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A Lower Bound for the Pyjama Problem

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Abstract

Let $N(\varepsilon)$ be the least number of rotations of the periodic stripe $\{(x, y) : \operatorname{dist}(x, \mathbb{Z}) \le \varepsilon\}$ that cover the plane. We prove that $\varepsilon N(\varepsilon) \to \infty$ as $\varepsilon \to 0$, answering a question of Li. More precisely, for each fixed $C > 0$, n stripe families in distinct directions with half-width C/n leave an area fraction $e^{-2C} + o(1)$ uncovered. The error is uniform in the directions and their translations.

1 Introduction

Write $\|x\| = \operatorname{dist}(x, \mathbb{Z})$. The pyjama stripe $E(\varepsilon) = \{(x, y) \in \mathbb{R}^2 : \|x\| \le \varepsilon\}$ consists of vertical strips of width 2ε, centred on the lines with integer first coordinate. For $0 < \varepsilon < 1/2$, let $N(\varepsilon)$ be the least number of rotations of $E(\varepsilon)$ about the origin whose union is the plane. Each family occupies the fraction 2ε of a large disk in the limit, so $N(\varepsilon) \ge 1/(2\varepsilon)$.

Theorem 1.1. *As $\varepsilon \downarrow 0$, one has $\varepsilon N(\varepsilon) \longrightarrow \infty$.*

Iosevich, Kolountzakis and Matolcsi [1] posed the problem in 2006 while studying rotations of disks centred at lattice points. Three families at angles $0, 2\pi/3, 4\pi/3$ cover the plane at half-width $1/3$. Indeed, the three projections of any point sum to zero. If all three missed the stripes, their fractional parts would lie strictly between $1/3$ and $2/3$ and could not have integer sum. Malikiosis, Matolcsi and Ruzsa [2] gave an explicit cover at half-width $1/3 - 1/48$ and proved the existence of covers for every $\varepsilon > 1/5$.

Manners [3] proved that finitely many rotations suffice for every $\varepsilon > 0$. His proof used two rational rotations and a version of Furstenberg's $\times 2, \times 3$ theorem. It gave no explicit bound on the number of rotations. Green [4, Problem 41] asked for such a bound. Kravitz and Leng [5] obtained the first explicit upper bound, showing that $\exp(\exp(\exp(\varepsilon^{-O(1)})))$ rotations suffice. Li [6] reduced this to $\exp((\log(1/\varepsilon))^{50000})$ for all sufficiently small ε.

On the lower-bound side, a circle of radius just below $1 - \varepsilon$ meets only the central strip of each family. Counting the available arc length gives $N(\varepsilon) \ge (\pi/2 + o(1))/\varepsilon$, an observation Li credits to Kravitz and Leng and, independently, to Gowers. Using a second circle, Li [6, Theorem 1.3] obtained $N(\varepsilon) \ge (\pi/2 + 1/300)/\varepsilon$ for small ε. His lower bound for open strips also holds for closed strips, by letting a larger half-width decrease to ε. He asked whether the constant could be replaced by a factor tending to infinity. Theorem 1.1 answers this question.

For a unit vector u and a phase $\theta \in \mathbb{R}/\mathbb{Z}$, put $E(u, \theta, \varepsilon) = \{z \in \mathbb{R}^2 : \|\langle z, u\rangle - \theta\| \le \varepsilon\}$. The phase

allows a translation in the normal direction. Write D_R for the disk of radius R centred at the origin. The area density of a set A is the limit of $\mathrm{area}(A \cap D_R)/(\pi R^2)$ as $R \to \infty$, when this limit exists.

Theorem 1.2. *Fix $C > 0$. Let $u_1, \dots, u_n$ be unit vectors, no two equal up to sign, and let $\theta_1, \dots, \theta_n \in \mathbb{R}/\mathbb{Z}$. The complement of $\bigcup_{i=1}^n E(u_i, \theta_i, C/n)$ has area density $e^{-2C} + o(1)$ as $n \to \infty$, uniformly in the vectors and phases.*

If the n stripe events were independent, the uncovered fraction would be $(1 - 2C/n)^n$, which tends to e^{-2C}. Theorem 1.2 gives this limit even when the directions satisfy exact integer relations. Distinctness of the directions matters when translations are allowed, since translates of one family can fill all its gaps.

