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THE DIVISOR CROSS-ENERGY HAS ALL ITS ZEROS ON THE CRITICAL LINE. GENEALOGY OF THE ZEROS OF THE REFLECTED CROSS-ENERGY AND THE EPSTEIN CONTROL

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Abstract. Cut by divisors, the reflected cross-energy of the Riemann zeta function has all its zeros on the critical line. For every m, with $M = m\,\mathrm{rad}(m)$, the divisor cross-energy $D_m(s) = M^{s-1}\sigma_{-s}(m) + M^{-s}\sigma_{s-1}(m)$ has all its zeros on $\mathrm{Re}\, s = \frac{1}{2}$ (Theorem 6). The Euler product factorizes the cut over the primes, and the Eneström–Kakeya theorem, applied prime by prime, gives $|E_m(s)| > |E_m(1-s)|$ for $\mathrm{Re}\, s > \frac{1}{2}$; the ratio $\sqrt{p}$ between consecutive coefficients is what places the line at $\frac{1}{2}$. With the coefficients of the Epstein zeta function of x^2+5y^2 on the same divisors (pole, functional equation, nonnegative coefficients, no Euler product) there are 12 zeros off the line. The cut by size behaves differently. The reflected cross-energy $C_2^{\mathrm{FR}}(N,s) = N^{-s}X_{N-1}(1-s) + N^{s-1}X_{N-1}(s)$, $X_N(s) = \sum_{n\le N} n^{-s}$, is a finite sum of hyperbolic cosines with positive coefficients, and for $N = 2$ all its zeros lie on the critical line. As N grows, zeros leave the line. For ζ they are transient: they sit at the heights of the zeros of ζ and their distance to the line decays like $1.1/\sqrt{N}$ (0.144 at $N \approx 40$, 0.033 at $N = 1200$). For the Davenport–Heilbronn function the same construction keeps a permanent zero at $0.81 + 85.70i$, and for the Epstein function it keeps permanent zeros near $\sigma = 0.9$. The passage from the divisor cut to the size cut is the convergence of partial Euler products in the strip, which is known to be equivalent to the Riemann hypothesis. The translation law at the zeros, $\lambda_\rho X_n(\rho)X_n(1-\rho) = n + \frac{1}{2} + \frac{\lambda_\rho}{12n} + O(n^{-2})$ with $\lambda_\rho = \rho(1-\rho)$, holds for every zero on or off the line, and the Epstein function realizes it with a complex slope: pole, functional equation, positivity and the translation law are compatible with zeros off the line. We add three tools: the exact recursion $E_{N+1} = (\frac{N+1}{N})^{s-1}(E_N + \frac{1}{N})$ of the Hermite–Biehler pair; a ladder that brackets each zero from the previous ones, $R_k^{-1/k} \le \lambda_{n+1} \le R_k/R_{k+1}$; and the correction of the harmonic-function criterion of [Ca17], whose limit is $\frac{1}{2\lambda}$ and not 0, which moves its zero-finding algorithm from 25.0083, 30.4356 to 25.0109, 30.4248 (true 25.010858, 30.424876). We also write the logarithm of the symmetrized Euler product as a cross-energy over the primes, $\sum_p 2p^{-1/2}\cosh((s-\frac{1}{2})\log p)$, and locate its singularities on a ladder of lines $\mathrm{Re}\, s = \frac{1}{2k}, 1-\frac{1}{2k}$. Finally, the arithmetic part of C_2^{FR} is $\zeta(s)$ times an explicit nonvanishing factor, so C_2^{FR} decides the Riemann hypothesis only through a Turán-type criterion for a finite sum T_N, whose zeros, unlike those of the partial sums, stay out of $\sigma > \frac{1}{2}$ up to $t \approx 3$–$6N$; and $\zeta(s) = \frac{1}{2}s(s+1)(J(s) - \frac{1}{\lambda_s})$ with J the Mellin transform of the positive weight $\{x\}(1-\{x\})$, so that the zeros are the points where $J = 1/\lambda$.

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1. Introduction

1.1. **Context.** The reflected cross-energy of [CE32] is

$$C_2^{\mathrm{FR}}(N,s) = X_N(s)X_N(1-s) - X_{N-1}(s)X_{N-1}(1-s) - \frac{1}{N} = N^{-s}X_{N-1}(1-s) + N^{s-1}X_{N-1}(s), \tag{1}$$

with $X_N(s) = \sum_{k\le N} k^{-s}$. Its closed form is

$$C_2^{\mathrm{FR}}(N,s) = \frac{1}{\lambda_s} - \frac{1}{N} - \frac{1}{12N^2} + \frac{2-3\lambda_s}{240N^4} + \zeta(s)N^{s-1} + \zeta(1-s)N^{-s} + O\big((1+|s|)^5 N^{-6}\big), \tag{2}$$

$\lambda_s = s(1-s)$. The closed form shows that the arithmetic content of C_2^{FR} is the pair $(\zeta(s), \zeta(1-s))$, that C_2^{FR} collapses to $\frac{1}{\lambda_\rho} - \frac{1}{N} + \cdots$ at every zero, and that at a zero on the line it changes sign once, at $N^* = \lambda_\rho + \frac{1}{12} + \frac{1}{180\lambda_\rho}$. In [CE42, Sect. 4] it was found that $C_2^{\mathrm{FR}}(N,\cdot)$ has zeros off the line. This paper studies those zeros as a function of N, compares ζ with two controls, and asks where the Euler product enters.

1.2. **Results and their status.**

(1) *Hermite–Biehler form and recursion* (Section 2). $C_2^{\mathrm{FR}} = E_N(s) + E_N(1-s)$ with $E_N(s) = N^{s-1}X_{N-1}(s)$, and $E_{N+1} = (\frac{N+1}{N})^{s-1}(E_N + \frac{1}{N})$ (Lemma 1: proved).
(2) *Genealogy* (Section 3). Off-line zeros of C_2^{FR} for ζ are transient with excursion $\approx 1.1/\sqrt{N}$; for the Davenport–Heilbronn function one is permanent (Table 1, Figure 1: numerical; the mechanism, Proposition 2: proved to first order).
(3) *Coefficient families* (Section 4). Positivity or monotonicity of the coefficients does not force the excursions to shrink (Table 2: numerical).
(4) *Translation law* (Section 5). Proposition 3 (proved) holds at every zero, on or off the line.
(5) *The Epstein control* (Section 6). Off-line zeros of the Epstein zeta function of x^2+5y^2, the translation law with complex slope, and permanent off-line zeros of its C_2^{FR} (Tables 3–4: numerical; the zeros are verified to 10^{-10}).
(6) *The divisor cross-energy* (Section 7). Theorem 6 (proved); comparison with the Epstein coefficients (numerical).

