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Lower Bounds for Self-Avoiding Walks

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Abstract

For a uniformly chosen n-step self-avoiding walk on the square lattice, we prove that the mean distance between its ends is at least $cn^{1/3}$ for every $n \geq 1$ and at least $c'n^{2/5}$ for infinitely many n, for absolute $c, c' > 0$.

1 Introduction

How far from its starting point does a long self-avoiding walk end? An n-step self-avoiding walk on the square lattice is a sequence $\omega_0, \ldots, \omega_n$ of distinct vertices in $\mathbb{Z}^2$ such that $\omega_0 = 0$ and consecutive vertices are nearest neighbours. Give the set $\mathcal{W}_n$ of these walks the uniform law $\mathbb{P}_n$. We study the mean Euclidean distance between the two ends, $d_2(n) = \mathbb{E}_n\left[\|\omega_n\|_2\right]$.

Theorem 1.1. *There is an absolute constant $c > 0$ such that $d_2(n) \geq cn^{1/3}$ for every integer $n \geq 1$.*

Theorem 1.2. *There is an absolute constant $c' > 0$ such that $d_2(n) \geq c'n^{2/5}$ for infinitely many integers n.*

In two dimensions the reflection argument of Madras [1, Proposition 1.1] gives a lower bound of order $n^{1/4}$ for $d_2(n)$ and $n^{1/3}$ for the root-mean-square distance. Theorem 1.1 reaches $1/3$ for the mean distance itself, at every length. By Theorem 1.2, if the exponent $\nu_1 = \lim_{n\to\infty} \log d_2(n)/\log n$ exists, then $\nu_1 \geq 2/5$. Whether the $2/5$ bound holds at every sufficiently large length remains open.

The question arose in the study of polymers. Orr [2] and Flory [3] investigated how the size of a molecular chain changes when two of its segments cannot occupy the same place. Self-avoidance imposes this restriction on a lattice. For simple random walk the root-mean-square distance between the ends is $\sqrt{n}$. Hara and Slade [4] proved that in dimensions $d \geq 5$ the root-mean-square distance for self-avoiding walk on $\mathbb{Z}^d$ is asymptotic to a constant times $\sqrt{n}$. In two dimensions Nienhuis's calculation [5] predicts the larger size exponent $3/4$. Jensen's exact square-lattice enumeration [6] supports that prediction.

Duminil-Copin and Hammond [7] proved that for each $v > 0$ the probability of reaching distance vn during the walk decays exponentially in n. In particular, $d_2(n) = o(n)$. Arikök, Panagiotis, Pischke and Powell [8] give an explicit decay rate in terms of v. For the honeycomb lattice, a recent preprint [9, Theorem 8.2] reports first-moment growth $n^{3/4+o(1)}$ on a set of lengths of natural density one.

The $n+1$ distinct vertices force a walk to reach distance at least a constant times $\sqrt{n}$ at some time. Yet a walk can visit every vertex of a large square and still end one step from its starting point. Madras unfolded a walk at an extremal vertex to turn a distant visit into a distant endpoint. He suggested that a better count of the walks giving the same image might improve his bound [1, p. 118]. We show that for candidate lines spanning $m \geq 1$ columns, the average number of valid folds is at most $C\sqrt{m}$. Once the walk is folded, its two portions must pass through successive matched crossings in the same vertical order.

The proof of Theorem 1.2, in Sections 4 and 5, starts with two walks ending at the same height. We slide one next to the other, a placement related to Madras's polygon joining argument [10]. Exchanging parts across a single edge gives two walks ending at a chosen height. Their individual lengths may change, but their combined length increases by exactly two. This is what allows us to average over lengths.

2 Unfolding a walk

Write $X(\omega) = \|\omega_n\|_\infty$ for the endpoint distance in the maximum norm and $b_n = \mathbb{E}_n[X]$. Since $d_2(n) \geq b_n$, it suffices to work with X. A walk that stays in $[-R, R]^2$ visits at most $(2R+1)^2$ vertices. Thus some vertex of every n-step walk reaches at least

$$L_n = \frac{\sqrt{n+1} - 1}{2} \tag{1}$$

in one of the four signed coordinate directions.

Consider a walk $\eta \in \mathcal{W}_N$ ending at horizontal coordinate $D > 0$. A *cut* is a half-integer a with $0 < a < D$ such that the line $x = a$ crosses exactly one edge of η. That edge goes east. To fold at a, delete the crossing edge and reflect the rest of the walk after that edge across $x = a$. The ends of the deleted edge now coincide and are identified. Call the fold *admissible* if the resulting walk is self-avoiding.