Proof of Theorem 1.1 from Theorem 1.2. Suppose there were a sequence $\varepsilon \downarrow 0$ on which $n = N(\varepsilon)$ satisfies $n\varepsilon \le C$ for some fixed C. A minimum cover has distinct directions, since repeated unshifted families coincide. The area bound gives $n \ge 1/(2\varepsilon)$, so $n \to \infty$. Widening the stripes to half-width C/n preserves the cover. Theorem 1.2 gives positive uncovered density for these enlarged stripes when n is large, a contradiction. □

2 The uncovered fraction

Averaging a stripe overlap leads to integer relations among its normals. We prove the following estimate in Section 3.

Lemma 2.1. *Let U contain at most M unit vectors, distinct up to sign. For each fixed $s \ge 1$, the number of ordered choices of distinct $u_1, \dots, u_s \in U$ and integers $0 < |a_i| \le M$ with $\sum_{i=1}^s a_i u_i = 0$ is $O_s(M^{s-c_s})$ for some $c_s > 0$.*

Proof of Theorem 1.2. Take $n > 2C$ and put $p = 2C/n$. Each stripe family covers an interval of length p in each period. If the families were independent, the sum of their j-fold overlaps would be $\binom{n}{j}p^j$. We will show that, for each fixed j, the actual sum has the same limit. Inclusion–exclusion will then give the uncovered fraction.

Smoothing the stripes. Lemma 2.1 counts relations with bounded integer coefficients. To use it, we first replace each stripe indicator by a function involving only bounded frequencies. We do this by giving each family a small random shift in its phase.

Choose the shifts independently, each with the Fejér density

$$F_h(t) = \frac{1}{h}\left|\sum_{a=0}^{h-1} e^{2\pi i a t}\right|^2 = \sum_{|k|<h}\left(1 - \frac{|k|}{h}\right)e^{2\pi i k t}, \qquad t \in \mathbb{R}/\mathbb{Z},$$

where h is a positive integer. F_h is a probability density involving only frequencies $|k| < h$.

Represent the shift by $t \in [-1/2, 1/2)$, so $\|t\| = |t|$. The sum defining F_h has absolute value at most

h. For $t \neq 0$, the geometric-series formula also gives

$$\left|\sum_{a=0}^{h-1} e^{2\pi i a t}\right| = \frac{|\sin(\pi h t)|}{|\sin(\pi t)|} \leq \frac{1}{2|t|},$$

since $|\sin(\pi t)| \geq 2|t|$ on this interval. Squaring and dividing by h gives $F_h(t) \leq h$ and, for $t \neq 0$, $F_h(t) \leq 1/(4ht^2)$. The two bounds agree at $t = 1/(2h)$. Thus, if T has density F_h, evenness gives

$$\mathbb{E}\|T\| = 2\int_0^{1/2} tF_h(t)\,dt \leq 2h\int_0^{1/(2h)} t\,dt + \frac{1}{2h}\int_{1/(2h)}^{1/2} \frac{dt}{t} = \frac{1+2\log h}{4h}.$$

Write q_R for the original uncovered fraction of D_R and $q_{R,h}$ for its expected value after shifting. Moving an interval by t can change membership only where one of its two endpoints sweeps past. The changed length in one period is therefore at most $2\|t\|$. This also bounds the changed fraction of a large disk, up to $O(1/R)$: slice the disk along lines parallel to the stripe normal and split each chord into full periods and at most two end pieces. The end pieces contribute only $O(R)$ in total. A point can change from covered to uncovered, or conversely, only if at least one family changes there. Adding the bounds for the n families gives

$$|q_R - q_{R,h}| \leq 2n\,\mathbb{E}\|T\| + O(n/R).$$

Averaging over the disk. Let $I = \{x \in \mathbb{R}/\mathbb{Z} : \|x\| \leq C/n\}$. A point with phase x is hit after a shift T precisely when $x - T \in I$. Its hit probability is therefore

$$v(x) = \int_I F_h(x-y)\,dy = \sum_{|k|<h} c_k e^{2\pi i k x}, \qquad c_k = \left(1 - \frac{|k|}{h}\right)\int_I e^{-2\pi i k y}\,dy.$$

