hexagon:2611.00983v2

CONVERGENCE OF THE ONE-PARTICLE QUANTUM SYMMETRIC SIMPLE EXCLUSION PROCESS TO THE KPZ EQUATION

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Abstract. The quantum symmetric simple exclusion process (QSSEP) is a random quantum system describing fermions hopping on a one-dimensional lattice. The dynamics depend on a family of independent complex Brownian motions. When the system contains a single particle, the model boils down to the unitary evolution of a wave function in random environment. We prove that, in a window of size $t^{1/2}$ located at a distance of order $t^{3/4}$ from the origin, the random probability distribution of the quantum particle converges, after a suitable rescaling, to the solution of the multiplicative noise stochastic heat equation, whose logarithm solves the Kardar-Parisi-Zhang equation. Such a convergence is well-understood for random walks in space-time random environment, but the one-particle QSSEP is not a random walk in a random environment. Our proof relies on a Markov duality between the one-particle QSSEP and the symmetric inclusion process.

Acknowledgment of AI use

The main result of this text, Theorem 1.2, was conjectured by the author in September 2026, and was used to compare the mathematical performance of several Large Language Models. The proof presented below is the one found by ChatGPT 6 Astra. We used concurrently ChatGPT 6 Astra and Claude Fable 5.2 to check and improve the rigor of the proof. Only minor issues were found during these back-and-forth exchanges. The paper was written by Claude Fable 5.2, following instructions from the author about the content and the structure of Section 1. A Lean formalization was performed using Claude code. The text was finally lightly edited by the author.

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1. Introduction

1.1. **Preface.** Random unitary dynamics are minimal models for the evolution of quantum systems subject to noise. One of the simplest is the quantum symmetric simple exclusion process (QSSEP), introduced in [BBJ17] (see also [BBJ19; BJ19]): fermions hop between neighbouring sites of a one-dimensional lattice, and the hopping amplitudes depend on complex Brownian motions, independent from one bond of the lattice to another. For each realization of the noise the evolution is unitary, and after averaging over the noise one recovers the classical symmetric simple exclusion process, hence the name. The fluctuations around this mean behaviour have a remarkably rich mathematical structure, related in particular to free probability theory [Bia25; HB23; BH24]; we refer to the lecture notes [BB25] for an introduction.

In this article, we study the QSSEP in the simplest possible situation, when the system contains only one particle. The model then boils down to the unitary evolution of a wave function ψ_t on the lattice $\mathbb{Z}$,

$$\psi_{t+dt} = e^{\mathbf{i}\, dH_t}\psi_t,$$

where H_t is a tridiagonal Hermitian matrix whose non-zero entries are complex Brownian motions. We are interested in the random probability distribution

$$P(t,x) = |\psi_t(x)|^2,$$

which describes the position of the particle at time t. On average, the particle simply performs a random walk: if the particle starts from the origin, $\mathbb{E}[P(t,x)]$ is the probability that a continuous-time simple random walk started from the origin is at x at time t (see Figure 2).

Our main result, Theorem 1.2, shows that when x is at a distance of order $t^{3/4}$ from the origin, the fluctuations of $P(t,x)$ are described by the Kardar-Parisi-Zhang (KPZ) equation [KPZ86]. More precisely, $P(t,x)$ converges, in a sense that will be made precise in Section 1.4, to the solution of the stochastic heat equation with multiplicative noise, whose logarithm solves the KPZ equation.

The occurrence of the KPZ equation at distance $t^{3/4}$ is by now well-understood for another class of models: random walks in space-time random environment (RWRE). It was first predicted in [LT17], and proved in increasing generality in [BL20; DDP25; Par26] (see Section 1.5 for more details and references). The relation with quantum dynamics is that several models of unitary evolution in a noisy medium turn out to be equivalent to RWRE. This was first observed by Saul, Kardar and Read [SKR92] for a discrete model of directed waves in random media. The Kipnis-Marchioro-Presutti model, which was recently shown to converge to the KPZ equation [BC25], can be interpreted in a similar way. These analogies suggest that the same should be true for the one-particle QSSEP. However, in these examples, the equivalence with RWRE relies on the fact that the unitary matrices which drive the dynamics are Haar distributed, and the one-particle QSSEP does not have such a random walk representation (see Section 1.5): the randomness of $P(t,x)$ does not come from a random environment in which a classical particle would evolve, but from the interferences between the possible trajectories of a quantum particle. Hence, the results available for RWRE do not apply.

Our proof relies instead on the fact that the process $t \mapsto P(t,\cdot)$ satisfies a closed system of stochastic differential equations – it is an instance of the Brownian energy process of [GKRV09] – and that this process is dual to a system of random walks with an attractive interaction, the symmetric inclusion process (see Appendix A, where this duality is stated for an arbitrary

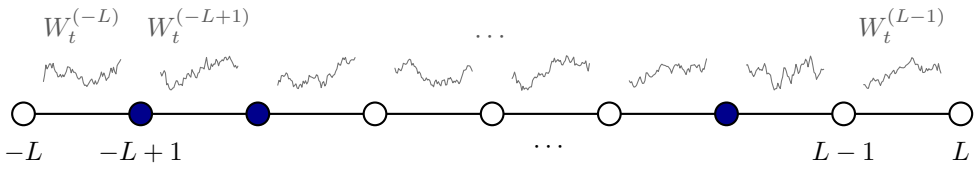


FIGURE 1. The QSSEP on $\Lambda_L = \{-L, \dots, L\}$. Fermions (filled circles) hop between neighbouring sites, and the hopping amplitude through the bond $(x, x+1)$ during the time interval $[t, t+dt]$ is the increment $dW_t^{(x)}$ of a complex Brownian motion.

number of particles). We only use this duality for one and two particles. The one-particle duality yields a microscopic version of the stochastic heat equation, in the form of a martingale problem. The two-particle duality is used to identify the limit of the quadratic variation of the martingales. Such use of Markov duality is reminiscent of [CGST20]. The main novelty of the present proof is the following: we solve explicitly a Poisson equation (i.e. an equation of the form $L\phi = f$ where L is some Markovian generator) which allows to control the distance between two dual particles. The method is explained in more details in Section 1.6.

1.2. **The quantum symmetric simple exclusion process.** Let us start with an informal description of the QSSEP, in order to explain where the objects studied in this article come from. Precise definitions are given in Section 1.3, and the reader only interested in mathematical statements may safely skip the present section.

Consider a chain of N sites, that we label by the elements of $\Lambda_L = \{-L, \dots, L\}$, with $N = 2L+1$. Each site may be either empty or occupied by one fermion. The pure states of the system are unit vectors in the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$, which has a basis indexed by particle configurations, i.e. by the elements of $\{0,1\}^{\Lambda_L}$. For $x \in \Lambda_L$, let $c_x^\dagger$ and c_x be the operators on $\mathcal{H}$ which respectively create and annihilate a fermion at site x. They satisfy anticommutation relations

$$c_x c_y^\dagger + c_y^\dagger c_x = \mathbb{1}_{x=y}, \qquad c_x c_y + c_y c_x = 0,$$

and the operator $c_{x+1}^\dagger c_x$ moves a particle from site x to site $x+1$, whenever this is possible.

The state of the system at time t is a random density matrix ϱ_t, that is a positive semi-definite Hermitian operator on $\mathcal{H}$ with unit trace. It evolves as

$$\varrho_{t+dt} = e^{-\mathbf{i}\, d\mathbf{H}_t}\, \varrho_t\, e^{\mathbf{i}\, d\mathbf{H}_t}, \qquad d\mathbf{H}_t = \sum_{x=-L}^{L-1} \left(c_{x+1}^\dagger c_x\, dW_t^{(x)} + c_x^\dagger c_{x+1}\, d\overline{W}_t^{(x)} \right), \tag{1.1}$$

where the $W^{(x)}$ are independent complex Brownian motions attached to the bonds $(x, x+1)$ of the chain, see Figure 1. In words, the system evolves unitarily, under a Hamiltonian which makes particles hop to neighbouring sites with amplitudes given by (derivatives of) complex Brownian motions. The model is called QSSEP for the following reason [BBJ17; BJ19]: the average $\mathbb{E}[\varrho_t]$ over the Brownian motions satisfies a closed linear equation which, when restricted to diagonal matrices, is exactly the master equation of the symmetric simple exclusion process. The mean behaviour of the QSSEP is thus classical, while the quantum coherences – the off-diagonal entries of ϱ_t – only show up in the fluctuations.

Since the Hamiltonian increments $d\mathbf{H}_t$ are quadratic in the fermionic operators, the dynamics can be encoded using matrices of size N instead of 2^N. The central object is the

two-point function, that is the $N \times N$ Hermitian matrix

$$G_t(x,y) = \mathrm{Tr}\left(\varrho_t\, c_x^\dagger c_y\right), \qquad x,y \in \Lambda_L,$$

whose diagonal entry $G_t(x,x)$ is the probability, for a given realization of the Brownian motions, to observe a particle at site x at time t. Using the anticommutation relations, one finds that the matrix G_t evolves autonomously: $G_{t+dt} = e^{\mathbf{i}\,dH_t} G_t e^{-\mathbf{i}\,dH_t}$, where H_t is now a $N \times N$ matrix, given in (1.3) below. The fluctuations of G_t have been thoroughly studied, in particular in the stationary state of the open QSSEP, where reservoirs at both ends of the chain drive the system out of equilibrium [BJ19; BJ21]. As N goes to infinity, these fluctuations have a rich combinatorial structure, which turns out to be governed by free probability theory [Bia25; HB23; BH24].

In this article, we consider a different regime. We start with the simplest initial condition, a single particle located at the origin, and we study the distribution of the position of this particle on a system so large that the particle never feels its boundaries.

1.3. **The two-point function.** We now give precise definitions, starting directly from matrices of size N. Let $L \geqslant 1$ be an integer, $\Lambda_L = \{-L, \dots, L\}$ and $N = 2L+1$. Let $(W_t^{(x)})_{t\geqslant 0}$, for $-L \leqslant x \leqslant L-1$, be independent standard complex Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ satisfying the usual conditions. This means that

$$W_t^{(x)} = \frac{1}{\sqrt{2}}\left(B_1^{(x)}(t) + \mathbf{i} B_2^{(x)}(t)\right),$$

where the $B_1^{(x)}, B_2^{(x)}$ are independent standard real $(\mathcal{F}_t)$-Brownian motions. In terms of Itô's calculus,

$$dW_t^{(x)}\, d\overline{W}_t^{(y)} = \mathbb{1}_{x=y}\, dt, \qquad dW_t^{(x)}\, dW_t^{(y)} = 0. \tag{1.2}$$

Let H_t be the $N \times N$ Hermitian matrix, with rows and columns indexed by Λ_L,

$$H_t = \begin{pmatrix} 0 & W_t^{(-L)} & 0 & \cdots & 0 \\ \overline{W}_t^{(-L)} & 0 & W_t^{(-L+1)} & & \vdots \\ 0 & \overline{W}_t^{(-L+1)} & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & W_t^{(L-1)} \\ 0 & \cdots & 0 & \overline{W}_t^{(L-1)} & 0 \end{pmatrix}, \tag{1.3}$$

that is $H_t(x,x+1) = W_t^{(x)}$, $H_t(x+1,x) = \overline{W}_t^{(x)}$, and all other entries are zero.

Definition 1.1. Let $(U_t)_{t\geqslant 0}$ be the solution of the stochastic differential equation

$$dU_t = \mathbf{i}\, dH_t\, U_t - \frac{1}{2}(dH_t)^2\, U_t, \qquad U_0 = I, \tag{1.4}$$

which is the Itô form of the evolution $U_{t+dt} = e^{\mathbf{i}\,dH_t} U_t$. By (1.2), $(dH_t)^2 = D\,dt$, where D is the diagonal matrix such that $D(x,x)$ is the number of neighbours of x in Λ_L, i.e. $D(x,x) = 2 - \mathbb{1}_{|x|=L}$. Given a $N \times N$ Hermitian matrix G_0, the two-point function of the QSSEP on Λ_L started from G_0 is the matrix-valued process

$$G_t = U_t\, G_0\, U_t^*. \tag{1.5}$$

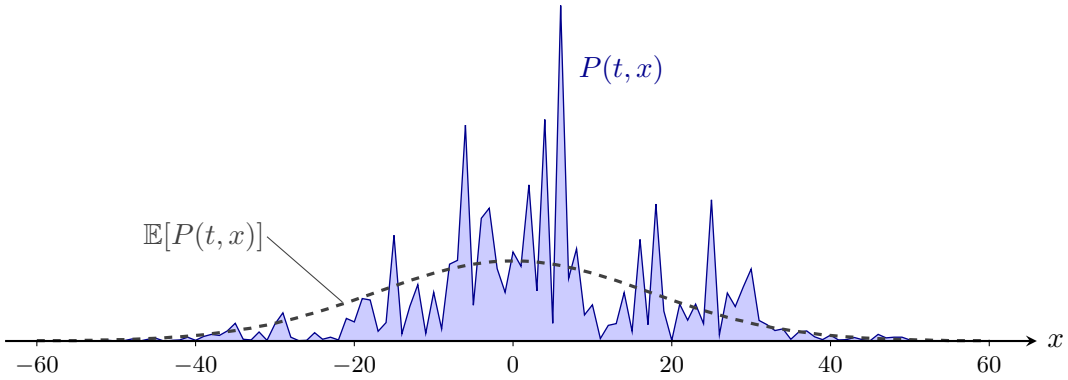


FIGURE 2. A simulation of the random probability distribution $P(t,x) = |\psi_t(x)|^2$ at time $t = 150$ (in blue), together with its expectation, the distribution $\mathsf{p}_t(x)$ of the simple random walk (dashed curve).

It follows from (1.4) and (1.2) that $d(U_t^* U_t) = 0$, so that the matrix U_t is unitary for all $t \geqslant 0$. Let us stress that this diffusion on the unitary group $U(N)$ is different from the Brownian motion on $U(N)$, which would correspond to a matrix H_t whose entries are all Brownian motions. Nevertheless, as t goes to infinity, the distribution of G_t converges to that of UG_0U^*, where U is distributed according to the Haar measure on $U(N)$ [BBJ19].

1.3.1. *One particle.* When the system initially contains a single particle, located at the origin, the two-point function at time 0 is the rank one projector $G_0 = e_0 e_0^*$, where $(e_x)_{x\in\Lambda_L}$ denotes the canonical basis of $\mathbb{C}^{\Lambda_L}$. In this case,

$$G_t = \psi_t \psi_t^*, \quad \text{where } \psi_t = U_t e_0 \in \mathbb{C}^{\Lambda_L}. \tag{1.6}$$

The vector ψ_t is the wave function of the particle. By (1.4), its coordinates solve

$$d\psi_t(x) = \mathbf{i}\,\psi_t(x+1)\,dW_t^{(x)} + \mathbf{i}\,\psi_t(x-1)\,d\overline{W}_t^{(x-1)} - \frac{D(x,x)}{2}\psi_t(x)\,dt, \qquad \psi_0(x) = \mathbb{1}_{x=0}, \tag{1.7}$$

with the convention that the terms involving $x \pm 1 \notin \Lambda_L$ are omitted. Our main object of study is

$$P(t,x) := G_t(x,x) = |\psi_t(x)|^2, \qquad x \in \Lambda_L, \tag{1.8}$$

which is the probability, for a given realization of the Brownian motions, that the particle is at site x at time t. Since U_t is unitary, $P(t,\cdot)$ is a random probability distribution on Λ_L. As for random walks in random environment, there are thus two levels of randomness: the Brownian motions play the role of an environment, and for each realization of this environment, $P(t,\cdot)$ is the probability distribution of the location of the particle. We will see in Section 2 that

$$\mathbb{E}\left[P(t,x)\right] = \mathsf{p}_t(x), \tag{1.9}$$

where $\mathsf{p}_t(x)$ is the probability that the continuous-time simple random walk on Λ_L, which jumps at rate 1 to each of its neighbours and starts from the origin, is at x at time t. In this sense, $P(t,\cdot)$ is a random perturbation of the distribution of the simple random walk, see Figure 2.

1.3.2. *Infinite lattice.* All the definitions above make sense when Λ_L is replaced by $\mathbb{Z}$, i.e. when $L = \infty$. We simply define the wave function $(\psi_t)_{t\geqslant 0}$ as the unique continuous $\ell^2(\mathbb{Z})$-valued solution of the infinite system of equations (1.7), where the family $(W^{(x)})_{x\in\mathbb{Z}}$ of independent complex Brownian motions is now indexed by $\mathbb{Z}$ and $D(x,x) = 2$ for all $x \in \mathbb{Z}$ (see Proposition 2.1). Then $P(t,x) = |\psi_t(x)|^2$ is a random probability distribution on $\mathbb{Z}$, and (1.9) holds, where p_t is the distribution of the simple random walk on $\mathbb{Z}$. When we need to emphasize the dependence on the size of the system, we will write $P^L(t,x)$ instead of $P(t,x)$, with $L \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$ and the convention that $\Lambda_\infty = \mathbb{Z}$.

1.4. **Main result.** The Kardar-Parisi-Zhang equation is the stochastic partial differential equation

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}\left(\partial_x h\right)^2 + \xi, \qquad t > 0, x \in \mathbb{R}, \tag{1.10}$$

where ξ is a space-time white noise, that is a centered Gaussian field on $\mathbb{R}_+ \times \mathbb{R}$ with covariance $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$. We will use the notion of Cole-Hopf solution, defined by

$$h(t,x) := \log Z(t,x), \tag{1.11}$$

where Z is a solution of the stochastic heat equation with multiplicative noise

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + Z\xi, \qquad t > 0, x \in \mathbb{R}. \tag{1.12}$$

In this article, we only consider the solution of (1.12) with initial condition $Z(0,\cdot) = \delta_0$, a Dirac mass at the origin; the corresponding initial condition for the KPZ equation is called narrow wedge. This solution is defined as the random field $Z(t,x)$, adapted to the filtration generated by the white noise, which satisfies

$$Z(t,x) = g_t(x) + \int_0^t \int_{\mathbb{R}} g_{t-s}(x-y) Z(s,y) \xi(ds\, dy), \qquad g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \tag{1.13}$$

for all $t > 0$ and $x \in \mathbb{R}$, where the stochastic integral is understood in the sense of Itô and Walsh [Wal86]. Existence and uniqueness (in a suitable class of random fields) are proved in [BC95; ACQ11; CD15], and the solution $Z(t,x)$ is continuous and almost surely positive on $(0,\infty)\times\mathbb{R}$ [Mue91; Mor14], so that (1.11) makes sense. More details are given in Section 7. Notice that $\mathbb{E}[Z(t,x)] = g_t(x)$.

Let us now explain how to rescale the one-particle QSSEP so that it converges to Z. Let $n \geqslant 1$ be a scaling parameter, that will go to infinity. We look at the system at times of order n, in a window of size $n^{1/2}$ around a location at distance of order $n^{3/4}$ from the origin. More precisely, for $t \geqslant 0$, let

$$\mathbb{L}_n(t) = \left\{ x \in \mathbb{R} : n^{3/4} t + n^{1/2} x \in \Lambda_L \right\},$$

and define the rescaled field

$$Z_n(t,x) := n^{1/2} \exp\left(\frac{n^{1/2} t}{2} + n^{1/4} x - \frac{t}{24} \right) P\left(\frac{nt}{2}, n^{3/4} t + n^{1/2} x \right), \qquad x \in \mathbb{L}_n(t). \tag{1.14}$$

Let us comment on the scalings. Time is scaled as $nt/2$ so that, by (1.9), the mean distribution $\mathbb{E}[P(nt/2,\cdot)]$ has variance nt. The factor $n^{1/2}$ accounts for the fact that the lattice $\mathbb{L}_n(t)$ has mesh $n^{-1/2}$, so that $Z_n(t,\cdot)$ should be thought of as a density. Finally, the exponential factor compensates the decay of the mean distribution far from the origin: one can check that $\mathbb{E}[Z_n(t,x)]$ converges to $g_t(x)$. The constant $1/24$ comes from the fact that at distance $n^{3/4}$, the distribution of the simple random walk at time of order n is not exactly Gaussian. Since

$Z_n(t,\cdot)$ is only defined on a lattice, and will converge in a weak sense only, it is convenient to associate to it the random measure

$$\mathcal{Z}_n(t) := n^{-1/2} \sum_{x \in \mathbb{L}_n(t)} Z_n(t,x)\, \delta_x. \tag{1.15}$$

We denote by $\mathcal{M}_+(\mathbb{R})$ the space of finite nonnegative Borel measures on $\mathbb{R}$, equipped with the topology of weak convergence, and by $C([0,T], \mathcal{M}_+(\mathbb{R}))$ the space of continuous functions from $[0,T]$ to $\mathcal{M}_+(\mathbb{R})$, equipped with the topology of uniform convergence. We may now state our main result.

Theorem 1.2. *Let $(L_n)_{n\geqslant 1}$ be a family of elements of $\mathbb{Z}_{\geqslant 1} \cup \{\infty\}$ such that $L_n/n^{3/4}$ goes to infinity as n goes to infinity. For each n, let $P = P^{L_n}$ be the probability distribution of the position of the particle in the one-particle QSSEP on Λ_{L_n}, defined in* (1.8)*, and let $\mathcal{Z}_n$ be the associated measure-valued process defined by* (1.14) *and* (1.15)*. Then, for any $T > 0$, we have the convergence in distribution*

$$\Big(\mathcal{Z}_n(t)\Big)_{t\in[0,T]} \Longrightarrow_{n\to\infty} \Big(\mathcal{Z}(t)\Big)_{t\in[0,T]} \tag{1.16}$$

in the space $C([0,T], \mathcal{M}_+(\mathbb{R}))$, where $\mathcal{Z}(0) = \delta_0$ and for $t > 0$,

$$\mathcal{Z}(t)(dx) = Z(t,x)\, dx = e^{h(t,x)} dx,$$

where Z is the solution of the stochastic heat equation (1.12) *with initial condition δ_0, and h is the Cole-Hopf solution of the KPZ equation* (1.10) *with narrow wedge initial condition.*

In particular, for any fixed $t > 0$ and any bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$n^{-1/2} \sum_{x \in \mathbb{L}_n(t)} Z_n(t,x)\varphi(x) \Longrightarrow_{n\to\infty} \int_{\mathbb{R}} Z(t,x)\varphi(x)\, dx.$$

Let us make a few remarks on the statement of Theorem 1.2.

Remark 1.3 (Pointwise convergence and tail probabilities)**.** Theorem 1.2 is a statement about measures, and the field $Z_n(t,x)$ itself is not expected to converge to $Z(t,x)$. Indeed, $P(t,x)$ fluctuates wildly from one site to the next, as can be seen in Figure 2: around a given site, the variables $P(t,x)$ should behave as independent exponential random variables with slowly varying means, because products of exponential distributions are invariant for the dynamics of P (see Appendix A). This suggests that if $x_n \in \mathbb{L}_n(t)$ converges to x, then $Z_n(t,x_n)$ converges in distribution to $Z(t,x)\,\mathfrak{e}$, where $\mathfrak{e}$ is an exponential random variable with mean 1, independent of Z, with independent factors at distinct points. These local fluctuations are at the heart of the difficulties in the proof of Theorem 1.2, see Section 1.6.

In order to obtain a pointwise convergence, and thus a convergence of logarithms towards the solution of the KPZ equation, one should consider the slightly different observable

$$H(t,x) := \sum_{y\geqslant x} P(t,y).$$

In the regime of Theorem 1.2, the terms of this sum decay as $e^{-n^{-1/4}(y-x)}$, up to the local fluctuations. Hence, the sum is dominated by of the order of $n^{1/4}$ sites: this is enough for the local fluctuations to average out, and much less than the $n^{1/2}$ sites over which the field varies macroscopically. One thus expects that for all $t > 0$ and $x \in \mathbb{R}$,

$$n^{1/4} \exp\left(\frac{n^{1/2}t}{2} + n^{1/4}x - \frac{t}{24}\right) H\left(\frac{nt}{2}, n^{3/4}t + n^{1/2}x\right) \Longrightarrow_{n\to\infty} Z(t,x),$$

so that the logarithm of the left-hand side converges to $h(t,x)$.

Remark 1.4 (Size of the system)**.** The assumption that $L_n/n^{3/4}$ goes to infinity ensures that the particle does not feel the boundaries of the system: we will show that under this assumption, the QSSEP on Λ_{L_n} and the QSSEP on $\mathbb{Z}$ can be coupled so that the total variation distance between the corresponding measures $\mathcal{Z}_n(t)$ goes to zero (Proposition 8.1). Since the observation window is located at distance $n^{3/4}t$ from the origin, some condition of this type is necessary, but we have not tried to find the optimal one.

Remark 1.5 (Large time behaviour)**.** As t goes to infinity, the fluctuations of $h(t,x)+t/24$ are of order $t^{1/3}$ and are governed by the Tracy-Widom GUE distribution [ACQ11]. By analogy with the results of [BC17] recalled in Section 1.5, this suggests that $\log P(t,x)$ has Tracy-Widom fluctuations on the scale $t^{1/3}$ when x is proportional to t. This question is beyond the reach of the methods of the present article.

1.5. **Why the KPZ equation? Analogies with random walks in random environment.** The scalings in Theorem 1.2 may seem surprising at first sight. In this section we explain why such a result could be expected, based on analogies with other models of unitary dynamics in random media, which turn out to be equivalent to random walks in random environment, for which the convergence to the KPZ equation is known.

