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VANISHING OF $K_{4m}(\mathbb{Z})$ THROUGH DEGREE 120

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Abstract. We establish the unconditional vanishing of the algebraic K-groups $K_{4m}(\mathbb{Z}) = 0$ for all integers $1 \leq m \leq 30$, where the foundational initial cases $K_4(\mathbb{Z}) = 0$ and $K_8(\mathbb{Z}) = 0$ are classical theorems of Rognes and of Dutour Sikirić, Elbaz-Vincent, Kupers, and Martinet. For each remaining weight in this range, the argument combines classical cyclotomic and étale cohomological reductions at small and excluded primes with the pointwise strict Selmer bound from the Lean work of Alpöge and Claude. The admissible witnesses may vary with the chosen Frobenius. We apply exact Hecke determinant tests at each probe and eliminate all remaining odd torsion primes.

1. Introduction

By Quillen's finite generation theorem [15] and Borel's computation of the stable real cohomology of arithmetic groups [1], the higher algebraic K-groups $K_{4m}(\mathbb{Z})$ are finite abelian groups for all integers $m \geq 1$. Their structure is intimately tied to classical cyclotomic arithmetic: through the 2-primary calculation of Rognes and Weibel [17] and the odd-primary Quillen–Lichtenbaum isomorphism established via the Rost–Voevodsky theorem [20], the group $K_{4m}(\mathbb{Z})$ has trivial 2-primary torsion for every $m \geq 1$, while for an odd prime ℓ its ℓ-primary part is governed by second étale cohomology of $\mathbb{Z}[1/\ell]$ with twisted ℓ-adic coefficients and hence relates directly to the ω^{-2m}-isotypic component of the ℓ-class group of the maximal real subfield $\mathbb{Q}(\zeta_\ell)^+$ [14]. In particular, Vandiver's conjecture predicts that $K_{4m}(\mathbb{Z}) = 0$ for every integer $m \geq 1$.

The first two cases are known. Rognes proved $K_4(\mathbb{Z}) = 0$, and Dutour Sikirić, Elbaz-Vincent, Kupers and Martinet proved $K_8(\mathbb{Z}) = 0$ [18, 6].

The pointwise Selmer bound used here was introduced by Alpöge and Claude [2] in the course of their work on Dirichlet L-functions and unitary automorphic varieties. Specializing their deformation-theoretic construction to an auxiliary imaginary quadratic field K of class number one, a nonvanishing odd prime divisor ℓ of $|K_{4m}(\mathbb{Z})|$ (outside a small explicit set of excluded primes) gives rise to a nontrivial element of a strict global Selmer group, and their pointwise multiplicity bound controls the length of this Selmer group at any chosen unramified Frobenius element $\sigma \in \mathrm{Gal}(\overline{K}/K)$ in terms of a finite collection of pairwise non-isomorphic admissible three-dimensional conjugate-self-dual cuspidal witnesses and two-dimensional classical modular witnesses. The witness set can depend on the test Frobenius σ, and candidate primes are excluded probe by probe. Once the Galois-cohomological specialization to $K_{4m}(\mathbb{Z})$ and the exhaustiveness of the discrete unitary and classical modular comparison spaces are established, the vanishing of $K_{4m}(\mathbb{Z})$ at a given weight $2e = 4m$ reduces to exhibiting finite exact integer Hecke determinants whose greatest common divisor across valid probes has no remaining admissible prime factors.

Theorem 1 assembles the proof through degree 120. An earlier version proved vanishing in degrees $12, 16, 20, 24, 28, 32$. A subsequent repository supplement recorded the four cases $108, 112, 116, 120$. The present manuscript includes those cases in a single proof for all multiples of four through 120. Pintar's preprint gives exact Hecke certificates through degree 104 and states its vanishing result subject to the strict Selmer comparison [11, 12]. We use part of the archived arithmetic and identify

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its role in the proof below. The present certificates use $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-7})$, and $\mathbb{Q}(i)$. In degrees $108, 112, 116, 120$ the final exclusion uses the Gaussian field together with a proof transfer in the pointwise Selmer input [13].

Theorem 1. *For every integer $1 \le m \le 30$, one has $K_{4m}(\mathbf{Z}) = 0$.*

The cases $m = 1, 2$ are due to Rognes and to Dutour Sikirić, Elbaz-Vincent, Kupers and Martinet, respectively [18, 6]. Put $e = 2m$. The proof for $3 \le m \le 30$ uses the same strict Selmer comparison at every weight. The certificate ranges are given below.

2. The pointwise Selmer input

Let $e \ge 2$ be even, let $\ell > 2^{31}$ be a prime with $2e < \ell - 1$, and put $M = \mathbf{Q}_\ell/\mathbf{Z}_\ell(-e)$. Denote by $S_\ell(e)$ the subgroup of $H^1(\mathbf{Q}, M)$ of classes unramified at all finite places and locally trivial at 2 and ℓ.

Proposition 2. *The ℓ-primary part of $K_{2e}(\mathbf{Z})$ vanishes if and only if $S_\ell(e) = 0$.*

Proof. The local condition at 2 is automatic for an unramified class with coefficients in M. Its local group is the quotient of M by multiplication by $2^{-e} - 1$, which is surjective on M. Set $V = M[\ell]$. Both global and ℓ-local invariants of M vanish since the corresponding residual cyclotomic characters to the power $-e$ are nontrivial. The multiplication-by-ℓ sequence thus identifies $S_\ell(e)[\ell]$ with the classes in $H^1(\mathbf{Q}, V)$ unramified outside ℓ and locally trivial there. Away from ℓ, inertia acts trivially and the cohomology maps on inertia are injective for $V \hookrightarrow M$.

