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RESURGENCE OF THE SQUEEZED KERR OSCILLATOR: TUNNEL SPLITTINGS, KUMMER CONNECTION, AND A PERTURBATIVE/NON-PERTURBATIVE RELATION

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Abstract. The squeezed (two-photon-driven) Kerr oscillator $H = a^{\dagger 2}a^2 - \epsilon_2(a^{\dagger 2} + a^2) - \Delta a^\dagger a$, in units of the Kerr constant, underlies the Kerr-cat qubit. Its two wells at $a = \pm\alpha$, $\alpha^2 = \epsilon_2 + \Delta/2$, carry doublets of nearly degenerate cat states. We give the resurgent structure of these doublets at fixed detuning Δ as $\epsilon_2 \to \infty$. The splitting of the nth doublet is

$$E_{\mathrm{odd},n} - E_{\mathrm{even},n} = \frac{8\epsilon_2(4\epsilon_2)^{2n-\Delta/2}\,\mathrm{e}^{-2\epsilon_2}}{n!\,\Gamma(n-\Delta/2)}\left(1 + O(\epsilon_2^{-1})\right).$$

In the Bargmann representation, the factor $1/\Gamma(n-\Delta/2)$ is a Kummer connection coefficient, and its zeros are the known exact degeneracies at $\Delta = 2m \geq 2n$. At fixed Δ and leading order, the known large-detuning WKB formula is the same expression with $\Gamma(1+\Delta/2)$ replaced by its Stirling approximation. The Rayleigh–Schrödinger series of each level in $1/\alpha^2$, computed exactly through order 150, diverges as $\Gamma(j + 4n - \Delta)4^{-j}$ with the prefactor fixed by the splitting, and converges at $\Delta = 2$ for $n = 0$. The median Borel sum of the $n = 0$ series reproduces the center of the exact ground doublet up to a real two-instanton term, which a quantization condition obtained from the same matching predicts at leading order. Finally, the corrections $1 + k_1/\epsilon_2 + k_2/\epsilon_2^2 + \cdots$ to the splitting obey a relation of Dunne–Ünsal type with the perturbative energies: the predicted closed forms—for k_1 in n and Δ (checked for $n \leq 2$ and at $(n, \Delta) = (3, 2)$), k_2 (checked for $n \leq 2$), and k_3 (checked for $n = 0$)—agree with the values extracted from exact splittings, including k_2 for $n = 2$ and k_3 for $n = 0$, which were not used to find the relation. With three correction terms, the formula agrees with exact diagonalization to a relative error below 2×10^{-5} for $-1 \leq \Delta \leq 3$ at $\epsilon_2 \geq 10$ (including the limit at the exact zeros), and below 2×10^{-4} for $-1 \leq \Delta \leq 11$ at $\epsilon_2 = 30$.

1. Introduction

A Kerr-nonlinear resonator driven at twice its frequency has two classical steady states of opposite phase. Its quantum counterpart, the squeezed Kerr oscillator, has a spectrum of nearly degenerate doublets whose two members are even and odd superpositions (cat states) of states localized in the two wells [22, 6, 17]. The Kerr-cat qubit is built on the lowest doublet [11]. Its splitting is a coherent tunneling amplitude: it sets the rate of coherent phase flips, and it vanishes at special parameter values. Frattini et al. [10] observed pairwise level degeneracies as the squeezing amplitude is increased. Venkatraman et al. [20] and Ruiz et al. [18] showed that when the detuning is an even multiple $2m$ of the Kerr constant, the lowest $m+1$ doublets are exactly degenerate through a finite block structure of the Hamiltonian in a displaced frame. A quasi-spin symmetry at integer detuning and the relation of these coalescences to excited-state quantum phase transitions are discussed in [12, 5].

Several approximations to the splitting exist. Wielinga and Milburn [22] estimated its exponential dependence by degenerate perturbation theory at one value of the detuning. Puri,

Boutin, and Blais [17] treated detuning to leading order. Marthaler and Dykman obtained the tunneling exponent [14] and derived the splitting by WKB [15], including a prefactor that oscillates with the detuning and vanishes at the degeneracy points; their formula, reproduced in [20] and compared with exact diagonalization and experiment in [21, §3.2 and Ch. 5], assumes a detuning large compared with the Kerr constant. Su et al. [19] computed the splittings of all doublets at zero detuning by WKB.

This note treats the problem as a resurgence problem [9, 2]: the perturbative series of each level about one well, the exponentially small splitting, and the relations between them. Our results are as follows.

(1) The splitting of every doublet at fixed detuning, to leading order (Result 3.1), with corrections checked for $n \le 2$ (Result 6.1). In the Bargmann representation, in which resurgence has been studied for the quartic anharmonic oscillator [1], the eigenproblem becomes a confluent Heun equation with regular singular points at the two wells, and the factor $1/\Gamma(n - \Delta/2)$ arises as a Kummer connection coefficient. The Marthaler–Dykman formula agrees with Result 3.1 for $n = 0$ in the overlap $1 \ll \Delta \ll \epsilon_2$, where $\Gamma(1 + \Delta/2)$ may be replaced by its Stirling approximation.
(2) The large-order behavior of the perturbative series, with Borel singularity at twice the instanton action and prefactor fixed by the splitting, and convergence at a degeneracy point (Section 4).
(3) The median Borel sum, the cancellation of its ambiguity against the instanton–anti-instanton term, and a real two-instanton term predicted by a quantization condition of the type conjectured by Zinn-Justin and Jentschura for the double well [23, eq. (2.22)] (Section 5).
(4) A relation of the Dunne–Ünsal type [7, 8], which determines the corrections to the splitting from the perturbative energies, checked numerically through the second correction for $n \le 2$ and the third for $n = 0$ (Section 6).

