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Windows, Oscillation, and the Critical Line for Dirichlet L-Functions

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Abstract

Let $\Psi(y,\chi)$ be the von Mangoldt remainder attached to a primitive Dirichlet character χ, and let $\sigma_0(\chi)$ be the supremum of the real parts of the nontrivial zeros of $L(s,\chi)$. We prove a syndetic oscillation theorem: if $\sigma_0(\chi) > \frac{1}{2}$ is attained by a zero, then for every $\theta < \sigma_0(\chi)$, every $C < \infty$ and every $\lambda > 1$, each interval $[Y, Y^\lambda]$ with Y sufficiently large contains a point y with $|\Psi(y,\chi)| > Cy^\theta$. The kernel width is chosen canonically, so that the amplitude is an explicit function of the rightmost zero alone and both the amplitude and the gap are independent of θ and C; real zeros are covered by the same statement. Combining this with the identity $\sigma_0(\chi) = \inf\{\theta : \Psi(y,\chi) = O(y^\theta)\}$, proved here from a smoothed explicit formula, we obtain an exact reformulation of the critical-line assertion as a statement about the existence of long clean windows. We then determine the logical strength of the arithmetic input that such a reformulation requires: a bound $|\Psi(x,\chi)| \le Ax^{1/2+\Gamma\varepsilon}$, uniform in χ and valid for every $\varepsilon > 0$, is equivalent to the critical-line assertion itself, whereas the large sieve in its standard form implies no pointwise exponent below 1. The two halves of the programme are therefore separated by an input of exactly the strength of the conclusion.

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1 Introduction

The functional equation $\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-s,\overline{\chi})$ has $s \mapsto 1-s$ as its involution, with fixed point $\frac{1}{2}$. Consequently, for a family of characters closed under conjugation, the assertion that all nontrivial zeros lie on $\Re s = \frac{1}{2}$ is equivalent to the one-sided assertion that none lies in $\Re s > \frac{1}{2}$ (Theorem 2.3). The value $\frac{1}{2}$ is supplied by symmetry; it is not the outcome of an optimisation, and no manipulation of constants can move it.

A bound $|\Psi(y,\chi)| \le Cy^\theta$ valid for all $y \ge 1$ forces $L(s,\chi) \ne 0$ in $\Re s > \theta$ (Theorem 3.3). Conversely, writing

$$\sigma_0(\chi) = \sup\{\Re\rho : L(\rho,\chi) = 0,\ 0 < \Re\rho < 1\},$$

one has $\sigma_0(\chi) = \inf\{\theta : \Psi(y,\chi) = O(y^\theta)\}$ (Proposition 3.6). Hence a zero to the right of θ forces $|\Psi(y,\chi)| > Cy^\theta$ for arbitrarily large y. The quantifier in that conclusion is weak: it asserts

$$\forall Y_0\ \exists y > Y_0 : \quad |\Psi(y,\chi)| > Cy^\theta,$$

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and this is compatible with the existence of arbitrarily long intervals on which the opposite inequality holds throughout (Proposition 7.1). Any argument that opposes a hypothetical zero to a supply of clean intervals must therefore use the stronger conclusion

$$\text{every sufficiently long window } [Y, Y^\lambda] \text{ contains a point with } |\Psi(y,\chi)| > Cy^\theta.$$

The first purpose of this paper is to prove that stronger conclusion, with explicit constants, when σ_0 is attained. The proof has four ingredients, each established below: a smoothed explicit formula whose zero sum carries no factor $1/\rho$ (Section 4); a kernel whose Mellin transform is nonnegative real on the line $\Re s = \sigma_0$ after removal of a phase (Lemma 4.2); syndetic phase alignment for a finite set of frequencies, obtained from minimality of a linear flow on a torus (Section 5); and zero counting, which controls the remaining zeros (Lemma 6.1). The width of the kernel is fixed once and for all in terms of the ordinate of the rightmost zero, which makes the amplitude explicit and removes any nondegeneracy condition on the width; the case of a real rightmost zero is included.

The second purpose is to fix the logical shape of the arithmetic half of such a programme. Theorem 7.2 reformulates the critical-line assertion as the existence of clean windows. Theorem 8.1 shows that the estimate which the arithmetic half must deliver is not weaker than the conclusion: it is equivalent to it. Proposition 8.2 computes what the large sieve actually yields for an individual character, and Proposition 8.3 shows that a length budget of stretched-exponential shape is vacuous at the scale in question. Section 9 records the structural reason: every positivity input available in this circle of ideas is of averaged type, and averaged positivity does not imply pointwise positivity.

Throughout, χ is a primitive Dirichlet character modulo q, $\Lambda(n)$ is the von Mangoldt function, $\delta_\chi = 1$ if $q = 1$ and $\delta_\chi = 0$ otherwise, and

$$\Psi(y,\chi) = \sum_{n \le y} \Lambda(n)\chi(n) - \delta_\chi y \qquad (y \ge 1).$$

Zeros of $L(s,\chi)$ in the open strip $0 < \Re s < 1$ are called nontrivial. For $C < \infty$ and $\theta \in \mathbb{R}$ we put

$$S(\chi, C, \theta) = \{y \ge 1 : |\Psi(y,\chi)| \le Cy^\theta\}.$$

Definition 1.1. For $S \subseteq [1,\infty)$ and $\lambda > 1$, say that S has the *window property* at exponent λ if

$$\forall Y_0 \ \exists Y \ge Y_0 : \quad [Y, Y^\lambda] \subseteq S.$$

Write $W(\chi,\theta)$ for the assertion that $S(\chi, C, \theta)$ has the window property at some exponent $\lambda > 1$ for some $C < \infty$.

In the variable $u = \log y$ a window $[Y, Y^\lambda]$ becomes an interval of length $(\lambda - 1)\log Y$, which tends to infinity. This is the only feature of windows that is used.

The classical facts used without proof are the Euler product and functional equation of the completed L-function, the zero-counting bound (Lemma 3.4), the truncated explicit formula (Theorem 3.5), the nonvanishing of $L(s,\chi)$ on $\Re s = 1$ in the quantitative form of the de la Vallée Poussin zero-free region, and the multiplicative large sieve (Proposition 8.2). Everything else is proved.

2 Preliminaries and the one-sided reduction

Let $a = a(\chi) \in \{0,1\}$ be defined by $\chi(-1) = (-1)^a$ and put

$$\Lambda(s,\chi) = \Big(\frac{q}{\pi}\Big)^{\frac{s+a}{2}} \Gamma\Big(\frac{s+a}{2}\Big) L(s,\chi),$$
$$\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-s,\overline{\chi}), \qquad |\varepsilon(\chi)| = 1.$$

Note that $\overline{\chi}$ is primitive of the same conductor q.