(7) *The bridge* (Section 8). The passage from divisors to sizes is the convergence of partial Euler products in the strip (Proposition 8: proved reduction; the equivalence with RH is classical [Co, KKK]).
(8) *The λ-ladder* (Section 9). Proposition 9 (proved) and Table 5 (numerical).
(9) *The harmonic-function criterion of* [Ca17] (Section 10). Proposition 10 (proved) corrects the limit; Table 6 (numerical); Proposition 11 (proved) identifies the off-line oscillation with $\zeta(s)$.
(10) *What C_2^{FR} can decide* (Section 12). Lemma 13 and Proposition 14 (proved) with the zeros of T_N (numerical); the nodal count of C_2 (numerical, with the argument principle); Proposition 15 (proved).
(11) *The prime cross-energy* (Section 11). The identity (5) (classical, by Möbius inversion) and the ladder of singular lines (Proposition 12: proved).

What is classical: the Eneström–Kakeya theorem [Ka, PS]; the Hermite–Biehler theorem [Le]; the Davenport–Heilbronn examples, including the Epstein zeta function of x^2+5y^2 [DH]; the representation of the prime zeta function by $\log\zeta(ks)$ and its singularities [LW]; the equivalence between convergence of partial Euler products in $\operatorname{Re} s>\frac12$ and the generalized Riemann hypothesis for $L(s,\chi)$, $\chi\neq1$ [Co, KKK]; the hybrid Euler–Hadamard product [GHK]; the sums $\sum_\rho\lambda_\rho^{-k}$ [Vo]. What is new: the recursion and the genealogy of the zeros of C_2^{FR} with the $1/\sqrt{N}$ law and the comparison with two controls; the use of the Epstein function of x^2+5y^2, with its complex off-line zeros, as a control for the translation law and for C_2^{FR} (the Epstein control itself is in [CE41, Sect. 2], with m^2+100n^2 and a real zero, for C_2 and its trichotomy); the divisor cross-energy and Theorem 6; the reduction of Section 8; the ladder as a bracket of each zero by the previous ones; the correction of [Ca17]; the reading of $\log(P(s)P(1-s))$ as a cross-energy over the primes and the band formula (6).

1.3. **Related work by the author.** This paper is Part 43 of the series *Cross-Energy of Dirichlet Partial Sums* [CE0]. The reflected cross-energy and its closed form are in [CE32], the Hurwitz route in [CE21], the controls (Davenport–Heilbronn, Beurling systems, function fields) in [CE46], the Epstein control for C_2 in [CE41], and the search for square roots of Weil's form, where the off-line zeros of C_2^{FR} first appeared, in [CE42]. The series is surveyed in [CE0]. Each of these is a self-contained preprint; the present paper can be read without them.

2. The Hermite–Biehler form of C_2^{FR}

Put $s=\frac12+z$. Since $N^{s-1}k^{-s}=(Nk)^{-1/2}(N/k)^z$ and $N^{-s}k^{s-1}=(Nk)^{-1/2}(N/k)^{-z}$,

$$C_2^{\mathrm{FR}}(N,\tfrac12+z)=\frac{2}{\sqrt{N}}\sum_{k<N}k^{-1/2}\cosh\Big(z\log\frac{N}{k}\Big). \tag{3}$$

It is an even entire function of z, real on the imaginary axis, with positive coefficients $k^{-1/2}$ that increase with the frequency $\log(N/k)$. For $N=2$ it is $\sqrt{2}\cosh(z\log2)$, whose zeros $z=i\pi(m+\frac12)/\log2$ all lie on the critical line.

Write $E_N(s)=N^{s-1}X_{N-1}(s)$. Then $C_2^{\mathrm{FR}}(N,s)=E_N(s)+E_N(1-s)$, the shape of a Hermite–Biehler pair: if $|E_N(s)|>|E_N(1-s)|$ for $\sigma>\frac12$, all zeros of $C_2^{\mathrm{FR}}(N,\cdot)$ lie on $\sigma=\frac12$ [Le, Ch. VII].

Lemma 1 (Recursion)**.** *$E_1=0$ and, for $N\geq1$,*

$$E_{N+1}(s)=\Big(\frac{N+1}{N}\Big)^{s-1}\Big(E_N(s)+\frac{1}{N}\Big).$$

At a zero ρ of ζ, $E_N(\rho)\to\frac{1}{1-\rho}$, which is the asymptotic fixed point of the recursion for every s; in general $E_N(s)-\big(\frac{1}{1-s}-\frac{1}{2N}-\frac{s}{12N^2}+\cdots\big)=\zeta(s)N^{s-1}$, so $\zeta(s)$ is the amplitude of the homogeneous mode N^{s-1}.

Proof. $E_N(s)+\frac{1}{N}=N^{s-1}\big(X_{N-1}(s)+N^{-s}\big)=N^{s-1}X_N(s)$, and multiplying by $(\frac{N+1}{N})^{s-1}$ gives $(N+1)^{s-1}X_N(s)=E_{N+1}(s)$. The second statement is the Euler–Maclaurin expansion $X_{N-1}(s)=\zeta(s)+\frac{N^{1-s}}{1-s}-\frac{1}{2}N^{-s}-\frac{s}{12}N^{-s-1}+\cdots$ multiplied by N^{s-1} [CE32]. The fixed point: with $r=1+\frac{1}{N}$, $r^{s-1}=1+\frac{s-1}{N}+O(N^{-2})$, so $E=r^{s-1}(E+\frac{1}{N})$ gives $(s-1)E+1=0$ to first order. □

The Hermite–Biehler inequality for E_N fails: C_2^{FR} has zeros off the line [CE42, Sect. 4]. The rest of this paper measures how, and why.

3. Genealogy of the zeros

For each N we locate the zeros of (3) with $0.004\le\operatorname{Re}z\le 0.45$ and $5<\operatorname{Im}z<100$ (grid minima of $|C_2^{\mathrm{FR}}|$ refined by Newton's method to $|C_2^{\mathrm{FR}}|<10^{-9}$). By symmetry it suffices to take $\operatorname{Re}z>0$. The same construction applies to any real coefficient sequence a_k, replacing $k^{-1/2}$ by $a_kk^{-1/2}$; for the Davenport–Heilbronn function f the coefficients are periodic modulo 5, $(a_0,\dots,a_4)=(0,1,\kappa_0,-\kappa_0,-1)$ [CE13, Sect. 9].

Table 1. Off-line zeros of $C_2^{\mathrm{FR}}(N,\cdot)$, $5<t<100$. Maximum of σ in each range of N; for f the permanent zero is listed separately.

range of N	ζ: max σ	f: max σ (transient)	f: permanent zero
2–30	0.914	0.940	intermittent
30–60	0.662	0.919	intermittent
60–100	0.608	0.726	0.80–$0.85+85.70i$ from $N=74$
100–160	0.592	0.625	0.79–$0.84+85.70i$

For ζ the off-line zeros appear and disappear from one N to the next, always near the ordinates of zeros of ζ (14.1, 21.0, 25.0, 30.4, 32.9, 37.6, 40.9, 43.3, 48.0, 49.8, …). Restricting to $5<t<60$:

$$\max(\sigma-\tfrac{1}{2})=0.144\ (N\approx 40),\quad 0.089\ (N\approx 120),\quad 0.063\ (N=300),$$

0.050 at $N\approx 360$ and 0.033 at $N=1200$, and $\sqrt{N}\max(\sigma-\frac{1}{2})=1.03, 1.09, 1.14$ at $N=130, 300, 1200$: the excursion decays like $1.1/\sqrt{N}$ (Figure 1). The number of off-line zeros in $5<t<60$ stays near one per value of N.

