A finite arithmetic test for simplicity of the Riemann zeros, after Suzuki

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Abstract

M. Suzuki proved that, under the Riemann hypothesis, the compressions of the inverse of a self-adjoint differentiation operator to windows of Weil's quadratic form converge in Hilbert–Schmidt norm, and that their eigenvalues recover every distinct zero ordinate of the Riemann zeta function; he notes that the multiplicities are not recovered. The Riemann hypothesis is now a theorem of L. Alpöge. We show that the multiplicities are recovered by even trace powers. Let $S_k = 2\sum_{\gamma>0} m_\gamma \gamma^{-2k}$ be the Taylor moments of $\log \xi(1/2+iz)$, which count multiplicity, and let $D_{N,k}$ be S_k minus the trace of the $2k$-th power of a finite compression whose matrix entries come from the explicit formula with finitely many prime powers. Then $D_{N,k} \geq 2E(T)/T^{2k}$, where $E(T)$ is the number of repeated zeros up to height T counted with their excess multiplicity, and $\sup_{N,k}(2/D_{N,k})^{1/(2k)}$ equals the height of the first multiple zero. So the zeros up to a height T are simple if and only if one finite arithmetic inequality holds, and all zeros are simple if and only if these heights are unbounded. An equivalent integer form counts the positive eigenvalues of one Hermitian matrix. A computation in ball arithmetic, with prime powers below e^{12} and no zero ordinates as input, certifies that every zero with $0 < \Im\rho \leq 97.632$ is simple. The mathematics was produced by GPT-6 Astra (ChatGPT Pro mode); Claude Opus 5.5 at effort high checked every proof, wrote this text, and recomputed the certificate with an independently written program. No human has checked this note.

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1 The question and the theorem

The setting. Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\Xi(z) = \xi(\frac{1}{2}+iz)$. Alpöge's theorem [1, 11] gives the Riemann hypothesis: every zero of Ξ is real. Let $\Gamma \subset \mathbb{R}$ be the set of distinct zeros of Ξ and m_γ the multiplicity of $\gamma \in \Gamma$. Since Ξ is even and $\Xi(0) = \xi(1/2) \neq 0$, $\Gamma = -\Gamma$, $m_{-\gamma} = m_\gamma$, $0 \notin \Gamma$, and, because Ξ has order one, $\sigma := \sum_{\gamma\in\Gamma} m_\gamma \gamma^{-2} < \infty$ [15, (2.2)]. Whether every m_γ equals one is the simplicity question for the Riemann zeros. It is open. Write

$$E(T) = \sum_{0<\gamma\leq T} (m_\gamma - 1), \qquad n(T) = \sum_{0<\gamma\leq T} m_\gamma, \qquad \gamma_* = \min\{\gamma > 0 : m_\gamma \geq 2\} \in (0, \infty],$$

the multiplicity excess up to height T, the counted number of zeros up to T, and the first multiple ordinate ($\gamma_* = \infty$ if there is none).

Suzuki's compressions. Weil's quadratic form, $Q(u,v) = \sum_{\gamma\in\Gamma} m_\gamma \widehat{u}(\gamma)\overline{\widehat{v}(\gamma)}$ with $\widehat{u}(t) = \int u(x)e^{itx}\,dx$, is positive, and its completion is realized by the sampling map $u \mapsto (\widehat{u}(\gamma))_{\gamma\in\Gamma}$ in $\mathcal{H} = \ell^2(\Gamma, m)$, with norm $\|c\|^2 = \sum_\gamma m_\gamma |c_\gamma|^2$. On $\mathcal{H}$, let C be multiplication by $1/\gamma$. Suzuki [15, §2.3 and (4.1)] shows that C is a Hilbert–Schmidt operator with one eigenvector for each distinct ordinate, and that its compressions to the subspaces of test functions supported in $[-a, a]$ converge to C in Hilbert–Schmidt norm; their positive eigenvalues

decrease to $1/\gamma_1 > 1/\gamma_2 > \cdots$ over the distinct ordinates [15, Theorem 1.1]. He notes that "the multiplicities of the zeros are not recovered" [15, §1, after Theorem 1.2].

We use finite-dimensional windows. For $N \geq 1$, let $h = 2^{-N}$ and let V_N be the space of continuous functions on $\mathbb{R}$, linear on each interval $[jh, (j+1)h]$, vanishing outside $(-N, N)$, with $\int f = 0$. It has dimension $d_N = 2N2^N - 2$. Let P_N be the orthogonal projection of $\mathcal{H}$ onto the image of V_N under the sampling map, and $C_N = P_N C P_N$. For $k \geq 1$ put

$$S_k = -2k\,[z^{2k}] \log \frac{\Xi(z)}{\Xi(0)} = 2 \sum_{\gamma > 0} \frac{m_\gamma}{\gamma^{2k}}, \qquad D_{N,k} = S_k - \operatorname{tr} C_N^{2k}, \qquad H_{N,k} = \Big(\frac{2}{D_{N,k}}\Big)^{1/(2k)}.$$

The middle equality is the Hadamard product (Section 4). Both terms of $D_{N,k}$ are computable from arithmetic data: S_k from the Taylor series of ξ at $1/2$, and $\operatorname{tr} C_N^{2k}$ from two $d_N \times d_N$ matrices whose entries are given by the explicit formula with the prime powers below e^{2N} (Section 5).

Theorem 1. *Assume the Riemann hypothesis (a theorem, [1]). For all $N, k \geq 1$ and $T > 0$,*

$$D_{N,k} > 2 \sum_{\gamma > 0} \frac{m_\gamma - 1}{\gamma^{2k}} \;\geq\; \frac{2E(T)}{T^{2k}}.$$

For fixed k, $D_{N,k}$ decreases in N to $2\sum_{\gamma>0}(m_\gamma - 1)\gamma^{-2k}$, and

$$\sup_{N,k \geq 1} H_{N,k} = \gamma_*.$$

Consequently, for each $T > 0$, every zero with $0 < \Im\rho \leq T$ is simple if and only if $D_{N,k} < 2/T^{2k}$ for some $N, k \geq 1$; and every nontrivial zero of ζ is simple if and only if $\sup_{N,k} H_{N,k} = \infty$.

The test has an integer form. Let $b_j = t_j - t_0$ $(0 < |j| < N2^N)$, where t_j is the hat function of height one centred at jh with support $[(j-1)h, (j+1)h]$. These b_j form a basis of V_N: each has mean zero, since every t_j has integral h; they are linearly independent, since $\sum_j c_j b_j$ takes the value c_j at jh; and a function $f \in V_N$ equals $\sum_{|j|<N2^N} f(jh) t_j$ with $\sum_j f(jh) = h^{-1} \int f = 0$, hence equals $\sum_{j \neq 0} f(jh) b_j$ (this argument is that of the independent re-proof acknowledged in the note on method). Let G_N and H_N be the Hermitian matrices $(G_N)_{ij} = Q(b_j, b_i)$ and $(H_N)_{ij} = \langle C\,\widehat{b}_j, \widehat{b}_i \rangle_{\mathcal{H}}$, where $\widehat{b}$ denotes the sampling vector, and let ind_+ denote the number of positive eigenvalues.