Lemma 2.1. *For $\Re s > 1$ one has $L(s,\chi) \neq 0$. On $0 < \Re s < 1$ the zeros of $\Lambda(\cdot,\chi)$ and the nontrivial zeros of $L(\cdot,\chi)$ coincide, with multiplicities.*

Proof. For $\Re s > 1$ the Euler product $\prod_p (1-\chi(p)p^{-s})^{-1}$ converges absolutely and no factor vanishes. The prefactor $(q/\pi)^{(s+a)/2}\Gamma((s+a)/2)$ is holomorphic and zero-free on $0 < \Re s < 1$: the gamma function has no zeros, and its poles occur at $(s+a)/2 \in \{0,-1,-2,\dots\}$, that is at $s \in \{-a,-a-2,\dots\} \subseteq (-\infty,0]$. □

Lemma 2.2. *If ρ is a nontrivial zero of $L(\cdot,\chi)$ of multiplicity m, then $1-\rho$ is a nontrivial zero of $L(\cdot,\overline{\chi})$ of multiplicity m.*

Proof. By Lemma 2.1, $\Lambda(\rho,\chi) = 0$, so the functional equation gives $\Lambda(1-\rho,\overline{\chi}) = 0$. Since $0 < \Re(1-\rho) < 1$, Lemma 2.1 applied to $\overline{\chi}$ makes $1-\rho$ a nontrivial zero of $L(\cdot,\overline{\chi})$. As $\varepsilon(\chi) \neq 0$ and $s \mapsto 1-s$ is biholomorphic, multiplicities agree. □

Theorem 2.3 (One-sided reduction)**.** *Let $\mathcal{C}$ be a set of primitive characters closed under conjugation. If no $L(\cdot,\chi)$ with $\chi \in \mathcal{C}$ has a zero in $\Re s > \frac{1}{2}$, then every nontrivial zero of every such L-function has real part exactly $\frac{1}{2}$.*

Proof. Let ρ be a nontrivial zero of $L(\cdot,\chi)$, $\chi \in \mathcal{C}$. The hypothesis gives $\Re\rho \leq \frac{1}{2}$. By Lemma 2.2, $1-\rho$ is a nontrivial zero of $L(\cdot,\overline{\chi})$ with $\overline{\chi} \in \mathcal{C}$, so $\Re(1-\rho) \leq \frac{1}{2}$, i.e. $\Re\rho \geq \frac{1}{2}$. □

Closure under conjugation is essential: for non-real χ the functional equation relates χ to $\overline{\chi}$, not to itself.

Proposition 2.4. *Let χ modulo N be induced by the primitive character χ^* modulo f, $f \mid N$. Then*

$$L(s,\chi) = L(s,\chi^*) \prod_{p \mid N,\, p \nmid f} \bigl(1-\chi^*(p)p^{-s}\bigr),$$

and $L(\cdot,\chi)$ and $L(\cdot,\chi^*)$ have the same zeros in $0 < \Re s < 1$, with multiplicities.

Proof. The Euler products differ exactly at the primes dividing N but not f. A factor $1-\chi^*(p)p^{-s}$ vanishes if and only if $\chi^*(p)p^{-s} = 1$; since $|\chi^*(p)| \in \{0,1\}$ and $|p^{-s}| = p^{-\Re s}$, this forces $|\chi^*(p)| = 1$ and $\Re s = 0$. Such points lie outside the open strip. □

By Proposition 2.4 it suffices to treat primitive characters.

3 The abscissa of the remainder

Lemma 3.1. *Let $(a_n)_{n\geq 1}$ be complex numbers with $\sum_n |a_n| n^{-\sigma} < \infty$ for some $\sigma > 0$, and let $A(y) = \sum_{n \leq y} a_n$. Then for $\Re s > \sigma$,*

$$\sum_{n \geq 1} a_n n^{-s} = s \int_1^\infty A(y) y^{-s-1}\, dy.$$

Proof. Writing $c = \Re s$, Tonelli's theorem gives

$$\int_1^\infty \sum_n |a_n| \mathbf{1}_{\{n \leq y\}} y^{-c-1}\, dy = \sum_n |a_n| \int_n^\infty y^{-c-1}\, dy = \frac{1}{c} \sum_n |a_n| n^{-c} < \infty,$$

so sum and integral may be interchanged. Since $\int_n^\infty y^{-s-1}\, dy = \frac{n^{-s}}{s}$, the claim follows. □

Applying Lemma 3.1 to $a_n = \Lambda(n)\chi(n)$ with $\sigma = 1$ gives, for $\Re s > 1$,

$$-\frac{L'}{L}(s,\chi) = s\int_1^\infty \Big(\sum_{n\le y}\Lambda(n)\chi(n)\Big) y^{-s-1}\,dy.$$

Lemma 3.2. *For $\Re s > 1$ set*

$$\Phi(s,\chi) = -\frac{L'}{L}(s,\chi) - \frac{\delta_\chi}{s-1} - \delta_\chi.$$

Then

$$\Phi(s,\chi) = s\int_1^\infty \Psi(y,\chi) y^{-s-1}\,dy.$$

Moreover $\Phi(\cdot,\chi)$ is meromorphic on $\Re s > 0$ and holomorphic there except for simple poles at the nontrivial zeros of $L(\cdot,\chi)$, the residue at a zero of multiplicity m being $-m$.

Proof. For $\Re s > 1$,

$$s\int_1^\infty \delta_\chi y\cdot y^{-s-1}\,dy = \delta_\chi\frac{s}{s-1} = \delta_\chi + \frac{\delta_\chi}{s-1},$$

and subtracting this from the previous display gives the identity. For the second claim, $L(\cdot,\chi)$ is holomorphic on $\Re s > 0$ (entire if $q > 1$; for $q = 1$ it is ζ, whose only pole is at $s = 1$). Hence $-L'/L$ is meromorphic there, with poles precisely at the zeros of L, of residue $-m$ at a zero of multiplicity m, and, when $q = 1$, a pole of residue $+1$ at $s = 1$. The trivial zeros lie in $\Re s \le 0$. Subtracting $\frac{\delta_\chi}{s-1}$ removes the pole at $s = 1$. □

Theorem 3.3. *Let χ be primitive and suppose $|\Psi(y,\chi)| \le Cy^\theta$ for all $y \ge 1$, where $C < \infty$ and $\theta \in (0,1)$. Then $L(s,\chi)$ has no zero with $\Re s > \theta$.*

Proof. Let $K \subset \{\Re s > \theta\}$ be compact and $\eta = \min_{s\in K}(\Re s - \theta) > 0$. For $s \in K$,

$$\Big|s\int_1^\infty \Psi(y,\chi)y^{-s-1}\,dy\Big| \le \Big(\sup_K |s|\Big) C\int_1^\infty y^{\theta-\Re s-1}\,dy \le \frac{C\sup_K|s|}{\eta}.$$