Lemma 2.1. *Fix N and $D > 0$ for which some walk in $\mathcal{W}_N$ ends at horizontal coordinate D. Choose such a walk uniformly, leaving its endpoint height free. If $I \subset (0, D)$ is an interval of length m, the expected number of admissible folds at cuts in I is at most $C\sqrt{m+1}$ for an absolute constant C.*

Section 3 proves the lemma. Its application to the endpoint bound takes only one unfolding.

Proof of Theorem 1.1. Fix $0 \leq m \leq L_n$. Consider the walks with $X \leq m$ that reach horizontal coordinate L_n or higher. Let h be the largest horizontal coordinate visited and let u be the horizontal coordinate of the endpoint. At the first vertex on $x = h$, insert an east edge and reflect the remaining portion across $x = h + \frac{1}{2}$, as in [11, Section 2.3]. The original portion lies in $x \leq h$, and the reflected portion lies in $x \geq h + 1$. The result is a self-avoiding walk of length $n+1$ ending at horizontal coordinate $D = 2h + 1 - u$. Folding it at the inserted edge recovers the original walk.

Knowing D confines the inverse fold to a short interval. Indeed, $a = h + \frac{1}{2} = (D+u)/2$ and $|u| \leq m$, so $a \in I_D = [(D-m)/2, (D+m)/2]$. Also $D \geq 2L_n + 1 - m > m$, so $I_D \subset (0, D)$. At each such

D, Lemma 2.1 bounds the number of possible original walks by $C\sqrt{m+1}$ times the number of $(n+1)$-step walks ending at that horizontal coordinate.

Sum over $D \geq 2L_n + 1 - m$. Deleting the last edge of any counted output leaves an n-step walk with endpoint horizontal coordinate at least $2L_n - m \geq L_n$. Each such walk has at most four possible last-edge extensions. By (1), the four choices of signed direction cover every original walk with $X \leq m$. Summing their bounds and dividing by $|\mathcal{W}_n|$ gives

$$\mathbb{P}_n(X \leq m) \leq C_1\sqrt{m+1}\,\mathbb{P}_n(X \geq L_n), \qquad 0 \leq m \leq L_n. \tag{2}$$

Take $m = 2b_n$. If $2b_n > L_n$, the claim follows from $L_n \geq c_0\sqrt{n}$. Otherwise Markov's inequality gives $\mathbb{P}_n(X \leq 2b_n) \geq 1/2$ and $\mathbb{P}_n(X \geq L_n) \leq b_n/L_n$. Every endpoint at positive length has $X \geq 1$. Substituting in (2) gives $L_n \leq C_2 b_n^{3/2}$. Hence $b_n \geq cn^{1/3}$. □

3 Counting the possible folds

Let $y(a)$ be the height of the crossing at a cut a. Suppose that $a - d$ and $a + d$ are also cuts, for some $d > 0$. After folding at a, the crossing at $a + d$ lands on the line $x = a - d$ at its original height. If the fold is admissible, the differences

$$y(a + d) - y(a - d) \tag{3}$$

must all be strictly positive or all strictly negative as d runs over such distances. Equal heights would put the two portions on the same edge. Figure 1 shows one such comparison.

To see why the order cannot reverse, take two matched lines on the left of a. Between them each portion of the folded walk joins the two sides of the strip and meets each side only at its crossing. If their order reversed, the endpoints of the two arcs would alternate around a rectangle enclosing them in the strip. The arcs would intersect and share a lattice vertex. Since d is a positive integer, the entire strip lies to the left of the identified vertex at $x = a - \frac{1}{2}$.

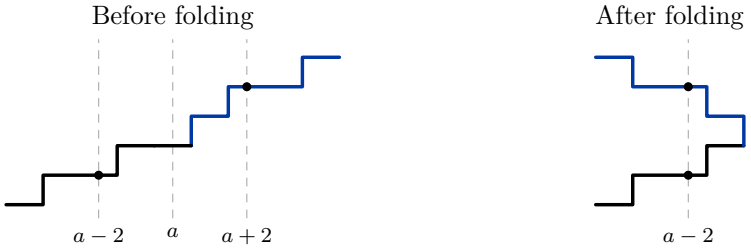


Figure 1: Folding brings the two marked crossings onto the line $x = a - 2$. The reflected portion crosses above the original portion.