In particular, $c_0 = p$ and $|c_k| \leq p$, since I has length p. For the ith family put $v_i(z) = v(\langle z, u_i\rangle - \theta_i)$. At a fixed point z the shifts are independent, so its probability of being missed is $\prod_i (1 - v_i(z))$. Averaging this product over D_R gives $q_{R,h}$.

Expanding the product gives a finite sum of waves $e^{2\pi i\langle z,\xi\rangle}$. A wave with $\xi = 0$ is constant. Every other wave has disk average tending to zero. Indeed, choose a fixed vector w with $\langle w, \xi\rangle = 1/2$. Translating D_R by w reverses the sign of the integral, yet the two disks differ by area only $O_w(R)$. After division by πR^2, the difference tends to zero.

It follows that $q_{R,h}$ has a limit as $R \to \infty$ for every fixed n, h. The bound on $|q_R - q_{R,h}|$ now gives existence of the original density as well: for fixed n, the gap between the limsup and liminf of q_R is at most $4n\,\mathbb{E}\|T\|$, which tends to zero as $h \to \infty$. From now on take $h = \lceil n\log^2(n)\rceil$. Then $n\,\mathbb{E}\|T\| = O(1/\log(n))$, so smoothing changes the density by $o(1)$.

Counting the overlaps. For fixed $j \geq 1$, let

$$S_j(z) = \sum_{\substack{J\subseteq\{1,\dots,n\}\\ |J|=j}} \prod_{i\in J} v_i(z), \qquad S_0(z) = 1.$$

Each product is the probability that all families in J hit z. On expanding it, a choice of frequencies $(k_i)_{i\in J}$ gives a wave with frequency vector $\sum_{i\in J} k_i u_i$. By the disk-averaging argument, this term

survives precisely when

$$\sum_{i\in J} k_i u_i = 0.$$

Taking every $k_i = 0$ gives p^j for each J, hence $\binom{n}{j}p^j$ in total.

Suppose exactly $s \geq 1$ of the frequencies are nonzero. Since $h \geq n$ and $|k_i| < h$, Lemma 2.1 with $M = h$ gives $O_s(h^{s-c_s}) = o(n^s)$ choices for those s directions and their frequencies. There are at most n^{j-s} choices for the remaining directions. Thus, for fixed j, only $o(n^j)$ terms besides the all-zero choices survive. Each has absolute value at most p^j, so together they contribute $o(n^j p^j) = o(1)$. The disk mean of S_j is consequently

$$\binom{n}{j}p^j + o(1) = \frac{(2C)^j}{j!} + o(1).$$

The error is uniform in the directions by the lemma, and the phases only multiply the terms by factors of absolute value one.

Finally, the inclusion–exclusion inequalities bound the probability of no hits between the odd and even partial sums:

$$\sum_{j=0}^{2m+1} (-1)^j S_j(z) \;\leq\; \prod_{i=1}^{n}(1 - v_i(z)) \;\leq\; \sum_{j=0}^{2m}(-1)^j S_j(z).$$

Average over the disk, then let $n \to \infty$ with m fixed. The two bounds tend to the corresponding partial sums of $\sum_{j\geq 0}(-2C)^j/j!$. Letting $m \to \infty$ sends both bounds to e^{-2C}. Since smoothing cost only $o(1)$, the original uncovered density is $e^{-2C} + o(1)$ as claimed. □

3 Counting short relations

Identify the plane with $\mathbb{C}$, and fix s. By a *box* we mean a set of the form

$$P = \left\{ \sum_{j=1}^{r} m_j z_j : m_j \in \mathbb{Z},\ |m_j| \leq H \right\},$$

where the number of generators is $r = O_s(1)$ and $H \leq M^{O_s(1)}$. The generators may be arbitrary complex numbers. We write $M^{o(1)}$ for a quantity that is $O_{s,\delta}(M^\delta)$ for every fixed $\delta > 0$.