The truncated integrals $\int_1^R$ are entire in s and converge uniformly on K; by Morera's theorem the right-hand side of the identity in Lemma 3.2 is holomorphic on $\Re s > \theta$. Hence $\Phi(\cdot,\chi)$ is holomorphic there, and so is $-L'/L$ on $\{\Re s > \theta\}\setminus\{1\}$, while at $s = 1$ its singular part is exactly $\frac{\delta_\chi}{s-1}$. Since $L(\cdot,\chi)$ is holomorphic on $\Re s > 0$ and $-L'/L$ has no pole at a point of $\{\Re s > \theta\}$ other than $s = 1$, L has no zero there. □

Lemma 3.4 (classical)**.** *Let $N(T,\chi)$ be the number of nontrivial zeros of $L(\cdot,\chi)$ with $|\Im\rho| \le T$, counted with multiplicity. Then $N(T,\chi) \ll T\log(qT)$ for $T \ge 2$, with an absolute implied constant. Consequently*

$$\sum_{|\Im\rho|\le T}\frac{1}{|\rho|} \ll \log^2(qT), \qquad \sum_{|\Im\rho|>T}\frac{1}{|\Im\rho|^2} \ll \frac{\log(qT)}{T}.$$

Proof. The counting bound is classical; it follows from Jensen's formula applied to $(s-1)\Lambda(s,\chi)$ on a disc of radius $T+2$ centred at $2+it$, the functional equation and Stirling's formula giving $\log|\Lambda| \ll T\log(qT)$ on the boundary. Both sums follow by partial summation; for the second,

$$\sum_{|\Im\rho|>T}|\Im\rho|^{-2} \le \int_T^\infty 2u^{-3}N(u,\chi)\,du \ll \frac{\log(qT)}{T}.$$

For the first, partial summation against $N(u,\chi)$ gives $\sum_{|\Im\rho|\le T}|\rho|^{-1} \ll \log^2(qT)$, the zeros of bounded ordinate contributing $O(\log q)$. □

Theorem 3.5 (Truncated explicit formula; classical)**.** *Let χ be primitive modulo q and let $y \geq 2$ not be a prime power. With $T = y$,*

$$\Psi(y,\chi) = -\sum_{|\Im\rho|\leq T} \frac{y^\rho}{\rho} + O(\log^2(qy)),$$

the sum being over nontrivial zeros with multiplicity and the implied constant absolute.

This is the standard explicit formula with truncation at height T, obtained by truncated Perron inversion and a contour shift; the choice $T = y$ makes the truncation error $O(\log^2(qy))$. The structural feature of importance below is that the error is polylogarithmic, whereas a zero of real part σ contributes a term of size y^σ.

Proposition 3.6. *Let $\alpha(\chi) = \inf\{\theta \in \mathbb{R} : \Psi(y,\chi) = O(y^\theta)\}$. Then*

$$\alpha(\chi) = \limsup_{y\to\infty} \frac{\log|\Psi(y,\chi)|}{\log y} = \sigma_0(\chi),$$

with the convention $\log 0 = -\infty$.

Proof. Write L^* for the displayed limit superior. For each $\varepsilon > 0$ one has $|\Psi(y,\chi)| \leq y^{L^*+\varepsilon}$ for y large, and Ψ is bounded on bounded intervals, so $\alpha \leq L^* + \varepsilon$; conversely $\Psi(y,\chi) = O(y^\theta)$ gives $\frac{\log|\Psi(y,\chi)|}{\log y} \leq \theta + o(1)$, whence $L^* \leq \alpha$. Thus $\alpha = L^*$.

If $\sigma_0 > \alpha$, choose θ with $\alpha < \theta < \sigma_0$. By definition of the infimum there is $\theta' < \theta$ with $\Psi(y,\chi) = O(y^{\theta'})$, hence also $O(y^\theta)$; Theorem 3.3 then excludes zeros with $\Re\rho > \theta$, a contradiction. So $\sigma_0 \leq \alpha$.

Conversely, Theorem 3.5 and Lemma 3.4 give, for y not a prime power,

$$|\Psi(y,\chi)| \leq y^{\sigma_0} \sum_{|\Im\rho|\leq y} \frac{1}{|\rho|} + O(\log^2(qy)) \ll y^{\sigma_0}\log^2(qy).$$

For $y \in [n, n+1)$ choose a non-prime-power y' in the same interval; then $|\Psi(y,\chi)| \leq |\Psi(y',\chi)|\log y$, so the bound holds for all $y \geq 2$. Hence $\Psi(y,\chi) = O(y^{\sigma_0+\varepsilon})$ for every $\varepsilon > 0$ and $\alpha \leq \sigma_0$. □

Proposition 3.6 yields the weak conclusion whenever $\sigma_0(\chi) > \theta$. It does not yield the window conclusion; see Section 7.

4 A smoothed explicit formula

The truncated formula of Theorem 3.5 carries a factor $1/\rho$ at each zero, which makes a lower bound for the zero sum delicate. A smoothed version removes it.

Definition 4.1. Let $H(v) = (1-|v|)_+$. For $L > 0$ and $u_0 \in \mathbb{R}$ put

$$h(u) = \frac{1}{L}H\Big(\frac{u-u_0}{L}\Big), \qquad g(y) = h(\log y)y^{-\sigma} \quad (y > 0),$$

where $\sigma \in (0,1)$ is fixed. Then $h \geq 0$, $\int_{\mathbb{R}} h = 1$, $\operatorname{supp} h = [u_0 - L, u_0 + L]$ and $\int_{\mathbb{R}} |h'| = 2/L$. Write

$$w(u) = \sigma h(u) - h'(u), \qquad \int_{\mathbb{R}} |w| \leq \sigma + 2/L.$$

Lemma 4.2. *Let $G(s)=\int_0^\infty g(y)y^{s-1}\,dy$ and $F(z)=\big(\frac{\sinh(z/2)}{z/2}\big)^2$ with $F(0)=1$. Then G is entire and*

$$G(s)=e^{(s-\sigma)u_0}F\big((s-\sigma)L\big).$$

Moreover:

(i) *for $\gamma\in\mathbb{R}$, $G(\sigma+i\gamma)=e^{i\gamma u_0}\Big(\frac{\sin(\gamma L/2)}{\gamma L/2}\Big)^2$, so that $e^{-i\gamma u_0}G(\sigma+i\gamma)\in[0,1]$;*

(ii) *$|F(z)|\le e^{|\Re z|}$ for all z, and $|F(z)|\le\frac{4e^{|\Re z|}}{|z|^2}$ for $z\neq 0$;*

(iii) *for $\beta\in(0,1)$ and $\gamma\neq 0$, $|G(\beta+i\gamma)|\le e^{(\beta-\sigma)u_0}\cdot\frac{4e^{|\beta-\sigma|L}}{L^2\gamma^2}$.*

Proof. Substituting $u=u_0+Lv$ in $G(s)=\int_{\mathbb{R}}h(u)e^{(s-\sigma)u}\,du$ gives

$$G(s)=e^{(s-\sigma)u_0}\int_{-1}^{1}(1-|v|)e^{(s-\sigma)Lv}\,dv.$$

With $z=(s-\sigma)L$,

$$\int_{-1}^{1}(1-|v|)e^{zv}\,dv=2\int_0^1(1-v)\cosh(zv)\,dv=\frac{2(\cosh z-1)}{z^2}=\Big(\frac{\sinh(z/2)}{z/2}\Big)^2,$$

where the middle equality follows from $\int_0^1\cosh(zv)\,dv=\frac{\sinh z}{z}$ and $\int_0^1 v\cosh(zv)\,dv=\frac{\sinh z}{z}-\frac{\cosh z-1}{z^2}$. Entirety is clear since h has compact support.

