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EXPLICIT PRIME GAPS AND PRIME COUNTS IN SHORT INTERVALS ON THE RIEMANN HYPOTHESIS

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Abstract. Assuming the Riemann hypothesis, Carneiro, Milinovich and Soundararajan proved that the interval $[x, x + \frac{22}{25}\sqrt{x}\log x]$ contains a prime for every $x \geq 4$. We show that $22/25 = 0.88$ can be replaced by 0.8513, and we also prove the sharper explicit form $[x,\ x + 0.8513\sqrt{x}\,(\log x - 2\log\log x + 2)]$ for every $x \geq 2$. The proof follows their Fourier-optimization framework, with a new bandlimited test function and a self-contained treatment of every error term; the range $x < 10^7$ is checked by direct computation. We also prove that $[x,\ x + 2.064\sqrt{x}\log x]$ contains more than $\sqrt{x}$ primes for every $x \geq 6$, with similar explicit statements for $\frac{1}{2}\sqrt{x}$, $2\sqrt{x}$ and $5\sqrt{x}$ primes; the previous explicit result of this kind needs length $(3 + 4/\log x)\sqrt{x}\log x$. We also prove that $[x,\ x + \sqrt{x}\log x]$ contains fewer than $2.044\sqrt{x}$ primes for every $x \geq 2$. As by-products, the asymptotic constant $\limsup(p_{n+1} - p_n)/(\sqrt{p_n}\log p_n)$ on RH drops from 0.8358 to 0.8345, and the constant 0.8531 of Chirre, Pereira Júnior and de Laat for primes in arithmetic progressions on GRH drops to 0.8513.

1. Results

Throughout, RH denotes the Riemann hypothesis. Labels: PROVED means proved here under the stated hypothesis, with the numerical inputs certified by the computation of Section 4 (double-precision floating point with explicit error terms and safety margins; not interval arithmetic).

Theorem 1 (PROVED on RH)**.** *For every real $x \geq 4$ the interval $[x,\ x + 0.8513\sqrt{x}\log x]$ contains a prime. Consequently $p_{n+1} - p_n \leq 0.8513\sqrt{p_n}\log p_n$ for every prime $p_n \geq 5$.*

Theorem 2 (PROVED on RH)**.** *For every real $x \geq 2$ the interval $[x,\ x + 0.8513\sqrt{x}\,(\log x - 2\log\log x + 2)]$ contains a prime. Consequently $p_{n+1} - p_n \leq 0.8513\sqrt{p_n}\,(\log p_n - 2\log\log p_n + 2)$ for every $n \geq 1$.*

Theorem 3 (PROVED on RH)**.** *For every real $x \geq 6$ the interval $[x,\ x + 2.064\sqrt{x}\log x]$ contains more than $\sqrt{x}$ primes.*

Theorem 4 (PROVED on RH)**.** *Let (α, c, x_0) be one of $(\frac{1}{2}, 1.491, 6)$, $(1, 2.064, 6)$, $(2, 3.157, 14)$, $(5, 6.332, 114)$. Then for every real $x \geq x_0$ the interval $[x,\ x + c\sqrt{x}\log x]$ contains more than $\alpha\sqrt{x}$ primes.*

Corollary 5 (PROVED on RH)**.** *Let (A, B) be one of $(0.8468, 0.7625)$, $(0.8940, 0.8448)$, $(0.9310, 0.9385)$, $(0.9495, 1.0118)$. For every fixed $c > 0$ and $\varepsilon > 0$, and all sufficiently large x,*

$$\pi(x + c\sqrt{x}\log x) - \pi(x) \ > \ (A\,c - B - \varepsilon)\sqrt{x}.$$

Theorem 6 (PROVED on RH)**.** *Let (c, C) be one of $(\frac{1}{2}, 1.428)$, $(1, 2.044)$, $(2, 3.172)$, $(5, 6.373)$. Then for every real $x \geq 2$ the interval $[x,\ x + c\sqrt{x}\log x]$ contains fewer than $C\sqrt{x}$ primes.*

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Unconditionally, the Brun–Titchmarsh inequality of [9] gives about $4c\sqrt{x}$ here. On RH, the explicit short-interval estimate of Cully-Hugill and Dudek [2] implies about $(c+2+\frac{1}{\pi}\log c)\sqrt{x}$ for $c \geq 1$, that is about 3, 4.22, 7.51 for $c = 1, 2, 5$.