Lemma 3.1 (Unit points in boxes)**.** *A box contains at most $M^{o(1)}$ points of the unit circle, uniformly in its generators.*

More generally, let P, Q be two boxes and fix a positive integer g. There are at most $M^{o(1)}$ unit vectors u for which

$$mu \in P, \qquad nu \in Q$$

for some positive integers $m, n \leq M^{O_s(1)}$ with $\gcd(m, n) = g$. This bound is uniform in g.

Proof. Adjoin the complex conjugates of the generators of P to obtain a box containing both P and $\overline{P}$. Each unit point $u \in P$ gives a factorization $u\bar{u} = 1$ in this enlarged box. Chang's factorization bound [7, Proposition 3 and Remark 1] gives at most $M^{o(1)}$ such choices for u.

For the second assertion, Bézout's identity gives integers k, ℓ of absolute value at most $\max(m, n)$ such that $km + \ell n = g$. Thus

$$gu = k(mu) + \ell(nu).$$

The right side lies in a box on the combined generators of P and Q. Its coefficient bound can be chosen uniformly over all the allowed m, n. Dividing its generators by g gives one box containing every possible u, so the first assertion applies. □

Proof of Lemma 2.1. Call a relation *short* if its coefficients have absolute value at most M. We induct on the number s of directions. For $s = 1$ there is no relation with a nonzero coefficient. For $s = 2$ such a relation would make the two directions parallel. Fix $s \geq 3$ and assume the lemma for all smaller values.

Counting at a larger scale. For an ordered tuple $V = (u_1, \ldots, u_j)$ of distinct directions, write

$$\mathcal{R}(V, t) = \left\{ a \in \mathbb{Z}^j : \ |a_i| \leq t, \ \sum_{i=1}^{j} a_i u_i = 0 \right\}, \qquad A_j(t) = \sum_{V} |\mathcal{R}(V, t)|.$$

Here we allow zero coefficients. Put $K = \lfloor M^\eta \rfloor$ and $h = KM$. Choose $\eta > 0$ small enough that $K^{s^2+1} \leq M$ and

$$(2h)^{r - c_r} = O_s(M^r) \qquad (1 \leq r < s).$$

This is possible because the induction hypothesis gives a fixed saving $c_r > 0$ at each smaller length.

We first bound $A_{s-1}(2h)$. The zero coefficient vector contributes at most M^{s-1}, one for each tuple. If exactly r coefficients are nonzero, choose their positions and apply the induction hypothesis with parameter $2h$. This gives $O_s((2h)^{r-c_r}) = O_s(M^r)$ choices for those directions and coefficients. The remaining directions have at most M^{s-1-r} choices, so $A_{s-1}(2h) = O_s(M^{s-1})$.

Now fix the first $s - 1$ directions $V = (u_1, \ldots, u_{s-1})$. If $a_s = 0$, there are at most M choices for u_s. If $a_s \neq 0$, fix its value. The relation requires

$$u_s = -\sum_{j<s} a_j \cdot \frac{u_j}{a_s}, \qquad |a_j| \leq h.$$

Lemma 3.1 leaves only $M^{o(1)}$ choices for u_s. Once a_s, u_s are fixed, subtract one solution from each of the others. The differences are distinct members of $\mathcal{R}(V, 2h)$, since their last coordinates cancel. Summing over V and the at most $2h$ nonzero values of a_s gives

$$A_s(h) \leq (M + 2hM^{o(1)}) A_{s-1}(2h) \leq KM^{s+o(1)}. \tag{1}$$

Tuples with two independent short relations. The point of enlarging the coefficient bound is that two independent relations can be combined in quadratically many ways. We need this gain relative to the number of short relations already present.