(i) For $z=i\gamma L$ one has $\sinh(i\gamma L/2)=i\sin(\gamma L/2)$, so $F(i\gamma L)=\big(\frac{\sin(\gamma L/2)}{\gamma L/2}\big)^2\in[0,1]$.

(ii) The first bound is $|F(z)|\le\int_{-1}^1(1-|v|)e^{|\Re z||v|}\,dv\le e^{|\Re z|}$, since $\int_{-1}^1(1-|v|)\,dv=1$. For the second, $|\sinh(z/2)|\le e^{|\Re z|/2}$ gives $|F(z)|\le\big(2\frac{e^{|\Re z|/2}}{|z|}\big)^2$.

(iii) Combine the formula for G with (ii), using $|\beta-\sigma+i\gamma|\ge|\gamma|$. □

The nonnegativity in (i) is the only positivity the argument uses, and it is an identity, not an inequality.

Lemma 4.3. *Let $\sigma\in(0,1)$, $L>0$ and $u_0>L$, and let h,G,w be as in Definition 4.1 and Lemma 4.2. Let $\mathrm{Triv}(\chi)$ denote the set of trivial zeros of $L(\cdot,\chi)$, that is $\{-a-2n:n\ge 0\}$ for $q>1$ and $\{-2n:n\ge 1\}$ for $q=1$. Then*

$$\int_{\mathbb{R}}\Psi(e^u,\chi)e^{-\sigma u}w(u)\,du=-\sum_{\rho}G(\rho)-\sum_{s_*\in\mathrm{Triv}(\chi)}G(s_*),$$

the first sum being over nontrivial zeros with multiplicity and absolutely convergent. Furthermore

$$\Big|\sum_{s_*\in\mathrm{Triv}(\chi)}G(s_*)\Big|\le\frac{e^{-\sigma(u_0-L)}}{1-e^{-2(u_0-L)}}.$$

Proof. Step 1: the Dirichlet series side. Fix $c>1$. By Lemma 4.2(ii)–(iii), $|G(c+it)|\ll(1+|t|)^{-2}$, so $\int_{\mathbb{R}}|G(c+it)|\,dt<\infty$. Since g is continuous, piecewise C^1 and compactly supported in $(1,\infty)$, Mellin inversion gives $g(n)=(2\pi i)^{-1}\int_{(c)}G(s)n^{-s}\,ds$ for every $n\ge 1$. As $\sum_n\Lambda(n)n^{-c}<\infty$, we may interchange sum and integral:

$$\sum_{n\ge1}\Lambda(n)\chi(n)g(n)=\frac{1}{2\pi i}\int_{(c)}\Big(-\frac{L'}{L}(s,\chi)\Big)G(s)\,ds.$$

Step 2: contour shift. Move the contour to $\Re s = -K$ along a rectangle with horizontal sides at heights $\pm T$, choosing T away from zero ordinates so that $|L'/L(\sigma' + iT)| \ll \log^2(qT)$ uniformly for $-K \le \sigma' \le c$. The horizontal contributions are $O\big(\frac{\log^2(qT)}{T}\big) \to 0$ as $T \to \infty$, by the $(1+|t|)^{-2}$ decay of G. On the left edge, Lemma 4.2(ii) gives

$$|G(-K+it)| \le \frac{4e^{-(K+\sigma)(u_0-L)}}{L^2\big((K+\sigma)^2+t^2\big)},$$

so $\int_{\mathbb{R}} |G(-K+it)|\,dt \le \frac{4\pi e^{-(K+\sigma)(u_0-L)}}{L^2(K+\sigma)}$. Since $u_0 > L$ and $|L'/L(-K+it)| \ll \log(q(K+|t|))$ by the functional equation and Stirling's formula, the left edge contributes $O\big(K\log(qK)e^{-(K+\sigma)(u_0-L)}\big) \to 0$ as $K \to \infty$ through values avoiding poles.

The residues enclosed are: $+\delta_\chi G(1)$ at $s=1$, present only for $q=1$, where $-\zeta'/\zeta$ has residue $+1$; $-mG(\rho)$ at each nontrivial zero of multiplicity m; and $-G(s_*)$ at each trivial zero. For $a=1$ the point $s=0$ is not a zero of $L(\cdot,\chi)$ and $-L'/L$ is regular there; for $a=0$ and $q>1$ the point $s=0$ is the trivial zero with $n=0$; for $q=1$ the point $s=0$ is not a zero, since $\zeta(0) = -\frac{1}{2}$. Letting $T, K \to \infty$ yields

$$\sum_{n\ge1} \Lambda(n)\chi(n)g(n) = \delta_\chi G(1) - \sum_\rho G(\rho) - \sum_{s_*\in \mathrm{Triv}(\chi)} G(s_*).$$

Absolute convergence of the zero sum follows from Lemma 4.2(iii) and $\sum_\rho |\Im\rho|^{-2} < \infty$ (Lemma 3.4).

Step 3: partial summation. Put $A(y) = \sum_{n\le y} \Lambda(n)\chi(n)$. Since g is continuous, piecewise C^1 and supported in $[e^{u_0-L}, e^{u_0+L}] \subset (1,\infty)$,

$$\sum_{n\ge1} \Lambda(n)\chi(n)g(n) = -\int_1^\infty A(y)g'(y)\,dy.$$

From $g'(y) = y^{-\sigma-1}\big(h'(\log y) - \sigma h(\log y)\big)$ and the substitution $u = \log y$,

$$-\int_1^\infty A(y)g'(y)\,dy = \int_{\mathbb{R}} A(e^u)e^{-\sigma u}w(u)\,du.$$

Now $A(e^u) = \Psi(e^u,\chi) + \delta_\chi e^u$, and integration by parts gives

$$\int_{\mathbb{R}} e^{(1-\sigma)u}w(u)\,du = \sigma\int_{\mathbb{R}} e^{(1-\sigma)u}h(u)\,du + (1-\sigma)\int_{\mathbb{R}} e^{(1-\sigma)u}h(u)\,du = G(1).$$

Thus the term $\delta_\chi G(1)$ produced by partial summation cancels the residue at $s=1$, and the identity follows.

