hexagon:2611.00990v1

A COUNTEREXAMPLE TO A QUESTION OF GEISS, LECLERC AND SCHRÖER

Abstract. We construct a dual canonical basis element of weight $4\alpha_0 + 4\alpha_1$ in affine type $A_1^{(1)}$ which is not the evaluation function of any finite-dimensional nilpotent preprojective module. We prove that every dual canonical basis element in Dynkin types A_5 and D_4 is the evaluation function of a finite-dimensional preprojective module, with all bases specialized at $q = 1$ over $\mathbb{C}$.

1. Introduction

Lusztig introduced canonical bases for quantized enveloping algebras in finite simply laced type [Lus90] and constructed them for symmetric Cartan data using varieties of quiver representations and perverse sheaves [Lus91]. Kashiwara constructed crystal bases and global crystal bases for symmetrizable Kac–Moody algebras by algebraic methods [Kas91]. The specialization of the canonical basis at $q = 1$ gives a basis of the corresponding ordinary enveloping algebra. A second geometric construction uses constructible functions on varieties of nilpotent preprojective modules. Lusztig extended this construction from finite and affine types to arbitrary symmetric Cartan data in [Lus00]. The resulting semicanonical basis is indexed by the irreducible components of these varieties. Canonical basis elements are represented by simple perverse sheaves arising from flag maps, whereas semicanonical elements are normalized by their generic values on irreducible components.

We work over $\mathbb{C}$. Let Q be a finite quiver without oriented cycles, with vertex set I and arrow set Q_1. Its double $\overline{Q}$ contains, for each arrow $a : i \to j$, a reverse arrow $a^* : j \to i$. The preprojective algebra is

$$\Pi(Q) = \mathbb{C}\overline{Q} \Big/ \Big\langle \sum_{a \in Q_1} (aa^* - a^*a) \Big\rangle,$$

where $\mathbb{C}\overline{Q}$ is the path algebra. For a finite-dimensional module M, write M_i for its vertex spaces. The module is nilpotent if all sufficiently long paths act by zero. Its dimension vector is $\underline{\dim} M = (\dim_{\mathbb{C}} M_i)_{i \in I}$, identified with the weight $\sum_i (\dim_{\mathbb{C}} M_i)\alpha_i$, where α_i are the simple roots. Write S_i for the one-dimensional simple module at vertex i. The underlying graph defines a symmetric generalized Cartan matrix: its diagonal entries are 2, and its ij entry for $i \neq j$ is minus the number of edges joining i and j. Let $\mathfrak{n}$ be the positive part of the associated Kac–Moody algebra, with Chevalley generators E_i. Its enveloping algebra has graded dual

$$U(\mathfrak{n})^*_{\mathrm{gr}} = \bigoplus_{\nu} U(\mathfrak{n})^*_{\nu}, \qquad \nu = \sum_{i \in I} \nu_i \alpha_i, \quad \nu_i \in \mathbb{Z}_{\geq 0}.$$

Lusztig's convolution algebra generated by the indicator functions of the S_i is isomorphic to $U(\mathfrak{n})$, with E_i corresponding to the indicator of S_i. For a nilpotent module M, view $u \in U(\mathfrak{n})_\nu$ as a constructible function and set $\delta_M(u) = u(M)$ when $\nu = \underline{\dim} M$ and $\delta_M(u) = 0$ otherwise. This defines $\delta_M \in U(\mathfrak{n})^*_{\mathrm{gr}}$ of weight $\underline{\dim} M$ [GLS11, §2.2]. Throughout, the dual canonical basis means the specialization at $q = 1$ of the integral dual canonical basis of the quantum half.

Kashiwara and Saito constructed the geometric crystal on the irreducible components and related its labels to those of the canonical basis [KS97]. They also exhibited reducible singular support in type A, answering a question arising from the perverse-sheaf construction. Subsequent

Date: November 24, 2026.

comparisons revealed differences in the bases themselves. Leclerc's imaginary vectors have squares which are not powers of q times a single dual canonical vector; his examples disproved the Berenstein–Zelevinsky multiplication conjecture [Lec03]. Geiss, Leclerc and Schröer proved that, among simple simply laced Dynkin types, the canonical and semicanonical bases coincide exactly in type A_n with $n \leq 4$ [GLS05, Theorem 1.2]. The dual semicanonical basis consists of generic evaluation functionals: on a dense open subset of each irreducible component, δ_M is constant. The same authors related products of these vectors to generic direct-sum decompositions of modules, linking multiplication to the geometry of irreducible components [GLS05, Theorem 1.1].

Geiss, Leclerc and Schröer asked whether every dual canonical basis element is δ_M for some finite-dimensional nilpotent preprojective module M [GLS07, 2008 author's version, §23.2], [GLS11, §18.4]. Their setting allows every symmetric generalized Cartan matrix, including multiple edges. The adjacent Open Orbit Conjecture concerns equality of the dual canonical and dual semicanonical vectors with the same crystal label for a component containing an open orbit [GLS11, §18.3]. The evaluation question ranges over all points of all nilpotent module varieties. Its authors already gave a Kronecker example of dimension $(2, 2)$: the two forward arrows are I_2 and a Jordan block $J_2(\lambda)$, while the reverse arrows are zero. Here I_2 is the identity matrix and $\lambda \in \mathbb{C}$. The evaluation is a dual canonical element, whereas the generic evaluation on its component is a different vector. A composition-flag variety has Euler characteristic 1 for this module and 2 for a generic direct sum of two regular simple modules [GLS11, §18.4]. Special points are therefore essential to the realization problem.

Explicit flag calculations in the same affine type were developed by Scott [Sco07]. For shape modules, he expressed composition-flag Euler characteristics as counts of chess tableaux and obtained block-Toeplitz determinant formulas for their evaluation functions. These results provide families for which evaluation can be computed directly from combinatorial data. Our counterexample is also specified by its complete composition-flag character; its nonrealizability is proved by classifying every module compatible with the support of that character.

We fix the conventions for this comparison. For a word $\mathbf{i} = i_1 \cdots i_d$ in I, let $[\mathbf{i}]$ denote its formal word symbol and $\alpha_{i_1} + \cdots + \alpha_{i_d}$ its weight. If M has total dimension $d = \sum_i \dim_{\mathbb{C}} M_i$, let $\mathcal{F}_{\mathbf{i}}(M)$ be the projective variety of submodule flags

$$0 = M^0 \subset M^1 \subset \cdots \subset M^d = M, \qquad M^r/M^{r-1} \simeq S_{i_r}.$$

Words thus record ascending flags, from socle to head. For a homogeneous functional f and a module M of the same weight, define

$$\begin{aligned} \mathrm{ch}_E(f) &= \sum_{\mathbf{i}} f(E_{i_1} \cdots E_{i_d})[\mathbf{i}], \\ \mathrm{ch}_E(M) &= \sum_{\mathbf{i}} \chi(\mathcal{F}_{\mathbf{i}}(M))[\mathbf{i}], \end{aligned} \tag{1.1}$$

where the sums run over words of that weight and χ is the topological Euler characteristic, equivalently the compactly supported Euler characteristic for these projective varieties. We use the convolution convention in which the first factor is evaluated on the submodule and the second on the quotient [GLS11, §2.2]. Hence

$$\delta_M(E_{i_1} \cdots E_{i_d}) = \chi(\mathcal{F}_{\mathbf{i}}(M)), \qquad \mathrm{ch}_E(\delta_M) = \mathrm{ch}_E(M). \tag{1.2}$$

Ordinary Chevalley monomials span each weight space, so these characters determine the corresponding functionals.

For the counterexample, take Q to be the Kronecker quiver with two arrows from 0 to 1, and put $\Pi = \Pi(Q)$. Its Cartan matrix and minimal positive imaginary root are

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \qquad \delta = \alpha_0 + \alpha_1.$$

Set

$$\begin{aligned} &a = [\mathtt{01010101}], \\ &P = 4\big([\mathtt{01101001}] + [\mathtt{01100101}] + [\mathtt{01011001}]\big), \qquad C = P + 2a. \end{aligned} \tag{1.3}$$

Theorem 1.1. *There is a specialized dual canonical basis element $b^* \in U(\mathfrak{n})^*_{4\delta}$ whose ordinary word character is C. For every finite-dimensional nilpotent Π-module M, one has $\delta_M \neq b^*$.*

Sections 2–3 construct this element from a simple Hecke module and determine its complete character. Sections 4–6 reduce a possible module realization to a local Gorenstein algebra of length eight, classify the resulting algebras, including nonhomogeneous relations, and compute their flag characters.

The two representation-infinite tame Dynkin preprojective algebras have a different outcome. The work of Geiss and Schröer on tubular module varieties [GS03] describes the generic decompositions for type A_5 and their analogue in type D_4. Geiss, Leclerc and Schröer used this description for a detailed semicanonical-basis analysis in type A_5 [GLS05, Sections 14–17 and 19.1]. The tubular models have weights $(2, 3, 6)$ and $(3, 3, 3)$, respectively. The cluster categories of tame canonical algebras have connected cluster-tilting graphs, as proved by Barot, Kussin and Lenzing [BKL10]. Barot and Geiss constructed categorifications of tubular cluster algebras and described their cluster variables using real Schur roots [BG12]. In these two Dynkin types, Baumann also computed canonical expansions and semicanonical pairings for explicit two-parameter crystal families in his study of the quantum Frobenius morphism [Bau12, §5].

The rank-two canonical calculation used here comes from Lampe's description of a Kronecker quantum unipotent cell [Lam11]. His distinguished family of dual canonical vectors specializes to a recurrence of Chebyshev polynomials of the second kind [Lam11, Corollary 4.4 and §5.1]. We use this recurrence with its projective coefficient to construct evaluation functions of indecomposables in ordinary tubes. Qin identifies dual canonical bases with common triangular bases for quantum unipotent cells after localization and rescaling [Qin23]. His freezing and coefficient-change results provide tools for transporting these bases [Qin25]. We apply them to the tubular construction at every slope and compare the resulting vectors with the entire dual semicanonical basis.

For a Dynkin diagram, let N denote a maximal unipotent subgroup of the simply connected complex algebraic group with that diagram, and set $\mathfrak{n} = \operatorname{Lie} N$. Its coordinate ring is $\mathbb{C}[N] = U(\mathfrak{n})^*_{\mathrm{gr}}$.

Theorem 1.2. *Let $\Lambda = \Pi(A_5)$ or $\Pi(D_4)$, and let $\mathbf{B}^*_\Lambda$ be the specialized dual canonical basis of $\mathbb{C}[N]$. For every $b \in \mathbf{B}^*_\Lambda$ there is a finite-dimensional Λ-module M such that $b = \delta_M$.*

Theorem 7.1 gives an explicit construction. A generic module in an irreducible component consists of a rigid summand and, when present, distinct ordinary mouths at one slope. We retain the rigid summand and replace these mouths by one indecomposable of the same total length in an ordinary tube. Sections 7–10 compute its character, prove canonical-basis membership with all projective coefficients, and use a unitriangular comparison to obtain every dual canonical basis element.

2. A simple Hecke module with complete character C

2.1. **Rank-one generators and the primitive exterior square.** Let $H_8(h)$ be the finite Hecke algebra with generators $T_1, \dots, T_7$. Fix $h = -1$, so that its relations are

$$(T_i + 1)^2 = 0, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_iT_j = T_jT_i \quad (|i-j| > 1).$$

On $K = \mathbb{C}^7$ with basis $e_1, \dots, e_7$, let δ_{ij} be 1 if $i = j$ and 0 otherwise, and set

$$\Omega(e_i, e_j) = \delta_{j,i+1} - \delta_{j,i-1}, \qquad N_i(z) = \Omega(e_i, z)e_i, \qquad T_i = -1 + N_i. \tag{2.1}$$

The radical of Ω is the line $\mathbb{C}(e_1 + e_3 + e_5 + e_7)$. It is killed by every N_i. Consequently Ω descends to a nondegenerate alternating form on

$$W = K/\operatorname{rad}\Omega, \qquad \dim W = 6.$$

Write v_i for the image of e_i. The same notation N_i, T_i denotes the induced endomorphisms of W.