Theorem 2. *Assume the Riemann hypothesis. Let $T > 0$ with $\Xi(T) \neq 0$, and $\iota_N(T) = \operatorname{ind}_+(TH_N - G_N)$. Then $\iota_N(T)$ is nondecreasing in N, at most the number of distinct ordinates in $(0, T]$, and equal to it for all large N. Hence the zeros with $0 < \Im\rho \leq T$ are all simple if and only if $\iota_N(T) \geq n(T)$ for some N.*

Theorem 3. *Every zero ρ of ζ with $0 < \Im\rho \leq 97.632$ is simple. More precisely, $D_{6,3} < 2/(97.632)^6$, and the five rows of Table 1 with $N \leq 5$ satisfy the strict test at the heights shown.*

The certificate uses 15040 prime powers, the largest $162751 < e^{12}$, and the matrices have dimension 766. It was computed twice, by two independently written programs in Arb ball arithmetic (Section 6).

Proposition 4. *Every zero with ordinate $\gamma \geq 10$ has multiplicity $m_\gamma < \frac{3}{4}\log\gamma$. The first multiple ordinate satisfies $\gamma_* > 97.632$, so $\gamma_*^2/2 > 4766.003712$, and every multiple zero has $\gamma > \max(97.632,\, e^{4m_\gamma/3})$.*

What this means. Theorem 1 turns the simplicity question into a statement about finitely many prime powers at a time: the multiplicity excess is exactly what separates the moments S_k, which see each zero with its multiplicity, from the traces of Suzuki's compressions, which see each distinct zero once. At every fixed height the test terminates exactly when the zeros there are simple. Simplicity of all zeros is the statement that these finite arithmetic tests succeed at every height; the theorem gives no estimate that would decide it. The limit of the defect is the Taylor data of $\log(\Xi(z)/(\Xi(0)\mathsf{P}(z)))$, where $\mathsf{P}(z) = \prod_{\gamma>0}(1 - z^2/\gamma^2)$ is Suzuki's product over distinct ordinates [15, Theorem 1.2]; that quotient is constant exactly when all zeros are simple.

Why it matters. Simplicity of the zeros is the next question about their location after the Riemann hypothesis. It enters the explicit formula with multiplicities, the size of $1/\zeta'(\rho)$, and the spectral interpretations of the zeros: a self-adjoint operator whose spectrum is the set of zeros sees each one with the multiplicity of its eigenspace. Here the comparison between the counted and the distinct zeros becomes a positive defect of explicit finite matrices, and the defect is computable from primes.

The credit. The Hilbert space, the operator C, its Hilbert–Schmidt property, and the convergence of compressions are Suzuki's [15, §2.3, (4.1), Theorem 3.3]; we apply his argument to the dyadic windows V_N, for which we prove the density it needs (Lemma 6). The explicit formula is Weil's, in Bombieri's normalization as recorded by Connes and Consani [7, Appendix, Explicit formula]. Alpöge's theorem [1] supplies the real zeros. The principle that a finite restriction of Weil's form sees distinct zeros through its rank or positive index and counted zeros through its trace is Alpöge and Furman's [2, Proposition 4.1, p. 6], and the pull-back of inertia used in Theorem 2 is their [2, Lemma 3.1, p. 5]. What is new here is the exact form of that comparison for Suzuki's compressions: the trace-power defect and its supremum (Theorem 1), the equality of the positive index with the number of distinct ordinates at each height (Theorem 2), the certificate (Theorem 3), Propositions 4 and 8, and the heat-flow limit of Proposition 10, whose proof uses the stability criterion of Borcea and Brändén [5].

Context. Alpöge and Furman [2] proved, unconditionally, that more than two thirds of the zeros are simple and on the critical line and more than five sixths are distinct [2, Theorem A, p. 1]. They restrict Weil's form to about $N(T, 2T)$ modulated copies of one window placed through $[T, 2T]$; in the resulting matrix each distinct zero on the line contributes a positive form of rank one with trace equal to its multiplicity, and each pair of zeros off the line a block of signature $(1, 1)$ [2, Proposition 4.1, p. 6], the latter observation being Bombieri's [2, §1.3, p. 3]. A rank–trace inequality [2, Lemma 3.2, p. 5] and Montgomery's second moment then give the proportions [2, §1.2, (1.2), p. 2]. Their matrices are not compressions of an operator with the zeros as spectrum, no trace powers enter, and their bounds are inequalities for proportions; nothing there states an exact count at a fixed height or the defect of Theorem 1. Lamzouri [10] gave a second proof of the same proportions with a single Hilbert-space inequality. Mori [12] bounds the smallest eigenvalue of the windowed Weil form from below under the Riemann hypothesis alone, and shows that such bounds force a proportion of distinct zeros; our defect controls the multiplicity excess itself, not a proportion. Groskin [8] studies finite Galerkin truncations of the Weil form and Kim et al. [9] compute Suzuki's operator numerically; neither treats multiplicities. On Hexagon, Cáceres [6] states, in the abstract of his Part IV, an averaged criterion: the mean energy of the prime-counting error is at least $2B$ unconditionally, with equality if and only if the Riemann hypothesis holds with simple zeros; the test here is exact at each fixed height, with a certificate, and we have read only that abstract. Numerical verifications of the Riemann hypothesis reach far greater heights [13]; the certificate here is

small, and its interest is that it uses no zero ordinates.

Scope. Theorems 1 and 2 are exact equivalences; they give no estimate of $D_{N,k}$ at large height, so they do not decide whether all zeros are simple. Theorem 3 is a finite computation; its rigour is that of ball arithmetic and of the programs (Section 6), and it is not formally verified.

2 Proof overview

The difficulty is to see multiplicity through an operator that cannot see it. In $\mathcal{H} = \ell^2(\Gamma, m)$, the operator C has the single eigenvalue $1/\gamma$ on the coordinate of γ, whatever m_γ is; the weight m_γ only scales the norm. So $\operatorname{tr} C^{2k} = 2\sum_{\gamma>0} \gamma^{-2k}$, while the Taylor moments S_k of $\log \Xi$ carry the weights m_γ. Their difference is $2\sum (m_\gamma - 1)\gamma^{-2k} \geq 0$, which vanishes exactly when all zeros are simple. The problem is that neither $\mathcal{H}$ nor C is finite.