The structure at the degeneracy points parallels that found by Kozçaz, Sulejmanpasic, Tanizaki, and Ünsal [13] for quasi-exactly solvable systems, where the imaginary ambiguity carries a factor $1/\Gamma(1 - \zeta)$ that vanishes at integer ζ and the low-lying perturbative series converge.

Code that reproduces every computation in this note is available at https://github.com/shankarsivarajan/resurgent-series.

2. Model, conventions, and exact data

Hamiltonian. We use units of the Kerr constant and the overall sign that makes the cat states the *lowest* levels:

$$H = a^{\dagger 2}a^2 - \epsilon_2(a^{\dagger 2} + a^2) - \Delta\, a^\dagger a, \qquad \epsilon_2 > 0,\ \Delta \in \mathbb{R}. \tag{1}$$

The usual laboratory form $H_{\mathrm{lab}} = \Delta_{\mathrm{lab}} a^\dagger a - K a^{\dagger 2}a^2 + \epsilon_{2,\mathrm{lab}}(a^{\dagger 2} + a^2)$ equals $-KH$ with $\Delta = \Delta_{\mathrm{lab}}/K$ and $\epsilon_2 = \epsilon_{2,\mathrm{lab}}/K$. The Hamiltonian commutes with the parity $(-1)^{a^\dagger a}$. In the Fock basis it couples $|k\rangle$ only to $|k \pm 2\rangle$, so each parity sector is a symmetric tridiagonal matrix. We write $E_{\mathrm{even},n}$ and $E_{\mathrm{odd},n}$ for the nth lowest eigenvalue ($n = 0, 1, \dots$) in the even and odd sectors, and

$$S_n = E_{\mathrm{odd},n} - E_{\mathrm{even},n} \tag{2}$$

for the splitting of the nth doublet. Exact eigenvalues are computed by Sturm-sequence bisection on truncations of the tridiagonal matrices, with truncation size and arithmetic precision increased until all quoted digits are stable.

Displaced expansion. The classical energy $|a|^4 - \epsilon_2(a^{*2} + a^2) - \Delta|a|^2$ has minima at $a = \pm\alpha$ with $\alpha^2 = \epsilon_2 + \Delta/2$ for $\epsilon_2 > -\Delta/2$ (in particular as $\epsilon_2 \to \infty$ at fixed Δ), where it equals $-\alpha^4$. Writing $a = \alpha + d$ removes the terms linear in d and gives, exactly,

$$\frac{H}{4\alpha^2} = -\frac{\alpha^2}{4} + d^\dagger d + \frac{\eta}{2}(d^{\dagger 2} d + d^\dagger d^2) + \frac{\eta^2}{4}\Big[\frac{\Delta}{2}(d^{\dagger 2} + d^2) - \Delta\, d^\dagger d + d^{\dagger 2} d^2\Big], \qquad \eta = \frac{1}{\alpha}. \tag{3}$$

Rayleigh–Schrödinger perturbation theory about the Fock state $d^{\dagger n}|0\rangle$ gives a formal series

$$E_n^{\mathrm{pert}} = -\alpha^4 + 4\alpha^2 \sum_{j\geq 0} c_j^{(n)} \alpha^{-2j}, \qquad c_0^{(n)} = n; \tag{4}$$

odd powers of η are absent because $\eta \to -\eta$, $d \to -d$ is a symmetry of (3). In the unnormalized basis $d^{\dagger k}|0\rangle$, every matrix element of (3) is a polynomial in k and Δ with dyadic-rational coefficients, so for rational Δ the $c_j^{(n)}$ are rational and can be computed exactly; we compute them through $j \leq 150$ (for $n = 0$, $c_1^{(0)} = 0$). The first coefficients are

$$c_1^{(n)} = -\tfrac{1}{4} n(2n + \Delta), \qquad c_2^{(n)} = -\tfrac{1}{64}(4n - \Delta)(4n^2 - 2\Delta n - \Delta). \tag{5}$$

The two wells give the same series, by the symmetry $a \to -a$.

Bargmann form. In the Bargmann representation [3], on the Hilbert space of entire functions $f(z)$ with the Gaussian inner product, $a^\dagger$ acts as multiplication by z and a as $\mathrm{d}/\mathrm{d}z$, and $Hf = Ef$ becomes

$$(z^2 - \epsilon_2) f'' - \Delta z f' - (\epsilon_2 z^2 + E) f = 0. \tag{6}$$

This is a confluent Heun equation [16, (31.12.1)] with regular singular points $z = \pm\sqrt{\epsilon_2}$, each with exponents 0 and $1 + \Delta/2$, and an irregular singular point at infinity. Eigenfunctions are the solutions that are entire, that is, analytic at both $z = \pm\sqrt{\epsilon_2}$ (every solution of (6) behaves as $\mathrm{e}^{\pm\sqrt{\epsilon_2} z} z^{\Delta/2}(1 + O(z^{-1}))$ at infinity, so any entire solution automatically has finite Bargmann norm).