One has $N_i^2 = 0$; for adjacent $A = N_i$, $B = N_{i+1}$, one has $ABA = -A$ and $BAB = -B$. Thus

$$(-1 + A)(-1 + B)(-1 + A) = -1 + A + B - AB - BA,$$

which is symmetric in A, B. Distant N_i, N_j have both products zero. This proves the relations on W.

Each $g_i = -T_i = 1 - N_i$ preserves Ω. Let $c : \Lambda^2 W \to \mathbb{C}$ be contraction, $c(u \wedge v) = \Omega(u, v)$, and let $\beta = \sum_{k=1}^{3} p_k \wedge q_k$ for a symplectic basis with $\Omega(p_k, q_l) = \delta_{kl}$. Then β is invariant and $c(\beta) = 3$, giving

$$\Lambda^2 W = \mathbb{C}\beta \oplus \ker c, \qquad L = \Lambda^2 W/\mathbb{C}\beta \simeq \ker c, \qquad \dim L = 14. \tag{2.2}$$

The Hecke generators on the exterior square and its quotient are

$$\mathsf{T}_i = -\Lambda^2 T_i = -\Lambda^2 g_i. \tag{2.3}$$

Since N_i has rank one, $\Lambda^2(1 - N_i) = 1 - D_i$, where $D_i(u \wedge v) = N_iu \wedge v + u \wedge N_iv$ and $D_i^2 = 0$. Hence $(\mathsf{T}_i + 1)^2 = 0$. The braid and distant commutation relations follow from functoriality of the exterior power, with the same scalar on both sides. Both the invariant line and the quotient are therefore Hecke modules.

Lemma 2.1. *The $H_8(-1)$-module L is simple over $\mathbb{C}$.*

Proof. Let $\mathrm{Sp}(W)$ be the group of linear automorphisms preserving Ω, and let G be the Zariski closure of the group generated by the g_i in $\mathrm{Sp}(W)$. For every integer m, $g_i^m = 1 - mN_i$, so G contains $1 - tN_i$ for all $t \in \mathbb{C}$. Its Lie algebra therefore contains all N_i. For $u, v \in W$, write

$$Q_{u,v}(z) = \Omega(v, z)u + \Omega(u, z)v.$$

Writing $\mathfrak{sp}(W)$ for the Lie algebra of $\mathrm{Sp}(W)$, the identification $\mathrm{Sym}^2 W \simeq \mathfrak{sp}(W)$ sends the symmetric tensors to these operators. Direct calculation gives

$$[N_i, N_{i+1}] = Q_{v_i, v_{i+1}}, \qquad [Q_{v_i, v_{j-1}}, N_j] = Q_{v_i, v_j} \quad (j \geq i + 2).$$

Together with $Q_{v_i, v_i} = 2N_i$, this gives every Q_{v_i, v_j} for $1 \leq i \leq j \leq 6$. The vectors $v_1, \dots, v_6$ form a basis, so these operators span $\mathfrak{sp}(W)$. Thus G is a closed subgroup of $\mathrm{Sp}(W)$ whose Lie algebra is $\mathfrak{sp}(W)$. Over $\mathbb{C}$ a closed algebraic subgroup has the dimension of its Lie algebra. It follows that the identity component of G has the same dimension as $\mathrm{Sp}(W)$, and hence is all of the connected group $\mathrm{Sp}(W)$. Therefore $G = \mathrm{Sp}(W)$.

For the diagonal symplectic torus, let ε_i be the weight of p_i, so that q_i has weight $-\varepsilon_i$. The twelve nonzero weights of $\ker c$ are $\pm\varepsilon_i \pm \varepsilon_j$ $(i<j)$, each with multiplicity one. Its zero-weight space is

$$\left\{ \sum_{k=1}^{3} a_k p_k \wedge q_k : \ \sum_{k=1}^{3} a_k = 0 \right\}.$$

A nonzero invariant subspace is a direct sum of its torus weight spaces. If it contains only zero-weight vectors, take a nonzero such vector and choose i,j with $a_i \neq a_j$. The symplectic root operator $p_j \mapsto p_i$, $q_i \mapsto -q_j$ sends it to $(a_j - a_i) p_i \wedge q_j \neq 0$. Thus the subspace contains a nonzero weight vector. The symplectic Weyl group is transitive on the twelve nonzero weights, so it contains all their weight spaces. The opposite root operator sends $p_i \wedge q_j$ to $p_j \wedge q_j - p_i \wedge q_i$; these differences span the remaining two-dimensional zero-weight space. The subspace is all of $\ker c$.

An T_i-invariant subspace is invariant under $\Lambda^2 g_i = -\mathsf{T}_i$ and their inverses, hence under their Zariski closure. The preceding irreducibility proves the assertion for L. □

2.2. **Jucys–Murphy operators.** The normalized Jucys–Murphy elements of $H_8(-1)$ are

$$J_1 = 1, \qquad J_{r+1} = -T_r J_r T_r. \tag{2.4}$$

For a finite-dimensional $H_8(-1)$-module V and $\mathbf{i} = i_1 \cdots i_8 \in \{0,1\}^8$, put

$$V_{\mathbf{i}} = \bigcap_{r=1}^{8} \ker\bigl((J_r - (-1)^{i_r})^{\dim V}\bigr),$$
$$\mathrm{ch}_{\mathrm{JM}}(V) = \sum_{\mathbf{i}} \dim V_{\mathbf{i}}[\mathbf{i}].$$

Thus residues $0,1$ correspond to generalized eigenvalues $+1,-1$. For the module W, set

$$a_0 = a_4 = 0, \qquad a_k = v_1 + v_3 + \cdots + v_{2k-1}, \qquad b_k = v_{2k}, \qquad W_k = \langle a_k, b_k \rangle \quad (1 \le k \le 3).$$

These vectors satisfy $\Omega(a_k, b_l) = \delta_{kl}$ and $\Omega(a_k, a_l) = \Omega(b_k, b_l) = 0$, so $W = W_1 \oplus W_2 \oplus W_3$ is an orthogonal sum of symplectic planes. In particular $\beta = \sum_k a_k \wedge b_k$. The complete generator action in this basis is

$$\begin{aligned} N_{2k} a_l &= -\delta_{kl} b_k, & N_{2k} b_l &= 0 && (1 \le k \le 3), \\ N_{2k-1} a_l &= 0, & N_{2k-1} b_l &= (\delta_{kl} - \delta_{k-1,l})(a_k - a_{k-1}) && (1 \le k \le 4). \end{aligned} \tag{2.5}$$

In (2.5), $1 \le l \le 3$. In the ordered basis (a_k, b_k), the recursion gives

$$J_r|_{W_k} = \begin{cases} (-1)^{r-1} I_2, & r < 2k, \\ \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, & r = 2k, \\ \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, & r = 2k+1, \\ (-1)^r I_2, & r > 2k+1. \end{cases} \tag{2.6}$$

At an internal odd step $r = 2k-1$, put $J = J_{2k-1}$ and $N = N_{2k-1}$, and set $(A,B) = (a_{k-1}, b_{k-1})$ and $(a,b) = (a_k, b_k)$. Before that step,

$$JA = -A, \quad JB = -B - 2A, \quad Ja = a, \quad Jb = b,$$
$$NB = A - a, \quad Nb = a - A, \quad NA = Na = 0.$$

Substitution into $-(1-N)J(1-N)$ gives $A \mapsto A$, $B \mapsto B$, $a \mapsto -a$, $b \mapsto -b+2a$. At an even step, the only nontrivial check is

$$-\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 2\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}-1 & -2\\ 0 & -1\end{pmatrix}.$$

The endpoint actions $N_1 b_1 = a_1$ and $N_7 b_3 = a_3$ give J_2 and $J_8 = I$. This proves (2.6) by induction, including the absence of off-diagonal blocks. The three full words on W_1, W_2, W_3 are

$$w_1 = \texttt{01101010}, \qquad w_2 = \texttt{01011010}, \qquad w_3 = \texttt{01010110}, \tag{2.7}$$

each with multiplicity two.

Let J_r^{ext} denote the normalized JM operators for the generators T_i of (2.3). Their recursion gives

$$J_r^{\mathrm{ext}} = (-1)^{r-1}\Lambda^2 J_r. \tag{2.8}$$

Equation (2.8) follows inductively from $\mathsf{T}_r = -\Lambda^2 T_r$:

$$-\mathsf{T}_r\big((-1)^{r-1}\Lambda^2 J_r\big)\mathsf{T}_r = (-1)^r \Lambda^2 (T_r J_r T_r) = (-1)^r \Lambda^2 J_{r+1}.$$

The final equality uses that $\Lambda^2(-A) = \Lambda^2 A$. Every block in (2.6) has determinant one. Thus each $\Lambda^2 W_k$ has word a. On $W_i \wedge W_j \simeq W_i \otimes W_j$, the word is the bitwise sum modulo two of the alternating word and w_i, w_j. This gives the entire character:

JM-stable exterior summand	Dimension	Word
$\bigoplus_{k=1}^{3} \Lambda^2 W_k$	3	01010101
$W_1 \wedge W_2$	4	01100101
$W_1 \wedge W_3$	4	01101001
$W_2 \wedge W_3$	4	01011001

Each summand has a single simultaneous generalized eigenvalue; its entire dimension counts, even when a JM operator has a nontrivial Jordan block. The invariant line $\mathbb{C}\beta$ has $\mathsf{T}_i = -1$, hence word a. Passing to the simple quotient removes one copy of a. We have proved

$$\mathrm{ch}_{\mathrm{JM}}(\Lambda^2 W) = P + 3a, \qquad \mathrm{ch}_{\mathrm{JM}}(L) = P + 2a = C. \tag{2.9}$$

In particular, all other word coefficients are zero.

3. Identification with a dual canonical basis element

Let $\alpha = 4\alpha_0 + 4\alpha_1$, and let R_α be the Khovanov–Lauda–Rouquier (quiver Hecke) algebra for the Cartan matrix in Section 1, with the parameters of Brundan–Kleshchev [BK09, Section 2.3]. Its word idempotents are denoted by $e(\mathbf{i})$. Let Λ_0 be the fundamental weight at vertex 0. The Hecke/KLR isomorphism [BK08, Main Theorem and §4.1], equivalently [BK09, Theorem 4.1], identifies the charge-zero level-one cyclotomic quotient $R_\alpha^{\Lambda_0}$ with the relevant block of $H_8(-1)$ and preserves every word idempotent $e(\mathbf{i})$. For the commuting affine-Hecke generators X_r, the convention is $T_r X_r T_r = h X_{r+1}$ with $X_1 = 1$; at $h = -1$ it is precisely (2.4). Thus $e(\mathbf{i})L$ is the simultaneous generalized eigenspace computed above.

The algebra has a graded anti-involution $*$ fixing its word idempotents, polynomial generators and crossing generators [BK09, Section 2.7]. The graded dual $V^{\circledast}$ of a finite-dimensional graded module V has degree-m part $\mathrm{Hom}_{\mathbb{C}}(V_{-m}, \mathbb{C})$ and action $(af)(v) = f(a^* v)$. Such a module is self-dual if $V \simeq V^{\circledast}$. By [BK09, Theorem 4.11], L has a self-dual graded lift. Inflate it along $R_\alpha \twoheadrightarrow R_\alpha^{\Lambda_0}$. The submodules are unchanged by this surjection, so the inflated module remains simple; its word spaces and self-duality are unchanged as well. The order of the Hecke parameter -1 is two, giving the affine $A_1^{(1)}$ Cartan matrix used here.