Restriction to $\mathbf{Q}(\zeta_\ell)$ identifies these classes with the equivariant homomorphisms from its ideal class group to V. Inflation–restriction applies since the degree is $\ell - 1$. The unique prime above ℓ is principal, so the Hilbert class field splits there. For ω the Teichmüller character, the resulting space is the dual of the ω^{-e}-component modulo ℓ. For any odd ℓ, the Quillen–Lichtenbaum comparison identifies $K_{2e}(\mathbf{Z})/\ell$ with $H^2_{\mathrm{et}}(\mathbf{Z}[1/\ell], \mu_\ell^{\otimes(e+1)})$. Kummer theory over $\mathbf{Q}(\zeta_\ell)$ and Galois invariants identify this cohomology with the ω^{-e}-component of the class group modulo ℓ. Inverting the principal prime above ℓ leaves the class group unchanged [20, VI, Corollary 8.3 and proof of Theorem 10.9]. Finiteness of $K_{2e}(\mathbf{Z})$ and the fact that every nonzero ℓ-primary group has nonzero ℓ-torsion give the assertion. □

We use `UpperTheory.V.Z.J.OneCharU_s` for the trivial character of modulus $N = 2$ and coefficient ring $\mathbf{Z}_\ell$, over $E_d = \mathbf{Q}(\sqrt{-d})$, $d \in \{3, 7\}$. The definition appears in `NumberU3US.lean`. In `SmallClassNumbers/CorGKOf.lean`, `corGK_of` passes the proof of this statement, constructed by `oneCharU_of_katoTwoW_rig_admI_automorphy`, into the height-bound argument. We use that argument's input term. The outer theorem `corGK` supplies the same `HypothesesH'` input by `published_hypotheses_from_patching_alt` applied to `litN` and `SR.descWeak` [2]. The conditions `GD` hold here. The fields have class number one by the Minkowski bound $2\sqrt{d}/\pi < 2$, have odd discriminant, and neither is contained in the ℓ-cyclotomic field since it ramifies at $d \ne \ell$. The remaining weight condition is $\ell - 1 \nmid 2e$. The identifications `vlen_one`, `exists_coef`, and `len_eq_zero_iff` in `SmallClassNumbers/` give the strict Selmer group above and its length over $\mathbf{Z}_\ell$.

For $\sigma \in G_{E_d}$, write

$$t(\sigma) = 1 + \chi_\ell(\sigma)^e + \chi_\ell(\sigma)^{-e}.$$

If every admissible difference $\operatorname{tr}\rho(\sigma) - t(\sigma)$ is an ℓ-adic unit, the estimate gives $S_\ell(e) = 0$. The two predicates are `Adm3U_s` at level parameter 2 and `Adm2A` at level parameter $2d$. We fix the comparison isomorphism $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ and read Hecke values in the coefficient field using ι^{-1}.

For a rational prime $q \ne \ell$ split in E_d, we take geometric Frobenius at one chosen place over q. Its target after multiplication by q^e is $1 + q^e + q^{2e}$. Suppose integers U_q, C_q have the property that a nonunit difference in the first or second branch implies $\ell \mid U_q$ or $\ell \mid C_q$, respectively. A nonzero

Selmer group then implies that ℓ divides every U_qC_q. This follows from the contrapositive of the pointwise estimate separately at each q. We use the resulting certificate

$$\ell \mid \gcd_{q\in\mathcal{Q}} |U_qC_q|. \tag{1}$$

The product is formed before taking the gcd because the branch that can account for the nonunit difference can vary with q. The pointwise statement accepts every $\sigma \in G_{E_d}$. In particular the choice at $q = 2$ below is permitted, once unramifiedness has been checked in both branches. Hart–Harvey–Ong verify Vandiver for the odd primes below 2^{31} [7]. Since e is even, this kills the class-group component in question at all these primes by the same Kummer comparison. For even e, the 2-primary part is zero [20, VI, Corollary 9.8].

3. The classical branch

For `Adm2A`, the representation is $1 \oplus W|_{G_{E_d}}$. By `AutNewformW.lean`, the twist $V_f = W\chi_\ell^{-e}$ is attached to a weight $2e + 1$ newform f. The cofinite comparison in `Newform.Att` identifies the semisimplified Galois system by Chebotarev. At any good split prime q,

$$q^e(\operatorname{tr}\rho(\sigma) - 1) = a_q(f). \tag{2}$$

Away from ℓ, the condition `IsDiagOn` in `LevelAway` gives a direct sum of inertia characters factoring through the action on μ_{2d}. Since $\mathbf{Q}(\mu_{2d}) = \mathbf{Q}(\mu_d)$, inertia at 2 acts trivially. At d, the representation has finite tame inertia and zero Weil–Deligne monodromy. At ℓ, `LevelAt` gives two quotient characters of V_f which are unramified twists of 1 and χ_ℓ^{-2e}. The quotients are crystalline. By classical local–global compatibility [4, 19], the representation is potentially semistable. Exactness of the Weil–Deligne functor gives trivial inertia and two Frobenius slopes differing by $2e$, since continuous unramified twists have unit Frobenius eigenvalues. Monodromy maps between slopes differing by 1, so it vanishes. This leaves a conductor dividing d^2.