3. The splitting

Result 3.1 (derived at leading order; confirmed numerically)**.** For fixed $n \geq 0$ and fixed real Δ, as $\epsilon_2 \to \infty$,

$$S_n = \frac{8\epsilon_2\, (4\epsilon_2)^{2n - \Delta/2}\, \mathrm{e}^{-2\epsilon_2}}{n!\, \Gamma(n - \Delta/2)} \Big(1 + O(\epsilon_2^{-1})\Big). \tag{7}$$

In particular, S_n vanishes to this order exactly when $\Delta \in \{2n, 2n + 2, 2n + 4, \dots\}$, and, for ϵ_2 large enough, S_n does not change sign for $\Delta < 2n$.

The zeros agree with the exact degeneracies proved in [20, 18]: at $\Delta = 2m$ the lowest $m + 1$ doublets are exactly degenerate, so doublet n is degenerate at every $\Delta = 2m$ with $m \geq n$.

Derivation. Put $z = \sqrt{\epsilon_2}\, w$ and $E = -\epsilon_2^2 - \Delta\epsilon_2 - 4\epsilon_2\kappa$, which defines the spectral parameter κ. Then (6) reads

$$(w^2 - 1) f_{ww} - \Delta w f_w - \big(\epsilon_2^2 (w^2 - 1) - (4\kappa + \Delta)\epsilon_2\big) f = 0. \tag{8}$$

Outer region. Setting $f = \mathrm{e}^{\epsilon_2 w} G$, at leading order in ϵ_2 we obtain $2(w^2 - 1)G' = (\Delta(w - 1) - 4\kappa)G$, so

$$F_+(w) = \mathrm{e}^{\epsilon_2 w} (w - 1)^{-\kappa} (w + 1)^{\kappa + \Delta/2} (1 + O(\epsilon_2^{-1})), \qquad F_-(w) = F_+(-w). \tag{9}$$

For $\kappa = -n$, F_+ is analytic at $w = +1$; it is the displaced Fock state of the $+$ well (the Bargmann function of a coherent state is $\mathrm{e}^{\alpha z}$). At $w = -1$ it has the exponent $\kappa + \Delta/2$, which is not a non-negative integer in general.

Inner region. Near $w = -1$, put $u = w + 1$, $f = \mathrm{e}^{-\epsilon_2 u} y$, and $t = 2\epsilon_2 u$. To leading order, (8) becomes Kummer's equation [16, (13.2.1)]

$$t\, y'' + (\beta - t)\, y' - \kappa\, y = 0, \qquad \beta = -\Delta/2. \tag{10}$$

Its solution analytic at $t = 0$ is Olver's $\mathbf{M}(\kappa, \beta, t) = M(\kappa, \beta, t)/\Gamma(\beta)$, which is defined for all β, including non-positive integers. For $t \to +\infty$ [16, (13.7.2)],

$$\mathbf{M}(\kappa, \beta, t) \sim \frac{1}{\Gamma(\kappa)}\, \mathrm{e}^{t} t^{\kappa - \beta} + \frac{\mathrm{e}^{\pm \mathrm{i}\pi\kappa}}{\Gamma(\beta - \kappa)}\, t^{-\kappa}, \tag{11}$$

where $t > 0$ lies on a Stokes line of Kummer's equation for the expansion (11): there the second term is recessive, and its phase $\mathrm{e}^{\pm \mathrm{i}\pi\kappa}$ depends on the side from which $t > 0$ is approached.

Matching. In the overlap $\epsilon_2^{-1} \ll u \ll 1$, F_+ and F_- are proportional to the first and second terms of (11):

$$F_+ \to (-2)^{-\kappa} \mathrm{e}^{-\epsilon_2} (2\epsilon_2)^{\beta - \kappa} \cdot \mathrm{e}^{t} t^{\kappa - \beta}, \qquad F_- \to \mathrm{e}^{\epsilon_2} 2^{\kappa - \beta} (2\epsilon_2)^{\kappa} \cdot (-t)^{-\kappa}, \tag{12}$$

both inside y, with $(-t)^{-\kappa} = \mathrm{e}^{\pm \mathrm{i}\pi\kappa} t^{-\kappa}$ on the same side. The combination $F_+ + \sigma F_-$, with $\sigma = +1$ for the even and $\sigma = -1$ for the odd state (it has parity σ under $w \to -w$), is analytic at $w = -1$ only if it is proportional to $\mathbf{M}$, that is, only if the ratio of the coefficients of $\mathrm{e}^{t} t^{\kappa - \beta}$ and $\mathrm{e}^{\pm \mathrm{i}\pi\kappa} t^{-\kappa}$ is $\Gamma(\beta - \kappa)/\Gamma(\kappa)$; by symmetry it is then also analytic at $w = +1$. This gives the quantization condition

$$\frac{1}{\Gamma(\kappa)} = \sigma\, (-1)^{-\kappa}\, \mathrm{e}^{-2\epsilon_2} (4\epsilon_2)^{\beta - 2\kappa}\, \frac{1}{\Gamma(\beta - \kappa)}\, (1 + O(\epsilon_2^{-1})), \tag{13}$$

where $(-1)^{-\kappa} = \mathrm{e}^{\pm \mathrm{i}\pi\kappa}$ and the $O(\epsilon_2^{-1})$ correction to the amplitudes is not computed here. Writing $\kappa = -n - \delta\kappa$, so that $(-1)^{-\kappa} = (-1)^n \mathrm{e}^{\pm \mathrm{i}\pi\delta\kappa}$, and using $1/\Gamma(-n - \delta\kappa) = (-1)^{n+1} n!\, \delta\kappa\, (1 + \psi(n+1)\, \delta\kappa + \cdots)$, to first order we obtain

$$\delta\kappa_\sigma = -\sigma\, \frac{\mathrm{e}^{-2\epsilon_2} (4\epsilon_2)^{2n - \Delta/2}}{n!\, \Gamma(n - \Delta/2)}, \tag{14}$$

and $S_n = 4\epsilon_2(\delta\kappa_- - \delta\kappa_+)$ yields (7). The steps that are not proved are the uniform validity of the outer approximation up to the overlap region and the neglect of the $O(\epsilon_2^{-1})$ corrections in both regions. The amplitude corrections enter Section 6.