1.5.1. *The model of Saul, Kardar and Read.* In [SKR92], Saul, Kardar and Read studied the propagation of directed waves in a random medium. Their motivation was to understand the wandering of a beam whose wave function $\Psi(t,x)$ formally satisfies a Schrödinger equation with a space-time random potential,

$$-\mathbf{i}\partial_t\Psi = (\Delta + \xi(t,x))\,\Psi,$$

and in particular to find the exponent ν such that the center $\langle x\rangle = \int x|\Psi(t,x)|^2dx$ of the beam fluctuates on the scale t^ν. To address this question, they introduced the following discrete model. The wave function at time $t\in\mathbb{Z}_{\geqslant 0}$ and position $x\in\mathbb{Z}$ has two components,

$$\psi_t(x) = \begin{pmatrix}\psi_t^+(x)\\ \psi_t^-(x)\end{pmatrix} \in \mathbb{C}^2,$$

and it evolves as

$$\begin{pmatrix}\psi_{t+1}^+(x-1)\\ \psi_{t+1}^-(x+1)\end{pmatrix} = u_{t,x}\begin{pmatrix}\psi_t^+(x)\\ \psi_t^-(x)\end{pmatrix}, \tag{1.17}$$

where the $u_{t,x}$ are independent 2×2 unitary matrices, distributed according to the Haar measure on $U(2)$, see Figure 3. In other words, $\psi_{t+1} = U_{t,t+1}\psi_t$ where, up to a relabelling of coordinates, $U_{t,t+1}$ is a block diagonal unitary matrix with independent Haar distributed blocks of size 2. This is a discrete-time analogue of the evolution $\psi_{t+dt} = e^{\mathbf{i}\,dH_t}\psi_t$ of the one-particle QSSEP, in which the infinitesimal unitary rotations $e^{\mathbf{i}\,dH_t}$ are replaced by Haar distributed rotations.

The key observation of [SKR92] is that the squared norm of the wave function behaves as the quenched distribution of a random walk in random environment. Indeed, if u is a Haar distributed matrix in $U(2)$ and $v\in\mathbb{C}^2$ is a fixed vector, then $w=uv$ is uniformly distributed on the sphere of radius $\|v\|$ in $\mathbb{C}^2$, so that

$$|w_1|^2+|w_2|^2 = \|v\|^2 \quad \text{and} \quad \frac{|w_2|^2}{\|v\|^2} \sim \text{Uniform}([0,1]).$$

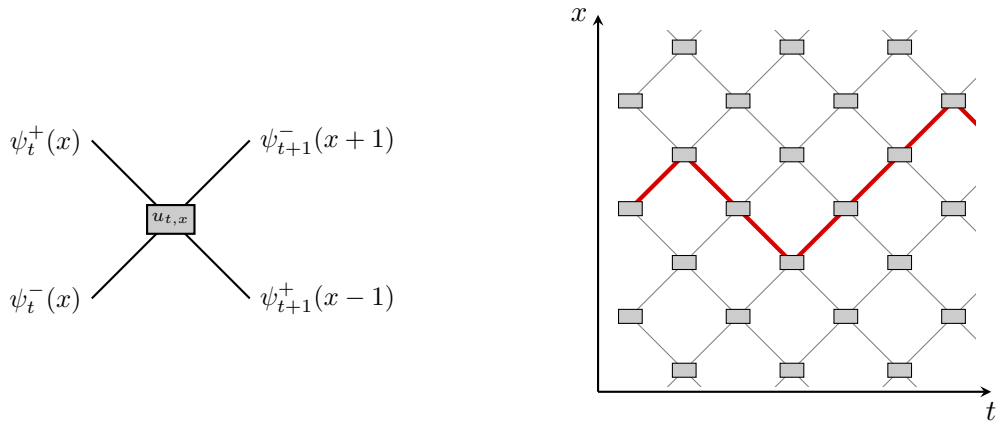


FIGURE 3. The model of Saul, Kardar and Read. Left: at each space-time point (t, x), the two components of the wave function are mixed by a Haar distributed unitary matrix $u_{t,x}$, and then sent to the sites $x+1$ and $x-1$. Right: the scatterers $u_{t,x}$ form a random medium on a space-time lattice. The squared norm of the wave function is distributed as the quenched distribution of a random walk on this lattice (one trajectory is depicted in blue), whose transition probabilities are independent and uniformly distributed on $[0, 1]$.

Hence, in (1.17), the squared norm $\|\psi_t(x)\|^2$ present at site x at time t is split between the sites $x+1$ and $x-1$ in proportions $p_{t,x}$ and $1-p_{t,x}$, where

$$p_{t,x} = \frac{|\psi_{t+1}^-(x+1)|^2}{\|\psi_t(x)\|^2}$$

is uniformly distributed on $[0, 1]$. Moreover, since the matrices $u_{t,x}$ are independent, the variables $p_{t,x}$ are independent over t and x. Now consider a random walk $(X_t)_{t\in\mathbb{Z}_{\geqslant 0}}$ on $\mathbb{Z}$ which, given the environment $(p_{t,x})$, jumps from x at time t to $x+1$ with probability $p_{t,x}$ and to $x-1$ with probability $1-p_{t,x}$. We denote by P the quenched probability distribution of the random walk, that is, its law given the environment. If X_0 is distributed according to $\|\psi_0(\cdot)\|^2$, we obtain the equality in distribution, jointly in (t, x),

$$\|\psi_t(x)\|^2 \overset{(d)}{=} \mathsf{P}\left(X_t = x\right). \tag{1.18}$$

1.5.2. *Three regimes for random walks in random environment.* Random walks in a random environment which is independent over space and time, such as in (1.18), are by now well understood. Three regimes must be distinguished, depending on the location where the quenched distribution $\mathsf{P}(X_t = \cdot)$ is observed.

- *Typical regime.* For almost every environment, the random walk satisfies the same central limit theorem as the simple random walk [BMP97; BMP04; RS05]. The environment only matters for lower order corrections, which are Gaussian in a window of size $t^{1/2}$ around any location $x \ll t^{3/4}$ [DP25].
- *Large deviations.* At distance $x \propto t$ from the origin, the quenched probabilities decay exponentially, $\log \mathsf{P}(X_t > xt) \approx -I(x)t$, with a deterministic rate function I [RSY13]. When the variables $p_{t,x}$ follow a Beta distribution, which includes the uniform distribution of (1.18), the model is exactly solvable. It was proved in [BC17] that the fluctuations of

$\log \mathsf{P}(X_t > xt)$ are of order $t^{1/3}$ and are governed by the Tracy-Widom GUE distribution, which is the signature of the KPZ universality class. This is expected to hold for more general distributions of the environment.

- *Moderate deviations.* Le Doussal and Thiery found in [LT17] that the crossover between these two regimes happens at distance $x \propto t^{3/4}$, and that at this scale, the quenched distribution is described by the KPZ equation itself. The convergence of the moments of the quenched distribution to those of the stochastic heat equation was obtained in [BL20] for Beta distributed environments (see also [HDCC25] for a more general computation). The convergence in distribution was then proved for nearest-neighbour random walks with arbitrary distribution of the $p_{t,x}$ in [DDP25]. Finally, a very general result was obtained by Parekh [Par26], who considers arbitrary stochastic flows of kernels satisfying a few natural assumptions (see also [DPS26] for an example where the critical scale is different). A different approach, in which the stochastic heat equation is characterized by its moments instead of a martingale problem, was developed in [Par24] for systems which are linear in their initial condition.

For instance, in the case of the uniform distribution, the result of [DDP25] shows that the squared norm of the wave function in the model of Saul, Kardar and Read satisfies the exact analogue of Theorem 1.2: when ψ_0 is supported at the origin, $\|\psi_t(x)\|^2$ converges, after a rescaling similar to (1.14), to the solution Z of the stochastic heat equation (1.12) with initial condition δ_0.

Let us mention that the KPZ equation also arises from random walks in random environment in a different scaling, when the randomness of the environment is tuned to zero [CG17], as for directed polymers in the intermediate disorder regime [AKQ14]. In the results discussed above, as in Theorem 1.2, the model is fixed and it is the location of the observation window which is tuned.

1.5.3. *The Kipnis-Marchioro-Presutti model.* The Kipnis-Marchioro-Presutti (KMP) model [KMP82] is a model of energy transport along a chain of oscillators. Each site $x \in \mathbb{Z}$ carries an energy $\eta_t(x) \geqslant 0$. Independent Poisson processes with rate 1 are attached to the bonds of $\mathbb{Z}$, and each time the Poisson process attached to the bond $(x, x+1)$ rings, the energies are redistributed as

$$\Big(\eta(x), \eta(x+1)\Big) \longrightarrow \Big(B\left(\eta(x)+\eta(x+1)\right), (1-B)\left(\eta(x)+\eta(x+1)\right)\Big), \tag{1.19}$$

where B is uniformly distributed on $[0,1]$, independently of everything else. This dynamics is linear in the initial condition: $\eta_t(x) = \sum_{y\in\mathbb{Z}} \eta_0(y) K_{0,t}(y,x)$, where the $K_{s,t}$ are the transition probabilities of a continuous-time random walk in random environment. When the Poisson process attached to $(x, x+1)$ rings, a walker located at x or $x+1$ goes to x with probability B and to $x+1$ with probability $1-B$. Using this fact and the result of [Par26], it was shown in [BC25] that the KMP model started from $\eta_0(x) = \mathbb{1}_{x=0}$ converges to the solution of a stochastic heat equation of the form (1.12), under the same scalings as in Theorem 1.2: time is of order n and the field is observed in a window of size $n^{1/2}$ at distance of order $n^{3/4}$ from the origin.

The KMP model can also be seen as the squared modulus of a wave function evolving under random unitary dynamics. Let $\psi_t \in \ell^2(\mathbb{Z})$ evolve as follows: each time the Poisson process attached to the bond $(x, x+1)$ rings, the vector $(\psi(x), \psi(x+1)) \in \mathbb{C}^2$ is multiplied by an independent Haar distributed matrix in $U(2)$. Then, by the same argument as in Section 1.5.1, the process $\eta_t(x) = |\psi_t(x)|^2$ is a KMP model (see [BC25] and [CLQ+26] for closely related statements).

1.5.4. *Back to the QSSEP.* The one-particle QSSEP belongs to the same family of models: instead of being multiplied by a Haar distributed unitary matrix at random times, the vector $(\psi_t(x), \psi_t(x+1))$ undergoes at all times an infinitesimal random unitary rotation. This analogy also has a classical counterpart: we will see in Section 2 that the process $P(t,x) = |\psi_t(x)|^2$ is a Brownian energy process, and the KMP model is the instantaneous thermalization of this process [GKRV09; CGGR13] (see Appendix A).

In view of the universality of the three regimes discussed above, it is thus natural to conjecture that the one-particle QSSEP converges to the KPZ equation in the moderate deviation regime. There is however an important difference with the models of Sections 1.5.1 and 1.5.3. For a general initial wave function ψ_0, we have

$$P(t,x) = \Big| \sum_y U_t(x,y)\psi_0(y) \Big|^2,$$

which is not a linear function of $P(0,\cdot) = |\psi_0(\cdot)|^2$, due to interferences between the contributions of the different sites y. This nonlinearity is intrinsic: the distribution of $P(t,\cdot)$ cannot be obtained by letting random transition probabilities act on $P(0,\cdot)$, as is the case for the quenched distribution of a random walk in a random environment (see Proposition C.1). Thus, unlike the models of Sections 1.5.1 and 1.5.3, the one-particle QSSEP is not a random walk in a random environment, and the results of [DDP25; Par26] cannot be applied. Theorem 1.2 shows that the convergence to the KPZ equation nevertheless holds.

It would be interesting to investigate the other scaling regimes of Section 1.5.2 for the one-particle QSSEP, as well as what happens for the QSSEP with several particles, and for more general random unitary dynamics.

1.6. **Ideas of the proof.** We now explain the main steps of the proof of Theorem 1.2. We assume here that the system is infinite, $L = \infty$, the case of finite systems being deduced from this one by a coupling argument.

1.6.1. *A stochastic heat equation with conservative noise.* The starting point is that the process $P(t,x)$ satisfies a closed system of stochastic differential equations. Applying Itô's formula to $|\psi_t(x)|^2$ (Proposition 2.2), we find that

$$dP(t,x) = \Delta P(t,x)\, dt + dJ_t(x-1) - dJ_t(x), \tag{1.20}$$

where $\Delta f(x) = f(x+1) + f(x-1) - 2f(x)$ is the discrete Laplacian and the $J_t(x)$ are martingales, which represent the fluctuations of the current of probability through the bond $(x, x+1)$. They satisfy

$$d\langle J(x), J(y)\rangle_t = 2\,\mathbb{1}_{x=y}\, P(t,x)P(t,x+1)\, dt. \tag{1.21}$$

The fact that only the diagonal of the matrix $G_t = \psi_t\psi_t^*$ appears in (1.21) is specific to the one-particle case, for which $|G_t(x,y)|^2 = G_t(x,x)G_t(y,y)$. Equation (1.20) is a discrete stochastic heat equation, whose noise is the discrete gradient of a multiplicative noise. Taking expectations, we see that $\mathbb{E}[P(t,x)]$ solves the discrete heat equation, which proves (1.9). Moreover, a noise in gradient form is too weak to survive in the diffusive scaling limit, which is why the particle behaves as a simple random walk in the typical regime.

1.6.2. *Tilting.* In order to probe the distribution $P(t,\cdot)$ far from the origin, we multiply it by an exponential weight: for $\theta \in \mathbb{R}$, let

$$\mathsf{z}^\theta(t,x) = e^{\theta x - 2(\cosh\theta - 1)t} P(t,x).$$

The constant $2(\cosh\theta - 1)$ is chosen so that the total mass $\sum_x \mathsf{z}^\theta(t,x)$ is a martingale. This transformation is classical in the context of random walks and directed polymers, and its effect on the mean $\mathbb{E}[P(t,x)]$ is merely to shift the Gaussian distribution by $2t\sinh\theta$. Its effect on the noise is more interesting. Indeed, for a test function φ, we deduce from (1.20) and a summation by parts that the martingale part of $\sum_x \mathsf{z}^\theta(t,x)\varphi(x)$ is

$$\sum_x \Big(e^\theta\varphi(x+1) - \varphi(x)\Big)\, e^{\theta x - 2(\cosh\theta - 1)t}\, dJ_t(x).$$

When $\theta = 0$, the noise is integrated against the discrete gradient of the test function. But when $\theta \neq 0$, we have $e^\theta\varphi(x+1) - \varphi(x) \approx \theta\varphi(x)$ for smooth test functions: the tilt turns the conservative noise into a non-conservative multiplicative noise of strength θ. Informally, in view of (1.21), the field z^θ solves an equation of the form

$$\partial_t \mathsf{z}^\theta = [\text{discrete Laplacian with drift}]\ \mathsf{z}^\theta + \theta\sqrt{2}\,\mathsf{z}^\theta\,\xi,$$

where ξ is a white noise in time and space. In a frame moving at velocity $2\sinh\theta \approx 2\theta$, under the diffusive scaling where time is multiplied by $n/2$ and space by $n^{1/2}$, the scaling properties of the white noise show that the noise coefficient becomes $\theta n^{1/4}$. Hence a nontrivial limit can only arise when θ is proportional to $n^{-1/4}$. For $\theta = n^{-1/4}$, the displacement of the frame at time $nt/2$ is approximately $n^{3/4}t$, and the limiting noise coefficient is 1: this explains the scalings of Theorem 1.2, and the same mechanism is at play for random walks in random environment.

1.6.3. *The martingale problem.* To turn these heuristics into a proof, we follow the strategy introduced by Bertini and Giacomin [BG97] to prove the convergence of the weakly asymmetric simple exclusion process to the KPZ equation. We consider the rescaled and tilted field $\mathsf{z}_n(t,x) = \mathsf{z}^\theta(nt/2, x)$ with $\theta = n^{-1/4}$, and the measure $\mathsf{Z}_n(t) = \sum_x \mathsf{z}_n(t,x)\delta_{y_x}$ where $y_x = n^{-1/2}(x - n^{3/4}t)$ is the position of the site x in the moving frame. This measure coincides with $\mathcal{Z}_n(t)$ up to a deterministic factor which goes to 1. For a smooth test function φ, the process

$$M_n^\varphi(t) = \mathsf{Z}_n(t;\varphi) - \varphi(0) - \int_0^t \mathsf{Z}_n(s;\mathcal{A}_n\varphi)ds$$

is a martingale, where $\mathsf{Z}_n(t;\varphi)$ denotes the integral of φ with respect to $\mathsf{Z}_n(t)$ and $\mathcal{A}_n$ is an explicit operator which converges to $\frac{1}{2}\partial_{xx}$ (Proposition 3.1). The stochastic heat equation (1.12) is characterized by a similar martingale problem [KS88; BG97]: for all test functions φ, the process

$$M^\varphi(t) = \int_{\mathbb{R}} Z(t,x)\varphi(x)\,dx - \varphi(0) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} Z(s,x)\varphi''(x)\,dx\,ds$$

is a martingale with quadratic variation

$$\langle M^\varphi\rangle_t = \int_0^t \int_{\mathbb{R}} Z(s,x)^2\varphi(x)^2\,dx\,ds. \tag{1.22}$$

The main difficulty is to prove that the quadratic variation of M_n^φ converges to (1.22). Up to negligible terms, we have

$$\langle M_n^\varphi\rangle_t \approx \int_0^t \mathsf{E}_n(s;\varphi)\,ds, \qquad \mathsf{E}_n(s;\varphi) = n^{1/2}\sum_x \mathsf{z}_n(s,x)\mathsf{z}_n(s,x+1)\varphi(y_x)\varphi(y_{x+1}). \tag{1.23}$$

Since $n^{1/2}\mathsf{z}_n$ plays the role of the density Z, the expression (1.23) formally looks like a Riemann sum for (1.22). However, the square of the density is replaced by a product of values at two neighbouring sites, and this cannot be neglected. Indeed, in the local equilibrium picture of

Remark 1.3, $\mathsf{z}_n(s,x) \approx \bar{\mathsf{z}}\,\mathfrak{e}_x$, where $\bar{\mathsf{z}}$ varies slowly and the $\mathfrak{e}_x$ are independent exponential random variables with mean 1: the average of $\mathsf{z}_n(s,x)\mathsf{z}_n(s,x+1)$ around a given site is close to $\bar{\mathsf{z}}^2$, as desired, while the average of $\mathsf{z}_n(s,x)^2$ is close to $2\bar{\mathsf{z}}^2$, because $\mathbb{E}[\mathfrak{e}_x^2] = 2$. Hence, what we have to show is that, after time integration, the product $\mathsf{z}_n(s,x)\mathsf{z}_n(s,x+1)$ can be replaced by the square of the average of $\mathsf{z}_n(s,\cdot)$ over a mesoscopic box containing $\varepsilon n^{1/2}$ sites, in the limit where n goes to infinity and then ε goes to zero. This is reminiscent of the replacement lemmas used in the theory of hydrodynamic limits [KL99], and of the key estimate in the proof of [BG97].

1.6.4. *Two-particle duality and a Poisson equation.* The replacement is performed using the two-particle duality of the model (the duality with an arbitrary number of particles is discussed in Appendix A). For a function f of two variables, it follows from (1.20) and (1.21) that

$$d \sum_{x_1,x_2} P(t,x_1)P(t,x_2)f(x_1,x_2) = \sum_{x_1,x_2} P(t,x_1)P(t,x_2)\,\mathcal{Q}f(x_1,x_2)\,dt + d(\text{martingale}), \quad (1.24)$$

where $\mathcal{Q}$ is the generator of the symmetric inclusion process with two particles: the two particles perform independent simple random walks on $\mathbb{Z}$, except that, when they are at neighbouring sites, each of them jumps on top of the other at an additional rate 1 (Lemma 2.3). After the tilt, the generator $\mathcal{Q}$ is replaced by a generator $\mathcal{Q}^\theta$ where the particles have a drift, but the distance $r = x_2 - x_1$ between the two particles is still a symmetric Markov chain: it is, up to a time change, a simple random walk on $\mathbb{Z}$ which is perturbed at the origin, with generator $\mathcal{R}$ given in (3.24). The reversible measure of this chain is

$$\pi(r) = 1 + \mathbb{1}_{r=0},$$

and the weight 2 of the origin is the counterpart of the identity $\mathbb{E}[\mathfrak{e}_x^2] = 2$ in the local equilibrium picture.

We wish to compare the functions $f_1(x_1,x_2) = \frac{1}{2}\mathbb{1}_{|x_2-x_1|=1}$, which corresponds to (1.23), and $f_2(x_1,x_2) = k_m(x_2-x_1)$, where k_m is a smooth nonnegative kernel spread over a box of size $m = \varepsilon n^{1/2}$, which corresponds to the square of a local average of the field. We normalize k_m so that these two functions of $r = x_2 - x_1$ have the same integral with respect to π. Then, there exists a bounded function g_m such that

$$\mathcal{R}g_m(r) = \frac{1}{2}\mathbb{1}_{|r|=1} - k_m(r).$$

An equation of this type, where one prescribes the image h of an unknown function g by the generator of a Markov process, is called a Poisson equation, by analogy with the case of Brownian motion, where it reads $\frac{1}{2}g'' = h$. It is useful for the following reason: if $(R_s)_{s\geqslant 0}$ denotes the Markov chain with generator $\mathcal{R}$, Dynkin's formula yields

$$\int_0^t \Big(\frac{1}{2}\mathbb{1}_{|R_s|=1} - k_m(R_s)\Big)ds = g_m(R_t) - g_m(R_0) - (\text{martingale}),$$

so that the time spent by the chain at the sites ± 1 is compared to the time that it spends at distance of order m from the origin, up to an error which is controlled by the size of g_m. It turns out that g_m is completely explicit and satisfies $0 \leqslant g_m \leqslant m$ (Lemma 6.3). The same mechanism operates at the level of the field: applying (1.24) to the function $g_m(x_2 - x_1)\varphi(y_{x_1})\varphi(y_{x_2})$, we obtain that, up to error terms, the time integral of the difference between (1.23) and its mesoscopic version is equal to the variation of the quantity

$$n^{-1/2} \sum_{x_1,x_2} \mathsf{z}_n(t,x_1)\mathsf{z}_n(t,x_2)g_m(x_2-x_1)\varphi(y_{x_1})\varphi(y_{x_2}),$$

which is of order $mn^{-1/2} = \varepsilon$, plus a martingale which is also small. In other words, g_m plays the role of a corrector. This is the content of Proposition 6.2. A very convenient feature of this argument is that it does not require any information on the transition probabilities of the inclusion process. The idea of using two-particle duality to control the quadratic variation in a martingale problem already appeared in [CGST20], for the stochastic six-vertex model, where it was implemented using exact contour integral formulas for the two-particle transition probabilities.

1.6.5. *A priori bounds and conclusion.* The same argument, applied with $\varphi \equiv 1$, has another important consequence: it bounds the L^2 norm of the smoothed field in terms of (1.23), that is in terms of the quadratic variation of the total mass $\mathsf{W}_n(t) = \sum_x \mathsf{z}_n(t,x)$. By duality, the second moment of $\mathsf{W}_n(t)$ is given by a Feynman-Kac formula for the Markov chain with generator $\mathcal{R}$, which is bounded uniformly in n thanks to an elementary estimate on the time spent by this chain at ± 1 (Section 4). These bounds imply tightness, and show that any limit point of Z_n has a density Z in $L^2(\Omega \times [0,T] \times \mathbb{R})$, which solves the martingale problem described above. Finally, we prove that this characterizes the solution of the stochastic heat equation. We need a statement slightly different from the classical uniqueness results of [KS88; BG97], since all the information we have on the limit points is an L^2 bound: the only estimates uniform in n that we use are first and second moment estimates, which come from the duality with one and two particles.

1.7. **Outline of the paper.** In Section 2, we show that the one-particle QSSEP is well-defined on $\mathbb{Z}$ and derive the stochastic differential equations satisfied by $P(t,x)$. In Section 3, we introduce the tilted and rescaled field and the associated martingales. Section 4 is devoted to second moment estimates. Tightness is proved in Section 5. In Section 6, we prove the replacement lemma, which is the heart of the proof. The limit points are identified in Section 7, which completes the proof of Theorem 1.2 when $L = \infty$. Finally, in Section 8, we deal with finite systems.

We placed in Appendix some useful material that is not needed for the proof of the main result. Appendix A explains the connection between the one-particle QSSEP and the Brownian energy process, states the Markov duality with the symmetric inclusion process for an arbitrary number of particles. Appendix B discusses some consequences. Appendix C shows that the one-particle QSSEP is not a random walk in a random environment.

1.8. **Formal verification.** Theorem 1.2 has been formalized and proved in the proof assistant Lean 4 [MU21], using its mathematical library Mathlib [The20] and no other library. The formal statement was first written by Claude Fable 5.2, and the formal proof by Claude Code (with the model Claude Opus 5.5), between November 11 and 13, 2026, from the text of the present paper. It consists of about 62 600 lines of Lean, and it is available at

`https://github.com/guillaumebarraquand/1particleQSSEPtoKPZ`

It is registered at Palomar under the identifier `PALOMAR-2026-11-19-000020 version 1`.

To know exactly what has been verified, one has to read the file `Challenge.lean` (about 550 lines, depending on Mathlib only), which contains 17 definitions and the statements of 8 theorems. The version of Mathlib that we use does not contain stochastic integrals, so that they are defined in this file.

The formal proof follows Sections 2 to 8. It departs from them where a classical theorem that we use is not available in Mathlib. These departures are listed in the file `DEVIATIONS.md`; no step of the proofs presented below was found to fail. The proofs were checked by Lean only: they rely on no axiom other than the three standard axioms of Lean, and the `comparator`

tool distributed with Lean confirmed that the theorems which are proved are exactly those of `Challenge.lean`.

Notations. We denote by $\mathbb{Z}_{\geqslant 1}$ the set of positive integers. For a measure μ on $\mathbb{R}$ and a function φ, we write $\mu(\varphi) = \int \varphi \, d\mu$; for the measure-valued processes $\mathcal{Z}_n(t)$, $\mathsf{Z}_n(t)$, $\mathcal{Z}(t)$, we use the notation $\mathcal{Z}(t;\varphi) = \int \varphi \, d\mathcal{Z}(t)$. We denote by $C_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$, by $C_b^k(\mathbb{R})$ the space of functions of class C^k which are bounded as well as their derivatives up to order k, and by $C_c^\infty(\mathbb{R})$ the space of smooth functions with compact support. We write $\|\cdot\|_\infty$ for the supremum norm and $\|\cdot\|_2$ for the norm of $L^2(\mathbb{R})$. The convolution of a measure $\mu \in \mathcal{M}_+(\mathbb{R})$ with a function ρ is the function $\mu * \rho(x) = \int \rho(x-y)\mu(dy)$. Throughout the paper, C denotes a positive constant whose value may change from line to line; the parameters on which it depends are specified when necessary.

Acknowledgments. G.B. was partially supported by Agence Nationale de la Recherche through the grant ANR-26-ERCC-0017.