Fix such a tuple V and put $A = \mathcal{R}(V, M)$. Project A linearly to the plane so that its points stay distinct and two independent vectors $b, c \in A$ stay independent. To see that this is possible, choose a first coordinate taking distinct values on the finite set A, and a second coordinate taking the

values $0, 1$ on b, c, respectively. Since $0 \in A$, the projected points have a two-dimensional convex hull.

Triangulate this hull using every projected point as a vertex. There are at least $|A|/3$ triangles, since each point belongs to a triangle and each triangle has three vertices. For a triangle with vertices x, y, z, consider the points

$$\frac{ix + jy + kz}{K}, \qquad i, j, k \geq 1, \qquad i + j + k = K.$$

There are $\binom{K-1}{2}$ of them. They are distinct because the three weights of a point in a nondegenerate triangle are unique, and they lie in the interior because all three weights are positive. Different triangles therefore give disjoint sets of points. Multiplying by K, each point is the projection of a sum of K short relations, hence of a relation of size at most KM. We have proved

$$|\mathcal{R}(V, KM)| \gg K^2 |\mathcal{R}(V, M)|.$$

Summing over these tuples and using (1), their total contribution at the original scale is at most $M^{s+o(1)}/K$.

Tuples with one independent short relation. On every remaining tuple that contributes to the lemma, the short relations lie on one line. Dividing a nonzero relation by the greatest common divisor of its entries gives a primitive vector $b = (b_1, \ldots, b_s)$, unique up to sign. All its entries are nonzero. Every integer relation on this line is an integer multiple of b, so if $D = \max_i |b_i|$, the tuple contributes at most $2M/D$ nonzero multiples.

First suppose $D \leq K^s$. There are $O_s(K^{s^2})$ choices for b. Fix b and the first $s - 2$ directions. The last two must satisfy

$$b_{s-1} u_{s-1} + b_s u_s = w$$

for a known vector w. If $w = 0$, the two directions would be parallel. Otherwise $b_{s-1} u_{s-1}$ lies both on the circle of radius $|b_{s-1}|$ about 0 and on the circle of radius $|b_s|$ about w. These circles meet in at most two points, each determining both directions. Allowing at most $2M$ multiples per tuple gives a total of

$$O_s(K^{s^2} M^{s-2} M) = O_s(M^s/K).$$

Grouping the tuples with $D > K^s$. Call any $s - 1$ directions of a tuple a *base*. We may now treat tuples and bases as unordered sets, restoring the order at a cost of at most $s!$. Given a base, the relation puts one integer multiple of the missing direction in a box on that base. We group tuples to obtain a second multiple in a box fixed for the whole group. This will let us use the second part of Lemma 3.1.

Start with any base among the remaining tuples and mark its directions. In each round, mark the missing direction of every remaining tuple whose other $s - 1$ directions were marked before that round. Let $\mathcal{B}_r$ be the set of bases of remaining tuples entirely marked by round r, with the starting base marked at round zero. For some $r = O_s(1)$ we have

$$|\mathcal{B}_{r+1}| \leq \sqrt{K}\, |\mathcal{B}_r|.$$

Indeed, the sets start with one base and can never contain more than M^{s-1}, while $\sqrt{K}$ grows as a fixed positive power of M.

Form a group from all tuples having a base in $\mathcal{B}_r$, remove them, and repeat with a fresh starting base. Every direction of a grouped tuple is marked by round $r+1$, so all its bases lie in $\mathcal{B}_{r+1}$. Moreover, no base in $\mathcal{B}_r$ survives the removal, since every tuple containing it has been removed. Thus the sets $\mathcal{B}_r$ from different groups are disjoint. By the displayed inequality, the total number of bases used by the groups is at most $\sqrt{K}\,M^{s-1}$, counting a base once in each group that uses it.