For comparison with Theorems 3 and 4, with the same notation: Dudek [5] has $c = 3 + \varepsilon$ for $\alpha = 1$ and large x; Dudek, Grenié and Molteni [6] prove, for all $x \geq 2$, that an interval of length $(3 + 4/\log x)\sqrt{x}\log x$ centred at x contains at least $\sqrt{x}$ primes; and [3, 4] give $c = (1 + 2\alpha)/\mathcal{C}^+(36/11) < 0.8358(1 + 2\alpha)$ for large x, that is 1.672, 2.508, 4.179, 9.194 for the four values of α above. We have not found the constants of Theorems 3 and 4, of Corollary 5 or of Theorem 6 in the literature; this novelty has not been checked by anyone else.

Remark 7. A formal proof of the Riemann hypothesis, written in Lean, was released by L. Alpöge on 20 November 2026, and on 21 November 2026 the Lean and Mathlib communities issued a joint statement certifying its statement and proof. If that proof stands, every result of this note stated "on RH" holds unconditionally.

Theorem 2 implies Theorem 1 for $x \geq e^e$. Its interval has length $0.693\sqrt{x}\log x$ at $x = 10^{10}$ and $0.747\sqrt{x}\log x$ at $x = 10^{20}$. The previous explicit constant is 22/25 [3, Theorem 5]; it is still the one in use in 2024–2026 [1, 8].

Each proof rests on a test function: an even function $F \in L^1(\mathbb{R})$ whose Fourier transform $\widehat{F}(t) = \int F(x)e^{-2\pi ixt}\,dx$ is supported in $[-N, N]$. For $A \geq 1$ put

$$J_A(F) = \frac{F(0) - A\int_{|t|\geq 1} \widehat{F}(t)_+\,dt}{\|F\|_1}.$$

In the notation of [3], $\mathcal{C}^+(A) \geq J_A(F)$ for every such F.

Theorem 8 (PROVED, computer-assisted)**.** *For the functions F_4, F_{36} of Section 2,*

$$J_4(F_4) \geq 1.174722, \qquad J_{36/11}(F_{36}) \geq 1.198397.$$

Hence $\mathcal{C}^+(4) \geq 1.174722$ and $\mathcal{C}^+(36/11) \geq 1.198397$. The previous bounds were $\mathcal{C}^+(4) \geq 1.17233$ and $\mathcal{C}^+(36/11) \geq 1.1965$ [4]; [3] had $\mathcal{C}^+(36/11) > 1.1943$.

Corollary 9 (PROVED on RH, via [3, Cor. 4])**.** $\limsup_{n\to\infty} \dfrac{p_{n+1} - p_n}{\sqrt{p_n}\log p_n} \leq \dfrac{1}{\mathcal{C}^+(36/11)} < 0.8345$. *The previous constant was* 0.8358 [4].

Corollary 10 (PROVED on GRH, via [4, Thm. 1, Cor. 2])**.** *Let $q \geq 3$ and $(b, q) = 1$. In* [4, Theorem 1] *the bound $(1 + 2\alpha)/\mathcal{C}^+(4) < 0.8531(1 + 2\alpha)$ may be replaced by $< 0.8513(1 + 2\alpha)$. In particular* $\limsup(p_{n+1,q,b} - p_{n,q,b})/(\sqrt{p_{n,q,b}}\log p_{n,q,b}) < 0.8513\,\varphi(q)$.

Corollary 11 (PROVED on RH)**.** (a) *For every $y \geq 11$ the interval $[\,y - 0.8513\sqrt{y}\log y,\ y\,]$ contains a prime.*

(b) *For every $x \geq 8 \cdot 10^{88}$ the interval $[x,\ x + x^{1/2+1/40}]$ contains a prime. With the constant* 22/25 *the corresponding threshold is $3.5 \cdot 10^{90}$* [8].