We thank Denis Bernard for inspiring discussions and many explanations about the QSSEP. We also thank Pierre Le Doussal and Andrea De Luca for useful conversations about [SKR92]. Moreover, we thank the organizers of the conference in honor of Philippe Biane, *At the intersection of probability, algebra and combinatorics*, October 19-23, 2026, where the main result was first presented, as a conjecture.

2. The one-particle QSSEP: a stochastic heat equation with conservative noise

In this section, we show that the wave function ψ_t is well-defined on the infinite lattice, and we derive the stochastic differential equations satisfied by $P(t,x) = |\psi_t(x)|^2$. Throughout the section, $L \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$, with the convention that $\Lambda_\infty = \mathbb{Z}$, and $(W^{(x)})_{x,x+1\in\Lambda_L}$ is a family of independent standard complex $(\mathcal{F}_t)$-Brownian motions.

2.1. Well-posedness.

For $a \geqslant 0$, let $\ell^2_a(\mathbb{Z})$ be the Hilbert space of functions $\psi : \mathbb{Z} \to \mathbb{C}$ such that

$$\|\psi\|_a^2 := \sum_{x\in\mathbb{Z}} e^{a|x|}|\psi(x)|^2 < \infty,$$

so that $\ell^2_0(\mathbb{Z}) = \ell^2(\mathbb{Z})$.

Proposition 2.1. *Assume that $L = \infty$. There exists a unique continuous $(\mathcal{F}_t)$-adapted process $(\psi_t)_{t\geqslant 0}$ with values in $\ell^2(\mathbb{Z})$ such that for all $x \in \mathbb{Z}$ and $t \geqslant 0$,*

$$\psi_t(x) = \mathbb{1}_{x=0} + \mathbf{i}\int_0^t \psi_s(x+1)\, dW_s^{(x)} + \mathbf{i}\int_0^t \psi_s(x-1)\, d\overline{W}_s^{(x-1)} - \int_0^t \psi_s(x)\, ds. \tag{2.1}$$

Moreover, for all $a \geqslant 0$, $T > 0$ and $k \geqslant 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\psi_t\|_a^{2k}\right] < \infty. \tag{2.2}$$

Proof. Writing $W^{(x)} = (B_1^{(x)} + \mathbf{i}B_2^{(x)})/\sqrt{2}$, the system (2.1) is a linear stochastic differential equation in the (real) Hilbert space $\ell^2_a(\mathbb{Z})$,

$$d\psi_t = -\psi_t\, dt + \sum_{x\in\mathbb{Z}}\sum_{j\in\{1,2\}} \sigma_{x,j}(\psi_t)\, dB_j^{(x)}(t),$$

where, denoting by $(e_x)_{x\in\mathbb{Z}}$ the canonical basis,

$$\sigma_{x,1}(\psi) = \frac{\mathbf{i}}{\sqrt{2}}(\psi(x+1)e_x + \psi(x)e_{x+1}), \qquad \sigma_{x,2}(\psi) = \frac{1}{\sqrt{2}}(-\psi(x+1)e_x + \psi(x)e_{x+1}).$$

The maps $\sigma_{x,j}$ are linear and satisfy

$$\sum_{x\in\mathbb{Z}}\sum_{j\in\{1,2\}} \|\sigma_{x,j}(\psi)\|_a^2 = \sum_{x\in\mathbb{Z}} \left(e^{a|x|}|\psi(x+1)|^2 + e^{a|x+1|}|\psi(x)|^2\right) \leqslant 2e^a\|\psi\|_a^2.$$

Hence the diffusion coefficient is a bounded linear map from $\ell^2_a(\mathbb{Z})$ to the space of Hilbert-Schmidt operators from $\ell^2(\mathbb{Z}\times\{1,2\};\mathbb{R})$ to $\ell^2_a(\mathbb{Z})$. The drift being bounded and linear as well, the existence and uniqueness of a continuous solution in $\ell^2_a(\mathbb{Z})$, as well as the bound (2.2), follow from the standard theory of stochastic differential equations with Lipschitz coefficients in Hilbert spaces [DZ14, Theorem 7.2]. Since $\ell^2_a(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ and the initial condition e_0 belongs to all these spaces, uniqueness in $\ell^2(\mathbb{Z})$ shows that the solution does not depend on a. □

When $L < \infty$, (1.7) is a linear stochastic differential equation in finite dimension, which has a unique solution $\psi_t = U_t e_0$. In both cases, we let $P(t,x) = |\psi_t(x)|^2$ for $x \in \Lambda_L$.

2.2. **The equation satisfied by P.** Let Δ_L be the Laplacian on Λ_L with nearest-neighbour edges, that is

$$\Delta_L f(x) = \sum_{y\in\Lambda_L : |y-x|=1} (f(y) - f(x)).$$

We simply write Δ when $L = \infty$. Let also $\mathsf{p}_t(x)$ be the probability that the continuous-time random walk on Λ_L with generator Δ_L, started from 0, is at x at time t.

Proposition 2.2. *For $x \in \Lambda_L$ such that $x+1 \in \Lambda_L$, let*

$$J_t(x) = 2\int_0^t \mathfrak{Im}\left(\overline{\psi_s(x)}\psi_s(x+1)\, dW_s^{(x)}\right), \tag{2.3}$$

and let $J_t(x) = 0$ if $x \notin \Lambda_L$ or $x+1 \notin \Lambda_L$. Then, for all $x \in \Lambda_L$ and $t \geqslant 0$,

$$P(t,x) = \mathbb{1}_{x=0} + \int_0^t \Delta_L P(s,x)\, ds + J_t(x-1) - J_t(x). \tag{2.4}$$

The processes $J(x)$ are continuous square-integrable martingales, with

$$\langle J(x), J(y)\rangle_t = 2\,\mathbb{1}_{x=y}\int_0^t P(s,x)P(s,x+1)\, ds. \tag{2.5}$$

In particular, for all $t \geqslant 0$ and $x \in \Lambda_L$, $\mathbb{E}[P(t,x)] = \mathsf{p}_t(x)$.

The martingale $J_t(x)$ is the fluctuating part of the current of probability through the bond $(x, x+1)$, and (2.4) is a continuity equation.

Proof. By Itô's formula, $d|\psi_t(x)|^2 = \overline{\psi_t(x)}\, d\psi_t(x) + \psi_t(x)\, d\overline{\psi_t(x)} + d\psi_t(x)\, d\overline{\psi_t(x)}$. Using (1.7) and the rules (1.2), we have

$$d\psi_t(x)\, d\overline{\psi_t(x)} = \left(|\psi_t(x+1)|^2 + |\psi_t(x-1)|^2\right) dt,$$

with the convention that the terms corresponding to sites outside Λ_L are omitted, and

$$\begin{aligned}\overline{\psi_t(x)}\, d\psi_t(x) + \psi_t(x)\, d\overline{\psi_t(x)} = {}& 2\mathfrak{Re}\left(\mathbf{i}\,\overline{\psi_t(x)}\psi_t(x+1)\, dW_t^{(x)}\right)\\ &+ 2\mathfrak{Re}\left(\mathbf{i}\,\overline{\psi_t(x)}\psi_t(x-1)\, d\overline{W}_t^{(x-1)}\right) - D(x,x)|\psi_t(x)|^2 dt.\end{aligned}$$

Since $\mathfrak{Re}(\mathbf{i}z) = -\mathfrak{Im}(z) = \mathfrak{Im}(\overline{z})$, the first term equals $-dJ_t(x)$ and the second term equals $dJ_t(x-1)$. Collecting the terms in dt yields (2.4). For a complex number a, the quadratic variation of $\mathfrak{Im}(a\,dW_t^{(x)})$ is $\frac{1}{2}|a|^2dt$, which implies (2.5) for $x = y$. When $x \neq y$, the bracket vanishes since the Brownian motions attached to different bonds are independent. It remains to check that $\mathbb{E}[\langle J(x)\rangle_t] < \infty$ for all $t \geqslant 0$, so that the local martingales $J(x)$ are square-integrable martingales. When $L < \infty$, this follows from the fact that $P(s,x) \leqslant 1$, since U_s is unitary. When $L = \infty$, the fact that $P(s,\cdot)$ is a probability distribution will only be proved after Lemma 2.3; we use instead the inequality $P(s,x)P(s,x+1) \leqslant \|\psi_s\|_0^4$ and the bound (2.2).

Finally, let $m_t(x) = \mathbb{E}[P(t,x)]$. By the same arguments, the function m is bounded on $[0,T]\times\Lambda_L$ for all $T > 0$. Taking expectations in (2.4), we find that m is a solution of the equation $\partial_t m_t = \Delta_L m_t$ with $m_0(x) = \mathbb{1}_{x=0}$. Since Δ_L is a bounded operator on $\ell^\infty(\Lambda_L)$, this equation has a unique solution which is bounded on $[0,T]\times\Lambda_L$ for all $T > 0$, and this solution is p_t. □

The equation (2.4) and the brackets (2.5) show that the drift and the diffusion coefficients of the process $P(t,\cdot)$ are functions of $P(t,\cdot)$ itself. This closed system of stochastic differential equations is that of the Brownian energy process [GKRV09]. We refer to Appendix A for a discussion of this connection, on which the proofs below do not rely.

2.3. **Linear and quadratic functionals of P.** We now assume that $L = \infty$. In the sequel, we will apply Itô's formula to linear and quadratic functionals of $P(t,\cdot)$ involving infinite sums. This is justified by the next lemma. For a symmetric function $f : \mathbb{Z}^2 \to \mathbb{R}$, let

$$\mathcal{Q}f(x_1,x_2) = (\Delta_1+\Delta_2)\,f(x_1,x_2) + \mathbb{1}_{|x_1-x_2|=1}\Big[f(x_1,x_1)+f(x_2,x_2)-2f(x_1,x_2)\Big], \qquad (2.6)$$

where Δ_i denotes the discrete Laplacian acting on the variable x_i. The operator $\mathcal{Q}$ is the generator of the symmetric inclusion process with two particles described in Section 1.6.

Lemma 2.3. *Assume that $L = \infty$. Let $T > 0$ and $a \geqslant 0$.*

(1) Let $\alpha : [0,T]\times\mathbb{Z} \to \mathbb{R}$ be a function which is C^1 in the time variable and such that $|\alpha_t(x)| + |\partial_t\alpha_t(x)| \leqslant Ce^{a|x|}$ for some constant C. Then, almost surely, for all $t \in [0,T]$,

$$\sum_{x\in\mathbb{Z}} \alpha_t(x)P(t,x) = \alpha_0(0) + \int_0^t \sum_{x\in\mathbb{Z}} (\partial_s\alpha_s + \Delta\alpha_s)(x)P(s,x)\,ds + \int_0^t \sum_{x\in\mathbb{Z}} (\alpha_s(x+1)-\alpha_s(x))\,dJ_s(x). \qquad (2.7)$$

(2) Let $f : [0,T]\times\mathbb{Z}^2 \to \mathbb{R}$ be a function such that $f_t(x_1,x_2) = f_t(x_2,x_1)$, which is C^1 in the time variable and such that $|f_t(x_1,x_2)| + |\partial_t f_t(x_1,x_2)| \leqslant Ce^{a(|x_1|+|x_2|)}$. Then, almost surely, for all $t \in [0,T]$,

$$\begin{aligned}\sum_{x_1,x_2\in\mathbb{Z}} f_t(x_1,x_2)P(t,x_1)P(t,x_2) &= f_0(0,0)\\ &+ \int_0^t \sum_{x_1,x_2\in\mathbb{Z}} (\partial_s f_s + \mathcal{Q}f_s)(x_1,x_2)P(s,x_1)P(s,x_2)\,ds\\ &+ 2\int_0^t \sum_{x\in\mathbb{Z}} \Big[\sum_{x'\in\mathbb{Z}} (f_s(x+1,x')-f_s(x,x'))P(s,x')\Big]\,dJ_s(x). \qquad (2.8)\end{aligned}$$

In both cases, all the series converge absolutely, and the last term is a continuous square-integrable martingale.

In particular, applying (2.7) with $\alpha \equiv 1$, we obtain that $\sum_{x\in\mathbb{Z}} P(t,x) = 1$ for all $t \geqslant 0$: as in finite volume, ψ_t is a unit vector of $\ell^2(\mathbb{Z})$ and $P(t,\cdot)$ is a probability distribution.

Proof. Let us first explain the formal computation leading to (2.8), the case of (2.7) being similar and simpler. By Proposition 2.2,

$$\begin{aligned} d\langle P(\cdot,x), P(\cdot,x)\rangle_t &= 2\big(P(t,x-1)P(t,x) + P(t,x)P(t,x+1)\big)dt, \\ d\langle P(\cdot,x), P(\cdot,x+1)\rangle_t &= -2P(t,x)P(t,x+1)dt, \end{aligned}$$

and $d\langle P(\cdot,x_1), P(\cdot,x_2)\rangle_t = 0$ when $|x_1 - x_2| \geqslant 2$. Hence, by Itô's formula and the symmetry of f, the drift of $\sum_{x_1,x_2} f_t(x_1,x_2)P(t,x_1)P(t,x_2)$ is the sum of $\sum_{x_1,x_2} \partial_t f_t(x_1,x_2)P(t,x_1)P(t,x_2)$ and of

$$\begin{aligned} 2\sum_{x_1,x_2} f_t(x_1,x_2)P(t,x_2)\Delta P(t,x_1) \\ + 2\sum_x P(t,x)P(t,x+1)\Big[f_t(x,x) + f_t(x+1,x+1) - 2f_t(x,x+1)\Big]. \end{aligned}$$

After a summation by parts in the first sum, this is exactly $\sum_{x_1,x_2} \mathcal{Q}f_t(x_1,x_2)P(t,x_1)P(t,x_2)$. The martingale part is $2\sum_{x_1,x_2} f_t(x_1,x_2)P(t,x_2)\,(dJ_t(x_1-1) - dJ_t(x_1))$, which, after a summation by parts, is the last term in (2.8).

To make this rigorous, we first apply Itô's formula to the sums restricted to $|x_1|, |x_2| \leqslant K$, which only involve finitely many semimartingales. The summations by parts then produce boundary terms, which involve the values of $P(t,x)$ for $|x| \in \{K, K+1\}$, multiplied by coefficients bounded by $Ce^{a(K+1)}\sum_{x'} e^{a|x'|}P(t,x')$. Let

$$\mathsf{N}_a = \sup_{t\in[0,T]} \sum_{x\in\mathbb{Z}} e^{a|x|}P(t,x) = \sup_{t\in[0,T]} \|\psi_t\|_a^2,$$

which has finite moments of all orders by (2.2). Since $e^{a|x|}P(t,x) \leqslant e^{-|x|}\mathsf{N}_{a+1}$, all the series in (2.8) converge absolutely, the sums restricted to $|x_1|, |x_2| \leqslant K$ converge almost surely as K goes to infinity, uniformly on $[0,T]$, to the corresponding infinite sums, and the boundary terms in the drift vanish in the limit. Using moreover (2.5), the quadratic variations of the martingale terms are controlled by powers of N_{a+1}. This shows that the truncated martingales converge in L^2, uniformly on $[0,T]$, to the last term of (2.8), which is thus a continuous square-integrable martingale, and that the boundary martingale terms vanish in the limit. □

Lemma 2.3 expresses the Markov duality between the process $P(t,\cdot)$ and the symmetric inclusion process, with one particle in (2.7) and with two particles in (2.8). The general statement, with k particles, is given in Appendix A (Proposition A.2).

3. The tilted field

From now on and until the end of Section 7, we assume that $L = \infty$. In this section we introduce the tilted and rescaled field, and the martingales that will be used throughout the proof.

3.1. **Definitions.** Let $n \geqslant 1$ be a real number. We set

$$\theta_n = n^{-1/4}, \qquad c_n = \cosh\theta_n, \qquad \lambda_n = 2(\cosh\theta_n - 1), \qquad a_n = n^{1/2}\lambda_n. \tag{3.1}$$

Using the elementary inequalities $\theta^2 \leqslant 2(\cosh\theta - 1) \leqslant \theta^2 \cosh\theta$ and $0 \leqslant \sinh\theta - \theta \leqslant \frac{\theta^3}{6}\cosh\theta$, valid for all $\theta \geqslant 0$, we have for all $n \geqslant 1$,

$$n^{-1/2} \leqslant \lambda_n \leqslant c_n n^{-1/2}, \qquad 1 \leqslant a_n \leqslant c_n \leqslant \cosh(1) < 2. \tag{3.2}$$

As n goes to infinity, c_n and a_n converge to 1. For $t \geqslant 0$ and $x \in \mathbb{Z}$, we define the tilted and rescaled field

$$\mathsf{z}_n(t,x) := \exp\left(\theta_n x - \frac{n\lambda_n}{2}t\right) P\left(\frac{nt}{2}, x\right), \tag{3.3}$$

the position of the site x in the moving frame

$$y_n(t,x) := n^{-1/2}\left(x - n^{3/4}t\right), \tag{3.4}$$

and the random measure

$$\mathsf{Z}_n(t) := \sum_{x\in\mathbb{Z}} \mathsf{z}_n(t,x)\,\delta_{y_n(t,x)}. \tag{3.5}$$

For a function $\varphi : \mathbb{R} \to \mathbb{R}$, we use the notation $\mathsf{Z}_n(t;\varphi) = \sum_{x\in\mathbb{Z}} \mathsf{z}_n(t,x)\varphi(y_n(t,x))$. The total mass of the measure $\mathsf{Z}_n(t)$ is denoted by

$$\mathsf{W}_n(t) := \mathsf{Z}_n(t;1) = \sum_{x\in\mathbb{Z}} \mathsf{z}_n(t,x). \tag{3.6}$$

By (2.2), we have for all $T > 0$ and $k \geqslant 1$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \mathsf{W}_n(t)^k\Big] < \infty, \tag{3.7}$$

but at this stage this bound is not uniform in n. Comparing with (1.14) and (1.15), the measure $\mathcal{Z}_n(t)$ of Theorem 1.2 is given by

$$\mathcal{Z}_n(t) = \varrho_n(t)\,\mathsf{Z}_n(t), \qquad \text{where } \varrho_n(t) = \exp\left(\left(\frac{n\lambda_n}{2} - \frac{n^{1/2}}{2} - \frac{1}{24}\right)t\right). \tag{3.8}$$

Since $\frac{n\lambda_n}{2} = n(\cosh\theta_n - 1) = \frac{n^{1/2}}{2} + \frac{1}{24} + n\sum_{k\geqslant 3}\frac{\theta_n^{2k}}{(2k)!}$, we have

$$1 \leqslant \varrho_n(t) \leqslant \exp\left(tn^{-1/2}\right). \tag{3.9}$$

Hence, it will be sufficient to prove the convergence of Z_n. The advantage of the normalization chosen in (3.3) is that W_n is a martingale. We work with the time-changed filtration $\mathcal{F}_t^n := \mathcal{F}_{nt/2}$. For $x \in \mathbb{Z}$, we define the rescaled currents

$$\mathsf{J}_n(t,x) := \int_0^{nt/2} e^{\theta_n(x+\frac{1}{2})-\lambda_n s}\, dJ_s(x), \tag{3.10}$$

where $J(x)$ is the martingale defined in (2.3). By Proposition 2.2, the processes $\mathsf{J}_n(\cdot,x)$ are continuous $(\mathcal{F}_t^n)$-martingales with

$$d\langle \mathsf{J}_n(\cdot,x), \mathsf{J}_n(\cdot,x')\rangle_t = \mathbb{1}_{x=x'}\, n\,\mathsf{z}_n(t,x)\mathsf{z}_n(t,x+1)\,dt. \tag{3.11}$$

A central role will be played by the quantity

$$\mathsf{E}_n(t;\varphi) := n^{1/2}\sum_{x\in\mathbb{Z}} \mathsf{z}_n(t,x)\mathsf{z}_n(t,x+1)\,\varphi\big(y_n(t,x)\big)\varphi\big(y_n(t,x+1)\big), \tag{3.12}$$

defined for bounded functions φ. We simply write $\mathsf{E}_n(t)$ when $\varphi \equiv 1$. Since $n^{1/2}\mathsf{z}_n$ plays the role of a density with respect to the Lebesgue measure, $\mathsf{E}_n(t;\varphi)$ is a discrete analogue of the integral of φ^2 against the square of this density, where the square is replaced by a product of values at neighbouring sites.

3.2. **The discrete stochastic heat equation.** In terms of the notation above, (2.4) becomes

$$d\mathsf{z}_n(t,x) = \frac{n}{2}\Big[e^{-\theta_n}\mathsf{z}_n(t,x+1) + e^{\theta_n}\mathsf{z}_n(t,x-1) - 2c_n\mathsf{z}_n(t,x)\Big]\,dt \\ + e^{\theta_n/2}\,d\mathsf{J}_n(t,x-1) - e^{-\theta_n/2}\,d\mathsf{J}_n(t,x),$$

which is a discrete version of the stochastic heat equation (1.12). We will use it in the following weak form. For $T>0$, let $C_b^{1,3}([0,T]\times\mathbb{R})$ be the space of functions $\phi:(t,x)\mapsto\phi_t(x)$ such that $\partial_t\phi$, $\partial_x\phi$, $\partial_x^2\phi$ and $\partial_x^3\phi$ exist and are continuous and bounded on $[0,T]\times\mathbb{R}$, as well as ϕ itself. We let

$$\|\phi\|_{1,3} := \|\phi\|_\infty + \|\partial_t\phi\|_\infty + \sum_{k=1}^3 \|\partial_x^k\phi\|_\infty.$$

A function $\varphi\in C_b^3(\mathbb{R})$ is seen as an element of $C_b^{1,3}([0,T]\times\mathbb{R})$ which does not depend on time. Finally, for $\varphi\in C_b^3(\mathbb{R})$, we define

$$\mathcal{A}_n\varphi(y) := \frac{n}{2}\Big[e^{\theta_n}\varphi(y+n^{-1/2}) + e^{-\theta_n}\varphi(y-n^{-1/2}) - 2c_n\varphi(y)\Big] - n^{1/4}\varphi'(y). \tag{3.13}$$

Proposition 3.1. *Let $T>0$ and $\phi\in C_b^{1,3}([0,T]\times\mathbb{R})$. The process*

$$M_n^\phi(t) := \mathsf{Z}_n(t;\phi_t) - \phi_0(0) - \int_0^t \mathsf{Z}_n(s;\partial_s\phi_s + \mathcal{A}_n\phi_s)\,ds, \qquad t\in[0,T], \tag{3.14}$$

is a continuous square-integrable $(\mathcal{F}_t^n)$-martingale, with quadratic variation

$$\langle M_n^\phi\rangle_t = n\int_0^t \sum_{x\in\mathbb{Z}} \mathsf{z}_n(s,x)\mathsf{z}_n(s,x+1)\Big[e^{\theta_n/2}\phi_s(y_n(s,x+1)) - e^{-\theta_n/2}\phi_s(y_n(s,x))\Big]^2 ds. \tag{3.15}$$

Moreover, for all $n\geqslant 1$,

(1) for all $\varphi\in C_b^3(\mathbb{R})$, $\Big\|\mathcal{A}_n\varphi - \frac{1}{2}\varphi''\Big\|_\infty \leqslant n^{-1/4}\,\|\varphi\|_{1,3}$;

(2) for all $0\leqslant s\leqslant t\leqslant T$, $\langle M_n^\phi\rangle_t - \langle M_n^\phi\rangle_s \leqslant 2\|\phi\|_{1,3}^2\,(\langle \mathsf{W}_n\rangle_t - \langle \mathsf{W}_n\rangle_s)$;

(3) for all $t\in[0,T]$, $\Big|\langle M_n^\phi\rangle_t - a_n\int_0^t \mathsf{E}_n(s;\phi_s)\,ds\Big| \leqslant 5n^{-1/4}\,\|\phi\|_{1,3}^2\int_0^t \mathsf{E}_n(s)\,ds$.

In particular, the total mass $\mathsf{W}_n = 1 + M_n^1$ is a continuous square-integrable martingale with

$$\langle \mathsf{W}_n\rangle_t = a_n\int_0^t \mathsf{E}_n(s)\,ds. \tag{3.16}$$

Proof. We apply the first part of Lemma 2.3 on the time interval $[0,nT/2]$, to the function

$$\alpha_s(x) = e^{\theta_n x - \lambda_n s}\,\phi_{2s/n}(y_n(2s/n,x)),$$

so that $\mathsf{Z}_n(t;\phi_t) = \sum_x \alpha_{nt/2}(x)P(nt/2,x)$. Since $\frac{d}{ds}y_n(2s/n,x) = -2n^{-3/4}$ and $2+\lambda_n = 2c_n$, we find that

$$(\partial_s\alpha_s + \Delta\alpha_s)(x) = \frac{2}{n}\,e^{\theta_n x - \lambda_n s}\Big[\partial_t\phi_t + \mathcal{A}_n\phi_t\Big]\big(y_n(t,x)\big) \quad \text{with } t = 2s/n,$$

and

$$\alpha_s(x+1) - \alpha_s(x) = e^{\theta_n(x+\frac{1}{2}) - \lambda_n s}\Big[e^{\theta_n/2}\phi_t\big(y_n(t,x+1)\big) - e^{-\theta_n/2}\phi_t\big(y_n(t,x)\big)\Big].$$

After the change of variables $s=nu/2$ in the time integrals, (2.7) shows that

$$M_n^\phi(t) = \int_0^t \sum_{x\in\mathbb{Z}} \Big[e^{\theta_n/2}\phi_u(y_n(u,x+1)) - e^{-\theta_n/2}\phi_u(y_n(u,x))\Big]\,d\mathsf{J}_n(u,x), \tag{3.17}$$

which is a continuous square-integrable martingale, and (3.15) follows from (3.11).

