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Local coverings for frangible prime gaps

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Abstract

Gafni and Tao ask whether, for all sufficiently large even h, one can cover the interior of an interval of length h using one residue class per prime below h, without covering the endpoints. We give an unconditional construction and, with an exact finite check, obtain the result for every even $h \geq 4$. More generally, the construction works with prescribed admissible protected sets of polylogarithmic size and a prime cutoff $o(h)$. We give two-sided bounds on the proportion of legal coverings: it decays faster than every fixed inverse power but not at a geometric rate. A classical upper-bound sieve transfers the stretched-exponential estimate to an unconditional upper bound on the number of genuine frangible prime gaps, uniformly for lengths up to a fixed fraction of the logarithm of the counting interval length. This does not prove that prime pairs occur. We also give exact local computations through $h = 100$.

1 Statements

Write $P(t) = \prod_{p<t} p$, where p always denotes a prime. For an integer $h \geq 4$, a set $E \subseteq \{0, \dots, h\}$ is *admissible* if $\nu_p(E) := |E \bmod p| < p$ for every prime p. Let $A_h(E)$ be the classes $b \bmod P(h)$ for which $\gcd(b+e, P(h)) = 1$ for every $e \in E$. Let $\Omega_h(E) \subseteq A_h(E)$ also satisfy $\gcd(b+j, P(h)) > 1$ for every $j \in \{0, \dots, h\} \setminus E$. Put $q_h(E) = |\Omega_h(E)|/|A_h(E)|$. When $E = \{0, h\}$, write Ω_h and q_h. This is the set Ω_h of Gafni–Tao [1, Remark 4.1] for even $h \geq 4$.

Theorem 1. *For every even integer $h \geq 4$, the set Ω_h is nonempty. The proof uses an analytic argument for $h \geq 2{,}000{,}000$ and an exact finite check below that threshold.*

Theorem 2. *Fix $D \geq 0$, and let $L = \log h$, $l = \log L$, and $m = \log l$. For all sufficiently large h, uniformly over admissible $E \subseteq \{0, \dots, h\}$ with $0, h \in E$ and $|E| \leq 2 + L^D$, there is a translation that protects E and covers its complement using only primes below $C_D h l^3/(Lm)$. Furthermore, for suitable constants $C'_D, a > 0$,*

$$\exp\left(-C'_D \frac{hl^3}{Lm}\right) \;\leq\; q_h(E) \;\leq\; \exp\left(-a \frac{\sqrt{h}\, l^2}{L^{3/2}}\right).$$

The largeness threshold may depend on D.

Thus $q_h^{1/h} \to 1$ along even h, and by the prime number theorem $\log |\Omega_h| \sim h$. In particular, the lower bound answers the geometric-decay question in Hales [2] in the negative. In the notation of [1, Section 4],

$$c_h = \mathfrak{S} \frac{|\Omega_h|}{\prod_{2<p<h}(p-2)}, \qquad \mathfrak{S} = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Equivalently $c_h = \Sigma(h) q_h$, where $\Sigma(h) = \mathfrak{S} \prod_{2<p|h}(p-1)/(p-2)$ for even $h \geq 4$. Mertens's estimate gives $\mathfrak{S} \leq \Sigma(h) \ll L^2$, so the same two decay rates hold for c_h after adjusting

constants. The upper bound also holds for $\sum_{h\ge H,\ 2\mid h} c_h$, with a smaller a: on successive dyadic blocks the exponent $\sqrt{H}(\log\log H)^2/(\log H)^{3/2}$ grows by a factor bounded away from one and absorbs the number of indices in each block. Uğurdoğan's fixed-moment and conditioning estimates already imply $c_h \ll_A h^{-A}$ for every fixed A [7, Lemmas 2.2–2.3]; the upper bound here instead comes from a conditional concentration argument.

For any fixed even $h \ge 4$ and $b \in \Omega_h$, the two linear forms $P(h)t + b$ and $P(h)t + b + h$ are admissible: primes below h exclude no t, and the others exclude at most two classes. Dickson's conjecture would therefore turn this local construction into infinitely many genuine frangible prime gaps of length h; the interior integers are composite for all sufficiently large t. The unconditional result for actual prime gaps below is an upper bound only.