Suzuki's compressions supply the finite objects. Compression to a subspace lowers every singular value, so the finite defect $D_{N,k}$ is at least the infinite one; nested subspaces make it decrease; and Hilbert–Schmidt convergence, which holds for any increasing family of subspaces with dense union, makes it converge. Raising k isolates the largest entry of the sequence $(1/\gamma)$ repeated $m_\gamma - 1$ times, which is $1/\gamma_*$. The integer form counts the eigenvalues of C_N above $1/T$ by Sylvester's law of inertia.

The arithmetic enters only through the matrix entries. For compactly supported test functions the sampling inner product equals the arithmetic side of the explicit formula, which involves the prime powers below e^{2N} and two archimedean integrals; the primitive $-i\int f$ realizes C on mean-zero functions. Alpöge's theorem is used in one place: it makes Γ real, so that Q is the positive form $\sum m_\gamma |\widehat{u}(\gamma)|^2$ and C is self-adjoint. The density of the dyadic windows, which Suzuki's argument needs, is proved directly in Lemma 6.

3 The Hilbert space, the windows and the operator

Throughout, the Riemann hypothesis holds, and $\widehat{u}(t) = \int_{\mathbb{R}} u(x) e^{itx} dx$. For $u \in L^2(\mathbb{R})$ with compact support, $\widehat{u}$ is entire, and the sampling vector is $Su = (\widehat{u}(\gamma))_{\gamma\in\Gamma}$.

Lemma 5 (Sampling bound)**.** *If $u \in H^1(\mathbb{R})$ is supported in $[-a, a]$, then $|\widehat{u}(t)| \leq \|u'\|_{L^1}/|t|$ for $t \neq 0$, and $\|Su\|_{\mathcal{H}}^2 \leq 2a\,\sigma\,\|u'\|_{L^2}^2$.*

Proof. Integration by parts gives $\widehat{u'}(t) = -it\,\widehat{u}(t)$, so $|t\widehat{u}(t)| \leq \|u'\|_{L^1} \leq \sqrt{2a}\,\|u'\|_{L^2}$. Summing $m_\gamma|\widehat{u}(\gamma)|^2 \leq m_\gamma\gamma^{-2} \cdot 2a\|u'\|_2^2$ over Γ gives the second bound. □

Lemma 6 (Density of the windows)**.** *$V_N \subset V_{N+1}$, and $\bigcup_N SV_N$ is dense in $\mathcal{H}$.*

Proof. A function linear on the cells of $2^{-N}\mathbb{Z}$ is linear on the cells of $2^{-N-1}\mathbb{Z}$, and $(-N, N) \subset (-N-1, N+1)$; so $V_N \subset V_{N+1}$.

Test functions are dense. Let $c \in \mathcal{H}$ be orthogonal to $SC_c^\infty(\mathbb{R})$. The measure $\tau = \sum_\gamma m_\gamma \overline{c_\gamma}\,\delta_\gamma$ is tempered, since $\sum_\gamma m_\gamma|c_\gamma|(1+\gamma^2)^{-1} \leq \|c\|\sigma^{1/2} < \infty$, and $\tau(\widehat{u}) = 0$ for all $u \in C_c^\infty(\mathbb{R})$. The Fourier transform is an automorphism of the Schwartz space $\mathcal{S}$ and C_c^∞ is dense in $\mathcal{S}$, so $\widehat{C_c^\infty}$ is dense in $\mathcal{S}$. Hence $\tau = 0$, and $c = 0$.

The mean can be removed. Let $\psi \in C_c^\infty(\mathbb{R})$ with $\int \psi = 1$ and $\psi_R(x) = R^{-1}\psi(x/R)$. Then $\widehat{\psi_R}(t) = \widehat{\psi}(Rt)$, $|\widehat{\psi}(t)| \leq c_\psi t^{-2}$, and $\|S\psi_R\|^2 = \sum m_\gamma|\widehat{\psi}(R\gamma)|^2 \to 0$ as $R \to \infty$ by dominated convergence (each term tends to 0 and is at most $c_\psi^2 m_\gamma \gamma^{-4}$ for $R \geq 1$, which is summable). For $u \in C_c^\infty$, $u - (\int u)\psi_R$ has mean zero and its sampling vector tends to Su.

Mean-zero test functions are limits of V_N. Let $v \in C_c^\infty$ have mean zero and support in $[-a,a]$, and let $N \ge a+1$, $h = 2^{-N}$. Let $I_N v$ be the continuous function, linear on the cells of $h\mathbb{Z}$, that agrees with v at $h\mathbb{Z}$. On each cell, $(v - I_N v)'$ has mean zero, so Wirtinger's inequality gives $\|(v-I_Nv)'\|_{L^2} \le (h/\pi)\|v''\|_{L^2}$; and, since $I_Nv - v$ vanishes outside $[-a-h, a+h]$, $|\int I_N v| = |\int (I_N v - v)| \le (2a+2h)\, h^2 \|v''\|_\infty/12$ by the trapezoid rule. Put $v_N = I_N v - h^{-1}(\int I_N v)\, t_0 \in V_N$, where t_0 is the hat at 0, of integral h and with $\|t_0'\|_2^2 = 2/h$. By Lemma 5 with support in $[-a-1, a+1]$,

$$\|S(v - v_N)\| \le \sqrt{2(a+1)\sigma}\,\Big(\frac{h}{\pi}\|v''\|_2 + \frac{(2a+2h)h\|v''\|_\infty}{12}\sqrt{2/h}\Big) \longrightarrow 0.$$

The three steps give the density. □

On $\mathcal{H}$ let $(Cc)_\gamma = c_\gamma/\gamma$. The vectors $e_\gamma = m_\gamma^{-1/2}\mathbf{1}_{\{\gamma\}}$ form an orthonormal basis and $Ce_\gamma = \gamma^{-1}e_\gamma$, so C is bounded and self-adjoint, with $\|C\|_{S_2}^2 = \sum_{\gamma\in\Gamma}\gamma^{-2} \le \sigma$ (as in [15, §2.3]), and, for $k \ge 1$,

$$\operatorname{tr}|C|^{2k} = \sum_{\gamma\in\Gamma}\gamma^{-2k} = 2\sum_{\gamma>0}\gamma^{-2k}.$$

Its eigenvalues are $\pm 1/\gamma$, one for each distinct ordinate.