Let us prove (1). Let $h = n^{-1/2}$ and $s_n = \sinh\theta_n$. Writing $e^{\pm\theta_n} = c_n \pm s_n$, we have

$$\mathcal{A}_n\varphi(y) = \frac{nc_n}{2}\big[\varphi(y+h) + \varphi(y-h) - 2\varphi(y)\big] + \frac{ns_n}{2}\big[\varphi(y+h) - \varphi(y-h)\big] - n^{1/4}\varphi'(y).$$

By Taylor's formula,

$$\Big|\varphi(y+h) + \varphi(y-h) - 2\varphi(y) - h^2\varphi''(y)\Big| \leqslant \frac{h^3}{3}\|\varphi'''\|_\infty,$$
$$\big|\varphi(y+h) - \varphi(y-h) - 2h\varphi'(y)\big| \leqslant \frac{h^3}{3}\|\varphi'''\|_\infty.$$

Hence

$$\Big|\mathcal{A}_n\varphi(y) - \frac{1}{2}\varphi''(y)\Big| \leqslant \frac{c_n - 1}{2}\|\varphi''\|_\infty + \Big(n^{1/2}s_n - n^{1/4}\Big)\|\varphi'\|_\infty + \frac{c_n + s_n}{6}n^{-1/2}\|\varphi'''\|_\infty.$$

By the inequalities recalled above (3.2), $c_n - 1 \leqslant \cosh(1)n^{-1/2}/2$, $0 \leqslant n^{1/2}s_n - n^{1/4} \leqslant \cosh(1)n^{-1/4}/6$ and $c_n + s_n \leqslant e$, which yields (1).

To prove (2) and (3), we use the identity, valid for any real numbers u, v and θ,

$$\big(e^{\theta/2}u - e^{-\theta/2}v\big)^2 = 2(\cosh\theta - 1)\,uv + (u - v)\big(e^{\theta}u - e^{-\theta}v\big). \tag{3.18}$$

We apply it with $u = \phi_s(y_n(s, x+1))$ and $v = \phi_s(y_n(s,x))$, so that $|u - v| \leqslant n^{-1/2}\|\phi\|_{1,3}$ and $|e^{\theta_n}u - e^{-\theta_n}v| \leqslant (2\sinh\theta_n + n^{-1/2})\|\phi\|_{1,3} \leqslant 5n^{-1/4}\|\phi\|_{1,3}$. Hence

$$\Big|n\Big[e^{\theta_n/2}u - e^{-\theta_n/2}v\Big]^2 - n\lambda_n uv\Big| \leqslant 5n^{1/4}\|\phi\|_{1,3}^2.$$

Recall that u and v depend on x and s. By the definition (3.12),

$$n\lambda_n\sum_{x\in\mathbb{Z}}\mathsf{z}_n(s,x)\mathsf{z}_n(s,x+1)\,uv = a_n\mathsf{E}_n(s;\phi_s) \quad\text{and}\quad n^{1/4}\sum_{x\in\mathbb{Z}}\mathsf{z}_n(s,x)\mathsf{z}_n(s,x+1) = n^{-1/4}\mathsf{E}_n(s).$$

Hence, summing over x and integrating over time yields (3). When $\phi \equiv 1$, we have $\mathcal{A}_n 1 = 0$, so that $M_n^1 = \mathsf{W}_n - 1$, and (3.15) becomes (3.16) since $(e^{\theta_n/2} - e^{-\theta_n/2})^2 = \lambda_n$. Finally, for general ϕ,

$$\begin{aligned}\big(e^{\theta_n/2}u - e^{-\theta_n/2}v\big)^2 &\leqslant 2\big(e^{\theta_n/2} - e^{-\theta_n/2}\big)^2u^2 + 2e^{-\theta_n}(u-v)^2\\ &\leqslant 2\lambda_n\|\phi\|_\infty^2 + 2n^{-1}\|\partial_x\phi\|_\infty^2 \leqslant 2\lambda_n\|\phi\|_{1,3}^2,\end{aligned}$$

where we used that $n^{-1} \leqslant \lambda_n$ by (3.2). Hence $d\langle M_n^\phi\rangle_t \leqslant 2\|\phi\|_{1,3}^2\, n\lambda_n\sum_x \mathsf{z}_n(t,x)\mathsf{z}_n(t,x+1)\,dt = 2\|\phi\|_{1,3}^2\, d\langle\mathsf{W}_n\rangle_t$, which is (2). □

3.3. **Quadratic functionals and the two-particle inclusion process.** We now turn to quadratic functionals of the field. For a symmetric function $f : \mathbb{Z}^2 \to \mathbb{R}$ and $\theta \in \mathbb{R}$, let

$$\begin{aligned}\mathcal{Q}^\theta f(x_1,x_2) = \sum_{\sigma\in\{-1,1\}} e^{\sigma\theta}\Big[f(x_1+\sigma,x_2) + f(x_1,x_2+\sigma) - 2f(x_1,x_2)\Big]\\ + \mathbb{1}_{x_2=x_1+1}\Big[e^{\theta}\big(f(x_2,x_2) - f(x_1,x_2)\big) + e^{-\theta}\big(f(x_1,x_1) - f(x_1,x_2)\big)\Big]\\ + \mathbb{1}_{x_2=x_1-1}\Big[e^{-\theta}\big(f(x_2,x_2) - f(x_1,x_2)\big) + e^{\theta}\big(f(x_1,x_1) - f(x_1,x_2)\big)\Big].\end{aligned} \tag{3.19}$$

When $\theta = 0$, this is the generator $\mathcal{Q}$ of the two-particle symmetric inclusion process defined in (2.6). In general, $\mathcal{Q}^\theta$ is the generator of an asymmetric version of this process, where the rates of all the jumps to the right are multiplied by e^θ, and the rates of all the jumps to the

left by $e^{-\theta}$. Considering the weight $w_\theta(x_1, x_2) = e^{\theta(x_1+x_2)}$, a direct computation shows that for any symmetric function f,

$$\mathcal{Q}(w_\theta f) = w_\theta \Big[\mathcal{Q}^\theta f + 4(\cosh\theta - 1)f + 2(\cosh\theta - 1)\mathbb{1}_{|x_1-x_2|=1} f\Big]. \tag{3.20}$$

Indeed, a jump of one particle by $\sigma \in \{-1, 1\}$ multiplies w_θ by $e^{\sigma\theta}$, and the sum of $(e^{\sigma\theta} - 1)$ over all possible jumps, weighted by their rates in $\mathcal{Q}$, equals $4(\cosh\theta - 1)$ in general, plus $2(\cosh\theta - 1)$ when the particles are at neighbouring sites, since there are two additional possible jumps, in opposite directions.

For a bounded function $f : \mathbb{Z}^2 \to \mathbb{R}$, we use the notation

$$\mathsf{Z}_n^{\otimes 2}(t; f) := \sum_{x_1,x_2\in\mathbb{Z}} \mathsf{z}_n(t, x_1)\mathsf{z}_n(t, x_2) f(x_1, x_2).$$

Lemma 3.2. *Let $T > 0$ and let $f : [0, T]\times\mathbb{Z}^2 \to \mathbb{R}$ be a bounded function such that $f_t(x_1, x_2) = f_t(x_2, x_1)$, which is C^1 in the time variable, with a bounded derivative. Then, for all $t \in [0, T]$,*

$$\mathsf{Z}_n^{\otimes 2}(t; f_t) = f_0(0, 0) + \int_0^t \mathsf{Z}_n^{\otimes 2}\Big(s; \partial_s f_s + \frac{n}{2}\mathcal{Q}^{\theta_n} f_s + \frac{n\lambda_n}{2}\mathbb{1}_{|x_1-x_2|=1} f_s\Big)\, ds + N_n^f(t), \tag{3.21}$$

where N_n^f is the continuous square-integrable $(\mathcal{F}_t^n)$-martingale

$$N_n^f(t) = 2\int_0^t \sum_{x\in\mathbb{Z}} \Big[\sum_{x'\in\mathbb{Z}} \mathsf{z}_n(s, x')\Big(e^{\theta_n/2} f_s(x + 1, x') - e^{-\theta_n/2} f_s(x, x')\Big)\Big]\, d\mathsf{J}_n(s, x). \tag{3.22}$$

Proof. We apply the second part of Lemma 2.3 on the time interval $[0, nT/2]$ to the function

$$F_s(x_1, x_2) = e^{\theta_n(x_1+x_2)-2\lambda_n s} f_{2s/n}(x_1, x_2),$$

so that $\mathsf{Z}_n^{\otimes 2}(t; f_t) = \sum_{x_1,x_2} F_{nt/2}(x_1, x_2)P(nt/2, x_1)P(nt/2, x_2)$. By (3.20) with $\theta = \theta_n$, for which $2(\cosh\theta_n - 1) = \lambda_n$, we have, with $t = 2s/n$,

$$\partial_s F_s + \mathcal{Q}F_s = \frac{2}{n} e^{\theta_n(x_1+x_2)-2\lambda_n s}\Big[\partial_t f_t + \frac{n}{2}\mathcal{Q}^{\theta_n} f_t + \frac{n\lambda_n}{2}\mathbb{1}_{|x_1-x_2|=1} f_t\Big],$$

and (3.21) follows from (2.8) after the change of variables $s = nu/2$ in the time integrals. The expression (3.22) is obtained from the last term of (2.8) using the definitions (3.3) and (3.10). □

A crucial feature of the operators $\mathcal{Q}^\theta$ is their action on functions of the distance between the two particles. If $f(x_1, x_2) = g(x_2 - x_1)$, where $g : \mathbb{Z} \to \mathbb{R}$ is an even function, we deduce from (3.19) that

$$\mathcal{Q}^\theta f(x_1, x_2) = \cosh\theta\, \mathcal{R}g(x_2 - x_1), \tag{3.23}$$

where

$$\mathcal{R}g(r) := 2\big[g(r + 1) + g(r - 1) - 2g(r)\big] + 2\,\mathbb{1}_{|r|=1}\big[g(0) - g(r)\big]. \tag{3.24}$$

In other words, even though the two particles have a drift under $\mathcal{Q}^\theta$, the difference of their positions is still a symmetric Markov chain, whose generator does not depend on θ up to the multiplicative factor $\cosh\theta$. This Markov chain is studied in the next section.

4. Second moment estimates

The goal of this section is to bound the second moment of the total mass $\mathsf{W}_n(t)$ uniformly in n, and to control the increments of its quadratic variation. By Lemma 3.2 and (3.23), this reduces to the study of a Markov chain on $\mathbb{Z}$.

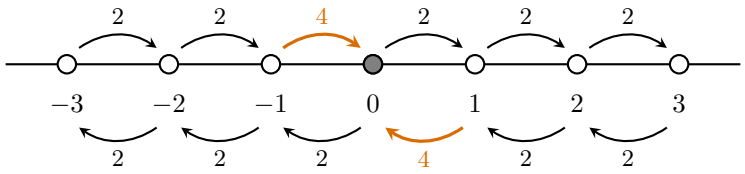


FIGURE 4. Jump rates of the Markov chain $(R_v)_{v\geqslant 0}$ with generator $\mathcal{R}$. The chain is a simple random walk, except that the jumps towards the origin from its two neighbours occur at a higher rate.

4.1. **A random walk with a defect at the origin.** Let $(R_v)_{v\geqslant 0}$ be the continuous-time Markov chain on $\mathbb{Z}$ with generator $\mathcal{R}$ defined in (3.24), and denote by P_r and E_r the probability and the expectation for this chain started from $r \in \mathbb{Z}$. From any site, the chain jumps to each of its two neighbours at rate 2, except for the jumps from 1 and -1 to 0, which occur at rate 4 (see Figure 4). It describes, up to a time change, the difference between the positions of two inclusion particles. It is reversible with respect to the measure

$$\pi(r) = 1 + \mathbb{1}_{r=0}, \qquad r \in \mathbb{Z}. \tag{4.1}$$

Lemma 4.1. *For all $r \in \mathbb{Z}$ and $S \geqslant 0$,*

$$\mathsf{E}_r\left[\int_0^S \mathbb{1}_{|R_v|=1}\, dv\right] \leqslant \min\left\{S, \frac{1}{4} + \sqrt{S}\right\} \leqslant \frac{5}{4}\sqrt{S}. \tag{4.2}$$

Proof. The bound by S is trivial, and the last inequality is easily checked by distinguishing the cases $S \leqslant 1$ and $S \geqslant 1$. Consider the functions $f_1(r) = \max\{|r|, 1\}$ and $f_2(r) = r^2$. A direct computation from (3.24) shows that for all $r \in \mathbb{Z}$,

$$\mathcal{R}f_1(r) = 2\,\mathbb{1}_{|r|=1}, \qquad \mathcal{R}f_2(r) = 4 - 2\,\mathbb{1}_{|r|=1}. \tag{4.3}$$

Since the jump rates of the chain are bounded, the number of jumps before time S is stochastically dominated by a Poisson random variable, so that $\sup_{v\leqslant S}|R_v|$ has finite moments of all orders and Dynkin's formula applies to f_1 and f_2. Using the second identity in (4.3), we obtain $\mathsf{E}_r[R_S^2] \leqslant r^2 + 4S$. Using the first one, the inequality $f_1(r) \leqslant \sqrt{1+r^2}$ and Jensen's inequality, we obtain

$$2\,\mathsf{E}_r\left[\int_0^S \mathbb{1}_{|R_v|=1}\, dv\right] = \mathsf{E}_r\left[f_1(R_S)\right] - f_1(r) \leqslant \sqrt{1 + r^2 + 4S} - \max\{|r|, 1\} \leqslant \frac{1}{2} + 2\sqrt{S},$$

where in the last inequality we used that $\sqrt{1+r^2+4S} \leqslant \sqrt{1+r^2} + 2\sqrt{S}$ and that $\sqrt{1+r^2} - \max\{|r|,1\} \leqslant \sqrt{2} - 1$ for all $r \in \mathbb{Z}$. □

The function f_1 in the proof above plays the same role as the absolute value in Tanaka's formula for the local time of Brownian motion. We now consider the function

$$u_n(t, r) := \mathsf{E}_r\left[\exp\left(\frac{\lambda_n}{c_n}\int_0^{nc_nt/2} \mathbb{1}_{|R_v|=1}\, dv\right)\right], \qquad t \geqslant 0, r \in \mathbb{Z}. \tag{4.4}$$

Lemma 4.2. *Let $n \geqslant 1$.*

(1) The function u_n is even in the variable r, of class C^1 in the variable t, and u_n and $\partial_t u_n$ are bounded on $[0,T]\times\mathbb{Z}$ for all $T>0$. It solves

$$\partial_t u_n(t,r) = \frac{nc_n}{2}\mathcal{R}u_n(t,r) + \frac{n\lambda_n}{2}\mathbb{1}_{|r|=1}u_n(t,r), \qquad u_n(0,r)=1. \tag{4.5}$$

(2) For all $t\geqslant 0$ and $r\in\mathbb{Z}$,

$$1\leqslant u_n(t,r)\leqslant 2^{1+16t}, \qquad \text{and} \qquad u_n(t,r)\leqslant 1+4\sqrt{t} \quad \text{if } t\leqslant \tfrac{1}{16}. \tag{4.6}$$

Proof. The operator $\mathcal{R}$ and the multiplication by $\mathbb{1}_{|r|=1}$ are bounded operators on $\ell^\infty(\mathbb{Z})$. Hence, the unique bounded solution of (4.5) is $u(t,\cdot)=\exp\big(t(\frac{nc_n}{2}\mathcal{R}+\frac{n\lambda_n}{2}\mathbb{1}_{|r|=1})\big)\mathbf{1}$, which is smooth in time with values in $\ell^\infty(\mathbb{Z})$. By the Feynman-Kac formula, $u(t,r)$ is the expectation of $\exp\big(\frac{n\lambda_n}{2}\int_0^t \mathbb{1}_{|R_{nc_n s/2}|=1}\,ds\big)$ under P_r, which is (4.4) after a change of variables. The function u_n is even since the law of the chain is invariant by the symmetry $r\mapsto -r$. This proves (1).

To prove (2), we use Khas'minskii's lemma [Kha59], that we recall: let $A_S=\int_0^S \mathbb{1}_{|R_v|=1}\,dv$ and let $\beta\geqslant 0$ be such that $\alpha:=\beta\sup_r \mathsf{E}_r[A_S]<1$. Expanding the exponential, and applying the Markov property at the successive times $v_1\leqslant\ldots\leqslant v_{k-1}$, we have

$$\mathsf{E}_r\left[e^{\beta A_S}\right] = \sum_{k\geqslant 0}\beta^k\,\mathsf{E}_r\left[\int_{0\leqslant v_1\leqslant\ldots\leqslant v_k\leqslant S}\prod_{i=1}^k \mathbb{1}_{|R_{v_i}|=1}\,dv_1\ldots dv_k\right] \leqslant \sum_{k\geqslant 0}\alpha^k = \frac{1}{1-\alpha}.$$

We apply this with $\beta=\lambda_n/c_n$ and $S=nc_nt/2$. By Lemma 4.1 and (3.2),

$$\alpha\leqslant \frac{\lambda_n}{c_n}\cdot\frac{5}{4}\sqrt{\frac{nc_nt}{2}} = \frac{5}{4\sqrt{2}}\frac{a_n}{\sqrt{c_n}}\sqrt{t}\leqslant \frac{5}{4\sqrt{2}}\sqrt{\cosh(1)}\,\sqrt{t}\leqslant 2\sqrt{t}.$$

For $t\leqslant 1/16$, we have $\alpha\leqslant 1/2$, hence $u_n(t,r)\leqslant(1-\alpha)^{-1}\leqslant 1+2\alpha\leqslant 1+4\sqrt{t}\leqslant 2$. Finally, by the Markov property, $\sup_r u_n(t+s,r)\leqslant\sup_r u_n(t,r)\,\sup_r u_n(s,r)$ for all $s,t\geqslant 0$, so that $\sup_r u_n(t,r)\leqslant 2^{\lceil 16t\rceil}\leqslant 2^{1+16t}$. □

4.2. **The total mass.** We may now state the main estimates on the total mass $\mathsf{W}_n(t)$ defined in (3.6). Recall that by Proposition 3.1, W_n is a continuous square-integrable martingale, with $\mathsf{W}_n(0)=1$ and quadratic variation given by (3.16).

Proposition 4.3. *Let $n\geqslant 1$.*

(1) For all $s,t\geqslant 0$,

$$\mathbb{E}\left[\mathsf{W}_n(t+s)^2\,\middle|\,\mathcal{F}_t^n\right] = \sum_{x_1,x_2\in\mathbb{Z}}\mathsf{z}_n(t,x_1)\mathsf{z}_n(t,x_2)\,u_n(s,x_2-x_1). \tag{4.7}$$

The same identity holds when t is replaced by a bounded $(\mathcal{F}_t^n)$-stopping time τ.

(2) For all $t\geqslant 0$,

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\mathsf{W}_n(s)^2\Big]\leqslant 2^{3+16t} \qquad \text{and} \qquad \mathbb{E}\left[\int_0^t \mathsf{E}_n(s)\,ds\right]\leqslant 2^{1+16t}. \tag{4.8}$$

(3) For any bounded $(\mathcal{F}_t^n)$-stopping time τ and any $s\in[0,\frac{1}{16}]$,

$$\mathbb{E}\left[\langle\mathsf{W}_n\rangle_{\tau+s}-\langle\mathsf{W}_n\rangle_\tau\,\middle|\,\mathcal{F}_\tau^n\right]\leqslant 4\sqrt{s}\,\mathsf{W}_n(\tau)^2. \tag{4.9}$$

Proof. Let $t_0 = t + s$ and consider the function $f_u(x_1, x_2) = u_n(t_0 - u, x_2 - x_1)$ for $u \in [0, t_0]$. By Lemma 4.2, it satisfies the assumptions of Lemma 3.2. By (3.23) and (4.5), the integrand in the drift term of (3.21) vanishes identically, so that $\mathsf{Z}_n^{\otimes 2}(u; f_u)$ is a martingale on $[0, t_0]$. Since $u_n(0, \cdot) = 1$, its value at $u = t_0$ is $\mathsf{W}_n(t_0)^2$, which proves (4.7). If τ is a stopping time taking finitely many values, (4.7) with t replaced by τ follows by decomposing over the values of τ. A general bounded stopping time τ is the decreasing limit of a sequence of stopping times τ_k taking finitely many values. The processes $t \mapsto \mathsf{W}_n(t + s)^2$ and $t \mapsto \mathsf{Z}_n^{\otimes 2}(t; u_n(s, \cdot))$, where we write $u_n(s, \cdot)$ for the function $(x_1, x_2) \mapsto u_n(s, x_2 - x_1)$, are continuous, and they are dominated on bounded time intervals by $2^{1+16s} \sup_{u \leqslant T} \mathsf{W}_n(u)^2$ for some T, which is integrable by (3.7). Hence we may pass to the limit in the identity $\mathbb{E}[\mathsf{W}_n(\tau_k + s)^2 \mathbb{1}_A] = \mathbb{E}[\mathsf{Z}_n^{\otimes 2}(\tau_k; u_n(s, \cdot))\mathbb{1}_A]$, valid for $A \in \mathcal{F}_\tau^n \subset \mathcal{F}_{\tau_k}^n$. This proves (1).

Taking $t = 0$ in (4.7), we get $\mathbb{E}[\mathsf{W}_n(s)^2] = u_n(s, 0) \leqslant 2^{1+16s}$ by (4.6). The first bound in (4.8) follows by Doob's inequality. Moreover, by (3.16), $a_n \mathbb{E}[\int_0^t \mathsf{E}_n(s) ds] = \mathbb{E}[\mathsf{W}_n(t)^2] - 1$, and $a_n \geqslant 1$, which gives the second bound. Finally, since $\mathsf{W}_n^2 - \langle \mathsf{W}_n \rangle$ is a martingale, the left-hand side of (4.9) equals

$$\mathbb{E}\left[\mathsf{W}_n(\tau + s)^2 \,\middle|\, \mathcal{F}_\tau^n\right] - \mathsf{W}_n(\tau)^2 = \sum_{x_1, x_2 \in \mathbb{Z}} \mathsf{z}_n(\tau, x_1) \mathsf{z}_n(\tau, x_2) \big(u_n(s, x_2 - x_1) - 1\big),$$

which is bounded by $4\sqrt{s}\, \mathsf{W}_n(\tau)^2$ by (4.6). □

Remark 4.4. The identity (4.7) shows that $\mathbb{E}[\mathsf{W}_n(t)^2] = u_n(t, 0)$. Using the explicit expression of the resolvent of the generator $\mathcal{R}$, one can check that

$$\lim_{n \to \infty} \mathbb{E}\left[\mathsf{W}_n(t)^2\right] = 2e^{t/4} \int_{-\infty}^{\sqrt{t/2}} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du,$$

which is the second moment of the total mass $\int_{\mathbb{R}} Z(t, x) dx$ of the solution of the stochastic heat equation (1.12) with initial condition δ_0. This computation, which we do not reproduce here since it is not needed in the sequel, provides a simple check that the noise coefficient in (1.12) is the correct one.

5. Tightness

Recall that $\mathcal{M}_+(\mathbb{R})$ denotes the space of finite nonnegative measures on $\mathbb{R}$, equipped with the topology of weak convergence, which makes it a Polish space. The goal of this section is to prove the following.

Proposition 5.1. *For any $T > 0$, the processes $(\mathsf{Z}_n(t))_{t \in [0,T]}$ have continuous trajectories in $\mathcal{M}_+(\mathbb{R})$, and the family of their laws, indexed by $n \geqslant 1$, is tight in $C([0, T], \mathcal{M}_+(\mathbb{R}))$.*

The estimates of Section 4 control the fluctuations of $\mathsf{Z}_n(t; \varphi)$ for bounded functions φ. In order to obtain tightness for the topology of weak convergence, we also need to make sure that the mass of $\mathsf{Z}_n(t)$ does not escape to infinity. This is the purpose of the next lemma.

Lemma 5.2. *For all $n \geqslant 1$, $T > 0$ and $K > 0$,*

$$\mathbb{P}\Big(\sup_{t \in [0,T]} \mathsf{Z}_n(t; \cosh) \geqslant K\Big) \leqslant \frac{2e^{2T}}{K}. \tag{5.1}$$

Proof. We first consider the exponential function instead of cosh. Let $\delta = n^{-1/2}$ and $\theta' = \theta_n + \delta$, so that $e^{y_n(t,x)} = e^{\delta x - n\theta_n \delta t}$. By definition,

$$\mathsf{Z}_n(t;\exp) = \sum_{x\in\mathbb{Z}} e^{\theta_n x - \frac{n\lambda_n}{2}t + y_n(t,x)} P\Big(\frac{nt}{2}, x\Big) = e^{\zeta t} \sum_{x\in\mathbb{Z}} e^{\theta' x - n(\cosh\theta' - 1)t} P\Big(\frac{nt}{2}, x\Big),$$

where $\zeta = n(\cosh\theta' - \cosh\theta_n) - n\theta_n\delta$. By the first part of Lemma 2.3, applied to the function $\alpha_s(x) = e^{\theta' x - 2(\cosh\theta' - 1)s}$, which satisfies $\partial_s\alpha_s + \Delta\alpha_s = 0$, the sum in the right-hand side is a nonnegative continuous martingale, equal to 1 at time 0. Since $\cosh(\theta_n + \delta) - \cosh\theta_n = \sinh\theta_n \sinh\delta + \cosh\theta_n(\cosh\delta - 1)$,

$$\zeta = n\big(\sinh\theta_n \sinh\delta - \theta_n\delta\big) + n\cosh\theta_n(\cosh\delta - 1).$$

Using that $\theta_n \leqslant 1$ and $|\delta| = n^{-1/2} \leqslant 1$, and the inequalities recalled above (3.2), one finds that $|\zeta| \leqslant 2$. Hence, by Doob's maximal inequality for nonnegative martingales, $\mathbb{P}(\sup_{t\leqslant T} \mathsf{Z}_n(t;\exp) \geqslant K) \leqslant e^{2T}/K$. The same bound holds when the exponential function is replaced by $y \mapsto e^{-y}$: it suffices to take $\delta = -n^{-1/2}$ in the argument above. Finally, since $\cosh y = \frac{1}{2}(e^y + e^{-y})$, the supremum over $[0,T]$ of $\mathsf{Z}_n(t;\cosh)$ can be larger than K only if this is the case for at least one of the two exponential functions, which yields (5.1). □

Proof of Proposition 5.1. We first show that the trajectories are continuous. For $\varphi \in C_b(\mathbb{R})$, the function $t \mapsto \mathsf{Z}_n(t;\varphi) = \sum_x \mathsf{z}_n(t,x)\varphi(y_n(t,x))$ is a sum of continuous functions, and this sum converges uniformly on $[0,T]$ by (2.2). Hence $t \mapsto \mathsf{Z}_n(t)$ is almost surely continuous for the topology of weak convergence.