Proposition 2 (Mechanism)**.** *Let $A_N(z)=\zeta(\frac{1}{2}+z)N^z+\zeta(\frac{1}{2}-z)N^{-z}$. Then*

$$C_2^{\mathrm{FR}}(N,\tfrac{1}{2}+z)=\Big(\frac{1}{\lambda}-\frac{1}{N}+\cdots\Big)+N^{-1/2}A_N(z),\qquad A_N(z)=\Xi(z)\Big(\frac{N^z}{G(\frac{1}{2}+z)}+\frac{N^{-z}}{G(\frac{1}{2}-z)}\Big),$$

with $G(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Xi(z)=\xi(\frac{1}{2}+z)$. The second factor has all its zeros on the line for $|t|<2\pi N^2$ approximately (Hermite–Biehler for $E(z)=N^z/G(\frac{1}{2}+z)$). Hence the zeros of the arithmetic part are those of ξ together with a family of points on the line which move with N. At a simple zero ρ on the line, the first-order displacement of the nearby zero of C_2^{FR} is $\delta=-(\frac{1}{\lambda}-\frac{1}{N})/\partial_sC_2^{\mathrm{FR}}(N,\rho)$, which is purely imaginary, so the zero stays on the line; it leaves the line only near a collision with a moving zero, where $\partial_sC_2^{\mathrm{FR}}$ is small, and the size of the perturbation relative to A_N is $N^{1/2}\cdot N^{-1}=N^{-1/2}$.

Proof. The decomposition is (2) with $\zeta(s)=\xi(s)/G(s)$ and $\xi(\frac{1}{2}+z)=\xi(\frac{1}{2}-z)$. For $E(z)=N^z/G(\frac{1}{2}+z)$, $|E(z)/E(-z)|=N^{2x}|G(\frac{1}{2}-z)/G(\frac{1}{2}+z)|\approx(2\pi N^2/t)^x$ by Stirling, which exceeds 1 for $x>0$ when $t<2\pi N^2$. On the line $\frac{1}{\lambda}-\frac{1}{N}$ is real. Up to the Bernoulli terms, $\partial_sC_2^{\mathrm{FR}}(N,\rho)=N^{-1/2}\big(\zeta'(\rho)N^{i\gamma}-\zeta'(1-\rho)N^{-i\gamma}\big)+\frac{2s-1}{\lambda_s^2}\big|_{s=\rho}$. Since $\zeta'(1-\rho)=\zeta'(\bar\rho)=\overline{\zeta'(\rho)}$, the first term is $2iN^{-1/2}\operatorname{Im}\big(\zeta'(\rho)N^{i\gamma}\big)$, and the second is $2i\gamma/\lambda^2$. Both are purely imaginary, so δ is purely imaginary. □

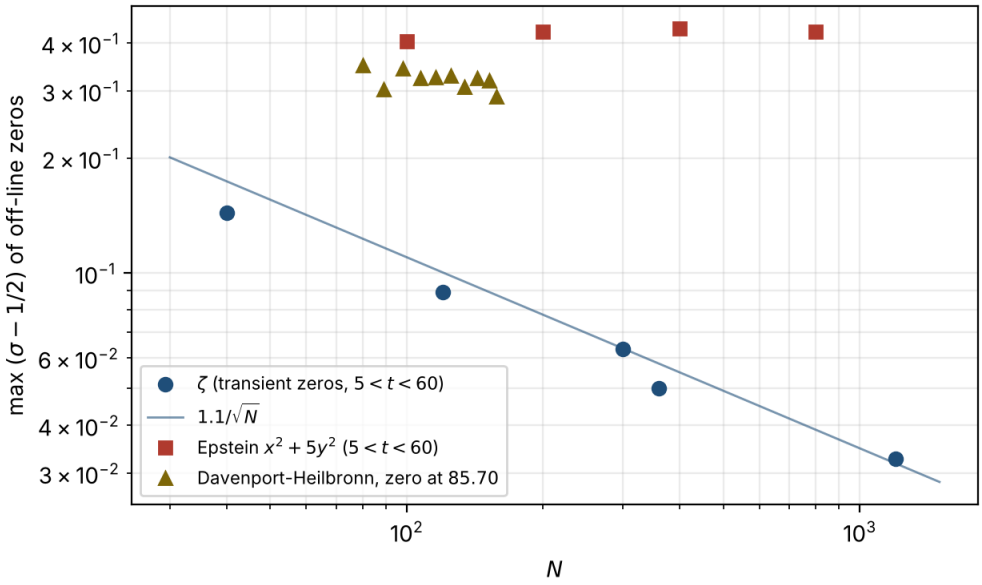

FIGURE 1. Largest distance to the line of the off-line zeros of $C_2^{\mathrm{FR}}(N,\cdot)$. For ζ the zeros are transient and approach the line like $1.1/\sqrt{N}$; for the Davenport–Heilbronn function and for the Epstein zeta function of x^2+5y^2 they do not.

The picture is a stream of zeros moving down the line as N grows (the zeros of the Gamma factor) and fixed rocks (the zeros of ξ). At each passage the term $-\frac{1}{N}$, the shadow of the pole, can push a pair off the line by $O(N^{-1/2})$. A rock off the line, as for f, is not moved by any passage.

4. Coefficient families

To see whether the shrinking is a property of the coefficients, we keep the frequencies $\log(N/k)$ and change a_k.

TABLE 2. $\max(\sigma-\frac{1}{2})$ over off-line zeros of the cosine sum with coefficients $a_k k^{-1/2}$, $5<t<60$, six consecutive values of N.

coefficients a_k	$N\approx 40$	$N\approx 120$	$N\approx 360$
1 (ζ)	0.144	0.089	0.050
Davenport–Heilbronn (no off-line zero below $t=60$)	0.277	0.092	0.075
positive, random in $[1,1.6]$	0.302	0.218	0.259
k (decreasing with the frequency)	0.419	0.085	0.434

Positive coefficients that increase with the frequency up to noise do not shrink. The excursions shrink when, and only when, the limit function has its zeros on the line in the window. The Pólya–Kakeya direction of (3) (coefficients increasing with the frequency) is not the mechanism; the density of frequencies, $\propto e^{-u/2}$ in $u=\log(N/k)$, decreases, and no positive average over N changes this, since each $C_2^{\mathrm{FR}}(N,\cdot)$ contributes a decreasing density.