3.1 Rearranging the pieces

Fix N, D, I as in Lemma 2.1. If there are at least two cuts in I, list all of them in increasing order as $a_0, \ldots, a_B$. Split the walk at the midpoint of each crossing edge. The B pieces between consecutive

cuts have positive integer widths $w_i = a_i - a_{i-1}$, total width $a_B - a_0 \le m$, and mean width $\bar{w}$. Each piece lies in the open strip between its cuts except at its two ends.

Keep the portion before a_0 fixed. Put the B pieces back in any order, optionally reflecting each in the horizontal line through its starting point. Translate the pieces to join them and translate the portion after a_B to meet the last piece. Neighbouring half-edges form a single east edge, and the interiors of the pieces occupy disjoint strips. Thus the result is still a lattice walk, self-avoiding, of length N, with horizontal endpoint D. Its height may change. The cuts in I are exactly the new boundaries. Every half-integer line internal to a piece retains its crossing count, as do the lines outside $[a_0, a_B]$.

These operations partition the walks into classes. To sample uniformly within a class, label the pieces of one representative, put them in a uniformly random order and reflect each independently with probability $1/2$. Then forget the labels. Even when some pieces agree, each walk in the class occurs equally often, since composition with any rearrangement taking one walk to another is a bijection between the corresponding choices.

3.2 Finding cuts at equal distances

In one such class, fix an internal index $1 \le j < B$. Starting from a_j, read the nearest $\ell = \min(j, B-j)$ pieces on each side in order outwards. The partial sums of the widths give the distances to the cuts on that side. Let K count the positive distances common to the two lists. For example, widths $(1, 2, 1)$ on the left and $(2, 1, 1)$ on the right give $K = 2$. The height comparisons in (3) take place at distances 3 and 4 since $1 + 2 = 2 + 1$ and $1 + 2 + 1 = 2 + 1 + 1$.

Lemma 3.1. *Within the rearrangement class, for every integer $k \ge 1$ we have*

$$\mathbb{P}(K < k) \le \min\{1, 2k\bar{w}/\ell\}.$$

Proof. Fix the set of ℓ labels on each side and the cyclic order in each list. Each list still starts at a uniformly chosen boundary, independently on the two sides. Write H_i for the total width on side $i \in \{1, 2\}$. Arrange list i on a circle of H_i integer positions, with a boundary at the end of each piece. There are ℓ boundaries on either circle.

Start one marker at a boundary on each circle and advance both by one unit at each step. Count one whenever a marker lands on a boundary, counting two when both do. Let J_k be the count up to and including the kth positive time when both markers are again at boundaries. The starting time contributes zero.

There are ℓ^2 pairs of starting boundaries. Summed over these pairs, J_1 is at most $(H_1 + H_2)\ell$. To see this, advancing the two markers permutes the $H_1 H_2$ pairs of positions in cycles. On each cycle that visits a pair of boundaries, the segments from one such visit to the next partition the cycle if we omit the start and include the end of each segment. The sum of the counts on these cycles is at most the count over all pairs of positions. A boundary on the first circle contributes once for each of the H_2 positions on the second, and conversely. Passing from one simultaneous visit to the next also permutes the ℓ^2 boundary pairs. Consequently $\mathbb{E}[J_k] = k\mathbb{E}[J_1] \le k(H_1 + H_2)/\ell$ for the two uniform starting boundaries.

The positive simultaneous visits up to distance $\min(H_1, H_2)$ are precisely the K matches of the original lists. If $K < k$, the kth visit occurs after one marker has completed its first circuit. That marker is at a boundary again, so it has finished more than ℓ pieces and $J_k > \ell$. Markov's inequality gives $\mathbb{P}(K < k) \le k(H_1 + H_2)/\ell^2$ under the conditioning above. Averaging over the labels gives the stated bound, since each list has expected total width $\ell\bar{w}$. □

3.3 Comparing the crossing heights

List the common distances as $0 = d_0 < d_1 < \cdots < d_K$. Between consecutive common distances, collect the pieces traversed on the left and those traversed on the right. These two groups have equal total width and no common boundary in their interiors. For the sth pair of groups let Z_s be the sum of the vertical displacements of all its pieces, each measured from west to east. Moving outwards reads the left pieces backwards. It follows that

$$y(a_j + d_s) - y(a_j - d_s) = Z_1 + \cdots + Z_s. \tag{4}$$

Within the rearrangement class, now condition only on $K = k$. The law of $(Z_1, \ldots, Z_k)$ is unchanged by any permutation and by changing any of its signs. For the permutation, reorder the k pairs of groups in the same way on both sides, leaving the pieces beyond the last match in their original order. Equal widths within each pair give matches at the new group boundaries. There are no extra matches inside a pair, and the two remainders still start at the same distance. Thus the matches recover the reordered groups, making the operation a bijection on labelled arrangements with $K = k$. To change the sign of one Z_s, reverse the vertical reflection choices of all pieces in that pair of groups. This changes no width.