The factor $1/\Gamma(n - \Delta/2)$ is thus the connection coefficient $1/\Gamma(\beta - \kappa)$ at the singular point of (6) belonging to the opposite well, whose non-trivial exponent is $1 + \Delta/2$. Its zeros (the poles of $\Gamma(n - \Delta/2)$) lie at $\Delta = 2m \geq 2n$, where the opposite well imposes no condition at this order. At $\Delta = 2m$ with $m < n$, β is a non-positive integer but $\beta - \kappa = n - m > 0$, and the argument goes through unchanged with $\mathbf{M}$, the unique solution analytic at $t = 0$ when $\beta = -m$ and $\kappa \notin \{0, -1, \ldots, -m\}$ (the other solution then has a logarithm). Numerically, for $(n, \Delta) = (1, 0)$, $(2, 0)$, $(2, 2)$, and $(3, 2)$, the Richardson limit of the ratio R_n defined below is 1 (to within 4×10^{-7} for the first three on the grid $\alpha^2 = 10, 12, \ldots, 36$, and within 3×10^{-9} for $(3, 2)$ on the $\epsilon_2 = 16, 18, \ldots, 50$ grid of Section 6, or 5×10^{-6} on $\alpha^2 \leq 36$), and $\epsilon_2(R_n - 1)$ extrapolates to the k_1 of (24).

Numerical confirmation. Let $R_n(\epsilon_2)$ be the exact S_n divided by the right side of (7) without the factor $1 + O(\epsilon_2^{-1})$. Richardson extrapolation of R_n in $1/\alpha^2$, from exact splittings at $\alpha^2 = \epsilon_2 + \Delta/2 = 10, 12, \ldots, 36$, gives 1 for $\Delta \in \{-1, \frac{1}{2}, 1, 3\}$ to the accuracy shown in Table 1. The accuracy decreases with n because the correction coefficients grow (Section 6).

	$n=0$	$n=1$	$n=2$	$n=3$
$\max_\Delta \lvert \lim R_n - 1\rvert$	2×10^{-11}	6×10^{-9}	5×10^{-7}	10^{-5}

TABLE 1. Richardson-extrapolated ratio of exact to predicted splitting (7), worst case over $\Delta \in \{-1, \frac{1}{2}, 1, 3\}$.

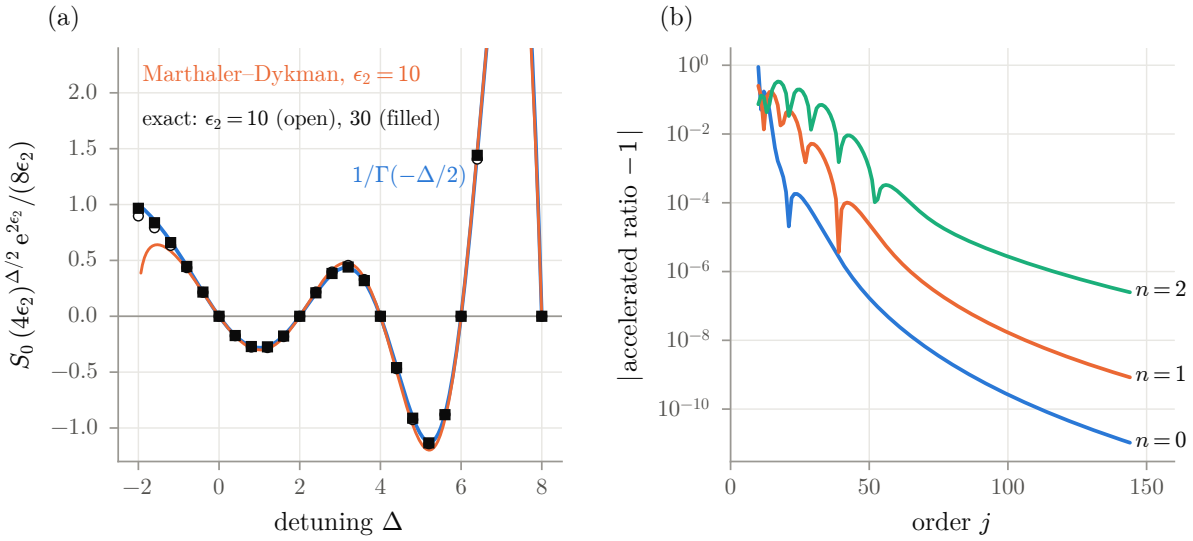

FIGURE 1. (a) Ground-doublet splitting divided by $8\epsilon_2(4\epsilon_2)^{-\Delta/2}e^{-2\epsilon_2}$, that is, by all of (7) except $1/\Gamma(-\Delta/2)$, so that the leading term is $1/\Gamma(-\Delta/2)$ (blue). Points: exact diagonalization at $\epsilon_2 = 10$ (open) and $\epsilon_2 = 30$ (filled). Orange: the Marthaler–Dykman formula [15] at $\epsilon_2 = 10$, in the same normalization. The zeros at $\Delta = 0, 2, 4, 6, 8$ are exact. (b) Convergence of the perturbative coefficients to the large-order prediction (16) at $\Delta = 1$: the ratio $c_j^{(n)}$/(right side of (16)) after order-6 Richardson acceleration in j, minus 1, in absolute value.

