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Forcing and Avoiding Monochromatic Pairs $\{x+y, xy\}$

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Abstract

Let $R(r)$ be the least N for which every r-coloring of $[N]$ contains a monochromatic pair $\{x+y, xy\}$ with $x > y > 2$. We prove that

$$\exp\left(c \cdot \frac{r \log r}{\log \log r}\right) \leq R(r) \leq \exp\left(2^r r^C\right)$$

for absolute $c, C > 0$ and all sufficiently large r.

1 Introduction

In 1979, Hindman [1] investigated when a finite coloring of the positive integers forces sums and products to have the same color. The two-element case of his finite sums and products conjecture asks for distinct x, y such that $x, y, x+y, xy$ all have one color. Moreira [2] proved in 2017 that every finite coloring contains infinitely many monochromatic triples $\{x, x+y, xy\}$. A 2026 preprint of OpenAI [3] gives a proof of the finite sums and products conjecture for sets of every prescribed size.

We ask how large an interval must be to force the sum and the product to have the same color. Write $[N] = \{1, \ldots, N\}$. Following Green and Sawhney [4], let $R(r)$ be the least N such that every map $\chi : [N] \to [r]$ admits integers $x > y > 2$ with $x+y, xy \in [N]$ and $\chi(x+y) = \chi(xy)$.

Theorem 1.1. *There are absolute constants $c, C > 0$ such that, for all sufficiently large r,*

$$\exp\left(c \cdot \frac{r \log r}{\log \log r}\right) \leq R(r) \leq \exp\left(2^r r^C\right).$$

The coloring for the lower bound distinguishes $x+y$ from xy for all $x \geq y \geq 1$ except $x = y = 2$.

Green and Sawhney [4] constructed an r-coloring of $[(3^r+7)/2]$ with no such pair. They also proved that $N \geq \exp(\exp(r^{50}))$ suffices for large r. Theorem 1.1 gives $R(r)^{1/r} \to \infty$ and brings the upper bound for $\log \log R(r)$ down to $r \log 2 + O(\log r)$. It remains open, for instance, whether $\log R(r)$ is at most a polynomial in r.

2 The lower bound

Green and Sawhney use one color on each of a sequence of intervals whose endpoints grow by a factor close to three. We instead cover intervals that grow by a factor B with $O(\log \log B)$

colors. Suppose that $x \geq y$ and that both $x+y$ and xy lie in $[X, BX)$. Then $x \geq X/2$, and $y/2 \leq xy/(x+y) = y/(1+y/x) < B$, so $y < 2B$. For a fixed such y, the displacement on the logarithmic scale is nearly $\log y$, since

$$\log(xy) - \log(x+y) = \log y - \log(1+y/x), \qquad 0 \leq \log(1+y/x) \leq \frac{4B}{X}.$$

We first try to distinguish $x+y$ from xy by counting factors of two. Write $v(n)$ for the exponent of two in n. If $v(x) \neq v(y)$, then $v(x+y) = \min\{v(x), v(y)\}$, while $v(xy) = v(x) + v(y)$. But the sum and product can have the same exponent. For example, when $x = 14$ and $y = 6$, the sum is $20 = 4 \cdot 5$ and the product is $84 = 4 \cdot 21$. Both contain exactly two factors of two.

For pairs that this first test misses, we use the estimate above: $\log(xy) - \log(x+y)$ is close to $\log y$ when x is large compared with y. We give n a second color according to which quarter of the unit circle contains $t \log n$ modulo one. To separate the sum and product, it suffices to choose t so that their positions on the circle are more than a quarter-turn apart. The next lemma lets us choose one t for which $t \log y$ is close to $\alpha v(y)$ modulo one for every small y. We will choose α separately for each range of exponents so that this gives the required separation. Write $\|z\|$ for the distance from z to the nearest integer.

