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# THE CONSTANT TERM IN CRAMÉR'S MEAN-SQUARE THEOREM FOR THE PRIMES. SECOND-ORDER ASYMPTOTICS AND LIMITING FLUCTUATIONS OF THE PRIME ENERGY

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ABSTRACT. Let $I(X) = \int_1^X (\psi(x) - x)^2 x^{-2}\, dx$ be the energy of the primes. Under the Riemann hypothesis (RH) and simple zeros its slope is $C_B = 2 + \gamma_E - \log 4\pi$ (Cramér). We study the next terms. Under RH, simple zeros and an additional spacing condition (H*), $I(X) = C_B \log X + K + \Phi(\log X) + O(X^{-1/2})$, where $\Phi$ is an absolutely convergent almost periodic sum over pairs of zeros. The constant has a closed form without zeros, $K = \frac{1}{2}C_B^2 - C_B - \sigma_2 + 2(1+\gamma_E)\log 2\pi - (\log 4\pi - 1)^2 - 1 + 2\sum_n \Lambda(n)\varphi(1/n)$, with $\sigma_2 = \sum_\rho \rho^{-2}$ and $\varphi(y) = y - \operatorname{artanh} y - \frac{y}{2}\log(1-y^2)$; under RH, $K = 2.512479790006\ldots$, certified to $6 \cdot 10^{-13}$ in interval arithmetic. Under RH and simple zeros alone, $K$ is the Cesàro mean in $\log X$ of $I(X) - C_B \log X$ if and only if $\sum_{0<|\gamma-\gamma'|<\varepsilon} |\rho\rho'|^{-1} = o(\varepsilon)$: a pair of zeros at distance $\lambda$ acts like a double zero for $\log X \ll 1/\lambda$. Under (H*), $I(X) - C_B \log X$ has a limiting logarithmic distribution with mean $K$ and variance $V$, an explicit double sum over zeros under linear independence, whose factors we check on finite sets of zeros; numerically $V = 1.2502 \cdot 10^{-5}$ under LI. The primes up to $10^9$ agree with the zero side to $1.3 \cdot 10^{-5}$ (r.m.s.) and give $K = 2.512484$. In the random models of the primes and of $\mu$ that we examine the median energy is 8 to 150 times larger. A Gaussian-window identity isolates the principal mode of each zero in the prime signal: a closed ellipse on the line, an expanding spiral off it. A final section assembles the slope, the intercept and the oscillation as one calibrated instrument, the energy gauge, with a multiplicity-meter reading of the slope and a ledger reading of the intercept, and runs it on the primes to $10^8$ beside 38 zeros computed from scratch.

## 1. INTRODUCTION

1.1. **Context.** Let $X_n(s) = \sum_{k \le n} k^{-s}$, $s = \sigma + it$, and let

$$C_2(n,s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma} = 2\operatorname{Re}\left(X_{n-1}(s)\, n^{-\bar{s}}\right)$$

be the cross-energy of consecutive partial sums. The author's paper [CE1] shows that $\lim_n C_2(n,s)$ is $1/(\frac{1}{4}+t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$ (Theorem A, [CE1, Thm. 3.2]), and [CE5] shows that the mean energy of the prime signal

$$E(u) = e^{-u/2}\Big(\psi(e^u) - e^u + \log 2\pi + \tfrac{1}{2}\log\left(1 - e^{-2u}\right)\Big) = -\sum_\rho \frac{e^{(\rho-\frac{1}{2})u}}{\rho}$$

is $\sum_\rho \lim_n C_2(n,\rho)$, at least $C_B = 2 + \gamma_E - \log 4\pi = 0.0461914\ldots$, with equality if and only if RH holds and all zeros are simple (Theorem B and its corollary in [CE5, Sect. 2]). Throughout, $\rho = \beta + i\gamma$ runs over the nontrivial zeros of $\zeta$ with multiplicity, sums over zeros are taken in symmetric order unless they converge absolutely, $\gamma_E$ is Euler's constant and $\gamma_1$ the first Stieltjes constant, $\zeta(s) = \frac{1}{s-1} + \gamma_E - \gamma_1(s-1) + \cdots$.

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That result concerns the slope of the energy of the primes,

$$I(X) = \int_1^X \frac{(\psi(x)-x)^2}{x^2}\,dx = \int_0^{\log X} E_0(u)^2\,du, \qquad E_0(u) = e^{-u/2}\big(\psi(e^u)-e^u\big),$$

whose asymptotic $I(X) \sim C_B \log X$ under RH is Cramér's theorem [Cr]. This paper studies what comes after the slope: the constant term and the oscillation of $I(X)$, the hypotheses they need, their limiting distribution, and the resolution of the prime signal into the modes of single zeros. The large values of short Dirichlet polynomials, which govern what the energy can say about zeros off the line and were treated in an earlier version of this paper, are now in [CE7].

1.2. **Results.** The results, with their hypotheses and status, are the following.

(1) *Second-order energy law* (Theorem 5; under RH, simplicity of the zeros and the additional spacing condition (H*) of Sect. 3; proved): $I(X) = C_B \log X + K + \Phi(\log X) + O(X^{-1/2})$, with $\Phi$ an absolutely convergent trigonometric series over the differences $\gamma - \gamma'$ (Proposition 4) and $K$ an explicit constant. The passage from finite sets of zeros to all zeros rests on two lemmas stated and proved here: the zero expansion converges in $L^2$ on bounded intervals (Lemma 1, unconditional) and its energy in unit windows is bounded uniformly (Lemma 2, under RH).

(2) *The constant without zeros* (Propositions 7 and 8, Corollary 9; under RH; proved, and the value is certified in interval arithmetic): $K$ is given by a closed formula in $\gamma_E$, $\gamma_1$, $\log 2\pi$, $\pi^2$ and a rapidly convergent sum over prime powers, and $K \in [2.512479790006 \pm 6\cdot 10^{-13}]$. The trivial part of the explicit formula contributes $(\log 4\pi - 1)^2 + 1 - \pi^2/12 = 2.5215682\ldots$; the interval $1 \le x < 2$ alone contributes exactly 1.

(3) *The strength of the spacing condition* (Sect. 3.2; under RH and simple zeros, without (H*); proved): $K$ is the Cesàro mean of $I(e^u) - C_B u$ if and only if the weighted number of pairs of zeros at distance less than $\varepsilon$ is $o(\varepsilon)$ (Proposition 11). A pair at distance $\lambda$ acts like a double zero on the scale $u \ll 1/\lambda$, which is why some condition on small gaps is needed. If $I(e^u) - C_B u - K$ is approximated in mean square by the truncations of $\Phi$, it has a limiting distribution without absolute convergence (Proposition 12); under linear independence this approximation forces both a quadratic spacing condition ($H_2$) and the condition $\nu(\varepsilon) = o(\varepsilon)$ above, which are independent of each other. Whether the two together suffice is our main open problem (Problem 13).

(4) *Limiting distribution* (Theorem 14; under (H*), and under the linear independence hypothesis LI for part 3; proved): $I(X) - C_B \log X$ has a limiting distribution in $\log X$ with mean $K$, variance $V = \sum_\lambda |a_\lambda|^2$ and support in $[K-H, K+H]$; under LI, $V$ is an explicit double sum over zeros. Its combinatorial factors are checked against a direct computation with 1 to 5 pairs of zeros (numerical). Numerically, under LI, $V = 1.2502\cdot 10^{-5}$; the truncation $V_{10^4}$ is a numerical lower bound, the part of the tail coming from pairs of distant zeros is bounded rigorously, and the part coming from close pairs only under an assumption on the gaps. The primes up to $10^9$ confirm the second-order law: the residual is $1.2$–$1.3\cdot 10^{-5}$ (r.m.s.) and $K_{\text{primes}} = 2.512484$ (numerical).

(5) *Comparison with random models* (Sect. 5, numerical): in Cramér's model the energy is 46 to 150 times that of the primes, and in random models of $\mu$ it is 6 to 20 (independent signs, smallest realization to median) and typically 8 (random multiplicative functions) times that of $\mu$; no realization out of 100 reaches the arithmetic value. After the zeros up to height $T$ are subtracted, the measured residual energy agrees with the contribution predicted from the zeros above $T$ to within 2%, for $50 \le T \le 10^4$; no additional stochastic component is detected at the precision and scales investigated.

(6) *Principal modes of the zeros* (Proposition 16 and Corollary 17, proved; Sect. 6): the Gaussian demodulation of the prime signal at the frequency $\gamma$ of a zero is an exact sum over

all zeros; it splits into the principal mode of the zero itself, $-e^{(\beta-\frac12)u+(\beta-\frac12)^2s^2/2}/\rho$, and an interference term damped by $e^{-(\gamma'-\gamma)^2s^2/2}$, which we bound. The principal mode is a closed ellipse for a zero on the line and an expanding spiral off it; the measured orbit is that curve up to the interference. Computations with the primes up to $10^8$ and with the Davenport–Heilbronn function are numerical.

1.3. **What is classical and what is new.** We separate three kinds of contribution.

*Known results, cited as such:* Cramér's theorem [Cr]; the equivalence of bounded mean energy with RH [CE5]; the explicit formula and the Hadamard product [Da, In]; the theorem of Wintner and of Jessen and Wintner on limiting distributions of almost periodic functions [Wi35, ANS], and the fact that $B^2$-almost periodic functions have limiting distributions [ANS, Thm. 2.9]; the Montgomery–Vaughan inequalities [MV74]; Trudgian's bound for $N(T)$ [Tr14]. Lemmas 1–3 are standard in nature; we prove them because the passage to the limit in Theorem 5 depends on them.

*New results (proved here):* Theorem 5 with its constant $K$; the closed formula of Proposition 8 and the certified value of $K$; Proposition 11, which characterizes when $K$ is the Cesàro mean of the deviation without assuming (H$^*$); Proposition 12 (a direct consequence of [ANS, Thm. 2.9]); the identification of the mean and the explicit variance in Theorem 14 (the existence of the distribution under (H$^*$) is standard); the orbit identity of Proposition 16 (elementary).

*New formulations of known objects:* the zero–zero part of $K$ as the second elementary symmetric function of the limits of $C_2$; the variance as a weighted Rodgers–Tao energy; the reading of single zeros through their principal modes.

*Numerical observations (reproducible, not theorems):* the values of $V$, the shape of the distribution, the agreement with the primes up to $10^9$, the comparison with random models, and the orbits and phases measured from the primes and from the Davenport–Heilbronn function.

On priority: Brent, Platt and Trudgian [BPT] (and the related talks of Brent) study the mean square of the prime error and double sums over zeros, and give the leading term of the logarithmic mean but no constant term; Akbary, Ng and Shahabi [ANS] give general criteria for limiting distributions of the classical error terms, not this one. A targeted search did not find $K$, its closed form or the distribution of $I(X)-C_B\log X$; we cannot exclude that they appear elsewhere. The numbers 2.481, 2.504, 2.510, 2.513, 2.511 reported in [CE18] (in the remark on the signed form of the energy identity) for the difference between $I(N)$ and $C_B\log N$, $N=10^2,\dots,10^6$, are values of $K+\Phi(\log N)$.

1.4. **Organization.** Sect. 2 fixes notation and proves the three lemmas. Sect. 3 contains the second-order law, the closed form of $K$ and the discussion of the spacing condition. Sect. 4 treats the limiting distribution and its variance, Sect. 5 the comparison with random models, and Sect. 6 the principal modes of single zeros. Sect. 8 lists what is and is not achieved, and the open problems. Related work by the author. This paper is Part 6 of the series *Cross-Energy of Dirichlet Partial Sums* [CE1]. It builds on [CE1, CE4, CE5, CE7, CE8] (the cross-energy, the circles of the primes, the energy law of the primes, large values and the Euler product). The energy gauge of Section 7 is used again for the Chebyshev race between the classes $6k\pm1$ in [CE38], and the controls are in [CE46]. Each of these is a self-contained preprint; the present paper can be read without them.

## 2. Setting

By the explicit formula [In, T], for $u>0$,

$$E_0(u)=S(u)+r(u),\qquad S(u)=E(u)=-\sum_\rho \frac{e^{(\rho-\frac12)u}}{\rho},$$
$$r(u)=-e^{-u/2}\Big(\log 2\pi+\tfrac12\log\big(1-e^{-2u}\big)\Big), \tag{1}$$

the sum over zeros converging boundedly in symmetric order at points $e^u$ that are not prime powers. We use (1) only as the definition $S := E_0 - r$ on $(0,\infty)$ (as an element of $L^2_{\mathrm{loc}}$, so the values of $\psi$ at its jumps are irrelevant) and prove below the form of convergence that we need. The trivial part $r$ lies in $L^1 \cap L^2(0,\infty)$ and $|r(u)| \le 2e^{-u/2}$ for $u \ge 1$. With $r(u) = \sum_{j\ge 0} c_j e^{(a_j - \frac12)u}$, $a_0 = 0$, $c_0 = -\log 2\pi$, $a_k = -2k$, $c_k = 1/(2k)$ $(k \ge 1)$, its Laplace transform is

$$R(s) = \int_0^\infty r(u) e^{-(s-\frac12)u}\, du = \sum_{j\ge 0} \frac{c_j}{s - a_j} = \frac{1}{s}\Big(\frac12 \frac{\Gamma'}{\Gamma}\Big(1 + \frac{s}{2}\Big) + \frac{\gamma_E}{2} - \log 2\pi\Big), \tag{2}$$

and the Mellin transform of the prime error is

$$\mathcal{M}(s) := \int_0^\infty E_0(u) e^{-(s-\frac12)u}\, du = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = -\sum_\rho \frac{1}{\rho(s-\rho)} + R(s) \qquad (\mathrm{Re}\, s > 1), \tag{3}$$

the last equality being the partial fraction expansion of $\zeta'/\zeta$ [Da, Ch. 12] divided by $s$; the series converges absolutely. We use two facts from [CE5, Sect. 2]. First, Theorem B: in $[0,\infty]$, $\limsup_U \frac{1}{U}\int_0^U E(u)^2\, du = \sum_\rho \lim_n C_2(n,\rho)$ when the zeros are simple, the two sides are finite exactly under RH, and then both equal $\sum_\rho |\rho|^{-2} = C_B$. Second, the local energy law: for $U \in \mathbb{R}$ and $h > 0$,

$$\frac{1}{h}\int_{-\infty}^\infty S(u)^2 e^{-\pi(u-U)^2/h^2}\, du = \sum_{\rho,\rho'} \frac{e^{zU + z^2h^2/(4\pi)}}{\rho\,\overline{\rho'}}, \qquad z = \rho + \overline{\rho'} - 1, \tag{4}$$

with absolute convergence. The energy matrix of the zeros, $M(U)_{\rho\rho'} = (\rho\bar\rho')^{-1} U^{-1} \int_0^U e^{i(\gamma-\gamma')u}\, du$, whose diagonal consists of the limits $\lim_n C_2(n,\rho) = |\rho|^{-2}$ and whose trace is $C_B$, is also in [CE5, Sect. 2]; we refer to it below only to interpret the constant term.

For a finite set $F$ of zeros let $S_F(u) = -\sum_{\rho\in F} e^{(\rho-\frac12)u}/\rho$, and let $F_T = \{\rho : |\gamma| < T\}$, $S_T = S_{F_T}$. For $\rho$ a zero let $\delta_\rho = \min_{\rho'\ne\rho} |\gamma - \gamma'|$ be the distance from $\gamma$ to the nearest other ordinate. We write $u = \log X$ and

$$\mathcal{D}(u) = I(e^u) - C_B u,$$

the deviation of the energy from its slope.