The Sparre Andersen inequality in the form of Berger and Béthencourt [12, Theorem 2.1] now applies. For a vector of k real increments whose law has these two symmetries, the probability that every partial sum is strictly positive is at most $4^{-k}\binom{2k}{k} \le (k+1)^{-1/2}$. The same bound holds for strictly negative sums. By (3) and (4), the conditional probability of an admissible fold is at most $2(k+1)^{-1/2}$, which also covers $k = 0$. Averaging gives

$$\mathbb{P}(\text{the fold at } a_j \text{ is admissible}) \le 2\mathbb{E}\left[(K+1)^{-1/2}\right]. \tag{5}$$

Completion of the proof of Lemma 2.1. For $f(k) = (k+1)^{-1/2}$ we have $f(k-1) - f(k) \le Ck^{-3/2}$. Writing $f(K)$ as a telescoping sum from $k = K+1$ to infinity and using Lemma 3.1 gives

$$\mathbb{E}[f(K)] \le C \sum_{k \ge 1} k^{-3/2} \min\{1, 2k\bar{w}/\ell\} \le C'\sqrt{\bar{w}/\ell}.$$

For the last inequality split the sum at $\ell/(2\bar{w})$ when this is at least one, and otherwise use $f(K) \le 1$.

Let F_I be the number of admissible folds in I. Allow one fold at each of the two outermost cuts and sum (5) over the inner cuts. Within the class this gives

$$\mathbb{E}[F_I] \le 2 + C\sqrt{\bar{w}} \sum_{j=1}^{B-1} \frac{1}{\sqrt{\min(j, B-j)}} \le 2 + C'\sqrt{B\bar{w}}.$$

The total width $B\bar{w} = a_B - a_0$ is at most m. Walks with fewer than two cuts contribute at most one. Averaging over the classes proves the lemma. □

4 Exchanging parts of two walks

We turn to Theorem 1.2. The exchange below can change the lengths of its two inputs by different amounts, but the sum of their lengths always increases by two. This suggests giving each walk of length n weight z^n.

Let $c_n = |\mathcal{W}_n|$, with $c_0 = 1$. Splitting a walk after its ith step gives $c_{i+j} \le c_i c_j$, so $\mu = \lim_{n\to\infty} c_n^{1/n} = \inf_{n\ge 1} c_n^{1/n}$. For $0 < z < \mu^{-1}$ put $Z(z) = \sum_{n\ge 0} c_n z^n$. Give each finite rooted walk ω probability $z^{|\omega|}/Z(z)$, where $|\omega|$ is its number of steps. Under this law $\mathbb{P}_z$, denote the length by T, the endpoint height by Y, and the highest row visited by H^+. As before, $X = \|\omega_T\|_\infty$. Conditional on $T = n$ the walk is uniform in $\mathcal{W}_n$.

Lemma 4.1. *For integers $m \ge 0$ and $1 \le r \le H$,*

$$\mathbb{P}_z(Y = r) \ge \frac{z}{(2m+1)^{3/2}}\,\mathbb{P}_z(X \le m,\ H^+ \ge H). \tag{6}$$

Proof. Put $S = \mathbb{P}_z(X \le m, H^+ \ge H)$. We may assume $S > 0$. Among the walks in this event, choose an endpoint height s whose walks have probability $p_s \ge S/(2m+1)$. Take an ordered pair A, B of these walks, with endpoints u, v. Translate B by $w = (h, r)$, where

$$h = 1 + \max\{a_x - b_x : a \in A,\ b \in B,\ a_y = b_y + r\}.$$

Here the maximum is over vertices. It exists because A visits row r and B contains the origin. On every shared row, all vertices of A are strictly to the left of those of $B + w$. A maximizing pair gives an east edge e from a to $b' = b + w$. Break ties by choosing the earliest visit in A and then in B.

Form P by going along A from 0 to a, crossing e, and reading $B + w$ backwards from b' to w. For the other walk, start at u, read A backwards to a, cross e, and continue along $B + w$ from b' to $w + v$. Translate this second walk by $-u$ and call it Q. Both walks are self-avoiding since A and $B + w$ are disjoint. Their rooted endpoints are w and $w + v - u$. Both have height r because $u_y = v_y = s$. Every input edge is used once in total and the new edge is used twice, so $|P| + |Q| = |A| + |B| + 2$.