Put $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Let ${}_{\mathcal{A}}\mathbf{f}$ be the divided-power integral form of the quantum half for this Cartan matrix, with generators θ_0, θ_1. For each positive-root-lattice weight ν, let $\operatorname{Proj} R_\nu$ denote the category of finitely generated graded projective R_ν-modules. The categorification isomorphism is

$$\gamma : {}_{\mathcal{A}}\mathbf{f} \longrightarrow \bigoplus_{\nu} K_0(\operatorname{Proj} R_\nu).$$

Here K_0 is the split Grothendieck group and the grading shift gives its $\mathcal{A}$-module structure. Multiplication corresponds to induction, so [BK09, Theorem 2.7] gives, in the stated order,

$$\gamma(\theta_{i_1} \cdots \theta_{i_8}) = [R_\alpha e(\mathbf{i})]. \tag{3.1}$$

For graded modules P, V, let $\operatorname{HOM}(P, V)$ be the graded vector space whose degree-m part consists of module maps sending P_k to V_{k+m} for every k. For a finite-dimensional graded vector space V, put $\operatorname{qdim} V = \sum_m q^m \dim V_m$. For a finitely generated graded projective P, its projective dual is $P^{\#} = \operatorname{HOM}_{R_\alpha}(P, R_\alpha)$ with action $(af)(p) = f(p)a^*$. Corollary 2.6 of [BK09] shows that the projective cover of a self-dual simple module is self-dual under $\#$. Over $\mathbb{C}$, Theorem 5.19 of [BK09] identifies these self-dual indecomposable projectives with the canonical basis under γ. The graded Cartan pairing is $\langle [P], [L] \rangle = \operatorname{qdim} \operatorname{HOM}_{R_\alpha}(P, L)$. If P_b is the self-dual projective corresponding to a canonical vector b, this pairing is 1 when its head is L and 0 otherwise. Hence $[L]$ gives the functional dual to a canonical basis vector. Moreover evaluation at the idempotent gives a graded isomorphism

$$\operatorname{HOM}_{R_\alpha}(R_\alpha e(\mathbf{i}), L) \simeq e(\mathbf{i})L.$$

After specializing the grading variable q to 1 and identifying θ_i with E_i, the corresponding dual canonical functional is therefore characterized by

$$\boxed{b^*(E_{i_1} \cdots E_{i_8}) = \dim_{\mathbb{C}} e(\mathbf{i})L.} \tag{3.2}$$

The pairing takes values in $\mathcal{A}$. Its semilinearity in the first argument specializes to ordinary bilinearity at $q = 1$. Both (1.2) and (3.2) evaluate the same ordered ordinary Chevalley monomial. Complete-flag Euler characteristics include the factorial contributed by refinements of an isotypic semisimple block. The specialization changes the grading parameter q, while the Hecke parameter remains $h = -1$. The character formula [BK09, (2.28)] and (2.9) now prove the existence assertion in Theorem 1.1, with exactly the coefficients in C.

4. Consequences of the word support

4.1. **Socle multiplicities and word support.**

Lemma 4.1. *Let M be a finite-dimensional nilpotent module over a quiver algebra without vertex loops. For a vertex i, let $s_i = \dim \operatorname{Hom}(S_i, M)$. The largest integer $r \geq 0$ for which some word beginning with i^r has nonzero flag Euler characteristic is exactly s_i. Every nonzero nilpotent module has a nonzero word coefficient.*

Proof. A submodule supported only at i is semisimple, since there are no loops. A flag beginning with i^r therefore requires $r \leq s_i$. For the reverse inequality, let $\operatorname{soc}_i M$ be the sum of all simple submodules isomorphic to S_i, and put $S = \operatorname{soc}_i M \simeq S_i^{s_i}$. It is the unique submodule supported at i of that dimension. Write $\operatorname{Flag}(S)$ for the complete flag variety of the vector space S. For every suffix v there is an isomorphism of varieties

$$\mathcal{F}_{i^{s_i} v}(M) \simeq \operatorname{Flag}(S) \times \mathcal{F}_v(M/S), \qquad \chi\mathcal{F}_{i^{s_i} v}(M) = s_i! \, \chi\mathcal{F}_v(M/S). \tag{4.1}$$

Every nonzero nilpotent module has nonzero socle: a nonzero last term of its radical filtration is annihilated by all arrows. Induction on dimension, repeatedly removing a nonzero entire isotypic socle and using (4.1), therefore produces a word with Euler characteristic a positive product of

factorials. For the zero module, the empty word has coefficient one. Apply this assertion to M/S to obtain a suffix with nonzero coefficient, proving the reverse inequality. □

4.2. **Ordinary commuting odd operators and the cyclic dual.** Write the two arrows as $\mathsf{a}_1, \mathsf{a}_2 : 0 \to 1$ and their reverses as $\mathsf{a}_1^*, \mathsf{a}_2^* : 1 \to 0$, with relations

$$\mathsf{a}_1^*\mathsf{a}_1 + \mathsf{a}_2^*\mathsf{a}_2 = 0, \qquad \mathsf{a}_1\mathsf{a}_1^* + \mathsf{a}_2\mathsf{a}_2^* = 0.$$

On the parity-graded vector space $M_0 \oplus M_1$ define

$$x|_{M_0} = \mathsf{a}_1, \quad x|_{M_1} = \mathsf{a}_2^*, \qquad y|_{M_0} = \mathsf{a}_2, \quad y|_{M_1} = -\mathsf{a}_1^*. \tag{4.2}$$

The two relations are exactly $xy = yx$. Conversely these formulas recover all four arrows from any pair of commuting odd operators. A graded subspace is a submodule exactly when it is x, y-stable. Nilpotence of all sufficiently long paths is equivalent to $(x, y)^N M = 0$ for some N: restricting a word in x, y to a fixed source parity gives the corresponding arrow path, up to sign, and every arrow path arises in this way. An odd operator interchanges M_0 and M_1. Let $(M^\vee)_i = \operatorname{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$, with the ordinary transpose actions

$$(xf)(m) = f(xm), \qquad (yf)(m) = f(ym).$$

These actions commute, so they define another module of the same kind. For $\operatorname{rev}(i_1 \cdots i_d) = i_d \cdots i_1$, annihilators give

$$\mathcal{F}_{\mathbf{i}}(M^\vee) \simeq \mathcal{F}_{\operatorname{rev}(\mathbf{i})}(M). \tag{4.3}$$

Here $\operatorname{soc} M = \ker x \cap \ker y$ and $\operatorname{top} M = M/(xM + yM)$. Suppose now that $\operatorname{ch}_E(M) = C$. Its support has no word beginning in 1 or 00. Lemma 4.1 gives $\operatorname{soc} M = S_0$. Its support has no word ending in 0 or 11; the same lemma and (4.3) give $\operatorname{top} M = S_1$. Thus $M^\vee$ has an even one-dimensional head and an odd one-dimensional socle. Choose an even lift v of a head generator. Writing $\mathfrak{r} = (x, y)$, one has $M^\vee = \mathbb{C}v + \mathfrak{r}M^\vee$. Iteration and $\mathfrak{r}^N M^\vee = 0$ give $M^\vee = \mathbb{C}[x, y]v$. The surjective map $\mathbb{C}[[x, y]] \to M^\vee$, $f \mapsto f(x, y)v$, has a parity-stable ideal kernel I containing $\mathfrak{r}^N$. It gives

$$M^\vee \simeq A := \mathbb{C}[[x, y]]/I. \tag{4.4}$$

The algebra A is Artinian local of length eight, with maximal ideal $\mathfrak{m} = (x, y)/I$. Its cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is purely odd and has dimension at most two. Its one-dimensional socle is the Artinian Gorenstein condition.

5. Length-eight algebras with odd socle

Proposition 5.1. *Let A be a commutative local $\mathbb{C}$-algebra of length eight, with a $\mathbb{Z}/2$-grading, even residue field, purely odd cotangent space of dimension at most two, and one-dimensional odd socle. As a graded algebra it is isomorphic to one of*

$$\begin{aligned} T &= \mathbb{C}[t]/(t^8), && |t| = 1, \\ B &= \mathbb{C}[u, v]/(uv, u^5 - v^3), && |u| = |v| = 1, \\ D &= \mathbb{C}[x, y]/(x^2 - y^4, xy^2), && |x| = |y| = 1. \end{aligned} \tag{5.1}$$

5.1. **The Hilbert function.** Write $\mathfrak{m}$ for the maximal ideal, $h_j = \dim \mathfrak{m}^j/\mathfrak{m}^{j+1}$, and $s = \max\{j : \mathfrak{m}^j \neq 0\}$. Every associated-graded piece is purely of parity j mod 2, since products of j odd cotangent generators span it. Also $\mathfrak{m}^s \subseteq \operatorname{soc} A$, so $h_s = 1$ and s is odd. All h_j for $0 \leq j \leq s$ are positive, by Nakayama.

If $h_1 = 1$, an odd lift t generates the algebra; its length gives $A \simeq T$. The case $h_1 = 0$ has length one and is impossible. Assume $h_1 = 2$. Then length eight gives $s \leq 6$. The value $s = 1$ would put a two-dimensional maximal ideal in the socle. For $s = 3$ the length is at most $1 + 2 + 3 + 1 = 7$. Thus $s = 5$ and $h_2 + h_3 + h_4 = 4$. If $h_2 = 1$, the initial ideal contains two independent binary quadratics. If they have no common linear factor, a linear syzygy between them would force a common factor; hence their four linear multiples are independent. They span the cubic forms, giving $h_3 = 0$. If they share a linear factor, their span becomes $\langle x^2, xy\rangle$ after a linear change. In every degree at least two, only the pure power of y can survive. The length is then at most $1 + 2 + 1 + 1 + 1 + 1 = 7$. Both cases contradict length eight. Consequently

$$(h_0, h_1, h_2, h_3, h_4, h_5) = (1, 2, 2, 1, 1, 1), \qquad \mathfrak{m}^6 = 0. \tag{5.2}$$

Take $R = \mathbb{C}[[x, y]]$, $\mathfrak{r} = (x, y)$ and $A = R/I$. Now $\mathfrak{r}^6 \subset I$. The quadratic initial relations form a line, generated by a binary quadratic q_2. The cubic initial relations have dimension three, two of which are the linear multiples of q_2. Over $\mathbb{C}$ there are two cases for q_2, according as its two factors are distinct or equal. All substitutions below have odd coordinate functions and invertible linear part. They are invertible formal coordinate changes preserving parity and $\mathfrak{r}^6$; on the finite quotient they and their inverses are polynomial substitutions.

5.2. **Distinct quadratic factors.** Take odd linear coordinates u, v with $q_2 = uv$. By parity, a lift of this relation is $Q = uv + q_4$ modulo $\mathfrak{r}^6$, where q_4 is a quartic form. Choose cubic forms f_3, g_3 with $vf_3 + ug_3 = q_4$; every quartic monomial lies in (u, v), so this is possible. The odd coordinates $u + f_3, v + g_3$ have product Q modulo $\mathfrak{r}^6$. Hence we may assume $uv \in I$.