**Lemma 1** (local $L^2$ convergence of the zero expansion)**.** *Let $\sigma_1 > 1$, $\eta = \sigma_1 - 1$, and for zeros $\rho, \rho'$ let*

$$J(\rho,\rho') = \frac{1}{|\rho||\rho'|}\int_{-\infty}^\infty \frac{dt}{|\sigma_1 + it - \rho|\,|\sigma_1 + it - \rho'|}.$$

*Unconditionally, $\sum_{\rho,\rho'} J(\rho,\rho') < \infty$, and for every finite set $F$ of zeros*

$$\int_0^\infty |S(u) - S_F(u)|^2 e^{-(2\sigma_1-1)u}\, du \le \frac{1}{2\pi}\sum_{\rho,\rho'\notin F} J(\rho,\rho').$$

*Hence, for every $U > 0$, $S_T \to S$ in $L^2(0,U)$ as $T \to \infty$, and $\int_0^U S_T^2 \to \int_0^U S^2$. The same holds for any increasing sequence of finite sets exhausting the zeros.*

*Proof.* Put $f(u) = S(u)e^{-(\sigma_1-\frac12)u}$ for $u > 0$ and $f(u) = 0$ for $u \le 0$. Since $|\psi(x) - x| \le x$ for $x \ge 1$, $|E_0(u)| \le e^{u/2}$; and $r$ is in $L^1 \cap L^2$. So $f \in L^1 \cap L^2(\mathbb{R})$, and by (2) and (3) its Fourier transform is

$$\hat f(t) = \int_0^\infty S(u) e^{-(\sigma_1 - \frac12 + it)u}\, du = \mathcal{M}(s) - R(s) = -\sum_\rho g_\rho(t), \qquad g_\rho(t) = \frac{1}{\rho(\sigma_1 + it - \rho)}, \quad s = \sigma_1 + it.$$

For the finite sum, $\int_0^\infty e^{(\rho-\frac12)u}e^{-(\sigma_1-\frac12+it)u}du = 1/(s-\rho)$ because $\mathrm{Re}(s-\rho)=\sigma_1-\beta>0$, so the transform of $f_F = S_F e^{-(\sigma_1-\frac12)u}\mathbf{1}_{u>0}$ is $-\sum_{\rho\in F} g_\rho$. By Plancherel and Tonelli,

$$\|f-f_F\|_2^2 = \frac{1}{2\pi}\int \Big|\sum_{\rho\notin F} g_\rho(t)\Big|^2 dt \le \frac{1}{2\pi}\sum_{\rho,\rho'\notin F}\int |g_\rho||g_{\rho'}|\,dt = \frac{1}{2\pi}\sum_{\rho,\rho'\notin F} J(\rho,\rho'),$$

which is the inequality. For the convergence of $\sum J$: $|\sigma_1+it-\rho|\ge \max(\eta,|t-\gamma|)\ge \frac12(\eta+|t-\gamma|)$, and with $d=|\gamma-\gamma'|$, splitting the integral at the midpoint of $\gamma$ and $\gamma'$,

$$\int_{-\infty}^{\infty}\frac{dt}{(\eta+|t-\gamma|)(\eta+|t-\gamma'|)} \le \frac{4\log(1+d/(2\eta))+2}{\eta+d/2}.$$

Hence $J(\rho,\rho')\ll_\eta \log(2+d)/((1+d)|\rho||\rho'|)$. Since a unit interval at height $t$ contains $\ll\log(|t|+2)$ ordinates, the sum over $\rho'$ is $\ll\log^3(|\gamma|+2)/(|\gamma|+1)$, as in the proof of Proposition 4 below, and $\sum_\rho \log^3(|\gamma|+2)/|\gamma|^2<\infty$. The tail of a convergent double series of nonnegative terms tends to 0 along any exhaustion; finally $\int_0^U|S-S_F|^2\le e^{(2\sigma_1-1)U}\|f-f_F\|_2^2$, and $\big|\|S_T\|^2_{L^2(0,U)}-\|S\|^2_{L^2(0,U)}\big|\le \|S_T-S\|_{L^2(0,U)}\big(\|S_T\|_{L^2(0,U)}+\|S\|_{L^2(0,U)}\big)$. □

Status: proved, unconditional. Under RH, $S_F(u)=-\sum_{\rho\in F}e^{i\gamma u}/\rho$ is defined for all real $u$, and it is real when $F$ is closed under conjugation.

**Lemma 2** (uniform window energy)**.** *Assume RH, and let* $M=\sum_{\rho,\rho'} e^{-(\gamma-\gamma')^2/(4\pi)}/(|\rho||\rho'|)$. *Then* $M<\infty$, *and for every finite set* $F$ *of zeros and every* $V\in\mathbb{R}$,

$$\int_V^{V+1}|S_F(u)|^2\,du \le e^{\pi/4}M;$$

*the same bound holds for* $S$ *on* $[V,V+1]$, $V\ge 0$.

*Proof.* Since $\int e^{i\lambda u}e^{-\pi(u-V')^2}du = e^{i\lambda V'-\lambda^2/(4\pi)}$, for $V'=V+\frac12$

$$\int_{-\infty}^{\infty}|S_F(u)|^2e^{-\pi(u-V')^2}du = \sum_{\rho,\rho'\in F}\frac{e^{i(\gamma-\gamma')V'-(\gamma-\gamma')^2/(4\pi)}}{\rho\,\bar\rho'}\le M,$$

and the weight is at least $e^{-\pi/4}$ on $[V,V+1]$. For fixed $\rho$, the zeros $\rho'$ in unit windows at distance $j$ from $\gamma$ number $\ll\log(|\gamma|+j+2)$ and carry the factor $e^{-j^2/(4\pi)}$, so $\sum_{\rho'}e^{-(\gamma-\gamma')^2/(4\pi)}/|\rho'|\ll \log(|\gamma|+2)/(|\gamma|+1)$ and $M\ll\sum_\rho\log(|\gamma|+2)/|\gamma|^2<\infty$. For $S$ use Lemma 1 on $[V,V+1]$. □

Status: proved, under RH. It is the case $h=1$ of (4) written for finite sets of zeros, which is what makes the bound uniform in $F$.

**Lemma 3** (the cross term)**.** *Assume RH. Then* $\int_0^\infty|S(u)r(u)|\,du<\infty$; *for* $U\ge1$,

$$\int_U^\infty|S(u)r(u)|\,du \le \sum_{j\ge0}\Big(\int_{U+j}^{U+j+1}S^2\Big)^{1/2}\Big(\int_{U+j}^{U+j+1}r^2\Big)^{1/2} \le \frac{2e^{\pi/8}M^{1/2}}{1-e^{-1/2}}\,e^{-U/2};$$

*and* $\int_0^\infty S(u)r(u)\,du=-\sum_\rho R(1-\rho)/\rho$, *where the series converges absolutely.*

*Proof.* The first inequality is Cauchy–Schwarz on each interval $[U+j,U+j+1]$; the second follows from Lemma 2 and $|r(u)|\le 2e^{-u/2}$ for $u\ge1$. On $(0,1)$, $\int_0^1|Sr|\le\|S\|_{L^2(0,1)}\|r\|_{L^2(0,1)}<\infty$. For $F=F_T$, $\int_0^\infty S_Fr=-\sum_{\rho\in F}R(1-\rho)/\rho$, because $r\in L^1$ and $\int_0^\infty e^{i\gamma u}r(u)\,du=R(\frac12-i\gamma)=R(1-\rho)$ under RH. For $U\ge1$, by Lemma 1 and the bound just proved, which holds for $S_F$ and for $S$ with the same constant,

$$\Big|\int_0^\infty(S-S_F)r\Big|\le\|S-S_F\|_{L^2(0,U)}\|r\|_2+\frac{4e^{\pi/8}M^{1/2}}{1-e^{-1/2}}e^{-U/2};$$

letting $T \to \infty$ and then $U \to \infty$ gives the identity. From (2) and $\frac{\Gamma'}{\Gamma}(z) = \log z + O(1/|z|)$ in $\operatorname{Re} z \geq 1$, $|R(\frac{1}{2}+it)| \ll \log(|t|+2)/(|t|+1)$, so $\sum_\rho |R(1-\rho)/\rho| \ll \sum_\rho \log(|\gamma|+2)/|\gamma|^2 < \infty$. □

Status: proved, under RH.

## 3. The constant term and the oscillation of the energy

The hypothesis of Theorem 5 is

(H*) RH holds, all zeros are simple, and $\displaystyle\sum_\rho \frac{\log(|\gamma|+2)}{|\rho|^2\,\delta_\rho} < \infty.$

The spacing condition in (H*) does not follow from RH and simplicity; Sect. 3.2 discusses its strength and what survives without it. Write

$$\Phi(u) = \sum_{\rho \neq \rho'} c_{\rho\rho'}\, e^{i(\gamma-\gamma')u}, \qquad c_{\rho\rho'} = \frac{1}{i(\gamma-\gamma')\,\rho\,\bar\rho'}, \qquad H = \sum_{\rho\neq\rho'} |c_{\rho\rho'}|. \tag{5}$$

**Proposition 4.** *Under (H*), $H < \infty$. Hence $\Phi$ is a uniformly convergent trigonometric series, and in particular a Bohr almost periodic function with mean zero and $|\Phi| \leq H$.*

*Proof.* Fix $\rho$ with $|\gamma|$ large and split the sum over $\rho'$. For $|\gamma-\gamma'| \geq 1$, the Riemann–von Mangoldt formula gives $\sum_{\rho'} |\gamma-\gamma'|^{-1}|\rho'|^{-1} \ll \log^2|\gamma|/|\gamma|$: the zeros with $|\gamma'| \leq |\gamma|/2$ or $|\gamma'| \geq 2|\gamma|$ contribute $\ll \log^2|\gamma|/|\gamma|$, and those with $\frac{1}{2}|\gamma| < |\gamma'| < 2|\gamma|$ contribute $\ll |\gamma|^{-1}\sum_{1\leq k\leq 2|\gamma|} k^{-1}\log|\gamma|$. For $|\gamma-\gamma'|<1$ there are $O(\log|\gamma|)$ zeros, each contributing at most $\delta_\rho^{-1}|\rho'|^{-1} \ll \delta_\rho^{-1}|\gamma|^{-1}$. Multiplying by $|\rho|^{-1}$ and summing over $\rho$,

$$H \ll \sum_\rho \frac{\log^2(|\gamma|+2)}{|\gamma|^2} + \sum_\rho \frac{\log(|\gamma|+2)}{|\gamma|^2\delta_\rho} < \infty.$$

Since the zeros are simple, $\gamma = \gamma'$ with $\rho \neq \rho'$ is impossible, so $\Phi$ has no constant term. □

Status: proved. Truncated to the zeros below height 600, 3000 and $10^4$, the sum is 0.136, 0.179 and 0.195 (`variance.py`); all its terms are positive, so these are lower bounds for $H$, and the tail decays slowly.

**Theorem 5** (second-order energy law)**.** *Assume (H*). Then, with $u = \log X$,*

$$I(X) = C_B\, u + K + \Phi(u) + O\big(e^{-u/2}\big),$$

*where $\Phi$ is given by (5) and*

$$K = \frac{1}{2}\Big(C_B^2 - \sum_\rho \frac{1}{|\rho|^4}\Big) - 2\sum_\rho \frac{R(1-\rho)}{\rho} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|R(\tfrac{1}{2}+it)\big|^2\,dt. \tag{6}$$

*Proof.* By (1), $I(e^U) = \int_0^U S^2 + 2\int_0^U Sr + \int_0^U r^2$.

*The trivial part.* Since $r \in L^2(0,\infty)$, Plancherel's theorem for the Laplace transform on $\operatorname{Re} s = \frac{1}{2}$ gives $\int_0^\infty r^2 = \frac{1}{2\pi}\int |R(\frac{1}{2}+it)|^2 dt$, and $\int_U^\infty r^2 \leq 4e^{-U}$ for $U \geq 1$, since $|r(u)| \leq 2e^{-u/2}$ there.

*The cross term.* By Lemma 3, $\int_0^U Sr = \int_0^\infty Sr + O(e^{-U/2})$ and $2\int_0^\infty Sr = -2\sum_\rho R(1-\rho)/\rho$.

*The zero part.* For a finite set $F$ of zeros closed under conjugation, $S_F$ is real and

$$\int_0^U S_F^2\,du = \int_0^U \Big|\sum_{\rho\in F}\frac{e^{i\gamma u}}{\rho}\Big|^2 du = U\sum_{\rho\in F}\frac{1}{|\rho|^2} + \sum_{\substack{\rho\neq\rho'\\ \rho,\rho'\in F}} \frac{e^{i(\gamma-\gamma')U}-1}{i(\gamma-\gamma')\,\rho\,\bar\rho'}.$$

Take $F = F_T$ and let $T \to \infty$ with $U$ fixed. The left side tends to $\int_0^U S^2$ by Lemma 1. On the right, $\sum_{F_T} |\rho|^{-2} \to C_B$, and by Proposition 4 the double sum converges absolutely, so it tends to $\Phi(U) - \sum_{\rho \neq \rho'} c_{\rho\rho'}$. In the constant part the terms $(\rho, \rho')$ and $(\rho', \rho)$ add up to

$$-\frac{1}{i(\gamma - \gamma')}\Big(\frac{1}{\rho\bar{\rho}'} - \frac{1}{\rho'\bar{\rho}}\Big) = -\frac{2\,\mathrm{Im}(1/\rho\bar{\rho}')}{\gamma - \gamma'} = \frac{1}{|\rho|^2|\rho'|^2},$$

since $\mathrm{Im}(1/\rho\bar{\rho}') = \mathrm{Im}(\bar{\rho}\rho')/(|\rho|^2|\rho'|^2) = (\gamma' - \gamma)/(2|\rho|^2|\rho'|^2)$ when $\beta = \beta' = \frac{1}{2}$. Hence the constant part is the sum of $|\rho|^{-2}|\rho'|^{-2}$ over unordered pairs of distinct zeros, which is $\frac{1}{2}\big((\sum_\rho |\rho|^{-2})^2 - \sum_\rho |\rho|^{-4}\big) = \frac{1}{2}(C_B^2 - \sum_\rho |\rho|^{-4})$. Adding the three parts proves the theorem. □

Status: under RH, simplicity of the zeros and the spacing condition of (H*); proved. The leading term is Cramér's [Cr]. Only the oscillating part $\Phi$ uses the spacing condition: the three terms of $K$ are defined, and Lemmas 1–3 hold, under RH alone (the zero–zero term is an absolutely convergent sum over unordered pairs).

In the language of [CE5], the constant $\frac{1}{2}(C_B^2 - \sum |\rho|^{-4})$ is the second elementary symmetric function of the limits $\lim_n C_2(n, \rho)$: it is what the off-diagonal entries of the energy matrix $M(U)$, decaying like $1/U$, leave after integration in time, and their oscillating remainder is $\Phi$.

*Remark* 6 (the Montgomery–Vaughan inequality). The generalized Hilbert inequality of Montgomery and Vaughan [MV74], $|\sum_{m\neq n} a_m\bar{a}_n/(\lambda_m - \lambda_n)| \le \frac{3\pi}{2}\sum_n |a_n|^2\delta_n^{-1}$, applied with $a_\rho = e^{i\gamma u}/\rho$, bounds every finite truncation of $\Phi$ uniformly: $|\Phi_T(u)| \le \frac{3\pi}{2}\sum_\rho |\rho|^{-2}\delta_\rho^{-1}$. This is how the second-order law was first stated, with the weaker hypothesis $\sum_\rho |\rho|^{-2}\delta_\rho^{-1} < \infty$ and a proof in outline. The inequality alone does not show that the truncations converge uniformly, because the cross terms between a fixed set of zeros and the zeros added at a larger height are not bounded by the tail of $\sum |\rho|^{-2}\delta_\rho^{-1}$. We therefore assume (H*), which gives absolute convergence; removing the logarithm is open problem 2 of Sect. 8.