5. The translation law

Proposition 3 (All zeros are one)**.** *Let ρ be any zero of ζ in the strip and $\lambda = \rho(1-\rho)$. Then*

$$X_n(\rho)X_n(1-\rho) = \frac{n}{\lambda} + \frac{1}{2\lambda} + \frac{1}{12n} + O_\rho(n^{-2}),$$

equivalently $\lambda X_n(\rho)X_n(1-\rho) = n + \frac{1}{2} + \frac{\lambda}{12n} + \cdots$. On the line $X_n(1-\rho) = \overline{X_n(\rho)}$, the left side is $|X_n(\rho)|^2$, $\lambda = \frac{1}{4} + \gamma^2$ is real, and in the variable $u = n/\lambda$ every zero follows the same profile $u + \frac{1}{2\lambda} + \cdots$; at C_2^{FR} level the crossing is $N^* = \lambda + \frac{1}{12} + \frac{1}{180\lambda}$.

Proof. With $X_n(s) = \zeta(s) + \frac{n^{1-s}}{1-s} + \frac{1}{2}n^{-s} - \frac{s}{12}n^{-s-1} + \cdots$ and $\zeta(\rho) = \zeta(1-\rho) = 0$, multiply the expansions at ρ and $1-\rho$: the leading term is $\frac{n}{\rho(1-\rho)}$; the constant term is $\frac{1}{2}\big(\frac{1}{1-\rho} + \frac{1}{\rho}\big) = \frac{1}{2\lambda}$; the n^{-1} terms are $\frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \frac{1}{12}$. □

The proof uses only $\zeta(\rho) = \zeta(1-\rho) = 0$. The line enters only through $X_n(1-\rho) = \overline{X_n(\rho)}$. For a zero off the line the law holds with complex λ, and the rescaling $n \mapsto n\lambda'/\lambda$ between two zeros is real exactly when both λ are real. A law of this kind cannot by itself exclude zeros off the line; Section 6 shows a function that realizes it off the line.

6. The Epstein control

Let $E(s) = \zeta(s)L(s,\chi_{-20}) + L(s,\chi_{-4})L(s,\chi_5)$, a multiple of the Epstein zeta function of x^2+5y^2. It has a simple pole at $s = 1$ with residue $c = L(1,\chi_{-20}) = 2\pi/\sqrt{20}$, the functional equation of a degree-two function of conductor 20, nonnegative coefficients $a(n) = \sum_{d|n}\big(\chi_{-20}(d) + \chi_{-4}(d)\chi_5(n/d)\big)$ $(2,0,0,2,2,4,0,0,6,\dots)$, and no Euler product; it is the classical example of Davenport and Heilbronn [DH]. The paper [CE41, Sect. 2] already used an Epstein function, that of $m^2 + 100n^2$ with a real zero, as a control for C_2; here we need complex zeros off the line, to test the translation law and the zeros of C_2^{FR} at a given height.

Table 3. Zeros of $E(s)$ off the line with $5 < t < 100$ found by grid search and `findroot`, $|E| < 10^{-10}$.

$0.93767 + 29.98340i$	$0.82319 + 44.00011i$	$0.63491 + 46.75841i$	$0.59744 + 52.74327i$
$0.79978 + 61.14026i$	$0.72485 + 64.99194i$	$0.84547 + 75.34001i$	$0.80772 + 79.41954i$
$0.90799 + 89.55983i$	$0.85028 + 96.10717i$		

With $X_n^E(s) = \sum_{k\le n} a(k)k^{-s}$ and $Q_n = X_n^E(\rho)X_n^E(1-\rho)$, the analogue of Proposition 3 holds with c^2:

Table 4. The translation law at two off-line zeros of E: $\lambda Q_n/(c^2 n) \to 1$ and $\arg Q_n \to -\arg\lambda$.

zero	n	$\lambda Q_n/(c^2n)$	$\arg Q_n$	$-\arg\lambda$
$0.938 + 29.98i$	10^4	$1.0063 - 0.0002i$	0.0290	0.0292
	10^5	$1.0004 - 0.0000i$	0.0292	0.0292
	$2\cdot 10^5$	$0.9996 + 0.0000i$	0.0292	0.0292
$0.635 + 46.76i$	10^4	$1.0180 - 0.0001i$	0.0056	0.0058
	$2\cdot 10^5$	$0.9993 + 0.0000i$	0.0058	0.0058

The C_2^{FR} built from $a(n)$ keeps off-line zeros that do not shrink: $\max(\sigma - \frac{1}{2})$ is 0.404, 0.429, 0.436, 0.428 at $N = 100, 200, 400, 800$ for $5 < t < 60$, with zeros at 0.84–$0.92 + 44.0i$ and 0.79–$0.90 + 58.5i$ present at every N tested (Figure 1).

Corollary 4. *A pole at $s = 1$, the functional equation $s \leftrightarrow 1 - s$, nonnegative coefficients and the translation law of Proposition 3 are compatible with zeros off the line. Any argument built from C_2^{FR} that does not use the Euler product cannot prove the Riemann hypothesis.*

7. The divisor cross-energy

The sum X_{N-1} cuts the Dirichlet series by size, $n < N$. With an Euler product one may cut by divisors.

Definition 5. Let $m = \prod_p p^{K_p - 1}$ ($K_p \geq 1$, finitely many $K_p > 1$) and $M = \prod_{K_p \geq 2} p^{K_p} = m \operatorname{rad}(m)$. Put

$$E_m(s) = M^{s-1} \sum_{d \mid m} d^{-s} = \prod_{p \mid m} E_p(s), \qquad E_p(s) = p^{K_p(s-1)} \sum_{k < K_p} p^{-ks},$$

and $D_m(s) = E_m(s) + E_m(1 - s)$, the *divisor cross-energy*. The factorization is the Euler product of ζ restricted to the divisors of m; E_p is the cross-energy of a single prime, C_2^{FR} with $N = p^{K_p}$ and the sum restricted to powers of p.

Theorem 6. *For every m, all the zeros of D_m lie on $\operatorname{Re} s = \frac{1}{2}$.*

Proof. Fix p and $K = K_p$, and put $\zeta_p = p^{s-1/2}$, so that $|\zeta_p| = p^{\sigma - 1/2}$. Then

$$E_p(s) = p^{-K} \sum_{j=1}^{K} p^{j/2} \zeta_p^{\,j} = p^{-K} R(\zeta_p), \qquad E_p(1-s) = p^{-K} R(1/\zeta_p), \qquad R(w) = \sum_{j=1}^{K} p^{j/2} w^j,$$

where $j = K - k$. The coefficients of $R(w)/w = \sum_{j=0}^{K-1} p^{(j+1)/2} w^j$ are positive and increasing, so by the Eneström–Kakeya theorem [Ka, PS] all zeros of R lie in $|w| \leq 1$. Let $R^*(w) = w^K \overline{R(1/\bar{w})}$. Since the zeros of R lie in the closed unit disc, R/R^* is a finite Blaschke-type quotient with $|R(w)| \geq |R^*(w)|$ for $|w| \geq 1$, and R has no zeros in $|w| > 1$. As the coefficients of R are real, $|R(1/w)| = |R(1/\bar{w})| = |w|^{-K} |R^*(w)|$. Hence for $|w| > 1$

$$|R(1/w)| = |w|^{-K} |R^*(w)| < |R^*(w)| \leq |R(w)|,$$

with strict inequality also when $R^*(w) = 0$, since then $R(w) \neq 0$. With $w = \zeta_p$ this is $|E_p(s)| > |E_p(1-s)|$ for $\sigma > \frac{1}{2}$. Multiplying over $p \mid m$, $|E_m(s)| > |E_m(1-s)|$ for $\sigma > \frac{1}{2}$, so $D_m(s) \neq 0$ there; by the symmetry $D_m(s) = D_m(1-s)$, also for $\sigma < \frac{1}{2}$. □

