# A counterexample to Terao's conjecture for free hyperplane arrangements over $\mathbb{F}_{121}$

Shuhei Tsujie *

We give a counterexample to Terao's conjecture for free hyperplane arrangements defined over the finite field $\mathbb{F}_{121}$.

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## 1 Introduction

Let $V$ be an $\ell$-dimensional vector space over a field $\mathbb{K}$. A central hyperplane arrangement $\mathcal{A}$ is a finite collection of linear hyperplanes in $V$. Let $S = \mathbb{K}[x_1, \ldots, x_\ell]$ be the coordinate ring of $V$. A logarithmic derivation module $D(\mathcal{A})$ is the set of $\mathbb{K}$-derivations $\theta\colon S \to S$ such that $\theta(\alpha_H) \in \alpha_H S$ for all $H \in \mathcal{A}$, where $\alpha_H$ denotes a fixed linear form such that $\operatorname{Ker} \alpha_H = H$. We say that $\mathcal{A}$ is free with exponents $(d_1, \ldots, d_\ell)$ if $D(\mathcal{A})$ is a free $S$-module with a homogeneous basis whose degrees are $(d_1, \ldots, d_\ell)$. Let $L(\mathcal{A})$ denote the intersection lattice of $\mathcal{A}$, that is, the set of all intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion.

**Conjecture 1.1** (Terao [3, Problem 1], [2, Conjecture 4.138])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be arrangements over a fixed field $\mathbb{K}$. Suppose that $\mathcal{A}$ is free and the intersection lattices $L(\mathcal{A})$ and $L(\mathcal{B})$ are isomorphic. Then $\mathcal{B}$ is free.

For arrangements of rank 3 with at most 14 hyperplanes, this conjecture was verified in arbitrary characteristic by Barakat and Kühne [1]. Note that the condition that two arrangements $\mathcal{A}$ and $\mathcal{B}$ are defined over the same field is necessary. In fact, Ziegler [5, Example 4.1] constructed the following example:

- $\mathcal{A}$ has a defining polynomial with coefficients in the ring of Eisenstein integers $\mathbb{Z}[\omega]$, where $\omega$ is a primitive cube root of unity, and is free with exponents $(1, 4, 4)$.
- Let $\mathfrak{p} = (1 - \omega)$ be the prime ideal of $\mathbb{Z}[\omega]$ dividing 3. Let $\mathcal{A}_\mathfrak{p}$ be the arrangement over $\mathbb{F}_3 \simeq \mathbb{Z}[\omega]/\mathfrak{p}$ obtained by reducing the defining form of each hyperplane in $\mathcal{A}$ modulo $\mathfrak{p}$. The arrangement $\mathcal{A}_\mathfrak{p}$ is non-free.
- $L(\mathcal{A})$ and $L(\mathcal{A}_\mathfrak{p})$ are isomorphic.

It is a key point that the Frobenius derivation of degree 3 belongs to $D(\mathcal{A}_\mathfrak{p})$ since $\mathcal{A}_\mathfrak{p}$ is defined over $\mathbb{F}_3$. If $\mathcal{A}_\mathfrak{p}$ were free, Terao's factorization theorem would imply that the exponents of $\mathcal{A}_\mathfrak{p}$ are $(1, 4, 4)$. Every homogeneous derivation of degree 3 in $D(\mathcal{A}_\mathfrak{p})$ would be a multiple of the Euler derivation. However, the Frobenius derivation of degree 3 is not a multiple of the Euler derivation. This contradiction shows that $\mathcal{A}_\mathfrak{p}$ is non-free.

In this article, we will construct the following arrangements:

*Department of Mathematics, Hokkaido University of Education, Asahikawa, Hokkaido 070-8621, Japan.
E-mail: tsujie.shuhei@a.hokkyodai.ac.jp

- $\mathcal{A}$ has a defining polynomial with coefficients in the ring $\mathbb{Z}[\psi]$, where $\psi$ is the supergolden ratio, and $\mathcal{A}$ is free with exponents $(1, 13, 13)$.
- In $\mathbb{Z}[\psi]$, the ideal $(11)$ factors as $(11) = \mathfrak{p}_1\mathfrak{p}_2$. Let $\mathcal{A}_{\mathfrak{p}_i}$ be the arrangement obtained by reducing the defining form of each hyperplane in $\mathcal{A}$ modulo $\mathfrak{p}_i$ for $i \in \{1, 2\}$.
- $\mathcal{A}_{\mathfrak{p}_1}$ is defined over $\mathbb{F}_{11} \simeq \mathbb{Z}[\psi]/\mathfrak{p}_1$ and non-free. We can regard $\mathcal{A}_{\mathfrak{p}_1}$ as an arrangement over $\mathbb{F}_{121}$.
- $\mathcal{A}_{\mathfrak{p}_2}$ is defined over $\mathbb{F}_{121} \simeq \mathbb{Z}[\psi]/\mathfrak{p}_2$ and free.
- $L(\mathcal{A}), L(\mathcal{A}_{\mathfrak{p}_1})$, and $L(\mathcal{A}_{\mathfrak{p}_2})$ are isomorphic.