3.1. **The value of $K$.** We first write the three terms of (6) as convergent sums, stating the conventions of convergence, and then remove the zeros altogether.

**Proposition 7** ($K$ from three sums)**.** *Let $G(a) = \sum_\rho \frac{1}{\rho(a-\rho)}$ for $a > 1$, an absolutely convergent series. Unconditionally,*

$$G(a) = \frac{1}{a}\Big(\frac{C_B}{2} + \frac{\xi'}{\xi}(a)\Big), \qquad \frac{\xi'}{\xi}(a) = \frac{1}{a} + \frac{1}{a-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{a}{2}\Big) + \frac{\zeta'}{\zeta}(a), \tag{7}$$

*and $G(1) := \sum_\rho \frac{1}{\rho(1-\rho)} = C_B$. Moreover*

$$-2\sum_\rho \frac{R(1-\rho)}{\rho} = 2C_B\log 2\pi - \sum_{k\ge 1}\frac{G(1+2k)}{k}, \tag{8}$$

*and, unconditionally,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \big|R(\tfrac{1}{2}+it)\big|^2 dt = \sum_{j,l\ge 0}\frac{c_j c_l}{1 - a_j - a_l} = \log^2 2\pi - \log 2\pi \sum_{k\ge 1}\frac{1}{k(2k+1)} + \sum_{k,l\ge 1}\frac{1}{4kl(2k+2l+1)}. \tag{9}$$

*Proof. Conventions.* The Hadamard product $\xi(s) = e^{A+Bs}\prod_\rho (1 - s/\rho)e^{s/\rho}$ of $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ gives

$$\frac{\xi'}{\xi}(s) = B + \sum_\rho\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big),$$

an absolutely convergent series, since its terms are $s/(\rho(s-\rho)) \ll_s |\gamma|^{-2}$; here $B = -\frac{1}{2}\gamma_E - 1 + \frac{1}{2}\log 4\pi = -\frac{C_B}{2}$ and $B = -\sum_\rho \mathrm{Re}(1/\rho)$ [Da, Ch. 12]. The series $\sum_\rho 1/\rho$ does not converge

absolutely, since $\sum_\rho |\rho|^{-1} = \infty$. In symmetric order it converges: the terms of $\rho$ and $\bar\rho$ add up to $2\operatorname{Re}(1/\rho) = 2\beta/|\rho|^2 \geq 0$, so $\sum_{|\gamma|<T} 1/\rho = \sum_{|\gamma|<T} \operatorname{Re}(1/\rho)$, which increases to the absolutely convergent sum $\sum_\rho \operatorname{Re}(1/\rho) = -B = \frac{C_B}{2}$. Consequently, in symmetric order,

$$\sum_\rho \frac{1}{a-\rho} = \lim_{T\to\infty} \sum_{|\gamma|<T} \Big[\Big(\frac{1}{a-\rho} + \frac{1}{\rho}\Big) - \frac{1}{\rho}\Big] = \Big(\frac{\xi'}{\xi}(a) - B\Big) - \frac{C_B}{2} = \frac{\xi'}{\xi}(a).$$

(7). Since $\frac{1}{\rho(a-\rho)} = \frac{1}{a}\big(\frac{1}{\rho} + \frac{1}{a-\rho}\big)$ and the left side is absolutely summable, its sum equals the limit of its symmetric partial sums, which is $\frac{1}{a}\big(\frac{C_B}{2} + \frac{\xi'}{\xi}(a)\big)$. The formula for $\xi'/\xi$ is the logarithmic derivative of the definition of $\xi$. For $a = 1$, $\frac{\xi'}{\xi}(1) = -\frac{\xi'}{\xi}(0) = -B$ by the functional equation, so $G(1) = \frac{C_B}{2} - B = C_B$.

(8). By (2), $R(s) = -\frac{\log 2\pi}{s} + \sum_{k\geq 1} \frac{1}{2k(s+2k)}$, so $R(1-\rho) = -\frac{\log 2\pi}{1-\rho} + \sum_k \frac{1}{2k(2k+1-\rho)}$. Dividing by $\rho$ and summing over $\rho$ is legitimate because the double series converges absolutely: for $a = 2k+1$ the zeros with $|\gamma| \leq a$ contribute $\ll a^{-1}\sum_{|\gamma|\leq a} |\rho|^{-1} \ll a^{-1}\log^2 a$ to $\sum_\rho |\rho|^{-1}|a-\rho|^{-1}$ and the others $\ll \sum_{|\gamma|>a} \gamma^{-2} \ll a^{-1}\log a$, and $\sum_k k^{-2}\log^2 k < \infty$. With $G(1) = C_B$ this gives (8).

(9). $\int_0^\infty r^2 = \sum_{j,l} c_j c_l \int_0^\infty e^{(a_j+a_l-1)u}du$ (the double series of $|c_jc_l|e^{(a_j+a_l-1)u}$ is integrable), and Plancherel gives the first form. □

Status: proved. The two series over $k$ converge only like $(\log k)/k$; truncated at $k \leq 4000, \dots, 64000$ they give $-0.009606, \dots, -0.010075$ for (8) and $2.521224, \dots, 2.521536$ for (9), and Richardson extrapolation gave $K = 2.512480$. The next proposition replaces these sums by a formula without zeros that converges like $\sum_n n^{-3}\log n$.

**Proposition 8** ($K$ without zeros)**.** *Assume RH. Let $\sigma_2 = \sum_\rho \rho^{-2}$ (an absolutely convergent series), $\varphi(y) = \sum_{k\geq 1} \frac{y^{2k+1}}{2k(2k+1)} = y - \operatorname{artanh} y - \frac{y}{2}\log(1-y^2)$ for $0 \leq y < 1$, and $P = \sum_{n\geq 2} \Lambda(n)\varphi(1/n)$. Then*

$$\sigma_2 = 1 + \gamma_E^2 + 2\gamma_1 - \frac{\pi^2}{8}, \qquad \frac{1}{2}\Big(C_B^2 - \sum_\rho \frac{1}{|\rho|^4}\Big) = \frac{C_B^2}{2} - C_B - \sigma_2, \tag{10}$$

$$\int_0^\infty r^2 = (\log 4\pi - 1)^2 + 1 - \frac{\pi^2}{12}, \tag{11}$$

$$2\int_0^\infty Sr = -2\sum_\rho \frac{R(1-\rho)}{\rho} = 2(1+\gamma_E)\log 2\pi - \frac{\pi^2}{12} + 2P - 2\int_0^\infty r^2, \tag{12}$$

*and consequently*

$$K = \frac{C_B^2}{2} - C_B - \sigma_2 + 2(1+\gamma_E)\log 2\pi - (\log 4\pi - 1)^2 - 1 + 2\sum_{n\geq 2} \Lambda(n)\varphi(1/n). \tag{13}$$

*Proof.* (10). Differentiating the Hadamard expansion of $\xi'/\xi$ gives $(\xi'/\xi)'(s) = -\sum_\rho (s-\rho)^{-2}$, so $\sigma_2 = -(\xi'/\xi)'(0) = -(\xi'/\xi)'(1)$, the second equality by $\xi(s) = \xi(1-s)$. From (7), $(\xi'/\xi)'(s) = -s^{-2} - (s-1)^{-2} + \frac{1}{4}(\Gamma'/\Gamma)'(\frac{s}{2}) + (\zeta'/\zeta)'(s)$, and the Laurent expansion $\zeta'/\zeta(s) = -\frac{1}{s-1} + \gamma_E - (\gamma_E^2 + 2\gamma_1)(s-1) + \cdots$ together with $(\Gamma'/\Gamma)'(\frac{1}{2}) = \pi^2/2$ gives $(\xi'/\xi)'(1) = -1 + \frac{\pi^2}{8} - \gamma_E^2 - 2\gamma_1$. Under RH $1-\rho = \bar\rho$, so $|\rho|^{-2} = \frac{1}{\rho(1-\rho)} = \frac{1}{\rho} + \frac{1}{1-\rho}$ and

$$\sum_\rho \frac{1}{|\rho|^4} = \sum_\rho \frac{1}{\rho^2} + \sum_\rho \frac{1}{(1-\rho)^2} + 2\sum_\rho \frac{1}{\rho(1-\rho)} = 2\sigma_2 + 2C_B,$$

using the bijection $\rho \mapsto 1-\rho$ of the zeros and $G(1) = C_B$; all three series converge absolutely.

(11). In (9), $\sum_k \frac{1}{k(2k+1)} = \sum_k \left(\frac{1}{k} - \frac{2}{2k+1}\right) = 2 - 2\log 2$, and since $\frac{1}{2k+2l+1} = \int_0^1 x^{2k+2l}dx$,

$$\sum_{k,l\ge 1} \frac{1}{kl(2k+2l+1)} = \int_0^1 \log^2(1-x^2)\,dx = 8 - 8\log 2 + 4\log^2 2 - \frac{\pi^2}{3},$$

from $\int_0^1 \log^2(1-x)dx = 2$, $\int_0^1 \log^2(1+x)dx = 2\log^2 2 - 4\log 2 + 2$ and $\int_0^1 \log(1-x)\log(1+x)dx = 2 - 2\log 2 + \log^2 2 - \frac{\pi^2}{6}$. Hence $\int_0^\infty r^2 = \log^2 2\pi - 2(1-\log 2)\log 2\pi + (1-\log 2)^2 + 1 - \frac{\pi^2}{12}$, which is (11).

(12). The first equality is Lemma 3. Since $S = E_0 - r$ and all the integrals converge absolutely, $\int_0^\infty Sr = \int_0^\infty E_0 r - \int_0^\infty r^2$. Expand $r = \sum_j c_j e^{(a_j - \frac12)u}$. For $j = k \ge 1$, $\int_0^\infty E_0(u)e^{-(2k+\frac12)u}du = \mathcal{M}(2k+1)$ by (3), and termwise integration is justified by $|E_0(u)| \le e^{u/2}$ and $\sum_k \frac{1}{2k}\int_0^\infty e^{-2ku}du < \infty$. For $j = 0$, $\int_0^\infty E_0(u)e^{-u/2}du = \int_1^\infty (\psi(x)-x)x^{-2}dx$ converges absolutely under RH, so by dominated convergence it equals $\lim_{s\to 1^+}\mathcal{M}(s) = \lim_{s\to 1^+}\left(-\frac{\zeta'}{s\zeta}(s) - \frac{1}{s-1}\right) = -1-\gamma_E$. Therefore

$$\int_0^\infty E_0 r = (1+\gamma_E)\log 2\pi + \sum_{k\ge1}\frac{1}{2k}\Big(\frac{-\zeta'/\zeta(2k+1)}{2k+1} - \frac{1}{2k}\Big) = (1+\gamma_E)\log 2\pi - \frac{\pi^2}{24} + \sum_{n\ge2}\Lambda(n)\sum_{k\ge1}\frac{n^{-2k-1}}{2k(2k+1)},$$

and the inner sum is $\varphi(1/n)$ (from $\sum_k \frac{y^{2k+1}}{2k} = -\frac{y}{2}\log(1-y^2)$ and $\sum_k \frac{y^{2k+1}}{2k+1} = \operatorname{artanh} y - y$). This is (12).

Finally, (6) is the sum of (10), (12) and (11), and $-\frac{\pi^2}{12}$ cancels. □

Status: under RH; proved. RH enters twice: in $|\rho|^2 = \rho(1-\rho)$, which turns $\sum|\rho|^{-4}$ into the Keiper constant $\sigma_2$, and in the absolute convergence of $\int_1^\infty(\psi(x)-x)x^{-2}dx$ used for the term $j=0$ (which converges also unconditionally, by the prime number theorem). The sum $P$ converges like $\sum_n n^{-3}\log n$, and $0 \le \varphi(y) \le y^3/(6(1-y^2))$.

**Corollary 9** (the value of $K$)**.** *Assume RH. Then*

$$K \in [2.512479790006 \pm 6\cdot 10^{-13}],$$

*with zero–zero term* $0.00102972290885806\ldots$*, cross term* $-0.010118144286\ldots$ *and trivial term* $2.52156821138377\ldots$*; moreover* $\sum_\rho |\rho|^{-4} = 7.42012728749\ldots\cdot 10^{-5}$ *and* $\sigma_2 = -0.0461543172958\ldots$.

*Proof.* Evaluate (13) in ball arithmetic (Arb [Jo], 128 bits; $\gamma_1$ from Arb's Stieltjes constants). $P$ is summed exactly over the prime powers $n \le N_0 = 10^6$, and the tail satisfies $0 \le \sum_{n>N_0}\Lambda(n)\varphi(1/n) \le \frac{1}{6(1-N_0^{-2})}\sum_{n>N_0}\Lambda(n)n^{-3} \le \frac{1.5583}{6(1-N_0^{-2})N_0^2}$, by partial summation and $\psi(x) < 1.03883x$ [RS62]. □

Status: computer-assisted, under RH (`K_closed_form_arb.py`, one minute). The value agrees with the truncations of Proposition 7: the trivial term 2.5215682 is the limit of the values $2.521224,\ldots,2.521536$ above, whose errors decrease like $1/k$, and similarly for the cross term $-0.0101181$; and $\sum_{|\gamma|<10^4}|\rho|^{-4}$ computed from the 10142 zeros below height $10^4$, plus its tail, agrees with $2\sigma_2 + 2C_B$ to about $5\cdot 10^{-13}$. It also agrees with $K_{\text{primes}} = 2.512484$ from the primes up to $10^9$ (Sect. 4.2) to $4\cdot 10^{-6}$, the size of the oscillation $\Phi$ left in that mean.

*Remark* 10 (what $K$ is made of). The constant is almost entirely the contribution of the trivial part of the explicit formula: the interval $1 \le x < 2$, where $\psi = 0$ and the integrand is 1, contributes exactly 1 to $I(X)$ for every $X \ge 2$, and the trivial term is $(\log 4\pi - 1)^2 + 1 - \pi^2/12$. The nontrivial zeros contribute only $0.0010297 - 0.0101181 = -0.0090884$, of which the zero–zero part is the second elementary symmetric function of the $C_2$ limits. In (13) the primes enter only through the rapidly convergent $P = 0.029014156649\ldots$, to which $n=2$ and $n=3$ contribute 0.0157 and 0.0070.