Lemma 2.1. *For every sufficiently large integer M and every real α, there is a real $t \in [0, \exp(M^4)]$ such that $\|t \log j - \alpha v(j)\| < 1/32$ for every $1 \leq j \leq M$.*

Proof. Apply Turán's simultaneous approximation lemma for prime logarithms (see Weber [6, p. 98]) to the first M primes, with $\omega = M - 1$. Prescribe the target α at 2 and zero at all the other primes. For large M, the theorem gives accuracy $1/(M-1)$ at some $0 \leq t \leq \exp(17(M-1)M(\log M)^2) \leq \exp(M^4)$. Every $j \leq M$ has at most $\log_2 M$ prime factors counted with multiplicity. The targets of its factors sum to $\alpha v(j)$, and their errors add to at most $(\log_2 M)/(M-1) < 1/32$. □

Lemma 2.2. *There is an absolute constant C_0 such that, for every sufficiently large integer B and every real $X \geq \exp(B^5)$, the integers in $[X, BX)$ can be colored with at most $C_0 \log\log B$ colors so that $x+y$ and xy have different colors whenever both lie in the interval. Here x, y may be any positive integers.*

Proof. Put $M = 2B$. Give the odd integers one color and the integers with $v(n) > 2\log_2 M$ another. Group the remaining integers according to the dyadic interval $[a, 2a)$ containing $v(n)$, where a runs through powers of two. Different groups will use disjoint sets of colors.

Take $x \geq y$ as above. If $v(x) \neq v(y)$, then $v(x+y) = \min\{v(x), v(y)\} \leq v(y) \leq \log_2 M$, whereas $v(xy) = v(x) + v(y) > 2v(x+y)$. If $v(x+y) = 0$, then $x+y$ is odd and xy even. If $v(x+y) \in [a, 2a)$, then $v(xy) \geq 2a$. Either way the groups differ. Two odd inputs also give opposite parities for $x+y$ and xy. The remaining case is $v(x) = v(y) = d \geq 1$. Since $v(xy) = 2d \leq 2\log_2 M$, a common group must be a dyadic interval $[a, 2a)$, and

$$a \leq 2d < 2a, \qquad \text{so} \qquad \frac{1}{3} \leq \frac{2d}{3a} < \frac{2}{3}.$$

For each band choose t_a by Lemma 2.1 with $\alpha = 2/(3a)$. Color n by the half-open quarter of $[0,1)$ containing the fractional part of $t_a \log n$. This takes four colors for each of $O(\log\log B)$ bands, in addition to the two colors already set aside. For a pair in the same band, $t_a \log y$ lies within $1/32$ of $2d/(3a)$ modulo one. Passing from $t_a \log y$ to $t_a \log(xy/(x+y))$ subtracts $t_a \log(1+y/x)$. By the estimate at the start of the section this is at most $4B\exp((2B)^4)/X < 1/32$ for large B, since $X \geq \exp(B^5)$. The distance from $2d/(3a)$ to the nearest integer is at least $1/3$. The two fractional parts lie in different quarters, since

$$\|t_a \log(xy) - t_a \log(x+y)\| \geq \frac{1}{3} - \frac{2}{32} > \frac{1}{4}. \qquad \square$$

Proof of the lower bound in Theorem 1.1. Take $B = \lfloor r^{1/10} \rfloor$ and $X = \lceil \exp(B^5) \rceil$. Below X, color n by its parity and by $\lfloor \log_{6/5} n \rfloor$. This uses $O(\log X) = O(\sqrt{r})$ colors. If $y = 1$, the sum and the product are consecutive. If $x \geq y \geq 2$ and $(x,y) \neq (2,2)$, then $x \geq 3$ and $xy/(x+y) \geq 6/5$, its value at $(3,2)$, since the ratio increases with each factor. The two logarithms in base $6/5$ therefore differ by at least one..