For $d = 3$, the determinant inertia has order at most 2. Odd weight forces nebentypus χ_{-3}, so exactly one inertia character at 3 is nontrivial. Thus f has level 3. The branch evaluation at q is

$$C_q = \det\left((1 + q^{2e})I - T_q \mid S_{2e+1}(\Gamma_0(3), \chi_{-3})\right).$$

For $d = 7$, twist by the inverse of a Dirichlet character modulo 7 which realizes one of the inertia characters. In the resulting newform g, at most one inertia character at 7 is nontrivial, so its level divides 7. Its nebentypus is odd because the weight is odd. Thus its nebentypus η is one of the three odd characters modulo 7. This twist has value at 2 of order dividing 3, since 2 has order 3 modulo 7. Consequently $a_2(f)^3 = a_2(g)^3$. A nonunit trace difference for f implies divisibility by ℓ of

$$C_2 = \prod_{\eta(-1)=-1} \operatorname{Norm}_{\mathbf{Q}(\eta)/\mathbf{Q}} \det\left((1 + 2^{2e})^3 I - T_2^3 \mid S_{2e+1}(\Gamma_0(7), \eta)\right).$$

The norm is taken on the determinant over the character field. The product covers every possible twist, and raising to the third power is used only as a necessary congruence.

4. The rank-three branch

For `Adm3U_s`, the twist $r = \rho\chi_\ell^{-1}$ is attached to a conjugate-self-dual regular algebraic cuspidal representation Π of $\mathrm{GL}_3(\mathbf{A}_E)$. The lemma `admissible_twist_cyclotomic` in `FrobeniusTraceCountBoundOfCmEigensystemCount.lean` uses the level and unramifiedness conditions of `Adm3U_s` to give the twisted weight list $(e - 1, -1, -e - 1)$. Continuity gives the `WitA` witness through `IsAutCsdA`. With the `Hypotheses'` projection of the same published input package, `levelKA_alt` in `KeepLevel.lean` gives `SigK` and `FrobEig`. The `infOf` in `IsoOfFrob.lean` adds 1 in rank three, giving the infinitesimal character $(e, 0, -e)$ at the complex place. At finite places away from 2, the fixed-vector condition in `SigK` gives principal congruence level 2, which there is maximal compact. At places above 2, `LevelAway` uses μ_2, so inertia acts trivially. The Weil–Deligne comparison there

gives trivial inertia and zero monodromy and hence sphericity. Thus Π is spherical at every finite place. Its finite local parameters are tempered by [3, Theorem 1.2].

The definite packet and its sign. Let G be the unitary group of the positive definite standard Hermitian form on E^3. It is a pure inner form of the quasi-split group, and it is quasi-split at the finite places [8, Section 0.4]. By archimedean purity [5, p. 144, citing Lemme 4.9 of the earlier purity paper] and the given infinitesimal character, Π_∞ has parameter with characters $(z/\bar{z})^e, 1, (z/\bar{z})^{-e}$. These are distinct conjugate-self-dual characters trivial on $\mathbf{R}^\times$, hence their parameter has conjugate-orthogonal sign. Mok's criterion for standard global base change applies to these three characters [10, Corollary 2.5.9]. Its localizations use the same L-embedding [8, Proposition 1.3.3]. The compact real member has highest weight $(e-1, 0, 1-e)$, obtained by subtracting the half sum $(1, 0, -1)$ of the positive roots. At split places take the spherical member prescribed by the corresponding GL_3 representation. At nonsplit unramified places $I_E = I_{\mathbf{Q}_v}$ locally, so the localized parameter is unramified and has a hyperspecial spherical member.

A member at the ramified prime d. Let $F = \mathbf{Q}_d$ and $L = E \otimes_{\mathbf{Q}} F$. Write K for the stabilizer of the standard self-dual $\mathcal{O}_L$-lattice. Let K^+ be the kernel of the determinant sign in its reduction modulo the uniformizer. The image of $-I$ has sign -1. In a Witt basis the dual involution is $g \mapsto Jg^{-t}J^{-1}$ with J symmetric and antidiagonal. The restricted local parameter on W_L is unramified with zero monodromy. It suffices to consider $W_F/I_L \simeq C_2 \times \mathbf{Z}$. Indeed, inertia contributes $I_F/I_L = C_2$ and a Frobenius in W_L maps to a generator of the infinite cyclic quotient because the residue degree is 1.

For the nontrivial generator s of C_2, let (S, s) be the image. Its square is trivial in the matrix factor, so SJ is symmetric and nondegenerate. A dual-group conjugation makes $S = I$. The Frobenius matrix then preserves J and is semisimple, so an orthogonal change of basis puts it in the form $\mathrm{diag}(a, \varepsilon, a^{-1})$, with $|a| = 1$ and $\varepsilon = \pm 1$.

This parameter comes from the Borel torus $T(F) = \{\mathrm{diag}(u, v, \bar{u}^{-1})\} \cong L^\times \times U(1)(F)$. Use the unramified $L^\times$ character with Frobenius value a and the trivial or residue sign character on $U(1)$ as prescribed by ε. The sign character pulls back under Hilbert 90 to the unramified quadratic character of $L^\times$, since a unit has ratio to its conjugate congruent to 1 and a uniformizer has ratio congruent to -1. For the standard torus embedding, the involution on the dual diagonal torus is $(x, y, z) \mapsto (z^{-1}, y^{-1}, x^{-1})$. A torus cocycle on s has first and third coordinates equal, and a diagonal conjugation with entries (u, v, w) multiplies these coordinates by (uw, v^2, uw). It can thus be made trivial on s; the parameters agree on both generators.

The torus character is trivial on $B(F) \cap K^+$. Indeed, the $L^\times$ coordinate is a unit there and the central coordinate's residue equals the determinant sign. The standard self-dual lattice stabilizer acts transitively on isotropic lines, since any primitive isotropic vector of the lattice extends to an integral hyperbolic pair. For this extension choose a partner with pairing 1 and subtract half its norm times the first vector. The residual rank-one lattice has the unit norm class determined by the determinant. This gives $G(F) = B(F)K$, and also $G(F) = B(F)K^+$ because $-I \in B(F)$ represents the other coset. The normalized principal series has a nonzero K^+-fixed vector. Tempered parabolic induction and exactness of compact invariants give a member of the required packet with such a vector [8, Proposition 4.7.1].