Relation to the WKB formula. Marthaler and Dykman [15] give the splitting for detuning large compared with the Kerr constant, in the scaled Hamiltonian $g_{\rm MD} = \frac{1}{4}(P^2+Q^2)^2 + \frac{1}{2}(1-\mu)P^2 - \frac{1}{2}(1+\mu)Q^2$ with $[P,Q] = -i\lambda$. With a written in quadratures scaled so that $\lambda = 1/(2\epsilon_2)$, the classical energy of (1) is $4\epsilon_2^2 g_{\rm MD}$, with $\mu = \Delta/(2\epsilon_2)$ classically. Following [20, eq. (S13)], we use $\mu = (\Delta+2)/(2\epsilon_2)$, where the shift reflects operator ordering; this identification is confirmed by the comparison with exact splittings at large Δ in Table 3. Their eq. (16) (for $\Delta > -2$) then reads

(15)
$$S_0^{\rm MD} = -\frac{64\epsilon_2^2}{\sqrt{\pi(\Delta+2)}}\,(1+\mu)^{5/4}\sin\frac{\pi\Delta}{2}\,e^{-2\epsilon_2\mathcal{A}(\mu)}, \qquad \mathcal{A}(\mu) = \sqrt{1+\mu} + \mu\log\frac{1+\sqrt{1+\mu}}{\sqrt{\mu}},$$

which reproduces the oscillating factor and the scale of the exact splitting (Figure 1(a)). At fixed Δ and $\epsilon_2 \to \infty$, (15) is (7) for $n = 0$ with $\Gamma(1+\Delta/2)$, which enters through $1/\Gamma(-\Delta/2) = -\sin(\pi\Delta/2)\Gamma(1+\Delta/2)/\pi$, replaced by its Stirling approximation $\sqrt{2\pi/\nu}\,(\nu/e)^\nu$, $\nu = 1 + \Delta/2$ [16, (5.11.3)]. The two agree as $\Delta \to \infty$ and differ by 14% at $\Delta = -1$ and 5% at $\Delta = 1$.

Conversely, the leading term of (7) alone converges slowly at large Δ, where the Marthaler–Dykman formula, which keeps Δ/ϵ_2 fixed, does better. The corrections of Section 6 remove this limitation (Table 3). Figure 1(a) shows the oscillating factor $1/\Gamma(-\Delta/2)$, whose amplitude grows like $\Gamma(1+\Delta/2)$, against exact data.

4. Large order and the Borel plane

Terminology. For a series $\sum_j c_j x^j$ with factorially growing c_j, the Borel transform $\sum_j c_j \zeta^j/\Gamma(j+\rho)$, for a suitable shift ρ, is continued from the origin and Laplace-transformed back along a ray. When a singularity lies on the ray, the integrals just above and below it (the lateral sums $\mathcal{S}_\pm$) differ by an exponentially small amount, the Stokes jump, and their average is the median sum $\mathcal{S}_{\mathrm{med}}$. A transseries is a sum of such series multiplied by distinct exponentials.

Result 4.1 (numerical, with the prefactor derived from Result 3.1)**.** For fixed n and $\Delta \notin \{2n, 2n+2, \dots\}$, the coefficients of (4) satisfy, as $j \to \infty$,

$$c_j^{(n)} \simeq -\frac{\mathrm{e}^{2\Delta}}{\left(n!\,\Gamma(n-\Delta/2)\right)^2}\,\Gamma(j+4n-\Delta)\,4^{-j}\big(1+O(j^{-1})\big). \tag{16}$$

The Borel transform in $x = 1/\alpha^2$ is therefore singular at $\zeta = 4$, on the positive real axis, so the series is not Borel summable along it; the exponent and prefactor are those of the instanton–anti-instanton pair, $\mathrm{e}^{-4\alpha^2} = \mathrm{e}^{-4\epsilon_2 - 2\Delta}$, whose amplitude is the square of (14). Equivalently, the half-jump $|\operatorname{Im}\mathcal{S}_\pm|$ of the Borel sum of $\sum_j c_j^{(n)} x^j$ is $\pi(\delta\kappa_\sigma)^2$ at leading order (where $\delta\kappa_\sigma^2$ and $(S_n/(8\alpha^2))^2$ agree); this is the cancellation between the Borel ambiguity and the imaginary part of the instanton–anti-instanton contribution familiar from the double well [23, 24], checked directly in Section 5. Numerically, from the orders $j \le 150$ at $\Delta \in \{-1, \frac{1}{2}, 1, \frac{3}{2}\}$, Richardson extrapolation gives the action 4 to within 10^{-15}, 3×10^{-12}, 4×10^{-10}, and 3×10^{-8} for $n = 0, 1, 2, 3$, and the exponent $4n-\Delta$ to within 5×10^{-15}, 2×10^{-11}, 4×10^{-9}, and 3×10^{-7}. With these values fixed, the extrapolated prefactor agrees with (16) to a relative error below 2×10^{-17}, 9×10^{-14}, and 2×10^{-11} for $n = 0, 1, 2$, respectively; for $n = 0$ the agreement holds at fourteen values of $\Delta \in [-\frac{3}{2}, 5]$, with relative errors below 10^{-10}, comparable to the extrapolation error estimate (5×10^{-10} at $\Delta = 5$). Figure 1(b) shows the convergence for $n = 0, 1, 2$.