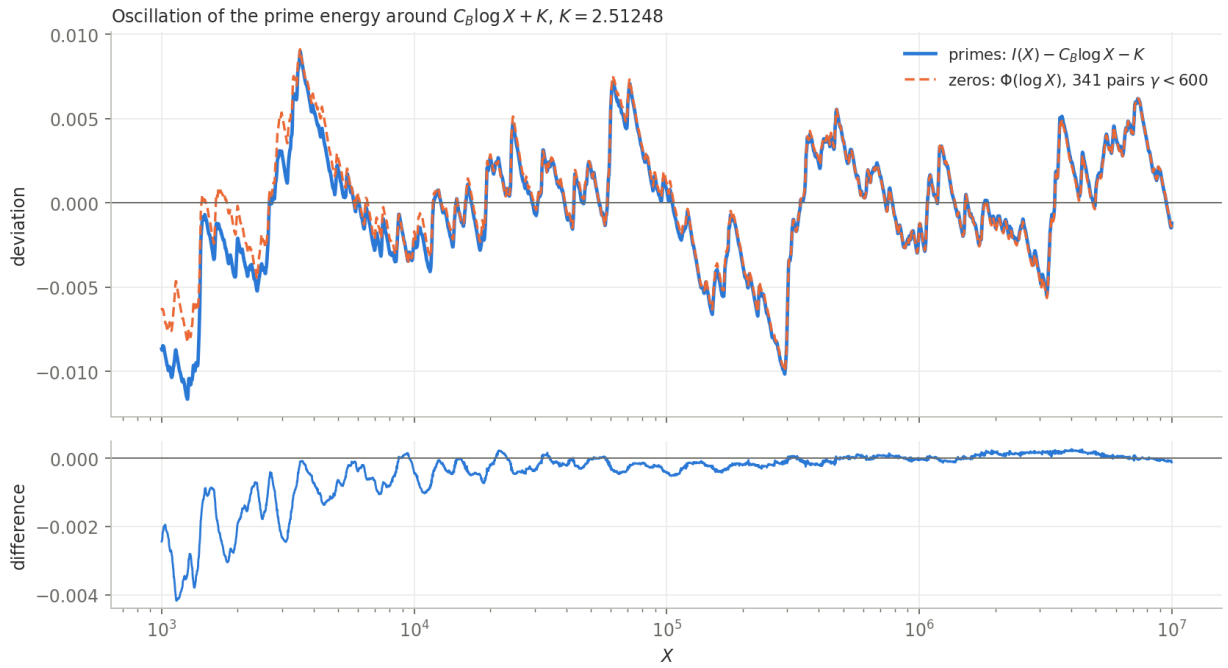


FIGURE 1. The oscillation of the prime energy. Top: $I(X) - C_B \log X - K$ from the primes up to $10^7$ (line) and $\Phi(\log X)$ from the zeros with $0 < \gamma < 600$ (dashed). Bottom: their difference.

From the primes, $I(X) - C_B \log X$ equals 2.5038, 2.5104, 2.5133, 2.5110, 2.5113, 2.5114, 2.5112 at $X = 10^3, 10^4, \ldots, 10^9$; the values for $X \le 10^6$ are those of [CE18]. The deviations from $K$ are the oscillation $\Phi$. On the decades from $[10^3, 10^4]$ to $[10^6, 10^7]$ the correlation of $I(X) - C_B \log X - K$ with $\Phi(\log X)$ computed from the 341 zeros with $0 < \gamma < 600$ is 0.979, 0.995, 0.999, 0.999, and the r.m.s. residual decreases from $1.7 \cdot 10^{-3}$ to $1.2 \cdot 10^{-4}$ (Figure 1). Status: numerical (`QX.py`, `Phi.py`, `figphi.py`; `primes_1e9.py`). Sect. 4.2 extends the comparison to $10^9$.

3.2. **On the strength of the spacing assumption.** Theorem 5 rests on RH, on the simplicity of the zeros, and on the spacing condition $\sum_\rho \log(|\gamma| + 2)|\rho|^{-2}\delta_\rho^{-1} < \infty$. The first two are needed already at first order: the slope is $C_B$ exactly when they hold [CE5, Sect. 2]. The third is an additional assumption. It does not follow from RH and simplicity, and nothing is known about it: there is no lower bound for the gaps $\delta_\rho$, even under RH, and the simplicity of all zeros is itself unknown. We ask three questions.

*(a) What would imply* (H*)*?* Any bound $\delta_\rho \gg |\gamma|^{-1+\varepsilon}$, together with RH and simplicity. Heuristically (H*) is expected: under the GUE model the density of normalized gaps vanishes quadratically at 0, so the expected value of $\delta_\rho^{-1}$ at height $\gamma$ is of order $\log \gamma$, the expected size of the sum in (H*) is of order $\sum_\rho \log^2 \gamma/\gamma^2 < \infty$, and the smallest gaps up to height $T$ are expected to be of size $T^{-1/3+o(1)}$. This is a heuristic, not a proof.

*(b) What weaker condition gives the constant $K$?* For ordered pairs of distinct zeros put

$$\nu(\varepsilon) = \sum_{\substack{\rho \ne \rho' \\ 0 < |\gamma - \gamma'| < \varepsilon}} \frac{1}{|\rho||\rho'|} \qquad (0 < \varepsilon \le 1).$$

By the Riemann–von Mangoldt formula $\nu(1) < \infty$ unconditionally.

**Proposition 11** (the constant as a Cesàro mean)**.** *Assume RH and that the zeros are simple. Then for $U \ge 1$*

$$\frac{1}{U}\int_0^U \mathcal{D}(u)\,du = K + \mathcal{E}(U) + O\Big(\frac{1}{U}\Big), \qquad \mathcal{E}(U) = \sum_{\{\rho,\rho'\}} \frac{2}{(\gamma-\gamma')^2 U}\,\mathrm{Re}\Big(\frac{1-e^{i(\gamma-\gamma')U}}{\rho\,\bar\rho'}\Big),$$

*where $K$ is the constant (6) and the sum runs over unordered pairs of distinct zeros and converges absolutely. Moreover $\mathcal{E}(U)\to 0$ if and only if $\nu(\varepsilon)=o(\varepsilon)$ as $\varepsilon\to 0$. Hence $\lim_{U\to\infty}\frac{1}{U}\int_0^U (I(e^u)-C_B u)\,du = K$ if and only if $\nu(\varepsilon)=o(\varepsilon)$.*

*Proof.* By (1) and Lemma 3, $\mathcal{D}(u)=\int_0^u S^2 - C_B u + (K-Z) + \epsilon(u)$ with $Z=\frac12(C_B^2-\sum|\rho|^{-4})$, $\epsilon(u) = -2\int_u^\infty Sr - \int_u^\infty r^2$, $|\epsilon(u)|\ll e^{-u/2}$ for $u\ge 1$ and $\epsilon$ bounded on $[0,1]$; so the mean of $\epsilon$ over $[0,U]$ is $O(1/U)$. Next, $\frac1U\int_0^U\int_0^u S^2\,dv\,du = \int_0^U(1-\frac{v}{U})S(v)^2dv$. For a finite set $F$ closed under conjugation, with $\lambda=\gamma-\gamma'$ and $\int_0^U(1-\frac vU)e^{i\lambda v}dv = -\frac{1}{i\lambda}+\frac{1-e^{i\lambda U}}{\lambda^2 U}$ for $\lambda\neq 0$,

$$\int_0^U \Big(1-\frac{v}{U}\Big)S_F^2\,dv = \frac{U}{2}\sum_{\rho\in F}\frac{1}{|\rho|^2} + Z_F + \sum_{\substack{\rho\neq\rho'\\ \rho,\rho'\in F}}\frac{1-e^{i\lambda U}}{\lambda^2 U\,\rho\bar\rho'},$$

where $Z_F$ is the sum of $|\rho|^{-2}|\rho'|^{-2}$ over unordered pairs in $F$, by the computation in the proof of Theorem 5. Combining the ordered pairs $(\rho,\rho')$ and $(\rho',\rho)$, the last sum is the sum over unordered pairs of $\frac{2}{\lambda^2 U}\,\mathrm{Re}((1-e^{i\lambda U})A)$, $A=1/(\rho\bar\rho')$. Under RH, $\mathrm{Re}\,A=(\frac14+\gamma\gamma')/(|\rho|^2|\rho'|^2)$, $|A|=1/(|\rho||\rho'|)$ and $\mathrm{Im}\,A=-\lambda/(2|\rho|^2|\rho'|^2)$, so

$$\frac{2}{\lambda^2 U}\big|\,\mathrm{Re}((1-e^{i\lambda U})A)\big| \le \frac{2(1-\cos\lambda U)}{\lambda^2 U}\,|\,\mathrm{Re}\,A| + \frac{2|\sin\lambda U|}{\lambda^2 U}\,|\,\mathrm{Im}\,A| \le \frac{\min(4/(\lambda^2U),\,U)}{|\rho||\rho'|} + \frac{\min(1,\,1/(|\lambda|U))}{|\rho|^2|\rho'|^2}.$$

For fixed $U$ the right side is summable over all pairs: the pairs with $|\lambda|<1$ give at most $U\nu(1)$, the others $\frac4U\sum_{|\lambda|\ge1}\lambda^{-2}|\rho\rho'|^{-1}<\infty$, and $\sum|\rho|^{-2}|\rho'|^{-2}=C_B^2$. Taking $F=F_T$, $T\to\infty$, Lemma 1 and dominated convergence give the formula for $\frac1U\int_0^U\mathcal{D}$, since $\frac{U}{2}C_B$ cancels the mean of $C_Bu$.

Now let $U\to\infty$. The terms $\frac{2\sin(\lambda U)}{\lambda^2 U}\,\mathrm{Im}\,A$ are dominated by $|\rho|^{-2}|\rho'|^{-2}$ and tend to 0 for each pair, and the terms of the pairs with $|\lambda|\ge1$ are $O(1/U)$ in total. The remaining part is $\mathcal{E}_0(U)=\sum_{|\lambda|<1}\frac{2(1-\cos\lambda U)}{\lambda^2 U}\,\mathrm{Re}\,A$, whose terms are nonnegative, because $|\gamma-\gamma'|<1$ and $|\gamma|,|\gamma'|>14$ force $\gamma\gamma'>0$; in fact $\mathrm{Re}\,A\ge\frac{0.99}{|\rho||\rho'|}$ for these pairs. Since the sum over unordered pairs with $|\lambda|<\varepsilon$ of $1/(|\rho||\rho'|)$ is $\frac12\nu(\varepsilon)$, if $\nu(\varepsilon)=o(\varepsilon)$ then

$$\mathcal{E}_0(U)\le\sum_{|\lambda|<1}\frac{\min(4/(\lambda^2U),U)}{|\rho||\rho'|}\le\frac12\Big(U\nu(1/U)+\frac4U\int_{[1/U,1)}\frac{d\nu(\lambda)}{\lambda^2}\Big),$$

and integrating by parts, $\frac4U\int_{[1/U,1)}\lambda^{-2}d\nu\le\frac{4\nu(1)}{U}+\frac8U\int_{1/U}^1\nu(\lambda)\lambda^{-3}d\lambda$, which is $o(1)$ by splitting the last integral at a fixed $\lambda_0$ with $\nu(\lambda)\le\eta\lambda$ for $\lambda<\lambda_0$. Conversely, $1-\cos x\ge\frac{11}{24}x^2$ for $|x|\le1$ gives $\mathcal{E}_0(U)\ge\frac12\cdot\frac{11}{12}\cdot0.99\,U\nu(1/U)$, so $\mathcal{E}_0(U)\to0$ forces $\nu(\varepsilon)=o(\varepsilon)$. □

Status: under RH and simple zeros, without the spacing condition of (H*); proved. The condition $\nu(\varepsilon)=o(\varepsilon)$ is implied by (H*), since $\nu(\varepsilon)\le\varepsilon\sum_{0<|\lambda|<\varepsilon}|c_{\rho\rho'}|$ and the right side is $\varepsilon$ times a tail of the convergent series $H$; it is much weaker, being a weak-type condition on the distribution of $\delta_\rho^{-1}$ instead of a summability condition, without the logarithm. Under the GUE model one expects $\nu(\varepsilon)\asymp\varepsilon^3$. The proof also explains why some condition of this kind is needed: by (5) two zeros at distance $\lambda$ contribute $\frac{2\sin\lambda u}{\lambda|\rho|^2}+O(|\rho|^{-4})$ to $\Phi(u)$, which for $u\ll1/\lambda$ is the linear term $2u/|\rho|^2$ that a double zero would add to the slope. A positive proportion of such near-coincidences on every scale would shift the mean of $\mathcal{D}$ by an amount that does not tend to 0.

*(c) Is absolute convergence of $\Phi$ necessary for the distribution?* No: mean-square approximation suffices. Write $\Phi_T$ for the sum (5) restricted to $\rho, \rho' \in F_T$, and consider

$$\lim_{T\to\infty} \limsup_{U\to\infty} \frac{1}{U} \int_0^U \left|\mathcal{D}(u) - K - \Phi_T(u)\right|^2 du = 0. \tag{Q}$$

**Proposition 12** (the limiting distribution from mean-square approximation)**.** *Assume RH, simple zeros and (Q). Then $\mathcal{D}$ is $B^2$-almost periodic and has a limiting distribution $\mu$ with mean $K$ and variance $\lim_{T\to\infty}\lim_{U\to\infty}\frac{1}{U}\int_0^U|\Phi_T|^2$. Moreover (H*) implies (Q), and (Q) implies $\nu(\varepsilon) = o(\varepsilon)$.*

*Proof.* Each $K + \Phi_T$ is a trigonometric polynomial, so (Q) says that $\mathcal{D}$ is $B^2$-almost periodic, and [ANS, Thm. 2.9] gives the limiting distribution. Since $\Phi_T$ has mean 0 (its frequencies are nonzero), Cauchy–Schwarz and (Q) give $\frac{1}{U}\int_0^U \mathcal{D} \to K$, and (Q) also bounds $\frac{1}{U}\int_0^U \mathcal{D}^2$; so the time averages of $\mathcal{D}$ are uniformly integrable and $\int x\, d\mu = K$, For the variance, couple the time averages of $\mathcal{D} - K$ and of $\Phi_T$ through the same $u$: the Wasserstein distance $W_2$ between their limiting laws is at most $\limsup_U(\frac{1}{U}\int_0^U|\mathcal{D} - K - \Phi_T|^2)^{1/2}$, which tends to 0 by (Q); the law of $\Phi_T$ has second moment $\|\Phi_T\|_{B^2}^2$ (Parseval for a trigonometric polynomial), $W_2$-convergence preserves second moments, and $\|\Phi_T\|_{B^2} \to \|\mathcal{D} - K\|_{B^2}$ under (Q). Under (H*), Theorem 5 and Proposition 4 give $\sup_u|\mathcal{D} - K - \Phi_T| \le \sum_{\rho\neq\rho',\ \{\rho,\rho'\}\not\subset F_T}|c_{\rho\rho'}| + o(1)$, which tends to 0. Finally (Q) gives $\frac{1}{U}\int_0^U \mathcal{D} \to K$, hence $\nu(\varepsilon) = o(\varepsilon)$ by Proposition 11. □

Status: under RH, simple zeros and (Q); proved, from the standard theory of $B^2$-almost periodic functions. The general criterion [ANS, Thm. 1.2] does not apply directly: it asks for bounds on $\sum|c|$ over unit intervals of frequencies, while the difficulty here is the accumulation of the frequencies $\gamma - \gamma'$ at 0, with large coefficients.

What does (Q) require? First, an exact identity: for every $T$, by the finite-set identity in the proof of Theorem 5 and the decomposition used for Proposition 11,

$$\mathcal{D}(u) - K - \Phi_T(u) = \int_0^u \big(S^2 - S_T^2\big)\, dv - u\big(C_B - C_T\big) + \big(Z_T - Z\big) + O\big(e^{-u/2}\big), \tag{14}$$

with $C_T = \sum_{F_T}|\rho|^{-2}$; so (Q) is a statement about the centred energy of the zeros above height $T$ and their interaction with the zeros below. Second, under LI the frequencies of $\Phi_T$ are distinct up to the pairing of Theorem 14, so $\frac{1}{U}\int_0^U|\Phi_{T_2} - \Phi_{T_1}|^2$ tends to the part of the double sum (15) below over pairs with at least one zero in $F_{T_2}\setminus F_{T_1}$. Hence, under LI, (Q) forces the quadratic spacing condition

$$\sum_{\rho\neq\rho'} \frac{1}{(\gamma-\gamma')^2|\rho|^2|\rho'|^2} < \infty, \tag{H$_2$}$$

and conversely (H$_2$) with LI makes $(\Phi_T)$ a Cauchy sequence in $B^2$. Condition (H$_2$) is implied by (H*) (its terms are the squares of the $|c_{\rho\rho'}|$, which are bounded) and by itself allows $\delta_\rho$ as small as $|\gamma|^{-3/2+\varepsilon}$, against $|\gamma|^{-1+\varepsilon}$ for (H*). But by Proposition 12, (Q) also forces $\nu(\varepsilon) = o(\varepsilon)$, and the two conditions are independent: if every zero had a neighbour at distance $|\gamma|^{-a}$ with $1 \le a < \frac{3}{2}$, (H$_2$) would hold while $\nu(\varepsilon) \asymp \varepsilon^{1/a}\log(1/\varepsilon)$ would not be $o(\varepsilon)$, so the Cesàro mean of $\mathcal{D}$ would not be $K$ and (Q) would fail. In such regular configurations the pair of conditions (H$_2$) and $\nu(\varepsilon) = o(\varepsilon)$ holds exactly when $a < 1$, the same threshold as (H*); the weaker conditions gain only for irregular distributions of small gaps (a weak-type instead of a summability condition, without the logarithm). What is missing, beyond these necessary conditions, is the passage from the $B^2$ limit of $\Phi_T$ to $\mathcal{D} - K$, that is, the mean-square smallness of the right side of (14).