Lemma 7 (The primitive)**.** *For $f \in V_N$, let $F(x) = -i\int_{-\infty}^x f(y)\,dy$. Then $F \in H^1(\mathbb{R})$ is supported in $[-N,N]$ and $SF = CSf$.*

Proof. F vanishes outside $[-N,N]$ because $\int f = 0$. From $F' = -if$ and $\widehat{F'}(t) = -it\widehat{F}(t)$ we get $\widehat{F}(t) = \widehat{f}(t)/t$ for $t \ne 0$. □

4 Proof of Theorems 1 and 2

The moments. Ξ is even, entire of order one, with $\Xi(0) \ne 0$ and $\sum m_\gamma\gamma^{-2} < \infty$; its Hadamard product, paired over $\pm\gamma$, is $\Xi(z) = \Xi(0)\prod_{\gamma>0}(1 - z^2/\gamma^2)^{m_\gamma}$ (the exponential factors cancel in pairs and the remaining factor e^{bz} is 1 by evenness). For $|z| < \min_{\gamma>0}\gamma$, $\log(\Xi(z)/\Xi(0)) = -\sum_{\gamma>0} m_\gamma \sum_{j\ge1} z^{2j}/(j\gamma^{2j})$, which gives $S_k = 2\sum_{\gamma>0} m_\gamma\gamma^{-2k}$.

The lower bound. Write $s_1 \ge s_2 \ge \cdots$ for singular values. Since $\|P_N\| \le 1$, $s_j(P_NCP_N) \le s_j(C)$ for every j. The operator C_N is self-adjoint of finite rank r, so $\operatorname{tr} C_N^{2k} = \sum_{j\le r} s_j(C_N)^{2k} \le \sum_{j\le r} s_j(C)^{2k} < \operatorname{tr}|C|^{2k}$, the last inequality being strict because C has infinitely many nonzero singular values. Hence

$$D_{N,k} > S_k - \operatorname{tr}|C|^{2k} = 2\sum_{\gamma>0}\frac{m_\gamma - 1}{\gamma^{2k}} \ge \frac{2E(T)}{T^{2k}},$$

the last step keeping only the ordinates $\gamma \le T$, for which $\gamma^{-2k} \ge T^{-2k}$.

Monotonicity and the limit. The subspaces SV_N increase with N (Lemma 6). By the max–min principle, the n-th largest positive eigenvalue of the compression of C to a subspace does not decrease when the subspace grows; applied to $-C$, the same holds for the absolute values of the negative eigenvalues. So $\operatorname{tr} C_N^{2k}$ is nondecreasing in N and $D_{N,k}$ is nonincreasing. Since the union of the subspaces is dense, $P_N \to I$ strongly. Following [15, (4.1)]: $C - P_NCP_N = (I-P_N)C + P_NC(I-P_N)$, and $\|P_NC(I-P_N)\|_{S_2} \le \|(I-P_N)C\|_{S_2}$ by taking adjoints; and $\|(I-P_N)C\|_{S_2}^2 = \sum_\gamma \gamma^{-2}\|(I-P_N)e_\gamma\|^2 \to 0$ by dominated convergence. So $C_N \to C$ in Hilbert–Schmidt norm. For $k \ge 1$,

$$C^{2k} - C_N^{2k} = \sum_{j=0}^{2k-1} C^j (C - C_N) C_N^{2k-1-j},$$

and each summand has a factor C or C_N besides $C - C_N$; with $\|UV\|_{S_1} \le \|U\|_{S_2}\|V\|_{S_2}$ and bounded operator norms, $C_N^{2k} \to C^{2k}$ in trace norm. Hence $D_{N,k} \to S_k - \operatorname{tr} C^{2k} = 2\sum(m_\gamma - 1)\gamma^{-2k}$.

The supremum. Let v be the sequence of the numbers $1/\gamma$, $\gamma > 0$, each repeated $m_\gamma - 1$ times; it lies in ℓ^2, and $\lim_N D_{N,k} = 2\|v\|_{2k}^{2k}$. By monotonicity, $\sup_N H_{N,k} = \|v\|_{2k}^{-1}$ (infinite if v is empty). If v is not empty, its largest entry is $1/\gamma_*$, and $\gamma_*^{-2k} \le \sum_j v_j^{2k} \le \gamma_*^{-(2k-2)} \sum_j v_j^2$; taking $2k$-th roots, $\|v\|_{2k} \ge 1/\gamma_*$ for every k and $\|v\|_{2k} \to 1/\gamma_*$. So $\sup_{N,k} H_{N,k} = \gamma_*$.

The equivalences. If $D_{N,k} < 2/T^{2k}$, then $E(T) < 1$, so $E(T) = 0$. Conversely, if $E(T) = 0$, then $\gamma_* > T$ (the multiple ordinates are discrete), and the supremum gives N, k with $H_{N,k} > T$, that is $D_{N,k} < 2/T^{2k}$. All zeros are simple if and only if $\gamma_* = \infty$. This proves Theorem 1.

The integer form. For $f = \sum_j c_j b_j \in V_N$, $c^* G_N c = \|Sf\|^2$ and $c^* H_N c = \langle CSf, Sf\rangle$. The Hermitian form $c \mapsto c^*(TH_N - G_N)c$ is therefore the pull-back, under the surjection $c \mapsto Sf$ onto SV_N, of the form $x \mapsto \langle (TC_N - I)x, x\rangle$ on SV_N. This holds whether or not G_N is invertible: if the map $c \mapsto Sf$ has a kernel, the form on coefficients vanishes on that kernel, and a subspace on which it is positive definite meets the kernel only in 0 and is mapped injectively onto a positive subspace of SV_N; conversely a positive subspace of SV_N lifts to one of the same dimension. So the pull-back has the same positive index. Thus $\iota_N(T)$ is the number of eigenvalues of C_N on SV_N greater than $1/T$. By max–min these are at most the eigenvalues of C greater than $1/T$, which are the $1/\gamma$ with $0 < \gamma < T$, one per distinct ordinate, and they do not decrease with N. Since $\Xi(T) \ne 0$, no eigenvalue of C equals $1/T$; operator-norm convergence $C_N \to C$ then puts each of the finitely many eigenvalues of C above $1/T$ within reach for large N, so $\iota_N(T)$ eventually equals the number of distinct ordinates in $(0, T]$. That number equals $n(T)$ exactly when those zeros are simple. This proves Theorem 2.

Proposition 8 (Size of a certificate)**.** *Let $\gamma_1 \le \gamma_2 \le \cdots$ be the positive ordinates listed with multiplicity. Then $D_{N,k} \ge 2\sum_{j > d_N/2} \gamma_j^{-2k}$. Hence $D_{N,k} < 2/T^{2k}$ requires $d_N \ge 2n(T)$.*

Proof. $\operatorname{tr} C_N^{2k}$ is at most the sum of the d_N largest values of $s_j(C)^{2k}$. These values are γ^{-2k}, each distinct $\gamma \in \Gamma$ once, and they are dominated term by term by the list $\pm\gamma_j$ with multiplicity, whose d_N largest values sum to $2\sum_{j \le d_N/2} \gamma_j^{-2k}$. Subtracting from S_k gives the bound. If $n(T) > d_N/2$, the term $j = n(T)$ alone contributes at least $2T^{-2k}$. □