Modulo uv, the extra cubic initial relation is $cu^3 + dv^3$. If both coefficients were nonzero, the homogeneous quotient by $(uv, cu^3 + dv^3)$ would have Hilbert function $(1, 2, 2, 1)$ and length six: its fourth powers vanish after multiplying the cubic relation by u, v. It maps onto the associated graded algebra of A, contradicting length eight. Interchanging the variables and rescaling, the extra relation therefore lifts v^3. Modulo uv and $\mathfrak{r}^6$ its only further odd terms are $\alpha u^5 + \beta v^5$. Multiplication by the even unit $1 - \beta v^2$ removes the last term. We obtain

$$J = (uv, v^3 + \alpha u^5, \mathfrak{r}^6) \subset I. \tag{5.3}$$

The quotient R/J has the eight-element basis $1, u, u^2, u^3, u^4, u^5, v, v^2$ for every α: mixed terms vanish, $v^3 = -\alpha u^5$, and all remaining powers reduce or vanish. Put $f_1 = uv$, $f_2 = v^3 + \alpha u^5$ and $f_3 = u^6$. These generate J: mixed degree-six monomials are multiples of uv, while $u^6 = f_3$ and $v^6 \equiv \alpha^2 u^{10} \equiv 0$ modulo (f_1, f_2, f_3). In lexicographic order $v > u$, the three S-polynomials are

$$v^2 f_1 - u f_2 = -\alpha f_3, \qquad u^5 f_1 - v f_3 = 0, \qquad u^6 f_2 - v^3 f_3 = \alpha u^{11}.$$

All reduce to zero, the last by f_3. Hence these polynomials form a Gröbner basis with leading monomials uv, v^3, u^6. Its eight standard monomials are the vectors displayed above. Since R/I also has length eight, $J = I$. If $\alpha = 0$, the independent vectors u^5, v^2 lie in the socle, which is impossible. If $\alpha \neq 0$, a constant rescaling gives $(uv, u^5 - v^3)$. Its relations already imply $\mathfrak{r}^6 = 0$. This is B.

5.3. **A double quadratic factor.** Take $q_2 = x^2$, and write a parity-homogeneous lift modulo $\mathfrak{r}^6$ as

$$Q = x^2 + A_4 x^4 + B_4 x^3 y + C_4 x^2 y^2 + D_4 x y^3 + E_4 y^4.$$

Multiplication by the even unit $1 - A_4x^2 - B_4xy - C_4y^2$ removes the first three quartic terms modulo $\mathfrak{r}^6$. In the new odd coordinate $x' = x + (D_4/2)y^3$, substitution of $x = x' - (D_4/2)y^3$ gives $Q = (x')^2 + E_4y^4$ modulo $\mathfrak{r}^6$. Using this new coordinate as x, write $Q = x^2 + Ey^4 \in I$.

Modulo $\langle x^3, x^2y\rangle$, the extra cubic initial relation is $cxy^2 + dy^3$. If $d \neq 0$, the homogeneous quotient has $x^2 = 0$ and $y^3 = -(c/d)xy^2$. Multiplication first by x gives $xy^3 = 0$ and then by y gives $y^4 = 0$. Its Hilbert function is $(1, 2, 2, 1)$, contradicting length eight. Thus the extra relation lifts xy^2. Reducing its quintic terms with Q leaves only $\alpha xy^4 + \beta y^5$. Multiplication by $1 - \alpha y^2$ removes the first, giving $F = xy^2 + \beta y^5 \in I$.

Set $X = x + \beta y^3$, an odd change. Then $F = Xy^2$, while Q becomes $X^2 - 2\beta Xy^3 + Ey^4$ modulo degree six. Adding $2\beta yF$ gives

$$J = (X^2 + Ey^4, Xy^2, (X, y)^6) \subset I. \tag{5.4}$$

The quotient R/J has the eight-element basis

$$1, y, y^2, y^3, y^4, y^5, X, Xy.$$

Indeed, $X^2 = -Ey^4$, $Xy^2 = 0$, and the degree-six truncation reduce every monomial to this list. The ideal in (5.4) is also generated by $X^2 + Ey^4$, Xy^2, and y^6. Put $g_1 = X^2 + Ey^4$, $g_2 = Xy^2$ and $g_3 = y^6$. In lexicographic order $X > y$, their S-polynomials are

$$y^2g_1 - Xg_2 = Eg_3, \qquad y^6g_1 - X^2g_3 = Ey^{10}, \qquad y^4g_2 - Xg_3 = 0.$$

They reduce to zero, so the three polynomials form a Gröbner basis. The listed vectors are its eight standard monomials and are independent. Length comparison again gives $J = I$. For $E = 0$, both Xy and y^5 lie in the socle, which is impossible. For $E \neq 0$, scaling X gives D. The relations of D imply $y^6 = x^2y^2 = 0$, so the truncation is automatic.

This completes the proof of Proposition 5.1. The socle of T is $\mathbb{C}t^7$. In B, put $z = u^5 = v^3$. Multiplication by u has kernel $\langle v, v^2, z\rangle$. Multiplication by v sends v to v^2, sends v^2 to z, and kills z, so the common kernel is $\mathbb{C}z$. In D, multiplication by y has kernel $\langle y^5, xy\rangle$, and $x(xy) = y^5$, so its socle is $\mathbb{C}y^5$. These socles are odd. Each displayed basis contains four even and four odd elements.

Remark 5.2. In the regular module R/I, graded submodules are graded ideals. A graded algebra isomorphism therefore induces isomorphisms of every ideal flag variety. Equivalently, a subspace is stable under the original coordinates if and only if it is stable under the new coordinates, since both substitutions are polynomial on the nilpotent algebra. Duality then preserves all the colored flag varieties under consideration.

6. The complete flag characters and the contradiction

6.1. **Why coordinate flags give the Euler characteristics.** The algebras B, D admit parity-preserving $\mathbb{C}^*$-actions with weights

$$B:\ \mathrm{wt}(u) = 3,\ \mathrm{wt}(v) = 5; \qquad D:\ \mathrm{wt}(x) = 2,\ \mathrm{wt}(y) = 1.$$

For B, put $z = u^5 = v^3$. The basis and its distinct weights are

$$(1, u, u^2, u^3, u^4, v, v^2, z), \qquad (0, 3, 6, 9, 12, 5, 10, 15).$$

For D the joint weight and color data are

Vector	1	y	y^2	y^3	y^4	y^5	x	xy
Weight	0	1	2	3	4	5	2	3
Color	0	1	0	1	0	1	1	0

Every joint weight-and-color space is one-dimensional, including in D: its repeated weights occur only between opposite colors. Thus the torus-fixed *graded* subspaces, and hence its fixed colored flags, are coordinate subspaces and coordinate flags.

For a fixed word, the ambient variety of graded flags is a product of ordinary complete flag varieties in the two parity spaces. Its attracting Schubert cells are affine spaces contracted by the torus to coordinate flags, since the weights are distinct within each parity. The submodule flag variety is closed and torus-invariant. Its intersection with a cell is closed in that cell; when nonempty, it contains the cell's fixed point and contracts to it. The intersection has Euler characteristic one. Using the equality of ordinary and compactly supported Euler characteristics for complex algebraic varieties and additivity over the cells, the required Euler characteristic equals the number of fixed flags.

In the following calculation, order basis vectors from the unit toward the socle. Removing them in that order gives a descending ideal flag of A and, by annihilators, the ascending flag of $A^\vee$. Thus the colors in this order give $\mathrm{ch}_E(A^\vee)$.

6.2. **The three flag characters.** The algebra T has the unique ideal chain $(t^7) \subset (t^6) \subset \cdots \subset (t) \subset T$. Consequently

$$\mathrm{ch}_E(T^\vee) = a. \tag{6.1}$$

For B, the multiplication order consists of two chains meeting at their ends:

$$1 < u < u^2 < u^3 < u^4 < z, \qquad 1 < v < v^2 < z.$$

The unit is first and z is last. The internal chains of lengths four and two have exactly $\binom{6}{2} = 15$ shuffles.

For D, the complete nonzero multiplication arrows are

$$\begin{aligned} y:&\quad 1 \longmapsto y \longmapsto y^2 \longmapsto y^3 \longmapsto y^4 \longmapsto y^5, \qquad x \longmapsto xy,\\ x:&\quad 1 \longmapsto x, \quad y \longmapsto xy, \quad x \longmapsto y^4, \quad xy \longmapsto y^5. \end{aligned}$$

Put 1 first and y^5 last. Start with the same fifteen shuffles of

$$y < y^2 < y^3 < y^4, \qquad x < xy.$$

The additional arrows impose exactly $x < y^4$ and $y < xy$. If (p, q) are the two internal positions occupied by x, xy, the first inequality excludes only $(5, 6)$ and the second excludes only $(1, 2)$. There are therefore thirteen fixed colored flags.

Both pairs of internal chains have color strings 1010 and 10. Table 1 lists all fifteen shuffles, grouped by their complete word, with initial 0 and final 1 included. It also lists all thirteen flags for D by deleting the indicated two positions.

Full word	Short-chain positions (p, q)	$T^\vee$	$B^\vee$	$D^\vee$	C
`01010101`	$(1,2),(3,4),(5,6)$	1	3	1	2
`01011001`	$(3,5),(3,6),(4,5),(4,6)$	0	4	4	4
`01100101`	$(1,3),(1,4),(2,3),(2,4)$	0	4	4	4
`01101001`	$(1,5),(1,6),(2,5),(2,6)$	0	4	4	4
Any other word	none	0	0	0	0
Total		1	15	13	14

TABLE 1. Complete Euler word characters. For D, remove $(1, 2)$ and $(5, 6)$ from the first row.

Both deleted flags have the alternating word. We have thus proved, including all vanishing coefficients,

$$\boxed{\mathrm{ch}_E(T^\vee) = a, \qquad \mathrm{ch}_E(B^\vee) = P + 3a, \qquad \mathrm{ch}_E(D^\vee) = P + a.} \tag{6.2}$$

Proof of Theorem 1.1. The simple module of Lemma 2.1 and (2.9), through the full-algebra Cartan pairing of Section 3, give a specialized dual canonical basis element b^* with word character C. If $\delta_M = b^*$, homogeneity forces $\underline{\dim} M = (4, 4)$, and (1.2)–(3.2) force $\mathrm{ch}_E(M) = C$ on every word. Section 4 then makes $M^\vee$ an algebra satisfying Proposition 5.1. By that proposition and Remark 5.2, its dual has one of the three characters in (6.2). Their alternating coefficients are $1, 3, 1$, respectively, whereas that of C is 2. This is a contradiction. □

7. Realization in types A_5 and D_4

Throughout Sections 7–10, let $\Lambda = \Pi(A_5)$ or $\Pi(D_4)$ over $\mathbb{C}$. Let N be the maximal unipotent subgroup of the corresponding simply connected group, and set $\mathfrak{n} = \mathrm{Lie}\, N$. Identify $U(\mathfrak{n})^*_{\mathrm{gr}}$ with $\mathbb{C}[N]$ and write $\mathbf{B}^*$ for its specialized dual canonical basis. The arrow ideals of these Dynkin preprojective algebras are nilpotent, so every finite-dimensional Λ-module is nilpotent. With the normalization in (1.2),

$$\delta_{M \oplus M'} = \delta_M \delta_{M'}, \qquad \delta_0 = 1.$$

For an irreducible component Z of a Λ-module variety, let ρ_Z be its generic evaluation. These functions form the dual semicanonical basis.

A module M is *rigid* if $\mathrm{Ext}^1_\Lambda(M, M) = 0$. The stable category $\underline{\mathrm{mod}}\Lambda$ has the same objects as $\mathrm{mod}\,\Lambda$; its morphisms are module homomorphisms modulo those factoring through projective modules. We write $\underline{\mathrm{End}}_\Lambda(M)$ for its endomorphism algebra. A module is *projective-free* if it has no nonzero projective direct summand. Each stable isomorphism class has a projective-free representative, unique up to isomorphism by Krull–Schmidt. Here Λ is self-injective, so projective and injective modules coincide. The suspension Σ in the stable category is induced by the cosyzygy $I(M)/M$, where $I(M)$ is the minimal injective envelope; τ denotes Auslander–Reiten translation.