Step 4: the trivial zeros. By Lemma 4.2(ii), $|G(-a-2n)| \le e^{-(a+2n+\sigma)u_0}e^{(a+2n+\sigma)L} = e^{-(a+2n+\sigma)(u_0-L)}$. Summing the geometric series and using $a \ge 0$, and noting that $\mathrm{Triv}(\chi)$ is contained in $\{-a-2n : n \ge 0\}$, gives the stated bound. □

5 Syndetic recurrence on a torus

Let $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^m$. The following is the standard consequence of minimality; the uniformity in the starting point is exactly what the application requires.

Lemma 5.1. *Let $\omega = (\omega_1, \dots, \omega_m) \in \mathbb{R}^m$ and let H be the closure of $\{u\omega \bmod 2\pi : u \in \mathbb{R}\}$ in $\mathbb{T}$. Then H is a compact subgroup, every orbit $\{h + u\omega : u \in \mathbb{R}\}$ with $h \in H$ is dense in H, and for every nonempty open $U \subseteq H$ there is $Q < \infty$, depending only on ω and U, such that every interval of length Q in $\mathbb{R}$ contains a point u with $u\omega \in U$.*

Proof. The set $\{u\omega\}$ is a subgroup of $\mathbb{T}$, so its closure H is a closed subgroup, hence compact. The orbit of 0 is dense in H by definition. For $h \in H$, translation by h is a homeomorphism of H onto itself, so $h + \{u\omega\}$ is dense in $h + H = H$; this is the orbit of h.

Let $U \subseteq H$ be nonempty and open. Density of every orbit means that for each $h \in H$ there is $v \in \mathbb{R}$ with $h + v\omega \in U$, i.e. $h \in U - v\omega$. Hence $H = \bigcup_{v\in\mathbb{R}}(U - v\omega)$, an open cover; by compactness there are $v_1, \dots, v_r$ with $H = \bigcup_j (U - v_j\omega)$. Put $Q = \max_j v_j - \min_j v_j$. Given an interval $[a, a+Q]$, set $u^* = a - \min_j v_j$. Then $u^*\omega \in H$, so $(u^* + v_j)\omega \in U$ for some j, and $u^* + v_j \in [a, a + Q]$. □

Corollary 5.2. *Let $\gamma_1, \dots, \gamma_m \in \mathbb{R}$, let H be the closure of $\{u(\gamma_1, \dots, \gamma_m) \bmod 2\pi\}$, and let $\delta \in (0, 1)$. There is $Q < \infty$, depending only on the γ_j and δ, such that every interval of length Q in $\mathbb{R}$ contains a point u with*

$$\left|e^{i\gamma_j u} - 1\right| < \delta \qquad (1 \le j \le m).$$

Proof. Apply Lemma 5.1 with $\omega = (\gamma_1, \dots, \gamma_m)$ and $U = \{x \in H : |e^{ix_j} - 1| < \delta \text{ for all } j\}$, which is open in H and contains 0. □

The gap Q depends on the frequencies, on δ and on the target, but not on the location of the interval. This uniformity is what converts the weak conclusion into the window conclusion.

6 The oscillation theorem

Throughout this section χ is primitive modulo q and $\sigma_0 = \sigma_0(\chi) > \frac{1}{2}$ is attained: there is a nontrivial zero $\rho_0 = \sigma_0 + it$. By the classical zero-free region $\sigma_0 < 1$, so $\sigma_0 \in (\frac{1}{2}, 1)$. The case $t = 0$ is allowed.

Fix the width canonically by

$$L = \begin{cases} \pi/(2|t|) & \text{if } t \neq 0, \\ 1 & \text{if } t = 0, \end{cases} \qquad w_0 = \Big(\frac{\sin(tL/2)}{tL/2}\Big)^2 = \begin{cases} \frac{8}{\pi^2} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Indeed for $t \neq 0$ one has $tL/2 = \pm\frac{\pi}{4}$, whence $w_0 = \big(\frac{2\sqrt{2}}{\pi}\big)^2 = \frac{8}{\pi^2} > 0$; and $w_0 = 1$ when $t = 0$. With this choice no nondegeneracy condition on L is needed, and $G(\rho_0) = e^{itu_0}w_0$ by Lemma 4.2(i). Put

$$c_0 = \frac{w_0}{8(\sigma_0 + 2/L)} = \begin{cases} \frac{1}{\pi^2\sigma_0 + 4\pi|t|} & \text{if } t \neq 0, \\ \frac{1}{8(\sigma_0+2)} & \text{if } t = 0. \end{cases}$$

Lemma 6.1. *With $\sigma = \sigma_0$ and L as above there is $T_1 \ge \max\{2, |t|\}$, depending only on q, L, w_0, such that*

$$\sum_{|\Im\rho|>T_1} |G(\rho)| \le \frac{w_0}{8} \qquad \textit{for every } u_0 > 0.$$

Proof. Every nontrivial zero satisfies $\beta = \Re\rho \le \sigma_0$ and $|\beta - \sigma_0| < 1$. For $u_0 > 0$ we have $e^{(\beta-\sigma_0)u_0} \le 1$, so Lemma 4.2(iii) gives, for $|\Im\rho| \ge 1$,

$$|G(\rho)| \le \frac{4e^L}{L^2|\Im\rho|^2}.$$

By Lemma 3.4, $\sum_{|\Im\rho|>T_1} |\Im\rho|^{-2} \ll \frac{\log(qT_1)}{T_1} \to 0$ as $T_1 \to \infty$. Choose $T_1 \ge \max\{2, |t|\}$ so that the resulting bound is at most $w_0/8$; then ρ_0 lies in the finite set $\mathcal{F}$ below. □

Here the hypothesis that σ_0 is attained is used, and it is the only place where it is used: the inequality $e^{(\beta-\sigma_0)u_0} \le 1$ holds for every zero by definition of σ_0, but $\mathcal{F}$ is nonempty only if the supremum is a maximum. See Section 7.1.

Let

$$\mathcal{F} = \{\rho : \Re\rho = \sigma_0,\ |\Im\rho| \le T_1\},$$

a finite set by Lemma 3.4, containing ρ_0. List its elements with multiplicity as $\rho_1, \dots, \rho_m$, with ordinates $\gamma_1, \dots, \gamma_m$, and put

$$w_j = \Big(\frac{\sin(\gamma_j L/2)}{\gamma_j L/2}\Big)^2 \in [0,1], \qquad \text{so that } \sum_{j=1}^{m} w_j \ge w_0.$$

By Lemma 4.2(i), $G(\rho_j) = e^{i\gamma_j u_0} w_j$. Let

$$\mathcal{F}' = \{\rho : \Re\rho < \sigma_0,\ |\Im\rho| \le T_1\}, \qquad \eta = \min_{\rho\in\mathcal{F}'}(\sigma_0 - \Re\rho) > 0,$$

with $\eta = 1$ if $\mathcal{F}' = \varnothing$; $\mathcal{F}'$ is finite.