Remark 7. The line is placed at $\frac{1}{2}$ by the ratio $\sqrt{p}$ between consecutive coefficients of R: Eneström–Kakeya puts the zeros of R in $|w| \leq 1$, and $|w| = p^{\sigma - 1/2}$. Primes with $K_p = 1$ may be added as factors p^{s-1}, for which $|p^{s-1}| > |p^{-s}|$ when $\sigma > \frac{1}{2}$.

For $m = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ together with the factors of 11 and 13 ($M = 151\,351\,200$, 60 divisors), on $0.505 \leq \sigma \leq 1.5$, $0.5 \leq t \leq 60$:

coefficients on the divisors	$\min \lvert E(s)\rvert / \lvert E(1-s)\rvert$	off-line zeros found
1 (Euler product of ζ)	1.178	0
$a(d)$ of the Epstein function	0.072	12 (e.g. $0.893 + 1.38i$, $0.546 + 7.17i$)

The Epstein coefficients do not factor over the primes, and the divisor cut loses the inequality. Theorem 6 separates ζ from the Epstein control, which Corollary 4 shows that pole, functional equation and positivity cannot do.

8. The bridge from divisors to sizes

The size cut and the divisor cut agree on $n < N$ when m is a multiple of $\operatorname{lcm}(1, \dots, N-1)$; the size cut is the divisor object with the large divisors removed and the scale M replaced by N.

Proposition 8. (1) *For $\sigma > \frac{1}{2}$ and $K_p \to \infty$, $E_m(1-s)/E_m(s) \to 0$, and $M^{1-s}E_m(s) = \prod_{p|m} \frac{1-p^{-K_p s}}{1-p^{-s}}$ is a truncated partial Euler product.*

(2) *For the size cut, $E_N(s) = \zeta(s)N^{s-1} + \frac{1}{1-s} + O(N^{-1})$; the arithmetic part $F(s) = \zeta(s)N^{s-1}$ has $F(1-s)/F(s) = N^{1-2s}\chi(1-s)$, whose modulus is < 1 for $\sigma > \frac{1}{2}$ and $t < 2\pi N^2$ approximately. The Hermite–Biehler inequality $|F(s)| > |F(1-s)|$ fails exactly where $F(s) = 0$, that is, at zeros of ζ.*

Hence a passage from Theorem 6 to the size cut which keeps the inequality amounts to partial Euler products representing ζ in $\frac{1}{2} < \sigma < 1$.

Proof. (1) $|E_p(1-s)/E_p(s)| = |R(1/w)|/|R(w)| \le |w|^{-K} \to 0$. (2) The functional equation gives $\zeta(1-s) = \chi(1-s)\zeta(s)$ identically, so the quotient is $N^{1-2s}\chi(1-s)$; its modulus is $(2\pi N^2/t)^{1/2-\sigma}$ up to $1 + O(1/t)$ by Stirling. □

For a nontrivial character, convergence of the Euler product of $L(s,\chi)$ in $\operatorname{Re} s > \frac{1}{2}$ is equivalent to the generalized Riemann hypothesis [Co, KKK]; for ζ the pole requires a modified product (the deep Riemann hypothesis). The divisor-to-size passage therefore lands on a known equivalence. A remark on any attempt to bridge the gap by a correction: if $\zeta = P_X \cdot C_X$ with a partial Euler product P_X, then C_X vanishes exactly at the zeros of ζ, since $P_X \neq 0$; and the functional equation fixes only $C_X(s)/C_X(1-s) = \chi(s)P_X(1-s)/P_X(s)$, which is blind to a pair of zeros $\rho, 1-\rho$, where both sides are $0/0$. Numerically $|\log(\zeta/P_X)|$ at $0.52 + 14.13i$ is $2.04, 1.48, 0.65, 1.42$ for $X = 10^2, \dots, 10^5$, and $|\log P_X(s)P_X(1-s) - \log\zeta(s)\zeta(1-s)|$ at $0.75 + 20i$ grows from 0.76 to 114.8 between $X = 10^2$ and 10^6, because $1-s$ lies in the left half of the strip. The hybrid product of [GHK] supplies C_X as a product over the zeros.

9. The λ-ladder

The function $\widehat{\Xi}(\lambda) = \xi(s)$, $\lambda = s(1-s)$, is entire of order $\frac{1}{2}$ in λ, with zeros λ_ρ, one for each pair $\{\rho, 1-\rho\}$, and $\widehat{\Xi}(\lambda)/\widehat{\Xi}(0) = \prod_\rho (1 - \lambda/\lambda_\rho)$. Its Taylor coefficients at $\lambda = 0$ give, by Newton's identities, the power sums

$$\sigma_k = \sum_\rho \lambda_\rho^{-k} \qquad (k \ge 1),$$

without using any zero; $\sigma_1 = \frac{1}{2}(2 + \gamma_E - \log 4\pi) = 0.0230957089661210338\ldots$ (checked to 20 digits). These are values of the zeta function over the zeros studied by Voros [Vo].

Proposition 9 (Ladder)**.** *Let $\lambda_1 \le \lambda_2 \le \cdots$ be the λ_ρ and $R_k^{(n)} = \sigma_k - \sum_{j \le n} \lambda_j^{-k}$. If λ_j is real for every $j > n$, then for every $k \ge 1$*

$$\big(R_k^{(n)}\big)^{-1/k} \le \lambda_{n+1} \le R_k^{(n)}/R_{k+1}^{(n)},$$

and both bounds converge to λ_{n+1} as $k \to \infty$. An off-line pair with $\lambda = |\lambda|e^{i\theta}$, $\theta \approx (2\beta-1)/\gamma$, contributes $2|\lambda|^{-k}\cos k\theta$, which changes sign near $k \approx \pi\gamma/(2(2\beta-1))$, where the bracket fails.