Within a group, each marked direction u has a positive multiplier $d(u)$ such that $d(u)u$ is an integer combination of the starting base. Start with $d(u)=1$ on that base. When a relation first marks u_i, it gives

$$b_i u_i = -\sum_{j\neq i} b_j u_j.$$

We may take $d(u_i) = |b_i| \prod_{j\neq i} d(u_j)$, which clears all the denominators on the right. Since $|b_j| \le M$ and there are only $O_s(1)$ rounds, all the multipliers and all the resulting integer coefficients are bounded by $M^{O_s(1)}$. Fix one such $d(u)$ for each direction in the group. There is now a single box P on the starting base containing every $d(u)u$ in the group.

Recovering the missing direction. Fix a group, and for a tuple in it put $g_i = \gcd(|b_i|, d(u_i))$. There are two ways to count the tuple.

If $|b_i| > Kg_i$ for some i, record the base omitting u_i and the integer g_i. The relation puts $|b_i|u_i$ in a box on the recorded base, and $d(u_i)u_i$ lies in the group box P. Thus for each recorded g_i, Lemma 3.1 gives only $M^{o(1)}$ choices for u_i. Once u_i is known, the tuple is complete and its primitive relation is fixed up to sign. It contributes at most $2M/D \le 2M/(Kg_i)$ multiples. Summing over $g_i \le M$, the contribution for a recorded base is at most

$$M^{o(1)} \sum_{g=1}^{M} \frac{2M}{Kg} \le \frac{M^{1+o(1)}}{K}.$$

Otherwise $|b_j| \le Kg_j$ at every direction. Record any base. At each of its directions u_j, the number $d(u_j)$ is known, and $|b_j|$ is a divisor g_j of $d(u_j)$ times an integer at most K. By the elementary divisor bound, an integer of size $M^{O_s(1)}$ has $M^{o(1)}$ divisors. Allowing either sign, we obtain at most $K^{s-1}M^{o(1)}$ coefficient lists on the base. Each list gives the vector

$$-\sum_{j\neq i} b_j u_j = b_i u_i.$$

This vector is nonzero, so its direction determines the missing u_i up to sign. Since $D > K^s$, each completed tuple contributes at most $2M/K^s$ multiples. This again gives at most $M^{1+o(1)}/K$ relations per recorded base.

Finally, multiply this bound by the $\sqrt{K}\,M^{s-1}$ bases used across the groups. The contribution for $D > K^s$ is at most $M^{s+o(1)}/\sqrt{K}$. The earlier cases satisfy the same bound. Since $K = \lfloor M^\eta \rfloor$, we obtain $O_s(M^{s-c_s})$ relations, for example with $c_s = \eta/4$. This completes the induction. □

Remark 3.2. *GPT Astra proposed a refinement of the argument above that would give*

$$N(\varepsilon) \geq \frac{1}{\varepsilon} \cdot \big(\log\log(1/\varepsilon)\big)^{1/2-o(1)} \qquad (\varepsilon \to 0).$$

The refinement requires considerably more technical estimates, so we omit the details here.

References

[1] A. Iosevich, M. N. Kolountzakis and M. Matolcsi, Covering the plane by rotations of a lattice arrangement of disks, in *Complex and Harmonic Analysis*, Proceedings of the International Conference, Thessaloniki, 2006, DEStech Publications, 2007. arXiv:math/0611800.

[2] R. D. Malikiosis, M. Matolcsi and I. Z. Ruzsa, A note on the pyjama problem, *European J. Combin.* **34** (2013), 1071–1077. doi:10.1016/j.ejc.2013.03.001.

[3] F. Manners, A solution to the Pyjama Problem, *Invent. Math.* **202** (2015), 239–270. doi:10.1007/s00222-014-0571-7.

[4] B. Green, *100 open problems*, Problem 41. Author's manuscript.

[5] N. Kravitz and J. Leng, Quantitative pyjama, arXiv:2510.17744, 2025.

[6] S. Li, Quasipolynomial pyjamas, arXiv:2611.26622, 2026.

[7] M.-C. Chang, Factorization in generalized arithmetic progressions and application to the Erdős–Szemerédi sum-product problems, *Geom. Funct. Anal.* **13** (2003), 720–736. doi:10.1007/s00039-003-0428-5.