2 Covering every even length

Fix even $h \ge 6$. For each $p < h$ not dividing $3(h+3)$ choose $b \equiv 3 \pmod p$; at the other primes choose any legal residue. Denote the choice at 3 by $\rho \in \{1, 2\}$. The prime 2 covers every odd offset. Let S be the prime divisors of $3(h+3)$. If an even interior offset j is still uncovered, every prime divisor of $j+3$ below h belongs to S. Thus $j+3$ is S-smooth (all its prime divisors lie in S), or is itself the prime $h+1$.

A donor is a prime $\ell \in (h/2+2, h)$ with $\ell \equiv -\rho \pmod 3$. It does not divide $3(h+3)$. Its only possible original interior hit is $\ell - 3$, which is permanently covered by 3. A distinct donor can be reassigned to each uncovered even j by choosing $b \equiv -j \pmod \ell$. Neither j nor $h - j$ is divisible by ℓ, since each is positive, even and $< 2\ell$. These changes leave $0, h$ protected and lose no coverage. The Chinese remainder theorem then gives the desired translation.

Here is a bound that makes the argument effective. For $h \ge 2{,}000{,}000$, put $x = h/2+2$. Bennett–Martin–O'Bryant–Rechnitzer [4, Corollary 1.6, $q = 3$] give

$$\frac{v}{2\log v} < \pi(v; 3, a) < \frac{v}{2\log v}\left(1 + \frac{5}{2\log v}\right) \qquad (v \ge 450,\ a = 1, 2).$$

Using $x \le 0.501h$, $\log x \ge 0.95\log h$, and $1 + 5/(2\log x) < 1.19$ gives more than $h/(6\log h)$ donors. Rankin's bound at exponent $1/3$ bounds the number of holes by

$$1 + (h+1)^{1/3}\prod_{p\in S}(1 - p^{-1/3})^{-1}.$$

For $p \ge 19$, $(1-p^{-1/3})^{-1} \le p^{1/6}$, as follows from $p^{1/6} - p^{-1/6} \ge 1$. Over $p = 3, 5, 7, 11, 13, 17$, the product of $\max(1, (1-p^{-1/3})^{-1}/p^{1/6})$, multiplied by $3^{1/6}$, is less than 14. For instance, lower bounds for the corresponding sixth roots are $1.200, 1.307, 1.383, 1.491, 1.533, 1.603$, by taking sixth powers. The correction $x/(x^2-1)$ decreases for $x > 1$, and the six upper bounds $2.73, 1.85, 1.52, 1.22, 1.14, 1.03$ give the claim with $3^{1/6} < 1.21$. Since $\prod_{p\in S} p \le 3(h+3)$, the number of holes is below $14\sqrt{h+3} + 1 < 15\sqrt{h}$. At the starting point $h = 2{,}000{,}000$ we have $\sqrt{h}/(6\log h) > 15$; this ratio is increasing thereafter.

For $6 \le h < 2{,}000{,}000$, the companion C++17 verifier enumerates the candidate holes using a smallest-prime-factor table, tries both legal choices at 3 where applicable, and counts donors directly. At exceptional $p \ne 3$ it uses $b = 2$ if $h \equiv -1 \pmod p$ and $b = 1$ otherwise. Precisely 999,972 values pass the donor check; for the remaining 25 the verifier checks explicit residue vectors, prime by prime, against both endpoint and all interior requirements. For $h = 4$ use $b = 1 \bmod 6$. This proves Theorem 1.

3 Prescribed protected offsets with a smaller cutoff

Use the notation of Theorem 2, and let

$$w = L^{D+3}, \qquad z = \exp(Lm/(10l)), \qquad y = C_D h l^3/(Lm), \qquad K = h/y.$$

Take C_D large enough below. Let F comprise the prime divisors of $\prod_{e\in E}(1+e)$. At primes $p < y$ use the baseline $b \equiv 1 \pmod p$ if $p \notin F$, and any legal choice otherwise. Replace the choices in the band $w < p \le z$ independently by uniform residues outside $\{1\}\cup\{-e : e \in E\}$ (mod p), and keep them fixed for the rest of the construction. There is always a choice in this band for large h, as $w > |E| + 1$.