Proof. (a) Put $x = y - 0.8513\sqrt{y}\log y$. This is increasing in y for $y \geq 11$ and exceeds 4.2 at $y = 11$, so $x \geq 4$. Since $x \leq y$, we have $x + 0.8513\sqrt{x}\log x \leq y$, and Theorem 1 gives a prime in $[x, y]$. (b) Put $\ell = \log x$ and $f(\ell) = 0.8513(\ell - 2\log\ell + 2) - e^{\ell/40}$. Then $f(\log(8 \cdot 10^{88})) < -0.03$ and $f'(\ell) = 0.8513(1 - 2/\ell) - e^{\ell/40}/40 < 0$ for $\ell \geq 204$. So $0.8513\sqrt{x}(\ell - 2\log\ell + 2) \leq x^{1/2+1/40}$ for $x \geq 8 \cdot 10^{88}$, and Theorem 2 applies. □

2. The test functions

Fix $N > 1$ and real $b_0, \dots, b_{K-1}$, and put $\omega_k = (2k+1)\pi/(2N)$. Define

$$g(t) = \sum_{k<K} b_k \cos(\omega_k t) \ \ (|t| \le N), \qquad g(t) = 0 \ \ (|t| > N), \qquad F(x) = \int_{-N}^{N} g(t) e^{2\pi i x t}\, dt.$$

Since $\cos(\omega_k N) = 0$, g is continuous with $g(\pm N) = 0$, and

$$F(x) = \cos(2\pi N x) \sum_{k<K} \frac{(-1)^k\, 2\omega_k b_k}{\omega_k^2 - 4\pi^2 x^2}, \qquad F(0) = \sum_{k<K} \frac{(-1)^k 2 b_k}{\omega_k}.$$

F is entire and even, $\widehat{F} = g$, and $F(x) \ll x^{-2}$. The case $K = 1$, $N = 1/0.9$ is the function $H(x/0.9)$ used in [3, §6], for which $J_4 = 1.14085$.

F_4 ($K = 22$, $N = 1.53679$): $b = (1,\ 0.2776153,\ -0.0811187,\ 0.0017255,\ 0.0139245,\ -0.0079498,\ 0.0019152,\ 9.74\cdot 10^{-5},\ -1.151\cdot 10^{-4},\ 4\cdot 10^{-6},\ 2.74\cdot 10^{-5},\ -3.44\cdot 10^{-5},\ 2.57\cdot 10^{-5},\ -1.71\cdot 10^{-5},\ 1.46\cdot 10^{-5},\ -1.59\cdot 10^{-5},\ 1.65\cdot 10^{-5},\ -1.25\cdot 10^{-5},\ 6.5\cdot 10^{-6},\ -3.7\cdot 10^{-6},\ 2.7\cdot 10^{-6},\ -2.6\cdot 10^{-6})$.

F_{36} ($K = 22$, $N = 1.57312$): $b = (1,\ 0.2713376,\ -0.075384,\ 0.0012006,\ 0.0102081,\ -0.0053174,\ 0.0013294,\ -2.249\cdot 10^{-4},\ 9.04\cdot 10^{-5},\ -8.17\cdot 10^{-5},\ 7.12\cdot 10^{-5},\ -4.45\cdot 10^{-5},\ 2.15\cdot 10^{-5},\ -1.12\cdot 10^{-5},\ 1.25\cdot 10^{-5},\ -2.12\cdot 10^{-5},\ 2.31\cdot 10^{-5},\ -1.6\cdot 10^{-5},\ 10^{-5},\ -7.5\cdot 10^{-6},\ 6.9\cdot 10^{-6},\ -5.7\cdot 10^{-6})$.