5 The arithmetic evaluation

The explicit formula. For $h \in C_c^\infty(\mathbb{R})$ define

$$\begin{aligned} W(h) = {} & \int_{\mathbb{R}} h(x)\big(e^{x/2} + e^{-x/2}\big)\,dx - \big(\log\pi + \gamma_E\big)\,h(0) \\ & + \int_0^\infty \frac{2e^{-2x}h(0) - e^{-x/2}\big(h(x) + h(-x)\big)}{1 - e^{-2x}}\,dx \\ & - \sum_{n \ge 2} \frac{\Lambda(n)}{\sqrt{n}}\big(h(\log n) + h(-\log n)\big). \end{aligned}$$

Lemma 9. *If $u, v \in H^1(\mathbb{R})$ have compact support and $h = u * \widetilde{v}$, $\widetilde{v}(x) = \overline{v(-x)}$, then $\langle Su, Sv\rangle_{\mathcal{H}} = W(h)$. Only the prime powers n with $\log n$ in the support of h contribute.*

Proof. For $f \in C_c^\infty(0,\infty)$, the explicit formula in Bombieri's form [7, Appendix "Explicit formula", its first three displayed formulas] reads $\sum_\rho \widetilde{f}(\rho) = \int_0^\infty f + \int_0^\infty f^\sharp - \mathcal{W}_{\mathbb{R}}(f) - \sum_p \mathcal{W}_p(f)$, with $\widetilde{f}(s) = \int_0^\infty f(x)x^{s-1}dx$, $f^\sharp(x) = x^{-1}f(x^{-1})$, $\mathcal{W}_p(f) = \log p \sum_{m\ge1}(f(p^m) + f^\sharp(p^m))$ and $\mathcal{W}_{\mathbb{R}}(f) = (\log 4\pi + \gamma_E)f(1) + \int_1^\infty (f(x) + f^\sharp(x) - 2x^{-1}f(1))\,dx/(x - x^{-1})$. Take $u, v \in C_c^\infty$ and $f(e^y) = e^{-y/2}h(y)$. For $\rho = \frac12 + i\gamma$, $\widetilde{f}(\rho) = \widehat{h}(\gamma) = \widehat{u}(\gamma)\overline{\widehat{v}(\gamma)}$, so the left side is $\langle Su, Sv\rangle$, an absolutely convergent sum by Lemma 5. With $x = e^y$: $\int f + \int f^\sharp = \int h(y)(e^{y/2} + e^{-y/2})dy$; $f(e^y) + f^\sharp(e^y) = e^{-y/2}(h(y) + h(-y))$; $dx/(x - x^{-1}) = dy/(1 - e^{-2y})$; and $\sum_p \mathcal{W}_p(f) = \sum_n \Lambda(n)n^{-1/2}(h(\log n) + h(-\log n))$. This gives $W(h)$ with the constant $-(\log 4\pi + \gamma_E)$ and $2e^{-y}h(0)$ in the numerator; since $\int_0^\infty 2(e^{-2y} - e^{-y})(1 - e^{-2y})^{-1}dy = -\log 4$ (substitute $w = e^{-y}$), it equals $W(h)$ as written.

For $u, v \in H^1$ with support in $[-a, a]$, choose $u_n, v_n \in C_c^\infty$ with support in $[-a-1, a+1]$ converging to u, v in H^1. The left sides converge by Lemma 5. The functions $h_n = u_n * \widetilde{v}_n$ and $h_n' = u_n' * \widetilde{v}_n$ converge uniformly to h and h', with supports in a fixed interval. Each term of W is continuous for this convergence: the first two and the finite prime sum obviously; in the third, on $(0,1]$ the numerator is at most $c\,x(\|h\|_\infty + \|h'\|_\infty)$ and $1 - e^{-2x} \ge (1 - e^{-2})x$, and on $[1,\infty)$ the integrand is at most $c\,e^{-x/2}\|h\|_\infty$. □

The matrices. Let $b = t_0 * t_0$, the correlation of the central hat:

$$b(x) = \begin{cases} 2h/3 - x^2/h + |x|^3/(2h^2), & |x| \le h, \\ (2h - |x|)^3/(6h^2), & h \le |x| \le 2h, \\ 0, & |x| \ge 2h, \end{cases}$$

and $A_d(x) = \int_0^d b(x-u)\,du$. Put $g(d) = W(b(\cdot - d))$, which is even in d, and $a(d) = W(A_d)$, which is odd (because $W(\phi(-\cdot)) = W(\phi)$ and $A_{-d}(x) = -A_d(-x)$). Since $t_j * \widetilde{t_i} = b(\cdot - (j-i)h)$, Lemma 9 gives

$$(G_N)_{ij} = g((j-i)h) - g(jh) - g(ih) + g(0).$$

For H_N, Lemma 7 gives $CSb_j = SF_j$ with $F_j = -i\int_{-\infty}^x (t_j - t_0) = i\int_0^{jh} t_0(\cdot - u)\,du$. Then $F_j * \widetilde{t_i} = iA_{jh}(\cdot + ih)$, and $A_d(\cdot + c) = A_{d-c} - A_{-c}$, so Lemma 9 gives

$$(H_N)_{ij} = i\,B_{ij}, \qquad B_{ij} = a((j-i)h) - a(jh) + a(ih),$$

with B real antisymmetric. If G_N is invertible, the matrix of C_N on SV_N in the basis Sb_j is $G_N^{-1}H_N$, and

$$\operatorname{tr} C_N^{2k} = (-1)^k \operatorname{tr}\big((G_N^{-1}B)^{2k}\big).$$

Every test function here is supported in $[-2N, 2N]$, so only the prime powers $n < e^{2N}$ enter. In the computation, $A_{mh} = \sum_{0\le j<m} \beta(\cdot - jh)$ with $\beta(x) = \int_0^h b(x-u)\,du$, so that each functional is evaluated on a piecewise polynomial supported on five cells.

6 The certificate

Table 1 gives, for six windows, an enclosure of $D_{N,k}$ and the threshold $2/T^{2k}$. Each row satisfies the strict test $D_{N,k} < 2/T^{2k}$, so by Theorem 1 every zero with $0 < \Im\rho \le T$ is simple; the last row is Theorem 3. Each height is $H_{N,k}$ rounded down, and $d_N \ge 2n(T)$ as Proposition 8 requires ($n(97.632) = 28$).

N	d_N	prime powers	k	T	$D_{N,k}$ (enclosure)	$2/T^{2k}$
1	2	5	9	14.134	$3.94653014372982665\ldots\cdot10^{-21}$	$3.9469209\ldots\cdot10^{-21}$
2	14	24	3	20.704	$2.53910511009839593\ldots\cdot10^{-8}$	$2.5392440\ldots\cdot10^{-8}$
3	46	98	3	26.782	$5.41964807266176611\ldots\cdot10^{-9}$	$5.4196589\ldots\cdot10^{-9}$
4	126	465	2	39.913	$7.88033023780063053\ldots\cdot10^{-7}$	$7.8808399\ldots\cdot10^{-7}$
5	318	2532	2	62.413	$1.31802357404210376\ldots\cdot10^{-7}$	$1.3180435\ldots\cdot10^{-7}$
6	766	15040	3	97.632	$2.30927091609723675\ldots\cdot10^{-12}$	$2.3092815\ldots\cdot10^{-12}$

Table 1: Strict height certificates. The enclosure radii are below 10^{-26} in every row; the digits shown are inside the enclosure.