For $j \notin E$ the band can cover j at a prime p with probability at least $1/p$, unless p divides $(j+1)\prod_{e\in E}(j-e)$. This nonzero integer has $O_D(L^{D+1})$ distinct prime factors, so its factors above w have reciprocal sum $o(1)$, uniformly in j. By independence and Mertens's product estimate, the probability j misses the entire band is

$$\ll_D \frac{\log w}{\log z} \ll_D \frac{l^2}{Lm} =: \eta.$$

Indeed, omitting the exceptional primes increases the sieve product by at most $1 + o(1)$.

There are $N \sim y/(2L)$ donor candidates, the primes $q \in (y/2, y) \setminus F$, by the prime number theorem and $|F| = O_D(L^{D+1})$. Call a donor bad if any of its $O(K)$ original hits j with $q \mid j + 1$ misses the band. All such hits are outside E. The expected number of bad donors is $O(\eta KN) = o(N)$. Let A count the unprotected offsets that miss the band and have some prime divisor $q \ge y$ of $j + 1$. By the union bound and dyadic summation using $\pi(t) \ll t/\log t$,

$$\mathbb{E}A \ll \eta h \sum_{y\le q\le h+1} q^{-1} \ll \eta h \frac{\log(2K)}{L}.$$

Since $\eta K \log(2K) = O_D(1/C_D)$, we can choose C_D and then h so that there is an outcome with A plus the number of bad donors at most $N/4$.

Any other uncovered offset has all prime factors of $j + 1$ in $\{p : p \le z\} \cup F$, since a factor in $(z, y) \setminus F$ would cover it at baseline. With $\sigma = 1 - 4l/L$, Rankin's bound for those integers is

$$(h + 1)^\sigma \prod_{p\le z \text{ or } p\in F} (1 - p^{-\sigma})^{-1}.$$

The logarithm of the product over $p \le z$ is $\log\log z + o(l)$: higher powers contribute $O(1)$, and

$$\sum_{p\le z}(p^{-\sigma} - p^{-1}) \ll (1 - \sigma)e^{(1-\sigma)\log z} \sum_{p\le z} \frac{\log p}{p} \ll ml^{2/5} = o(l).$$

The extra primes from F above z contribute $o(1)$ to the logarithm. Thus the smooth remainder is $h/L^{3-o(1)} = o(N)$.

Reassign a distinct good donor q to each hole j by choosing $b = -j \pmod q$. At most $|E|$ candidates are forbidden for each j: each nonzero $j - e$ has absolute value at most h and at most one prime divisor above $y/2 > \sqrt{h}$. There are enough donors for greedy assignment. Good donors' old hits are already covered by the unchanged band, and the Chinese remainder theorem finishes the construction. This uses the small-prime and smooth-number decomposition familiar from large-gap constructions, as in [5], but targets a prescribed interval and protects a prescribed admissible set.

Leaving all legal choices for $y \le p < h$ free gives $q_h(E) \ge \prod_{p<y}(p - \nu_p(E))^{-1} \ge \exp(-O(y))$. This proves the lower bound in Theorem 2.

4 Upper bound by concentration

In the uniform distribution on $A_h(E)$, the choices of residue at each $p < h$ are independent and uniform among those that do not cover E. Freeze any choices at $p \le u = \sqrt{h/L}$. The uniform Iwaniec lower sieve bound as stated in Banks–Ford–Tao [3, (1.11)–(1.12)] leaves $\gg hl/L^2$ integers unsieved in $\{1, \dots, h\}$, regardless of the frozen residue classes. Remove E and call the resulting set S_0; the same lower bound holds. Let X count those that are not covered by any remaining prime. For each fixed member of S_0, its conditional probability of survival is at least

$$\prod_{u<p<h} \left(1 - \frac{1}{p - |E|}\right) = \tfrac{1}{2} + o(1),$$

by Mertens and the error $O(|E| \sum_{p>u} p^{-2}) = o(1)$ in the logarithm of the product. In particular $\mathbb{E}(X \mid \text{frozen choices}) \gg hl/L^2$.