Proof. $R_k^{(n)} = \sum_{j>n} \lambda_j^{-k}$ is the k-th moment of a positive measure on $(0, \lambda_{n+1}^{-1}]$. It is at least λ_{n+1}^{-k}, which gives the lower bound; R_{k+1}/R_k is a weighted mean of the λ_j^{-1}, at most λ_{n+1}^{-1}, which gives the upper bound; it increases with k by Cauchy–Schwarz and tends to the largest atom. □

By the crossing law $\lambda_\rho = N_\rho^* - \frac{1}{12} - \frac{1}{180\lambda_\rho}$, Table 5 is the ladder of the crossing points of C_2^{FR}: each new crossing is fixed, up to the bracket, by the previous ones. The Riemann hypothesis is

TABLE 5. The ladder with $k = 23$: each zero bracketed from the constants σ_k and the previous zeros (computed at 80 digits; Taylor coefficients by a Cauchy integral on $|\lambda| = 120$).

n	lower bound	upper bound	true λ_{n+1}	bracket for γ_{n+1}
0	200.040455	200.040456	200.040455	$[14.1347, 14.1347]$
1	442.1696	442.2202	442.1762	$[21.0219, 21.0231]$
2	625.7896	625.8186	625.7930	$[25.0108, 25.0114]$
3	924.884	929.408	925.923	$[30.408, 30.482]$
4	1084.858	1085.565	1084.968	$[32.933, 32.944]$
5	1411.66	1417.83	1412.97	$[37.569, 37.651]$
6	1669.48	1687.18	1674.59	$[40.856, 41.072]$
7	1876.61	1881.40	1877.49	$[43.317, 43.372]$
8	2286.48	2334.56	2304.74	$[47.81, 48.31]$
9	2471.36	2495.31	2477.68	$[49.71, 49.95]$

equivalent to the residues $R_k^{(n)}$ being Stieltjes moment sequences for all k; a finite ladder excludes only zeros with $\beta - \frac{1}{2} \gtrsim \gamma/k$.

10. The harmonic-function criterion revisited

Part V of [Ca17] uses the exact identity, valid for every real β,

$$|X(\tfrac{1}{2} + i\beta, n)|^2 = H_n + \sum_{k=1}^{n} \sum_{j \neq k} (kj)^{-1/2} \cos\Big(\beta \log \frac{k}{j}\Big), \qquad H_n = \sum_{k \le n} \frac{1}{k}, \tag{4}$$

and states that at a zero $H_n - O(n) - P(n) \to 0$, with $O(n) = n/(\beta^2 + \frac{1}{4})$ and $P(n)$ the double sum with a minus sign.

Proposition 10. *At a zero $\frac{1}{2} + i\beta$,*

$$H_n - O(n) - P(n) = |X_n|^2 - \frac{n}{\lambda} \longrightarrow \frac{1}{2\lambda}, \qquad \lambda = \tfrac{1}{4} + \beta^2.$$

Consequently the comparison between two zeros in [Ca17, (24)] *must be made with $H(\beta, n) + \frac{1}{2\lambda_\beta}$.*

Proof. Proposition 3. Numerically $|X_n|^2 - n/\lambda = 0.002500$ at $n = 10^5, 10^6$ for γ_1 against $\frac{1}{2\lambda} = 0.002499$, and 0.001131 for γ_2 against 0.001131. □

TABLE 6. Algorithm H2 of [Ca17] from the known zero $\beta_1 = 14.134725142$: roots of $H(\beta, n) - H(\beta_1, n)$ (original) and of the corrected comparison.

	$n = 60$ original	$n = 60$ corrected	$n = 400$ corrected	true
β_3	25.0083	25.0109	25.0109	25.010858
β_4	30.4356	30.4248	30.4249	30.424876

Spurious matches at non-zeros remain at finite n (for instance 25.38 and 30.39 at $n = 60$). Off the line, the object that carries the law is $X_n(s)X_n(1-s)$, not $|X_n|^2$. With $x = \alpha - \frac{1}{2}$ and $L = \log(j/k)$,

$$X_n(s)X_n(1-s) = H_n + \sum_{j \neq k} (jk)^{-1/2} \Big(\cosh(xL)\cos(\beta L) + i \sinh(xL)\sin(\beta L) \Big).$$

Proposition 11 (The oscillation is ζ)**.** *For $0 < \sigma < 1$, $s \neq \frac{1}{2} + i\beta$ with $\zeta(1-s) \neq \infty$,*

$$X_n(s)X_n(1-s) = \frac{n}{\lambda_s} + \frac{1}{2\lambda_s} + \zeta(s)\frac{n^s}{s} + \zeta(1-s)\frac{n^{1-s}}{1-s} + \zeta(s)\zeta(1-s) + o(1).$$

For $\sigma > \frac{1}{2}$ the term $\zeta(s)n^s/s$ is the oscillation of order n^σ; it vanishes exactly when $\zeta(s) = 0$.

Proof. Multiply the Euler–Maclaurin expansions at s and $1-s$. Numerically, at $s = 0.7 + 20i$ the residual divided by n^s/s is $0.55728 - 0.88299i$ at $n = 10^6$, against $\zeta(s) = 0.55727 - 0.88300i$. □

So the statement that the $\sinh \cdot \sin$ sum keeps its n^σ oscillation off the line is, word for word, $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$. In [Ca17, Part IV] the step "$|X|^2$ is a straight line if and only if $\operatorname{Re} z = \frac{1}{2}$" uses $|X|^2$, which is the right object only on the line.

11. The cross-energy of the primes

For the partial Euler product, $\log P_X(s)P_X(1-s) = \sum_{p \le X} \left(p^{-s} + p^{s-1}\right) + \cdots$, and

$$p^{-s} + p^{s-1} = 2p^{-1/2}\cosh\left((s - \tfrac{1}{2})\log p\right).$$

This is (3) with frequencies $\log p$ in place of $\log(N/k)$: a cross-energy of the primes. With the prime zeta function $P(s) = \sum_p p^{-s} = \sum_{k \ge 1} \frac{\mu(k)}{k} \log \zeta(ks)$ [LW], let $F(s) = P(s) + P(1-s)$. Möbius inversion gives

$$\log\zeta(s) + \log\zeta(1-s) = F(s) - \sum_{k \ge 2} \frac{\mu(k)}{k}\Big(\log\zeta(ks) + \log\zeta(k(1-s))\Big) \pmod{2\pi i}, \tag{5}$$

checked numerically to $5.8 \cdot 10^{-8}$ at $0.7 + 20i$ and $8.7 \cdot 10^{-10}$ at $0.6 + 7.3i$ with 60 terms. In the strip $P(s)$ is defined by this continuation; $\sum_{p \le X} p^{-s}$ does not converge there unconditionally.