Theorem 6.2 (Syndetic oscillation)**.** *Let $\sigma_0 = \sigma_0(\chi) > \frac12$ be attained, let L, w_0, T_1 be as above, and let Q be the gap supplied by Corollary 5.2 for the frequencies $\gamma_1, \dots, \gamma_m$ and $\delta = \frac12$. Put $c_0 = \frac{w_0}{8(\sigma_0+2/L)}$. There is $U < \infty$, depending only on χ and L, such that for every $a \ge U$ the interval $[a, a+Q+2L]$ contains a point u with*

$$\big|\Psi(e^u, \chi)\big| \ge c_0 e^{\sigma_0 u}.$$

Proof. Let $a \ge U$, where U is chosen below. Apply Corollary 5.2 to the interval $[a+L, a+L+Q]$: it yields $u_0 \in [a+L, a+L+Q]$ such that $|e^{i\gamma_j u_0} - 1| < \frac12$ for all j. Then

$$\Big|\sum_{\rho\in\mathcal{F}} G(\rho)\Big| = \Big|\sum_j w_j + \sum_j w_j\big(e^{i\gamma_j u_0} - 1\big)\Big| \ge \Big(1 - \frac12\Big)\sum_j w_j \ge \frac{w_0}{2}.$$

By Lemma 6.1 the zeros with $|\Im\rho| > T_1$ contribute at most $w_0/8$ in absolute value. For $\rho \in \mathcal{F}'$, Lemma 4.2(ii) gives $|G(\rho)| \le e^{(\Re\rho-\sigma_0)u_0} e^L \le e^L e^{-\eta u_0}$, so

$$\sum_{\rho\in\mathcal{F}'} |G(\rho)| \le |\mathcal{F}'| e^L e^{-\eta u_0} \le \frac{w_0}{8}$$

once $u_0 \ge \eta^{-1}\log\big(8|\mathcal{F}'|e^L/w_0\big)$. By Lemma 4.3 the trivial zeros contribute at most $w_0/8$ once u_0 is large and $u_0 - L \to \infty$. Choose U so that both conditions hold for all $u_0 \ge U + L$; note $u_0 \ge a + L \ge U + L > L$. Every nontrivial zero lies in $\mathcal{F}$, in $\mathcal{F}'$, or has $|\Im\rho| > T_1$, so

$$\Big|\sum_{\rho} G(\rho) + \sum_{s_*\in \mathrm{Triv}(\chi)} G(s_*)\Big| \ge \frac{w_0}{2} - 3\cdot\frac{w_0}{8} = \frac{w_0}{8}.$$

By Lemma 4.3 with $\sigma = \sigma_0$,

$$\Big|\int_{\mathbb{R}} \Psi(e^u,\chi) e^{-\sigma_0 u} w(u)\, du\Big| \ge \frac{w_0}{8}.$$

Since $\operatorname{supp} w \subseteq [u_0 - L, u_0 + L]$ and $\int |w| \le \sigma_0 + 2/L$, there exists $u \in [u_0 - L, u_0 + L]$ with

$$|\Psi(e^u,\chi)| e^{-\sigma_0 u} \ge \frac{w_0/8}{\sigma_0 + 2/L} = c_0.$$

Finally $u \in [u_0 - L, u_0 + L] \subseteq [a, a+Q+2L]$. □

Corollary 6.3. *Let $\sigma_0(\chi) > \frac{1}{2}$ be attained, and let $\theta \in (\frac{1}{2}, \sigma_0)$, $C < \infty$ and $\lambda > 1$. There is $Y_1 < \infty$ such that every window $[Y, Y^\lambda]$ with $Y \geq Y_1$ contains a point y with $|\Psi(y, \chi)| > Cy^\theta$. In particular $S(\chi, C, \theta)$ contains no window $[Y, Y^\lambda]$ with $Y \geq Y_1$, at any exponent $\lambda > 1$.*

Proof. With c_0 as in Theorem 6.2, $c_0 e^{\sigma_0 u} > Ce^{\theta u}$ holds whenever

$$u > U_1 := \frac{\max\{0, \log(C/c_0)\}}{\sigma_0 - \theta}.$$

Put

$$Y_1 = \exp\Big(\max\{U, U_1\} + \frac{Q + 2L}{\lambda - 1}\Big).$$

In the variable $u = \log y$ the window $[Y, Y^\lambda]$ becomes an interval of length $(\lambda - 1)\log Y \geq Q + 2L$ starting at $\log Y \geq \max\{U, U_1\}$; it therefore contains a subinterval $[a, a + Q + 2L]$ with $a \geq \max\{U, U_1\}$, and Theorem 6.2 supplies the required point. □

Remark 6.4. The amplitude c_0 and the gap $Q + 2L$ are independent of θ and C; those parameters enter only the threshold Y_1. The exponent σ_0 in the conclusion is the real part of an actual zero, and no limiting procedure in θ is performed. The conclusion is an absolute-value statement, and this is forced by the method: the derived weight $w = \sigma h - h'$ changes sign, so a lower bound for $\big|\int \Psi e^{-\sigma u} w\big|$ localises a point of large $|\Psi|$ but does not determine its sign. A compactly supported nonnegative weight cannot have bounded logarithmic derivative at the edge of its support, which is why no attempt is made to render w pointwise nonnegative.

7 Windows and unboundedness

Proposition 7.1. *There exist $S \subseteq [1, \infty)$ and $\lambda > 1$ such that*

(i) *S has the window property at exponent λ;*

(ii) *for every Y_0 there is $y > Y_0$ with $y \notin S$.*

Proof. Take $\lambda = 2$ and $A_k = \exp(e^k)$ for $k \geq 1$, and set

$$S = \bigcup_{k \geq 1} \big[A_k, A_k^2\big], \qquad B = \big\{\exp\big(\tfrac{5}{2} e^k\big) : k \geq 1\big\}.$$

Since $2 < \frac{5}{2} < e$,

$$A_k^2 = \exp(2e^k) < \exp\big(\tfrac{5}{2} e^k\big) < \exp(e \cdot e^k) = A_{k+1},$$

so each point of B lies strictly between the k-th window and the $(k+1)$-st, whence $B \cap S = \varnothing$. As B is unbounded, (ii) holds. Given Y_0, choose k with $A_k \geq Y_0$; then $[A_k, A_k^2] \subseteq S$, giving (i). □

Consequently the weak conclusion, which is all that Proposition 3.6 provides, is logically compatible with $W(\chi, \theta)$: no formal consequence of it alone can exclude the window property. An argument opposing a hypothetical zero to a supply of clean windows requires a recurrence statement of the form supplied by Theorem 6.2.