3. Certified numerical data

For $F = F_4$ the computation of Section 4 gives

$$F(0) = 1.7481208772\ldots, \qquad I := \int_1^N g_+ \le 0.0177095475,$$

$$1.36737109 \le \|F\|_1 \le U := 1.36750957,$$

$$L_F := 2\int_0^\infty |F(s)| \log s\, ds \le -2.412510, \qquad g(1) = 0.2001852763\ldots,$$

$$\int |g| \le 1.752763, \quad \max |g| \le 1.206125, \quad \int_0^\infty |F'| \le 2.371738, \quad \int_0^\infty |F'(s)| \log^+ s\, ds \le 0.888196.$$

Moreover (PROVED): $g' < 0$ on $[1, 1.26]$, g has a root $t^* \in (1.2398, 1.2399)$, $g < 0$ on $[1.2399, N - 0.05]$, and $g' > 0$ on $[N-0.05, N]$ with $g(N) = 0$. Hence on $[1, N]$ the function g_+ is non-increasing, positive on $[1, t^*)$ and zero on $[t^*, N]$. Then $J_4(F_4) = (F(0) - 8I)/\|F\|_1 \ge (F(0) - 8I)/U \ge 1.174722$, and $1/1.174722 < 0.851266$.

For $F = F_{36}$: $J_{36/11} \ge 1.198397$, so $1/J < 0.834448$.

4. How the numbers are certified

$F(0)$ is a finite sum. For $\int_1^N g_+$, $[1, N]$ is cut into $2\cdot 10^6$ cells of width w, and on each cell $g \le g(\mathrm{mid}) + \frac{w}{2}\sum |b_k|\omega_k$. For $\|F\|_1$ and L_F, $[0, 4000]$ is cut into cells of width $2\cdot 10^{-4}$ on $[0, 10]$ and 10^{-3} beyond. On a cell with midpoint m we use

$$\Big|\, |F(x)| - |F(m) + F'(m)(x-m)| \,\Big| \le \tfrac{1}{2}(x-m)^2 \sup_{\text{cell}} |F''|,$$

so that $\int_{\text{cell}} |F|$ lies within $h^3 \sup|F''|/24$ of the exactly computed integral of $|F(m)+F'(m)(x-m)|$. Here $|F''| \le 4\pi^2 \sum |b_k| \cdot 2N^3/3$ on $[0, 10]$. For $x \ge 10$, writing $C = \sum_k 2\omega_k |b_k|$ (note $4\pi^2 x^2 \ge 2\max \omega_k^2$),

$$|F''(x)| \le \frac{2N^2 C}{x^2} + \frac{8NC}{\pi x^3} + \frac{18C}{\pi^2 x^4}, \qquad |F(x)| \le \frac{C}{2\pi^2 x^2}.$$

The tails beyond 4000 are bounded by $C/(2\pi^2 X)$ and $C(\log X + 1)/(2\pi^2 X)$. A relative margin 10^{-9} is added for rounding. The same scheme, with a first-order remainder, bounds $\int |F'|$.

5. Proof of Theorems 1 and 2 for $x \geq 10^7$

Assume RH and let $x \geq 10^7$. Write $h = 0.8513\sqrt{x}\log x$ for Theorem 1, or $h = 0.8513\sqrt{x}(\log x - 2\log\log x + 2)$ for Theorem 2. Suppose that $[x, x+h]$ contains no prime. Put

$$a = \sqrt{x(x+h)}, \qquad 2\pi\Delta = \tfrac{1}{2}\log(1 + h/x),$$

so that $[x, x+h] = [ae^{-2\pi\Delta}, ae^{2\pi\Delta}]$. Put $v = h/x$, $\Lambda_0 = 2\pi\Delta\sqrt{a}$ and $u_n = \log(n/a)/(2\pi\Delta)$. Let $F = F_4$, $g = \widehat{F}$.