To prove tightness, we use Jakubowski's criterion [Jak86, Theorem 3.1], which ensures that the laws of Z_n are tight in the Skorokhod space $D([0,T], \mathcal{M}_+(\mathbb{R}))$ provided that the following two conditions hold:

(i) for all $\eta > 0$, there exists a compact subset $\mathcal{K}_\eta$ of $\mathcal{M}_+(\mathbb{R})$ such that for all $n \geqslant 1$,

$$\mathbb{P}(\mathsf{Z}_n(t) \in \mathcal{K}_\eta \text{ for all } t \in [0,T]) \geqslant 1 - \eta;$$

(ii) there exists a family $\mathbb{F}$ of continuous functions on $\mathcal{M}_+(\mathbb{R})$, which separates points and is closed under addition, such that for all $F \in \mathbb{F}$, the laws of the processes $(F(\mathsf{Z}_n(t)))_{t\in[0,T]}$ are tight in $D([0,T],\mathbb{R})$.

Since the processes Z_n are continuous, and since the Skorokhod topology coincides with the topology of uniform convergence on the closed subset $C([0,T], \mathcal{M}_+(\mathbb{R}))$ of $D([0,T], \mathcal{M}_+(\mathbb{R}))$, tightness in $D([0,T], \mathcal{M}_+(\mathbb{R}))$ implies tightness in $C([0,T], \mathcal{M}_+(\mathbb{R}))$.

For condition (i), we take $\mathcal{K}_\eta = \{\mu \in \mathcal{M}_+(\mathbb{R}) : \mu(\cosh) \leqslant K\}$ with $K = 2e^{2T}/\eta$. This set is compact by Prokhorov's theorem, since it is closed and its elements have a total mass bounded by K and satisfy $\mu(\mathbb{R} \setminus [-\ell,\ell]) \leqslant K/\cosh(\ell)$ for all $\ell > 0$. Condition (i) then follows from Lemma 5.2.

For condition (ii), we consider the family of functions $\mu \mapsto \mu(\varphi)$, for $\varphi \in C_c^\infty(\mathbb{R})$. We have to prove that for a fixed $\varphi \in C_c^\infty(\mathbb{R})$, the laws of the real-valued processes $\mathsf{Z}_n(\cdot;\varphi)$ are tight. We use Aldous' criterion [Ald78] (see also [Bil99, Theorem 16.10]). First, $\sup_{t\leqslant T} |\mathsf{Z}_n(t;\varphi)| \leqslant \|\varphi\|_\infty \sup_{t\leqslant T} \mathsf{W}_n(t)$, which is bounded in L^2, uniformly in n, by (4.8). It remains to show that for all $\eta > 0$,

$$\lim_{\delta\to 0} \sup_{n\geqslant 1} \sup_{\tau,s} \mathbb{P}\Big(|\mathsf{Z}_n(\tau+s;\varphi) - \mathsf{Z}_n(\tau;\varphi)| > \eta\Big) = 0, \tag{5.2}$$

where the supremum is over all $(\mathcal{F}_t^n)$-stopping times τ bounded by T and all $s \in [0, \delta]$ (to be precise, we consider the processes on the time interval $[0, T+1]$). By Proposition 3.1,

$$\mathsf{Z}_n(\tau + s; \varphi) - \mathsf{Z}_n(\tau; \varphi) = \int_\tau^{\tau+s} \mathsf{Z}_n(u; \mathcal{A}_n\varphi)\, du + M_n^\varphi(\tau + s) - M_n^\varphi(\tau).$$

Since $\|\mathcal{A}_n\varphi\|_\infty \leqslant 2\|\varphi\|_{1,3}$, the first term is bounded by $2s\|\varphi\|_{1,3} \sup_{u \leqslant T+1} \mathsf{W}_n(u)$, whose expectation is at most Cs by (4.8), for a constant C which does not depend on n. For the second term, the optional stopping theorem, Proposition 3.1(2) and Proposition 4.3 yield, for $s \leqslant 1/16$,

$$\mathbb{E}\left[\left(M_n^\varphi(\tau + s) - M_n^\varphi(\tau)\right)^2\right] = \mathbb{E}\left[\langle M_n^\varphi\rangle_{\tau+s} - \langle M_n^\varphi\rangle_\tau\right] \leqslant 2\|\varphi\|_{1,3}^2\, \mathbb{E}\left[\langle \mathsf{W}_n\rangle_{\tau+s} - \langle \mathsf{W}_n\rangle_\tau\right] \leqslant C\sqrt{s},$$

where again C does not depend on n. By Markov's inequality, the probability in (5.2) is bounded by $2C\delta/\eta + 4C\sqrt{\delta}/\eta^2$, which proves (5.2) and concludes the proof. □

6. The replacement lemma

In this section we prove that, after integration in time, the quantity $\mathsf{E}_n(t; \varphi)$ defined in (3.12), which governs the quadratic variation of the martingales M_n^φ, can be replaced by the squared L^2 norm of a smoothed version of the measure $\mathsf{Z}_n(t)$. We fix once and for all an even nonnegative function $\rho \in C_c^\infty(\mathbb{R})$, supported in $[-1, 1]$, such that $\int_{\mathbb{R}} \rho(x)dx = 1$, and for $\varepsilon > 0$ we let

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right), \qquad \kappa = \rho * \rho, \qquad \kappa_\varepsilon = \rho_\varepsilon * \rho_\varepsilon = \frac{1}{\varepsilon}\kappa\left(\frac{\cdot}{\varepsilon}\right).$$

The function κ is even, nonnegative, smooth, supported in $[-2, 2]$, with $\int_{\mathbb{R}} \kappa(x)dx = 1$ and $\|\kappa\|_\infty = \kappa(0) = \|\rho\|_2^2 > 0$. For a measure $\mu \in \mathcal{M}_+(\mathbb{R})$ and a function $\varphi \in C_b(\mathbb{R})$, we denote by $\varphi\mu$ the measure $\varphi(x)\mu(dx)$. Since ρ is even,

$$\|(\varphi\mu) * \rho_\varepsilon\|_2^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\varphi(y)\kappa_\varepsilon(x - y)\, \mu(dx)\mu(dy). \tag{6.1}$$

The main results of this section are the following two propositions.

Proposition 6.1. *There exists a constant C, depending only on ρ, such that for all $n \geqslant 1$, all integers $m \geqslant 1$ and all $T > 0$,*

$$\mathbb{E}\left[\int_0^T \|\mathsf{Z}_n(t) * \rho_{mn^{-1/2}}\|_2^2\, dt\right] \leqslant C\, 2^{16T}. \tag{6.2}$$

Proposition 6.2. *Let $T > 0$ and $\phi \in C_b^{1,3}([0, T] \times \mathbb{R})$. For all $\eta > 0$,*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty}\ \mathbb{P}\left(\sup_{t \in [0,T]} \left|\int_0^t \left(\mathsf{E}_n(s; \phi_s) - \|(\phi_s \mathsf{Z}_n(s)) * \rho_\varepsilon\|_2^2\right) ds\right| > \eta\right) = 0. \tag{6.3}$$

6.1. **A Poisson equation.** Recall the generator $\mathcal{R}$ defined in (3.24) and its reversible measure π defined in (4.1). For an integer $m \geqslant 1$, let

$$b_m = \frac{1}{m}\sum_{r \in \mathbb{Z}} \pi(r)\kappa\left(\frac{r}{m}\right), \qquad k_m(r) = \frac{1}{mb_m}\kappa\left(\frac{r}{m}\right). \tag{6.4}$$

The function $k_m : \mathbb{Z} \to \mathbb{R}$ is even, nonnegative, vanishes when $|r| \geqslant 2m$, and is normalized so that

$$\sum_{r\in\mathbb{Z}} \pi(r)k_m(r) = 1. \tag{6.5}$$

Moreover, $b_m > 0$ for all $m \geqslant 1$ and b_m converges to $\int_{\mathbb{R}} \kappa(x)dx = 1$ as m goes to infinity, so that

$$0 < \inf_{m\geqslant 1} b_m \leqslant \sup_{m\geqslant 1} b_m < \infty. \tag{6.6}$$

Lemma 6.3. *Let $m \geqslant 1$ and let $g_m : \mathbb{Z} \to \mathbb{R}$ be the even function defined by*

$$g_m(0) = \frac{k_m(0)}{4}, \qquad g_m(1) = 0, \qquad g_m(r+1) = g_m(r) + \frac{1}{2}\sum_{u>r} k_m(u) \quad \text{for } r \geqslant 1. \tag{6.7}$$

Then, for all $r \in \mathbb{Z}$,

$$\mathcal{R}g_m(r) = \frac{1}{2}\mathbb{1}_{|r|=1} - k_m(r). \tag{6.8}$$

Moreover, $0 \leqslant g_m(r) \leqslant \frac{m}{2}$ and $|g_m(r+1) - g_m(r)| \leqslant \frac{1}{4}$ for all $r \in \mathbb{Z}$, $g_m(r+1) = g_m(r-1)$ when $|r| \geqslant 2m$, and $g_m(0) \leqslant C/m$ for a constant C which depends only on ρ.

Proof. Since g_m and k_m are even, it suffices to check (6.8) for $r \geqslant 0$. For $r = 0$, $\mathcal{R}g_m(0) = 4(g_m(1) - g_m(0)) = -k_m(0)$. For $r \geqslant 2$,

$$\mathcal{R}g_m(r) = 2\big(g_m(r+1) - g_m(r)\big) - 2\big(g_m(r) - g_m(r-1)\big) = \sum_{u>r} k_m(u) - \sum_{u>r-1} k_m(u) = -k_m(r).$$

For $r = 1$, using that $g_m(1) = 0$,

$$\mathcal{R}g_m(1) = 2g_m(2) + 4g_m(0) = \sum_{u\geqslant 2} k_m(u) + k_m(0) = \frac{1}{2} - k_m(1),$$

where the last equality follows from (6.5), which reads $2k_m(0) + 2\sum_{u\geqslant 1} k_m(u) = 1$. This proves (6.8). By (6.5) again, the increments $g_m(r+1) - g_m(r)$ belong to $[0, \frac{1}{4}]$ for $r \geqslant 1$, and vanish for $r \geqslant 2m-1$, since $k_m(u) = 0$ for $u \geqslant 2m$. This implies that $0 \leqslant g_m(r) \leqslant \frac{m}{2}$ for $|r| \geqslant 1$ and that $g_m(r+1) = g_m(r-1)$ for $|r| \geqslant 2m$. Finally, $0 \leqslant g_m(0) = \frac{\kappa(0)}{4mb_m} \leqslant \min\{\frac{1}{8}, \frac{C}{m}\}$ by (6.5) and (6.6). □

Let us comment on Lemma 6.3. For a function h with finite support, the Poisson equation $\mathcal{R}g = h$ can only have a bounded solution if $\sum_r \pi(r)h(r) = 0$, and (6.5) ensures that this is the case for the right-hand side of (6.8). The bounded solution is then unique up to an additive constant, which we have fixed by imposing that $g_m(\pm 1) = 0$. This choice will be convenient below, since it exactly cancels the last term in the integrand of (3.21). The important point is that g_m is bounded by m, while its discrete gradient remains bounded.

6.2. **The corrector.** For $T > 0$, $\phi \in C_b^{1,3}([0,T] \times \mathbb{R})$ and an integer $m \geqslant 1$, we define

$$\mathsf{F}_n(t) := \frac{2}{c_n n^{1/2}} \sum_{x_1,x_2\in\mathbb{Z}} \mathsf{z}_n(t,x_1)\mathsf{z}_n(t,x_2)\, g_m(x_2 - x_1)\, \phi_t(y_n(t,x_1))\phi_t(y_n(t,x_2)), \tag{6.9}$$

and, for $\varphi \in C_b(\mathbb{R})$,

$$\mathsf{B}_n(t;\varphi) := n^{1/2} \sum_{x_1,x_2\in\mathbb{Z}} \mathsf{z}_n(t,x_1)\mathsf{z}_n(t,x_2)\, k_m(x_2 - x_1)\, \varphi(y_n(t,x_1))\varphi(y_n(t,x_2)). \tag{6.10}$$

These quantities depend on m and ϕ, but we omit this dependence in the notation. Since $y_n(t,x_1) - y_n(t,x_2) = n^{-1/2}(x_1 - x_2)$, we have $\kappa_\varepsilon(y_n(t,x_1) - y_n(t,x_2)) = n^{1/2} b_m k_m(x_2 - x_1)$ when $\varepsilon = mn^{-1/2}$, so that by (6.1),

$$\mathsf{B}_n(t;\varphi) = \frac{1}{b_m} \left\| (\varphi \mathsf{Z}_n(t)) * \rho_{mn^{-1/2}} \right\|_2^2. \tag{6.11}$$

Lemma 6.4. *Let $T > 0$, $\phi \in C_b^{1,3}([0,T] \times \mathbb{R})$, $n \geqslant 1$ and $m \geqslant 1$. For all $t \in [0,T]$,*

$$\mathsf{F}_n(t) = \mathsf{F}_n(0) + \int_0^t \Big(\mathsf{E}_n(s;\phi_s) - \mathsf{B}_n(s;\phi_s) + \mathsf{R}_n(s) \Big)\, ds + N_n(t), \tag{6.12}$$

where N_n is a continuous square-integrable $(\mathcal{F}_t^n)$-martingale and, for a constant C which depends only on ρ, and for all $t \in [0,T]$,

$$|\mathsf{F}_n(t)| \leqslant \frac{m}{n^{1/2}} \|\phi\|_{1,3}^2 \, \mathsf{W}_n(t)^2, \tag{6.13}$$

$$|\mathsf{R}_n(t)| \leqslant C \left(\frac{m}{n^{1/2}} + \frac{1}{n^{1/4}} + \frac{1}{m} \right) \|\phi\|_{1,3}^2 \, \mathsf{W}_n(t)^2, \tag{6.14}$$

$$d\langle N_n \rangle_t \leqslant C \left(\frac{m^2}{n} + \frac{1}{n^{1/2}} \right) \|\phi\|_{1,3}^4 \, \mathsf{W}_n(t)^2 \, d\langle \mathsf{W}_n \rangle_t. \tag{6.15}$$

Moreover, when $\phi \equiv 1$, we have $\mathsf{R}_n \equiv 0$ and $\mathsf{F}_n(t) \geqslant 0$.

Proof. To simplify the notation, we write θ, c, g and h instead of θ_n, c_n, g_m and $n^{-1/2}$, and we let $s_n = \sinh \theta_n$. We also let

$$G(x_1,x_2) = g(x_2 - x_1), \qquad \Phi_t(x_1,x_2) = \phi_t(y_n(t,x_1))\phi_t(y_n(t,x_2)),$$

so that $\mathsf{F}_n(t) = \mathsf{Z}_n^{\otimes 2}(t; f_t)$ with $f_t = \frac{2}{cn^{1/2}} G\Phi_t$. The function f is symmetric and satisfies the assumptions of Lemma 3.2. We apply (3.21). The last term in the integrand vanishes because $g(\pm 1) = 0$. For the other terms, we use the product rule for generators of Markov jump processes,

$$\mathcal{Q}^\theta(G\Phi_t) = \Phi_t \, \mathcal{Q}^\theta G + G \, \mathcal{Q}^\theta \Phi_t + \Gamma^\theta(G, \Phi_t),$$

where $\Gamma^\theta(G,\Phi_t)(x_1,x_2)$ is the sum, over all possible jumps from (x_1,x_2) to some (x_1',x_2'), of the rate of the jump in $\mathcal{Q}^\theta$ multiplied by $\big(G(x_1',x_2') - G(x_1,x_2)\big)\big(\Phi_t(x_1',x_2') - \Phi_t(x_1,x_2)\big)$. By (3.23) and (6.8),

$$\frac{2}{cn^{1/2}} \cdot \frac{n}{2} \, \Phi_t \, \mathcal{Q}^\theta G(x_1,x_2) = n^{1/2} \Phi_t(x_1,x_2) \left(\frac{1}{2} \mathbb{1}_{|x_1 - x_2| = 1} - k_m(x_2 - x_1) \right),$$

and the integral of this function with respect to $\mathsf{Z}_n^{\otimes 2}(t)$ is exactly $\mathsf{E}_n(t;\phi_t) - \mathsf{B}_n(t;\phi_t)$. This proves (6.12) with $N_n = N_n^f$ and

$$\mathsf{R}_n(t) = \mathsf{Z}_n^{\otimes 2} \left(t; \frac{2}{cn^{1/2}} \, G \left(\partial_t \Phi_t + \frac{n}{2} \mathcal{Q}^\theta \Phi_t \right) + \frac{n^{1/2}}{c} \, \Gamma^\theta(G, \Phi_t) \right). \tag{6.16}$$

The bound (6.13) follows from the bound $0 \leqslant g \leqslant m/2$ of Lemma 6.3. Let us now bound the two terms in (6.16).

First term. When $|x_1 - x_2| \neq 1$, the two particles jump independently under $\mathcal{Q}^\theta$, and a computation similar to the proof of Proposition 3.1 shows that

$$\begin{aligned} \partial_t \Phi_t(x_1,x_2) + \frac{n}{2} \mathcal{Q}^\theta \Phi_t(x_1,x_2) = {} & \big[(\partial_t + \mathcal{A}_n)\phi_t \big](y_n(t,x_1)) \, \phi_t(y_n(t,x_2)) \\ & + \phi_t(y_n(t,x_1)) \, \big[(\partial_t + \mathcal{A}_n)\phi_t \big](y_n(t,x_2)). \end{aligned}$$

This is bounded by $4\|\phi\|_{1,3}^2$ since $\|(\partial_t + \mathcal{A}_n)\phi_t\|_\infty \leqslant 2\|\phi\|_{1,3}$ by Proposition 3.1(1). When $|x_1 - x_2| = 1$, we have $G(x_1, x_2) = 0$. Hence, the contribution of the first term to $|\mathsf{R}_n(t)|$ is at most $4mn^{-1/2}\|\phi\|_{1,3}^2\,\mathsf{W}_n(t)^2$.

Second term. We decompose $\Gamma^\theta(G, \Phi_t) = \Gamma_{\text{free}} + \Gamma_{\text{incl}}$, where Γ_{free} corresponds to the jumps in the first line of (3.19), and Γ_{incl} to the additional jumps present when $|x_1 - x_2| = 1$. Let $r = x_2 - x_1$ and $y_i = y_n(t, x_i)$. We have

$$\Gamma_{\text{free}} = \sum_{\sigma\in\{-1,1\}} e^{\sigma\theta}\Big\{ \big[g(r-\sigma) - g(r)\big]\big[\phi_t(y_1 + \sigma h) - \phi_t(y_1)\big]\phi_t(y_2) \\ + \big[g(r+\sigma) - g(r)\big]\phi_t(y_1)\big[\phi_t(y_2 + \sigma h) - \phi_t(y_2)\big]\Big\}.$$

We write $\phi_t(y + \sigma h) - \phi_t(y) = \sigma h\,\partial_x\phi_t(y) + O(h^2)$, where the error term is bounded by $\frac{h^2}{2}\|\phi\|_{1,3}$, and $e^{\sigma\theta} = c + \sigma s_n$. Since the increments of g are bounded by $\frac{1}{4}$, this yields

$$\Gamma_{\text{free}} = hc\,\big[g(r+1) - g(r-1)\big]\Big(\phi_t(y_1)\partial_x\phi_t(y_2) - \partial_x\phi_t(y_1)\phi_t(y_2)\Big) + \Gamma',$$

where $|\Gamma'| \leqslant \Big(hs_n + \frac{ch^2}{2}\Big)\,\|\phi\|_{1,3}^2$. By Lemma 6.3, $|g(r+1) - g(r-1)| \leqslant \frac{1}{2}\mathbb{1}_{|r|<2m}$. Moreover, the function $(y_1, y_2) \mapsto \phi_t(y_1)\partial_x\phi_t(y_2) - \partial_x\phi_t(y_1)\phi_t(y_2)$ vanishes on the diagonal, hence it is bounded by $2\|\phi\|_{1,3}^2|y_2 - y_1| = 2\|\phi\|_{1,3}^2|r|h$. Therefore,

$$|\Gamma_{\text{free}}| \leqslant \Big(2cmh^2 + hs_n + \frac{ch^2}{2}\Big)\|\phi\|_{1,3}^2,$$

and, since $s_n \leqslant c\,\theta$ and $h = n^{-1/2}$, the contribution of Γ_{free} to $|\mathsf{R}_n(t)|$ is at most

$$\Big(3mn^{-1/2} + n^{-1/4}\Big)\,\|\phi\|_{1,3}^2\,\mathsf{W}_n(t)^2.$$

Let us stress that the product structure of the function Φ_t is important here: the term of order h in Γ_{free} involves an antisymmetric function of (y_1, y_2), which is small because $g(r+1) - g(r-1)$ vanishes unless y_1 and y_2 are at distance at most $2mh$.

When $x_2 = x_1 + 1$, the two additional jumps lead to (x_2, x_2) and (x_1, x_1), at rates e^θ and $e^{-\theta}$. For both of them, the increment of G is $g(0) - g(1) = g(0)$ and the increment of Φ_t is bounded by $h\|\phi\|_{1,3}^2$. The case $x_2 = x_1 - 1$ is similar. Hence

$$|\Gamma_{\text{incl}}| \leqslant 2c\,g(0)\,h\,\|\phi\|_{1,3}^2\,\mathbb{1}_{|x_1 - x_2| = 1},$$

and since $g(0) \leqslant C/m$, the contribution of Γ_{incl} to $|\mathsf{R}_n(t)|$ is at most $\frac{2C}{m}\|\phi\|_{1,3}^2\,\mathsf{W}_n(t)^2$. This proves (6.14).

The martingale. By (3.22) and (3.11),

$$d\langle N_n\rangle_t = 4n\sum_{x\in\mathbb{Z}} \mathsf{z}_n(t, x)\mathsf{z}_n(t, x+1)\,\mathsf{D}_t(x)^2\,dt,$$

where

$$\mathsf{D}_t(x) = \sum_{x'\in\mathbb{Z}} \mathsf{z}_n(t, x')\Big(e^{\theta/2}f_t(x+1, x') - e^{-\theta/2}f_t(x, x')\Big).$$

Writing $y = y_n(t, x)$ and $y' = y_n(t, x')$, we have

$$e^{\theta/2}f_t(x+1, x') - e^{-\theta/2}f_t(x, x') = \frac{2}{cn^{1/2}}\phi_t(y')\Big[e^{\theta/2}g(x' - x - 1)\phi_t(y+h) - e^{-\theta/2}g(x' - x)\phi_t(y)\Big].$$

We decompose the term between brackets as

$$(e^{\theta/2}-e^{-\theta/2})g(x'-x-1)\phi_t(y+h)+e^{-\theta/2}(g(x'-x-1)-g(x'-x))\phi_t(y+h) \\ +e^{-\theta/2}g(x'-x)(\phi_t(y+h)-\phi_t(y)).$$

Since $(e^{\theta/2}-e^{-\theta/2})^2=\lambda_n$ and $h\leqslant\lambda_n^{1/2}$, the bounds of Lemma 6.3 show that it is at most $(m\lambda_n^{1/2}+\frac{1}{4})\|\phi\|_{1,3}$ in absolute value. Hence $|\mathsf{D}_t(x)|\leqslant 2n^{-1/2}(m\lambda_n^{1/2}+\frac{1}{4})\|\phi\|_{1,3}^2\,\mathsf{W}_n(t)$, and

$$d\langle N_n\rangle_t\leqslant 16\Big(m\lambda_n^{1/2}+\frac{1}{4}\Big)^2\|\phi\|_{1,3}^4\,\mathsf{W}_n(t)^2\sum_{x\in\mathbb{Z}}\mathsf{z}_n(t,x)\mathsf{z}_n(t,x+1)\,dt.$$

By (3.16), $\sum_x \mathsf{z}_n(t,x)\mathsf{z}_n(t,x+1)dt=(n\lambda_n)^{-1}d\langle\mathsf{W}_n\rangle_t$, and $n\lambda_n\geqslant n^{1/2}$ by (3.2). This yields (6.15).

Finally, when $\phi\equiv 1$, the function Φ_t is constant, so that both terms in (6.16) vanish, and $\mathsf{F}_n(t)\geqslant 0$ because $g\geqslant 0$. □

6.3. **Proof of Proposition 6.1.** We apply Lemma 6.4 with $\phi\equiv 1$. In this case $\mathsf{R}_n\equiv 0$ and $\mathsf{F}_n\geqslant 0$. Since N_n is a square-integrable martingale, taking expectations in (6.12) at time T yields

$$\mathbb{E}\left[\int_0^T\mathsf{B}_n(s;1)\,ds\right]=\mathbb{E}\left[\int_0^T\mathsf{E}_n(s)\,ds\right]+\mathsf{F}_n(0)-\mathbb{E}\left[\mathsf{F}_n(T)\right]\leqslant\mathbb{E}\left[\int_0^T\mathsf{E}_n(s)\,ds\right]+\mathsf{F}_n(0).$$

Since $\mathsf{Z}_n(0)=\delta_0$, we have $\mathsf{F}_n(0)=\frac{2}{c_n n^{1/2}}g_m(0)\leqslant 1$, and the first term is bounded by 2^{1+16T} by Proposition 4.3. Hence $\mathbb{E}[\int_0^T\mathsf{B}_n(s;1)ds]\leqslant 2^{1+16T}+1$, and we conclude using (6.11) and (6.6).