*Problem* 13 (main open problem). Assume RH, simple zeros, ($H_2$) and $\nu(\varepsilon) = o(\varepsilon)$ (and, if needed, LI); under LI both are necessary for (Q). Prove (Q). Separately, prove

$$\lim_{T_1,T_2\to\infty} \limsup_{U\to\infty} \frac{1}{U}\int_0^U |\Phi_{T_1}(u) - \Phi_{T_2}(u)|^2\,du = 0$$

under ($H_2$) without LI (with LI it holds), which requires control of the coincidences among the differences $\gamma - \gamma'$. By Proposition 12, (Q) would give the limiting distribution of $I(X) - C_B \log X$ without ($H^*$).

## 4. The limiting distribution of the energy fluctuations

Theorem 5 says that $\mathcal{D} = K + \Phi + o(1)$. Collect the coefficients of $\Phi$ by frequency:

$$\Phi(u) = \sum_{\lambda\neq 0} a_\lambda e^{i\lambda u}, \qquad a_\lambda = \sum_{\gamma-\gamma'=\lambda} c_{\rho\rho'}.$$

**Theorem 14.** *Assume* ($H^*$).

(1) *$\mathcal{D}$ has a limiting distribution $\mu$: for every bounded continuous $f$,*

$$\lim_{U\to\infty} \frac{1}{U}\int_0^U f\big(\mathcal{D}(u)\big)\,du = \int f\,d\mu.$$

(2) *$\int x\,d\mu = K$, $\int (x-K)^2\,d\mu = V := \sum_\lambda |a_\lambda|^2$, and $\operatorname{supp}\mu \subset [K-H, K+H]$.*

(3) *If in addition the positive ordinates are linearly independent over $\mathbb{Q}$* (LI)*, then*

$$V = 2\sum_{\substack{\rho\neq\rho'\\ \rho'\neq\bar\rho}} \frac{1}{(\gamma-\gamma')^2|\rho|^2|\rho'|^2} + \sum_\rho \frac{1}{4\gamma^2|\rho|^4}, \tag{15}$$

*and $\mu$ is the law of $K + \sum_{\rho\neq\rho'} c_{\rho\rho'} e^{i(\theta_\gamma - \theta_{\gamma'})}$, where the $\theta_\gamma$, $\gamma > 0$, are independent and uniform on $[0, 2\pi)$, and $\theta_{-\gamma} = -\theta_\gamma$.*

*Proof. Part 1.* The $o(1)$ term does not affect limiting distributions. By Proposition 4, $\Phi$ is a uniform limit of trigonometric polynomials, and every Bohr almost periodic function has a limiting distribution (Wintner [Wi35], Jessen–Wintner; see [ANS] for the form used in prime number theory).

*Part 2.* $\Phi$ has no constant term (Proposition 4), which gives the mean. The variance is Parseval's identity for Bohr almost periodic functions; $\Phi$ is real. The bound $|\Phi| \le H$ gives the support.

*Part 3.* Each frequency is $\lambda = \gamma - \gamma'$, with both ordinates possibly negative. Under LI, $\gamma_1 - \gamma_2 = \gamma_3 - \gamma_4$ with $\gamma_1 \neq \gamma_2$ forces $(\gamma_3, \gamma_4) = (\gamma_1, \gamma_2)$ or $(\gamma_3, \gamma_4) = (-\gamma_2, -\gamma_1)$; in terms of zeros, $(\rho_3, \rho_4) = (\bar\rho_2, \bar\rho_1)$, and $c_{\bar\rho_2\bar\rho_1} = c_{\rho_1\rho_2}$. The two ordered pairs coincide exactly when $\rho_2 = \bar\rho_1$, that is $\lambda = 2\gamma_1$. Hence $a_\lambda$ is $2c_{\rho\rho'}$ for a class of two ordered pairs and $c_{\rho\bar\rho}$ for $\lambda = 2\gamma$. Parseval gives $\sum_\lambda |a_\lambda|^2$: each class of two ordered pairs contributes $4|c_{\rho\rho'}|^2$, that is $2|c_{\rho\rho'}|^2$ for each of its ordered pairs, and each $\lambda = 2\gamma$ contributes $|c_{\rho\bar\rho}|^2 = 1/(4\gamma^2|\rho|^4)$; this is (15). By the Kronecker–Weyl theorem, under LI the vector $(\gamma_1 u, \dots, \gamma_m u)$ is equidistributed modulo $2\pi$ for every $m$, so every truncation of $\Phi$ has the stated law, and uniform convergence passes it to the limit. □

Status: conditional on ($H^*$), and on LI for part 3; proved. The existence of the limiting distribution follows from the standard framework; what is new is the object, the identification of the mean with $K$, and the explicit variance. Without LI, (15) is replaced by $V = \sum_\lambda |a_\lambda|^2$, which also counts coincidences of frequencies. Without ($H^*$), see Proposition 12.

*The variance as a weighted Rodgers–Tao energy.* Under RH the weights in (15) are the limits of the cross-energy at the zeros, $w_\rho = \lim_n C_2(n, \rho) = |\rho|^{-2}$ (Theorem A of [CE1]). Since $\gamma - \gamma' = 2\gamma$

when $\rho' = \bar\rho$, (15) reads

$$V = \sum_{\rho \neq \rho'} \kappa_{\rho\rho'} \, \frac{w_\rho \, w_{\rho'}}{(\gamma - \gamma')^2}, \qquad \kappa_{\rho\rho'} = 2 \ \ (\rho' \neq \bar\rho), \qquad \kappa_{\rho\bar\rho} = 1.$$

In the variable $x = 2\gamma$ of the zeros of $H_0(x) = \frac{1}{8}\xi(\frac{1}{2} + \frac{ix}{2})$ this is $4\sum \kappa_{\rho\rho'} w_\rho w_{\rho'} (x - x')^{-2}$, a weighted form of the energy $E(t) = \sum_{j \neq k} |x_j(t) - x_k(t)|^{-2}$ that Rodgers and Tao use for the zeros of the de Bruijn–Newman deformation $H_t$ [RT]. These zeros move by $\partial_t x_k = 2\sum_{j \neq k} (x_k - x_j)^{-1}$, and $\partial_t \mathcal{H} = -4E$ for $\mathcal{H} = \sum_{j \neq k} \log |x_j - x_k|^{-1}$. Over all the zeros $E$ diverges, and [RT] work with spatial cutoffs and a renormalization. With the limits of $C_2$ as weights, the sum converges exactly under ($H_2$), which is unproved, and the weighted sum is then the variance of the energy of the primes. A real zero can leave the real axis under the backward flow only by colliding with another one, and at a collision both $E$ and its weighted form blow up; the heat flow in the language of the cross-energy is in [CE8, Sect. 6]. Status: the identity is a rewriting of (15) (under RH and LI); the reading is ours, and we draw no consequence from it.

*Remark* 15 (a distributional signature of RH). If a zero had $\beta > \frac{1}{2}$, then $\mathcal{D}(u_n) \geq c e^{(2\beta-1)u_n}$ along a sequence $u_n \to \infty$, by the growth-rate theorem of [CE5, Sect. 2]; since $I$ is nondecreasing, $\mathcal{D}(u) \geq c e^{(2\beta-1)u_n} - 2C_B u_n$ on $[u_n, 2u_n]$, a set of relative density at least $\frac{1}{2}$ in $[0, 2u_n]$, so the time averages are not tight and no limiting distribution exists. Under RH and ($H^*$), Theorem 14 shows that $\mu$ exists and has bounded support. This is what RH looks like at second order; it is a consequence of RH, not a criterion that can be verified in finite range.

4.1. **Numerical results.** *Zeros.* We use the 1517 zeros below height 2000 to twelve decimals and compute those up to height $10^4$ with the Riemann–Siegel formula, including the first two correction terms, by bisection on sign changes of $Z(t)$ on a grid of step 0.01; the count is $10142 = N(10^4)$, and the Riemann–Siegel values agree with the tabulated ones below 2000 to $1.2 \cdot 10^{-4}$ (to $8 \cdot 10^{-7}$ above height 1500). Status: numerical (`rs_zeros.py`).

*A finite check of the factors in* (15). The pairing in part 3 is where factors 2 or 4 could be lost, so we checked (15) against a direct computation that does not use it. Take the first $m$ pairs of zeros $\rho_j, \bar\rho_j$, the real signal $S_F(u) = -\sum_{\rho \in F} e^{i\gamma u}/\rho$, and compute $\Phi_F(u) = \int_0^u S_F^2 - u\sum_F |\rho|^{-2} - K_0$, with $K_0$ the sum of $|\rho|^{-2}|\rho'|^{-2}$ over unordered pairs in $F$, in two ways: by quadrature of $S_F^2$, and from the exact trigonometric expansion $\sum_{\rho \neq \rho'} c_{\rho\rho'} e^{i(\gamma - \gamma')u}$ without any grouping of frequencies. The time averages $\frac{1}{U}\int_0^U \Phi_F^2$, $U = 4 \cdot 10^4$, are compared with (15) restricted to $F$:

| $m$ | 1 | 2 | 3 | 5 |
|---|---|---|---|---|
| $10^6\, V$ from the expansion | 0.0625403 | 2.048178 | 4.462907 | 7.053590 |
| $10^6\, V$ by quadrature | 0.0625070 | 2.048017 | 4.462619 | 7.053100 |
| $10^6\, V_F$ from (15) | 0.0625402 | 2.048178 | 4.462900 | 7.053447 |
| relative difference, expansion | $4 \cdot 10^{-7}$ | $1 \cdot 10^{-7}$ | $2 \cdot 10^{-6}$ | $2 \cdot 10^{-5}$ |

The differences for the expansion are of the size $O(1/(\lambda_{\min} U))$ of the time average, $\lambda_{\min}$ the smallest frequency; those of the quadrature ($-5 \cdot 10^{-4}$ to $-5 \cdot 10^{-5}$) are the discretization error of the cumulative trapezoid rule. The mean of $\Phi_F$ is below $2 \cdot 10^{-8}$, which confirms that $K_0$ is the constant term. A wrong factor in either sum of (15) would give a relative error of order 1: for $m = 1$ only the diagonal term $1/(4\gamma^2|\rho|^4)$ is present, and for $m \geq 2$ the pair terms carry 97–99% of $V_F$. Status: numerical (`variance_finite_check.py`); the ordinates used are linearly independent as far as is known.

*Variance.* Formula (15), truncated to the zeros with $|\gamma|, |\gamma'| < T$, gives:

| $T$ | 100 | 300 | 600 | 1000 | 2000 | 3000 | 10000 |
|---|---|---|---|---|---|---|---|
| $10^5 V_T$ | 1.16763 | 1.24271 | 1.24865 | 1.24972 | 1.25010 | 1.25015 | 1.25018 |

Under LI all terms of (15) are positive, so $V \geq V_{10^4} = 1.25018 \cdot 10^{-5}$, a numerical lower bound (double precision, with zeros that are not certified); without LI, $\sum_\lambda |a_\lambda|^2$ could be smaller. The tail $V - V_{10^4}$ splits into pairs at distance $|\gamma - \gamma'| \geq 1$ and close pairs. Under RH the first part is at most $3.5 \cdot 10^{-9}$, rigorously, by Trudgian's bound $|N(t) - \frac{t}{2\pi}\log\frac{t}{2\pi e} - \frac{7}{8}| \leq 0.112\log t + 0.278\log\log t + 2.510 + 0.2/t$ [Tr14]. The close pairs cannot be bounded without a lower bound for the gaps: if $\delta_\rho \geq |\gamma|^{-0.4}$ for $|\gamma| > 10^4$ they contribute at most $5.7 \cdot 10^{-7}$ (at most $1.6 \cdot 10^{-7}$ if $\delta_\rho \geq |\gamma|^{-1/3}$, an assumption that is borderline, since under the GUE model the smallest gaps up to height $T$ are expected to be of size $T^{-1/3}(\log T)^{-4/3}$), while their expected size under Montgomery's pair correlation is $5 \cdot 10^{-12}$. Thus

$$V = 1.2502 \cdot 10^{-5}, \qquad \sqrt{V} = 0.0035358,$$

under LI, where the lower bound is a computation, the upper bound $V \leq 1.31 \cdot 10^{-5}$ holds under the gap assumption $\delta_\rho \geq |\gamma|^{-0.4}$, and the last digits rest on the pair-correlation estimate. The variance is carried by the lowest zeros: the terms of (15) involving only the first 1, 2, 3, 5 and 10 zeros give 0.5%, 16.4%, 35.7%, 56.4% and 76.7% of it, and the 138 zeros below height 300 give 99.4%. Status: numerical (`variance.py`; tail bounds `tail_bounds.py`).

*Shape.* We sampled the LI model with $3 \cdot 10^5$ draws of random phases on the 138 zeros below height 300, and $\Phi$ along the actual zeros at $3 \cdot 10^5$ random $u \in [10^2, 10^6]$. Both give standard deviation 0.00353, skewness of absolute value below 0.004, excess kurtosis $-0.10$ and $-0.09$, and $P(\Phi > 0) = 0.50$; their Kolmogorov–Smirnov distance is 0.0015, at the level of sampling noise. The limiting distribution is slightly platykurtic, very close to Gaussian, and supported in a narrow band around $K$ (Figure 2b). Status: numerical (`variance.py`).

4.2. **The primes up to $10^9$.** We computed $I(X)$ exactly from a segmented sieve of the prime powers up to $10^9$, integrating the step function in extended precision (with a series expansion of the integrand when $\psi$ is close to $x$), at $4 \cdot 10^4$ points $X \in [10^3, 10^9]$ equally spaced in $\log X$, and compared $\mathcal{D}(\log X) - K$ with $\Phi(\log X)$ truncated to the 2469 zeros below height 3000:

| range of $X$ | $[10^3, 10^5]$ | $[10^5, 10^7]$ | $[10^7, 10^8]$ | $[10^8, 10^9]$ |
|---|---|---|---|---|
| r.m.s. of $\mathcal{D} - K$ | $4.0 \cdot 10^{-3}$ | $3.4 \cdot 10^{-3}$ | $5.0 \cdot 10^{-3}$ | $2.9 \cdot 10^{-3}$ |
| r.m.s. of $\mathcal{D} - K - \Phi$ | $1.3 \cdot 10^{-3}$ | $6.1 \cdot 10^{-5}$ | $1.2 \cdot 10^{-5}$ | $1.3 \cdot 10^{-5}$ |

With only the zeros below height 600, the residual stalls at $1.4$–$1.6 \cdot 10^{-4}$ for $X \geq 10^5$, which is of the size $\sqrt{V - V_{600}} = 1.2 \cdot 10^{-4}$ predicted for the truncation: the residual is dominated by the zeros left out, not by the $O(X^{-1/2})$ term. The mean of $\mathcal{D} - \Phi$ over $[10^7, 10^9]$ gives the empirical value

$$K_{\text{primes}} = 2.512484, \qquad \text{against} \qquad K = 2.5124798 \text{ (Corollary 9)}.$$

The two values agree to $4 \cdot 10^{-6}$. Status: numerical (`primes_1e9.py`; Figure 2a from `fig_fluct.py`).