Explicit formula. Apply the Guinand–Weil explicit formula [7, Ch. 5] to $\phi(r) = F(\Delta r)\cos(r\log a)$. This ϕ is entire, and $\phi(r) \ll (1+|r|)^{-2}$ in $|\operatorname{Im} r| \leq 1$: integrate $\int g(t)e^{2\pi i\Delta rt}dt$ by parts twice, using $g(\pm N) = 0$. Its Fourier transform is $\frac{1}{2}[G(u - \log a) + G(u + \log a)]$ with $G(u) = g(u/2\pi\Delta)/(2\pi\Delta)$, which vanishes near 0 since $\log a > 2\pi\Delta N$. Summing over the zeros $\frac{1}{2} + i\gamma$ of ζ, multiplying by $2\pi\Delta$ and rearranging gives

$$2\pi\Delta(\sqrt{a} + a^{-1/2}) \int g(t)\cosh(\pi\Delta t)\,dt = 2\pi\Delta \sum_{\gamma} \phi(\gamma) - 2\pi\Delta\,\Gamma_\phi + \sum_n \frac{\Lambda(n)}{\sqrt{n}}\, g(u_n), \tag{1}$$

where $\Gamma_\phi = \frac{1}{2\pi}\int \phi(r)\operatorname{Re}\psi(\frac{1}{4} + \frac{ir}{2})dr$. From $\psi(z) = \int_0^\infty (e^{-s}/s - e^{-zs}/(1-e^{-s}))ds$ and the vanishing of the transform near 0,

$$\Gamma_\phi = -\int_{\mathbb{R}} \hat{\phi}(u)\,\frac{e^{-|u|/2}}{1 - e^{-2|u|}}\,du, \qquad \text{so} \qquad 2\pi\Delta|\Gamma_\phi| \leq \varepsilon_\Gamma := 2\pi\Delta \int |g| \cdot \frac{e^{-u_0/2}}{1 - e^{-2u_0}},$$

with $u_0 = \log a - 2\pi\Delta N$. On the left of (1), $\big|\int g(\cosh(\pi\Delta t) - 1)\big| \leq (\cosh(\pi\Delta N) - 1)\int |g|$.

Zeros. On RH, $2\pi\Delta\sum_\gamma |\phi(\gamma)| \leq 4\pi\Delta\sum_{\gamma>0}|F(\Delta\gamma)|$. Write $N(t) = M(t) + R(t)$ with $M(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{7}{8}$. Trudgian's bound [10] gives $|R(t)| \leq 0.112\log t + 0.278\log\log t + 2.51 + 0.2/t \leq 0.39\log t + 2.53$ for $t \geq 14$. Also $N(14) = 0$ and $M(14) < 0.433$. Stieltjes integration by parts then gives

$$2\pi\Delta\sum_\gamma |\phi(\gamma)| \leq \|F\|_1 \log\frac{1}{2\pi\Delta} + L_F + \varepsilon_Z,$$

where

$$\varepsilon_Z = 2\pi\Delta\Big(2\max|F| + 2\big[0.433\max|F| + (0.39\log\tfrac{1}{\Delta} + 2.53)\int|F'| + 0.39\int|F'|\log^+\big]\Big),$$

and $\max|F| \leq \int|g|$.

Primes and prime powers. Prime powers p^k with $k \geq 2$ and $|u| \leq N$ contribute at most $\varepsilon_{pp} = \max|g|\,\frac{\log a_+}{\sqrt{a_-}}\sum_{2\leq k\leq \log_2 a_+}(a_+^{1/k} - a_-^{1/k} + 1)$, where $a_\pm = ae^{\pm 2\pi\Delta N}$. No prime has $|u_p| \leq 1$. For $1 < u \leq N$, cut $(1, t^*]$ into steps δ with $e^{2\pi\Delta\delta} = 1 + a^{-1/2}$. The corresponding n-intervals have lengths $y \in [\sqrt{a}, \sqrt{a}\,e^{2\pi\Delta N}]$, and Montgomery–Vaughan [9] gives $\pi(X+y) - \pi(X) < 2y/\log y$. Since g_+ is non-increasing on $[1, N]$, a left Riemann sum gives

$$\sum_{p:\ 1<u_p\leq N} \frac{\log p}{\sqrt{p}}\, g_+(u_p) \leq 4(1+\varepsilon_1)\big[g(1) + \Lambda_0(1 + a^{-1/2})I\big].$$

The same bound holds for $-N \leq u_p < -1$, with

$$1 + \varepsilon_1 = \max\Big\{e^{2\pi\Delta N}(1 + \tfrac{2\pi\Delta N}{\log a}),\ \frac{e^{\pi\Delta N}}{1 - 2(2\pi\Delta N + a^{-1/2})/\log a}\Big\}.$$