6.4. **Proof of Proposition 6.2.** Let $\varepsilon\in(0,1)$. For $n\geqslant\varepsilon^{-2}$, let $m=\lfloor\varepsilon n^{1/2}\rfloor\geqslant 1$ and $\varepsilon_n=mn^{-1/2}$, so that $\varepsilon-n^{-1/2}<\varepsilon_n\leqslant\varepsilon$. For $K>1$, let $\tau_K=\inf\{t\geqslant 0:\mathsf{W}_n(t)\geqslant K\}\wedge T$. Since W_n is a nonnegative martingale with $\mathsf{W}_n(0)=1$, Doob's maximal inequality shows that

$$\mathbb{P}\left(\tau_K<T\right)\leqslant\frac{1}{K}. \tag{6.17}$$

By Lemma 6.4, for all $t\in[0,T]$,

$$\int_0^{t\wedge\tau_K}\left(\mathsf{E}_n(s;\phi_s)-\mathsf{B}_n(s;\phi_s)\right)ds=\mathsf{F}_n(t\wedge\tau_K)-\mathsf{F}_n(0)-\int_0^{t\wedge\tau_K}\mathsf{R}_n(s)\,ds-N_n(t\wedge\tau_K).$$

On the time interval $[0,\tau_K]$, we have $\mathsf{W}_n\leqslant K$. Hence, by (6.13) and (6.14), and since $mn^{-1/2}\leqslant\varepsilon$,

$$\sup_{t\leqslant T}\left|\mathsf{F}_n(t\wedge\tau_K)-\mathsf{F}_n(0)-\int_0^{t\wedge\tau_K}\mathsf{R}_n(s)\,ds\right|\leqslant C(1+T)\left(\varepsilon+\frac{1}{n^{1/4}}+\frac{1}{m}\right)\|\phi\|_{1,3}^2K^2.$$

Moreover, by Doob's inequality, (6.15) and the fact that $\mathbb{E}[\langle\mathsf{W}_n\rangle_{\tau_K}]=\mathbb{E}[\mathsf{W}_n(\tau_K)^2]-1\leqslant K^2$,

$$\mathbb{E}\Big[\sup_{t\leqslant T}N_n(t\wedge\tau_K)^2\Big]\leqslant 4\,\mathbb{E}\left[\langle N_n\rangle_{\tau_K}\right]\leqslant C\left(\varepsilon^2+\frac{1}{n^{1/2}}\right)\|\phi\|_{1,3}^4K^4.$$

It remains to compare $\mathsf{B}_n(s;\phi_s)$ with $\|(\phi_s\mathsf{Z}_n(s))*\rho_\varepsilon\|_2^2$. By (6.11) and (6.1),

$$\left|\mathsf{B}_n(s;\phi_s)-\|(\phi_s\mathsf{Z}_n(s))*\rho_\varepsilon\|_2^2\right|\leqslant\left(\left|\frac{1}{b_m}-1\right|\,\|\kappa_{\varepsilon_n}\|_\infty+\|\kappa_{\varepsilon_n}-\kappa_\varepsilon\|_\infty\right)\|\phi\|_{1,3}^2\,\mathsf{W}_n(s)^2.$$

As n goes to infinity with ε fixed, m goes to infinity and ε_n converges to ε, so that the quantity between parentheses, which is deterministic, goes to zero. Collecting these estimates, and

using Markov's inequality together with (6.17), we obtain that for all $\eta > 0$, all $\varepsilon \in (0,1)$ and all $K > 1$ such that $C(1+T)\varepsilon\|\phi\|_{1,3}^2 K^2 < \eta/4$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left| \int_0^t \left(\mathsf{E}_n(s;\phi_s) - \|(\phi_s \mathsf{Z}_n(s)) * \rho_\varepsilon\|_2^2 \right) ds \right| > \eta \right) \leqslant \frac{1}{K} + \frac{C\varepsilon^2 \|\phi\|_{1,3}^4 K^4}{\eta^2}.$$

Letting ε go to zero for a fixed K, and then K go to infinity, we obtain (6.3).

7. Identification of the limit

In this section, we complete the proof of Theorem 1.2 in the case $L = \infty$. We fix $T > 0$. By Proposition 5.1, the family of the laws of $(\mathsf{Z}_n(t))_{t\in[0,T]}$ is tight in $C([0,T], \mathcal{M}_+(\mathbb{R}))$. We first show that all the limit points satisfy a martingale problem (Section 7.1), then that this martingale problem can be reformulated as a mild form of the stochastic heat equation (Section 7.2), which has a unique solution (Section 7.3).

7.1. **Properties of the limit points.** Let $(n_k)_{k\geqslant 1}$ be a sequence going to infinity such that Z_{n_k} converges in distribution in $C([0,T], \mathcal{M}_+(\mathbb{R}))$. Since this space is Polish, Skorokhod's representation theorem [Bil99, Theorem 6.7] allows us to assume that the processes Z_{n_k} and their limit $\mathcal{Z}$ are defined on the same probability space, and that almost surely, Z_{n_k} converges to $\mathcal{Z}$ in $C([0,T], \mathcal{M}_+(\mathbb{R}))$. This changes the probability space on which Z_n is defined, but all the quantities introduced in the previous sections, in particular W_n, E_n, M_n^ϕ and $\langle M_n^\phi \rangle$, are measurable functions of the trajectory $(\mathsf{Z}_n(t))_{t\in[0,T]}$, because $\mathsf{z}_n(t,x)$ is the mass of the atom of $\mathsf{Z}_n(t)$ located at $y_n(t,x)$. Hence, all the estimates of the previous sections remain valid, and the processes M_n^ϕ and $(M_n^\phi)^2 - \langle M_n^\phi \rangle$, which are adapted to the natural filtration of Z_n, are martingales with respect to this filtration. To simplify the notation, we write n instead of n_k. We denote by $(\mathcal{G}_t^0)_{t\in[0,T]}$ the natural filtration of the process $\mathcal{Z}$, that is $\mathcal{G}_t^0 = \sigma\left(\mathcal{Z}(s) : s \leqslant t\right)$, and by $(\mathcal{G}_t)_{t\in[0,T]}$ the smallest filtration containing $(\mathcal{G}_t^0)$ which is right-continuous and complete.

Proposition 7.1. *The process $\mathcal{Z}$ satisfies the following properties.*

(1) $\mathcal{Z}(0) = \delta_0$.

(2) There exists a $(\mathcal{G}_t)$-predictable random field $Z : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}_+$ such that

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} Z(t,x)^2 \, dx \, dt \right] < \infty, \tag{7.1}$$

and such that, almost surely, $\mathcal{Z}(t)(dx) = Z(t,x)\,dx$ for almost every $t \in [0,T]$.

(3) For all $\phi \in C_b^{1,3}([0,T]\times\mathbb{R})$, the process

$$M^\phi(t) := \mathcal{Z}(t;\phi_t) - \phi_0(0) - \int_0^t \mathcal{Z}\Big(s; \partial_s\phi_s + \frac{1}{2}\partial_x^2\phi_s\Big)\, ds, \qquad t \in [0,T], \tag{7.2}$$

is a continuous square-integrable $(\mathcal{G}_t)$-martingale, with quadratic variation

$$\langle M^\phi \rangle_t = \int_0^t \int_{\mathbb{R}} Z(s,x)^2 \phi_s(x)^2 \, dx \, ds. \tag{7.3}$$

Proof. Property (1) is clear since $\mathsf{Z}_n(0) = \delta_0$ for all n.

Proof of (2). Let $\varepsilon > 0$ and $\varepsilon_n = \lfloor \varepsilon n^{1/2} \rfloor n^{-1/2}$, which converges to ε. For all $t \in [0,T]$ and $x \in \mathbb{R}$, the functions $\rho_{\varepsilon_n}(x - \cdot)$ converge uniformly to $\rho_\varepsilon(x - \cdot)$, hence $\mathsf{Z}_n(t) * \rho_{\varepsilon_n}(x)$ converges almost surely to $\mathcal{Z}(t) * \rho_\varepsilon(x)$. By Fatou's lemma and Proposition 6.1,

$$\mathbb{E}\left[\int_0^T \|\mathcal{Z}(t) * \rho_\varepsilon\|_2^2 \, dt\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T \|\mathsf{Z}_n(t) * \rho_{\varepsilon_n}\|_2^2 \, dt\right] \leqslant C 2^{16T}, \tag{7.4}$$

where C does not depend on ε. Applying Fatou's lemma once more,

$$\mathbb{E}\left[\int_0^T \liminf_{k\to\infty} \left\|\mathcal{Z}(t) * \rho_{1/k}\right\|_2^2 \, dt\right] \leqslant C 2^{16T}.$$

Hence, almost surely, for almost every $t \in [0,T]$, the sequence $(\mathcal{Z}(t) * \rho_{1/k})_{k \geqslant 1}$ has a subsequence which is bounded in $L^2(\mathbb{R})$. Since $\mathcal{Z}(t) * \rho_{1/k}(x)dx$ converges weakly to $\mathcal{Z}(t)$ as k goes to infinity, this implies that the measure $\mathcal{Z}(t)$ has a density in $L^2(\mathbb{R})$, whose norm is at most $\liminf_k \|\mathcal{Z}(t) * \rho_{1/k}\|_2$. We define

$$Z(t,x) := \liminf_{k\to\infty} \mathcal{Z}(t) * \rho_{1/k}(x)$$

if this quantity is finite, and $Z(t,x) = 0$ otherwise. This defines a predictable random field, since $(t,x) \mapsto \mathcal{Z}(t) * \rho_{1/k}(x)$ is continuous and adapted. By Lebesgue's differentiation theorem, when $\mathcal{Z}(t)$ has a density, this density coincides with $Z(t,\cdot)$ almost everywhere. This proves (2).

Proof of (3). Let $\phi \in C_b^{1,3}([0,T] \times \mathbb{R})$. By Proposition 3.1(1), $\|\mathcal{A}_n \phi_s - \frac{1}{2}\partial_x^2 \phi_s\|_\infty \leqslant n^{-1/4} \|\phi\|_{1,3}$. Moreover, Z_n converges almost surely to $\mathcal{Z}$ in $C([0,T], \mathcal{M}_+(\mathbb{R}))$. In particular, $\sup_{t \leqslant T} \mathsf{W}_n(t)$ converges to $\sup_{t \leqslant T} \mathcal{Z}(t;1)$, so that the random variable $\sup_n \sup_{t \leqslant T} \mathsf{W}_n(t)$ is almost surely finite. Since $t \mapsto \phi_t$ is continuous for the supremum norm, $\mathsf{Z}_n(t;\phi_t)$ converges to $\mathcal{Z}(t;\phi_t)$ uniformly in $t \in [0,T]$ (indeed, if t_n converges to t, then $\mathsf{Z}_n(t_n)$ converges weakly to $\mathcal{Z}(t)$ and ϕ_{t_n} converges uniformly to ϕ_t, so that both $\mathsf{Z}_n(t_n;\phi_{t_n})$ and $\mathcal{Z}(t_n;\phi_{t_n})$ converge to $\mathcal{Z}(t;\phi_t)$). Furthermore, for all $s \in [0,T]$, the function $\partial_s \phi_s + \frac{1}{2}\partial_x^2 \phi_s$ is bounded and continuous, so that $\mathsf{Z}_n(s; \partial_s \phi_s + \frac{1}{2}\partial_x^2 \phi_s)$ converges to $\mathcal{Z}(s; \partial_s \phi_s + \frac{1}{2}\partial_x^2 \phi_s)$, and these quantities are bounded by $\|\phi\|_{1,3} \sup_n \sup_{t \leqslant T} \mathsf{W}_n(t)$. Using the dominated convergence theorem for the integral over s, we deduce from (3.14) that almost surely,

$$\sup_{t \in [0,T]} \left| M_n^\phi(t) - M^\phi(t) \right| \xrightarrow[n\to\infty]{} 0. \tag{7.5}$$

Let us denote

$$Q_n(t) = \langle M_n^\phi \rangle_t, \qquad Q(t) = \int_0^t \int_{\mathbb{R}} Z(s,x)^2 \phi_s(x)^2 \, dx \, ds, \qquad Q^\varepsilon(t) = \int_0^t \|(\phi_s \mathcal{Z}(s)) * \rho_\varepsilon\|_2^2 \, ds.$$

We claim that

$$\sup_{t \in [0,T]} |Q_n(t) - Q(t)| \xrightarrow[n\to\infty]{} 0 \quad \text{in probability.} \tag{7.6}$$

Indeed, we may write $Q_n - Q$ as the sum of the following four terms.

(1) The first one is $Q_n(t) - a_n \int_0^t \mathsf{E}_n(s;\phi_s) ds$, which converges to 0 in L^1, uniformly in t, by Proposition 3.1(3) and (4.8).
(2) The second one is $a_n \int_0^t \left(\mathsf{E}_n(s;\phi_s) - \|(\phi_s \mathsf{Z}_n(s)) * \rho_\varepsilon\|_2^2\right) ds$, which is controlled by Proposition 6.2.
(3) The third one is $a_n \int_0^t \|(\phi_s \mathsf{Z}_n(s)) * \rho_\varepsilon\|_2^2 ds - Q^\varepsilon(t)$. By (6.1), for a fixed ε, $\|(\phi_s \mathsf{Z}_n(s)) * \rho_\varepsilon\|_2^2$ is the integral of a bounded continuous function on $\mathbb{R}^2$ against $\mathsf{Z}_n(s) \otimes \mathsf{Z}_n(s)$, so that it converges almost surely to $\|(\phi_s \mathcal{Z}(s)) * \rho_\varepsilon\|_2^2$ for all $s \in [0,T]$. Moreover, it is

bounded by $\|\kappa_\varepsilon\|_\infty \|\phi\|_{1,3}^2 \sup_{s\leqslant T} \mathsf{W}_n(s)^2$, and $\sup_{s\leqslant T} \mathsf{W}_n(s)$ converges almost surely to $\sup_{s\leqslant T} \mathcal{Z}(s;1) < \infty$. Since a_n converges to 1, the dominated convergence theorem shows that for a fixed ε, the third term converges almost surely to 0 as n goes to infinity, uniformly in t.

(4) The fourth one is $Q^\varepsilon(t) - Q(t)$, which does not depend on n. For almost every (ω, s), $\mathcal{Z}(s)$ has the density $Z(s,\cdot) \in L^2(\mathbb{R})$, so that $(\phi_s \mathcal{Z}(s)) * \rho_\varepsilon$ converges to $\phi_s Z(s,\cdot)$ in $L^2(\mathbb{R})$ as ε goes to zero, and its norm is at most $\|\phi_s Z(s,\cdot)\|_2$. By (7.1) and the dominated convergence theorem, $\sup_{t\leqslant T} |Q^\varepsilon(t) - Q(t)|$ converges to 0 in L^1 as ε goes to zero.

Letting n go to infinity and then ε go to zero, we obtain (7.6). Up to extracting a further subsequence, we may assume that the convergence in (7.6) holds almost surely.

We now prove that M^ϕ is a martingale. Let $0 \leqslant s < t \leqslant T$ and let

$$\Theta = \vartheta\big(\mathcal{Z}(s_1;\varphi_1), \dots, \mathcal{Z}(s_j;\varphi_j)\big),$$

where $j \geqslant 1$, $s_1, \dots, s_j \in [0,s]$, $\varphi_1, \dots, \varphi_j \in C_b(\mathbb{R})$ and $\vartheta : \mathbb{R}^j \to \mathbb{R}$ is bounded and continuous. Let Θ_n be the same random variable with $\mathcal{Z}$ replaced by Z_n, so that Θ_n is bounded and converges almost surely to Θ. Since M_n^ϕ is a martingale,

$$\mathbb{E}\Big[\Big(M_n^\phi(t) - M_n^\phi(s)\Big)\,\Theta_n\Big] = 0. \tag{7.7}$$

By Proposition 3.1(2) and (4.8), $\mathbb{E}[M_n^\phi(t)^2] = \mathbb{E}[Q_n(t)]$ is bounded uniformly in n, so that the random variables $M_n^\phi(t)$ are uniformly integrable. Using (7.5), we may pass to the limit in (7.7) and obtain that $\mathbb{E}[(M^\phi(t) - M^\phi(s))\Theta] = 0$. Since the random variables Θ of this form are stable under multiplication and generate $\mathcal{G}_s^0$, the process M^ϕ, which is continuous and square-integrable by Fatou's lemma, is a $(\mathcal{G}_t^0)$-martingale. Since it is continuous, it is also a $(\mathcal{G}_t)$-martingale.

It remains to identify its quadratic variation. For $K > 0$, let

$$\sigma_n^K = \inf\{t \in [0,T] : Q_n(t) \geqslant K\}, \qquad \sigma^K = \inf\{t \in [0,T] : Q(t) \geqslant K\},$$

with the convention that $\inf \emptyset = T$. Since Q_n converges uniformly to Q, and since these functions are continuous and nondecreasing, σ_n^K converges to σ^K as soon as $\sigma^K = \lim_{K'\downarrow K} \sigma^{K'}$. The function $K \mapsto \mathbb{E}[\sigma^K]$ is nondecreasing, so that it has at most countably many discontinuities, and when K is not one of them, we have $\sigma^K = \lim_{K'\downarrow K} \sigma^{K'}$ almost surely. For such a K, the random variables $M_n^\phi(t\wedge\sigma_n^K)^2 - Q_n(t\wedge\sigma_n^K)$ converge almost surely to $M^\phi(t\wedge\sigma^K)^2 - Q(t\wedge\sigma^K)$. They are bounded in L^2, since $Q_n(t\wedge\sigma_n^K) \leqslant K$ and, by the Burkholder-Davis-Gundy inequality, $\mathbb{E}[M_n^\phi(t\wedge\sigma_n^K)^4] \leqslant CK^2$. Hence they are uniformly integrable, and we may pass to the limit in the identity

$$\mathbb{E}\Big[\Big(M_n^\phi(t\wedge\sigma_n^K)^2 - Q_n(t\wedge\sigma_n^K) - M_n^\phi(s\wedge\sigma_n^K)^2 + Q_n(s\wedge\sigma_n^K)\Big)\Theta_n\Big] = 0,$$

which holds because σ_n^K is a stopping time for the natural filtration of Z_n. We obtain as above that the process $M^\phi(t\wedge\sigma^K)^2 - Q(t\wedge\sigma^K)$ is a $(\mathcal{G}_t)$-martingale. Since $Q(T) < \infty$ almost surely, we have $\sigma^K = T$ for K large enough, so that $(M^\phi)^2 - Q$ is a local martingale. As Q is continuous, nondecreasing and adapted, this proves (7.3). □

7.2. **Representation with a white noise.** We now reformulate the martingale problem of Proposition 7.1 as a stochastic integral equation. Let us first recall a few facts about stochastic integration with respect to a space-time white noise, referring to [Wal86] for details. Let $(\overline{\mathcal{G}}_t)_{t\in[0,T]}$ be a filtration satisfying the usual conditions. A white noise on $[0,T]\times\mathbb{R}$ with respect to $(\overline{\mathcal{G}}_t)$ is a family $\xi = (\xi_t(A))$, indexed by $t \in [0,T]$ and by the bounded Borel subsets A of $\mathbb{R}$, of random variables forming a centered Gaussian family with covariances

$\mathbb{E}[\xi_t(A)\xi_s(B)] = (t \wedge s)|A \cap B|$, where $|\cdot|$ denotes the Lebesgue measure, and such that for all A, the process $t \mapsto \xi_t(A)$ is a continuous $(\overline{\mathcal{G}}_t)$-martingale. We denote by $\mathcal{H}_T$ the space of random fields $h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ which are predictable with respect to $(\overline{\mathcal{G}}_t)$ and such that

$$\|h\|_{\mathcal{H}_T}^2 := \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} h(s,y)^2 \, dy \, ds\right] < \infty.$$

For $h \in \mathcal{H}_T$, the stochastic integral $\int_0^t \int_{\mathbb{R}} h(s,y)\xi(ds\, dy)$ is a continuous square-integrable martingale with quadratic variation $\int_0^t \int_{\mathbb{R}} h(s,y)^2 dyds$.

Proposition 7.2 (see e.g. [KS88] or [EM90])**.** *Let $\mathcal{Z}$ and Z be as in Proposition 7.1. Possibly after enlarging the probability space, there exist a filtration $(\overline{\mathcal{G}}_t)_{t\in[0,T]}$ containing $(\mathcal{G}_t)_{t\in[0,T]}$ and a white noise ξ on $[0,T]\times\mathbb{R}$ with respect to $(\overline{\mathcal{G}}_t)$ such that for all $f \in C_c^\infty(\mathbb{R})$ and all $t \in (0,T]$, almost surely,*

$$\mathcal{Z}(t;f) = \int_{\mathbb{R}} g_t(x) f(x)\, dx + \int_0^t \int_{\mathbb{R}} (g_{t-s} * f)(y)\, Z(s,y)\, \xi(ds\, dy), \tag{7.8}$$

where g_t is the heat kernel defined in (1.13)*.*

Proof. The argument is classical. We explain the main steps for completeness (this was needed in the Lean formalization of the main result).

Step 1: stochastic integrals with respect to the martingales M^φ. By Proposition 7.1(3) and polarization, for all $\varphi_1, \varphi_2 \in C_b^3(\mathbb{R})$,

$$\langle M^{\varphi_1}, M^{\varphi_2}\rangle_t = \int_0^t \int_{\mathbb{R}} Z(s,x)^2 \varphi_1(x)\varphi_2(x)\, dx\, ds. \tag{7.9}$$

For a simple function of the form $h(s,x) = \sum_{i=1}^k a_i \mathbb{1}_{(s_i,t_i]}(s)\varphi_i(x)$, where $\varphi_i \in C_b^3(\mathbb{R})$ and a_i is a bounded $\mathcal{G}_{s_i}$-measurable random variable, we define

$$h \cdot M(t) := \sum_{i=1}^k a_i \left(M^{\varphi_i}(t \wedge t_i) - M^{\varphi_i}(t \wedge s_i)\right).$$

By (7.9), $h \cdot M$ is a continuous martingale and for two simple functions h, h',

$$\langle h \cdot M, h' \cdot M\rangle_t = \int_0^t \int_{\mathbb{R}} h(s,x) h'(s,x) Z(s,x)^2\, dx\, ds. \tag{7.10}$$

Let $\mathscr{L}_Z^2$ be the space of $(\mathcal{G}_t)$-predictable random fields h such that

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} h(s,x)^2 Z(s,x)^2\, dx\, ds\right] < \infty.$$

Since the measure $Z(s,x)^2\, d\mathbb{P}\, ds\, dx$ is finite by (7.1), simple functions are dense in $\mathscr{L}_Z^2$. Hence the map $h \mapsto h \cdot M$ extends by isometry to $\mathscr{L}_Z^2$, and (7.10) remains valid.

Step 2: construction of the white noise. Enlarging the probability space if necessary, let ξ' be a white noise on $[0,T]\times\mathbb{R}$ which is independent of $\mathcal{Z}$, and let $(\overline{\mathcal{G}}_t)$ be the filtration generated by $(\mathcal{G}_t)$ and ξ', augmented in the usual way. For $t \in [0,T]$ and a bounded Borel set $A \subset \mathbb{R}$, let

$$\xi_t(A) := \left(\frac{\mathbb{1}_{Z>0}}{Z}\mathbb{1}_A\right) \cdot M(t) + \int_0^t \int_A \mathbb{1}_{Z(s,y)=0}\, \xi'(ds\, dy).$$

By (7.10) and the independence of ξ' and $\mathcal{Z}$, the processes $\xi(A)$ are continuous $(\overline{\mathcal{G}}_t)$-martingales with

$$\langle \xi(A), \xi(B)\rangle_t = \int_0^t \int_{A\cap B} \left(\mathbb{1}_{Z(s,y)>0} + \mathbb{1}_{Z(s,y)=0}\right) dy\, ds = t\,|A \cap B|.$$

By Lévy's characterization of Brownian motion, ξ is a white noise with respect to $(\overline{\mathcal{G}}_t)$. By construction, for any $(\mathcal{G}_t)$-predictable random field k of the form $k(s,y) = a\mathbb{1}_{(s_1,s_2]}(s)\mathbb{1}_A(y)$, where a is a bounded $\mathcal{G}_{s_1}$-measurable random variable,

$$\int_0^t \int_{\mathbb{R}} k(s,y)\,\xi(ds\,dy) = \left(\frac{\mathbb{1}_{Z>0}}{Z}\,k\right)\cdot M(t) + \int_0^t \int_{\mathbb{R}} k(s,y)\mathbb{1}_{Z(s,y)=0}\,\xi'(ds\,dy),$$

and since both sides are isometries, this remains true for all $(\mathcal{G}_t)$-predictable k such that $\mathbb{E}[\int_0^T \int_{\mathbb{R}} k^2 dyds] < \infty$. Applying this to $k = hZ$, we obtain that for any $h \in \mathscr{L}^2_Z$,

$$\int_0^t \int_{\mathbb{R}} h(s,y)Z(s,y)\,\xi(ds\,dy) = (h\mathbb{1}_{Z>0})\cdot M(t) = h\cdot M(t), \tag{7.11}$$

where the last equality holds because $(h\mathbb{1}_{Z=0})\cdot M$ has zero quadratic variation.

Step 3: time-dependent test functions. Let $\phi \in C_b^{1,3}([0,T]\times\mathbb{R})$. We claim that

$$M^\phi(t) = \phi\cdot M(t) = \int_0^t \int_{\mathbb{R}} \phi_s(y)Z(s,y)\,\xi(ds\,dy). \tag{7.12}$$

Consider a subdivision $0 = s_0 < s_1 < \cdots < s_N = T$ of mesh δ, and the simple function $\phi^{(N)}(s,x) = \sum_i \mathbb{1}_{(s_i,s_{i+1}]}(s)\phi_{s_i}(x)$. Since M^ϕ depends linearly on ϕ, the difference $M^\phi(t) - \phi^{(N)}\cdot M(t)$ is the sum over i of the increments, between the times $t\wedge s_i$ and $t\wedge s_{i+1}$, of the martingales $M^{\phi-\phi_{s_i}}$ associated to the functions $(s,x)\mapsto \phi_s(x)-\phi_{s_i}(x)$. These increments are orthogonal, and by (7.3),

$$\begin{aligned}\mathbb{E}\left[\left(M^\phi(t) - \phi^{(N)}\cdot M(t)\right)^2\right] &= \sum_i \mathbb{E}\left[\int_{t\wedge s_i}^{t\wedge s_{i+1}} \int_{\mathbb{R}} Z(s,x)^2\left(\phi_s(x)-\phi_{s_i}(x)\right)^2 dx\,ds\right]\\ &\leqslant \delta^2\,\|\phi\|_{1,3}^2\,\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} Z(s,x)^2\,dx\,ds\right].\end{aligned}$$

As δ goes to zero, the right-hand side goes to zero, and $\phi^{(N)}\cdot M(t)$ converges in L^2 to $\phi\cdot M(t)$ by (7.10). This proves the first equality in (7.12), and the second one is (7.11).

