THE KNOT $17nh_{0010647}$ IS NOT SMOOTHLY SLICE

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Abstract. Dunfield and Gong listed three topologically slice knots, $16n_{68278}$, $17nh_{0010647}$ and $18nh_{00098198}$, such that the smooth sliceness of any one of them would produce an exotic smooth 4-sphere. We show that $17nh_{0010647}$ is not smoothly slice. The argument combines the super-special RBG link given by Dunfield and Gong for one of the 0-friends of this knot, the trace-embedding argument of Nakamura, and the adjunction inequality of Ren for the even Sq^1-refinement of Rasmussen's invariant. The input from computation is the value $s_+^{\mathrm{Sq}^1} = 2$ for that 0-friend, together with checks of the RBG link. Nothing in this note has been checked by a professional mathematician or by any third party.

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1. Statement

A knot is *slice* if it bounds a smooth properly embedded disk in B^4. Two knots are 0-*friends* if their 0-surgeries are homeomorphic. In [1, Theorem 5.14] Dunfield and Gong show that if any of $16n_{68278}$, $17nh_{0010647}$, $18nh_{00098198}$ is smoothly slice, then there is an exotic smooth 4-sphere; these three knots have trivial Alexander polynomial and are topologically slice. For each of them, [1, Table 11] gives a super-special RBG link relating it to a 0-friend K' that is not smoothly slice; for $17nh_{0010647}$ two such links, with distinct 0-friends, are given.

Theorem 1.1. *The knot $17nh_{0010647}$ is not smoothly slice.*

Consequently, among the three knots of [1, Theorem 5.14], only $16n_{68278}$ and $18nh_{00098198}$ remain as possible sources of an exotic 4-sphere through that theorem.

The proof has two parts. Section 2 shows that the trace-embedding argument behind [1, Theorem 5.9] remains valid when Rasmussen's invariant is replaced by the even Sq^1-refinement $s_+^{\mathrm{Sq}^1}$ of Lipshitz and Sarkar [4], using the adjunction inequality of Ren [8]. Section 3 records the computation of $s_+^{\mathrm{Sq}^1}$ for the second 0-friend of $17nh_{0010647}$ in [1, Table 11].

2. A refined version of the obstruction

2.1. **Refined invariants.** For a knot K let $s(K)$ denote Rasmussen's invariant over $\mathbb{F}_2$, and let $r_\pm^{\mathrm{Sq}^1}(K)$, $s_+^{\mathrm{Sq}^1}(K)$, $s_-^{\mathrm{Sq}^1}(K)$ be the refinements of [4] associated with the Steenrod square Sq^1 on even Khovanov homology. By definition $s_-^{\mathrm{Sq}^1}(K) = -s_+^{\mathrm{Sq}^1}(\overline{K})$, where $\overline{K}$ is the mirror of K, and one has

$$s_+^{\mathrm{Sq}^1}(K) \in \{s(K),\, s(K)+2\}, \qquad s_-^{\mathrm{Sq}^1}(K) \in \{s(K)-2,\, s(K)\}, \tag{1}$$

so that $s_-^{\mathrm{Sq}^1}(K) \le s(K) \le s_+^{\mathrm{Sq}^1}(K)$. The unknot has $s_+^{\mathrm{Sq}^1} = s_-^{\mathrm{Sq}^1} = 0$.

2.2. **Ren's adjunction inequality.** Ren proved [8, Theorem 1] that the adjunction-type inequality

$$s(L_1) \le s(L_0) - \chi(\Sigma) - [\Sigma]^2 - |[\Sigma]|$$

of [7, Corollary 1.4] holds with s replaced by $s_+^{\mathrm{Sq}^1}$, for oriented link cobordisms $\Sigma\colon L_0 \to L_1$ in $k\overline{\mathbb{CP}}^2$ minus two balls with $\pi_0(L_1) \to \pi_0(\Sigma)$ surjective; here $|[\Sigma]|$ is the L^1-norm of $[\Sigma]$ in the standard

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basis of $H_2(k\overline{\mathbb{CP}}^2)$. Removing a small disk from the interior of a connected surface bounded by a knot and taking L_0 to be the unknot, we obtain:

Lemma 2.1. *Let D be a smooth properly embedded disk in $(\#^k\overline{\mathbb{CP}}^2) \setminus \mathring{B}^4$ with boundary a knot K. Then $s_+^{\mathrm{Sq}^1}(K) \leq -[D]^2 - |[D]|$. In particular $s_+^{\mathrm{Sq}^1}(K) \leq 0$ if $[D] = 0$, and also if $[D]^2 = -1$.*

Proof. The first inequality is the specialization just described, with $\chi(D) = 1$ and $s_+^{\mathrm{Sq}^1}(\text{unknot}) = 0$. If $[D]^2 = -1$, then $[D] = \sum a_i e_i$ with $\sum a_i^2 = 1$, so exactly one a_i equals ± 1 and $|[D]| = 1$. □

