A MACHINE-CERTIFIED DEGREE-7,299 COUNTEREXAMPLE TO SMALE'S MEAN VALUE CONJECTURE

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Abstract. We construct and rigorously machine-certify an explicit polynomial $p(u) \in \mathbb{Q}(i)[u]$ of exact degree $d = 7{,}299$ satisfying $p(0) = 0$, $p'(0) = 1$, and

$$\min_{p'(\zeta)=0} \left| \frac{p(\zeta)}{\zeta} \right| \in \left[\, 1.00002719655422710959,\ \ 1.00002719655422710960 \,\right] > 1,$$

disproving Smale's 1981 Mean Value Conjecture ($K \leq 1$). Compared to the October 2026 disproof of Adamczewski and Colbrook (`arXiv:2610.36158`, degree 29,984, minimum ≈ 1.000000495958), our explicit construction reduces the polynomial degree by a factor of 4.11× while increasing the certified margin above 1 by 54.8×. By standard leading-term perturbation, this establishes explicit counterexamples for every degree $n \geq 7{,}299$. All 7,298 critical points are certified in 413.7 s using a standalone zero-float-assumption Rust verifier based on exact arbitrary-precision integer arithmetic, Rouché's theorem, exact pairwise disc disjointness, and outward-rounded Taylor bounds, independently cross-checked at 900-digit precision in `mpmath` and in `pandroots v0.31.0`.

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1. Main Certified Theorem

In 1981, Smale [3] proved that every complex polynomial p of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) = 1$ possesses a critical point $\zeta \in \mathbb{C}$ ($p'(\zeta) = 0$) satisfying $|p(\zeta)/\zeta| \leq 4$, and conjectured that the constant 4 could be sharpened to 1 (or $(d-1)/d$). While the conjecture holds for $d \leq 5$, Adamczewski and Colbrook [1] disproved the global $K \leq 1$ conjecture in October 2026 by constructing a certified counterexample of degree 29,984. Here we lower the certified counterexample threshold by more than a factor of four.

Theorem 1 (Machine-Certified Counterexample)**.** *Let $P(z) = \sum_{k=0}^{7299} c_k z^k \in \mathbb{Z}[i][z]$ be the explicit Gaussian-integer polynomial archived in `data/champion_deg7299.txt.gpoly` (SHA-256: `e91dd6758e7d72b58db4a4f5702b627aa85c4d968faf892c28a87678fc3b786f`, leading coefficient $c_{7299} \neq 0$, maximum coefficient bit-length 2,311), and let*

$$a = \frac{6485183463413514 - 3422735716801577\, i}{2^{53}} \in \mathbb{Q}(i) \approx 0.72 - 0.38\, i.$$

Define the normalized polynomial

$$p(u) = \frac{P(u+a) - P(a)}{P'(a)} \in \mathbb{Q}(i)[u], \qquad \deg(p) = 7{,}299, \quad p(0) = 0, \quad p'(0) = 1.$$

Then all 7,298 critical points of P are simple, lie in $|c_j| < 0.942$, and satisfy the rigorously certified global enclosure

$$\min_{p'(\zeta)=0} \left| \frac{p(\zeta)}{\zeta} \right| = \min_{P'(c)=0} \left| \frac{P(c) - P(a)}{(c-a)\, P'(a)} \right| \in \left[\, 1.00002719655422710959,\ \ 1.00002719655422710960 \,\right] > 1.$$

Corollary 2. *By adding εu^n with sufficiently small $\varepsilon > 0$ (via Rouché's theorem, as in [1]), counterexamples to Smale's Mean Value Conjecture ($K \leq 1$) exist for every polynomial degree $n \geq 7{,}299$.*

Table 1 compares our certified counterexample against the October 2026 benchmark [1].

Date: November 24, 2026.

TABLE 1. Comparison of rigorously certified counterexamples to Smale's Mean Value Conjecture.

Metric	Adamczewski–Colbrook	**This Work (Nov 2026)**	Improvement
Exact degree d	29,984	**7,299**	**4.11**× smaller
Certified min	$(1.00000049595801, \ldots 04)$	$[\mathbf{1.00002719655422710959}, \ldots \mathbf{60}]$	**54.8**× margin
Critical points	29,983/29,983	**7,298/7,298** (all simple)	max 1.031936
Artifact size	—	7.6 MB (`.gpoly`)	z-centered
Cert. time	—	**413.72** s (10 cores, Rust)	zero C/GMP

2. Analytic Construction and Exact Monomial Expansion

(1) **Fourier-Smoothed Analytic Pac-Man Domain (`src/bin/find_champion.rs`):** To avoid sharp slit-vertex singularities that stall polynomial convergence on polygonal or slit domains, we construct a smooth analytic horizontal-slot Pac-Man boundary via Gaussian-damped Fourier filtering ($h = 0.14$, $x_c = 0.32$, $\sigma = 5.0$, $k_{\max} = 28$). Using Lightning rational approximation for the interior conformal map (boundary error 2.32×10^{-4}, base point $a = 0.72 - 0.38\,i$, conformal margin 1.002027) and the exterior conformal map (error 2.54×10^{-7}, logarithmic capacity 0.925161), we form the exact degree-7,299 Fejér–Lagrange interpolant at $M = 7{,}300$ Fejér nodes $\Phi_{\text{ext}}(\zeta_j) = e^{2\pi i j/M}$ with $\varepsilon = 0$.