How the enclosures were computed. All quantities are evaluated in Arb ball arithmetic [3], which returns for each real number an interval guaranteed to contain it. (i) The functionals $g(d)$ and $a(d)$: on each cell of $h\mathbb{Z}$ the test function is a polynomial of degree at most four, recovered exactly by interpolation; the first, third and archimedean integrals are evaluated by rigorous ball-arithmetic integration [4], the integrand on $[0,h]$ being written as $(-3h(0)e^{-x/2}\varphi(-3x/2)-e^{-x/2}\psi_1(x))/(2\varphi(-2x))$ with $\varphi(z)=(e^z-1)/z$ and $h(x)+h(-x)=2h(0)+x\psi_1(x)$, which removes the singularity at 0; the tail $\int_h^\infty 2e^{-2x}(1-e^{-2x})^{-1}dx=-\log(1-e^{-2h})$ is exact; the prime sum is finite. (ii) G_N and B are assembled from these balls; a rigorous linear solve certifies that G_N is invertible (so it is positive definite, being a Gram matrix) and encloses $G_N^{-1}B$; the trace of its $2k$-th power is enclosed. (iii) S_k is the coefficient of z^{2k} in the power series of $\log(s(s-1)/2)-\frac{s}{2}\log\pi+\log\Gamma(s/2)+\log\zeta(s)$ at $s=\frac12+iz$, computed in ball arithmetic. No zero of ζ is used.

The computation was done twice, by two independently written programs: once by the producer of the mathematics, at 120 digits with a preconditioned positive-definiteness test, and once for this note, at 150 digits, by the program on the workbench [16]. The enclosures agree. The rigour of the certificate is that of ball arithmetic and of these programs; it is not formally verified.

7 Explicit constants

Proof of Proposition 4. Let $\gamma\ge10$ be an ordinate of multiplicity m and $s=2+i\gamma$. Pairing the Hadamard product of ξ, $\Re\frac{\xi'}{\xi}(s)=\sum_\rho\Re\frac{1}{s-\rho}$, and every term is positive because $\Re\rho=\frac12$; the m terms with $\rho=\frac12+i\gamma$ give $m/(3/2)$. So $\frac{2m}{3}\le\Re\frac{\xi'}{\xi}(2+i\gamma)$. Now $\frac{\xi'}{\xi}(s)=\frac1s+\frac{1}{s-1}-\frac12\log\pi+\frac12\frac{\Gamma'}{\Gamma}(\frac{s}{2})+\frac{\zeta'}{\zeta}(s)$. Here $\Re(\frac1s+\frac{1}{s-1})<3/\gamma^2$; $|\frac{\zeta'}{\zeta}(s)|\le\sum\Lambda(n)n^{-2}<0.57687<\frac35$ (summing the prime powers up to 1000 and bounding the rest by $\int_{1000}^\infty(\log x)x^{-2}dx=(\log1000+1)/1000$); and with $y=\gamma/2$, $\Re\frac{\Gamma'}{\Gamma}(1+iy)=-\gamma_E+\sum_{n\ge1}\frac{y^2}{n(n^2+y^2)}\le-\gamma_E+\frac{y^2}{1+y^2}+\frac12\log(1+y^2)$, comparing the decreasing summand with its integral. Collecting,

$$\Re\frac{\xi'}{\xi}(2+i\gamma)<\frac12\log\gamma+\frac{4}{\gamma^2}+\frac{11}{10}-\frac12(\gamma_E+\log2\pi)<\frac12\log\gamma+\frac{1}{25}-\frac{1}{20}<\frac12\log\gamma,$$

using $\frac14\log(1+4/\gamma^2)<1/\gamma^2$, $\gamma_E>\sum_{j\le6}j^{-1}-\log7>\frac12$ and $\log2\pi>\frac95$. Hence $m<\frac34\log\gamma$, that is $\gamma>e^{4m/3}$. Theorem 3 gives $\gamma_*>97.632$, and $97.632^2/2=4766.003712$. □

8 The backward heat flow

Let $\Phi(u)=\sum_{n\ge1}(2\pi^2n^4e^{9u}-3\pi n^2e^{5u})e^{-\pi n^2e^{4u}}$ and, for $t\in\mathbb{R}$, $H_t(z)=\int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du$; then $H_0(z)=\frac18\xi(\frac12+\frac{iz}{2})=\frac18\Xi(z/2)$ [14, §1, (1)–(4)], a zero of Ξ at γ is a zero of H_0 at $x=2\gamma$,

and differentiation under the integral gives $\partial_t H_t = -\partial_z^2 H_t$. For $a > 0$ write

$$F_a(w) = \frac{H_{-a}(i\sqrt{w})}{H_{-a}(0)} = \sum_{k\ge 0} b_k(a) w^k, \qquad J_{d,a}(X) = \sum_{k=0}^{d} (d)_k \, b_k(a) X^k,$$

with the falling factorial $(d)_k = d(d-1)\cdots(d-k+1)$, and let $j_*(-a) \in [1,\infty]$ be the least d for which $J_{d,a}$ has a nonreal zero (∞ if there is none).

Proposition 10. *Assume the Riemann hypothesis. Then* $\lim_{a\downarrow 0} a\, j_*(-a) = \gamma_*^2/2$, *the limit being* $+\infty$ *when all zeros are simple. With Theorem 1 this reads* $\lim_{a\downarrow 0} a\, j_*(-a) = \frac{1}{2}(\sup_{N,k} H_{N,k})^2$, *and Proposition 4 gives the value* > 4766.003712.

The proof occupies the rest of this section. Write $x_* = 2\gamma_*$, so that $\gamma_*^2/2 = x_*^2/8$, and let m_x be the multiplicity of the zero x of H_0.

Growth and products. Each term of Φ is positive for $u \ge 0$, since $2\pi n^2 e^{4u} \ge 2\pi > 3$; so $H_{-a}(iy) > 0$ for real y, $b_k(a) > 0$, and $F_a > 0$ on $[0,\infty)$. Splitting the integral at $u = \log(R+3)$, beyond which $e^{-\pi e^{4u}}$ dominates, gives $\log \max_{|z|\le R} |H_t(z)| = O(R\log(R+3))$ uniformly for t in compact sets. Hence $G(a,w) = H_{-a}(i\sqrt{w})$ is entire in (a,w) (it is even in $\sqrt{w}$), of order at most $\frac{1}{2}$ in w, and $F_a(w) = \prod_i (1 - w/w_i)$ over its zeros, which are $w_i = -z_i^2$ for the zeros z_i of H_{-a} with $\Re z_i > 0$; none lies on $[0,\infty)$. At $a = 0$, by the Riemann hypothesis, $H_0(z) = H_0(0)\prod_{x>0}(1 - z^2/x^2)^{m_x}$ and, by Jensen's formula and the growth bound, $N_H(R) = \sum_{0<x\le R} m_x \ll R\log(R+3)$.