## 5. Rigidity: comparison with random models

Pointwise, $E_0(u)$ fluctuates by $O(1)$; by Wintner's theorem it has, under RH, a limiting distribution, with variance $C_B$ when the zeros are simple [Wi35]. The integrated energy is far more rigid: its fluctuations have standard deviation 0.0035 around the line $C_B \log X + K$, because the coefficients $c_{\rho\rho'}$ carry the factor $1/(\gamma - \gamma')$, which damps all high frequencies. The random models examined below do not share this rigidity.

We replaced the primes by Cramér's model [Cr36] (each $n \geq 3$ is "prime" with probability $1/\log n$, with weight $\log n$), and the Möbius function by independent signs on the squarefree integers (Denjoy [Dj]) and by a Rademacher random multiplicative function, with independent signs at the primes and supported on the squarefree integers (Wintner [Wi]; see [Hp]), with 100 realizations of each up to $10^7$. The logarithmic mean of the energy on $[100, X]$, that is $(\log X - \log 100)^{-1} \int_{100}^X F(x)^2 x^{-2}\, dx$ with $F = \psi(x) - x$ or $F = M(x) + 2$, is at $X = 10^7$:

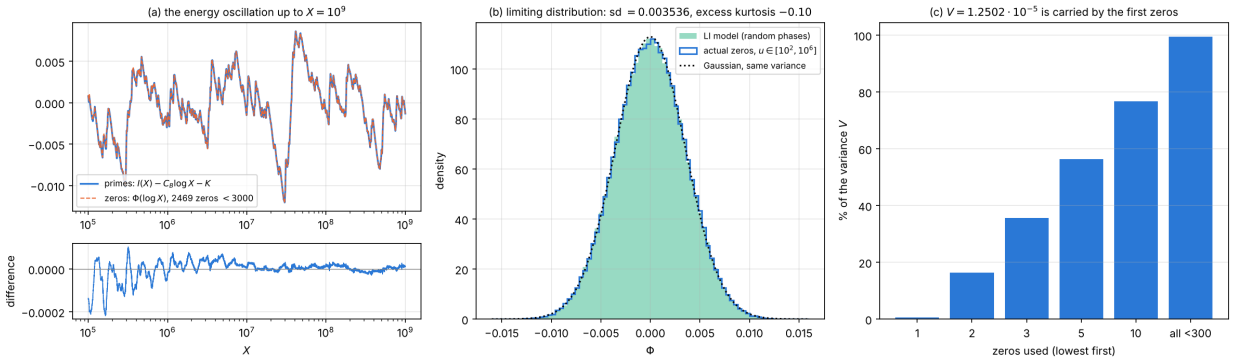


FIGURE 2. (a) $I(X) - C_B \log X - K$ from the primes up to $10^9$ against $\Phi(\log X)$ from the zeros below 3000, with their difference below. (b) Limiting distribution: LI model, actual zeros, and a Gaussian with the same variance. (c) Share of $V$ carried by the lowest zeros.

| | arithmetic | median | smallest of 100 | ratio median/arithmetic |
|---|---|---|---|---|
| $\psi(x) - x$: primes; Cramér's model | 0.0488 | 7.24 | 2.23 | 148 |
| $M(x) + 2$: $\mu$; independent signs | 0.0289 | 0.570 | 0.175 | 20 |
| $M(x) + 2$: $\mu$; random multiplicative | 0.0289 | 0.242 | 0.050 | 8.4 |

The mean for independent signs is $0.627 \approx 6/\pi^2$, as the variance $\sum_{n \le x} \mu(n)^2 \sim 6x/\pi^2$ predicts; the mean for the random multiplicative function, 1.27, is dominated by a few realizations with very large energy, as expected from the heavy tails of multiplicative sums [CE7]. No realization reaches the arithmetic value (Figure 3). For Cramér's model the energy grows like $\frac{1}{2}(u + \log 100) - 2$, as its variance $\sum_{n \le x} (\log n - 1) \approx x \log x - 2x$ predicts, although the model satisfies $\psi(x) - x = O(x^{1/2+\varepsilon})$ almost surely. Square-root cancellation is therefore weaker than the energy law [CE5, Thm. B]: bounded mean energy requires the discrete spectrum supplied by the zeros, while a model with independent noise has a continuous one. In terms of ratios, in these experiments the energy of the primes is 46 to 150 times smaller than in Cramér's model (smallest realization to median), and that of $\mu$ about 20 times smaller than for independent signs and typically 8 times smaller than for a random multiplicative function, whose smallest realization comes within a factor 1.7. Status: numerical (`mc.py`, seed 20261008; `fig.py`).

*Local structure.* Cramér's model ignores divisibility by small primes. Granville's modification [Gr95] keeps the primes $p \le z$ and declares each $n > z$ coprime to all of them "prime" with probability $\min\big(1, (\log n \prod_{p \le z} (1 - p^{-1}))^{-1}\big)$, again with weight $\log n$. With 100 realizations up to $10^7$ for each $z$, the medians of the logarithmic mean of the energy on $[100, 10^7]$ are

| $z$ | 1 (Cramér) | 2 | 3 | 5 | 7 | 13 | 31 | primes |
|---|---|---|---|---|---|---|---|---|
| median | 7.00 | 6.62 | 5.04 | 4.65 | 4.24 | 4.33 | 4.44 | 0.0488 |
| smallest | 2.21 | 2.60 | 1.31 | 1.07 | 1.69 | 1.38 | 1.68 | |

Local structure lowers the energy by a factor of at most about 1.7. The primes stay 87 to 136 times below the sieved medians, and 22 times below the smallest of all 700 realizations. (For $z \ge 101$ the probability exceeds 1 for small $n$ and the model loses its mean.) Sieving does not explain the rigidity: in every model with independent noise that we tried the energy grows like $u$. Status: numerical (`granville.py`, seed 20261019). An earlier run in the author's working notes, with 20 realizations, gave medians from 7.26 ($z = 1$) down to 3.54 ($z = 13$), with the same conclusion.

*Residual after subtracting the known zeros.* The mean energy of $E(u)$ is $\sum_\rho |\rho|^{-2} = C_B$ under RH (Theorem B of [CE5]). Subtracting the zeros up to height $T$ should therefore leave the energy

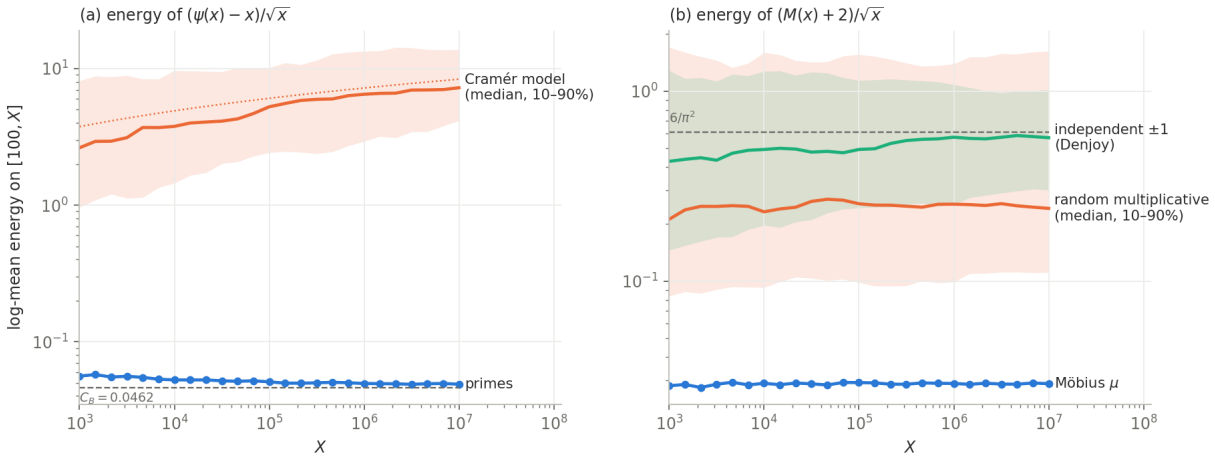


FIGURE 3. Logarithmic mean of the energy on $[100, X]$. (a) $(\psi(x)-x)/\sqrt{x}$ for the primes and for Cramér's model (median and 10–90% band of 100 realizations; dotted: $\frac{1}{2}(u+\log 100)-2$). (b) $(M(x)+2)/\sqrt{x}$ for $\mu$, for independent signs and for a random multiplicative function on the squarefree integers.

of the zeros above $T$. Let $R_T(u) = E(u) + \sum_{|\gamma|<T} e^{i\gamma u}/\rho$, with $E$ computed from the prime powers up to $10^7$ at $4\cdot 10^5$ points equally spaced in $u$, and the zeros of Sect. 4. The mean of $R_T^2$ over $10^5 \le x \le 10^7$ is compared with $\sum_{|\gamma|>T} |\rho|^{-2} = C_B - 2\sum_{0<\gamma<T} (\frac{1}{4}+\gamma^2)^{-1}$:

| $T$ | 50 | 100 | 200 | 400 | 600 | 1000 | 2000 | $10^4$ |
|---|---|---|---|---|---|---|---|---|
| $10^3\cdot$mean of $R_T^2$ | 19.43 | 12.21 | 7.14 | 4.14 | 3.00 | 1.953 | 1.089 | 0.267 |
| $10^3 \cdot \sum_{\lvert\gamma\rvert>T} \lvert\rho\rvert^{-2}$ | 19.12 | 11.99 | 7.11 | 4.10 | 2.95 | 1.931 | 1.076 | 0.267 |

The total mean energy is 0.0458 (against $C_B = 0.0462$). The residual does not grow with $x$. With $T = 600$ its mean square is 0.00297, 0.00286, 0.00309, 0.00292 on the four decades from $[10^3, 10^4]$ to $[10^6, 10^7]$, and with $T = 10^4$ it is 0.00025, 0.00027, 0.00027, 0.00027. The excess of the measured over the predicted residual is at most 2% of the residual for every $T$, and 0.3% at $T = 10^4$, that is $2\cdot 10^{-5}$ of the total energy. So the measured residual energy agrees with the contribution predicted from the zeros above $T$, within the numerical accuracy of the experiment and at every scale of $x$ tested, and no additional stochastic component is detected at that precision; in Cramér's model the noise has mean square about $u-2$, between 9.5 and 14 on this range. A finite computation cannot exclude a component below its accuracy, and a decomposition of the prime error into "signal" and "noise" is not defined mathematically; the experiment is the measured counterpart of the discrete spectrum discussed above. It confirms the explicit formula in mean square with the known zeros, but it cannot see a zero above the verified height: a zero with $\beta = 0.75$ at height $3\cdot 10^{12}$ would add an amplitude $2x^{\beta-1/2}/|\rho| \approx 4\cdot 10^{-11}$ at $x = 10^7$. Status: numerical (`residual.py`); the same comparison up to $T = 600$ is in the author's working notes, with the same values.

## 6. Principal modes and orbits of the zeros in the prime signal

The energy can be resolved zero by zero. Let $w_s(y) = (2\pi s^2)^{-1/2} e^{-y^2/(2s^2)}$ be the Gaussian of standard deviation $s > 0$. For real $\gamma_0$ define the demodulated signal

$$A_{\gamma_0}(u) = \lim_{T\to\infty} \int_{-\infty}^{\infty} S_T(v)\, e^{-i\gamma_0 v}\, w_s(u-v)\, dv, \qquad E_{\gamma_0}(u) = 2\,\mathrm{Re}\left(A_{\gamma_0}(u) e^{i\gamma_0 u}\right),$$

with $S_T(v) = -\sum_{|\gamma|<T} e^{(\rho-\frac{1}{2})v}/\rho$ for $v \in \mathbb{R}$.

**Proposition 16** (the orbit identity)**.** *For every $s>0$ and real $u$, $\gamma_0$, the limit exists and*

$$A_{\gamma_0}(u) = -\sum_{\rho} \frac{e^{z_\rho u + z_\rho^2 s^2/2}}{\rho}, \qquad z_\rho = \rho - \tfrac12 - i\gamma_0, \tag{16}$$

*where the series converges absolutely, the zeros counted with multiplicity.*

*Proof.* For each zero, $\int e^{zv} w_s(u-v)\,dv = e^{zu}\int e^{-zy} w_s(y)\,dy = e^{zu+z^2s^2/2}$, which gives (16) for $S_T$, with the sum restricted to $|\gamma|<T$. Since $|\beta-\frac12|\le\frac12$, the modulus of the term of $\rho$ is $e^{(\beta-\frac12)u+((\beta-\frac12)^2-(\gamma-\gamma_0)^2)s^2/2}/|\rho| \le e^{|u|/2+s^2/8}e^{-(\gamma-\gamma_0)^2s^2/2}/|\rho|$, and $\sum_\rho e^{-(\gamma-\gamma_0)^2s^2/2}/|\rho|<\infty$ because $N(T+1)-N(T)\ll\log T$. Hence the right side converges absolutely and is the limit of the truncations. □

Status: proved. For $u>0$, $S_T\to S=E$ in $L^2$ on bounded intervals (Lemma 1), so for $u$ at distance several $s$ from the ends of a range $(0,\log x)$ the quantity $A_{\gamma_0}(u)$ is, up to the Gaussian tails of the window, the demodulation of the prime signal computed from the primes up to $x$.

When $\rho_0=\beta_0+i\gamma_0$ is a simple zero, (16) is an exact decomposition

$$A_{\gamma_0} = A^{\mathrm{pr}}_{\gamma_0} + A^{\mathrm{int}}_{\gamma_0}, \qquad A^{\mathrm{pr}}_{\gamma_0}(u) = -\frac{e^{(\beta_0-\frac12)u+\frac12(\beta_0-\frac12)^2s^2}}{\rho_0}, \qquad A^{\mathrm{int}}_{\gamma_0}(u) = -\sum_{\rho\ne\rho_0}\frac{e^{z_\rho u+z_\rho^2s^2/2}}{\rho},$$

into the *principal mode* of $\rho_0$, the term of $\rho_0$ itself, and the *interference* of all the other zeros. The geometric statements below are statements about the principal mode; the measured orbit is the principal one up to the interference, which the next corollary bounds.

**Corollary 17** (principal modes and their interference)**.** *Let $u\ge0$, $s>0$, $\Gamma>0$, let $\rho_0=\beta_0+i\gamma_0$ be a simple zero, and suppose that all zeros $\rho\ne\rho_0,1-\bar\rho_0$ with $|\gamma-\gamma_0|\le\Gamma$ lie on the critical line. Put*

$$D_{\gamma_0}(s) = \sum_{\substack{\rho\ne\rho_0,1-\bar\rho_0\\ |\gamma-\gamma_0|\le\Gamma}} \frac{e^{-(\gamma-\gamma_0)^2s^2/2}}{|\rho|}, \qquad T_{\gamma_0}(u,s) = e^{u/2+s^2/8}\sum_{|\gamma-\gamma_0|>\Gamma}\frac{e^{-(\gamma-\gamma_0)^2s^2/2}}{|\rho|}.$$

(a) *If $\beta_0=\frac12$, then $A^{\mathrm{pr}}_{\gamma_0}=-1/\rho_0$ and $|A^{\mathrm{int}}_{\gamma_0}(u)|\le D_{\gamma_0}(s)+T_{\gamma_0}(u,s)$. Under RH the same holds with $T_{\gamma_0}$ replaced by the sum over $|\gamma-\gamma_0|>\Gamma$ without the factor $e^{u/2+s^2/8}$.*

(b) *If $\beta_0>\frac12$ and $\delta=\beta_0-\frac12$, then $A^{\mathrm{pr}}_{\gamma_0}(u)=-e^{\delta u+\delta^2s^2/2}/\rho_0$ and*

$$|A^{\mathrm{int}}_{\gamma_0}(u)| \le \frac{e^{-\delta u+\delta^2s^2/2}}{|1-\bar\rho_0|} + D_{\gamma_0}(s) + T_{\gamma_0}(u,s).$$

*Proof.* In (16), the term of $\rho_0$ has $z=\beta_0-\frac12$, real; that of $1-\bar\rho_0$ has $z=\frac12-\beta_0$. A zero on the line has $z=i(\gamma-\gamma_0)$ and $|e^{zu+z^2s^2/2}|=e^{-(\gamma-\gamma_0)^2s^2/2}$. The remaining zeros are bounded as in the proof of Proposition 16. □

Status: proved.