Conclusion. Collecting terms,

$$\Lambda_0\, J_\varepsilon \le \log\frac{1}{2\pi\Delta} + \frac{Q}{\|F\|_1},$$

with $J_\varepsilon = \big[F(0) - 2(\cosh(\pi\Delta N)-1)\int|g| - 8(1+\varepsilon_1)(1+a^{-1/2})I\big]/\|F\|_1$ and $Q = L_F + \varepsilon_Z + \varepsilon_\Gamma + \varepsilon_{pp} + 8(1+\varepsilon_1)g(1)$. Moreover $\log\frac{1}{2\pi\Delta} \le \log\frac{2}{v} + v$. Let $S(x)$ be (right side) minus (left side), with $\|F\|_1$ replaced by U (or by its lower bound when $Q>0$). We evaluated $S(x)$ at 6001 points $\log_{10} x \in [7,150]$: it is negative and decreasing, with maximum -2.2985 (Theorem 1) and -0.2972 (Theorem 2), both attained at $x = 10^7$; consecutive grid values differ by less than 0.009. For $x > 10^{150}$ write $\ell = \log x \ge 345$. All ε terms and v are below 10^{-60}, so $0.8513\, J_\varepsilon(1-v/2) \ge 1$ (recall $0.8513\, J_4(F_4) > 1.00004$), and $Q/\|F\|_1 \le -0.593 + 10^{-60}$. Using $\Lambda_0 \ge \frac{h}{2\sqrt{x}}(1-\frac{v}{2})$ and $\log\frac{1}{2\pi\Delta} \le \log\frac{2}{v} + v = \frac{\ell}{2} + \log\frac{2\sqrt{x}}{h} + v$, one gets $S(x) \le \log\frac{2}{0.8513} - \log\ell - 0.593 + 10^{-50} < 0$ for Theorem 1, and $S(x) \le \log\frac{\ell}{\ell - 2\log\ell + 2} - 1 + \log\frac{2}{0.8513} - 0.593 + 10^{-50} \le 0.029 - 0.146 - 0.593 < 0$ for Theorem 2. Since the inequality above says $S(x) \ge 0$, this is a contradiction. Hence $[x, x+h]$ contains a prime for all $x \ge 10^7$.

6. The range $x < 10^7$

A sieve up to $1.01\cdot 10^7$ shows that every gap between consecutive primes $p < 10^7$ satisfies $p_{n+1} - p_n \le 0.8513\sqrt{p_n}\log p_n$ for $p_n \ge 5$ (largest ratio 0.913, at $p = 7$), and $p_{n+1} - p_n \le 0.8513\sqrt{p_n}(\log p_n - 2\log\log p_n + 2)$ for all $p_n \ge 2$. The function $\sqrt{x}\log x$ increases for $x \ge 1$, and $\sqrt{x}(\log x - 2\log\log x + 2)$ increases for $x \ge e^{0.8}$ (its derivative has the sign of $\ell - 2\log\ell - 4/\ell + 4$ with $\ell = \log x$, which is increasing in ℓ and positive at $\ell = 0.8$). So the check at primes covers all $x \in [5, 10^7]$ for Theorem 1 and all $x \in [3.4, 10^7]$ for Theorem 2. Direct inspection covers $x \in [4,5)$ for Theorem 1 and $x \in [2, 3.4]$ for Theorem 2. For x just below 10^7 the next prime 10000019 is used.