Proposition 12 (Ladder of singular lines)**.** *In $0 < \sigma < 1$, $t \neq 0$, the singularities of F are the points ρ/k and $1 - \rho/k$, k squarefree, ρ a zero of ζ. Under RH they lie on the lines $\operatorname{Re} s = \frac{1}{2}$ and $\operatorname{Re} s = \frac{1}{2k}, 1 - \frac{1}{2k}$ ($k \ge 2$ squarefree). In the band $\frac{1}{2} < \sigma < \frac{3}{4}$,*

$$F(s) = 2\log\zeta(s) - \log\chi(s) - \tfrac{1}{2}\log\zeta(2-2s) - \tfrac{1}{3}\log\zeta(3-3s) + R(s), \tag{6}$$

with $R(s) = \sum_{k \ge 2} \frac{\mu(k)}{k} \log\zeta(ks) + \sum_{k \ge 5} \frac{\mu(k)}{k} \log\zeta(k(1-s))$ absolutely convergent. A zero $\beta + i\gamma$ with $\beta > \frac{1}{2}$ produces a singularity of F at $1 - \rho/2$, with real part $1 - \beta/2 \in (\frac{1}{2}, \frac{3}{4})$; hence, barring exact cancellation of multiplicities, RH holds if and only if F is holomorphic in $\frac{1}{2} < \sigma < \frac{3}{4}$, $t \neq 0$.

Proof. $\log\zeta(ks)$ is singular exactly at $ks \in \{\rho, 1\}$, and $\log\zeta(k(1-s))$ at $k(1-s) \in \{\rho, 1\}$. For $\sigma > \frac{1}{2}$ and $k \ge 2$, $\operatorname{Re} ks > 1$; for $k \ge 5$ and $\sigma < \frac{3}{4}$, $\operatorname{Re} k(1-s) > 1$. The term $k = 1$ of $P(1-s)$ is $\log\zeta(1-s) = \log\zeta(s) - \log\chi(s)$, and $\mu(2) = \mu(3) = -1$. The only singularity of the active terms in the band under RH is the pole of $\zeta(3-3s)$ at $s = \frac{2}{3}$, on the real axis. □

The equivalence is immediate: holomorphy of F in the band is the absence of zeros of $\zeta(2-2s)$ there. What (6) adds is the coupling: the map $s \mapsto 2 - 2s$ sends the thin band $(\frac{1}{2}, \frac{3}{4})$ onto the whole right half $(\frac{1}{2}, 1)$, a hypothetical zero at β reappears at distance $\frac{1-\beta}{2}$ from the line, and the iteration $s \mapsto 1 - \frac{s}{2}$ has its fixed point at $\frac{2}{3}$, the pole of $\zeta(3-3s)$. Since (5) is an identity valid for every configuration of zeros, it cannot by itself exclude any; it would become a constraint only with an independent analytic description of $\sum_p p^{-s}$ in the strip.

12. What C_2^{FR} can decide analytically

12.1. **Factorization.**

Lemma 13 (Factorization)**.** *For $0 < \sigma < 1$ and $N \geq 2$, with $H_N(s) = N^{s-1} + \chi(1-s)N^{-s}$,*

$$T_N(s) := C_2^{\mathrm{FR}}(N,s) - \Big[\frac{1}{\lambda_s} - \frac{1}{N} - \frac{1}{12N^2} + \frac{2-3\lambda_s}{240N^4}\Big] = \zeta(s)H_N(s) + O\big((1+|s|)^5 N^{-6}\big).$$

H_N has no zeros in $\sigma > \frac{1}{2}$ for $t < 2\pi N^2$ approximately, and $R_X = \zeta/P_X$, with P_X a partial Euler product that never vanishes. Hence, in such a region, the arithmetic part of C_2^{FR}, the rough zeta function R_X and ζ vanish at the same points.

Proof. Insert $\zeta(1-s) = \chi(1-s)\zeta(s)$ into (2); the size of H_N is Proposition 8(2). □

The left side of the first identity is a finite sum and explicit terms, without ζ. Any lower bound for it in $\sigma > \frac{1}{2}$ is a lower bound for $|\zeta(s)H_N(s)|$; neither C_2^{FR} nor R_X contains information on $|\zeta|$ beyond ζ itself.

12.2. **A Turán-type criterion.**

Proposition 14. *$T_N(s)/H_N(s) \to \zeta(s)$ uniformly on every compact $K \subset \{\frac{1}{2} < \sigma < 1,\ t > 0\}$. Consequently the Riemann hypothesis holds if and only if, for every such K, T_N has no zeros on K for all $N \geq N_0(K)$.*

Proof. Uniform convergence is Lemma 13. If $T_N \neq 0$ on K for large N, Hurwitz's theorem gives $\zeta \neq 0$ on K (as $\zeta \not\equiv 0$). Conversely, if $\zeta \neq 0$ on K, $|\zeta H_N|$ is bounded below on K for large N and the error tends to 0. □

Turán proposed the analogous criterion for the partial sums X_N; it fails because X_N has zeros in $\sigma > 1$ for large N [Mo]. The sum T_N behaves differently. Its zeros with $\sigma > \frac{1}{2}$ (search on $0.502 \leq \sigma \leq 1$):

N	10	20	50	100
zeros, $\sigma > \frac{1}{2}$, $2 < t < 60$	1	0	0	0
first one at	$0.562 + 43.37i$	$0.514 + 67.09i$	$0.562 + 266.60i$	$0.519 + 559.35i$

For comparison, X_{19} has 12 zeros with $\sigma > \frac{1}{2}$ and $t < 60$. The frontier $t^*(N) \approx 3$–$6N$ is set by the truncation of the Euler–Maclaurin terms, not by ζ. On the line T_N has many more sign changes than ζ ($55, 73, 85$ for $N = 20, 50, 100$ and $2 < t < 60$, against 10): they are the zeros of H_N.

12.3. **The nodal count of C_2.** The cross-energy $C_2(N,s) = |X_N|^2 - |X_{N-1}|^2 - N^{-2\sigma} = 2N^{-\sigma}\sum_{n<N} n^{-\sigma}\cos(t\log\frac{N}{n})$ is real and not holomorphic; its zeros are curves. Write $C_2 = 2N^{1-2\sigma}(1-\sigma)/|1-s|^2 + 2\,\mathrm{Re}\,\big(A_N(s)N^{-\bar{s}}\big)$ with $A_N(s) = X_{N-1}(s) - N^{1-s}/(1-s)$. Where the wave dominates, the number of sign changes on a vertical segment $T_0 < t < T_1$ is $\big((T_1-T_0)\log N + \Delta_t \arg A_N\big)/\pi$, so by the argument principle it changes by 2 each time σ crosses a zero of A_N. With $N = 2000$ and $10 < t < 60$ (full count 121), the count is 121 for every $\sigma \in [0.52, 1.20]$ with the coefficients of ζ. With the Epstein coefficients of Section 6 it falls in steps of 2 ($121, 119, 117, 115, 113, 111, 109, 105$ at $\sigma = 0.94, 0.92, 0.88, 0.80, 0.78, 0.72, 0.64, 0.56$) as σ crosses its zeros $0.938 + 29.98i$, $0.933 + 15.67i$, $0.823 + 44.00i$, $0.811 + 58.51i$, $0.635 + 46.76i$, $0.597 + 52.74i$ (Figure 2; at finite N the steps are shifted, and the step at 0.72 matches none of these zeros). This reads RH as the constancy in $\sigma > \frac{1}{2}$ of the number of sign changes of real cosine sums.