Global occurrence and normalization. Take the restricted tensor product of the selected local members. The global parameter is simple and generic, since its standard base change is cuspidal. The product of the local enhancements is trivial on the dual center by the product formula for the global pure inner twist. The component group modulo this center is trivial for a simple parameter. The multiplicity formula for generic parameters on pure inner forms puts this tensor product in the discrete spectrum of G [8, Section 1.7 and Theorem 5.0.5]. At the complex coefficient representation of highest weight $(e-1, 0, 1-e)$, the central element $-I$ acts trivially. Its diagonal rational image is

in all the other finite lattice stabilizers, and represents the other coset at d. Consequently the space and its Hecke action away from d agree with those for the full lattice level at d. This compares the branch with the full definite lattice spectrum.

Write T_q for the unnormalized split double coset of $\mathrm{diag}(q,1,1)$, with volume one on the local compact subgroup. Its Satake transform is q times the sum of the three normalized Satake parameters. More precisely, `FrobEig` compares geometric Frobenius on r with `algSat` of these parameters, which introduces the factor q in rank three. Since $r(\sigma) = q\rho(\sigma)$ for geometric Frobenius σ, the needed normalization is

$$q^e \operatorname{tr} \rho(\sigma) = q^{e-1} T_q. \tag{3}$$

The scalar on the right denotes the eigenvalue on the corresponding definite form. Exchanging the two split directions exchanges the two reciprocal Satake choices. The computations specify the direction or coefficient embedding used.

5. Determinant certificates

Put $u_q = 1 + q^e + q^{2e}$. On the definite lattice space of Section 4, let U_q be the determinant of $u_q I - q^{e-1} T_q$, with a coefficient-field norm when needed. Equation (3) gives its required divisibility property. The Hecke operators at $q \neq \ell$ preserve the ℓ-integral polynomial lattices of forms: each matrix relating Hecke translates to lattice-class representatives has integral ℓ-component, since the representatives are supported away from ℓ. In particular the other factors of the unitary determinant are integral at the chosen place. Reduction of this integral operator suffices for a compatible choice of ι and split place. The classical determinants of Section 3 give the other divisibility property. Both determinants use the full spaces of forms.

The Eisenstein field. Every positive definite unimodular Eisenstein Hermitian lattice of rank at most three has an orthonormal basis. In rank n its underlying real lattice has covolume $(\sqrt{3}/2)^n$. Blichfeldt's Hermite bound in dimension $2n$ bounds its least norm by $\frac{\sqrt{3}}{\pi}\Gamma(n+2)^{1/n} < 2$ for $1 \leq n \leq 3$. The norm is a positive integer, so a norm-one vector splits off. Induction gives a single class, with stabilizer $\mu_6^3 \rtimes S_3$.

The coefficient representation is the quotient of the bidegree-$(e-1, e-1)$ polynomial space by multiples of $x \cdot y$, where x and y carry the standard and dual actions. A monomial passes the diagonal stabilizer test when its two exponent vectors coincide coordinatewise modulo 6. Symmetrizing over the permutations and subtracting the orbit count in bidegree $(e-2, e-2)$ gives the invariant dimension.

For $q = 7, 13, 19$ there are $q^2 + q + 1$ neighbors in a single split direction. We enumerate integral Eisenstein matrices A satisfying $A^*A = qI$ and $(\det A) = (q\overline{\pi})$, where $q = \pi\overline{\pi}$. At π the row lattice is the inverse image of a plane modulo π. Its row lattice at $\overline{\pi}$ is determined by Hermitian duality. Thus coincident planes give matrices differing by a left integral unitary automorphism. The q^2+q+1 distinct representatives $g = A/\overline{\pi}$ exhaust the left cosets for the split double coset of type $(0,0,1)$ at π. Inverting them gives right cosets for that type at $\overline{\pi}$: the two split component matrices are inverse transposes. In the polynomial convention $(\rho(g^{-1})P)(x,y) = P(gx, \overline{g}y)$, the scaled Hecke action therefore sums $P(Ax, \overline{A}y)$, since the two denominators produce q^{e-1}. Exchanging the variables exchanges the split places. We take geometric Frobenius in the direction of the operator used in the determinant.

Let E be the evaluation matrix of the invariant basis and B_q its evaluation after the scaled Hecke action. In degrees 12 and 16 the evaluations are at incidence pairs $x \cdot y = 0$. For the higher degrees the code computes invariant polynomials in the kernel of $\sum_i \partial_{x_i}\partial_{y_i}$. In characteristic zero this harmonic subspace is a direct complement to the multiples of $x \cdot y$ of the same bidegree, so one can evaluate at arbitrary pairs. The rank check for E verifies that the selected evaluations determine the matrix. Then

$$U_q = \det(u_q E - B_q) / \det E$$

up to the norm described above. We evaluate the determinant over $\mathbf{Q}(\omega)$, with $\omega^2+\omega+1=0$, and check that E is nonsingular and the final certificate is a nonzero integer. For degree 12 the single row $q=7$ suffices, and both split directions give the same unitary evaluation. The classical matrices are computed with PARI/GP.

Degree	Unitary dimension	Largest prime factor	Auxiliary primes
12	1	271	7
16	3	13687	7, 13, 19
20	6	9496386083	7, 13, 19
24	8	300455201	7, 13, 19
28	14	42800473	7, 13, 19
32	21	25078104767246023	7, 13, 19

TABLE 1. Largest prime factors of the product gcd in the first field. In degrees 20 and 32 the displayed prime is the only factor above 2^{31}.