*The principal orbits.* In case (a) the principal component $E^{\mathrm{pr}}_{\gamma_0}(u) = 2\,\mathrm{Re}(A^{\mathrm{pr}}_{\gamma_0}e^{i\gamma_0u}) = -\frac{2}{|\rho_0|}\cos(\gamma_0u-\arg\rho_0)$, and the curve $u\mapsto(E^{\mathrm{pr}}_{\gamma_0}(u), E^{\mathrm{pr}}_{\gamma_0}(u+\frac{\pi}{3\gamma_0}))$ is a fixed ellipse, both coordinates of amplitude $2/|\rho_0|$, traversed once for every increase of $u$ by $2\pi/\gamma_0$; it is exactly closed. In case (b) the same curve is an exact spiral whose amplitude grows at the rate $\beta_0-\frac12$. The measured curve, built from $E_{\gamma_0}=E^{\mathrm{pr}}_{\gamma_0}+2\,\mathrm{Re}(A^{\mathrm{int}}_{\gamma_0}e^{i\gamma_0u})$, lies within $2|A^{\mathrm{int}}_{\gamma_0}|$ of it in each coordinate, and is closed only up to that error. In this language RH says that every principal orbit of the prime signal is closed, and Theorem B of [CE5, Sect. 2] says that the mean squares $2|A^{\mathrm{pr}}_\gamma|^2=2/|\rho|^2$ of the principal modes, $\gamma>0$, add up to the energy $\sum_\rho \lim_n C_2(n,\rho)$. A located zero is resolved as soon as $s$ exceeds a few times the inverse distance to its neighbours, which makes $D_{\gamma_0}(s)$ small, but the

orbit says nothing about a zero that has not been located, and an off-line zero high up becomes visible only for $u$ of order $\log|\gamma_0|/\delta$, as in the local energy law [CE5, Thm. 2.11 and Cor. 2.12].

*Numerics.* We computed $E(u)$ from the prime powers up to $10^8$ and, for the Davenport–Heilbronn function $f$ (the standard control without Euler product, which has zeros off the line [DH]; see [CE46]), the analogous signal $E_f(u) = e^{-u/2}\sum_{n\le e^u} b(n)$, $b(n)$ the coefficients of $-f'/f$, from $n \le 2\cdot 10^7$; the window is $s = 1$ for $\zeta$ and $s = 1.5$ for $f$. Table 1 gives the results. For $\zeta$ at $\gamma_1 = 14.135$ the measured orbit is traversed about 22 times and closes on itself to the resolution of Figure 4a: the interference bound $D_{\gamma_1}(1)$ of Corollary 17(a), computed from the zeros below $10^4$ as if all were on the line, is below $10^{-5}$, against the principal amplitude $1/|\rho_1| = 0.0707$. On $8 \le u \le 12$ the mean of $|A_{\gamma_1}|$ is 0.07069, against $1/|\rho_1| = 0.07070$, and the slightly smaller mean over the whole range comes from the window reaching the ends of the computed range. For $f$, the isolated on-line zero at $\gamma = 43.0813$ gives a closed ellipse up to the interference, and the off-line zeros give spirals with the predicted growth rates (Figure 4b–d). Status: numerical (`orbits.py`).

| zero | mean $\lvert A_\gamma\rvert$ | $1/\lvert\rho\rvert$ | growth rate of $\log\lvert A_\gamma\rvert$ | $\beta - \frac12$ |
|---|---|---|---|---|
| $\zeta$: $\frac12 + 14.1347i$ | 0.07059 | 0.07070 | 0.0001 | 0 |
| $\zeta$: $\frac12 + 21.0220i$ | 0.04756 | 0.04756 | −0.0011 | 0 |
| $\zeta$: $\frac12 + 25.0109i$ | 0.04009 | 0.03997 | 0.0007 | 0 |
| $f$: $\frac12 + 43.0813i$ | 0.02335 | 0.02321 | 0.0066 | 0 |
| $f$: $0.8085 + 85.6993i$ | – | – | 0.3057 | 0.3085 |
| $f$: $0.8695 + 240.4047i$ | – | – | 0.3716 | 0.3695 |

TABLE 1. Measured demodulations of single zeros: $\zeta$ from the primes up to $10^8$ ($s = 1$), the Davenport–Heilbronn function $f$ from its coefficients up to $2\cdot 10^7$ ($s = 1.5$). The columns $1/|\rho|$ and $\beta - \frac12$ are the predictions of the principal mode; the differences include the interference. For an off-line zero the amplitude grows, and only its rate is meaningful.

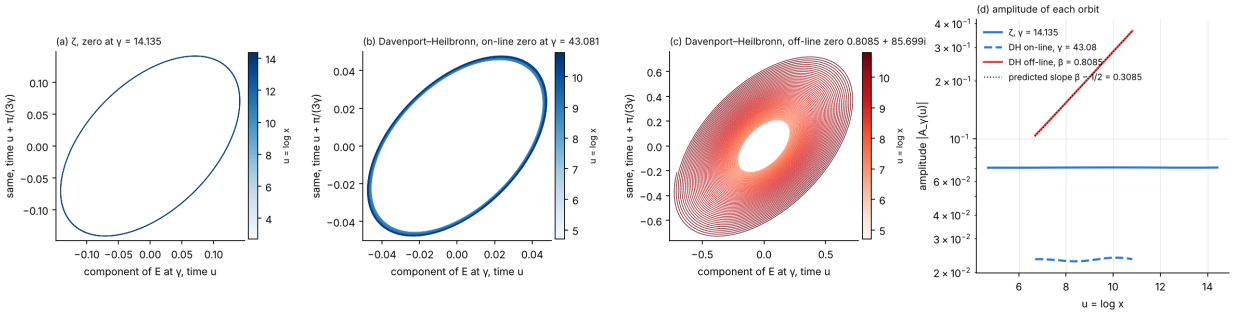


FIGURE 4. Measured orbits of single zeros in the prime signal (principal mode plus interference): the component of the signal at the frequency $\gamma$ at time $u$ against its value at $u + \pi/(3\gamma)$, coloured by $u$. (a) $\zeta$, zero at $\gamma = 14.135$, from the primes up to $10^8$: an ellipse, closed up to the interference. (b) Davenport–Heilbronn function, on-line zero at $\gamma = 43.081$: an ellipse. (c) Its off-line zero $0.8085+85.699i$: an expanding spiral. (d) Amplitude $|A_\gamma(u)|$; the dotted line has the slope $\beta - \frac12$ predicted by the principal mode.

*Remark* 18 (the phase of each orbit). By Corollary 17 the principal mode of $-A_{\gamma_0}$ is a positive multiple of $1/\rho_0$, so up to the interference $\arg(-A_{\gamma_0}) \approx -\arg\rho_0 = -\arctan(\gamma_0/\beta_0)$, and $\beta_0 =$

$\gamma_0/\tan\arg\rho_0$ can be read from the phase in a finite window, without waiting for any growth. On the critical line all principal modes are phase-locked, $\arg\rho=\arctan 2\gamma$, and $E(u)=-\sum_{\gamma>0}(\cos\gamma u+2\gamma\sin\gamma u)/(\frac14+\gamma^2)$ is close to a sine series. From the primes up to $10^8$ the phase gives $\beta=0.499\pm0.023$, $0.480\pm0.029$ and $0.505\pm0.059$ for the first three zeros of $\zeta$, where the error is $\gamma$ times the standard deviation of the measured phase along the window. The sensitivity decreases with height, since the phase difference between $\beta$ and $\frac12$ is about $(\beta-\frac12)/\gamma$: the error is 0.23 at $\gamma=30.4$ and 0.44 at $\gamma=49.8$. For the off-line zero $0.8085+85.699i$ of $f$ the phase gives $0.930\pm0.054$, a miss of more than two standard deviations, against the growth rate 0.3057, that is $\beta=0.8057$. The bias has a clear source: the partner zero $1-\bar\rho_0$ has the same ordinate, and its term in Corollary 17(b), of relative size about $e^{-2\delta u}$, shifts the phase; since $\beta$ is read as $\gamma$ times the phase, a relative error of $10^{-3}$ at $\gamma=85.7$ already moves $\beta$ by about 0.09. For an off-line zero the phase is reliable only for $u\gg\log\gamma/(2\delta)$, so the statement "without waiting for any growth" applies to zeros on the line. The phase sees $\beta$ in a short window but only at low height, and the growth sees it at every height but only over a long range of $u$; neither constrains a zero that has not been located. Status: numerical (`orbits.py`).

*Remark* 19 (spiral times harmonic). The zero $\rho_n$ of the truncation $F_n(s)=X_{n-1}(s)-n^{1-s}/(1-s)$ near a simple zero $\rho$ satisfies $\rho_n-\rho=\frac12 n^{-\rho}/\zeta'(\rho)+O_\rho(n^{-2\beta}\log n+n^{-\beta-1})$ [CE4], and the principal harmonic of the prime signal at $\gamma$ is $h_\rho(u)=A^{\mathrm{pr}}_\gamma(u)e^{i\gamma u}=-e^{(\rho-\frac12)u}e^{(\beta-\frac12)^2s^2/2}/\rho$. Their product at $u=\log n$ does not depend on $\beta$ up to the factor $e^{(\beta-\frac12)^2s^2/2}$:

$$-2\sqrt{n}\,(\rho_n-\rho)\,h_\rho(\log n)\approx\frac{e^{(\beta-\frac12)^2s^2/2}}{\rho\,\zeta'(\rho)},$$

which on the line is the residue of $1/(s\zeta(s))$ at $\rho$, the coefficient of $\rho$ in the explicit formula for $M(x)=\sum_{n\le x}\mu(n)$. The integers supply $1/\zeta'(\rho)$ and the primes supply $1/\rho$. For the first three zeros of $\zeta$, with $n\le2\cdot10^5$ and the primes up to $10^8$, the product computed with the measured demodulation agrees with $1/(\rho\zeta'(\rho))$ to within 1–3%, and summed over the 341 zeros below 600 these modes rebuild $(M(x)+2)/\sqrt{x}$ with correlation 0.995 (Figure 5). The energy of the Mertens signal, the Möbius analogue of Theorem B with each zero weighted by $|\zeta'(\rho)|^{-2}$, is treated in [CE5, Sect. 2]. The product reads $1/\zeta'(\rho)$ at a located zero; it adds no constraint on $\beta$. Status: the approximation follows from [CE4] and Corollary 17 for a located zero; the numbers are numerical (`a.py`, `fig.py`, `mob.py`, `dhinv.py`).

## 7. The energy gauge: the prime energy as one calibrated instrument

This section was first written as a separate note, "The energy gauge" (31 October 2026); here it is a section of this paper, because its theorems are those of [CE5] and of the previous sections. What it adds is a way of reading them together, as three dials of one instrument, and a run of the instrument on the primes. Write $F(U)=I(e^U)=\int_0^U e(u)^2\,du$ with $e(u)=e^{-u/2}(\psi(e^u)-e^u)$, so that $e=E_0$ of (1) and $F$ is the energy of the primes in the variable $U=\log X$; on $0\le u<\log2$ one has $\psi=0$, so the transient contributes $\int_0^{\log2}e^u\,du=1$ exactly, and all measured quantities below are reported from $u=0$ with this convention. We write $C_B=2\widehat{B}$, with $\widehat{B}=1+\frac{\gamma_E}{2}-\frac12\log4\pi$ the constant of the margin programme [CE11, CE12, CE13, CE14].

The three dials (proved, in [CE5] and in this paper). *Dial A, the line:* unconditionally, $\Theta=\frac12+\frac12\limsup_{U\to\infty}U^{-1}\log F(U)$, where $\Theta$ is the supremum of the real parts of the zeros; this is the growth-rate theorem [CE5, Theorem 2.2], and subexponential energy is RH. *Dial B, the multiplicities:* unconditionally $\limsup F(U)/U\ge C_B$, and the limit exists and equals $C_B$ if and only if RH holds with all zeros simple; this is the energy law [CE5, Theorem B, Corollary 2.3]. *Dial C, the spacings:* under (H*), $F(U)-C_BU=K+\Phi(U)+o(1)$ (Theorem 5), and the remainder is a spacing-sensitive observable (Proposition 11 and Sect. 3.2).

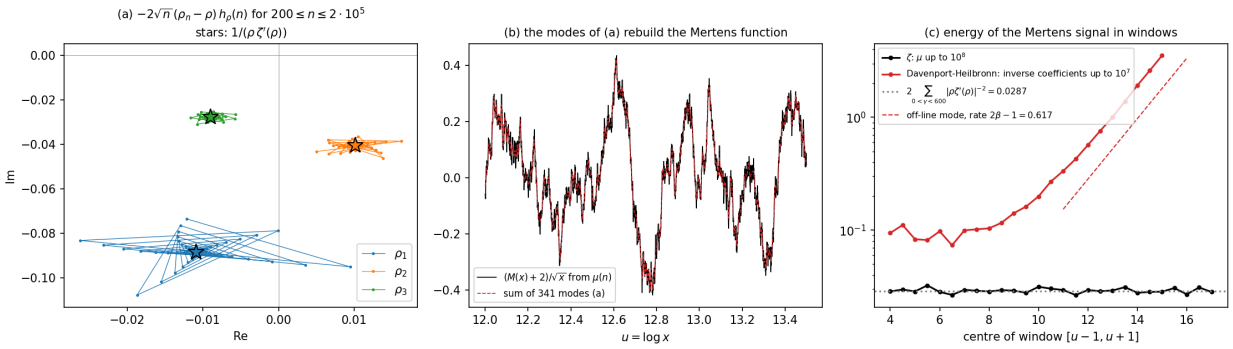


FIGURE 5. Spiral times harmonic. (a) $-2\sqrt{n}(\rho_n-\rho)h_\rho(\log n)$ for $200\le n\le 2\cdot 10^5$ at the first three zeros of $\zeta$, with the measured demodulation; stars: $1/(\rho\zeta'(\rho))$. (b) $(M(x)+2)/\sqrt{x}$ from $\mu(n)$ and the sum of the 341 modes. (c) Energy of the Mertens signal in windows of length 2, for $\zeta$ and for the Davenport–Heilbronn function; the dashed line has the slope $2\beta-1=0.617$ of its off-line zero.