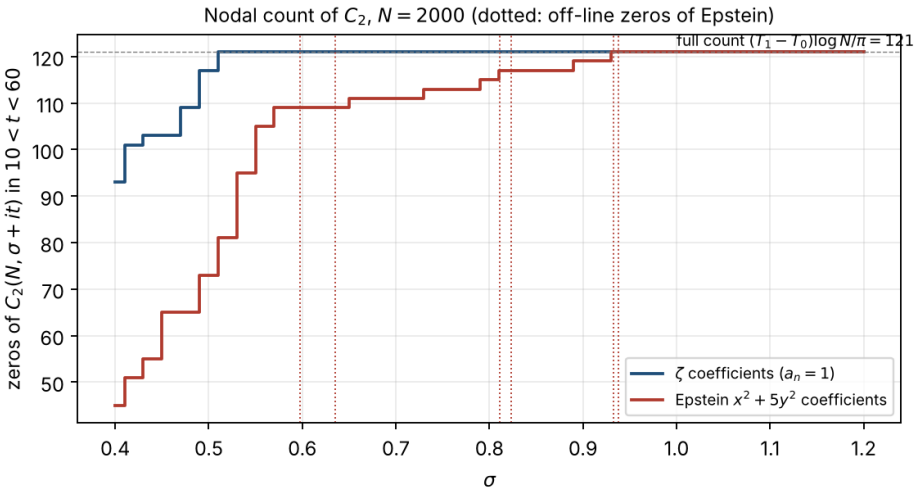

FIGURE 2. Number of zeros of $C_2(N, \sigma + it)$ in $10 < t < 60$ as a function of σ, $N = 2000$. Coefficients of ζ (blue) and of the Epstein function of $x^2 + 5y^2$ (red; dotted lines at its off-line zeros).

12.4. **The weighted trapezoid form.** Summing the trapezoid errors of x^{-s} on the arcs $[k, k+1]$, whose weight is $P(x) = \{x\}(1 - \{x\}) \geq 0$, gives an exact form of the collapse at the zeros.

Proposition 15. *For $0 < \sigma < 1$,*

$$\zeta(s) = \tfrac{1}{2}s(s+1)\Big(J(s) - \frac{1}{\lambda_s}\Big), \qquad J(s) = \int_1^\infty \{x\}(1 - \{x\})\, x^{-s-2}\, dx.$$

Hence $\zeta(s) = 0$ if and only if $J(s) = 1/\lambda_s$. Moreover $|J(s)| \leq J(\sigma) \leq \frac{1}{4(1+\sigma)}$, so there is no zero on $(0, 1)$, where $1/\lambda_\sigma \geq 4$, and none with $|s(1-s)| < 4(1+\sigma)$.

Proof. For $\sigma > 1$, the trapezoid rule on $[k, k+1]$ gives $\frac{f(k)+f(k+1)}{2} - \int_k^{k+1} f = \frac{1}{2}\int_k^{k+1} Pf''$; with $f(x) = x^{-s}$ and summation over $k \geq 1$, $\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s(s+1)}{2}J(s)$, and both sides continue to $\sigma > -1$. Since $\frac{1}{s-1} + \frac{1}{2} = \frac{s+1}{2(s-1)} = -\frac{s(s+1)}{2} \cdot \frac{1}{\lambda_s}$, the identity follows. The bounds use $0 \leq P \leq \frac{1}{4}$. Numerically $|J(\rho) - 1/\lambda_\rho| = 1.4 \cdot 10^{-17}$ and $2.0 \cdot 10^{-17}$ at the first two zeros. □

J is the Mellin transform of a positive weight, and the condition $J = 1/\lambda$ is the λ-collapse of C_2^{FR} in integral form. Splitting J at x_0, the tail $\int_{x_0}^\infty$ is of size $\lesssim x_0^{-\sigma-1}/(6t)$, negligible against $|1/\lambda| \approx t^{-2}$ once $x_0 \gtrsim t^{2/3}$; there the arcs are long and the trapezoid errors are controlled. The head $\int_1^{x_0}$, $x_0 \approx t^{2/3}$, contains the arcs on which $\cos(t \log x)$ oscillates faster than the integer grid. Convexity gives no sign there, and the whole question is whether this finite oscillatory integral can take the value $1/\lambda_s$ minus the tail when $\sigma > \frac{1}{2}$.

13. DISCUSSION

The results of this paper locate the difficulty in one place. The reflected cross-energy is a Hermite–Biehler pair $E_N(s) + E_N(1-s)$ whose inequality fails at the zeros of ζ and nowhere else (Proposition 8); its zeros off the line are transient for ζ and permanent for the two controls (Section 3). The Epstein control shows that pole, functional equation, positivity and the translation law do not decide (Corollary 4). The Euler product does decide at every finite stage when the Dirichlet sum is cut by divisors (Theorem 6), and the passage to the cut by size is the convergence of partial Euler products in the strip, an equivalent of the Riemann hypothesis. The ladder (Proposition 9)

and the correction of [Ca17] (Proposition 10) are tools that compute the zeros on the line from earlier data; Proposition 11 and Proposition 12 show that the off-line content of these formulas is exactly $\zeta(s) \neq 0$. We do not prove the Riemann hypothesis.

14. Open notes

(1) *Interpolating cuts.* Find weights $w(n)$ interpolating between the divisor cut and the size cut for which $|E(s)| > |E(1-s)|$ persists for $\sigma > \frac{1}{2}$; product weights $\prod_p w_p(v_p(n))$ with $w_p(k+1) \leq \sqrt{p}\, w_p(k)$ keep Theorem 6, size cuts do not.
(2) *The $1/\sqrt{N}$ law.* Prove that, if all zeros of ζ below T are simple and on the line, every zero of $C_2^{\mathrm{FR}}(N,\cdot)$ with $t \leq T$ satisfies $|\sigma - \frac{1}{2}| \leq K(T)/\sqrt{N}$ for large N, and compute $K(T)$ from $|\zeta'(\rho)|$ and the collision geometry of Proposition 2.
(3) *Genealogy of the Epstein zeros.* Trace back in N the permanent off-line zeros of the Epstein C_2^{FR} and find whether they are born from a collision on the line or enter from large σ.
(4) *Larger ladders.* Push Table 5 to $k = 200$ with 300-digit arithmetic and measure how the bracket width scales with n and k.

Data and code

Python scripts (NumPy, mpmath, Matplotlib) in the ancillary file: `gen.py` (genealogy of the zeros of C_2^{FR} for ζ and the Davenport–Heilbronn coefficients, Table 1); `fam.py` (Table 2); `epstein.py` (Table 3); `ep_gen.py` (Table 4 and the Epstein C_2^{FR}); `mult.py` (Theorem 6, numerical comparison); `ladder.py` (Table 5); the remaining checks are short scripts recorded in `NOTES_C2FR_genealogy.md`.

Use of AI assistance

The cross-energy and the research programme of this series are the author's. Claude (Anthropic) was used as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test statements against controls, to search the literature, and to draft and edit text and figures. The author checked all results and is responsible for the content.

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