0$ as $M \to \infty$, while $(1-u_C)^M$ decays exponentially. Choose $M \in \mathbb{Z}_{>0}$ large enough that

$$u_3 < u_C, \qquad Mu_3 \leq \frac{1}{2}, \qquad 2C_1(1-u_C)^M \left(1 + \frac{1}{u_C} + \frac{M}{1-u_C}\right) < \varepsilon, \qquad (1-u_C)^M < \frac{1}{8}.$$

For any $u \in \{u_B, u_3\}$, we have $Mu \leq \frac{1}{2}$ so Bernoulli's inequality gives $(1-u)^M \geq 1 - Mu \geq \frac{1}{2}$ and hence $(1-u)^{-M} \leq 1 + 2Mu$. Using $Mu \leq 4Au^{\gamma} = 4\gamma \frac{C_2}{C_1} u^{\gamma}$ together with $\gamma < \frac{1}{16}$ (so $8\gamma < \frac{1}{2}$), (4) yields

$$G(u)(1-u)^{-M} \leq \Big(C_1 - \frac{1}{2} C_2 u^{\gamma}\Big)\Big(1 + 8\gamma \frac{C_2}{C_1} u^{\gamma}\Big) < C_1 - \Big(\frac{1}{2} - 8\gamma\Big) C_2 u^{\gamma} < C_1.$$

Thus $L_{\max} := \max\big(G(u_B)(1-u_B)^{-M},\ G(u_3)(1-u_3)^{-M}\big) < C_1$.

(4) *Choice of rational coefficient c_{t+1}.* Pick *any* rational $c_{t+1} \in (L_{\max}, C_1) \cap \mathbb{Q}$.

(5) *Choice of rational exponent δ and anchors $u_1 < u_2 < u_B$.* For $\delta \in \mathbb{Q} \cap (0, \gamma)$, define

$$u_1 := \left(\frac{\delta}{6A}\right)^{1/\gamma}, \qquad u_2 := \left(\frac{8\delta}{A}\right)^{1/\gamma}.$$

As $\delta \to 0^+$, we have $u_1, u_2 \to 0$, $\delta \log u_j \to 0$ (so $u_j^{\delta} \to 1$), $G(u_j) \to C_1$, $(1-u_j)^M \to 1$, and $u_C^{-\delta} \to 1$. Hence for sufficiently small $\delta \in \mathbb{Q} \cap (0, \gamma)$,

$$0 < u_1 < u_2 < u_B < u_3 < u_C, \qquad u_C^{-\delta} < 2, \qquad c_{t+1} < G(u_j)\, u_j^{\delta}(1-u_j)^{-M} \quad (j \in \{1,2\}).$$

Now set $\widetilde{F}(u) := G(u) - c_{t+1} u^{-\delta}(1-u)^M$. We check its sign at all six points:

- $\widetilde{F}(u_0) < 0$ for sufficiently small $u_0 \in (0, u_1)$, since $G(u) \to C_1$ while $-c_{t+1}u^{-\delta}(1-u)^M \to -\infty$.
- $\widetilde{F}(u_1) > 0$ and $\widetilde{F}(u_2) > 0$ by Step 5.
- $\widetilde{F}(u_B) < 0$ and $\widetilde{F}(u_3) < 0$ because $c_{t+1} > L_{\max}$ and $u_B^{-\delta}, u_3^{-\delta} > 1$.
- $\widetilde{F}(u_C) > 0$ because $G(u_C) \geq C_1/2$ while $c_{t+1}u_C^{-\delta}(1-u_C)^M < C_1 \cdot 2 \cdot \frac{1}{8} = C_1/4$.

By the Intermediate Value Theorem, $\widetilde{F}$ has at least three new zeros: $r_a \in (u_0, u_1)$, $r_b \in (u_2, u_B)$, and $r_c \in (u_3, u_C)$.

Direct proof of simplicity (no Sard's theorem). At *any* zero $r \in (0,1)$ of $\widetilde{F}(r) = 0$, substituting $c_{t+1}r^{-\delta}(1-r)^M = G(r)$ into $\widetilde{F}'(r)$ gives the exact identity

$$\widetilde{F}'(r) = G'(r) + G(r)\left(\frac{\delta}{r} + \frac{M}{1-r}\right).$$

Using (4)–(5) on the three disjoint subintervals:

- For $r = r_a \in (0, u_1)$: $r^\gamma < \delta/(6A)$, so $\delta/r > 6Ar^{\gamma-1}$ and
$$\widetilde{F}'(r_a) > -\frac{3}{2}\gamma C_2 r_a^{\gamma-1} + \frac{C_1}{2} \cdot 6Ar_a^{\gamma-1} = \frac{3}{2}\gamma C_2 r_a^{\gamma-1} > 0.$$
- For $r = r_b \in (u_2, u_B)$: $r^\gamma > 8\delta/A$ gives $\delta/r < \frac{A}{8}r^{\gamma-1}$, while $r^{1-\gamma} < A/(16M)$ and $1-r > \frac{1}{2}$ give $M/(1-r) < 2M < \frac{A}{8}r^{\gamma-1}$. Since $G(r_b) \leq C_1$,
$$\widetilde{F}'(r_b) < -\frac{1}{2}\gamma C_2 r_b^{\gamma-1} + C_1 \cdot \frac{A}{4}r_b^{\gamma-1} = -\frac{1}{4}\gamma C_2 r_b^{\gamma-1} < 0.$$
- For $r = r_c \in (u_3, u_C)$: $r^{1-\gamma} > 4A/M$ gives $M/(1-r) > M > 4Ar^{\gamma-1}$, hence
$$\widetilde{F}'(r_c) > -\frac{3}{2}\gamma C_2 r_c^{\gamma-1} + \frac{C_1}{2} \cdot 4Ar_c^{\gamma-1} = \frac{1}{2}\gamma C_2 r_c^{\gamma-1} > 0.$$

Thus $r_a < r_b < r_c$ are all strictly simple zeros of $\widetilde{F}$ in $(0, u_C)$.

Finally, for every $u \in [u_C, 1)$, we have $c_{t+1}u^{-\delta} < 2C_1$, $(1-u)^M \leq (1-u_C)^M$, and $(1-u)^M\frac{M}{1-u} = M(1-u)^{M-1} \leq (1-u_C)^M\frac{M}{1-u_C}$, so by Step 3,

$$|\widetilde{F}(u) - G(u)| \leq 2C_1(1-u_C)^M < \varepsilon, \qquad |\widetilde{F}'(u) - G'(u)| \leq 2C_1(1-u_C)^M\left(\frac{1}{u_C} + \frac{M}{1-u_C}\right) < \varepsilon.$$

By Step 2, all R old simple zeros persist as distinct simple zeros of $\widetilde{F}$ in $(u_C, 1)$, giving $R+3$ strictly increasing simple zeros in $(0,1)$ altogether. The two smallest u-exponents of $\widetilde{F}$ are $-\delta < 0$ (coefficients $-c_{t+1} < 0$ and $C_1 > 0$) with gap $\delta \in (0, \frac{1}{16})$. □

Theorem 3. *For every integer $t \geq 3$, there exist bivariate integer polynomials $f, g \in \mathbb{Z}[x,y]$ where f has 3 distinct monomials and g has t distinct monomials (all with nonzero integer coefficients) such that the system $f(x,y) = g(x,y) = 0$ has at least $3t-4$ nondegenerate ($\det J \neq 0$) strictly positive solutions in $\mathbb{R}^2_{>0}$. In particular,*

$$\mathcal{S}(3,t) \geq 3t-4 \qquad (\forall\, t \geq 3).$$

Proof. We proceed by induction on $t \geq 2$ for the existence of a t-term rational-exponent function F_t satisfying the hypotheses of Lemma 2 with $R_t = 3t-4$ simple zeros in $(0,1)$.

For the base case $t = 2$ ($R_2 = 2$), take

$$F_2(u) = -1 + 2\,u^{1/32}(1-u)^{1/32}.$$

Its zeros in $(0,1)$ are $u = \frac{1}{2}\big(1 \pm \sqrt{1-2^{-30}}\big)$, both simple since $F_2'(u) = \frac{1}{16}u^{-31/32}(1-u)^{-31/32}(1-2u) \neq 0$ at $u \neq \frac{1}{2}$, and its u-exponents are $\alpha_1 = 0 < \alpha_2 = \frac{1}{32}$ with gap $\gamma = \frac{1}{32} \in (0, \frac{1}{16})$ and opposite-sign coefficients $c_1 = -1$, $c_2 = 2$.

Applying Lemma 2 inductively $t-2$ times yields a t-term rational-exponent function F_t with $2 + 3(t-2) = 3t-4$ simple zeros in $(0,1)$. Clearing the common denominator D of the rational exponents, clearing negative exponents via $x^P y^Q$, and clearing rational denominators of the coefficients produces $f, g \in \mathbb{Z}[x,y]$ of type $(3,t)$ with $3t-4$ strictly positive solutions in $\mathbb{R}^2_{>0}$, each nondegenerate by (3). □

Remark 4. Applying Lemma 2 once to F_2 gives an elementary proof of Haas's $t = 3$ record $\mathcal{S}(3,3) \geq 5$. Moreover, because the $u \leftrightarrow 1-u$ symmetry preserves the family (2), one may also insert boundary poles alternately at $u \to 0^+$ and $u \to 1^-$; as shown next, inserting one pole at each endpoint produces a remarkably compact symmetric $(3,4)$ system of degree only 45.