The tubular descriptions use weighted projective lines of weights $(p_1, p_2, p_3) = (2, 3, 6)$ for A_5 and $(3, 3, 3)$ for D_4. Their grading group and degree map are

$$\mathbb{L} = \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \mid p_1\vec{x}_1 = p_2\vec{x}_2 = p_3\vec{x}_3 = \vec{c} \rangle, \qquad \deg \vec{x}_i = \frac{\mathrm{lcm}(p_1, p_2, p_3)}{p_i}.$$

Thus $\vec{c}$ is the common multiple element. The slope of a positive-rank sheaf is its degree divided by its rank; torsion sheaves have slope ∞. Write $\mathcal{O}$ for the structure sheaf and $\mathcal{O}(\vec{a})$ for its twist by $\vec{a} \in \mathbb{L}$. An *ordinary point* is a nonweighted point of the line. Its torsion sheaves form a homogeneous tube, that is, a tube of rank one. The quasi-length of an object is its composition length within its tube; a mouth has quasi-length one. For $s \in \mathbb{Q} \cup \{\infty\}$, write $Q_{s,\xi}[r]$ for the projective-free Λ-module corresponding to the indecomposable of quasi-length r in the ordinary tube indexed by ξ at slope s, and put $Q_{s,\xi}[0] = 0$.

Theorem 7.1. *Every element of $\mathbf{B}^*$ is δ_M for a finite-dimensional Λ-module M. More precisely, let Z be an irreducible component. If its generic module is rigid, choose that module R and put $c_Z = \delta_R$. Otherwise its generic decomposition is*

$$R \oplus Q_{s,\xi_1}[1] \oplus \cdots \oplus Q_{s,\xi_n}[1], \tag{7.1}$$

where R is rigid, $n \geq 1$, and the ξ_i are pairwise distinct general ordinary points. For a general ordinary point ξ, put

$$c_Z = \delta_{R \oplus Q_{s,\xi}[n]}.$$

Then the c_Z, as Z runs through all irreducible components in all dimension vectors, are precisely the elements of $\mathbf{B}^*$, without repetitions.

7.1. **Tubular structure and generic components.** The stable categories of $\operatorname{mod}\Pi(A_5)$ and $\operatorname{mod}\Pi(D_4)$ are triangle equivalent to the cluster categories of the weighted projective lines above [BG12, Theorem 3.1(c), Sections 3.2.1 and 3.2.3]. The Frobenius subcategory in the cited theorem consists of modules whose restrictions to the original path algebra are direct sums of summands of a selected terminal module. In the two examples this terminal module contains all 15, respectively all 12, indecomposable path-algebra modules. Every Λ-module therefore belongs to that subcategory, so the equivalences apply to the whole stable categories. These categories are 2-Calabi–Yau: there are bifunctorial dualities

$$\operatorname{Hom}(X,\Sigma Y)\simeq D\operatorname{Hom}(Y,\Sigma X),\qquad D=\operatorname{Hom}_{\mathbb{C}}(-,\mathbb{C}).$$

Consequently $\tau\simeq\Sigma$. In an ordinary tube both functors fix each indecomposable isomorphism class.

The classification in [GS03, Theorem 6.6, Corollary 6.7 and pp. 181–182] gives the generic decompositions in Theorem 7.1. Indecomposable generic components are rigid or are families of ordinary mouths. When an ordinary family occurs, all nonprojective factors have the same slope, and the remaining rigid factors belong to exceptional tubes at that slope. Projective factors may occur in R. The criterion for a direct-sum family to be an irreducible component is generic Ext-orthogonality of its summands; the generic evaluation on that component is the product of the summands' generic evaluations [GLS05, Section 2.6 and Corollary 7.5].

Different tubes at one slope are Hom- and Ext-orthogonal in the hereditary category of semistable sheaves. The Ext space in the cluster category is the sum of the hereditary Ext space and the dual of the one in the reverse direction. Thus the rigid part of (7.1) satisfies

$$\operatorname{Ext}^1_\Lambda(R,Q_{s,\xi}[r])=0\quad\text{for every ordinary }\xi\text{ and every }r\geq 1. \tag{7.2}$$

Projective summands satisfy this equality as well.

For an object T, let $\operatorname{add}(T)$ consist of the direct summands of finite direct sums of copies of T; T is *basic* if its indecomposable summands are pairwise nonisomorphic. A rigid object T is cluster-tilting if $\operatorname{Ext}^1(T,X)=0$ is equivalent to $X\in\operatorname{add}(T)$. A cluster-tilting module is *reachable* if it is obtained from a fixed GLS cluster-tilting module by finitely many mutations. A tubular autoequivalence can carry any slope to the torsion slope [BKL10, Theorem 7.5]. At that slope let $\mathcal{S}_i$ be the unique simple sheaf in the ith exceptional tube for which $\operatorname{Hom}(\mathcal{O},\mathcal{S}_i)\neq 0$. Write $\mathcal{S}_i^{[j]}$ for the indecomposable of quasi-length j with top $\mathcal{S}_i$. The squid tilting object is

$$T_{\mathrm{sq}}=\mathcal{O}\oplus\mathcal{O}(\vec{c})\oplus R_{\mathrm{sq}},\qquad R_{\mathrm{sq}}=\bigoplus_{i=1}^{3}\bigoplus_{j=1}^{p_i-1}\mathcal{S}_i^{[j]}. \tag{7.3}$$

There are two arrows from $\mathcal{O}$ to $\mathcal{O}(\vec{c})$ in its endomorphism quiver [FG20, Sections 3.1 and 4.1]. Precomposition makes $\operatorname{Ext}^1(T_{\mathrm{sq}},-)$ a left $\operatorname{End}(T_{\mathrm{sq}})^{\mathrm{op}}$-module, reversing these arrows. Accordingly $\mathcal{O}$ is the sink and $\mathcal{O}(\vec{c})$ the source of the active Kronecker pair.

The Frobenius category $\operatorname{mod}\Lambda$ is stably 2-Calabi–Yau and has a GLS cluster-tilting object. Every rigid Λ-module is therefore a direct summand of a cluster-tilting module [BIRS07, Theorem I.1.8(b)]. Under the tubular equivalence, the stable image of such a completion corresponds to a tilting sheaf [BKL10, Proposition 2.3]. The tilting graph is connected [BKL10, Proposition 8.7 and Theorem 8.8]. Adjoining all projective-injective summands lifts its mutations to $\operatorname{mod}\Lambda$ [BIRS07, Proposition I.1.5]. Thus the completion and the pullback of T_{sq} are reachable from the fixed GLS object.

8. Actual characters in an ordinary tube

Fix an ordinary tube and abbreviate $Q_{s,\xi}[r]$ to $Q[r]$. Its Auslander–Reiten triangles have middle term $Q[r-1] \oplus Q[r+1]$ [BKL10, Proposition 2.4].

Lemma 8.1. *The Auslander–Reiten sequences in* $\operatorname{mod}\Lambda$ *are*

$$0 \longrightarrow Q[r] \longrightarrow Q[r-1] \oplus Q[r+1] \longrightarrow Q[r] \longrightarrow 0, \qquad r \geq 1. \tag{8.1}$$

Proof. Both the actual Auslander–Reiten translate $\tau Q[r]$ and $Q[r]$ are projective-free. Their stable isomorphism lifts to an actual isomorphism: a stable isomorphism gives an isomorphism after adjoining projective summands, which cancel by Krull–Schmidt. This identifies the left endpoint.

Only projective-injective summands can disappear from the middle term upon stabilization. Suppose an indecomposable such summand P occurs. A component of the minimal right-almost-split map is an irreducible map $P \to Q[r]$. An irreducible map of modules is either monic or epic. A monomorphism from the injective module P splits, so this map is epic. Its nonzero kernel contains the essential simple socle S of P, giving a factorization

$$P \twoheadrightarrow P/S \xrightarrow{h} Q[r].$$

The first map is not a section. Irreducibility forces h to be a retraction. Since P/S has simple top, it is indecomposable; the retraction onto the nonzero module $Q[r]$ is therefore an isomorphism.

The resulting exact sequence identifies $Q[r]$ with ΣS in the stable category. The doubled Dynkin quiver has no loops, so $\operatorname{Ext}^1_\Lambda(S,S) = 0$, and ΣS is rigid. But $\Sigma Q[r] \simeq Q[r]$, and the stable identity of the nonprojective module $Q[r]$ gives a nonzero element of $\operatorname{Ext}^1_\Lambda(Q[r],Q[r])$. This contradiction excludes P and proves (8.1). □

Lemma 8.2. *For every rigid module* T *and every* $r \geq 0$*,*

$$\dim \operatorname{Hom}_\Lambda(T, Q[r]) = r \dim \operatorname{Hom}_\Lambda(T, Q[1]). \tag{8.2}$$

Moreover,

$$I(Q[r]) \simeq I(Q[1])^{\oplus r}, \qquad 0 \longrightarrow Q[r] \longrightarrow I(Q[r]) \longrightarrow Q[r] \longrightarrow 0. \tag{8.3}$$

Proof. For $r \geq 1$ the nonrigid indecomposable $Q[r]$ cannot be a summand of T. Every map $T \to Q[r]$ is therefore a nonretraction and factors through the right-almost-split map in (8.1). Applying $\operatorname{Hom}_\Lambda(T,-)$ gives

$$0 \longrightarrow \operatorname{Hom}_\Lambda(T, Q[r]) \longrightarrow \operatorname{Hom}_\Lambda(T, Q[r-1] \oplus Q[r+1]) \longrightarrow \operatorname{Hom}_\Lambda(T, Q[r]) \longrightarrow 0.$$

If $h_r = \dim \operatorname{Hom}_\Lambda(T, Q[r])$, then $h_{r+1} - h_r = h_r - h_{r-1}$ and $h_0 = 0$. Hence every successive difference is h_1, which proves (8.2). Taking for T the projective covers of the vertex simples gives $\underline{\dim} Q[r] = r \underline{\dim} Q[1]$.

Every vertex simple is rigid because the quiver has no loops. Taking these simples for T shows that each socle multiplicity of $Q[r]$ is r times the corresponding multiplicity of $Q[1]$. The injective envelope is determined by these multiplicities, proving $I(Q[r]) \simeq I(Q[1])^{\oplus r}$. The quotient $I(Q[r])/Q[r]$ has no projective summand: such a summand would lift and split off from the envelope, giving a nonzero direct summand disjoint from its essential submodule $Q[r]$. Its stable class is $\Sigma Q[r] \simeq Q[r]$. Krull–Schmidt cancellation for projective-free modules now gives $I(Q[r])/Q[r] \simeq Q[r]$, proving the exact sequence. In particular,

$$\underline{\dim} I(Q[1]) = 2 \underline{\dim} Q[1]. \tag{8.4}$$

□

8.1. **Full indices and the character formula.** Let $T = \bigoplus_{i=1}^{m} T_i$ be a reachable basic cluster-tilting module containing every indecomposable projective. Order its nonprojective summands first, and let m_{mut} be their number. These are the mutable vertices; the remaining projective vertices are frozen. Put $x_i = \delta_{T_i}$ and

$$A_T = \mathrm{End}_\Lambda(T)^{\mathrm{op}}, \qquad H_T = (\dim \mathrm{Hom}_\Lambda(T_i, T_j))_{i,j=1}^{m}, \qquad B(T) = H_T^{-t}.$$

The algebra A_T has finite global dimension [GLS12, pp. 4–5]. Let $K_0(A_T)$ be the Grothendieck group of its finite-dimensional modules, with relations from short exact sequences. Its simple classes form an integral basis, and finite projective resolutions express them as integral combinations of the indecomposable projective classes. The (i,j) entry of the Cartan matrix is $\dim \epsilon_i A_T \epsilon_j = \dim \mathrm{Hom}_\Lambda(T_i, T_j)$, where ϵ_i is the primitive idempotent corresponding to T_i. Thus H_T gives a surjective endomorphism of $\mathbb{Z}^m$, so $H_T \in \mathrm{GL}_m(\mathbb{Z})$. The first m_{mut} columns of $B(T)$ form the extended exchange matrix $\widetilde{B}$; these columns are linearly independent.