Reversing orientations, a disk in $(\#^k\mathbb{CP}^2) \setminus \mathring{B}^4$ bounded by K gives a disk in $(\#^k\overline{\mathbb{CP}}^2) \setminus \mathring{B}^4$ bounded by $\overline{K}$, with opposite self-intersection. Hence:

Corollary 2.2. *If K bounds a smooth disk D in $(\#^k\mathbb{CP}^2) \setminus \mathring{B}^4$ with $[D] = 0$ or $[D]^2 = +1$, then $s_-^{\mathrm{Sq}^1}(K) \geq 0$.*

2.3. **RBG links.** We use the RBG links of Manolescu and Piccirillo [5], in the form of [1, Definition 5.7]: an integrally framed link $L = (R, r) \cup (B, 0) \cup (G, 0) \subset S^3$ determines two knots K_B, K_G with homeomorphic 0-surgeries, and L is *super-special* when $R \cup B$ and $R \cup G$ are both Hopf links. Then R is unknotted and L is special in the sense of [6, Definition 2.4]. The mirror image of L is a super-special RBG link with framings $(-r, 0, 0)$ whose associated knots are $\overline{K_B}$ and $\overline{K_G}$.

Proposition 2.3. *Let L be a super-special RBG link. If K_B is smoothly slice, then*

$$s_-^{\mathrm{Sq}^1}(K_G) \geq 0 \geq s_+^{\mathrm{Sq}^1}(K_G).$$

In particular, by (1)*, $s_+^{\mathrm{Sq}^1}(K_G) = s_-^{\mathrm{Sq}^1}(K_G) = s(K_G) = 0$.*

Proof. We follow the proof of [6, Theorem 3.13], in which Rasmussen's invariant enters only through two properties: a knot that is H-slice in some $\#^n\mathbb{CP}^2$ has $s \geq 0$ [6, Lemma 2.3], and a knot that is (-1)-slice in some $\#^n\mathbb{CP}^2$ has $s \geq 0$ [6, Conjecture 2.15], proved as [7, Corollary 1.5]. Here a framed knot (K, k) is slice in a closed manifold W if K bounds a disk $D \subset W \setminus \mathring{B}^4$ inducing the framing k, so $k = -[D]^2$; H-slice means $[D] = 0$. By Corollary 2.2, both properties hold with s replaced by $s_-^{\mathrm{Sq}^1}$. The topological steps are the following, for a special RBG link with K_B slice:

- if (R, r) is slice in a closed manifold W, then K_G is H-slice in $-W$ [6, Corollary 3.2];
- if $(R, r+1)$ is slice in a closed manifold W, then K_G is (-1)-slice in $W\#\mathbb{CP}^2$ [6, Lemma 3.11];
- since R is unknotted, (R, k) is slice in some $\#^n\overline{\mathbb{CP}}^2$ for $k \geq 0$ and in some $\#^n\mathbb{CP}^2$ for $k \leq 0$ [6, Corollary 2.11, Lemma 2.10].

If $r \geq 0$, (R, r) is slice in some $\#^n\overline{\mathbb{CP}}^2$, so K_G is H-slice in $\#^n\mathbb{CP}^2$. If $r \leq -1$, $(R, r+1)$ is slice in some $\#^n\mathbb{CP}^2$, so K_G is (-1)-slice in $\#^{n+1}\mathbb{CP}^2$. In both cases $s_-^{\mathrm{Sq}^1}(K_G) \geq 0$.

Applying this to the mirror of L, whose blue knot $\overline{K_B}$ is slice, gives $s_-^{\mathrm{Sq}^1}(\overline{K_G}) \geq 0$, that is $s_+^{\mathrm{Sq}^1}(K_G) \leq 0$. □

Remark 2.4. With s in place of $s_+^{\mathrm{Sq}^1}$ and $s_-^{\mathrm{Sq}^1}$ this is [1, Theorem 5.9]. The point of Proposition 2.3 is that [8, Theorem 1] supplies, for the s-version of the even Sq^1-refinement, the two properties used in [6]. Ren also shows that the analogous inequality fails for some other spatial refinements (odd Sq^1, Sq^2) [8]; Proposition 2.3 is stated only for $s_+^{\mathrm{Sq}^1}$ and $s_-^{\mathrm{Sq}^1}$.

3. The computation

Let L be the RBG link with alphabetical DT code

$$\texttt{xcdhlXqLTCvumPWsoAikRHbnEDfJg}, \qquad r = -1,$$

given in [1, Table 11] for $17nh_{0010647}$, with components R, B, G in this order. The following facts were established with SnapPy [9]; the ancillary script `anc/verify_17nh_0010647.py` repeats them.

(1) The three components are unknotted, and the sublinks $R \cup B$, $R \cup G$ have fundamental group $\mathbb{Z}^2$; hence they are Hopf links and L is super-special.
(2) The exterior of K_B is isometric to that of the 17-crossing diagram in `anc/K_17nh_0010647_PD.txt` (volume 15.95507376…), and the exterior of K_G to that of the 45-crossing diagram in `anc/friend_K1_PD.txt` (volume 18.34991644…).
(3) The 0-surgeries on these two knots are isometric hyperbolic manifolds with $H_1 = \mathbb{Z}$.