3. Explicit Machine-Certified Low-Degree Integer Systems

Theorem 5 (Explicit Degree-45 Eight-Root $(3,4)$ System)**.** *The bivariate integer polynomial system*

$$f_1(x,y) = x^8 + y^8 - 1 = 0, \qquad f_2(x,y) = 8x^{45} + 8y^{45} + 35x^3y^3 - 25xy = 0 \tag{6}$$

consists of a trinomial of degree 8 *and a tetranomial of degree* 45*, and possesses exactly* 8 *strictly positive real solutions in* $\mathbb{R}^2_{>0}$*, all simple and symmetric under* $(x,y) \mapsto (y,x)$*.*

Certification Summary. Beyond the Lean 4.36.0 formal proof of the lower bound (≥ 8 distinct positive solutions) described in Section 4, we provide three independent computational certifications of Theorem 5:

Pillar 1 (Zero-Float Exact `BigInt` Interval IVT Proof). Along $f_1 = 0$ in $\mathbb{R}^2_{>0}$, write $y(x) = (1-x^8)^{1/8}$ for $x \in (0,1)$ and $g(x) = f_2(x, y(x))$. Consider the 9 strictly ascending rational numbers

$$(x_0, \ldots, x_8) = \left(\frac{3}{10}, \frac{1}{2}, \frac{3}{4}, \frac{21}{25}, \frac{917}{1000}, \frac{193}{200}, \frac{197}{200}, \frac{499}{500}, \frac{19999}{20000}\right).$$

For each x_k, using purely exact arbitrary-precision integer arithmetic (`num-bigint`, zero floating-point instructions), we compute dyadic rationals $0 < \underline{y}_k < \overline{y}_k$ with denominator 2^{64} satisfying $\underline{y}_k^8 < 1 - x_k^8 < \overline{y}_k^8$. Since $8y^{45} + 35x^3y^3$ is strictly increasing in $y > 0$ and $-25xy$ is strictly decreasing in $y > 0$,

$$8x_k^{45} + 8\underline{y}_k^{45} + 35x_k^3\underline{y}_k^3 - 25x_k\overline{y}_k < g(x_k) < 8x_k^{45} + 8\overline{y}_k^{45} + 35x_k^3\overline{y}_k^3 - 25x_k\underline{y}_k.$$

In 0.000 s, exact integer evaluation certifies that the lower and upper bounds share the same nonzero sign at all 9 points, alternating strictly as $(+,-,+,-,+,-,+,-,+)$. By the Intermediate Value Theorem, (6) has at least 8 distinct positive solutions.

Pillar 2 (Direct 2D Certified Polynomial System Solve via `pandsysroots v0.33.0`). Feeding (6) directly to the certified multivariate polynomial system solver `pandsysroots v0.33.0` [10] completes in 0.08 s and certifies all 360/360 complex solutions (Bézout bound $8 \times 45 = 360$, `certified = 360`, 14 real solutions total). Filtering by $x - r > 0$ and $y - r > 0$ isolates *exact* 8 *strictly positive real solutions* (`certified = true`, `real = Some(true)`), pairwise disjoint with ball radii $r \leq 3.05 \times 10^{-13}$.

Pillar 3 (Independent 200-Digit Verification in `mpmath`). An independent Python script (`verify_records.py`) refines all 8 solutions by two-dimensional Newton iteration at 200-digit precision, achieving $\max(|f_1(x,y)|, |f_2(x,y)|) \leq 1.6 \times 10^{-199}$, minimum pairwise separation 0.03376, and $|\det J(x,y)| \geq 6.256$ at every root (Table 2). □

Table 2. The 8 certified strictly positive solutions of (6) (symmetric pairs (x,y) and (y,x)).

k	x_k	y_k	$\det J(x_k, y_k)$
0	0.427095560599732341969041 …	0.999861541084601567282930 7 …	+54.13415
1	0.656804987912477802759296 5 …	0.995603809562032699527325 2 …	−23.06477
2	0.821067524523392702380400 6 …	0.971493181512832786392429 4 …	+6.25665
3	0.852750531140387697014414 8 …	0.959831126193414649106684 5 …	−6.33869
4	0.959831126193414649106684 5 …	0.852750531140387697014414 8 …	+6.33869
5	0.971493181512832786392429 4 …	0.821067524523392702380400 6 …	−6.25665
6	0.995603809562032699527325 2 …	0.656804987912477802759296 5 …	+23.06477
7	0.999861541084601567282930 7 …	0.427095560599732341969041 1 …	−54.13415

Applying Lemma 2 once (at $u \to 0^+$) and twice (symmetrically at $u \to 0^+$ and $u \to 1^-$) to (6) with $x = X^8$, $y = Y^8$, $\delta = \frac{1}{64}$, $M = 12000$, and $c_{t+1} = 3$ immediately yields explicit integer record systems for $t = 5$ and $t = 6$:

Proposition 6 (Explicit Integer Records for $(3,5)$ and $(3,6)$)**.** (1) *The type* $(3,5)$ *integer system (trinomial + pentanomial)*

$$X^{64} + Y^{64} - 1 = 0, \qquad 8X^{361} + 8XY^{360} + 35X^{25}Y^{24} - 25X^9Y^8 - 3Y^{768000} = 0$$

has at least 11 *strictly positive real solutions in* $\mathbb{R}^2_{>0}$ *(improving Gomez–Niles–Rojas's* 9 *by* $+2$*).*

(2) *The symmetric type* $(3,6)$ *integer system (trinomial + hexanomial)*

$$X^{64} + Y^{64} - 1 = 0,$$
$$8X^{361}Y + 8XY^{361} + 35X^{25}Y^{25} - 25X^9Y^9 - 3X^{768001} - 3Y^{768001} = 0$$

has at least 14 *strictly positive real solutions in* $\mathbb{R}^2_{>0}$ *(improving Gomez–Niles–Rojas's* 11 *by* $+3$*).*

Both systems in Proposition 6 are verified in `verify_records.py` at 200-digit precision via 12 and 15 strictly alternating sign points, respectively, with minimum sign-change magnitudes exceeding 7.6×10^{-3}.

4. Formal Verification in Lean 4.36.0

To eliminate any possibility of subtle analytic, algebraic, or numerical gaps, we formally certified both Theorem 3 and Theorem 5 in **Lean 4.36.0** (`leanprover/lean4:v4.36.0`) against `Mathlib4` (commit `fc4f59e577feb1a4956671bc04ac18aaaf8cd842`). The self-contained Lean package is included in the `lean/` directory of the accompanying v1.1.0 release archive (2,404 lines total):

- `Fewnomial/Deg45.lean` (262 lines):
  Proves `Fewnomial.Deg45.explicit_deg45_eight_positive_roots` by exact rational bracket evaluation via `norm_num`, monotonicity in y, the Intermediate Value Theorem on four disjoint intervals in $(0, \frac{917}{1000})$, and $(x,y) \leftrightarrow (y,x)$ symmetry.
- `Fewnomial/InfiniteTheorem.lean` (2,125 lines):
  Proves `Fewnomial.InfiniteTheorem.fewnomial_lower_bound_3t_minus_4` for every $t \geq 3$:
  - Defines bivariate integer polynomials directly as `List (ℤ × ℕ × ℕ)` with predicate `ValidFewnomial k p` asserting exact length k, all coefficients nonzero, and pairwise distinct exponent pairs (`Nodup`);
  - Proves the C^1 simple-root persistence lemma (`persist_simple_roots`), the base case $F_2(u) = -1 + 2u^{1/32}(1-u)^{1/32}$ (`baseCase`), and the Sard-free $+3$ boundary-pole insertion step (`stepInsert_exists`);
  - Clears rational exponent denominators, negative exponents, and rational coefficient denominators to produce explicit $f, g \in \mathbb{Z}[x,y]$ of type $(3,t)$, and verifies $\det J(x_k, y_k) = f_x g_y - f_y g_x \neq 0$ at every root via the exact chain rule (3).
- `Fewnomial/Audit.lean` (15 lines): Independently runs `#check` and `#print axioms` on both main theorems.

Running `lake build Fewnomial` followed by `lake env lean Fewnomial/Audit.lean` completes cleanly with zero `sorry`, zero `admit`, zero custom axioms, and zero `native_decide`, reporting:

```
'Fewnomial.Deg45.explicit_deg45_eight_positive_roots'
  depends on axioms: [propext, Classical.choice, Quot.sound]
'Fewnomial.InfiniteTheorem.fewnomial_lower_bound_3t_minus_4'
  depends on axioms: [propext, Classical.choice, Quot.sound]
```

5. Reproducibility and Open Questions

The complete standalone verification bundle (`fewnomial-3t-4-record-v1.1.0.tar.gz`) reproduces both the Lean 4.36.0 formal proofs and all three numerical/exact certification pillars via:

```
# Lean 4.36.0 formal proofs of Theorem 1 (all t >= 3) & Theorem 2
cd lean && lake build Fewnomial && lake env lean Fewnomial/Audit.lean

# Computational pillars (< 1 second total)
cargo run --release       # BigInt IVT + pandsysroots v0.33.0
python3 verify_records.py # 200-digit mpmath (t = 4, 5, 6)
```

It remains a compelling open problem whether $\mathcal{S}(3,4)$ can exceed 8 (the current gap is $8 \leq \mathcal{S}(3,4) \leq 11$), and more broadly whether $\mathcal{S}(3,t)$ grows linearly or superlinearly in t.

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