7. Counting primes: proof of Theorems 3, 4 and Corollary 5

Let F be a test function as in Section 2, $g = \widehat{F}$, and put

$$m = \sup_{|t|\le 1} g(t)_+, \qquad I = \int_1^N g_+, \qquad V = \int_1^N |g'|, \qquad c^*(\alpha) = \frac{\|F\|_1 + 2\alpha m}{F(0) - 8I}.$$

Let $x \ge 10^7$, $h = c\sqrt{x}\log x$, and suppose that $[x, x+h]$ contains at most $\alpha\sqrt{x}$ primes. We repeat the argument of Section 5 with two changes. First, a prime $p \in [x, x+h]$ has $|u_p| \le 1$ and contributes at most $m\log p/\sqrt{p} \le m\log x/\sqrt{x}$ to the prime sum in (1), so these primes contribute at most $\alpha m \log x$ in total. Second, we no longer assume that g_+ is monotone on $[1, N]$: with the same cells as in Section 5, the sum over cells of $\sup g_+$ is at most $V + \Lambda_0(1+a^{-1/2})I$, so $g(1)$ is replaced by V. This gives

$$\Lambda_0\Big[F(0) - 2(\cosh(\pi\Delta N)-1)\int|g| - 8(1+\varepsilon_1)(1+a^{-1/2})I\Big] \le \|F\|_1\log\frac{1}{2\pi\Delta} + L_F + \varepsilon_Z + \varepsilon_\Gamma + \varepsilon_{pp} + 8(1+\varepsilon_1)V + \alpha m\log x.$$

Since $\Lambda_0 \sim \frac{c}{2}\log x$ and $\log\frac{1}{2\pi\Delta} = \frac{1}{2}\log x - \log\log x + O(1)$, this fails for all large x as soon as $c > c^*(\alpha)$, that is, as soon as $\alpha < A_F c - B_F$ with $A_F = (F(0) - 8I)/(2m)$ and $B_F = \|F\|_1/(2m)$. The ancillary file gives four test functions $F^{(1/2)}, F^{(1)}, F^{(2)}, F^{(5)}$ (all with $K = 22$, $N = 1.45$), for which, certified as in Section 4,

$$c^*(\tfrac{1}{2}) \le 1.490763, \quad c^*(1) \le 2.063334, \quad c^*(2) \le 3.156137, \quad c^*(5) \le 6.330937,$$

and (A_F, B_F) are bounded by the four pairs of Corollary 5 (A_F from below, B_F from above). This proves Corollary 5.

For Theorem 3 take $\alpha = 1$, $c = 2.064$ and the function $F^{(1)}$, with $K = 22$, $N = 1.45$ and $b = (1,\ 0.0477715,\ -0.2030288,\ 0.0955958,\ 0.030982,\ -0.0622397,\ 0.0247126,\ 0.01102,\ -0.0169132,\ 0.0076692,\ -2.641\cdot 10^{-4},\ -0.0026749,\ 0.0033963,\ -0.0022117,\ -3\cdot 10^{-5},\ 0.001212,\ -8.052\cdot 10^{-4},\ 1.108\cdot 10^{-4},\ 7.5\cdot 10^{-5},\ -4.65\cdot 10^{-5},\ 6.1\cdot 10^{-5},\ -4.2\cdot 10^{-5})$. The certified data are

$$F(0) = 1.7334716812\ldots, \qquad I \le 0.0078370044, \qquad \|F\|_1 \le 1.57863484, \qquad m \le 0.93436689,$$

$$L_F \le -2.160034, \quad V \le 0.221183, \quad \int |g| \le 1.735033, \quad \max|g| \le 0.934401,$$

$$\int_0^\infty |F'| \le 3.553063, \qquad \int_0^\infty |F'(s)| \log^+ s\, ds \le 1.772159,$$

so $c^*(1) \le 2.063334 < 2.064$. Let $S_3(x)$ be the right side minus the left side of the displayed inequality. On 3001 grid points $\log_{10} x \in [7, 150]$, S_3 is negative and decreasing, with maximum -4.2678 at $x = 10^7$ and consecutive values differing by less than 0.04. For $x > 10^{150}$ the same asymptotic argument as in Section 5 applies, since $c - c^*(1) > 0$ and the term $-\|F\|_1 \log\log x$ is negative. So the interval contains more than $\sqrt{x}$ primes for all $x \ge 10^7$.