All exponent and dimension vectors below are columns. Write $f_1, \dots, f_m$ for the standard basis of the full exponent lattice $\mathbb{Z}^m$, and $e_1, \dots, e_{m_{\mathrm{mut}}}$ for that of the mutable lattice $\mathbb{Z}^{m_{\mathrm{mut}}}$. For $a = (a_i)_{i=1}^m$, set $x^a = \prod_{i=1}^m x_i^{a_i}$. Define

$$h_M = (\dim \mathrm{Hom}_\Lambda(T_i, M))_{i=1}^{m}, \qquad g_M^T = H_T^{-1} h_M = B(T)^t h_M.$$

The left A_T-module $E_T(M) = \mathrm{Ext}^1_\Lambda(T, M)$ has the action induced by precomposition. Its projective-vertex coordinates vanish. For $e \in \mathbb{Z}_{\ge 0}^{m_{\mathrm{mut}}}$, let $\mathrm{Gr}_e E_T(M)$ be the variety of submodules of dimension vector e, with zero coordinates at projective vertices, and put

$$F_M^T(y) = \sum_{e \in \mathbb{Z}_{\ge 0}^{m_{\mathrm{mut}}}} \chi(\mathrm{Gr}_e E_T(M)) y^e, \qquad y^e = \prod_{j=1}^{m_{\mathrm{mut}}} y_j^{e_j}.$$

A Laurent expression $x^g F(\widehat{y})$ with $\widehat{y}_j = x^{\widetilde{B} e_j}$ and $F(0) = 1$ is *pointed of full degree* g. The full character formula [GLS12, Theorem 4 and equation (6.7)] is

$$\delta_M = x^{g_M^T} F_M^T(\widehat{y}), \qquad \widehat{y}_j = x^{\widetilde{B} e_j}. \tag{8.5}$$

Its constant F-coefficient is 1, since the zero submodule is a point. Lemma 8.2, applied to each T_i, gives

$$g_{Q[r]}^T = r\, g_{Q[1]}^T. \tag{8.6}$$

The full degree also has an actual-module presentation. Choose a right $\mathrm{add}(T)$-approximation $T^0 \to M$, meaning that every map from an object of $\mathrm{add}(T)$ to M factors through it. Adjoining a projective cover makes this map surjective; let T^1 be its kernel. Applying $\mathrm{Hom}_\Lambda(T, -)$ to $0 \to T^1 \to T^0 \to M \to 0$ gives $\mathrm{Ext}^1_\Lambda(T, T^1) = 0$, because the map on Hom spaces is onto and $\mathrm{Ext}^1_\Lambda(T, T^0) = 0$. The cluster-tilting property implies $T^1 \in \mathrm{add}(T)$. If $a_0, a_1 \in \mathbb{Z}_{\ge 0}^m$ record the multiplicities of the summands of T in T^0, T^1, respectively, the same Hom sequence gives

$$h_M = H_T(a_0 - a_1), \qquad g_M^T = a_0 - a_1. \tag{8.7}$$

After stabilization, the difference of the two summand-class vectors is its stable index. It is obtained from g_M^T by deleting the original projective coordinates.

Proposition 8.3. *Put $b = \delta_{Q[1]}$ and $p = \delta_{I(Q[1])}$, and define*

$$S_0(b,p) = 1, \quad S_1(b,p) = b, \quad S_{r+1}(b,p) = bS_r(b,p) - pS_{r-1}(b,p). \tag{8.8}$$

Then, for all $r \ge 0$,

$$\delta_{Q[r]} = S_r(b,p) = \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k \binom{r-k}{k} p^k b^{r-2k}. \tag{8.9}$$

Proof. The Dynkin preprojective module category satisfies the constructible-cones hypothesis of the Frobenius refined multiplication formula [KPQ23, Example 3.3(1)]. Apply that formula [KPQ23, Theorem 3.6] to the line spanned by the Auslander–Reiten extension (8.1). Its projectivization is one point, and its middle term contributes $\delta_{Q[r-1]}\delta_{Q[r+1]}$.

To compute the complementary locus, work in the stable category and write the Auslander–Reiten triangle for an indecomposable Z as

$$\Sigma Z \longrightarrow Y \xrightarrow{\beta} Z \xrightarrow{\varepsilon} \Sigma^2 Z.$$

Let $A = \operatorname{End}(\Sigma Z)$ in that category. The 2-Calabi–Yau pairing defines the linear form $\ell(\eta) = \langle \varepsilon, \eta\rangle_{Z,\Sigma Z}$ on A. If $\eta \in \operatorname{rad} A$, then $\Sigma^{-1}\eta : Z \to Z$ is not a retraction and has the form βf for a map $f : Z \to Y$. Naturality of the pairing gives

$$\ell(\eta) = \langle \varepsilon\beta, \Sigma f\rangle_{Y,\Sigma Z} = 0.$$

Here $\varepsilon\beta = 0$ because these maps are consecutive arrows of a triangle. The extension ε is nonzero, so nondegeneracy gives $\ell \neq 0$. The finite-dimensional local complex algebra A has $A/\operatorname{rad} A \simeq \mathbb{C}$; hence $\ker \ell = \operatorname{rad} A$. Its complementary projective locus is $\mathbb{P}(A^\times)$, where $A^\times$ is the group of units. Every unit line has a unique representative $1 + u$ with $u \in \operatorname{rad} A$. Therefore this locus is an affine space isomorphic to $\operatorname{rad} A$ and has Euler characteristic one.

Take $Z = Q[r]$, for which $\Sigma Z \simeq Z$. The units above are stable isomorphisms and have zero stable cones, so their actual extension middle terms are projective-injective. For such an extension

$$0 \longrightarrow Q[r] \longrightarrow P' \longrightarrow Q[r] \longrightarrow 0,$$

injectivity of P' extends the embedding of $Q[r]$ to a map $I(Q[r]) \to P'$. Essentiality of $Q[r] \subset I(Q[r])$ makes this extension injective, and injectivity of $I(Q[r])$ makes it split. Thus $P' = I(Q[r]) \oplus P''$, with quotient $(I(Q[r])/Q[r]) \oplus P'' \simeq Q[r] \oplus P''$. The prescribed quotient is the indecomposable nonprojective module $Q[r]$, so $P'' = 0$. The second term in the multiplication formula is consequently $\delta_{I(Q[r])} = p^r$.

Writing $C_r = \delta_{Q[r]}$, the formula gives

$$C_r^2 = C_{r-1}C_{r+1} + p^r, \qquad C_0 = 1, \quad C_1 = b. \tag{8.10}$$

For the polynomials defined in (8.8),

$$\begin{aligned} S_r^2 - S_{r-1}S_{r+1} &= S_r(S_r - bS_{r-1}) + pS_{r-1}^2 \\ &= p(S_{r-1}^2 - S_{r-2}S_r) \qquad (r \geq 2). \end{aligned}$$

At $r = 1$ this difference is $b^2 - (b^2 - p) = p$, so it equals p^r for every $r \geq 1$. Equation (8.10) at $r = 1$ gives $C_2 = b^2 - p = S_2$. If $C_{r-1} = S_{r-1}$ and $C_r = S_r$, subtracting the two identities yields

$$C_{r-1}(C_{r+1} - S_{r+1}) = 0.$$

The ring $\mathbb{C}[N]$ is a domain. Each C_{r-1} is nonzero by (8.5): the constant F-term has coefficient 1, and the independent exchange columns separate its exponent from the other terms. Thus induction gives $C_r = S_r$ for all r.

The recurrence and its initial values imply the formal identity

$$\sum_{r\geq 0} S_r(b,p)t^r = \frac{1}{1 - bt + pt^2} = \sum_{m\geq 0} (bt - pt^2)^m.$$

In the mth summand, choosing k factors $-pt^2$ gives $(-1)^k\binom{m}{k}p^k b^{m-k}t^{m+k}$. The coefficient of t^r has $m = r - k$ and $0 \leq k \leq \lfloor r/2 \rfloor$, proving the binomial formula in (8.9). □

9. Canonical basis membership

9.1. **The regular Kronecker polynomial.** Let $\mathcal{Q}_\xi[n]$ be the indecomposable torsion sheaf of length n at an ordinary point ξ. After carrying its slope to ∞, the stable image of $Q_{s,\xi}[n]$ is $\mathcal{Q}_\xi[n]$. Choose a local scalar coordinate λ_ξ for ξ. At the sink and source of the squid's Kronecker pair, the associated regular representation is $(I, J_n(\lambda_\xi))$, of dimension (n, n); the exceptional coordinates vanish, as the presentation (9.6) below verifies.

Lemma 9.1. *Order the two vertices as sink, source. The F-polynomial of the regular representation $(I, J_n(\lambda_\xi))$ is*

$$F_n(u, v) = \sum_{0 \le l \le e \le n} \binom{e}{l} \binom{n-l}{e-l} u^e v^l. \tag{9.1}$$

Proof. Write $J_n(\lambda_\xi) = \lambda_\xi I + N$ and choose a basis $v_1, \dots, v_n$ with $Nv_i = v_{i+1}$ for $i < n$ and $Nv_n = 0$. A subrepresentation is a pair $U \subseteq V \subseteq \mathbb{C}^n$ with $\dim U = l$, $\dim V = e$, and $NU \subseteq V$. The diagonal action $v_i \mapsto t^i v_i$ preserves these conditions, since it rescales N by t. Its fixed subspaces are exactly the coordinate subspaces.

The ambient product of Grassmannians is partitioned into attracting Schubert cells, each contracted by this action to one coordinate pair. The subrepresentation variety is closed and invariant. Its intersection with a cell is either empty or is contracted to that coordinate pair: the action on the cell extends to $t = 0$, and closedness keeps the limit in the intersection. The nonempty intersection therefore has Euler characteristic one. Additivity of the Euler characteristic reduces the calculation to counting fixed pairs. This argument uses the Euler characteristic of complex algebraic varieties; ordinary and compactly supported Euler characteristics agree.

Mark position i by A if $v_i \in U$, by B if $v_i \in V \setminus U$, and by C otherwise. There are l letters A, $e - l$ letters B, and $n - e$ letters C. The condition $NU \subseteq V$ excludes exactly the adjacent pattern AC; the last position imposes no further condition because $Nv_n = 0$. The $e - l$ letters B delimit $e - l + 1$ gaps. Within each gap every C precedes every A. Distributing the l copies of A and the $n - e$ copies of C independently among these gaps gives

$$\binom{l + (e-l)}{e-l} \binom{(n-e) + (e-l)}{e-l} = \binom{e}{l} \binom{n-l}{e-l}.$$

These counts include $l = 0$, $e = l$, and $e = n$. The empty representation gives $F_0 = 1$; outside $0 \le l \le e \le n$ the subrepresentation variety is empty. This proves every coefficient in (9.1). □

9.2. **Quantum bases, freezing, and valuations.** A quantum seed t consists of quantum variables $x_1, \dots, x_m$, mutable and frozen vertex sets, an extended exchange matrix $\widetilde{B}$, and an integral skew-symmetric matrix Ω_t. Order the m_{mut} mutable vertices first. We use the compatible convention

$$\Omega_t \widetilde{B} = \begin{pmatrix} D \\ 0 \end{pmatrix}, \qquad D = \mathrm{diag}(d_1, \dots, d_{m_{\mathrm{mut}}}), \quad d_i > 0.$$

In particular $\widetilde{B}$ has rank m_{mut}. With standard bases $f_1, \dots, f_m$ of $\mathbb{Z}^m$ and $e_1, \dots, e_{m_{\mathrm{mut}}}$ of $\mathbb{Z}^{m_{\mathrm{mut}}}$, the based quantum torus has monomials x^a and multiplication

$$x^a * x^b = q^{\lambda_t(a,b)/2} x^{a+b}, \qquad \lambda_t(a, b) = a^t \Omega_t b.$$

Its coefficient ring is $\mathbb{Z}[q^{\pm 1/2}]$. A dot denotes the commutative product on the underlying Laurent polynomial module, so $x^a \cdot x^b = x^{a+b}$. Quantum pointed expressions are written with this dot:

$$x^g \cdot F(\widehat{y}), \qquad \widehat{y}_j = x^{\widetilde{B} e_j}, \qquad F(0) = 1.$$

Their full degree is g. The order on degrees is given by $g' \preceq_t g$ if $g' = g + \widetilde{B}e$ for some $e \in \mathbb{Z}_{\geq 0}^{m_{\mathrm{mut}}}$; independence of the columns makes this a partial order. Thus g is the unique leading degree of the pointed expression. Its F-support is the set of indices j for which a monomial with nonzero coefficient in F has a positive jth exponent.