Set $k = \lfloor r/(2C_0 \log\log B) \rfloor$. Apply Lemma 2.2 to $[XB^j, XB^{j+1})$ for $0 \leq j < k$, using fresh colors each time. These intervals use at most $r/2$ colors in total, and the initial segment uses at most $r/2$ more for large r. Values in different intervals have different colors. The coloring of $[XB^k - 1]$ therefore separates every pair $\{x+y, xy\}$ with two distinct elements in that interval. In particular, $R(r) \geq XB^k$. Now $\log B \sim (\log r)/10$ and $k \gg r/\log\log r$, giving the asserted lower bound. $\square$

3 The upper bound

We follow the approach of Richter [5] and Green and Sawhney [4] by first finding a product xy of the same color as x. Choose $r+1$ coefficients, each dividing the next, and multiply them all by n. Two of these multiples, say cn and dn with $c \mid d$, have the same color. Taking $x = cn$ and $y = d/c$ gives

$$xy = dn, \qquad x + y = c\left(n + \frac{d}{c^2}\right).$$

We want the shifted value $c(n + d/c^2)$ to match the color of dn.

To try several factorizations of the same product, we let $c = A/s$, where s ranges from 1 to m and A is a common multiple of these integers. Increasing s makes $x = An/s$ smaller and $y = ds/A$ larger, while their product stays fixed at dn. The shift becomes $d/c^2 = (d/A^2)s^2$, so the shifts are square multiples of a common step. We choose d so that this step is $d/A^2 = qu$, where q is a fixed positive integer and u is a random multiplier. The next lemma chooses the distribution of u to change very little under multiplication by small positive integers, which lets us compare the shifts qs^2u after averaging.

Lemma 3.1. *Let $m \geq 3$ be an integer, $K = m^2 + m$ and $U = (K!)^m$. There is a probability distribution on the positive divisors of U with the following property. Fix a positive integer q and an interval I of $X \geq qU^2$ consecutive integers. Draw u from this distribution, n uniformly from I, and s uniformly from $[m]$, independently. For any functions $f_s : \mathbb{Z} \to [0,1]$ and $g : I \times [U] \to [0,1]$,*

$$\mathbb{E}\big[f_s(n + qs^2u)g(n,u)\big] \;\geq\; \big(\mathbb{E}[f_s(n)g(n,u)]\big)^2 - \frac{5}{\sqrt{m}}. \tag{1}$$

In our application $f_s(n)$ tests the color of $(A/s)n$ and $g(n,u)$ tests the color of A^2qun. The earlier test may depend on s. The later test is the same for every s, a fact used in the proof of Lemma 3.1 in Section 4.

Proof of the upper bound in Theorem 1.1. Take $m = r^{16}$ and U as in Lemma 3.1. Every possible u divides U, as does every $s \in [m]$. Choose deterministic integers

$$L_0 = U, \qquad L_{j+1} = U^3 L_j^2 \quad (0 \le j < r).$$

At each level $0 \le j \le r$ draw u_j from the lemma's distribution and s_j uniformly from $[m]$, with all choices independent. Put $c_j = L_j u_j / s_j$. These are integers since $s_j \mid U \mid L_j$. The factor U^3 in the recursion will supply the divisors s_b and u_a^2 when we compare levels $a < b$.

Write $H = UL_r$, so $1 \le c_j \le H$, and choose n independently and uniformly among the integers in $I = [H^3, 2H^3)$. Given an r-coloring χ of $[N]$, with $N = 3H^4$, consider the $r+1$ points $c_j n$. All lie in $[N]$, since $c_j n < 2H^4$.

At least two of these points have the same color. There are $\binom{r+1}{2}$ pairs of indices and r possible colors. Hence some fixed $a < b$ and color γ satisfy

$$\Pr(\chi(c_a n) = \chi(c_b n) = \gamma) \;\ge\; \frac{1}{r\binom{r+1}{2}} \;\ge\; r^{-3}.$$

By averaging, fix values of u_a and s_b for which this conditional probability is still at least r^{-3}. The variables s_a, u_b, n retain their independent laws. Since $L_{a+1} \mid L_b$ and s_b, u_a both divide U, we may take $A = L_a u_a$ and the positive integer

$$q = \frac{L_b}{s_b A^2} = \frac{L_b}{L_{a+1}} \cdot \frac{U}{s_b} \left(\frac{U}{u_a}\right)^2.$$

Then $c_a = A/s_a$ and $c_b = A^2 q u_b$. Also $q \le L_b \le H$ and $U \le H$, so $|I| = H^3 \ge qU^2$.