Changing a single later residue mod p affects only two classes mod p. For $u < p \le h^{2/3}$, each class contains $O(h/(pL))$ members of S_0 by [3, Lemma 3.1], applied after translating the interval by a CRT representative for the frozen choices. For $h^{2/3} < p < h$ the bound $O(h/p + 1)$ suffices. The sum of squares of the coordinate bounds is at most a constant times

$$\frac{h^2}{L^2} \sum_{p>u} p^{-2} + h^2 \sum_{n>h^{2/3}} n^{-2} \ll \frac{h^2}{L^2 u \log u} + h^{4/3} \ll \frac{h^{3/2}}{L^{5/2}}.$$

Expose the remaining independent residues one at a time. The conditional expectation of X is a Doob martingale whose increments are bounded by these coordinate bounds. Azuma's inequality [3, Lemma 3.2] therefore gives $\Pr(X = 0 \mid \text{frozen choices}) \le \exp(-a\sqrt{h}\, l^2/L^{3/2})$. The bound is uniform, so averaging over the frozen choices proves the upper bound of Theorem 2.

5 An unconditional bound for actual prime gaps

For even $h \ge 4$ and integer $r \ge 1$, let $B_{h,r}$ be the legal endpoint classes modulo $W = P(h)$ with fewer than r surviving interior offsets, and put $c_{h,r} = \Sigma(h)|B_{h,r}|/|A_h(\{0, h\})|$. Thus $c_{h,1} = c_h$. An interior integer survives if it has no prime factor below h. Write $N_{h,r}(x, Y)$ for the number of consecutive prime gaps of length h with left endpoint in $[x, x + Y]$ and fewer than r surviving interior integers.

Theorem 3. *Fix $0 < \varepsilon < 1$. Uniformly for sufficiently large Y, $x \ge Y$, even $h \ge 4$ with $P(h) \le Y^{1-\varepsilon}$, and all $r \ge 1$,*

$$N_{h,r}(x, Y) \ll_\varepsilon c_{h,r} \frac{Y}{(\log Y)^2}.$$

Consequently there are absolute $a, \delta > 0$ such that, for all sufficiently large h in this range and $1 \le r \le \delta h \log\log h/(\log h)^2$,

$$N_{h,r}(x, Y) \ll_\varepsilon \frac{Y}{(\log Y)^2} \exp\left(-a \frac{\sqrt{h}(\log\log h)^2}{(\log h)^{3/2}}\right).$$

For instance the modulus condition is satisfied for $h \le \frac{1}{2} \log Y$ with $\varepsilon = 1/4$ and sufficiently large Y.

Proof. Put $U = Y/W$, $R = U^{1/4}$, and $z = R^\eta$ with a sufficiently small absolute $\eta > 0$. The prime number theorem and the hypothesis give $h = O(\log Y)$ and $U \geq Y^\varepsilon$, so $h < z < x$ for large Y. Every counted pair has its left endpoint in $B_{h,r}$. For each class b in that set, count prime pairs $(b+kW, b+kW+h)$ in an interval of k of length U, dropping the consecutiveness condition. Each prime $h \leq p < z$ excludes two distinct classes of k, since $p \nmid Wh$ (h is even and composite). For squarefree d supported on these primes the excluded-class intersection has size $U2^{\omega(d)}/d + O(2^{\omega(d)})$.

Apply the finite Selberg upper bound [8, Theorem 30], where $\omega(d)$ counts distinct prime factors. If $w(d)$ is the squarefree multiplicative weight with $w(p) = 2/(p-2)$ on the sieving primes and zero at other primes, the bound per class is

$$\frac{U}{\sum_{d \leq R} w(d)} + O\left(\sum_{d \leq R^2} \tau_6(d)\right).$$

Here the remainder is bounded using $\tau_3(d)2^{\omega(d)} = \tau_6(d)$ at squarefree d, where τ_j counts ordered factorizations into j positive integers. Let $V = \prod_{h \leq p < z}(1 - 2/p)$; the full weight sum is V^{-1}. With $t = 1/\log z$, the proportion of weight from $d > R$ is at most

$$R^{-t} \prod_{h \leq p < z} \frac{1 + w(p)p^t}{1 + w(p)} \leq \exp(-1/\eta + O(1)).$$

Indeed $p^t - 1 \leq e \log(p)/\log z$ and $\sum_{p<z}(\log p)/p \ll \log z$ by Chebyshev's bound. Choose η so this proportion is at most $1/2$. The error $O(R^2(\log(2R))^5)$ is absorbed by UV, since $V \gg (\log z)^{-2}$ by Mertens's estimate and convergence of the twin-prime product. Thus each class contributes $O(UV)$.