The localized upper quantum cluster algebra $\mathcal{U}_q(t)$ is the intersection, in the ambient skew field, of the based Laurent tori of all seeds obtained from t by mutation at mutable vertices. Original frozen variables are invertible in this algebra. For the reachable unipotent seeds used here, the cluster and upper algebras agree and have a unique common triangular basis [Qin25, Theorems 1.1 and 1.7]; denote it by $\mathbf{L}$. This is the basis which is triangular in every reachable seed, with leading coefficient 1 and bar-invariant elements, where the bar sends $q^{1/2}$ to $q^{-1/2}$ and fixes each based monomial. By [Qin23, Theorem 9.5.1], $\mathbf{L}$ is the rescaled localized dual canonical basis. Each such seed t is injective-reachable: there is a mutation-equivalent seed whose mutable variables have leading degrees, computed in t, with mutable columns $-I_{m_{\mathrm{mut}}}$, up to permutation. Thus the freezing and similarity results cited below apply.

For a nonzero Laurent polynomial f in seed t, define

$$\nu_j^t(f) = \min\{a_j : \text{the coefficient of } x^a \text{ in } f \text{ is nonzero}\}, \qquad \nu_j^t(0) = +\infty.$$

Freezing a set J of mutable vertices gives a seed $t' = \mathfrak{f}_J t$ with the same variables and with the columns indexed by J deleted from $\widetilde{B}$. Its frozen set is the union of J and the original frozen set. On a pointed expression, $\mathfrak{f}_J$ retains the terms whose mutable exponent vector e satisfies $e_j = 0$ for every $j \in J$. The resulting elements $\mathfrak{f}_J\mathbf{L}$ form the common triangular basis of $\mathcal{U}_q(t')$ [Qin25, Theorem 3.17].

Define the partial compactification $\overline{\mathcal{U}}_q(t')$ to consist of the elements of $\mathcal{U}_q(t')$ having nonnegative valuation at every frozen vertex in every reachable seed of t'. Mutation at a vertex other than j preserves ν_j [Qin25, Lemma 2.12], so the frozen inequalities can be checked in one seed. Moreover, a pointed upper-cluster element whose F-support omits a mutable vertex j has $\nu_j \geq 0$ [Qin25, Lemma 2.13]. The comparison with the ambient basis is

$$(\mathfrak{f}_J\mathbf{L}) \cap \overline{\mathcal{U}}_q(t') = \mathbf{L} \cap \overline{\mathcal{U}}_q(t'), \qquad t' = \mathfrak{f}_J t, \tag{9.2}$$

by [Qin25, Corollary 3.19(3)]. The same valuation statements hold at $q = 1$.

Lemma 9.2. *Every $f \in \mathbb{C}[N]$ has nonnegative valuation at every original frozen vertex in every reachable seed.*

Proof. The coordinate ring $\mathbb{C}[N]$ is generated, as an ordinary algebra, by the reachable cluster variables and the original projective variables. The projective coefficients are polynomial coefficients; they are inverted only in the localized cell [GLS07, arXiv v4, Theorem 3.5 and Section 3.6]. Fix an original frozen vertex j. Each generator is a variable in some reachable seed, where its jth valuation is 0 or 1. Every mutation connecting this seed to any other reachable seed is at a vertex different from j. Valuation invariance therefore gives a nonnegative jth valuation for that generator in every seed. For Laurent polynomials,

$$\nu_j(fg) = \nu_j(f) + \nu_j(g), \qquad \nu_j(f+g) \geq \min\{\nu_j(f), \nu_j(g)\}.$$

The product equality follows by multiplying the nonzero lowest j-degree coefficients in the Laurent polynomial domain. The sum inequality follows by collecting equal exponents. Consequently every polynomial in the generators has nonnegative jth valuation, proving the assertion. □

9.3. **The full coefficient comparison.**

Lemma 9.3. *Let t be a reachable quantum seed associated with $\Pi(A_5)$ or $\Pi(D_4)$ whose stable cluster-tilting object is the squid T_{sq} in* (7.3)*. Let $Q = Q_{s,\xi}[n]$ be the projective-free module whose*

stable image is the ordinary torsion sheaf $\mathcal{Q}_\xi[n]$. Freeze the set J of vertices of R_{sq}, and write $t' = \mathfrak{f}_J t$. Order its two mutable vertices as $\mathcal{O}, \mathcal{O}(\vec{c})$ and put

$$\widetilde{B}' = \begin{pmatrix} B_K \\ C \end{pmatrix}, \qquad B_K = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

The rows of C include both the newly frozen vertices and the original projective vertices. Let $g_n = \binom{-n}{n}_{g_{n,\mathrm{fr}}}$ be the full degree of δ_Q, where $g_{n,\mathrm{fr}}$ records all frozen coordinates of t'. Then the common triangular basis of $\mathcal{U}_q(t')$ contains a g_n-pointed element c_n with

$$c_n\big|_{q^{1/2}=1} = \delta_Q.$$

The quantum and classical Laurent supports coincide. Moreover, c_n belongs to the ambient common triangular basis, and δ_Q belongs to the original dual canonical basis $\mathbf{B}^*$.

Proof. Use z^a for source based monomials and x^a for target based monomials. The source variables U_1, U_2, P_0, P_1 are the normalized versions of Lampe's Kronecker variables, with P_0, P_1 frozen. Let $f_1, f_2, f_{P_0}, f_{P_1}$ be the standard basis of its exponent lattice, in that order. Its matrices are

$$\widetilde{B}_0 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Omega_0 = \begin{pmatrix} 0 & -2 & 0 & -2 \\ 2 & 0 & 2 & 0 \\ 0 & -2 & 0 & -4 \\ 2 & 0 & 4 & 0 \end{pmatrix}. \tag{9.3}$$

The exchange matrix is given in [Lam11, Section 2.3], and the skew matrix follows from the commutation relations in [Lam11, Section 4.1]. Multiplying the displayed matrices gives

$$\Omega_0 \widetilde{B}_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\lambda_0(a,b) = a^t \Omega_0 b$ has compatibility multipliers $(2,2)$, and $\widetilde{B}_0$ has full column rank.

Let Ω' be the target skew matrix. If its active compatibility multipliers are d_1, d_2, then

$$(\widetilde{B}')^t \Omega' \widetilde{B}' = B_K^t \operatorname{diag}(d_1, d_2) = \begin{pmatrix} 0 & -2d_2 \\ 2d_1 & 0 \end{pmatrix}.$$

The left side is skew-symmetric, so $d_1 = d_2$. Write d for this common positive integer and set $\rho = d/2$. The source and target principal exchange matrices are B_K, with symmetrizers $(1,1)$. They satisfy the similarity conditions of [Qin25, Definition 4.1] with parameter change $q^{1/2} \mapsto q^{\rho/2}$. The target has full column rank, since $\det B_K = 4$.

Let D_n be the pointed normalization of Lampe's dual canonical element $B[n,0,0,n]$; the four entries are its exponents in Lampe's ordered Poincaré–Birkhoff–Witt root-vector parametrization. The source seed is a quantum unipotent subgroup seed for $w = \sigma_1\sigma_2\sigma_1\sigma_2$, where σ_1, σ_2 are the simple reflections for the Kronecker Cartan matrix. Applying [Qin23, Theorem 9.5.1] to this source algebra places D_n in its localized common triangular basis. Define

$$m_n = \begin{pmatrix} -n \\ n \\ n \\ 0 \end{pmatrix}, \qquad y_1 = z^{\widetilde{B}_0 e_1} = P_1 \cdot U_2^{-2}, \qquad y_2 = z^{\widetilde{B}_0 e_2} = U_1^2 \cdot P_0^{-1},$$

where e_1, e_2 are the standard basis of the common active lattice $\mathbb{Z}^2$. Formula (31) of [Lam11, Theorem 5.3], normalized at its $k = n, l = 0$ term, reads

$$D_n = z^{m_n} \cdot \sum_{0 \le l \le e \le n} a_{n;e,l}(q) y_1^e y_2^l. \tag{9.4}$$

Its coefficients are nonzero nonnegative Laurent polynomials in $q^{1/2}$, with $a_{n;0,0} = 1$, and satisfy

$$a_{n;e,l}(1) = \binom{e}{l}\binom{n-l}{e-l}. \tag{9.5}$$

Indeed, setting $e = n - k$ in Lampe's formula changes its exponent column to

$$\begin{pmatrix} -n+2l \\ -n+2k \\ n-l \\ n-k \end{pmatrix} = m_n + \widetilde{B}_0 \begin{pmatrix} e \\ l \end{pmatrix}.$$

Its two Gaussian binomials, in Lampe's source parameter, have upper and lower indices (e, l) and $(n-l, n-e)$. Both are nonzero and nonnegative for the stated indices. Passing from ordered products to based monomials, then making the leading coefficient 1, multiplies each coefficient by a power of q. At $q = 1$ the second Gaussian binomial becomes $\binom{n-l}{n-e} = \binom{n-l}{e-l}$, proving (9.5).

We compute the full degree of the actual module. Choose a basis s_0, s_1 of $H^0(\mathcal{O}(\vec{c}))$ with $s_0(\xi) \neq 0$, and now take $\lambda_\xi = s_1(\xi)/s_0(\xi)$. The pencil (s_0, s_1) identifies the coarse projective line, and the sheaf $\mathcal{Q}_\xi[n]$ has the presentation

$$0 \longrightarrow \mathcal{O}^n \xrightarrow{s_1 I - s_0 J_n(\lambda_\xi)} \mathcal{O}(\vec{c})^n \longrightarrow \mathcal{Q}_\xi[n] \longrightarrow 0. \tag{9.6}$$

The determinant is $(s_1 - \lambda_\xi s_0)^n$, a nonzero section, so the map is injective. It is invertible away from ξ. At this ordinary point put $\zeta = s_1/s_0 - \lambda_\xi$. Dividing the matrix by the unit s_0 gives $\zeta I - N$, with $N v_i = v_{i+1}$ and $N v_n = 0$. Its cokernel has generators $v_1, \dots, v_n$ with relations

$$\zeta v_i = v_{i+1} \quad (1 \le i < n), \qquad \zeta v_n = 0.$$

Every generator equals $\zeta^{i-1} v_1$, and the remaining relation is $\zeta^n v_1 = 0$. Conversely, mapping v_i to ζ^{i-1} defines the inverse identification with $\mathbb{C}[[\zeta]]/(\zeta^n)$ after completion. The cokernel is thus the length-n indecomposable at ξ, which proves (9.6).

Applying $\mathrm{Hom}(T_{\mathrm{sq}}, -)$ to its triangle is surjective at the last nonshifted term, since $\mathrm{Ext}^1(T_{\mathrm{sq}}, \mathcal{O}) = 0$. Hence this is an $\mathrm{add}(T_{\mathrm{sq}})$-presentation with stable index $n[\mathcal{O}(\vec{c})] - n[\mathcal{O}]$, where the brackets denote the standard summand classes in the split Grothendieck group of $\mathrm{add}(T_{\mathrm{sq}})$. By (8.7), deleting the original projective coordinates from g_Q^T gives this stable index. Its active coordinates are therefore $(-n, n)^t$, and its exceptional coordinates are zero. All remaining coordinates are the actual projective coordinates recorded in $g_{n,\mathrm{fr}}$.