Figure 2: Both new walks use the dashed edge. One goes from 0 to w and the other from u to $w + v$. In this example each gains two units of height.

For fixed s and r, the outputs (P, Q) and the single coordinate u_x determine the inputs. Indeed, $u = (u_x, s)$ locates $Q + u$ relative to P, as in Figure 2. Their portions from A meet only at a, and their portions from $B + w$ meet only at b'. Thus e is the unique common edge. Cutting there recovers the two parts of each input, and the endpoint of P gives the translation w. This recovery also works when e is incident to an input endpoint.

There are at most $2m+1$ choices for u_x. Under the product law on ordered pairs, each output pair has z^2 times the weight of its input pair. Counting preimages therefore gives $z^2 p_s^2 \le (2m+1)\mathbb{P}_z(Y=r)^2$. Taking square roots and using $p_s \ge S/(2m+1)$ proves (6). □

5 Recovering a bound at fixed lengths

To use Lemma 4.1 we choose the weight so that a walk has a fixed positive chance of lasting at least half its mean length. Write $z = \mu^{-1}e^{-t}$ for $t > 0$ and put $F(t) = Z(\mu^{-1}e^{-t})$, $M(t) = \mathbb{E}_z[T]$ and $V(t) = \mathrm{Var}_z(T)$.

Lemma 5.1. *There is a sequence $t_j \downarrow 0$ along which $M(t_j) \to \infty$ and $V(t_j) \le 2M(t_j)^2$.*

Proof. Since $c_n\mu^{-n} \ge 1$, we have $F(t) \ge (1-e^{-t})^{-1}$. The total weight of any fixed finite set of lengths stays bounded as $t \downarrow 0$, so its probability tends to zero and $M(t) \to \infty$. The defining series and all its derivatives converge for $t > 0$ by the root test. Differentiation gives

$$(\log F)' = -M, \qquad M' = -V, \qquad (1/M)' = V/M^2.$$

If $V(t) > 2M(t)^2$ for all sufficiently small t, integration from 0 to t gives $1/M(t) \ge 2t$. Integrating $M(s) \le 1/(2s)$ from t to a fixed small t_0 then gives $F(t) \le F(t_0)(t_0/t)^{1/2}$. This contradicts $F(t) \ge (1-e^{-t})^{-1}$ as $t \downarrow 0$. □

Proof of Theorem 1.2. Take t along the sequence in Lemma 5.1 and put $M = M(t)$. Then $\mathbb{E}_z[T^2] \le 3M^2$. Since $\mathbb{E}_z\left[T\mathbf{1}_{\{T \ge M/2\}}\right] \ge M/2$, Cauchy–Schwarz gives $\mathbb{P}_z(T \ge M/2) \ge 1/12$. Put $H = \lfloor \sqrt{M}/4 \rfloor$. For large M the square $[-H,H]^2$ contains fewer than $M/2$ vertices. Thus any walk of length at least $M/2$ reaches at least H in some signed coordinate direction. By symmetry, $\mathbb{P}_z(H^+ \ge H) \ge 1/48$.

Put $b = \mathbb{E}_z[X]$ and $m = \lceil 96b \rceil$. Markov's inequality gives $\mathbb{P}_z(X > m) \le 1/96$, so $\mathbb{P}_z(X \le m, H^+ \ge H) \ge 1/96$. Since $X \ge |Y|$, summing (6) over the positive rows gives

$$b \ge \sum_{r=1}^{H} r\,\mathbb{P}_z(Y=r) \ge \frac{zH(H+1)}{192(2m+1)^{3/2}}.$$

Now $z \to \mu^{-1} > 0$, $H^2 \ge c_1 M$ for large M, and $2m+1 \le C(1+b)$. Hence $b(1+b)^{3/2} \ge c_2 M$. This forces $b \to \infty$ and then gives $b \ge c_3 M^{2/5}$.

Finally, let $q_n = \mathbb{P}_z(T=n)$ and recall that $b_n = \mathbb{E}_n[X]$. Conditional uniformity gives $b = \sum_n q_n b_n$, while concavity gives $\sum_n q_n n^{2/5} \le M^{2/5}$. If $b_n < (c_3/2)n^{2/5}$ for all sufficiently large n, the finitely many smaller lengths would contribute at most a fixed constant, giving $b \le O(1) + (c_3/2)M^{2/5}$. This contradicts $b \ge c_3 M^{2/5}$ as $M \to \infty$. Thus $b_n \ge (c_3/2)n^{2/5}$ for infinitely many n, and $d_2(n) \ge b_n$ completes the proof. □

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