Writing $d_p = 1 - |\{0, h\} \bmod p|/p$, we have $|A_h(\{0,h\})|/W = \prod_{p<h} d_p$ and $\Sigma(h) = \prod_p d_p(1-1/p)^{-2}$. Consequently

$$\frac{|B_{h,r}|}{W} V = c_{h,r} \frac{\prod_{p<z}(1-1/p)^2}{\prod_{p \geq z}(1-(p-1)^{-2})}.$$

Summation over the classes, followed by Mertens's estimate and $\log z \geq (\eta\varepsilon/4)\log Y$, proves the first bound. For the second, apply the concentration argument above with $E = \{0, h\}$. The event $X < r$ still requires a downward deviation of at least half the conditional mean if δ is small enough. This gives the same exponential rate for $|B_{h,r}|/|A_h(\{0,h\})|$ as for q_h; absorb the factor $\Sigma(h) \ll (\log h)^2$ by reducing a. □

Gafni–Tao [1, Lemma 4.1] already obtain any fixed inverse power of h for $r = 1$ up to $h \leq (\log x)^{1/100}$ in a dyadic interval. Uğurdoğan [7, Lemma 3.2] gives a related fixed-r bound. The improvement here is the stretched-exponential rate, keeping the exact local coefficient and allowing r to grow within the indicated range. This is an upper bound, not an asymptotic or a proof of occurrence.

6 Exact counts and the partial sum

At 2 the legal choice covers all odd offsets. Enumerate the legal choices at the first few odd primes, grouping them by the surviving set U of even offsets. For a fixed U, inclusion–exclusion gives its number of completions as

$$\sum_{J \subseteq U} (-1)^{|J|} \prod_{\substack{p<h \\ p \text{ remaining}}} \big(p - \nu_p(\{0,h\} \cup J)\big).$$

Each term counts completions preserving all of J. For $p > h/2$ the even offsets in $\{0, h\} \cup J$ have distinct residues: any nonzero even difference that is a multiple of p is at least $2p > h$. Thus all such factors depend only on $|J|$. This is the large-prime symmetry also used in Hales's computation [2]. The implementation uses exact integers, exploiting this symmetry and reflecting $j \mapsto h - j$ to reduce the work.

The companion scripts reproduce the published exact table of Mossinghoff–Trudgian [6] for $4 \le h \le 46$, and the partial sum agrees with Hales's reported floor through 94. The next three terms are:

h	$\lvert\Omega_h\rvert$	c_h (approximate)
96	324675382201201651763496	$9.2244441990 \times 10^{-10}$
98	154030845380200297229138 76	$4.6065410043 \times 10^{-10}$
100	111742918284554988868281 92	$3.3418523007 \times 10^{-10}$

Summing all even terms through 100 and enclosing $\mathfrak{S}$ by directed integer arithmetic gives

$$2.741610743909460474043 \;<\; \sum_{\substack{h \le 100 \\ 2 \mid h}} c_h \;<\; 2.741610743909460474045.$$

Here $c_2 = 0$. The normalization uses the product $1 - 2x = \prod_{k \ge 1} (1 - x^k)^{b_k}$ with $\sum_{d \mid k} d b_d = 2^k$. The $k = 1$ factor cancels in $(1 - 2x)/(1 - x)^2$; summation over the odd primes turns the remaining factors into reciprocals of odd zeta sums. The script encloses these with the absolutely convergent Euler transform of the alternating series, and bounds the omitted product via $0 \le b_k \le 2^k$. The lower endpoint bounds the full constant from below. Neither the partial sum nor our asymptotic upper estimate certifies a numerical upper bound for the tail starting at 102.

Verification and AI assistance. OpenAI models assisted with derivation, drafting, code and separate technical checks. The finite results were replayed with executable scripts. The manuscript has not been independently verified by a human or checked in a proof assistant.

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