In particular, we have the following theorem.

**Theorem 1.2.** *The arrangements $\mathcal{A}_{\mathfrak{p}_1}$ and $\mathcal{A}_{\mathfrak{p}_2}$ give a counterexample to Conjecture 1.1 in the case $\mathbb{K} = \mathbb{F}_{121}$.*

## 2 Construction

Let $\psi$ be the supergolden ratio, that is, the real root of the cubic polynomial $x^3 - x^2 - 1$. Define the 3-dimensional arrangement $\mathcal{A}$ over $\mathbb{Z}[\psi]$ by

$$\begin{aligned} Q(\mathcal{A}) = xyz & \\ & \times (x-y)(y-z)(-\psi x+z) \\ & \times (x-z)(-\psi x+y)(-\psi y+z) \\ & \times (x-y+z)(\psi x+y-z)(-\psi x+\psi y+z) \\ & \times (x-\psi^2 y+z)(\psi x+y-\psi^2 z)(-\psi^3 x+\psi y+z) \\ & \times (x-\psi^2 y+\psi^2 z)(\psi^3 x+y-\psi^2 z)(-\psi^3 x+\psi^3 y+z) \\ & \times (x+\psi y-\psi z)(-\psi^2 x+y+\psi z)(\psi^2 x-\psi^2 y+z) \\ & \times (x+\psi y-\psi^3 z)(-\psi^4 x+y+\psi z)(\psi^2 x-\psi^4 y+z) \\ & \times (x+\psi^3 y-\psi^3 z)(-\psi^4 x+y+\psi^3 z)(\psi^4 x-\psi^4 y+z). \end{aligned}$$

Using SageMath [4], we can check that $\mathcal{A}$ is free with exponents $(1, 13, 13)$.

In the ring $\mathbb{Z}[\psi]$, the ideal $(11)$ factors as

$$(11) = \mathfrak{p}_1\mathfrak{p}_2, \text{ where } \mathfrak{p}_1 = (11, \psi - 5) \text{ and } \mathfrak{p}_2 = (11, \psi^2 + 4\psi + 9).$$

Therefore, the defining polynomial $Q(\mathcal{A}_{\mathfrak{p}_1}) \in \mathbb{F}_{11}[x, y, z]$ of $\mathcal{A}_{\mathfrak{p}_1}$ is

$$\begin{aligned} Q(\mathcal{A}_{\mathfrak{p}_1}) = xyz & \\ & \times (x-y)(y-z)(-5x+z) \\ & \times (x-z)(-5x+y)(-5y+z) \\ & \times (x-y+z)(5x+y-z)(-5x+5y+z) \\ & \times (x-5^2 y+z)(5x+y-5^2 z)(-5^3 x+5y+z) \\ & \times (x-5^2 y+5^2 z)(5^3 x+y-5^2 z)(-5^3 x+5^3 y+z) \\ & \times (x+5y-5z)(-5^2 x+y+5z)(5^2 x-5^2 y+z) \\ & \times (x+5y-5^3 z)(-5^4 x+y+5z)(5^2 x-5^4 y+z) \\ & \times (x+5^3 y-5^3 z)(-5^4 x+y+5^3 z)(5^4 x-5^4 y+z). \end{aligned}$$

Moreover, the defining polynomial $Q(\mathcal{A}_{\mathfrak{p}_2}) \in \mathbb{F}_{121}[x,y,z]$ of $\mathcal{A}_{\mathfrak{p}_2}$ is

$$\begin{aligned} Q(\mathcal{A}_{\mathfrak{p}_2}) = xyz & \\ & \times (x-y)(y-z)(-sx+z) \\ & \times (x-z)(-sx+y)(-sy+z) \\ & \times (x-y+z)(sx+y-z)(-sx+sy+z) \\ & \times (x-s^2y+z)(sx+y-s^2z)(-s^3x+sy+z) \\ & \times (x-s^2y+s^2z)(s^3x+y-s^2z)(-s^3x+s^3y+z) \\ & \times (x+sy-sz)(-s^2x+y+sz)(s^2x-s^2y+z) \\ & \times (x+sy-s^3z)(-s^4x+y+sz)(s^2x-s^4y+z) \\ & \times (x+s^3y-s^3z)(-s^4x+y+s^3z)(s^4x-s^4y+z), \end{aligned}$$

where $s$ is an element satisfying $\mathbb{F}_{121} = \mathbb{F}_{11}(s)$ and $s^2+4s+9=0$. Using SageMath [4], we can verify that $\mathcal{A}_{\mathfrak{p}_1}$ is non-free, $\mathcal{A}_{\mathfrak{p}_2}$ is free, and the three lattices $L(\mathcal{A}), L(\mathcal{A}_{\mathfrak{p}_1})$, and $L(\mathcal{A}_{\mathfrak{p}_2})$ are isomorphic.

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## AI Disclosure

The author used ChatGPT (GPT-6 Astra) to search for hyperplane arrangements. The author independently verified the results and takes responsibility for their correctness.

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