Extend χ to $\mathbb{Z}$ arbitrarily and set

$$f_s(z) = \mathbf{1}_{\{\chi((A/s)z)=\gamma\}}, \qquad g(n,u) = \mathbf{1}_{\{\chi(A^2qun)=\gamma\}}.$$

Here $s \mid A$ for every $s \in [m]$. The conditional probability above is exactly $\mathbb{E}[f_{s_a}(n) g(n, u_b)]$. By (1),

$$\mathbb{E}[f_{s_a}(n + qs_a^2 u_b) g(n, u_b)] \ge r^{-6} - 5r^{-8} > 0$$

for sufficiently large r. Both indicators therefore equal one for some choice of s_a, u_b, n. With $x = (A/s_a)n$ and $y = Aqs_a u_b$, the corresponding arguments of χ are

$$x + y = (A/s_a)(n + qs_a^2 u_b), \qquad xy = A^2 q u_b n = c_b n.$$

Thus both have color γ.

Finally, $y \ge A \ge U > 2$ and $y = c_b/c_a \le H$, while $H^3 \le x < 2H^4$. Hence $x > y > 2$, $xy < 2H^4$ and $x + y < 2H^4 + H \le N$. The recursion gives $L_r = U^{4 \cdot 2^r - 3}$, since $U^3 L_{j+1} = (U^3 L_j)^2$. As $\log U = m \log(K!) = O(m^3 \log m)$, we obtain $\log N = O(2^r m^3 \log m) \le 2^r r^C$ for an absolute C and all sufficiently large r. $\square$

4 Averaging the quadratic shifts

Two points $n + qku$ and $n + qju$ in the same arithmetic progression differ by $q(j-k)u$. Averaging over j, k therefore counts pairs of points in that progression. We first make this count and then choose u so that the square shifts qs^2u give nearly the same average.

Proof of Lemma 3.1. Identify I with the cyclic group $\mathbb{Z}/X\mathbb{Z}$, using the integers of I as representatives. For any fixed $s \le m$ and positive $u \le U$, adding qs^2u can cross the right endpoint of I for at most qm^2U starting points. Thus reducing the shifted point modulo X changes the left side of (1) by at most $qm^2U/X \le m^2/U \le 1/\sqrt{m}$. In the rest of the proof we add modulo X, so that every translate of uniform n is again uniform.

Write $\bar{f} = \mathbb{E}_s f_s$ and let j, k be independent and uniform on $[m]$. For an integer v, form the two averages

$$A(n, v) = \mathbb{E}_s f_s(n + qs^2v), \qquad B(n, v) = \mathbb{E}_{j,k} \bar{f}(n + q(j-k)v).$$

For the moment fix any positive divisor u of U. Put $h(n) = g(n, u)\bar{f}(n)$. Both $g(n, u)$ and $\bar{f}(n)$ dominate $h(n)$, since both take values in $[0, 1]$. It follows that

$$\begin{aligned}
\mathbb{E}_n[g(n, u)B(n, u)] &\ge \mathbb{E}_{n,j,k}\big[h(n)h(n + q(j-k)u)\big] \\
&= \mathbb{E}_{n,j,k}\big[h(n + qku)h(n + qju)\big] \\
&= \mathbb{E}_n\big(\mathbb{E}_j h(n + qju)\big)^2 \;\ge\; \big(\mathbb{E}_n h(n)\big)^2.
\end{aligned}$$