Step 4: conclusion. Let $f \in C_c^\infty(\mathbb{R})$ and $t\in(0,T]$. Consider the function ϕ defined by $\phi_s = g_{t-s}*f$ for $s\in[0,t)$ and $\phi_s = f - \frac{s-t}{2}f''$ for $s\in[t,T]$. Since $\partial_x^k\phi_s = g_{t-s}*f^{(k)}$ and $\partial_s\phi_s = -\frac{1}{2}g_{t-s}*f''$ for $s<t$, the function ϕ belongs to $C_b^{1,3}([0,T]\times\mathbb{R})$, and it solves $\partial_s\phi_s + \frac{1}{2}\partial_x^2\phi_s = 0$ on $[0,t]\times\mathbb{R}$. Hence (7.2) and (7.12) yield $\mathcal{Z}(t;f) - \phi_0(0) = \int_0^t\int_{\mathbb{R}}\phi_s(y)Z(s,y)\xi(ds\,dy)$, which is (7.8) since $\phi_0(0) = \int_{\mathbb{R}} g_t(x)f(x)dx$. □

7.3. **Uniqueness and conclusion.** The equation (7.8) is a weak formulation of the mild equation (1.13). The next proposition (again, well-known) shows that it uniquely characterizes Z among the elements of $\mathcal{H}_T$.

Proposition 7.3. *Let ξ be a white noise on $[0,T]\times\mathbb{R}$ with respect to a filtration $(\overline{\mathcal{G}}_t)$. Let $Z^1, Z^2 \in \mathcal{H}_T$ be such that for $i\in\{1,2\}$, for almost every $t\in[0,T]$ and for all $f\in C_c^\infty(\mathbb{R})$, almost surely,*

$$\int_{\mathbb{R}} Z^i(t,x)f(x)\,dx = \int_{\mathbb{R}} g_t(x)f(x)\,dx + \int_0^t\int_{\mathbb{R}} (g_{t-s}*f)(y)\,Z^i(s,y)\,\xi(ds\,dy). \tag{7.13}$$

Then $Z^1 = Z^2$ almost everywhere on $\Omega\times[0,T]\times\mathbb{R}$.

Proof. Let $D = Z^1 - Z^2 \in \mathcal{H}_T$ and $F(t) = \mathbb{E}[\|D(t,\cdot)\|_2^2]$, which is an integrable function on $[0,T]$. Let $(f_j)_{j\geqslant 1}$ be an orthonormal basis of $L^2(\mathbb{R})$ whose elements belong to $C_c^\infty(\mathbb{R})$ (such a basis is obtained by applying the Gram-Schmidt procedure to a countable subset of $C_c^\infty(\mathbb{R})$

which is dense in $L^2(\mathbb{R})$). For almost every t, (7.13) holds almost surely for $i \in \{1,2\}$ and all the functions f_j simultaneously. For such a t, by Parseval's identity and Itô's isometry,

$$F(t) = \sum_{j\geqslant 1} \mathbb{E}\left[\left(\int_{\mathbb{R}} D(t,x) f_j(x)\,dx\right)^2\right] = \sum_{j\geqslant 1} \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} (g_{t-s} * f_j)(y)^2\, D(s,y)^2\,dy\,ds\right]$$
$$= \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \|g_{t-s}\|_2^2\, D(s,y)^2\,dy\,ds\right],$$

where we used that $\sum_j (g_{t-s} * f_j)(y)^2 = \|g_{t-s}(y-\cdot)\|_2^2$, by Parseval's identity again. Since $\|g_r\|_2^2 = (4\pi r)^{-1/2}$, we obtain that for almost every $t \in [0,T]$,

$$F(t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{F(s)}{\sqrt{t-s}}\,ds.$$

Multiplying by e^{-at} for some $a > 0$ and integrating over $[0,T]$, we obtain by Fubini's theorem

$$\int_0^T e^{-at} F(t)\,dt \leqslant \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-ar}}{\sqrt{r}}\,dr \int_0^T e^{-as} F(s)\,ds = \frac{1}{2\sqrt{a}} \int_0^T e^{-as} F(s)\,ds.$$

Choosing $a > 1/4$, this implies that $F = 0$ almost everywhere. □

We may now conclude. Let us recall the known results about the stochastic heat equation that we need. Let ξ be a white noise with respect to a filtration $(\overline{\mathcal{G}}_t)$. By [CD15, Theorem 2.4] (see also [BC95; ACQ11]), there exists a random field $Z^\star(t,x)$, indexed by $(t,x) \in (0,T] \times \mathbb{R}$, which is jointly measurable, adapted to the filtration generated by ξ and continuous in $L^2(\Omega)$, such that for all $(t,x) \in (0,T] \times \mathbb{R}$, almost surely,

$$Z^\star(t,x) = g_t(x) + \int_0^t \int_{\mathbb{R}} g_{t-s}(x-y) Z^\star(s,y)\,\xi(ds\,dy). \tag{7.14}$$

This is the solution of the stochastic heat equation considered in Section 1.4. It has a version which is continuous on $(0,T] \times \mathbb{R}$ [BC95; ACQ11] and positive [Mue91; Mor14]; we always consider this version, which is in particular predictable. Its second moment is explicit [BC95], [CD15, Corollary 2.8]:

$$\mathbb{E}\left[Z^\star(t,x)^2\right] = g_t(x)^2 \left(1 + \sqrt{\pi t}\, e^{t/4} \int_{-\infty}^{\sqrt{t/2}} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du\right). \tag{7.15}$$

Since $\|g_t\|_2^2 = (4\pi t)^{-1/2}$ is integrable on $[0,T]$, (7.15) shows that $Z^\star \in \mathcal{H}_T$. Moreover, multiplying (7.14) by $f(x)$, with $f \in C_c^\infty(\mathbb{R})$, integrating over x and using the stochastic Fubini theorem [Wal86, Theorem 2.6], we see that $Z^\star$ satisfies (7.13) for all $t \in (0,T]$.

Proof of Theorem 1.2 when $L_n = \infty$ for all n. Let $\mathcal{Z}$ be a limit point of the family $(\mathsf{Z}_n)_{n\geqslant 1}$, obtained as in Section 7.1, let Z be its density, given by Proposition 7.1, and let ξ be the white noise given by Proposition 7.2. Let $Z^\star$ be the solution of the stochastic heat equation (7.14) driven by ξ. By Proposition 7.1(2) and Proposition 7.2, Z belongs to $\mathcal{H}_T$ and satisfies (7.13), and so does $Z^\star$. By Proposition 7.3, $Z = Z^\star$ almost everywhere. Hence, almost surely, $\mathcal{Z}(t)(dx) = Z^\star(t,x)dx$ for almost every $t \in [0,T]$. Since $t \mapsto \mathcal{Z}(t)$ is continuous for the weak topology, and since $Z^\star$ is continuous on $(0,T] \times \mathbb{R}$, this holds for all $t \in (0,T]$: indeed, for $\varphi \in C_c(\mathbb{R})$, both $\mathcal{Z}(t;\varphi)$ and $\int_{\mathbb{R}} Z^\star(t,x)\varphi(x)dx$ are continuous functions of $t \in (0,T]$ which coincide for almost every t. Recall also that $\mathcal{Z}(0) = \delta_0$. Consequently, the law of $\mathcal{Z}$ is the law of the process described in the statement of Theorem 1.2, and it does not depend on the choice of the subsequence. Together with the tightness proved in Proposition 5.1, this shows that Z_n converges in distribution to $\mathcal{Z}$ in $C([0,T], \mathcal{M}_+(\mathbb{R}))$. Finally, by (3.8) and (3.9),

$\mathcal{Z}_n(t) = \varrho_n(t)\mathsf{Z}_n(t)$ where ϱ_n is a deterministic function which converges to 1 uniformly on $[0,T]$. Hence $\mathcal{Z}_n$ converges in distribution to the same limit. □

8. Finite systems

In this section we complete the proof of Theorem 1.2, by showing that when $L_n/n^{3/4}$ goes to infinity, the one-particle QSSEP on Λ_{L_n} cannot be distinguished from the one-particle QSSEP on $\mathbb{Z}$ in the scaling window that we consider.

Let $L \geqslant 1$ be an integer. We construct both processes on the same probability space: let $(W^{(x)})_{x\in\mathbb{Z}}$ be independent standard complex $(\mathcal{F}_t)$-Brownian motions, let ψ_t^∞ be the wave function of the one-particle QSSEP on $\mathbb{Z}$ driven by these Brownian motions (Proposition 2.1), and let $\psi_t^L = U_t e_0$ be the wave function of the one-particle QSSEP on Λ_L, where U_t is the unitary matrix defined in (1.4), driven by the Brownian motions $W^{(x)}$ for $-L \leqslant x \leqslant L-1$. We let $P^\infty(t,x) = |\psi_t^\infty(x)|^2$ and $P^L(t,x) = |\psi_t^L(x)|^2$. With the notation of Section 3, let

$$\mathsf{Z}_n^L(t) := \sum_{x\in\Lambda_L} \exp\left(\theta_n x - \frac{n\lambda_n}{2}t\right) P^L\left(\frac{nt}{2}, x\right)\, \delta_{y_n(t,x)},$$

so that the measure $\mathcal{Z}_n(t)$ associated to the QSSEP on Λ_L by (1.14) and (1.15) is $\varrho_n(t)\mathsf{Z}_n^L(t)$, where ϱ_n is defined in (3.8). We keep the notation $\mathsf{Z}_n(t)$ for the measure (3.5) associated to the QSSEP on $\mathbb{Z}$. Finally, we denote by $\|\mu\|_{\mathrm{TV}}$ the total variation norm of a finite signed measure μ on $\mathbb{R}$.

Proposition 8.1. *Let $(L_n)_{n\geqslant 1}$ be a sequence of positive integers such that $L_n/n^{3/4}$ goes to infinity. Then, for all $T > 0$,*

$$\sup_{t\in[0,T]} \left\| \mathsf{Z}_n^{L_n}(t) - \mathsf{Z}_n(t) \right\|_{\mathrm{TV}} \xrightarrow[n\to\infty]{} 0 \qquad \textit{in probability.}$$

The proof relies on the following estimate, which does not involve any scaling.

Lemma 8.2. *Let $(X_s)_{s\geqslant 0}$ be the continuous-time simple random walk on $\mathbb{Z}$ which jumps at rate 1 to each of its neighbours, started from 0 under the probability measure P. For all integers $L \geqslant 1$ and all $S > 0$,*

$$\mathbb{E}\left[\sup_{s\in[0,S]} \Big(\sum_{x\in\Lambda_L} \left| P^\infty(s,x) - P^L(s,x) \right| \Big)^2 \right] \leqslant 4(8+S) \int_0^S \mathsf{P}\left(|X_s| \geqslant L\right) ds. \tag{8.1}$$

Proof. Let $\phi_s \in \mathbb{C}^{\Lambda_L}$ be the restriction of ψ_s^∞ to Λ_L. Let H_t and D be the matrices defined in Section 1.3, so that $dU_s = \mathbf{i}\, dH_s\, U_s - \frac{1}{2} D\, U_s\, ds$. Comparing the equations (2.1) and (1.7) satisfied by ψ^∞ and ψ^L, which only differ at the sites L and $-L$, we see that

$$d\phi_s = \mathbf{i}\, dH_s\, \phi_s - \frac{1}{2} D\, \phi_s\, ds + d\Upsilon_s,$$

where

$$d\Upsilon_s = \left(\mathbf{i}\, \psi_s^\infty(L+1)\, dW_s^{(L)} - \frac{1}{2}\psi_s^\infty(L)\, ds\right) e_L + \left(\mathbf{i}\, \psi_s^\infty(-L-1)\, d\overline{W}_s^{(-L-1)} - \frac{1}{2}\psi_s^\infty(-L)\, ds\right) e_{-L}.$$

Since the Brownian motions $W^{(L)}$ and $W^{(-L-1)}$ are independent of those appearing in H_s, and since $(dH_s)^2 = D\, ds$, Itô's formula yields $d\left(U_s^* \phi_s\right) = U_s^*\, d\Upsilon_s$. As $\phi_0 = \psi_0^L = e_0$ and U_s is

unitary, we obtain

$$\phi_s - \psi_s^L = U_s \int_0^s U_u^* \, d\Upsilon_u, \qquad \text{hence} \qquad \big\|\phi_s - \psi_s^L\big\| = \Big\| \int_0^s U_u^* \, d\Upsilon_u \Big\|, \tag{8.2}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^{\Lambda_L}$. We bound separately the contributions of the martingale part and of the finite variation part of Υ. Since $\|U_u^* e_{\pm L}\| = 1$, Doob's inequality and Itô's isometry yield

$$\mathbb{E}\left[\sup_{s\leqslant S}\Big\| \int_0^s \mathbf{i}\psi_u^\infty(L+1)\, U_u^* e_L \, dW_u^{(L)} + \int_0^s \mathbf{i}\psi_u^\infty(-L-1)\, U_u^* e_{-L} \, d\overline{W}_u^{(-L-1)}\Big\|^2\right] \leqslant 4\int_0^S \mathbb{E}\left[P^\infty(u,L+1) + P^\infty(u,-L-1)\right] du,$$

while by the Cauchy-Schwarz inequality,

$$\sup_{s\leqslant S}\Big\|\frac{1}{2}\int_0^s \left(\psi_u^\infty(L)\, U_u^* e_L + \psi_u^\infty(-L)\, U_u^* e_{-L}\right) du\Big\|^2 \leqslant \frac{S}{2}\int_0^S \left(P^\infty(u,L) + P^\infty(u,-L)\right) du.$$

By Proposition 2.2, $\mathbb{E}[P^\infty(u,x)] = \mathsf{P}(X_u = x)$. Hence, using (8.2),

$$\mathbb{E}\left[\sup_{s\in[0,S]}\big\|\phi_s - \psi_s^L\big\|^2\right] \leqslant (8+S)\int_0^S \mathsf{P}\left(|X_u| \geqslant L\right) du.$$

Finally, for all $x \in \Lambda_L$, we have $|P^\infty(s,x) - P^L(s,x)| \leqslant |\phi_s(x) - \psi_s^L(x)|\left(|\phi_s(x)| + |\psi_s^L(x)|\right)$, so that by the Cauchy-Schwarz inequality,

$$\sum_{x\in\Lambda_L}\left|P^\infty(s,x) - P^L(s,x)\right| \leqslant \big\|\phi_s - \psi_s^L\big\|\left(\|\phi_s\| + \|\psi_s^L\|\right) \leqslant 2\big\|\phi_s - \psi_s^L\big\|,$$

which concludes the proof of (8.1). □

Proof of Proposition 8.1. The measures $\mathsf{Z}_n^{L_n}(t)$ and $\mathsf{Z}_n(t)$ are supported on the points $y_n(t,x)$, $x \in \mathbb{Z}$. Hence, for all $t \in [0,T]$,

$$\big\|\mathsf{Z}_n^{L_n}(t) - \mathsf{Z}_n(t)\big\|_{\mathrm{TV}} \leqslant e^{\theta_n L_n}\sum_{x\in\Lambda_{L_n}}\left|P^{L_n}\Big(\frac{nt}{2},x\Big) - P^\infty\Big(\frac{nt}{2},x\Big)\right| + \sum_{x\in\mathbb{Z}\setminus\Lambda_{L_n}} \mathsf{z}_n(t,x). \tag{8.3}$$

Let us consider the first term. Let $r_n = \min\{1, L_n/(nT)\}$. For all $s \leqslant nT/2$, by Markov's inequality and the symmetry of the random walk,

$$\mathsf{P}\left(|X_s| \geqslant L_n\right) \leqslant 2e^{-r_n L_n}\,\mathsf{E}\left[e^{r_n X_s}\right] = 2\exp\big(-r_n L_n + 2s(\cosh r_n - 1)\big) \leqslant 2\exp\Big(-\frac{r_n L_n}{3}\Big),$$

where we used that $\cosh r - 1 \leqslant \frac{2}{3}r^2$ for $r \in [0,1]$, so that $2s(\cosh r_n - 1) \leqslant \frac{2}{3}nTr_n^2 \leqslant \frac{2}{3}r_nL_n$. By Lemma 8.2 applied with $S = nT/2$, the second moment of the supremum over $t \in [0,T]$ of the first term in (8.3) is at most

$$4nT\Big(8 + \frac{nT}{2}\Big)\exp\Big(2\theta_n L_n - \frac{r_n L_n}{3}\Big).$$

Since $r_n/\theta_n = \min\{n^{1/4}, L_n n^{-3/4}/T\}$ goes to infinity, we have $2\theta_n \leqslant r_n/6$ for n large enough, and $r_n L_n = \min\{L_n, L_n^2/(nT)\} \geqslant \min\{n^{3/4}, n^{1/2}/T\}$ for n large enough. Hence the quantity above goes to zero as n goes to infinity.

We now consider the second term in (8.3). Let $\ell_n = n^{-1/2}(L_n - n^{3/4}T)$, which goes to infinity. When $\ell_n > 0$, for all $t \in [0,T]$ and all $x \in \mathbb{Z} \setminus \Lambda_{L_n}$, we have $|y_n(t,x)| \geqslant \ell_n$, so that

$$\sup_{t\in[0,T]} \sum_{x\in\mathbb{Z}\setminus\Lambda_{L_n}} \mathsf{z}_n(t,x) \leqslant \frac{1}{\cosh(\ell_n)} \sup_{t\in[0,T]} \mathsf{Z}_n(t;\cosh),$$

which goes to zero in probability by Lemma 5.2. □

Proof of Theorem 1.2. Let $(L_n)_{n\geqslant 1}$ be as in the statement of the theorem. We have already proved, at the end of Section 7, that the processes Z_n and $\varrho_n\mathsf{Z}_n$ associated to the QSSEP on $\mathbb{Z}$ converge in distribution to $\mathcal{Z}$ in $C([0,T],\mathcal{M}_+(\mathbb{R}))$. This proves (1.16) along the values of n such that $L_n = \infty$, and we may now assume that L_n is finite for all n. Since the law of the process $\mathcal{Z}_n$ only depends on the law of P^{L_n}, we may assume that the QSSEP on Λ_{L_n} is coupled to the QSSEP on $\mathbb{Z}$ as above. The topology of weak convergence on $\mathcal{M}_+(\mathbb{R})$ is induced by the bounded Lipschitz distance (see e.g. [Dud02, Section 11.3])

$$d(\mu,\nu) = \sup\Big\{ |\mu(f) - \nu(f)| \,:\, f:\mathbb{R}\to\mathbb{R},\ \|f\|_\infty \leqslant 1,\ |f(x)-f(y)| \leqslant |x-y| \text{ for all } x,y\in\mathbb{R}\Big\},$$

which satisfies $d(\mu,\nu) \leqslant \|\mu-\nu\|_{\mathrm{TV}}$. The processes $\mathcal{Z}_n = \varrho_n\mathsf{Z}_n^{L_n}$ have continuous trajectories in $\mathcal{M}_+(\mathbb{R})$, and by Proposition 8.1 and (3.9),

$$\sup_{t\in[0,T]} d\left(\mathcal{Z}_n(t), \varrho_n(t)\mathsf{Z}_n(t)\right) \leqslant e^{Tn^{-1/2}} \sup_{t\in[0,T]} \left\| \mathsf{Z}_n^{L_n}(t) - \mathsf{Z}_n(t) \right\|_{\mathrm{TV}} \xrightarrow[n\to\infty]{} 0 \quad \text{in probability.}$$

Since $\varrho_n\mathsf{Z}_n$ converges in distribution to $\mathcal{Z}$ in $C([0,T],\mathcal{M}_+(\mathbb{R}))$, so does $\mathcal{Z}_n$ by [Bil99, Theorem 3.1]. This concludes the proof of Theorem 1.2. □

Appendix A. Connection to the Brownian energy process and Markov duality

The stochastic differential equations satisfied by $P(t,\cdot)$ are those of a well-studied system of interacting diffusions, the Brownian energy process, introduced in [GKRV09]. The two identities of Lemma 2.3 are the first two instances of a Markov duality between this process and a system of interacting random walks, the symmetric inclusion process. The purpose of this appendix is to explain this connection (Sections A.1 and A.2), and to state the duality for an arbitrary number k of particles (Section A.3). The proof of Theorem 1.2 only uses the cases $k = 1$ and $k = 2$, in the form of Lemma 2.3, and does not rely on the results of this appendix.

As in Section 2, we let $L \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$, with the convention that $\Lambda_\infty = \mathbb{Z}$. A bond of Λ_L is a pair $(x, x+1)$ such that $x \in \Lambda_L$ and $x+1 \in \Lambda_L$.

A.1. **The Brownian energy process.** We denote by $p = (p(x))_{x\in\Lambda_L}$ a generic element of $[0,\infty)^{\Lambda_L}$ and by ∂_x the partial derivative with respect to the coordinate $p(x)$. To each bond $(x,x+1)$ of Λ_L we associate the differential operator

$$\mathcal{L}_{x,x+1} = p(x)p(x+1)\left(\partial_x - \partial_{x+1}\right)^2 - \left(p(x) - p(x+1)\right)\left(\partial_x - \partial_{x+1}\right), \tag{A.1}$$

and we let $\mathcal{L} = \sum_x \mathcal{L}_{x,x+1}$, where the sum runs over all the bonds of Λ_L. By Proposition 2.2, the processes $P(\cdot,x)$ are continuous semimartingales whose drifts and brackets are explicit functions of $P(t,\cdot)$: the drift of $P(t,x)$ is $\Delta_L P(t,x)$, and the brackets are computed at the

beginning of the proof of Lemma 2.3. Hence, by Itô's formula, for every function F of class C^2 which depends on finitely many coordinates of p, the process

$$F\big(P(t,\cdot)\big) - \int_0^t \mathcal{L}F\big(P(s,\cdot)\big)\,ds, \qquad t \geqslant 0, \tag{A.2}$$

is a martingale. In other words, $P(t,\cdot)$ is a solution of the martingale problem associated with the operator $\mathcal{L}$ (when $L < \infty$, this solution is unique in law and is a Markov process, see Section B.2). Moreover, by Lévy's characterization, and possibly after enlarging the probability space, one may write $dJ_t(x) = \sqrt{2P(t,x)P(t,x+1)}\,dB_t(x)$, where the $B(x)$ are independent standard Brownian motions, and (2.4) becomes

$$dP(t,x) = \Delta_L P(t,x)\,dt + \sqrt{2P(t,x-1)P(t,x)}\,dB_t(x-1) - \sqrt{2P(t,x)P(t,x+1)}\,dB_t(x),$$

where the terms associated to pairs $(x-1,x)$ or $(x,x+1)$ which are not bonds of Λ_L are omitted. Thus, the random probability distribution $P(t,\cdot)$ has an autonomous evolution, which can be described without any reference to the phases of the wave function. It is crucial here that $G_t = \psi_t\psi_t^*$ has rank one, so that $|G_t(x,x+1)|^2 = P(t,x)P(t,x+1)$: for more general initial conditions G_0, the diagonal of the two-point function does not satisfy a closed equation.

The operator $\mathcal{L}$ is the generator of the Brownian energy process, which was introduced in [GKRV09] as a model of energy transport: $p(x)$ is interpreted as the energy of the site x, and $\mathcal{L}_{x,x+1}$ describes an exchange of energy between the sites x and $x+1$ which preserves $p(x)+p(x+1)$. More generally, the Brownian energy process with parameter $\alpha > 0$ is obtained by replacing (A.1) by

$$\mathcal{L}^{(\alpha)}_{x,x+1} = p(x)p(x+1)\,(\partial_x - \partial_{x+1})^2 - \alpha(p(x) - p(x+1))\,(\partial_x - \partial_{x+1})\,, \tag{A.3}$$

so that the process $P(t,\cdot)$ corresponds to $\alpha = 1$. This is the process denoted by BEP(α) in [CGGR13], where the parameter α is written $2k$. The process BEP(m) of [GKRV09] corresponds to $\alpha = m/2$, up to a multiplication of time by 4; it describes the energies in a chain where each site carries m real variables, the energy of a site being the sum of their squares. This is consistent with the value $\alpha = 1$ found here, since $P(t,x)$ is the sum of the squares of the real part and of the imaginary part of $\psi_t(x)$.

Equilibrium measures. Let $(x,x+1)$ be a bond and $s = p(x)+p(x+1)$. Since $(\partial_x - \partial_{x+1})\,s = 0$, a direct computation shows that for any $\theta > 0$,

$$\mathcal{L}_{x,x+1} = e^{s/\theta}\,(\partial_x - \partial_{x+1})\,\Big[p(x)p(x+1)\,e^{-s/\theta}\,(\partial_x - \partial_{x+1})\,\Big]. \tag{A.4}$$

Integrating by parts, we see that the operator $\mathcal{L}_{x,x+1}$ is symmetric with respect to the product of two exponential distributions with mean θ. Consequently, when $L < \infty$, the product measures whose marginals are exponential distributions with a common mean are reversible for the Brownian energy process (see [GKRV09, Section 6.1] and [CGGR13, Proposition 3.1]). For the process with parameter α, exponential distributions are replaced by Gamma distributions with shape parameter α.

The dynamics preserves the total mass $\sum_x p(x)$, which is equal to 1 for the process $P(t,\cdot)$. Hence, when $L < \infty$, the equilibrium measure which is relevant for us is obtained by conditioning such a product measure on the event that the total mass is equal to 1: this is the uniform distribution on the simplex

$$\Big\{p \in [0,\infty)^{\Lambda_L} \;:\; \sum_{x\in\Lambda_L} p(x) = 1\Big\}. \tag{A.5}$$

Indeed, $P(t,\cdot)$ converges in distribution to the uniform distribution on (A.5) as t goes to infinity. This follows from the convergence of the law of G_t to that of UG_0U^* recalled in Section 1.3 (see [BBJ19, Proposition 1], where the model is defined on a ring; the same argument applies on Λ_L), since the squared moduli of the entries of a column of a Haar distributed unitary matrix are uniformly distributed on the simplex.