For $x < 10^7$ we use the sieve. For consecutive primes $p_n < p_{n+1}$ with $11 \le p_n < 10^7$, let A_n be the number of primes in $(p_n,\ p_n + 2.064\sqrt{p_n}\log p_n]$. The computation gives $A_n > \sqrt{p_{n+1}}$ in every case (smallest ratio 1.83 for $p_n > 1000$). Since $x + 2.064\sqrt{x}\log x$ is increasing, this covers every $x \in [11, 10^7]$. The range $6 \le x \le 11$ is checked directly.

The other three cases of Theorem 4 are proved in the same way with $F^{(1/2)}, F^{(2)}, F^{(5)}$; all their data are printed by the ancillary script. The maximum of the corresponding function S_3 on $[10^7, 10^{150}]$ is -3.7503, -4.7377 and -4.7454 respectively, attained at $x = 10^7$. Below 10^7, the sieve condition $A_n > \alpha\sqrt{p_{n+1}}$ holds for all $p_n \ge 7$, 29, 127 respectively, and the ranges $[6, 7]$, $[14, 29]$, $[114, 127]$ are checked directly.

8. Upper bounds: proof of Theorem 6

Let F be a test function as in Section 2 with $g = \widehat{F} \ge 0$ on $[-N, N]$, and let $\mu = \min_{|t|\le 1} g(t) > 0$. Let $x \ge 10^7$, $h = c\sqrt{x}\log x$, and let M be the number of primes in $[x, x + h]$. In (1) every term of the sum over n is non-negative, and a prime $p \in [x, x + h]$ contributes at least $\mu \log p/\sqrt{p} \ge \mu \log(x + h)/\sqrt{x + h}$. Bounding the right side of (1) as in Section 5 gives

$$M\,\mu\,\frac{\log(x+h)}{\sqrt{x+h}} \le \Lambda_0(1 + a^{-1})\Big[F(0) + (\cosh(\pi\Delta N) - 1)\int g\Big] + \|F\|_1 \log\frac{1}{2\pi\Delta} + L_F + \varepsilon_Z + \varepsilon_\Gamma.$$

Write this as $M \le T(x)\sqrt{x}$. Since $\Lambda_0 \le \frac{c}{2}\log x\cdot(1+v)^{1/4}$ and $\log\frac{1}{2\pi\Delta} \le \frac{1}{2}\log x - \log\log x + \log\frac{2}{c} + v$, we get $\limsup T(x) \le C_\infty := (cF(0) + \|F\|_1)/(2\mu)$.

The ancillary file gives four such functions (with $K = 22$ and $N = 1.8$, 1.45, 1.3, 1.1 for $c = \frac{1}{2}, 1, 2, 5$). For each, the script certifies $g > 0$ on $[1, N - 0.05]$, $g' < 0$ on $[N - 0.05, N]$ (so $g \ge 0$ on $[1, N]$ because $g(N) = 0$), $\mu \ge 0.9999$, and

$$C_\infty \le 1.426727, \quad 2.042276, \quad 3.170473, \quad 6.371236$$

respectively. On 3001 grid points $\log_{10} x \in [7, 150]$ the maximum of T is 1.39851, 2.00863, 3.13073, 6.32142, each below the corresponding C. For $x > 10^{150}$ we have $\|F\|_1(\log\frac{2}{c} - \log\log x) + L_F < 0$ in all four cases, so $T(x) \le C_\infty(1 + 10^{-50}) < C$.

For $x < 10^7$: if $p_{n-1} < x \le p_n$, the number of primes in $[x, x + c\sqrt{x}\log x]$ is at most $1 + \pi(p_n + c\sqrt{p_n}\log p_n) - \pi(p_n)$. The sieve shows that this is less than $C\sqrt{p_{n-1}}$ for every prime $p_n < 10^7$ (with $p_0 = 1$), which covers all $x \in [2, 10^7]$.

Ancillary file

`VERIF_CRAMER_RH.py` (Python 3, numpy only) recomputes every number in Sections 3–6 with the error bounds of Section 4: calibration on $H(x/0.9)$, both J bounds, the shape of g, the inequality $S(x) < 0$ on $[10^7, 10^{150}]$, the sieve below 10^7, Corollary 11(b), and all the data of Sections 7 and 8, including the eight test functions. It runs in well under a minute on a recent machine.

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