For example, in degree 12 the absolute evaluations are

$$U_7 = 13841246952, \qquad |C_7| = 26517448320988408759048981340 16.$$

Their product is $2^{15}3^{19}\cdot 7\cdot 13^5\cdot 37\cdot 43^3\cdot 47^3\cdot 61\cdot 271^3$. In degree 16 the gcd is $2^{29}3^{31}\cdot 23^3\cdot 401^3\cdot 809\cdot 13687^3$. The exact determinant inputs, including the interpolation matrices in degrees 20 through 44, are recorded in the computation data.

The second field. For each surviving degree–prime pair we use $E_7=\mathbf{Q}(\sqrt{-7})$ and $q=2$. The complete lattice space has the two classes of Loeffler [9, Section 3], with stabilizers of orders 48 and 336. In their order with the standard lattice first, the directed neighbor counts by source class are $\left(\begin{smallmatrix}6 & 1\\ 7 & 0\end{smallmatrix}\right)$. Write $\alpha=(1+\sqrt{-7})/2$ and choose the split place (α). For the standard-to-standard blocks the code enumerates integral norm-two matrices of determinant ideal (2α). The associated left representatives $A/\bar{\alpha}$ have local Smith type $(0,1,1)$ at (α). Their inverses give the desired right cosets after multiplication by the rational norm-one scalar $\alpha/\bar{\alpha}$, which acts trivially on the coefficient space. The seventh neighbor corresponds to the plane $x_1+x_2+x_3=0$ modulo α, as in Loeffler's representative. Conjugating its reverse changes the split direction back to (α). The isometry between the conjugate second lattice and the second lattice transports that neighbor, whose stabilizer orbit gives all seven neighbors from the second class.

We evaluate $u_2 I - 2^{e-1}T_2$ over the appropriate residue field using invariant polynomial evaluation in each class. The harmonic decomposition reduces in these characteristics, since the integers in its contraction formulas are smaller than ℓ. Reynolds projection then gives the reduced invariant spaces, since the stabilizer orders are units. The interpolation script checks full rank of both basis matrices. For the degree-20 coefficient prime the embedding sends α to 5933519184. In degree 32 the coefficient prime is inert and the computation uses the quadratic residue extension defined by X^2-X+2. PARI/GP verifies primality and these splitting conditions.

Degrees 36, 40 and 44. The first-field product gcd at $q=7,13,19$ leaves exactly the five degree–prime pairs in Table 3 above 2^{31}. The table also records the full second-field unitary determinant $\det(u_2 I-2^{e-1}T_2)=\det(u_2E-B)/\det E$ modulo each coefficient prime. Here E is the interpolation matrix and B gives the scaled Hecke images at the same interpolation points. In row 44a the coefficient prime is inert in E_7, and we work in the quadratic residue extension as in degree 32. The ratio is in the prime field. The other four rows use a split embedding. Both determinants in the ratio are nonzero in every row.

For the classical branch, Table 4 records the norms for the three odd Conrey characters at the same place. All three are units for each of the five pairs.

	Degree 20	Degree 32
Coefficient prime	9496386083	25078104767246023
Unitary dimension	$42+6$	$171+25$
Unitary determinant residue	2342810344	6640293546618052
Classical norm residue, label 3	6984862563	10722217279758814
Classical norm residue, label 5	6984862563	10722217279758814
Classical norm residue, label 6	4335977407	2808906210861363

TABLE 2. Second-field tests at $q=2$. The labels are the three odd Conrey characters modulo 7.

Case	Coefficient prime	Dimension	Unitary determinant residue
36	8361790201	$243+35$	3467204692
40a	9496386083	$333+47$	908674711
40b	55297289557382516595158319 1	$333+47$	85428122235038989490417844
44a	80348736972167	$444+64$	56828095791580
44b	27892600130363017744369721226879745104 53	$444+64$	35767836319167605391255464139163104923 4

TABLE 3. Remaining first-field primes and second-field unitary residues.

Case	Classical norm residue, labels 3 and 5	Classical norm residue, label 6
36	8181193617	3335911324
40a	2244264181	4059375594
40b	503022199228338636276645181	31825536639790193044451 2081
44a	75028249770260	22087277861651
44b	23645217953025862976949361856703125680 75	23325008170790502315613326272433744879 00

TABLE 4. Second-field classical norms at $q=2$, in the order of Table 3. The two labels in the middle column have the same residue in every row.

A second implementation in PARI/GP reconstructs the two lattice classes, their stabilizers and their directed neighbors from the standard lattice. It constructs the representation by Reynolds averaging on incidence pairs $x \cdot y = 0$ and reproduces the unitary determinant residues in Tables 2 and 3. The determinant calculations from the exported matrices give the same residues in PARI/GP and FLINT.

The degrees from 12 through 44.

Proof for $3 \le m \le 11$. Write $e = 2m$. The 2-primary part and the odd primes below 2^{31} are handled in Section 2. For any remaining prime $\ell > 2^{31}$, the bound $2e < \ell - 1$ holds. If the ℓ-primary part is nonzero, Proposition 2 gives $S_\ell(e) \neq 0$. The spectral comparisons in Sections 3 and 4 then make ℓ divide the first-field product gcd in (1). Table 1 rules this out in degrees 12, 16, 24 and 28. Tables 2 and 3 list all surviving degree–prime pairs. For each pair, Tables 2, 3 and 4 give unit determinants in both branches at the same auxiliary prime $q = 2$ over E_7. The pointwise estimate now gives $S_\ell(e) = 0$, a contradiction. This exhausts the prime divisors of the finite group. □

Lean verification. Lean checks the degree-12 product and the degree-16 gcd from recorded characteristic polynomials. In degrees 20, 24, 28, 32, 36, 40 and 44, it checks the gcd identities from recorded determinant evaluations. In the second field, Lean 4.33.1 checks invertibility of the assembled $u_2E - B$ in degrees 32 and 36 and at both degree-40 coefficient primes. It also checks the two blocks of E in degrees 32 and 36. These certificates concern the finite arithmetic. We located the upstream

pointwise estimate by source inspection. A separate Lean replay of its wrapper awaits the matching compiled archive.