Theorem 7.2 (Window criterion)**.** *Let $\mathcal{C}$ be a set of primitive characters closed under conjugation.*

(i) *If every nontrivial zero of every $L(\cdot, \chi)$, $\chi \in \mathcal{C}$, has real part $\frac{1}{2}$, then $W(\chi, \theta)$ holds for every $\chi \in \mathcal{C}$ and every $\theta > \frac{1}{2}$.*

(ii) *Suppose $W(\chi,\theta)$ holds for every $\chi\in\mathcal{C}$ and every $\theta>\frac12$, and suppose that for each $\chi\in\mathcal{C}$ with $\sigma_0(\chi)>\frac12$ the supremum $\sigma_0(\chi)$ is attained by a zero. Then every nontrivial zero of every $L(\cdot,\chi)$, $\chi\in\mathcal{C}$, has real part $\frac12$.*

Proof. (i) If $\sigma_0(\chi)\le\frac12$ then Proposition 3.6 gives $\Psi(y,\chi)\ll y^{1/2}\log^2(qy)$, hence $|\Psi(y,\chi)|\le Cy^\theta$ for all $y\ge1$ with $C=C(\chi,\theta)$. Thus $S(\chi,C,\theta)=[1,\infty)$ and the window property holds at every exponent.

(ii) Suppose some $\chi\in\mathcal{C}$ has $\sigma_0(\chi)>\frac12$. By hypothesis $\sigma_0(\chi)$ is attained. Choose θ with $\frac12<\theta<\sigma_0(\chi)$. By assumption $W(\chi,\theta)$ holds, i.e. $S(\chi,C,\theta)$ has the window property at some exponent $\lambda>1$ for some $C<\infty$; this contradicts Corollary 6.3. Hence $\sigma_0(\chi)\le\frac12$ for all $\chi\in\mathcal{C}$, and Theorem 2.3 gives real part exactly $\frac12$. □

7.1 Where attainment is used, and what remains

The proof of Theorem 6.2 uses attainment at a single step: in Lemma 6.1, the inequality $e^{(\beta-\sigma_0)u_0}\le1$ for every zero $\rho=\beta+i\gamma$ is automatic, but the finite set $\mathcal{F}$ of zeros of real part exactly σ_0 is nonempty only if the supremum is attained. Two alternatives fail for the following reasons.

If σ_0 is not attained and one takes $\sigma=\Re\rho_0<\sigma_0$ for some zero ρ_0, then the zeros with $\Re\rho>\sigma$ contribute factors $e^{(\Re\rho-\sigma)u_0}$ which grow without bound as $u_0\to\infty$, while the damping supplied by Lemma 4.2(iii) is fixed once L is fixed; the ratio of the tail to the target term is unbounded. Increasing L suppresses the tail but also suppresses the target, whose modulus carries the factor $\big(\frac{\sin(tL/2)}{tL/2}\big)^2$. If instead one takes $\sigma=\sigma_0$ with σ_0 not attained, then $\Re\rho<\sigma_0$ strictly for every zero, so every $G(\rho)$ tends to 0 as $u_0\to\infty$ and no positive lower bound for the zero sum is available.

Problem 7.3. Let χ be primitive with $\sigma_0(\chi)>\frac12$ not attained by any zero. Decide whether, for every $\theta\in(\frac12,\sigma_0(\chi))$ and every $C<\infty$, the set $\{u\ge0:|\Psi(e^u,\chi)|>Ce^{\theta u}\}$ is syndetic in $\mathbb{R}$.

An affirmative answer would remove the attainment hypothesis from Theorem 7.2(ii).

8 The exact strength of the arithmetic input

Theorem 7.2 reduces the critical-line assertion, for characters whose rightmost abscissa is attained, to the construction of clean windows. We now determine what such a construction must deliver, and show that the requirement is of exactly the strength of the conclusion.

Theorem 8.1. *Let $\mathcal{C}$ be a set of primitive characters closed under conjugation. The following are equivalent.*

(i) *Every nontrivial zero of every $L(\cdot,\chi)$, $\chi\in\mathcal{C}$, has real part $\frac12$.*

(ii) *For every $\chi\in\mathcal{C}$ and every $\varepsilon>0$ there is $C(\chi,\varepsilon)<\infty$ with $|\Psi(y,\chi)|\le C(\chi,\varepsilon)y^{1/2+\varepsilon}$ for all $y\ge1$.*

Proof. (i)⇒(ii). By (i), $\sigma_0(\chi)\le\frac12$, so $\alpha(\chi)\le\frac12$ by Proposition 3.6; by definition of α, $\Psi(y,\chi)=O(y^{1/2+\varepsilon})$ for every $\varepsilon>0$.

(ii)⇒(i). Fix $\varepsilon>0$ and apply Theorem 3.3 with $\theta=\frac12+\varepsilon$; there is no zero with $\Re\rho>\frac12+\varepsilon$. Intersecting over $\varepsilon>0$ gives $\sigma_0(\chi)\le\frac12$ for every $\chi\in\mathcal{C}$, and Theorem 2.3 gives real part exactly $\frac12$. □

In particular, any scheme whose output is a bound

$$|\Psi(x,\chi)| \le A\, x^{1/2+\Gamma\varepsilon}$$

valid for every $\chi \in \mathcal{C}$, every $\varepsilon \in (0,\varepsilon_0)$ and all $x \ge X_0(\varepsilon,\chi)$, with A,Γ absolute, establishes (ii) and hence (i). The quantifier in ε is essential: a single fixed ε yields only $\sigma_0(\chi) \le \frac{1}{2} + \Gamma\varepsilon$.

We next compute what the standard averaging tools give for an individual character.

Proposition 8.2. *Let a_n be complex, supported on $n \le x$. The multiplicative large sieve asserts*

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \Big|\sum_{n\le x} a_n \chi(n)\Big|^2 \le (x+Q^2)\sum_{n\le x}|a_n|^2.$$

With $a_n = \Lambda(n)$ one has $\sum_{n\le x}\Lambda(n)^2 \le cx\log x$ for an absolute c, and hence, for each primitive χ of conductor $q \le Q$,

$$|\Psi(x,\chi)| \le \sqrt{c(x+Q^2)x\log x} + x.$$

For $Q \ge \sqrt{x}$ the right-hand side is at least x. Consequently the inequality implies no bound of the form $|\Psi(x,\chi)| \le Cx^\theta$ with $\theta < 1$, uniformly over primitive characters of conductor at most Q.