Convergence at a degeneracy point. The prefactor of (16) vanishes for $\Delta \in \{2n, 2n+2, \dots\}$. At $\Delta = 0$ the $n = 0$ series is identically zero beyond c_0 (the ground energy is exactly $-\epsilon_2^2$, by the factorization of [17]), and at $\Delta = 2$ the lowest two doublets are eigenvalues of finite matrices whose entries are polynomial in ϵ_2 [20, 18], so the ground energy is an algebraic function of ϵ_2 and its expansion at large ϵ_2 converges. Convergence rests on this argument; the computed $c_j^{(0)}$ at $\Delta = 2$ are consistent with it in showing no factorial growth, with $|c_j^{(0)}|^{1/j} \le 1.06$ for $j \le 150$ (it equals 0.87, 1.00, 1.04, 1.05 at $j = 30, 90, 120, 150$). This parallels [13]: at these points the leading Stokes jump, proportional to $1/\Gamma(-\Delta/2)^2$, vanishes, and the series converges.

5. Median resummation and the two-instanton term

Expanding (13) to second order in $\delta\kappa$, with $(-1)^{-\kappa} = (-1)^n \mathrm{e}^{\pm \mathrm{i}\pi\delta\kappa}$ (the sign depending on the side of the Stokes line), gives

$$\delta\kappa = \delta\kappa_\sigma + (\delta\kappa_\sigma)^2\big[2\log(4\epsilon_2) - \psi(n-\Delta/2) - \psi(n+1) \pm \mathrm{i}\pi\big] + \cdots . \tag{17}$$

The second-order term does not depend on σ: it shifts the center of the doublet. Its real part is the same on either side of the Stokes line, so the median prescription is unambiguous. Its imaginary part cancels the Stokes jump of the perturbative series (whose large order is that of Section 4; compare [4] for anharmonic oscillators), and its real part is a two-instanton correction that the median sum does not contain.

Result 5.1 (derived at leading order; confirmed numerically)**.** Let $h_{\mathrm{c}} = (\frac{1}{2}(E_{\mathrm{even},0}+E_{\mathrm{odd},0})+\alpha^4)/(4\alpha^2)$ be the center of the ground doublet in the units of (4), and $\delta_h = S_0/(4\alpha^2)$. Then, with $\mathcal{S}_{\mathrm{med}}$ the median Borel sum of $\sum_{j\ge1} c_j^{(0)}\alpha^{-2j}$,

$$h_{\mathrm{c}} - \mathcal{S}_{\mathrm{med}} = \Big(\frac{\delta_h}{2}\Big)^2\Big[2\log(4\epsilon_2) - \psi(-\Delta/2) + \gamma_{\mathrm{E}} + O(\epsilon_2^{-1}\log\epsilon_2)\Big], \tag{18}$$

and the half-jump $|\operatorname{Im}\mathcal{S}_{\pm}|$ of the lateral sums is $\pi(\delta_h/2)^2(1+O(\epsilon_2^{-1}))$.

Numerical confirmation. We use $\Delta = \pm\frac{1}{2}$, the coefficients $c_2^{(0)},\dots,c_{150}^{(0)}$ ($c_1^{(0)}=0$), and the Borel transform $\mathfrak{B}(\zeta) = \sum_{j\ge2} c_j^{(0)}\zeta^j/\Gamma(j-\Delta)$ (for which (16) makes the leading singularity a simple pole) in the Laplace integral $\alpha^{-2\Delta}\int_0^{\infty e^{\pm i\theta}} e^{-\zeta\alpha^2}\zeta^{-\Delta-1}\mathfrak{B}(\zeta)\,d\zeta$, replacing $\mathfrak{B}(\zeta)/\zeta^2$ by its [74/73] Padé approximant (which uses 148 of the coefficients, $c_2^{(0)},\dots,c_{149}^{(0)}$) along the rays $\theta = 0.1$. Table 2 and Figure 2 show the results. The median sum reproduces h_{c} to an error of the size of $(\delta_h/2)^2$, which is far below δ_h; the remaining difference is the bracket in (18), and the deviation from its leading value, multiplied by ϵ_2, grows slowly, consistent with $O(\epsilon_2^{-1}\log\epsilon_2)$. The $\psi(-\Delta/2)$ term accounts for most of the difference between $\Delta=\frac{1}{2}$ and $\Delta=-\frac{1}{2}$ (7.22 predicted at leading order, 6.98 observed at $\alpha^2=12$).