Since knots are determined by their complements up to mirror image [2], and since the quantities of Proposition 2.3 are exchanged with signs by mirroring, item 2 identifies K_G for the purpose of the computation below. We write K' for this knot.

Proposition 3.1. *For the knot K' given by the diagram `anc/knotjob_friend_diagram1.txt`,*

$$s_{\mathbb{Q}}(K') = s(K') = 0, \qquad \big(r_+^{\mathrm{Sq}^1},\ s_+^{\mathrm{Sq}^1},\ r_-^{\mathrm{Sq}^1},\ s_-^{\mathrm{Sq}^1}\big)(K') = (2,\,2,\,0,\,0).$$

These values were computed with KnotJob [3], which lists the even Sq^1-refinement as the vector (r_+, s_+, r_-, s_-). The same vector was obtained from a second diagram of K' (`anc/knotjob_friend_diagram2.txt`), and the mirror diagram (`anc/knotjob_friend_mirror.txt`) gives $(0, 0, -2, -2)$, as the definition $s_-^{\mathrm{Sq}^1}(K) = -s_+^{\mathrm{Sq}^1}(\overline{K})$ requires. The integral reduced Khovanov cohomology of K', also computed with KnotJob, contains $(\mathbb{Z}/2)^2$ in bidegree $(0, 0)$; some 2-torsion in this range is necessary for a nontrivial Sq^1-refinement. The odd Sq^1-refinement of K' vanishes. The outputs are in `anc/knotjob_output.log`.

Proof of Theorem 1.1. Suppose $17nh_{0010647} = K_B$ is smoothly slice. By Proposition 2.3, $s_+^{\mathrm{Sq}^1}(K_G) = s_-^{\mathrm{Sq}^1}(K_G) = 0$. If the diagram of Proposition 3.1 represents K_G, this contradicts $s_+^{\mathrm{Sq}^1}(K') = 2$; if it represents $\overline{K_G}$, then $s_-^{\mathrm{Sq}^1}(K_G) = -s_+^{\mathrm{Sq}^1}(K') = -2$, again a contradiction. □

4. Scope

(1) **Status.** Lemma 2.1, Corollary 2.2 and Proposition 2.3 are proved above from [8], [6] and [7]; the proof of Proposition 2.3 rests on our reading of the proof of [6, Theorem 3.13]. Proposition 3.1 and the facts listed in Section 3 are established by computer (SnapPy, KnotJob) and are not proved by hand. Theorem 1.1 depends on both.
(2) The identification of the knot K_B of the RBG link above with $17nh_{0010647}$ is taken from [1, Table 11].
(3) We have not found Theorem 1.1 in the literature; the novelty has not been checked further.
(4) The other two knots of [1, Theorem 5.14] are not concerned: for the 0-friends of $16n_{68278}$ and $18nh_{00098198}$ given in [1, Table 11], and for the first 0-friend of $17nh_{0010647}$, the same computations give $s = 0$ over $\mathbb{F}_2$ and vanishing even and odd Sq^1-refinements, so Proposition 2.3 gives nothing for them.
(5) **No external check.** Nothing in this note has been checked by a professional mathematician or by any third party: neither the proofs, nor the reading of the cited papers, nor the computations. The only checks are those carried out by the author with the AI model named below.

Ancillary files

`anc/verify_17nh_0010647.py` (SnapPy) checks items 1–3 of Section 3 and that the three KnotJob input files describe K' and its mirror; its output is in `anc/verify_output.log`. The KnotJob command reproducing Proposition 3.1 is

```
java -jar KnotJob.jar -s0 -s2 -sqe -sqo -nf knotjob_friend_diagram1.txt
```

and its outputs, together with those for the second diagram and the mirror, are in `anc/knotjob_output.log`.

Use of AI

This note was prepared with the assistance of an AI model (Claude, Anthropic), which carried out the reading of the cited papers, the computations and the drafting under the author's direction. The author takes responsibility for the content.

References

[1] N. M. Dunfield and S. Gong, *Ribbon concordances and slice obstructions: experiments and examples*, arXiv:2512.21825.
[2] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.
[3] D. Schütz, *KnotJob*, software for Khovanov homology and refinements of Rasmussen's invariant.
[4] R. Lipshitz and S. Sarkar, *A refinement of Rasmussen's s-invariant*, Duke Math. J. **163** (2014), 923–952.
[5] C. Manolescu and L. Piccirillo, *From zero surgeries to candidates for exotic definite four-manifolds*, J. London Math. Soc. **108** (2023).
[6] K. Nakamura, *Trace embeddings from zero surgery homeomorphisms*, J. Topol. **16** (2023), 1641–1664.
[7] Q. Ren, *Lee filtration structure of torus links*, Geom. Topol. **28** (2024), 3935–3960.
[8] Q. Ren, *Adjunction inequality for spatially refined s-invariants*, Pacific J. Math. **344** (2026), 145–150.
[9] M. Culler, N. M. Dunfield, M. Goerner and J. R. Weeks, *SnapPy*, a computer program for studying the geometry and topology of 3-manifolds.

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