When $L=\infty$, there is no such equilibrium measure, since the distribution $P(t,\cdot)$ spreads over a region of size $t^{1/2}$. However, the mass contained in a large box varies slowly, which leads to the local equilibrium picture of Remark 1.3.

Relation to the KMP model. The KMP model of Section 1.5.3 is the instantaneous thermalization of the Brownian energy process [GKRV09, Section 6.3], [CGGR13, Section 5]. This means that, each time the Poisson process attached to a bond $(x,x+1)$ rings, the pair $(p(x),p(x+1))$ is replaced by a sample of the stationary distribution of the diffusion with generator $\mathcal{L}_{x,x+1}$ started from this pair. By (A.4), this is the distribution of two independent exponential random variables conditioned on their sum being equal to $s=p(x)+p(x+1)$, that is, the distribution of $(Bs,(1-B)s)$ where B is uniformly distributed on $[0,1]$. This is exactly the redistribution rule (1.19). For the Brownian energy process with parameter α, the same procedure leads to a redistribution rule where B follows the Beta distribution with parameters (α,α).

A.2. **The symmetric inclusion process.** Let $k\geqslant 1$ be an integer. The symmetric inclusion process with k particles on Λ_L is the continuous-time Markov chain $\vec{X}(t)=(X_1(t),\dots,X_k(t))$ on Λ_L^k defined as follows: each particle jumps to each of the two neighbours of its position at rate 1 and, in addition, it jumps to a neighbouring site at rate 1 for each of the other particles which are located at this site. In other words, when the configuration is $\vec{x}=(x_1,\dots,x_k)$, the particle i jumps from x_i to $x_i+\sigma$, where $\sigma\in\{-1,1\}$ and $x_i+\sigma\in\Lambda_L$, at rate

$$c_{i,\sigma}(\vec{x})=1+\#\left\{j\neq i\,:\,x_j=x_i+\sigma\right\}. \tag{A.6}$$

The generator of this Markov chain acts on functions $f:\Lambda_L^k\to\mathbb{R}$ by

$$\mathcal{Q}_k f(\vec{x})=\sum_{i=1}^{k}\sum_{\sigma}c_{i,\sigma}(\vec{x})\Big(f(\vec{x}^{\,i,\sigma})-f(\vec{x})\Big), \tag{A.7}$$

where $\vec{x}^{\,i,\sigma}$ denotes the configuration obtained from $\vec{x}$ by replacing x_i with $x_i+\sigma$, and where the second sum runs over the $\sigma\in\{-1,1\}$ such that $x_i+\sigma\in\Lambda_L$. Since the rates (A.6) are at most k, the operator $\mathcal{Q}_k$ is bounded on $\ell^\infty(\Lambda_L^k)$, with $\|\mathcal{Q}_k f\|_\infty\leqslant 4k^2\|f\|_\infty$.

Thus, the particles attract each other: a particle is more likely to jump to a site which is already occupied. This process first appeared in [GKR07], as the dual of a model of heat conduction, and it was named the symmetric inclusion process in [GRV10], by analogy with the symmetric exclusion process. When $k=1$, it is the simple random walk with generator Δ_L. When $k=2$ and $L=\infty$, the restriction of $\mathcal{Q}_2$ to symmetric functions is the operator $\mathcal{Q}$ defined in (2.6). If one forgets the labels of the particles and only records the number $\eta_z(\vec{x})=\#\{i\,:\,x_i=z\}$ of particles at each site z, then a particle moves from a site x to a neighbouring site y at rate $\eta_x(1+\eta_y)$; this is the process denoted by SIP(1) in [CGGR13].

The symmetric inclusion process is reversible with respect to the measure

$$\pi_k(\vec{x})=\prod_{z\in\Lambda_L}\eta_z(\vec{x})!\,,\qquad \vec{x}\in\Lambda_L^k, \tag{A.8}$$

which is the form taken, for labelled particles, by the reversible product measures of [CGGR13, Proposition 3.1]. Indeed, if the sites x_i and $x_i+\sigma$ contain respectively a and b particles in

the configuration $\vec{x}$, then $c_{i,\sigma}(\vec{x}) = 1 + b$, while the rate of the reverse jump is $c_{i,-\sigma}(\vec{x}^{\,i,\sigma}) = a$ and $\pi_k(\vec{x}^{\,i,\sigma}) = \frac{b+1}{a}\,\pi_k(\vec{x})$. Hence

$$\pi_k(\vec{x})\, c_{i,\sigma}(\vec{x}) = \pi_k(\vec{x}^{\,i,\sigma})\, c_{i,-\sigma}(\vec{x}^{\,i,\sigma}). \tag{A.9}$$

For $k = 2$, we have $\pi_2(x_1, x_2) = 1 + \mathbb{1}_{x_1 = x_2}$, which is the measure π of (4.1), seen as a function of $x_2 - x_1$.

A.3. **Markov duality.** In this section, we consider a slightly more general initial condition than in the rest of the paper: we let ψ_0 be a deterministic unit vector of $\ell^2(\Lambda_L)$ with finite support, and we let $P(t, x) = |\psi_t(x)|^2$, where $\psi_t = U_t\psi_0$ when $L < \infty$, and where $(\psi_t)_{t\geqslant 0}$ is the solution of (2.1) with $\mathbb{1}_{x=0}$ replaced by $\psi_0(x)$ when $L = \infty$. The results of Section 2 remain valid, with the same proofs, provided that the initial values are modified accordingly: the terms $\mathbb{1}_{x=0}$, $\alpha_0(0)$ and $f_0(0, 0)$ appearing in (2.4), (2.7) and (2.8) are replaced by the values at time 0 of the corresponding left-hand sides, and p_t becomes the distribution at time t of the random walk with generator Δ_L started from the distribution $P(0, \cdot)$. For $p \in [0, \infty)^{\Lambda_L}$ and $\vec{x} \in \Lambda_L^k$, we let

$$p^{\vec{x}} = \prod_{i=1}^{k} p(x_i) \qquad \text{and} \qquad D(p, \vec{x}) = \frac{p^{\vec{x}}}{\pi_k(\vec{x})} = \prod_{z\in\Lambda_L} \frac{p(z)^{\eta_z(\vec{x})}}{\eta_z(\vec{x})!}. \tag{A.10}$$

The duality between the Brownian energy process and the symmetric inclusion process rests on the following algebraic identity. It is a reformulation, for labelled particles, of the relation between the generators of the two processes established in [GKRV09; CGGR13]; we include its short proof in the present notation.

Lemma A.1 ([GKRV09, Theorem 6.2], [CGGR13, Theorem 4.1])**.** *Let $k \geqslant 1$ and let $f : \Lambda_L^k \to \mathbb{R}$ be a function with finite support. Then, for all $p \in [0, \infty)^{\Lambda_L}$,*

$$\mathcal{L}\Big[\sum_{\vec{x}\in\Lambda_L^k} f(\vec{x})\, p^{\vec{x}}\Big] = \sum_{\vec{x}\in\Lambda_L^k} \mathcal{Q}_k f(\vec{x})\, p^{\vec{x}}. \tag{A.11}$$

Proof. Let us fix a bond $(x, x+1)$ and write $u = p(x)$, $v = p(x+1)$. For all integers $a, b \geqslant 0$, we have $(\partial_x - \partial_{x+1})\, u^a v^b = a u^{a-1} v^b - b u^a v^{b-1}$, and a direct computation yields

$$\mathcal{L}_{x,x+1}\left(u^a v^b\right) = a^2\, u^{a-1} v^{b+1} + b^2\, u^{a+1} v^{b-1} - \big(a(1+b) + b(1+a)\big)\, u^a v^b. \tag{A.12}$$

We apply this identity to the monomial $p^{\vec{x}}$, in which $a = \eta_x(\vec{x})$ and $b = \eta_{x+1}(\vec{x})$. For each of the a indices i such that $x_i = x$, the monomial obtained from $p^{\vec{x}}$ by replacing one factor u by a factor v is $p^{\vec{x}^{\,i,1}}$, and $c_{i,1}(\vec{x}) = 1 + b$. Similarly, for each of the b indices i such that $x_i = x+1$, the monomial obtained by replacing one factor v by a factor u is $p^{\vec{x}^{\,i,-1}}$, and $c_{i,-1}(\vec{x}) = 1 + a$. Hence (A.12) reads

$$\mathcal{L}_{x,x+1}\, p^{\vec{x}} = \sum_{i:x_i=x} \Big(\eta_{x_i}(\vec{x})\, p^{\vec{x}^{\,i,1}} - c_{i,1}(\vec{x})\, p^{\vec{x}}\Big) + \sum_{i:x_i=x+1} \Big(\eta_{x_i}(\vec{x})\, p^{\vec{x}^{\,i,-1}} - c_{i,-1}(\vec{x})\, p^{\vec{x}}\Big).$$

Summing over all the bonds of Λ_L, we obtain

$$\mathcal{L}\, p^{\vec{x}} = \sum_{i=1}^{k} \sum_{\sigma} \Big(\eta_{x_i}(\vec{x})\, p^{\vec{x}^{\,i,\sigma}} - c_{i,\sigma}(\vec{x})\, p^{\vec{x}}\Big), \tag{A.13}$$

where the second sum runs over the $\sigma \in \{-1,1\}$ such that $x_i + \sigma \in \Lambda_L$. We now multiply (A.13) by $f(\vec{x})$ and sum over $\vec{x}$. In the first term, we use the change of variables $\vec{y} = \vec{x}^{\,i,\sigma}$, for which $\vec{x} = \vec{y}^{\,i,-\sigma}$ and

$$\eta_{x_i}(\vec{x}) = 1 + \#\{j \neq i \,:\, y_j = y_i - \sigma\} = c_{i,-\sigma}(\vec{y}).$$

This yields

$$\sum_{\vec{x}} f(\vec{x})\, \mathcal{L}\, p^{\vec{x}} = \sum_{\vec{y}} p^{\vec{y}} \sum_{i=1}^{k} \sum_{\sigma} c_{i,-\sigma}(\vec{y}) f(\vec{y}^{\,i,-\sigma}) - \sum_{\vec{x}} p^{\vec{x}} \sum_{i=1}^{k} \sum_{\sigma} c_{i,\sigma}(\vec{x}) f(\vec{x}),$$

which is (A.11) in view of the definition (A.7) of $\mathcal{Q}_k$. □

Proposition A.2 (Markov duality, [GKRV09, Theorem 6.2], [CGGR13, Theorem 4.1])**.** *Let $k \geqslant 1$.*

(1) Let $f : \Lambda_L^k \to \mathbb{R}$ be a function such that $|f(\vec{x})| \leqslant Ce^{\gamma(|x_1|+\cdots+|x_k|)}$ for some constants $C, \gamma \geqslant 0$. Then the process

$$\sum_{\vec{x}\in\Lambda_L^k} f(\vec{x}) \prod_{i=1}^{k} P(t,x_i) - \int_0^t \sum_{\vec{x}\in\Lambda_L^k} \mathcal{Q}_k f(\vec{x}) \prod_{i=1}^{k} P(s,x_i)\, ds, \qquad t \geqslant 0, \tag{A.14}$$

is a continuous square-integrable martingale.

(2) For all $t \geqslant 0$ and $\vec{x} \in \Lambda_L^k$,

$$\mathbb{E}\Big[D\big(P(t,\cdot),\vec{x}\big)\Big] = \mathsf{E}_{\vec{x}}\Big[D\big(P(0,\cdot),\vec{X}(t)\big)\Big], \tag{A.15}$$

where $\mathsf{E}_{\vec{x}}$ denotes the expectation with respect to the symmetric inclusion process $(\vec{X}(t))_{t\geqslant 0}$ with k particles on Λ_L started from $\vec{X}(0) = \vec{x}$.

(3) Assume that $\psi_0 = e_0$, as in the rest of the paper. Then, for all $t \geqslant 0$ and $\vec{x} = (x_1,\dots,x_k) \in \Lambda_L^k$,

$$\mathbb{E}\Big[\prod_{i=1}^{k} P(t,x_i)\Big] = \mathsf{P}_{(0,\dots,0)}\big(X_1(t) = x_1, \dots, X_k(t) = x_k\big), \tag{A.16}$$

where $\mathsf{P}_{(0,\dots,0)}$ denotes the law of the symmetric inclusion process with k particles started from the origin.

In words, (A.16) states that the correlation function of order k of the random probability distribution $P(t,\cdot)$ is the distribution at time t of k inclusion particles started from the origin.

Proof. (1) Assume first that f has finite support. Then $F(p) = \sum_{\vec{x}} f(\vec{x}) p^{\vec{x}}$ is a polynomial in finitely many coordinates of p, so that (A.2) is a martingale, which coincides with (A.14) by Lemma A.1. It is bounded on bounded time intervals, since $P(t,x) \in [0,1]$. This proves the claim when $L < \infty$. Assume now that $L = \infty$ and let f be as in the statement. Let $T > 0$ and let $\mathsf{N}_\gamma = \sup_{t\leqslant T} \sum_{x\in\mathbb{Z}} e^{\gamma|x|} P(t,x)$, which has finite moments of all orders by (2.2). For $K \geqslant 1$, the function $f_K = f \mathbb{1}_{\{-K,\dots,K\}^k}$ has finite support. It coincides with f on $\{-K,\dots,K\}^k$, and $\mathcal{Q}_k f_K$ coincides with $\mathcal{Q}_k f$ on $\{-K+1,\dots,K-1\}^k$. Moreover, $|\mathcal{Q}_k f_K(\vec{x})|$ and $|\mathcal{Q}_k f(\vec{x})|$ are at most $4k^2 C e^{\gamma} e^{\gamma(|x_1|+\cdots+|x_k|)}$. Hence, the difference between the process (A.14) and the same process with f replaced by f_K is bounded, uniformly on $[0,T]$, by

$$8k^2 C e^{\gamma} (1+T) \sup_{t\leqslant T} \sum_{\vec{x}\,:\,\max_i |x_i| \geqslant K} \prod_{i=1}^{k} e^{\gamma|x_i|} P(t,x_i) \leqslant 8k^3 C e^{\gamma} (1+T)\, e^{-K}\, \mathsf{N}_{\gamma+1} \mathsf{N}_\gamma^{k-1},$$

where we used that $\sum_{|x|\geqslant K} e^{\gamma|x|}P(t,x) \leqslant e^{-K}\mathsf{N}_{\gamma+1}$. The right-hand side goes to zero in L^2 as K goes to infinity. Therefore, on $[0,T]$, the process (A.14) is the limit in L^2, uniformly in time, of a sequence of continuous martingales, so that it is a continuous square-integrable martingale.

(2) For $t \geqslant 0$ and $\vec{x} \in \Lambda_L^k$, let

$$F_t(\vec{x}) = \mathbb{E}\Big[\prod_{i=1}^{k} P(t,x_i)\Big].$$

This is a nonnegative function on Λ_L^k such that $\sum_{\vec{x}} F_t(\vec{x}) = 1$, since $\sum_x P(t,x) = 1$. Taking expectations in (A.14), we obtain that for every bounded function f and all $t \geqslant 0$,

$$\sum_{\vec{x}} f(\vec{x})F_t(\vec{x}) = \sum_{\vec{x}} f(\vec{x})F_0(\vec{x}) + \int_0^t \sum_{\vec{x}} \mathcal{Q}_k f(\vec{x})F_s(\vec{x})\,ds. \tag{A.17}$$

By Kolmogorov's equation, (A.17) is also satisfied by the function

$$G_t(\vec{x}) = \sum_{\vec{y}\in\Lambda_L^k} F_0(\vec{y})\,\mathsf{P}_{\vec{y}}(\vec{X}(t) = \vec{x}),$$

which is the distribution at time t of the symmetric inclusion process started from the distribution F_0. Let $\delta(t) = \sum_{\vec{x}} |F_t(\vec{x}) - G_t(\vec{x})|$. Subtracting the two equations, taking the supremum over the functions f such that $\|f\|_\infty \leqslant 1$, and using that $\|\mathcal{Q}_k f\|_\infty \leqslant 4k^2\|f\|_\infty$, we obtain $\delta(t) \leqslant 4k^2 \int_0^t \delta(s)ds$. Since δ is bounded by 2, Gronwall's lemma shows that $\delta \equiv 0$, that is

$$F_t(\vec{x}) = \sum_{\vec{y}\in\Lambda_L^k} F_0(\vec{y})\,\mathsf{P}_{\vec{y}}(\vec{X}(t) = \vec{x}). \tag{A.18}$$

By the detailed balance relation (A.9), the transition probabilities of the symmetric inclusion process satisfy $\pi_k(\vec{y})\,\mathsf{P}_{\vec{y}}(\vec{X}(t) = \vec{x}) = \pi_k(\vec{x})\,\mathsf{P}_{\vec{x}}(\vec{X}(t) = \vec{y})$. Dividing (A.18) by $\pi_k(\vec{x})$, we thus obtain

$$\frac{F_t(\vec{x})}{\pi_k(\vec{x})} = \sum_{\vec{y}\in\Lambda_L^k} \frac{F_0(\vec{y})}{\pi_k(\vec{y})}\,\mathsf{P}_{\vec{x}}(\vec{X}(t) = \vec{y}),$$

which is (A.15), since $F_0(\vec{y}) = P(0,\cdot)^{\vec{y}}$ is deterministic.

(3) When $\psi_0 = e_0$, we have $F_0(\vec{y}) = \mathbb{1}_{\vec{y}=(0,\dots,0)}$, and (A.18) becomes (A.16). □

Remark A.3 (Duality functions)**.** The relation (A.15) is a duality in the usual sense of the theory of Markov processes: the expectation of the function $D(\cdot,\vec{x})$ along the evolution of P is equal to the expectation of the function $D(P(0,\cdot),\cdot)$ along the evolution of the inclusion process. Since $D(p,\vec{x})$ depends on $\vec{x}$ only through the occupation numbers $\eta_z(\vec{x})$, this relation may be restated in terms of unlabelled particles. It then coincides, up to the normalizations of time and of the function D, with the duality established in [GKRV09, Theorem 6.2] and [CGGR13, Theorem 4.1] for the Brownian energy process, defined there as a Markov process. Proposition A.2 shows that it holds for the process $P(t,\cdot)$ constructed from the wave function, including on the infinite lattice. For the Brownian energy process with parameter α, the same computation as in the proof of Lemma A.1 shows that the dual process is the inclusion process in which a particle moves from a site x to a neighbouring site y at rate $\eta_x(\alpha+\eta_y)$, and that $\eta_z!$ must be replaced by $\alpha(\alpha+1)\cdots(\alpha+\eta_z-1)$ in the definition (A.10) of D.

Remark A.4 (The cases $k = 1$ and $k = 2$)**.** When $k = 1$, (A.16) is the last statement of Proposition 2.2. When $k = 2$ and $L = \infty$, it shows that for every bounded function f on $\mathbb{Z}^2$,

$$\mathbb{E}\Big[\sum_{x_1,x_2\in\mathbb{Z}} f(x_1,x_2)P(t,x_1)P(t,x_2)\Big] = \mathsf{E}_{(0,0)}\Big[f\big(X_1(t),X_2(t)\big)\Big],$$

where $(X_1(t), X_2(t))$ is the symmetric inclusion process with two particles. Lemma 2.3 is a refinement of the first statement of Proposition A.2 for $k \in \{1, 2\}$, where the function f may depend on time and where the martingale is explicit. In the proof of Theorem 1.2, the case $k = 1$ yields the martingales of Proposition 3.1, and the case $k = 2$ is used for the second moment estimates of Section 4 and for the replacement lemma of Section 6, through the Markov chain $X_2(t) - X_1(t)$, whose generator is the operator $\mathcal{R}$ defined in (3.24).

Appendix B. The Markov process $P(t,\cdot)$ (in finite volume)

When $L < \infty$, the duality has two consequences, which justify the statements made in Section A.1 about the equilibrium measure of the process $P(t,\cdot)$ and about the uniqueness of the solution of the martingale problem associated with $\mathcal{L}$.

B.1. **Equilibrium in finite volume.** Assume that $L < \infty$ and $\psi_0 = e_0$, and let $N = 2L+1$. The symmetric inclusion process with k particles is an irreducible Markov chain on the finite set Λ_L^k, whose stationary distribution is proportional to π_k. Each of the $\binom{N+k-1}{k}$ possible values of the family of occupation numbers $(\eta_z)_{z\in\Lambda_L}$ corresponds to $k!/\prod_z \eta_z!$ configurations $\vec{x}$, so that $\sum_{\vec{x}} \pi_k(\vec{x}) = k!\binom{N+k-1}{k}$. Letting t go to infinity in (A.16), we obtain

$$\lim_{t\to\infty} \mathbb{E}\Big[\prod_{i=1}^{k} P(t,x_i)\Big] = \frac{(N-1)!}{(N+k-1)!} \prod_{z\in\Lambda_L} \eta_z(\vec{x})!\,.$$

The right-hand side is the integral of $\prod_z p(z)^{\eta_z(\vec{x})}$ with respect to the uniform distribution on the simplex (A.5). Since the simplex is compact, the convergence of all these moments shows that $P(t,\cdot)$ converges in distribution to the uniform distribution on the simplex, as stated in Section A.1. This gives another proof, in the one-particle case, of the convergence to equilibrium established in [BBJ19, Proposition 1].

B.2. **Uniqueness in finite volume.** Assume that $L < \infty$. The proof of the first two statements of Proposition A.2 only uses the fact that (A.2) is a martingale for all polynomial functions F, and the fact that $P(t,\cdot)$ takes values in the simplex (A.5). Hence, the identity (A.18), which expresses the moments of $P(t,\cdot)$ in terms of those of $P(0,\cdot)$, holds for every solution of the martingale problem associated with $\mathcal{L}$ with values in this simplex, whatever its initial distribution. Since the simplex is compact, these moments determine the law of $P(t,\cdot)$. By a classical argument [EK86, Chapter 4, Theorem 4.2], the uniqueness of the one-dimensional marginals implies that the martingale problem is well posed and that its solutions are Markov processes. In this sense, when $L < \infty$, the process $P(t,\cdot)$ is the Brownian energy process started from $P(0,\cdot)$. When $L = \infty$, we do not address the uniqueness of the solutions of the infinite system of stochastic differential equations satisfied by $P(t,\cdot)$, which is not needed in this article.

Appendix C. Comparison with random walks in random environment

Let P be the quenched probability distribution of a random walk in a space-time random environment, as in Section 1.5.1. Then $\mathbb{E}\big[\prod_{i=1}^{k}\mathsf{P}(X_t = x_i)\big]$ is the probability that k random walks, which evolve independently in the same environment, are located at $x_1, \dots, x_k$ at time t. The positions of these k walks form a Markov chain, the k-point motion, in which the walks only interact when they are at the same site; it is a central object in [DDP25; Par26]. In view of (A.16), the symmetric inclusion process plays the role of the k-point motion for the one-particle QSSEP, the main difference being that the particles interact when they are at neighbouring sites.

The analogy stops here: the one-particle QSSEP is not a random walk in a random environment. Indeed, the quenched distribution of a random walk in a random environment is a linear function of its initial distribution: if $K_t(y,x)$ denotes the quenched probability that the walk started from y is at x at time t, then the quenched distribution at time t of the walk started from a distribution p_0 is $\sum_y p_0(y)K_t(y,\cdot)$. The models of Sections 1.5.1 and 1.5.3 have this structure. The one-particle QSSEP does not, even in distribution.

Proposition C.1. *Let $t \in (0, \frac{1}{8}]$. There is no family $(K_t(y,x))_{x,y\in\Lambda_L}$ of nonnegative random variables such that $\sum_x K_t(y,x) = 1$ for all y and such that, for every initial wave function ψ_0 as in Section A.3, the random variables $P(t,0)$ and $\sum_y P(0,y)K_t(y,0)$ have the same distribution.*

Proof. Assume that such a family exists. Taking $\psi_0 = e_{-1}$, we see that $\mathbb{E}[K_t(-1,0)] = \mathbb{E}[P(t,0)]$ is the probability that the random walk with generator Δ_L started from -1 is at 0 at time t. Since this walk jumps to the right at rate at most 1, this probability is at most $1-e^{-t}$. Let now $\psi_0 = \sqrt{a}\, e_{-1} + \sqrt{1-a}\, e_0$ with $a \in [0,1]$, so that $P(0,\cdot) = a\,\delta_{-1} + (1-a)\,\delta_0$. By (A.18) with $k = 2$,

$$\begin{aligned}\mathbb{E}\Big[\big(aK_t(-1,0) + (1-a)K_t(0,0)\big)^2\Big] &= \mathbb{E}\big[P(t,0)^2\big] \\ &= \sum_{y_1,y_2\in\{-1,0\}} P(0,y_1)P(0,y_2)\,\mathsf{P}_{(y_1,y_2)}\big(X_1(t) = X_2(t) = 0\big).\end{aligned}$$

Both sides are polynomials in a. Identifying the coefficients of $a(1-a)$, and using that the law of the inclusion process is invariant under the exchange of the two particles, we find

$$\mathbb{E}\big[K_t(-1,0)K_t(0,0)\big] = \mathsf{P}_{(-1,0)}\big(X_1(t) = X_2(t) = 0\big).$$

Since $K_t(0,0) \leqslant 1$, the left-hand side is at most $\mathbb{E}[K_t(-1,0)] \leqslant 1 - e^{-t}$. On the other hand, in the configuration $(-1,0)$, the first particle jumps onto the second one at rate 2, and the total jump rate is at most 6 in the configuration $(-1,0)$ and at most 4 in the configuration $(0,0)$. Hence, the right-hand side is at least

$$\int_0^t 2e^{-6s}e^{-4(t-s)}\,ds = e^{-4t}\big(1-e^{-2t}\big) = e^{-4t}\big(1+e^{-t}\big)\big(1-e^{-t}\big),$$

which is larger than $1-e^{-t}$ when $t \leqslant \frac{1}{8}$. This is a contradiction. □

The mechanism is the following. In a random environment, a random fraction of the mass located at -1 is sent to 0, whatever the mass already present at 0. In the one-particle QSSEP, by (2.5), the current of probability through a bond only fluctuates when both of its endpoints carry some mass, since these fluctuations come from the interference between the values of the wave function at the two sites. In terms of the dual processes, each walk of a k-point motion,

taken alone, is a random walk with the law of the one-point motion, while an inclusion particle jumps at a higher rate when another particle sits at a neighbouring site.

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