(2) **Bit-Reversed Karatsuba Product Tree (`src/bin/fejer_to_gpoly.rs`):** Expanding in centered z-coordinates (rather than shifted $u = z - a$) keeps every critical point strictly inside the unit disc ($\max|c_j| = 0.941507$) and bounds coefficient growth to 2,311 bits (`out_bits = 320`, `lead_bits = 1121`). Crucially, permuting the Fejér nodes in bit-reversed order before building the 3,200-bit parallel Karatsuba product tree guarantees that every intermediate subtree is equidistributed around the full boundary, eliminating half-plane binomial blowup ($\sim 2^{3650} \to\sim 2^{1000}$) and preserving full precision across the entire outer rim.

3. Zero-Float-Assumption Exact Integer Proof

The standalone verifier `src/bin/rigorous_interval_cert.rs` executes every proof inequality strictly in exact arbitrary-precision integer arithmetic (`num_bigint::BigInt`) with $F = 1{,}984$ fractional bits (≈ 597 decimal digits). Floating-point values are used solely to seed initial root guesses and for human-readable display; no proof step trusts IEEE-754 arithmetic.

(1) **Exact Degree:** Verifies $c_{7299} \neq 0$, establishing $\deg(P') = 7{,}298$.

(2) **Global Outward-Rounded Derivative Bounds:** On the closed disc $|w| \le R = 96/100$, computes exact integer upper bounds via ceiling-rounded powers of R:

$$M_1 \ge \sum_{k=1}^{d} k|c_k|R^{k-1} \le 2^{2048}, \qquad M_3 \ge \sum_{k=3}^{d} k(k-1)(k-2)|c_k|R^{k-3} \le 2^{2073}.$$

(3) **Exact Rouché Certification on Every Disc $D(z_j, r)$ ($r = 2^{-1863}$):** For each of the 7,298 dyadic centers $z_j \in 2^{-1984}\mathbb{Z}[i]$, exact simultaneous Horner evaluation of $P(z_j), P'(z_j), P''(z_j)$ with explicitly propagated integer error bounds (e_p, e_{dp}, e_{ddp}) proves $|z_j| + r \le 96/100$, $|P''(z_j)|_{\text{lo}} > 0$, and

$$|P'(z_j)|_{\text{hi}} + \frac{1}{2}M_3 r^2 < |P''(z_j)|_{\text{lo}} \cdot r.$$

By Rouché's theorem, each disc $D(z_j, r)$ contains exact multiplicity 1 (a single simple zero of P'). All 7,298/7,298 discs pass.

(4) **Exact Pairwise Disjointness and Completeness:** An exact integer sweep-line check confirms $|z_i - z_j|^2 > (2r)^2$ for every pair $i \neq j$. By the Fundamental Theorem of Algebra, these 7,298 disjoint discs account for every critical point of P in $\mathbb{C}$.

(5) **Exact Outward-Rounded Smale Enclosure:** Using $|P(w) - P(z_j)| \le M_1 r$ for all $w \in D(z_j, r)$, exact integer division proves $|(P(w) - P(a))/((w-a)P'(a))| > 1$ on all 7,298/7,298

discs, certifying the global minimum in $[1.00002719655422710959,\ 1.00002719655422710960]$ at $z^* \approx 0.5502519558 - 0.1147924148\,i$ (maximum critical value 1.03193594).

(6) **Independent Cross-Checks:** An independent 900-digit Python script (`verify_independent.py`) confirms $|P'(z^*)|/|P'(a)| = 1.78 \times 10^{-880}$ and Smale ratio $1.00002719655422710959640 7$, matching all 20 certified digits. Additionally, `pandroots v0.31.0` [2] independently certifies all 7,298/7,298 critical points directly in 199 s.

4. Reproducibility and Artifact Integrity

The complete polynomial, source code, and logs are archived on Zenodo (https://doi.org/10.5281/zenodo.24512942) and GitHub (https://github.com/ivan-fr/smale-counterexample-deg7299):

```
cargo build --release --bin rigorous_interval_cert
./target/release/rigorous_interval_cert \
    data/champion_deg7299.txt.gpoly data/champion_deg7299.roots_f64
python3 verify_independent.py
```

Table 2. SHA-256 checksums of the archived degree-7,299 counterexample artifacts.

File	Size	SHA-256
`champion_deg7299.txt.gpoly`	7.6 MB	`e91dd6758e7d72b58db4a4f5702b627aa85c4d968faf892c28a87678fc3b786f`
`champion_deg7299.txt`	670 KB	`319f9ab8f35cbccec305f1ff3e66e3b519705c61be5a43f51c711b123e89f579`
`champion_deg7299.roots_f64`	335 KB	`48fdfd58b78e31f04e95a3a6753450a65c818d6a20511d745d6b27ebadd27247`

References

[1] T. Adamczewski and M. J. Colbrook, *From boundary geometry to counterexamples in Smale's mean value problem*, arXiv:2610.36158 (2026).

[2] I. Besevic, ***pandroots:*** *Certified univariate polynomial root-finder in Rust (v0.31.0)*, https://crates.io/crates/pandroots (2026).

[3] S. Smale, *The fundamental theorem of algebra and complexity theory*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 1, 1–36.

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