The exceptional summands and $\mathcal{Q}_\xi[n]$ lie in different tubes, so the R_{sq}-coordinates of $\mathrm{Ext}^1(T_{\mathrm{sq}}, \mathcal{Q}_\xi[n])$ vanish. At the two line-bundle vertices, precomposition with s_0, s_1 gives $I, J_n(\lambda_\xi)$: multiplication by $s_1/s_0 = \lambda_\xi + \zeta$ on $\mathbb{C}[[\zeta]]/(\zeta^n)$ is that Jordan block. The extra orbit-category Hom term is $\mathrm{Ext}^1(T_{\mathrm{sq}}, \tau^{-1}\mathcal{Q}_\xi[n])$ in the hereditary category; it vanishes by local freeness of the line bundles and orthogonality of the torsion tubes. Thus the sheaf Hom representation is the cluster-category Hom representation. The identity $\Sigma \mathcal{Q}_\xi[n] \simeq \mathcal{Q}_\xi[n]$ identifies the Hom and Ext representations in the cluster category. Lemma 9.1 and (8.5) give

$$\delta_Q = x^{g_n} \cdot \sum_{0 \le l \le e \le n} \binom{e}{l}\binom{n-l}{e-l} (y_1')^e (y_2')^l, \qquad y_i' = x^{\widetilde{B}' e_i}. \tag{9.7}$$

Define

$$c_n = x^{g_n} \cdot \sum_{0 \le l \le e \le n} a_{n;e,l}(q^\rho)(y_1')^e (y_2')^l. \tag{9.8}$$

The active coordinates of g_n and m_n agree. With $\lambda'(a,b) = a^t \Omega' b$, compatibility gives, for every $e, h \in \mathbb{Z}^2$,

$$\begin{aligned} \lambda'(g_n, \widetilde{B}'e) &= d \begin{pmatrix} -n \\ n \end{pmatrix}^t e = \rho\, \lambda_0(m_n, \widetilde{B}_0 e), \\ \lambda'(\widetilde{B}'e, \widetilde{B}'h) &= d\, e^t B_K^t h = \rho\, \lambda_0(\widetilde{B}_0 e, \widetilde{B}_0 h), \\ \lambda'(a, \widetilde{B}'e) &= 0 \quad \text{if } a \text{ is supported on frozen coordinates.} \end{aligned}$$

Thus the quantum factors in the normalized pointed products agree after $q^{1/2} \mapsto q^{\rho/2}$; frozen shifts leave these factors unchanged. Formula (9.8) is the element similar to D_n with prescribed full degree g_n, in the sense of [Qin25, Definition 4.4]. The transfer theorem [Qin25, Lemma 4.17 and Proposition 4.18] places c_n in the common triangular basis of $\mathcal{U}_q(t')$. If ρ has a denominator, its proof extends the coefficient ring for the parameter change and descends the triangular basis to the original ring. Equations (9.5) and (9.7) give $c_n|_{q^{1/2}=1} = \delta_Q$.

The frozen correction can also be read on exponent lattices. Let $f_1', \dots, f_m'$ be the standard target basis, with active coordinates first, and regard the columns C_1, C_2 of C as vectors in its frozen sublattice. Define the integral map ℓ from the source exponent lattice to the target exponent lattice by

$$\ell(f_1) = f_1', \quad \ell(f_2) = f_2', \quad \ell(f_{P_0}) = -C_2, \quad \ell(f_{P_1}) = C_1.$$

Then

$$\ell\widetilde{B}_0 = \widetilde{B}', \qquad \ell(m_n) = \begin{pmatrix} -n \\ n \\ -nC_2 \end{pmatrix}, \qquad g_n - \ell(m_n) = \begin{pmatrix} 0 \\ 0 \\ g_{n,\mathrm{fr}} + nC_2 \end{pmatrix}.$$

Thus the prescribed degree introduces exactly the frozen shift $g_{n,\mathrm{fr}} + nC_2$. The compatibility calculation above shows why this shift preserves the normalized F-coefficients in the similar-element construction.

Every coefficient in (9.8) is nonnegative and remains nonzero at $q = 1$. The independent columns of $\widetilde{B}'$ give distinct exponents for distinct pairs (e, l). Hence the Laurent support of c_n is exactly that of δ_Q.

The character δ_Q is an element of $\mathbb{C}[N]$, hence of the classical upper cluster algebra. Its F-support in t is disjoint from J. Lemma 2.13 of [Qin25] gives nonnegative valuations at these newly frozen vertices, and Lemma 9.2 gives the same inequalities at the original frozen vertices. Equality of supports transfers all these inequalities to c_n. Valuation invariance under mutations at the two active vertices then places c_n in $\overline{\mathcal{U}}_q(t')$.

By uniqueness, the frozen common triangular basis containing c_n is $\mathfrak{f}_J\mathbf{L}$. Equation (9.2) gives $c_n \in \mathbf{L}$. Apply [Qin23, Theorem 9.5.1] to the ambient unipotent algebra. The specialized localized basis contains the original basis $\mathbf{B}^*$. Since the specialized element $c_n|_{q=1} = \delta_Q$ lies in $\mathbb{C}[N]$, it has a finite expansion in $\mathbf{B}^*$. Linear independence in the larger localized basis forces that expansion to consist of this single basis element. Thus $\delta_Q \in \mathbf{B}^*$. □

9.4. **Multiplication by an orthogonal rigid module.**

Lemma 9.4. *If $\delta_M \in \mathbf{B}^*$ and R is rigid with $\mathrm{Ext}^1_\Lambda(R, M) = 0$, then $\delta_{R\oplus M} \in \mathbf{B}^*$.*

Proof. Quantum dual canonical multiplication has nonnegative structure coefficients in symmetric type, and normalized quantum cluster monomials belong to this basis. These facts imply Laurent positivity in every reachable seed as follows [KK19, Introduction]. Expand a dual canonical element as a finite Laurent polynomial, and multiply by a sufficiently large cluster monomial from the chosen seed so that every exponent becomes nonnegative. The resulting Laurent expansion is a linear combination of distinct normalized cluster monomials, all in the dual canonical basis. Its expansion in this basis also has nonnegative coefficients, because it is a product of two basis elements. Linear independence identifies the two expansions and proves nonnegativity of the Laurent coefficients, up to powers of q introduced by normalization. For a localized basis element, first clear its original frozen denominators. Nonnegative Laurent coefficients are nonzero at $q = 1$, so specialization preserves their support.

Complete the distinct indecomposable summands of R to a reachable cluster-tilting module T, and let t be its seed. Let J be the mutable vertices whose summands occur in R. The J-coordinates of $E_T(M)$ vanish by Ext-orthogonality, so the F-support in (8.5) is disjoint from J. Let $\beta \in \mathbf{L}$ be the normalized quantum dual canonical lift of δ_M. Positivity shows that β and δ_M have the same Laurent support. The leading coefficient of β specializes to 1, and the common support has the same unique leading degree in the order $\preceq_t$. Since $\widetilde{B}$ has independent columns, each Laurent exponent has a unique mutable exponent relative to this degree. Thus the quantum F-support of β is also disjoint from J, and $\mathfrak{f}_J\beta = \beta$.

In $t' = \mathfrak{f}_J t$, all variables occurring in the cluster monomial for R are frozen. Multiplication by this monomial, normalized to have leading coefficient 1, preserves the frozen common triangular basis. Let d be the resulting basis element. Its specialization is

$$d|_{q^{1/2}=1} = \delta_R\delta_M = \delta_{R\oplus M},$$

and its Laurent coefficients remain nonnegative. For a summand T_j of R, rigidity gives $\mathrm{Ext}^1_\Lambda(T_j, R) = 0$, while the hypothesis gives $\mathrm{Ext}^1_\Lambda(T_j, M) = 0$. Hence the character of $R \oplus M$ has F-support disjoint from J. Lemma 2.13 of [Qin25] gives nonnegative valuations there, and Lemma 9.2 gives them at the original frozen vertices. Equality of quantum and classical supports transfers these inequalities to d. Mutation invariance then gives $d \in \overline{\mathcal{U}}_q(t')$.

Equation (9.2) puts d in the ambient common triangular basis. Its specialization is one localized dual canonical basis element lying in $\mathbb{C}[N]$. Expand it in the original basis $\mathbf{B}^*$; linear independence of the localized basis makes this a single-term expansion. Therefore $\delta_{R\oplus M} \in \mathbf{B}^*$. □

10. Exhaustion of the basis in the two tame types

Proof of Theorem 7.1. Fix a dimension vector $\mathbf{d}$. Write $\mathbb{C}[N]_{\mathbf{d}}$ for the corresponding homogeneous component in the original dimension-vector grading, and $\mathbf{B}^*_{\mathbf{d}} = \mathbf{B}^* \cap \mathbb{C}[N]_{\mathbf{d}}$. If an irreducible component Z has rigid generic module R, complete R to a reachable cluster-tilting module. Its character is a cluster monomial, which belongs to $\mathbf{B}^*$ by [Qin23, Theorem 9.5.2]. Set $n(Z) = 0$ and $c_Z = \rho_Z = \delta_R$.

Otherwise use (7.1) and put $n(Z) = n$. Choose ξ in a dense open subset of the ordinary parameter line on which the mouth character, injective envelope, and dimension vector are constant. Such a common open subset exists because these data are constructible in the one-parameter family: for the character this follows by applying constructibility to a finite basis of the relevant homogeneous component of $U(\mathfrak{n})$; for the injective envelope it follows from the socle multiplicities. Put

$$Q[r] = Q_{s,\xi}[r], \qquad b_s = \delta_{Q[1]}, \qquad p_s = \delta_{I(Q[1])}.$$

The generic evaluation is

$$\rho_Z = \delta_R b_s^n. \tag{10.1}$$

Lemma 9.3 gives $\delta_{Q[n]} \in \mathbf{B}^*$ at every slope. By (7.2) and Lemma 9.4,

$$c_Z = \delta_{R\oplus Q[n]} = \delta_R S_n(b_s, p_s) \in \mathbf{B}^*.$$

For $0 \le k \le \lfloor n/2 \rfloor$, let Z_k be the closure of the family whose general module is

$$R \oplus I(Q[1])^{\oplus k} \oplus Q_{s,\xi_1}[1] \oplus \cdots \oplus Q_{s,\xi_{n-2k}}[1], \tag{10.2}$$

where the points ξ_i are pairwise distinct and general in the open subset just chosen. Every Z_k is an irreducible component by the direct-sum criterion in Section 7: the summands of R are mutually Ext-orthogonal, its nonprojective summands lie in exceptional tubes, the ordinary mouths lie in distinct tubes, and projectives are Ext-orthogonal to every module. By (8.4),

$$\underline{\dim}\big(R \oplus I(Q[1])^{\oplus k} \oplus Q_{s,\xi_1}[1] \oplus \cdots \oplus Q_{s,\xi_{n-2k}}[1]\big) = \underline{\dim} R + 2k\underline{\dim} Q[1] + (n-2k)\underline{\dim} Q[1] = \mathbf{d}.$$

Multiplicativity of generic evaluations gives

$$\rho_{Z_k} = \delta_R p_s^k b_s^{n-2k}, \qquad Z_0 = Z.$$

Proposition 8.3 therefore yields

$$c_Z = \rho_Z + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \rho_{Z_k}. \tag{10.3}$$

For every displayed lower term the number of ordinary mouths is $n(Z_k) = n - 2k < n(Z)$.

The fixed-dimensional module variety has finitely many irreducible components. Order them by increasing $n(Z)$, choosing any order among equal values. Equation (10.3), together with the rigid case, gives a finite unitriangular matrix from the dual semicanonical basis (ρ_Z) of $\mathbb{C}[N]_{\mathbf{d}}$ to (c_Z). Its diagonal entries are 1, so the matrix is invertible and the c_Z are linearly independent and span $\mathbb{C}[N]_{\mathbf{d}}$. Each c_Z is already a dual canonical element. Their number equals $\dim \mathbb{C}[N]_{\mathbf{d}}$, hence they are exactly $\mathbf{B}^*_{\mathbf{d}}$, with no repetitions. Taking all dimension vectors proves the theorem. □

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