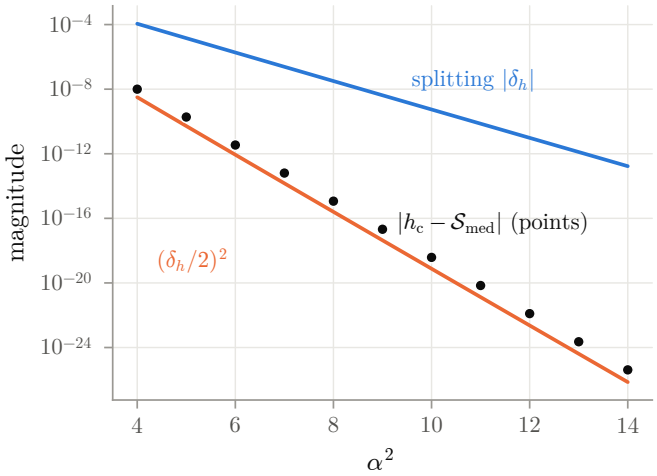

FIGURE 2. Ground doublet at $\Delta=\frac{1}{2}$: the splitting $|\delta_h|$, the two-instanton scale $(\delta_h/2)^2$, and the difference between the exact doublet center and the median Borel–Padé sum (points), as functions of $\alpha^2 = 4,5,\dots,14$, with the method of Table 2 (which lists $\alpha^2 = 6,8,10,12$).

6. Corrections to the splitting and a perturbative/non-perturbative relation

Write

$$S_n = \frac{8\epsilon_2\,(4\epsilon_2)^{2n-\Delta/2}\,e^{-2\epsilon_2}}{n!\,\Gamma(n-\Delta/2)}\,\mathcal{K}_n, \qquad \mathcal{K}_n = 1 + \frac{k_1}{\epsilon_2} + \frac{k_2}{\epsilon_2^2} + \frac{k_3}{\epsilon_2^3} + \cdots. \tag{19}$$

Δ	α^2	$\lvert\delta_h\rvert$	$\lvert h_{\rm c}-\mathcal{S}_{\rm med}\rvert$	$\dfrac{h_{\rm c}-\mathcal{S}_{\rm med}}{(\delta_h/2)^2}$	leading bracket of (18)
$\frac{1}{2}$	6	1.86×10^{-6}	3.47×10^{-12}	3.995	3.934
	8	3.18×10^{-8}	1.16×10^{-15}	4.586	4.531
	10	5.51×10^{-10}	3.82×10^{-19}	5.040	4.990
	12	9.64×10^{-12}	1.26×10^{-22}	5.409	5.363
$-\frac{1}{2}$	6	4.63×10^{-6}	5.82×10^{-11}	10.876	11.242
	8	9.07×10^{-8}	2.37×10^{-14}	11.512	11.798
	10	1.75×10^{-9}	9.21×10^{-18}	11.997	12.232
	12	3.35×10^{-11}	3.48×10^{-21}	12.388	12.588

TABLE 2. Median Borel–Padé sum of the ground-state series versus the exact center of the ground doublet.

Let $F_n(g)$, $g=1/\epsilon_2$, be the perturbative value of $-\kappa$ (Section 3); that is, $F_n=(E_n^{\rm pert}+\epsilon_2^2+\Delta\epsilon_2)/(4\epsilon_2)$ re-expanded in g. From (4),

$$F_n = n + g\Big(-\frac{n^2}{2}+\frac{\Delta n}{4}-\frac{\Delta^2}{16}\Big)+g^2\Big(-\frac{n^3}{4}+\frac{3\Delta n^2}{16}+\frac{\Delta n}{16}-\frac{\Delta^2 n}{32}-\frac{\Delta^2}{64}\Big)+O(g^3), \tag{20}$$

where the dependence on n is polynomial at every order, so that $\partial F_n/\partial n$ is defined.

Result 6.1 (numerical)**.** Define $A_n(g)$ by

$$\mathcal{K}_n = \frac{\partial F_n}{\partial n}\,\mathrm{e}^{-A_n(g)/2+2/g}. \tag{21}$$

Then, with $B=2n-\Delta/2$,

$$\frac{\partial F_n}{\partial n} = -\frac{g}{4}\Big(2B+g\,\frac{\partial A_n}{\partial g}\Big), \qquad A_n=\frac{4}{g}+O(g), \tag{22}$$

through the orders tested. At order g, (22) only restates $f_1=-B/2$ (below), which follows from $c_1^{(n)}$; its content begins at order g^2, together with the absence of logarithmic and constant terms in A_n. Equivalently, if $\partial F_n/\partial n=\sum_{j\ge0}f_j g^j$, then

$$\mathcal{K}_n = \frac{\partial F_n}{\partial n}\,\exp\Big(2f_2\,g+f_3\,g^2+\tfrac{2}{3}f_4\,g^3+\cdots\Big). \tag{23}$$

Relation (22) has the form found by Dunne and Ünsal for genus-one potentials in quantum mechanics [7, 8], where it was verified order by order and used to determine the fluctuations about the instanton from the perturbative energy: it is [8, eq. (108)] $\partial E/\partial B=-(g^2/S)(2B+g^2\partial A/\partial g^2)$, under the identification $g^2\to g$, $S=2$ (the instanton action: $\mathrm{e}^{-2\epsilon_2}=\mathrm{e}^{-S/g}$, with Borel singularity at $2S=4$), and $E\to 4F_n=(E_n^{\rm pert}+\epsilon_2^2+\Delta\epsilon_2)/\epsilon_2$ (since $\partial/\partial B=\frac{1}{2}\partial/\partial n$). Here the eigenproblem is not of Schrödinger type, and the role of B is played by the exponent of $4\epsilon_2$ in (7). From (23) and the perturbative coefficients through $c_4^{(n)}$,

$$k_1 = -\tfrac{3}{2}n^2-n+\tfrac{3}{8}(2n+1)\Delta-\tfrac{1}{16}\Delta^2, \tag{24}$$