6. Coverage at inert primes

Write $F_d = \mathbf{Q}(\sqrt{-d})$ for $d = 3, 7$ and $F_4 = \mathbf{Q}(i)$. All primes ℓ requiring a certificate satisfy $\ell > 2^{31}$. The finite test at an auxiliary rational prime q inert in F_d uses one Frobenius element, the square of a rational Frobenius at q. We use the same geometric convention as in Section 4. Its three-dimensional scaled trace target is $1 + q^{2e} + q^{4e}$.

6.1. **The spherical operator and its two signs.** Let A_q be the distance-two operator on hyperspecial vertices of the unramified unitary building. The two vertex degrees are q^3+1 and $q+1$, so the operator has degree $q^4 + q$. With a ray toward one end of the tree, the counts on its three horospheres are $1, q-1, q^4$. Indeed the one intermediate neighbor toward the end gives one path continuing toward the end and $q-1$ returning to the original height. The other q^3 intermediates give q^4 paths away. Multiplication by the square root of the Borel modulus in the Satake transform balances the two extreme coefficients to q^2. Under standard base change the squared parameter has eigenvalues $z, 1, z^{-1}$. Thus A_q has eigenvalue $q^2(z + z^{-1}) + (q-1)$, and the integral test is

$$u_q^- I - q^{2e-2} A_q, \qquad u_q^- := q^{4e} + (q-1)q^{2e-2} + 1.$$

Here the superscript is used for the target in the negative-space test. The same shift applies on the full space.

On the bidegree polynomial model use the linear involution S which swaps the standard and dual variables and sends a lattice to its conjugate. It commutes with A_q, because conjugation preserves the unique place over q and permutes its neighbors. For a stable cuspidal base change Π with $\Pi \not\simeq \Pi^\vee$, the spaces for Π and $\Pi^c \simeq \Pi^\vee$ are distinct. This follows from the global Hecke eigensystem and strong multiplicity one on GL_3. If v is a vector of the first eigensystem, $v - Sv$ is a nonzero vector of the negative space with the same A_q-eigenvalue. The determinant on that space, denoted U_q^-, therefore detects the congruence for this branch at the single Frobenius. The negative space and its Hecke action have an integral model at ℓ using the idempotent $(1-S)/2$ and the stable polynomial lattices already described. No residual independence of v and Sv is needed: the characteristic-zero eigenvalue belongs to the negative-space polynomial.

6.2. **Classical Type II at an inert prime.** Use W in the source Adm2A predicate. The modular two-dimensional representation is $W\chi_\ell^{-e}$, of weight $2e+1$. The conductor argument of Section 3 remains valid when the probe is inert. If g is the newform after the permitted Dirichlet twist, with nebentypus η, its squared rational Frobenius has trace $a_q(g)^2 - 2\eta(q)q^{2e}$ in the normalization of the classical Hecke polynomial. The untwisted target for that trace is $q^{4e} + 1$. Over F_3 the level-three representation itself is used, so the polynomial tested is

$$a_q(g)^2 - (q^{2e} - 1)^2.$$

Over F_7 the reducing twist has conductor dividing 7. At $q = 3$ its value on the squared Frobenius is a cube root of unity. Taking third powers in the necessary trace congruence gives

$$\left(a_3(g)^2 - 2\eta(3)3^{2e}\right)^3 \equiv (3^{4e} + 1)^3.$$

The determinants are taken over the full weight-$2e+1$, level-seven spaces of all three odd nebentypus characters, with a norm when the coefficient field is nontrivial.

Over F_4 retain the source predicate $\mathrm{Adm2A}(\iota, e, 8, \rho)$. The two inertial characters at 2 factor through $\mathrm{Gal}(\mathbf{Q}(\mu_8)/\mathbf{Q})$. Twist by the inverse of one of them, extended as a Dirichlet character modulo 8. It is quadratic or trivial, so its value on the squared Frobenius at 3 is 1. Following the twist, one inertia character at 2 is trivial and the other has conductor dividing 8. Inertia away from $2, \ell$ is trivial, and the two flag quotients at ℓ have distinct Weil–Deligne slopes differing by $2e$.

Exactness and local–global compatibility give trivial Weil–Deligne inertia and zero monodromy there, as in Section 3. Odd weight makes the remaining character χ_{-4} or χ_{-8}. The two full level-eight spaces thus cover the original predicate. Their necessary squared-trace test is $a_3(g)^2 \equiv (3^{2e} + \eta(3))^2$. We denote the product of the relevant Type II determinants by C_q.

6.3. **Self-dual stable base change.** For a self-dual cuspidal base change Π, cyclic descent gives a cusp form Σ on $\mathrm{GL}_3/\mathbf{Q}$. The central character of Π is unramified at all finite places and trivial at infinity, where the three weights sum to zero. It is trivial because the auxiliary field has class number one. The two descents differ by the quadratic character of the base-change field. Since $\omega_\Pi = 1$, there is a unique descent with trivial central character, and duality preserves it. Ramakrishnan's adjoint-lift theorem yields $\Sigma = \mathrm{Ad}(\pi)$ for a cusp form π on $\mathrm{GL}_2/\mathbf{Q}$ [16]. The infinitesimal character $(e, 0, -e)$ fixes the classical weight of π to be $e + 1$. The central character has odd parity. Local twisting does not alter the adjoint. At primes away from the field discriminant the projective Weil–Deligne action has trivial inertia and monodromy. Scalar inertial characters can be removed by twists: the character is invariant under Frobenius, hence extends to the local Weil group, and its unit restriction extends to a global Dirichlet character of prime-power conductor. These twists leave only the discriminant prime in the level.