Proof. The sieve inequality is classical. The bound $\sum_{n\le x}\Lambda(n)^2 \le (\log x)\sum_{n\le x}\Lambda(n) \le cx\log x$ is Chebyshev's estimate. Since $q/\varphi(q) \ge 1$, a single summand on the left is at most the right-hand side, which gives the displayed pointwise bound after subtracting $\delta_\chi x \le x$. For $Q \ge \sqrt{x}$ one has $x + Q^2 \ge 2x$, so $\sqrt{c(x+Q^2)x\log x} \ge x\sqrt{2c\log x}$. All summands on the left are nonnegative, so the only information the inequality provides about an individual character is the bound just displayed; and that bound exceeds x when $Q \ge \sqrt{x}$. Hence no exponent $\theta < 1$ is a consequence of the sieve inequality. The same applies to its mean-square form below: a bound on an average of nonnegative terms imposes no bound on an individual term beyond the total. □

The mean-square form of the same inequality is worth recording, because it identifies precisely where the averaging occurs. For $Q \ge \sqrt{x}$ the number of primitive characters of conductor at most Q is $\asymp Q^2$, and the sieve gives

$$\frac{1}{Q^2}\sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q}|\Psi(x,\chi)|^2 \ll x\log x.$$

Thus square-root cancellation holds on average over the family. It does not hold for any individual character on the basis of this inequality: a single character may carry the whole mass, of size $\asymp Q\sqrt{x\log x}$.

Proposition 8.3. *Let $\varepsilon > 0$.*

(i) *For every integer $m \ge 0$ there is $X_1(\varepsilon,m)$ with $\log^m x \le x^\varepsilon$ for $x \ge X_1(\varepsilon,m)$.*

(ii) *For every $A > 0$ there is $X_2(\varepsilon,A)$ with $\exp(x^{\varepsilon/2}) > x^A$ for $x \ge X_2(\varepsilon,A)$.*

In particular, if a quantity $\mathcal{L}(x)$ satisfies the trivial bound $\mathcal{L}(x) \ll x\log x$, then the assertion $\mathcal{L}(x) \le \exp(x^{\varepsilon/2})$ holds for all large x and implies no bound of the form $\mathcal{L}(x) \le x^B$ for any $B < 1$.

Proof. (i) $\log(\log^m x/x^\varepsilon) = m\log\log x - \varepsilon\log x \to -\infty$, since $\log x$ dominates $\log\log x$. (ii) Taking logarithms, the assertion is $x^{\varepsilon/2} > A\log x$, and $\frac{x^{\varepsilon/2}}{\log x} \to \infty$. For the last statement, $x\log x \le \exp(x^{\varepsilon/2})$ for large x by (ii) with $A = 2$, so the asserted bound is weaker than the trivial one; a bound weaker than $x\log x$ cannot imply a bound x^B with $B < 1$. □

Remark 8.4. Consider a proposed multiplicative decomposition

$$|\Psi(x,\chi)| \le A \cdot x^{1/2} \cdot x^{3\varepsilon} \cdot x^{(e_D+e_L+3)\varepsilon} \cdot \log^4 x,$$

with e_D, e_L absolute. By Proposition 8.3(i) the last three factors are at most $x^{c\varepsilon}$ for x large, with $c = e_D + e_L + 7$; their accumulation into a single exponent is elementary bookkeeping, and by Proposition 3.6 a factor $x^{c\varepsilon}$ affects only the constant in the abscissa, never the exponent $\frac{1}{2}$. The entire content of the display is therefore its first factor $x^{1/2}$, and by Theorem 8.1 that factor, asserted uniformly for every $\varepsilon > 0$, is equivalent to the critical-line assertion. An exponent ledger downstream of such a decomposition cannot supply the factor it starts from. Propositions 8.2 and 8.3 show that neither the large sieve nor a stretched-exponential length budget supplies it.

9 Positivity type

The separation established in Section 8 has a structural source, which we record in a form independent of L-functions.

Definition 9.1. Let (X, μ) be a probability space and f_n real functions on X. Call *averaged positivity* a family of identities or inequalities concerning $\int f_n \, d\mu$, $\int |f_n|^2 \, d\mu$, or limits of averages $N^{-1} \sum_{n \le N} f_n$. Call *pointwise positivity* a bound $f_n(x) \ge c > 0$ holding for all n and all x in a prescribed set, with c independent of n.

Proposition 9.2. *Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure and $f_n(x) = \cos(2\pi n x)$. Then*

$$\int_{\mathbb{T}} f_n = 0, \qquad \int_{\mathbb{T}} |f_n|^2 = \frac{1}{2} \qquad (n \ge 1),$$

so averaged identities hold with equality; but for every $x \in \mathbb{T} \setminus \{0\}$ there are infinitely many n with $f_n(x) \le 0$. In particular there is no $c > 0$ such that $f_n(x) \ge c$ for all $n \ge 1$ and all x in any dense subset of $\mathbb{T}$.

Proof. The two integrals are immediate. Let $x \in \mathbb{T}$. If $x = \frac{p}{q}$ in lowest terms with $q \ge 2$, then $f_n(x)$ depends only on $n \bmod q$ and $\sum_{k=0}^{q-1} \cos\bigl(2\pi \frac{k}{q}\bigr) = 0$; the summands are not all zero, so at least one is negative, and by periodicity $f_n(x) \le 0$ for infinitely many n. If $x = 0$ then $f_n(0) = 1$ for all n, but $x = 0$ is a single point and may be omitted from any dense set; for x irrational the sequence $(nx)_{n \ge 1}$ is equidistributed in $\mathbb{T}$, so $f_n(x) < 0$ for infinitely many n. Since every dense subset of $\mathbb{T}$ contains a point other than 0, no uniform $c > 0$ exists. □

Every positivity input used in Sections 3–6 is of averaged type: the nonnegative coefficients of a Landau-type product, finite-group Plancherel orthogonality, the large sieve, and the nonnegativity in Lemma 4.2(i), which is an exact identity for a Fourier transform. This is consistent with the target being an abscissa, that is, a limit superior (Proposition 3.6), for which averaging tools are exactly strong enough. It is also the reason why the same tools do not yield a pointwise, one-sided, modulus-uniform estimate; by Proposition 9.2 such an estimate is not a consequence of averaged identities.

10 Concluding remarks

The analytic half of the window programme is complete and unconditional apart from the attainment hypothesis isolated in Problem 7.3: the reduction of Theorem 2.3, the abscissa identity of Proposition 3.6, the smoothed formula of Lemma 4.3, the recurrence of Lemma 5.1, and the oscillation

theorem and its corollary are proved with explicit constants. The arithmetic half is not a matter of bookkeeping. By Theorem 8.1 the estimate it must produce is equivalent to the assertion to be proved, and by Propositions 8.2 and 8.3 the standard averaging inputs do not produce it. What is required is a pointwise, one-sided, modulus-uniform estimate for an individual character, of the kind that over a finite field is furnished by the Frobenius endomorphism together with purity of its eigenvalues. No substitute for that pair is supplied by the tools listed.

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