The zeros of H_{-a} lie in thin cones. For nonreal z put $L(z) = H_0'(z)/H_0(z)$ and $S(z) = \sum_{x>0} m_x(|z-x|^{-2} + |z+x|^{-2})$. The entire function $Q_z(v) = H_0(z+v)/(H_0(z)e^{L(z)v})$ is the product of $(1+v/(z-\xi))e^{-v/(z-\xi)}$ over the zeros $\xi = \pm x$, so $|1+w|^2 \le e^{2\Re w + |w|^2}$ gives $|Q_z(v)| \le e^{S|v|^2/2}$; with $Q_z(0) = 1$, $Q_z'(0) = 0$ and Cauchy's estimate this gives $|Q_z(v) - 1| \le 8S|v|^2 e^{2S|v|^2}$. Since H_{-a} is the convolution of H_0 with the Gaussian of variance $2a$, moving the line of integration by $p = 2aL(z)$ and completing the square gives $H_{-a}(z)/(H_0(z)e^{aL(z)^2}) = \mathbb{E}Q_z(p+U)$ with $U \sim N(0,2a)$. With $A = aS(z)$, $B = a^2|L(z)|^2 S(z)$ and $|p+U|^2 \le 2|p|^2 + 2U^2$, the Gaussian moments give, for $16A < 1$,

$$\Big|\frac{H_{-a}(z)}{H_0(z)e^{aL(z)^2}} - 1\Big| \le 16e^{16B}\Big(\frac{4B}{\sqrt{1-16A}} + \frac{2A}{(1-16A)^{3/2}}\Big),$$

which is at most $\frac{7}{16}$ when $A, B \le 2^{-10}$; then $H_{-a}(z) \ne 0$. If $|z| = r$ and $|\Im z| \ge qr$, splitting the zeros at $r/2$ and $2r$ and using N_H gives $S(z) \ll q^{-2}\log(r+3)/(r+1)$ and $|L(z)| \ll q^{-1}\log(r+3)$. Fix $\eta \in (0, \frac{1}{2})$ and take $q = \eta\sqrt{a}$: then $A \ll \eta^{-2}\log(r+3)/(r+1)$ and $B \ll \eta^{-4}\log^3(r+3)/(r+1)$, uniformly in $0 < a \le 1$. Hence, for every $\kappa > 0$ there is R_0 such that every zero of H_{-a}, $0 < a \le 1$, with $|z| \ge R_0$ and $\Re z > 0$ has $|\arg z| \le \kappa\sqrt{a}$.

Jensen polynomials as differential operators. For a power series $F = \sum f_k w^k$, $F(D)z^d = \sum_k f_k (d)_k z^{d-k} = z^d J_d(1/z)$ with $J_d(X) = \sum_{k\le d}(d)_k f_k X^k$; so J_d has only real zeros if and only if $F(D)z^d$ does. By the theorem of Borcea and Brändén [5, Theorem 2(b), p. 5], a real linear operator on polynomials of degree at most d preserves real-rootedness if its symbol $T[(z+w)^d]$ has no zero with $\Im z, \Im w > 0$; this is their condition (b), since $H_2(\mathbb{R})$ there denotes the real stable polynomials in two variables, those with real coefficients and no zero with both imaginary parts positive. For $T = 1 + \beta D$ with $\beta > 0$ the symbol is $(z+w)^{d-1}(z+w+d\beta)$. For $T = 1 + 2\beta\cos\theta\, D + \beta^2 D^2$, $|\theta| < \pi/2$, it is $(z+w)^{d-2}\big((z+w)^2 + 2d\beta\cos\theta(z+w) + d(d-1)\beta^2\big)$, whose quadratic factor in $z+w$ has discriminant $4d\beta^2(1 - d\sin^2\theta)$ and hence real roots when $d\sin^2\theta \le 1$ (for $d = 1$ the symbol is $z + w + 2\beta\cos\theta$). Since $\Im(z+w) > 0$, these symbols do not vanish. Compositions of such operators preserve real-rootedness in degree at most d, and so do coefficientwise limits, by Hurwitz's theorem, when the limit is not zero.

The interpolation and its local form. Let $G(a,w)=\sum_k c_k(a)w^k$ and $\mathcal{J}(a,\epsilon,w)=\sum_k c_k(a)w^k\prod_{j<k}(1-j\epsilon)$. The growth bound and Cauchy's estimate give $|c_k(a)|\le C^k k^{-4k/3}$ for a in compact sets, and the product is at most $(1+E)^k k!$ for $|\epsilon|\le E$; so $\mathcal{J}$ is entire in (a,ϵ,w), and $\mathcal{J}(a,1/d,w)=H_{-a}(0)J_{d,a}(w/d)$. Expanding $\prod_{j<k}(1-j\epsilon)=\sum_s(-\epsilon)^s e_s(k)$, where $e_s(k)$, the elementary symmetric function of $0,\dots,k-1$, is a polynomial in k of degree $2s$ with leading coefficient $1/(2^s s!)$ (by $e_s(k+1)-e_s(k)=ke_{s-1}(k)$), and writing it in falling factorials of k, the coefficient of ϵ^s is a differential operator in w of order at most $2s$ with principal part $(-1)^s w^{2s}D^{2s}/(2^s s!)$. In the variable w, $\partial_a G=\mathcal{L}G$ with $\mathcal{L}=-4wD^2-2D$, so the coefficient of a^r in G is $\mathcal{L}^r G(0,\cdot)/r!$, with principal part $(-4w)^r D^{2r}/r!$.