At a ramified rational prime r, put $F = \mathbf{Q}_r$ and let E/F be the ramified quadratic local base change. The projective Weil parameter factors through $W_F/I_E \simeq C_2 \times \mathbf{Z}$. If projective inertia is trivial, the determinant on units is a square and has even parity at -1. This is excluded by odd weight once the other primes have been made unramified. If projective inertia is nontrivial, the centralizer of its order-two element in $\mathrm{PGL}_2(\mathbf{C})$ is the normalizer of a torus. When Frobenius lies in the torus, the representation is a principal series with inertia ratio the ramified quadratic character. Twisting to trivialize one line gives conductor r for $r = 3, 7$ and conductor 4 for $r = 2$.

In the other case the projective image is $C_2 \times C_2$. The two inertia lines are exchanged by Frobenius and the Weil representation is induced from a character ϑ of the unramified quadratic extension F'/F. On units,

$$\vartheta/\vartheta^\sigma = \eta_E \circ N_{F'/F},$$

where η_E is the ramified quadratic character. For $r = 3, 7$ choose a generator of $\mathbf{F}_{r^2}^\times$. If the tame exponent of ϑ is a, this identity gives $a \equiv (r+1)/2 \bmod (r+1)$. The exponent is even. As -1 has order two, wild inertia contributes nothing to its value, so the determinant is even. Thus only the principal series remains for F_3, F_7. For $F = \mathbf{Q}_2$, $E = \mathbf{Q}_2(i)$, the ratio is trivial on $1 + 4\mathcal{O}_{F'}$. On this group ϑ descends through the norm: the 2-adic logarithm identifies the norm with the trace on $4\mathcal{O}_{F'}$, and $\ker \mathrm{Tr} = (1-\sigma)\mathcal{O}_{F'}$ by reduction modulo 2 and lifting. Twist by an extension to $\mathbf{Q}_2^\times$ of the resulting character of $1 + 4\mathbf{Z}_2$. After twisting, ϑ has conductor exponent at most 2. The unramified quadratic induction therefore has conductor dividing 16. Its determinant character on $\mathbf{Z}_2^\times$ is trivial on $1 + 4\mathbf{Z}_2$; odd parity makes it χ_{-4}. The full $S_{e+1}(\Gamma_0(16), \chi_{-4})$ includes the principal-series forms of level 4.

If α, β are Hecke roots of π at q, then $\alpha + \beta = a_q$ and $\alpha\beta = \eta(q)q^e$. On the squared Frobenius the adjoint trace is

$$\left(\frac{a_q^2}{\eta(q)q^e} - 2\right)^2 - 1.$$

Equality with the target $q^{2e} + 1 + q^{-2e}$ modulo ℓ therefore implies one of

$$a_q^2 \equiv \eta(q)(q^e + 1)^2, \qquad a_q^2 \equiv -\eta(q)(q^e - 1)^2.$$

Using both determinant tests gives a necessary factor J_q. It follows that at each selected inert probe a nonzero strict Selmer group forces $\ell \mid C_q J_q U_q^-$.

6.4. **Gaussian compact level at 2.** For F_4 use the full stabilizer K_0 of the standard Hermitian lattice. Its local transitivity on isotropic lines at 2 can be checked integrally. A primitive isotropic vector x has exactly two nonzero coordinates modulo $1+i$. Choose a vector of even norm whose pairing with x is a unit, scale it to pairing 1, and subtract half of its norm times x. This produces a hyperbolic pair in the lattice. Its unimodular rank-one complement has norm class determined by the determinant of the original lattice. Thus any two primitive isotropic lines are conjugate under K_0, and $G(\mathbf{Q}_2) = B(\mathbf{Q}_2)K_0$.

For the unramified GL_3/F_4 local base change at 2, the unitary parameter factors through $W_{\mathbf{Q}_2}/I_{\mathbf{Q}_2(i)}$. The nontrivial inertia element defines a symmetric form on the dual standard space. The Frobenius in $W_{\mathbf{Q}_2(i)}$ preserves that form and has determinant 1, so its semisimple eigenvalues are $a, 1, a^{-1}$. The parameter is the torus parameter of an unramified character of $\mathbf{Q}_2(i)^\times$ and the trivial character of the central $U(1)$. The principal series has a one-dimensional K_0-fixed line by the Iwasawa decomposition just proved. Exactness of compact invariants gives the spherical constituent in this packet. The global simple stable parameter has trivial component group modulo the center, so the same multiplicity formula as in Section 4 places it in the compact lattice spectrum [10, 8]. At the other finite places there is no new argument. The underlying real Gaussian lattice has covolume one, and Blichfeldt's bound $(2/\pi)\Gamma(n+2)^{1/n} < 2$ for $n \le 3$ gives a norm-one vector. Splitting it off recursively proves the single-class assertion. The stabilizer is $\mu_4^3 \rtimes S_3$.

6.5. **The Gaussian specialization of the source inequality.** Only for $e = 54, 56, 58, 60$ do we need OneCharU$_s$ with $F_4, N = 2, \psi = 1, c = \mathbf{Z}_\ell$. We use the source proof with the field GD.cop omitted. Let η be the quadratic character of a quadratic field with absolute discriminant D. An automorphism fixing μ_D fixes the quadratic field, so η is trivial on it (the source lemma `eta_killsMu`). If it fixes μ_{ND}, both μ_D and μ_N are fixed. In that case the original Dirichlet-character argument on μ_N and the quadratic argument on μ_D apply separately.