The multiplicity meter (proved; a reading of Theorem B).. Under RH, whenever $\lim F(U)/U$ exists it equals $\sum_\rho m_\rho^2/|\rho|^2$, where the sum runs over distinct zeros and $m_\rho$ is the multiplicity. Since $C_B=\sum_\rho m_\rho/|\rho|^2$, the excess slope

$$\lim_{U\to\infty}\frac{F(U)}{U}-C_B=\sum_\rho\frac{m_\rho^2-m_\rho}{|\rho|^2}\;\ge\;0$$

vanishes exactly when every zero is simple. *Proof.* Expand $F=\int_0^U|S+r|^2$ with $S$ and $r$ as in (1). The diagonal of $|S|^2$ is $U\sum_\rho m_\rho^2/|\rho|^2$; an off-diagonal pair is bounded both by $2m_\rho m_{\rho'}/(|\rho||\rho'||\gamma-\gamma'|)$ and by $Um_\rho m_{\rho'}/(|\rho||\rho'|)$, so the pairs inside a fixed finite set of zeros contribute $O(1)$ and the remaining pairs are $o(U)$ by Cauchy–Schwarz against the diagonal; the terms with $r$ are $O(1)$ because $r\in L^2(0,\infty)$. No spacing information is used. □

The ledger of the intercept (proved). The constant $K$ of Theorem 5 splits as in Proposition 7 into a trivial term, a cross term and a zero–zero term. The trivial term is $\int_0^\infty r(u)^2\,du$, and in closed form, with $L=\log 2\pi$,

$$\int_0^\infty r(u)^2\,du=L^2-(2-2\log 2)L+2-2\log 2+\log^2 2-\frac{\pi^2}{12}=2.5215682\ldots,$$

by expanding $\log(1-e^{-2u})$ and integrating term by term: $\sum_k 1/(k(2k+1))=2-2\log 2$, and $\sum_{j,k\ge1}1/(jk(2j+2k+1))=\int_0^1\log^2(1-x^2)\,dx=8-8\log 2+4\log^2 2-\pi^2/3$ (this is (9)). It carries all of $K=2.512479790006\ldots$ (Corollary 9) except $-0.0090884\ldots$: the cross term $-0.010118144286\ldots$ plus the zero–zero term $0.00102972290886\ldots$. The zeros adjust the intercept by 0.36%; the elementary bookkeeping of the explicit formula sets its size. The numerical proximity $K\approx\sqrt{2\pi}=2.5066\ldots$ is a coincidence: the certified $K$ differs from $\sqrt{2\pi}$ by 0.0058.

The gauge on the primes to $10^8$ (measured). We ran the instrument with the exact $\psi$, all prime powers included, to $10^8$, on grids of $2\cdot10^6$ points in $u$ (the trapezoidal $F$ is checked against exact piecewise integration of the step function $\psi$ on a subrange, agreeing to $10^{-7}$). Independently we recomputed the first 38 zeros of $\zeta$ from scratch (Euler–Maclaurin for $\zeta(\frac12+it)$, Stirling for the Riemann–Siegel $\vartheta$, bisection on sign changes of $Z$), obtaining $\gamma_1=14.134725$, $\gamma_2=21.022040$, …, within $4\cdot10^{-7}$ of the known values. Table 2 and Figures 6–7 give the readings. The slope ledger closes: the measured slope is 0.04630; the first 38 zeros give 0.03570, and the counting-function tail $\sum_{\gamma>120}2/(\frac14+\gamma^2)\approx0.0105$ supplies the rest. The measured intercept 2.5125(37) agrees with $K$, and

its measured zero-dependent part $-0.0091(37)$ with the certified $-0.0090884\ldots$. The oscillation of the remainder is reproduced by the first 38 zeros at correlation 0.988. The same prime-versus-zero comparison was run in Sect. 3 on the primes to $10^9$; the run here repeats it at lower depth, with the three readings side by side. The window $U \le 18.42$ is four decades of $x$ but a short interval of $u$: dial A at this depth excludes only gross violations, and all three readings are consistency checks of RH with simple zeros, not evidence at unexplored heights.

| dial | reads | measured on $[10^4, 10^8]$ |
|---|---|---|
| A (line) | $\Theta = \frac{1}{2} + \frac{1}{2}\limsup \frac{\log F}{U}$ | no exponential growth: $\Theta = \frac{1}{2}$ within the window |
| B (multiplicities) | slope of $F$ against $C_B$ | 0.04630 vs $C_B = 0.04619$ (+0.2%, within oscillation) |
| C (spacings) | ripple of $F - C_B U$ | flat: r.m.s. $\pm 0.004$, peak $\pm 0.012$; intercept 2.5125(37) |
| | | ripple vs 38-zero prediction: correlation 0.988 |

TABLE 2. The energy gauge on the primes to $10^8$.

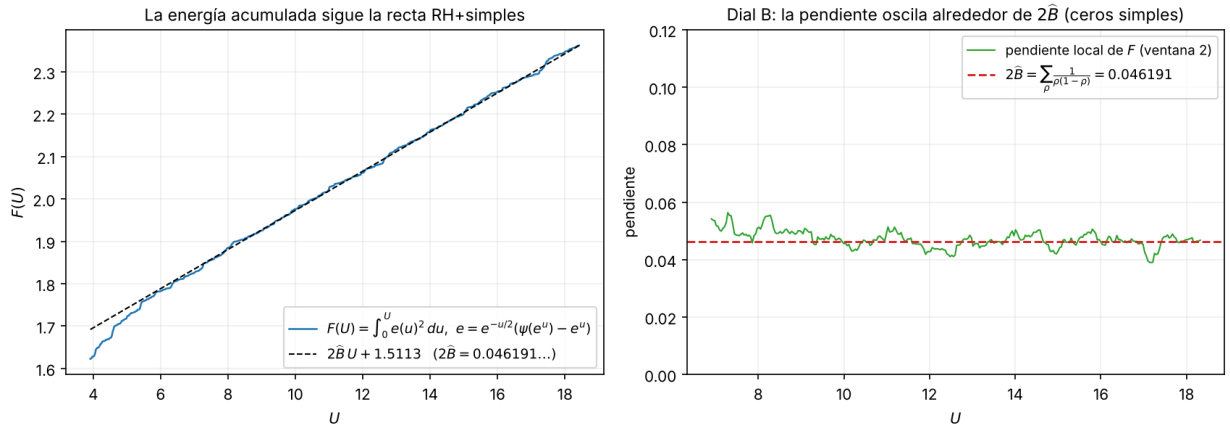


FIGURE 6. Left: the cumulative energy $F(U)$ against the line $C_B U + \text{const}$ over four decades. Right: the local slope of $F$ (sliding window of width 2) oscillating about $C_B$ (dashed); the dominant ripple has period $2\pi/\gamma_1 = 0.4445$ in $u$.

The Dirichlet column. Every dial carries over to a primitive character $\chi$ mod $q$ with $e_\chi(u) = e^{-u/2}\psi(e^u, \chi)$ (no pole to subtract): the slope constant becomes

$$\kappa_\chi = \sum_{\rho_\chi} \frac{1}{\rho(1-\rho)} = \log\frac{q}{\pi} + \frac{\Gamma'}{\Gamma}\Big(\frac{1+a}{2}\Big) + 2\,\mathrm{Re}\,\frac{L'}{L}(1,\chi),$$

with $a \in \{0,1\}$ the parity of $\chi$. For the character $\chi_3$ of the race between the classes $6k \pm 1$ the pair $(\kappa_3, C_3)$ is computed and measured in [CE38]: $\kappa_3 = 0.113229969857\ldots$. The instrument has one pair of constants per $L$-function, and $(C_B, K)$ is the row of $\zeta$.

## 8. Discussion and open problems

8.1. **What is achieved.** At second order the energy of the primes is $C_B \log X + K + \Phi(\log X)$: the slope is the trace of the Gram matrix of the zeros [CE5], the intercept $K$ is known in closed form and to twelve digits under RH alone, and the oscillation comes from the off-diagonal entries. The constant needs only RH and simple zeros, and is the Cesàro mean of the deviation exactly

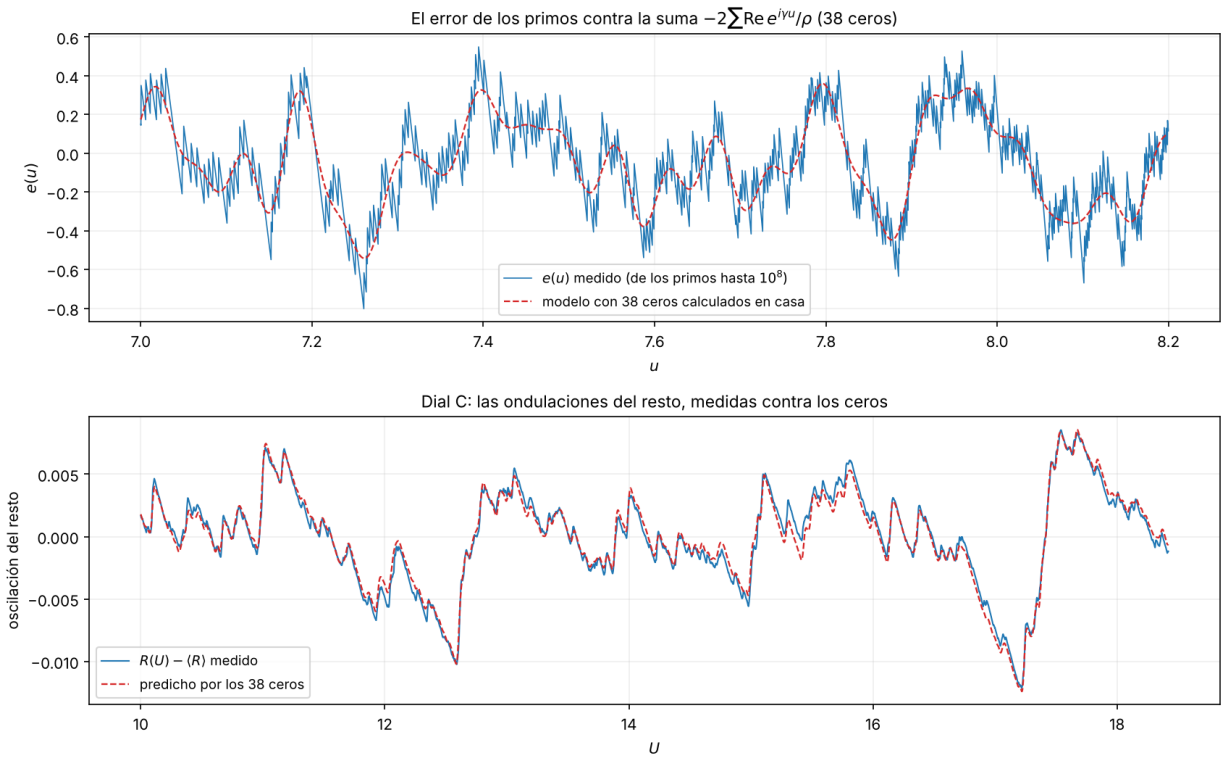


FIGURE 7. Top: the measured error $e(u)$ from the primes against the 38-zero model $-2\sum_{\gamma\le 120}\mathrm{Re}(e^{i\gamma u}/\rho)+r(u)$. Bottom: the oscillation of the remainder $F(U)-C_BU$, measured versus predicted by the same 38 zeros: correlation 0.988. The 38-zero diagonal is 0.03570, and the zero-counting tail $\sum_{\gamma>120}2/(\frac14+\gamma^2)\approx 0.0105$ restores $C_B=0.04619$.

when close pairs of zeros are rare in the weighted sense of Proposition 11. The pointwise law and the limiting distribution use in addition the spacing condition (H*); the distribution would follow from the mean-square approximation (Q), which under LI requires ($H_2$) together with $\nu(\varepsilon)=o(\varepsilon)$. The fluctuations are tiny, with standard deviation 0.0035, and the primes up to $10^9$ follow the zero side to $10^{-5}$. Resolved by frequency, each zero contributes a principal mode, a closed orbit on the line and a spiral off it, plus a controlled interference.

The same multiplicative structure that makes the integrated energy rigid produces, on short ranges $N\le n\le 2N$, heavy tails and, numerically, an excess of large values of Dirichlet polynomials at every computable size [CE7]. That other face of the Euler product, and what it means for counting zeros off the line, is the subject of [CE7].

8.2. **What is not achieved.** Nothing in this paper bears on the truth of RH. Theorems 5 and 14 describe what RH, with simple zeros and the additional spacing condition (H*), implies for the energy of the primes; Propositions 8–12 describe what RH and simple zeros imply with weaker or no spacing conditions. Proving that the energy is finite is RH [CE5, Cor. 2.3]. The orbit identity reads located zeros and cannot constrain a zero that has not been located. The hypotheses (H*), ($H_2$), (Q) and LI are unproved. The numerical value of $V$ assumes LI, has a numerical lower bound, and has an upper bound only under an assumption on the gaps.

8.3. **Open problems.**

(1) Problem 13: prove the mean-square approximation (Q) under RH, simple zeros, ($H_2$) and $\nu(\varepsilon) = o(\varepsilon)$, which would give the limiting distribution without (H*); and prove the $B^2$-Cauchy property of $\Phi_T$ under ($H_2$) without LI.
(2) Replace (H*) by $\sum_\rho |\rho|^{-2}\delta_\rho^{-1} < \infty$ in Theorems 5 and 14, using the Montgomery–Vaughan inequality for the truncations of $\Phi$ (Remark 6).
(3) A closed form for $V$, analogous to Proposition 8, and the symmetry of $\mu$. We observe a skewness of order $10^{-3}$, consistent with 0; under LI the third moment is a sum over the six-term relations $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 + \gamma_5 - \gamma_6 = 0$, which we have not evaluated in closed form. Unlike $K$, $V$ involves the differences $\gamma - \gamma'$ and has no evident expression through the Hadamard product.
(4) The constant term $K_M$ of the Möbius energy $\int_1^X M(x)^2 x^{-2}\,dx = C_M \log X + K_M + \dots$, for which [CE5, Sect. 2] reports the numerical value $K_M = 1.2603 \pm 0.0023$. The trivial and cross parts can be treated as in Proposition 8; the zero–zero part involves $\sum_\rho |\rho\zeta'(\rho)|^{-4}$, which has no closed form known to us.

## Data and code

The ancillary Python scripts (NumPy, SciPy, Matplotlib, mpmath; python-flint for interval arithmetic) are in the folder `anc`, whose file `README.txt` lists, for each script, the section, table or figure it reproduces and its output. `ancillary/energy_gauge.py` reproduces Section 7 (the exact $\psi$ to $10^8$, the in-house zeros, the readings and Figures 6–7); `zeros/rs_zeros.py` computes the zeros below $10^4$; `fluct/K_closed_form_arb.py` certifies the value of $K$ (Corollary 9); `fluct/K_formula.py`, `fluct/variance.py`, `fluct/variance_finite_check.py`, `fluct/tail_bounds.py`, `fluct/primes_1e9.py` and `fluct/fig_fluct.py` reproduce Sects. 3–4 and Figure 2; `K_constant/` and `energy/` reproduce Figure 1 and the values of $I(X)$ up to $10^7$; `montecarlo/` (with `granville.py`) and `fluct/residual.py` reproduce Sect. 5; `orbits/` and `mobius/` reproduce Sect. 6. Apart from Corollary 9 the computations are in double precision (extended precision for $I(X)$ up to $10^9$), without interval arithmetic; they illustrate the theorems and are not used in their proofs.

Provenance. This paper and [CE7] together supersede Parts II–IV of the preprint [LV] and the note [CE], which is contained in Part II of [LV]; Part I of [LV] (the trichotomy of $C_2$ and the energy law) is in [CE1, CE5]. It also contains the constant term, the random models and the orbits of the prime signal from version 3 of [CP]. An earlier version of this paper, titled "Fluctuations and large values of the prime energy", also contained the material on large values that is now in [CE7]. All numbers of Sects. 3–6 were recomputed for this paper.

Use of AI tools. The proofs, computations and drafting of this paper, and of the preprints [LV] and [CE] that it absorbs, were carried out with the assistance of an AI system (Claude, Anthropic), under the author's direction. The author checked the results and takes responsibility for them.

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