The first equality translates n by qku. The second uses independence of j and k, and the last inequality is Cauchy–Schwarz. Average in u and apply Cauchy–Schwarz once more. For any law of u this gives

$$\mathbb{E}[g(n, u)B(n, u)] \ge \big(\mathbb{E}[g(n, u)\bar{f}(n)]\big)^2 = \big(\mathbb{E}[g(n, u)f_s(n)]\big)^2. \tag{2}$$

To compare the two averages, consider the square shifts qv and $4qv$. Their gap is $3qv$, so after translating the first point to n, the second is $n + 3qv$. We want multiplication by 3, or by any other nonzero difference of coefficients, to have little effect on the distribution of the multiplier. Let $e_2, \dots, e_K$ be independent and uniform on $\{0, \dots, m-1\}$, and put

$$u = \prod_{d=2}^{K} d^{e_d}.$$

This is a positive divisor of $(K!)^m = U$. Give u an independent fair sign and write $V = \pm u$. For every integer b with $1 \le |b| \le K$ and every function $\phi : \mathbb{Z} \to [0, 1]$,

$$\big|\mathbb{E}\phi(bV) - \mathbb{E}\phi(V)\big| \le 1/m. \tag{3}$$

To see this for $2 \le b \le K$, fix the sign and all exponents except e_b. Multiplication by b changes its range from $\{0, \dots, m-1\}$ to $\{1, \dots, m\}$. The two averages have $m-1$ terms in common and their remaining terms differ by at most one. The case $b = 1$ is exact, and the fair sign gives the same estimate for negative b.

It remains to compare A and B. Write $\delta = j - k$, independent of s. Since $\bar{f} = \mathbb{E}_s f_s$,

$$A(n, v) - B(n, v) = \mathbb{E}_{s,\delta}[f_s(n + qs^2 v) - f_s(n + q\delta v)].$$

To square this average, take an independent copy (t, η) of (s, δ). For fixed s, t, abbreviate $C(a, b) = \mathbb{E}_{n,V}[f_s(n + qaV) f_t(n + qbV)]$. Expanding the product of the two brackets gives

$$C(s^2, t^2) - C(s^2, \eta) - C(\delta, t^2) + C(\delta, \eta). \tag{4}$$

The expectation of this expression over s, δ, t, η is $\mathbb{E}_{n,V}|A(n, V) - B(n, V)|^2$.

For distinct coefficients a, b in (4), the gap satisfies $1 \le |b - a| \le K$, since all coefficients lie between $1 - m$ and m^2. For fixed V, translate n by qaV. This gives

$$C(a, b) = \mathbb{E}_{n,V}[f_s(n) f_t(n + q(b - a)V)].$$

By (3), with $\phi(v) = \mathbb{E}_n[f_s(n) f_t(n + qv)]$, this differs from $\mathbb{E}_{n,V}[f_s(n) f_t(n + qV)]$ by at most $1/m$. That last quantity is the same for all four terms. If $\{s^2, \delta\} \cap \{t^2, \eta\} = \varnothing$, it cancels in (4), leaving an error of at most $4/m$.

Each value of s^2 has probability at most $1/m$, and the same is true of $\delta = j - k$, since for each k at most one j gives a specified difference. The two pairs of coefficients are independent. Each of the four possible equalities across the pairs therefore has probability at most $1/m$. On their union, (4) has absolute value at most one, since it is the mean of a product of two differences of $[0, 1]$-valued functions. We conclude that

$$\mathbb{E}_{n,V}|A(n, V) - B(n, V)|^2 \le 4/m + 4/m = 8/m.$$

Conditioning on $V > 0$ recovers the law of u and at most doubles this nonnegative mean. Since $0 \le g \le 1$, Cauchy–Schwarz now gives

$$\left|\mathbb{E}[g(n, u)(A(n, u) - B(n, u))]\right| \le 4/\sqrt{m}.$$

The average of $g(n, u)A(n, u)$ is the shifted average in (1) on the cyclic group. Combining the last inequality with (2) and allowing $1/\sqrt{m}$ for the endpoint of I proves (1). □

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