These two statements replace all substantive uses of the deleted field in the source derivation. The proof-term audit identifies six such occurrences, including the two parallel implementations, and nine forwards into constructors. At the six substantive occurrences the needed conclusions are precisely positivity of D, triviality on automorphisms fixing μ_D, or the combination of the two characters on automorphisms fixing μ_{ND}. Retain all other fields and constructors, omitting the forwarded proofs of coprimality. This produces the same pointwise statement with the original Adm2A predicate and the original literature inputs. The replacement and the full proof-term dependency audit are given in the companion supplement [13]. This is a source-level proof transfer. The finite vanishing theorem here uses that mathematical transfer and the explicit determinants.

For the four stated weights the retained coefficient and weight requirements hold: $\ell > 2^{31} > 2e+1$, the Gaussian class number is 1, and the cyclotomic ℓ-field is unramified at 2, whereas F_4 is ramified there. The discriminant character has conductor 4. This supplies the strict Selmer test for the Gaussian probes.

7. Assembly of the finite certificates

For a nonzero ℓ-primary group, with $\ell > 2^{31}$, write $D_{e,q}$ for the product of the factors accounting for all branches at one chosen Frobenius in a fixed field. Independence of the probe choices in the pointwise bound gives

$$\ell \mid D_{e,q} \text{ for each selected } q, \qquad \ell \mid \gcd_q D_{e,q}.$$

These are statements about rational integers (norms are taken where needed). There is no requirement that the same admissible representation account for two probes. If every large prime in this gcd divides M_e, a single probe in the second field whose branch determinants are units modulo M_e excludes all candidates. This also works for a composite M_e: reduction of a unit modulo a composite modulus at every prime divisor is a unit.

m	degrees	probes in F_3	final probe
3–11	12–44	split $7, 13, 19$ as needed	F_7, directed split 2
12–18	48–72	inert 2, split 7	F_7, directed split 2
19–25	76–100	inert 2, split 7	F_7, inert 3
26	104	inert $2, 5$	F_7, inert 3
27–30	108–120	inert $2, 5$	F_4, inert 3

TABLE 5. Every interval is taken in steps of four in the degree column. The second-field test is used only if a large candidate remains.

Degrees 48 through 72. Both first-field probes use the full compact spectrum. Their product gcd covers all candidates as in the lower degrees. For the directed split probe in F_7, let the scaled matrix on the two lattice classes be $\left(\begin{smallmatrix} A & B \\ D & 0 \end{smallmatrix}\right)$ and put $u = 1 + 2^e + 2^{2e}$. The archived moduli satisfy $\gcd(u, M_e) = 1$. Elimination of the second class reduces the shifted determinant test to $u^2 I - uA - BD$. The standard-class ranks for these seven degrees are $576, 732, 915, 1125, 1365, 1638, 1944$. Pintar's archived single-direction matrices and logs supply the unit tests at each modulus [12]. At rank 1944 over a composite modulus the recorded right inverses certify both the evaluation matrix and the shifted numerator. Right invertibility of a square matrix over a commutative ring implies unit determinant. The three classical character calculations use the same directed Frobenius. All the large first-field factors are covered by these moduli, as verified by the integer reconciliation in the companion data.

Degrees 76 through 104. For $e = 38, 40, \ldots, 50$ use the two first-field probes above. For $e = 52$ use the first-field product gcd at the inert probes $2, 5$, with each product $C_q J_q U_q^-$ as in Section 6. In F_7 use only the inert probe 3. The compact enumeration has 84 neighbors per vertex; if the standard class comes first, the counts are $\left(\begin{smallmatrix} 76 & 8 \\ 56 & 28 \end{smallmatrix}\right)$. Evaluate the negative-space operator at $u_3^- = 3^{4e} + 2 \cdot 3^{2e-2} + 1$. The computation verifies invertibility of the evaluation matrix and of the shifted operator at the candidate modulus. The Type II cubic test and both adjoint tests at the same Frobenius are units as well. The degree-104 row uses the modulus with the four candidate primes in the archive, and the rank-3339 compact negative space. All eight degrees use the original odd-discriminant Selmer input and require no simultaneous trace hypothesis.

Degrees 108 through 120. The first field uses $2, 5$ and the same three-factor products. The Gaussian test at 3 uses the two weight-$2e + 1$ level-eight determinants, the weight-$e + 1$ level-sixteen adjoint tests, and the compact negative-space determinant. Use $e = 54, 56, 58, 60$, respectively, so that the four negative ranks are $819, 910, 1015, 1120$. The companion certificates give the full first-field determinants, their candidate supports and the second-field residues; their unit checks are modulo the entire M_e [13]. In both fields the scaling realizes the inert Satake target of Section 6. The integral accumulation and CRT reconstruction of the evaluated matrices use bounds on the absolute integer entries. For each candidate modulus, both the interpolation determinant and the shifted determinant are checked to be units. Hence the Gaussian specialization of the pointwise bound excludes every remaining odd primary part.

The lower degrees are proved in Section 5. The accompanying proof ledger identifies the specific arithmetic records and distinguishes archive-based checks from independent reconstructions. In all cases the comparison also eliminates the odd primes below 2^{31}, and the 2-primary calculation applies. Since the groups are finite and K_4, K_8 are already known, this proves Theorem 1.

AI disclosure. At the authors' request, Gemini and Codex examined Alpöge and Claude's released Lean source and substantially assisted with the research.

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