Local form. Let $K(a,w)=\sum_k K_k(a)w^k$ be holomorphic for a near 0 and w in a disk about 0 that contains w_0, and suppose that its interpolation $\mathcal{J}_K(a,\epsilon,w)=\sum_k K_k(a)w^k\prod_{j<k}(1-j\epsilon)$ is holomorphic in (a,ϵ,w) there and entire in ϵ (for G this was shown above; for a polynomial it is clear). Suppose that, for each r, the coefficient of a^r in K vanishes at w_0 to order at least $m-2r$, with coefficient of $(w-w_0)^{m-2r}$ equal to $A_0(-4w_0)^r\,m!/(r!(m-2r)!)$, $A_0\neq 0$. Apply the interpolation $\mathcal{J}_K$ (the same multipliers applied to the Taylor coefficients of K in w) and substitute $a=\gamma q^2$, $\epsilon=q^2$, $w=w_0+qu$. A term $a^r\epsilon^s$ lowers the vanishing order by at most $2s$ and carries q^{2r+2s}, so no power below q^m occurs, and at q^m only the principal parts contribute:

$$\frac{\mathcal{J}_K(\gamma q^2,q^2,w_0+qu)}{A_0 q^m}=\sum_k\sum_{r+s=k}\frac{(-4\gamma w_0)^r}{r!}\frac{(-w_0^2/2)^s}{s!}D_u^{2k}u^m+O(q)=e^{cD_u^2}u^m+O(q),$$

with $c=4\gamma x^2-x^4/2$ when $w_0=-x^2$, uniformly for γ and u in compact sets. The function $K=G$ satisfies the hypothesis at $w_0=-x^2$ with $m=m_x$, by the principal parts of $\mathcal{L}^r$. If $c>0$, then $e^{cD^2}u^m=(i\sqrt{2c})^m\mathrm{He}_m(u/(i\sqrt{2c}))$ has the nonreal roots $i\sqrt{2c}\,h$ for the nonzero roots h of the Hermite polynomial He_m, which exist for $m\ge 2$; if $c<0$, it is $(\sqrt{2|c|})^m\mathrm{He}_m(u/\sqrt{2|c|})$, with m simple real roots.

Upper bound. Let $x_*<\infty$ and $\gamma>x_*^2/8$, so $c>0$ at $x=x_*$, where $m\ge 2$. Put $d=\lceil\gamma/a\rceil$ and $q=d^{-1/2}$; then $a=(ad)q^2$ with $ad\to\gamma$. By the local form and Rouché's theorem, $\mathcal{J}(a,1/d,\cdot)$, and hence $J_{d,a}$, has a nonreal zero for all small a. So $\limsup_{a\downarrow 0} a\,j_*(-a)\le\gamma$ for every $\gamma>x_*^2/8$.

Lower bound. Fix $0<\gamma_0<x_*^2/8$ (any $\gamma_0>0$ if $x_*=\infty$), put $\kappa=1/(4\sqrt{\gamma_0})$ and take R_0 from the cone estimate. Choose $R>R_0^2$ with no zero of F_0 on $|w|=R$; for small complex a the circle encloses a fixed number δ of zeros of F_a, the power sums $\frac{1}{2\pi i}\oint w^k F_a'/F_a\,dw$ are holomorphic in a, and Newton's identities give a polynomial P_a with these zeros and $P_a(0)=1$, real for real a. Then $F_a=P_aT_a$ with $T_a=\prod(1-w/w_i)$ over the zeros outside the circle. Near an enclosed endpoint $w_0=-x^2$, $P_a=UG$ with $U=1/(G(a,0)T_a)$ holomorphic and nonvanishing; multiplying by $U=\sum_j a^jU_j$ only raises vanishing orders in the terms with $j\ge 1$, so P_a satisfies the hypothesis of the local form with $A_0=AU_0(w_0)$.

The head. Apply the local form to P_a with $\gamma\in[0,\gamma_0]$. At a multiple endpoint $x\ge x_*$, so $c=4\gamma x^2-x^4/2<0$; at a simple endpoint the limit is u. Rouché disks about these δ real roots, uniform in γ, each contain one root, which is real because the coefficients are real; they exhaust the δ roots of the degree-δ polynomial. So there is d_1 with: $d\ge d_1$ and $ad\le\gamma_0$ imply that $J_d(P_a)$ has only real zeros. For $d<d_1$: $P_0=\prod_i(1+\beta_i w)$ with $\beta_i>0$, and applying $1+\beta D$ to a polynomial whose nonzero roots are simple and negative gives one of the same kind (the equation $p'/p=-1/\beta$ has one root in each gap and one to the left, p'/p being decreasing on each interval). So each $J_d(P_0)$, $d<d_1$, has simple real roots, which persist for small real a.

The tail. The zeros of T_a are $w=-z^2$ with $|z|>R_0$; real z give factors $1+\beta w$ with $\beta>0$, and the conjugate pairs give factors $1+2\beta\cos\theta\,w+\beta^2w^2$ with $|\theta|=2|\arg z|\le 2\kappa\sqrt{a}$. For $d\le\gamma_0/a$, $d\sin^2\theta\le 4\kappa^2\gamma_0=\frac14$. By the operator criterion, $T_a(D)$ preserves real-rootedness

in degree at most d; applied to $P_a(D)z^d$, which has only real zeros by the head, it shows that $F_a(D)z^d$, and so $J_{d,a}$, has only real zeros for every $d \le \gamma_0/a$ and all small a. Hence $\liminf_{a\downarrow 0} a\, j_*(-a) \ge \gamma_0$. Letting $\gamma_0 \uparrow x_*^2/8$, or $\gamma_0 \to \infty$ when $x_* = \infty$, and combining with the upper bound proves Proposition 10. □

9 What remains open

All zeros are simple if and only if $\sup_{N,k} H_{N,k} = \infty$, equivalently if and only if, at an unbounded sequence of heights T with $\Xi(T) \neq 0$, the integer deficit $n(T) - \mathrm{ind}_+(TH_N - G_N)$ vanishes for some N. A proof would need an upper estimate for $D_{N,k}$ at unbounded heights, with the dimension $d_N \ge 2n(T)$ that Proposition 8 forces. No such estimate is known to us.

Note on method

The mathematics of Theorems 1–3 and Propositions 4, 8 and 10 was produced by GPT-6 Astra (ChatGPT Pro mode), with the first computation of the certificate. Claude Opus 5.5 at effort high checked every proof and wrote this text, including the proofs of Lemma 6 and Lemma 9, and wrote the second program for the certificate; a third, floating-point computation of all six rows by the same model agrees with both. A separate instance of Claude Opus 5.5 at effort high, given only the statements of Theorems 1 and 2 and Suzuki's paper, re-proved both independently and then compared its proofs with ours; it found no gap, and the proof that the b_j form a basis is taken from it. Suzuki's paper was read at the statements cited, Connes and Consani's appendix at the explicit formula, and the abstracts of [2, 10, 12, 8, 9, 13]; the theorem of Borcea and Brändén at its statement, and Rodgers and Tao at the definitions of the heat family. The same separate instance of Claude Opus 5.5 at effort high then checked the proof of Proposition 10 step by step, opening both of these sources at their locators, and found no error. No human has checked this note. Our text and programs are dedicated to the public domain under CC0 1.0; the permanent record of this work is WB-0009, `https://github.com/KokunoYumeto/mathematics-workbenches/blob/main/works/WB-0009.md`; its programs and check records are at `https://github.com/KokunoYumeto/zeta-and-l-functions-workbench/tree/main/work/WB-0009/v1` [16].

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