$$k_2\big|_{n=0} = \tfrac{1}{512}\Delta^4-\tfrac{1}{64}\Delta^3-\tfrac{5}{128}\Delta^2+\tfrac{3}{32}\Delta, \tag{25}$$

$$k_3\big|_{n=0} = -\tfrac{1}{24576}\Delta^6+\tfrac{1}{4096}\Delta^5+\tfrac{23}{6144}\Delta^4+\tfrac{1}{1024}\Delta^3-\tfrac{25}{384}\Delta^2+\tfrac{3}{64}\Delta, \tag{26}$$

and, for general n,

$$512\,k_2 = \Delta^4 - 8(3n+1)\Delta^3 + 4(48n^2+20n-5)\Delta^2 - 16(36n^3+12n^2-14n-3)\Delta + 64n(9n^3+2n^2-6n-1), \tag{27}$$

while k_3 is a polynomial of total degree 6 in (n, Δ) (given in the code).

Numerical confirmation. We extract k_1, k_2, k_3 from exact splittings at $\epsilon_2 = 16, 18, \ldots, 50$ by Richardson extrapolation of $\epsilon_2(\mathcal{K}_n - 1)$ and its successors.

- k_1: the extracted values for $n = 0, 1, 2$ at $\Delta \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 3, 5\}$ agree with (24) to within 10^{-13} ($n = 0$), 3×10^{-10} ($n = 1$), and 3×10^{-8} ($n = 2$).
- k_2: for $n = 0$ and $n = 1$ the extracted values agree with (23) to within 10^{-10} ($n = 0$) and 2×10^{-7} ($n = 1$). For $n = 2$, which was not used in finding (22), the prediction (28.166016, 21.972778, 12.414185, 8.884766, 6.084106 at $\Delta = -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}$) agrees with the extracted values to within 1.1×10^{-5}.
- k_3 for $n = 0$: the prediction (-0.1094970703, -0.03960990906, 0.007524490356, -0.01330566406, -0.05253410339 at the same Δ) agrees to all ten digits with a separate extraction from the exact splittings, which uses the closed forms of k_1 and k_2 (so it tests k_3 given k_1, k_2).

Table 3 compares the resulting approximations for the ground doublet with exact diagonalization.

Δ	ϵ_2	leading (7)	with k_1, k_2, k_3: ratio $-$ 1	Marthaler–Dykman (15)	uniform
-1	10	1.04715	1.8×10^{-5}	0.948818	1.10615
	20	1.02268	9.9×10^{-7}	0.901911	1.05147
	30	1.01494	1.9×10^{-7}	0.886908	1.03398
$\frac{1}{2}$	10	0.982755	-1.0×10^{-7}	1.02685	1.09629
	20	0.991392	-6.9×10^{-9}	0.982622	1.04907
	30	0.994264	-1.4×10^{-9}	0.967496	1.03292
1	10	0.969325	1.4×10^{-6}	1.03507	1.09339
	20	0.984518	8.7×10^{-8}	0.992399	1.04831
	30	0.989647	1.7×10^{-8}	0.977503	1.03258
3	10	0.94981	-1.8×10^{-5}	1.048	1.08334
	20	0.973444	-1.1×10^{-6}	1.01141	1.04552
	30	0.981956	-2.3×10^{-7}	0.99765	1.03129
7	10	1.06167	8.9×10^{-4}	1.04899	1.06856
	20	1.02672	5.6×10^{-5}	1.02171	1.04077
	30	1.0168	1.1×10^{-5}	1.01014	1.02899
11	10	1.41943	0.012	1.04479	1.05826
	20	1.19086	7.8×10^{-4}	1.02372	1.03692
	30	1.12305	1.5×10^{-4}	1.01392	1.02699

TABLE 3. Approximations to S_0 divided by its exact value (for the fourth column, that ratio minus 1). "Uniform" is the Marthaler–Dykman formula [15] multiplied by $\Gamma(1 + \Delta/2)$ over its Stirling approximation, which reduces to the leading term of (7) at fixed Δ and to [15] as $\Delta \to \infty$.

7. Discussion

For the Kerr-cat qubit, the splitting of the lowest doublet is the coherent tunneling rate between the two cat basis states, and its zeros at even detuning are the protected operating points of [20]. Result 3.1 gives the exact leading behavior at fixed detuning, including the small and negative detunings where the WKB formula of [15] is not accurate, and Result 6.1 supplies the corrections from perturbation theory alone. With three correction terms, the relative error is below 2×10^{-5} for $-1 \leq \Delta \leq 3$ already at $\epsilon_2 = 10$, and below 2×10^{-4} up to $\Delta = 11$ at $\epsilon_2 = 30$ (Table 3).

Structurally, the problem behaves like a quantum-mechanical double well, with the perturbative series, the splitting, and the two-instanton term tied together by a quantization condition, although the Bargmann eigenproblem (6) is not a Schrödinger equation and the wells are regular singular points of it. The exact degeneracies, which are algebraic in origin [20, 18], appear in this picture as zeros of a connection coefficient, at which the Stokes jump vanishes and the series converges.

Two steps remain open. The relation (22) is established here only numerically, through the second correction for $n \leq 2$ and the third for $n = 0$; a derivation, for instance by exact WKB analysis of (6), would also give the $O(\epsilon_2^{-1})$ amplitude corrections in (13) and the corrections to (18). And the degeneracy points themselves, where $\beta - \kappa = n - \Delta/2$ is a non-positive integer and the connection coefficient vanishes, are described here only by the vanishing of the leading amplitude.

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