On reachability in Hernandez–Leclerc categories

Bob Smith

November 24, 2026

Abstract

We prove that a real simple module in a finite Hernandez–Leclerc window is reachable when every component of its high-defect subquiver is mutation-acyclic or has a dense cluster g-vector fan. We prove density for X_6 and obtain a finite-mutation criterion. We also prove reachability for simple modules with prehomogeneous separator PBW parameters. The resulting mutation paths belong to the original cluster structure and preserve its frozen variables.

hexagon:2611.00975v1

Contents

1 Introduction **2**
1.1 Real simple modules and cluster monomials 2
1.2 Known cases and categorical methods 3
1.3 Reachability criteria 3
1.4 Related results and the proofs 5

2 The original window and its bases **5**
2.1 The based cluster structure 5
2.2 Laurent positivity and nonnegative exponents 7

3 Strict support from real powers **7**
3.1 One fixed standard-element formula 8
3.2 Eliminating corrections with nonnegative pairing 9

4 From strict support to a reachable cone **10**
4.1 The leading theta function 10
4.2 A wall-free open set 11
4.3 Two ways to obtain one reachable chamber 12
4.4 Components, the actual simple class, and denominators 14

5 Unit defects and the high-defect subquiver **14**
5.1 Tensor products of defect one 14
5.2 A finite recursion with a fixed core 16
5.3 Finite mutation type and the four windows 16

6 The exceptional quiver X_6 **17**
6.1 The root and its negative leaf half-space 17
6.2 Return words and normalization 19
6.3 The closed polyhedral calculation 20
6.4 Simultaneous limits and completion of the proof 21

7 The separator model **22**
7.1 A reduced word and its hereditary quiver 23
7.2 The original ice matrix and auxiliary normalization 24
7.3 The basis comparison 26

8 Prehomogeneous PBW parameters **27**
8.1 An exact realization of the whole stratum 27
8.2 Open orbits give actual rigidity 29
8.3 Finite paths from the prescribed GLS root 30
8.4 Full degree labels and the return to the original window 33

A An exact certificate for the X_6 comparison **35**

References **38**

1 Introduction

1.1 Real simple modules and cluster monomials

Let $\mathfrak{g}$ be a simple Lie algebra of type ADE, and let $\varepsilon \in \mathbb{C}^\times$ be not a root of unity. We study finite-dimensional type 1 modules over $U_\varepsilon(\widehat{\mathfrak{g}})$. Let D be the Dynkin vertex set. Their simple objects are parametrized by dominant monomials in the fundamental ℓ-weights $Y_{i,a}$; we write $L(m)$ for the simple module of highest ℓ-weight m. The q-character of Frenkel and Reshetikhin records all ℓ-weights and embeds the Grothendieck ring into a Laurent polynomial ring [18]. A simple module S is *real* if $S \otimes S$ is simple. A nontrivial simple module is *prime* if it has no factorization as a tensor product of two nontrivial simple modules. Here nontrivial means different from the tensor unit.

The distinction between real and imaginary elements arose in the study of multiplication in the dual canonical basis. Leclerc called a basis element real when its square is another basis element up to a power of the quantum parameter, and imaginary otherwise [34]. His imaginary examples disproved the unrestricted Berenstein–Zelevinsky prediction that two quasi-commuting basis elements have a product proportional to a basis element. They also gave simple quantum affine modules with reducible tensor square. Leclerc then formulated a refined product conjecture with one real factor. Kang, Kashiwara, Kim and Oh proved that product conjecture for symmetric generalized Cartan data in characteristic zero [31, Section 4.2]. The question studied here is the Hernandez–Leclerc *multiplicative reachability* problem.

Fomin and Zelevinsky introduced cluster algebras as a framework for dual canonical bases and total positivity [17]. In geometric type, a seed consists of algebraically independent variables and an exchange matrix; mutation replaces one mutable variable using its exchange relation. A cluster monomial is a product of variables in one seed with nonnegative integral exponents. Throughout this paper, frozen variables may occur with nonnegative exponents as well. A simple module is *reachable* when its class is such a monomial in a seed obtained by a finite sequence of ordinary mutations from the prescribed initial seed.

Hernandez and Leclerc introduced finite spectral subcategories $\mathcal{C}_\ell$ and conjectured that their Grothendieck rings have specified cluster structures with Kirillov–Reshetikhin initial variables [27, Conjecture 13.2]. Their original definition of monoidal categorification requires all real simple classes to be cluster monomials and all real prime classes to be cluster variables [27, Definition 2.1]. Thus the original conjecture combines the ring identification, the realization of cluster monomials by simple modules, and the reachability of every real

simple module. The latter assertion is restated in [29, Conjecture 5.7] and, for common triangular bases, in [39, Conjectures 5.7–5.8].

1.2 Known cases and categorical methods

The level-one simply laced categories provide the first complete cases. Nakajima used graded quiver varieties and Fourier transform to identify the relevant geometry with representations of hereditary quivers [36]. In finite type ADE, every simple object of the level-one category is real, and the prime objects, apart from the frozen ones, are parametrized by almost-positive roots [36, Corollary 6.14]. Real Schur roots give the noninitial cluster variables, and the tensor-product criterion identifies the factorizations belonging to one cluster [36, Propositions 7.5–7.6]. The use of an open orbit in a hereditary representation space is an antecedent of the Poincaré–Birkhoff–Witt (PBW) criterion considered below.

On the additive side, Geiss, Leclerc and Schröer constructed cluster structures using rigid modules over preprojective algebras and placed cluster monomials in the dual semicanonical basis [21]. Their reduced-word categories provide cluster-tilting objects, layer filtrations and finite mutation sequences [22]; subsequent work relates these constructions to generic bases, PBW parameters and quantum coordinate rings [23, 24]. Here an object is rigid when its self-extension space $\operatorname{Ext}^1(X,X)$ vanishes.

Kang, Kashiwara, Kim and Oh established monoidal categorifications of quantum unipotent coordinate rings using symmetric quiver Hecke algebras [31]. In particular, quantum cluster monomials belong to the upper global basis. Qin's common triangular bases give compatible degree labels in every seed and identify the quantum affine simple basis in the relevant categories [38]. These results establish the passage from cluster monomials to real simple modules. The converse is expressed by the multiplicative reachability conjectures in [39]. The exact word-seed identifications and basis constructions of [40, 41] will be used here with all original frozen coordinates retained.

Several explicit families admit finite mutation constructions. Hernandez and Leclerc gave an algorithm for the q-characters of Kirillov–Reshetikhin modules in every untwisted type [28, Theorem 3.1 and Remark 3.3]. Duan, Li and Luo proved reachability of prime snake modules in types A and B through their S-systems [13]. Duan and Schiffler constructed the categories associated with arbitrary height functions, described the level-one real simples, and proved the proposed correspondences in the cases of finite cluster type [14]. Their later cluster modules form another family of reachable real modules, including Kirillov–Reshetikhin and Hernandez–Leclerc modules [15, Theorems 4.10 and 4.12].

Recent work also treats categories of infinite cluster type. Sakamoto classifies real and imaginary modules and proves the real-simple/cluster-monomial correspondence for the affine-type interval categories specified in [43, Theorem 1.4], including families of affine D and E cluster type. Baur, Fu and Li prove connectivity of the exchange graphs of the tame Grassmannian categories for $\mathrm{Gr}(3,9)$ and $\mathrm{Gr}(4,8)$, and formulate a conjectural correspondence between additive and monoidal categorifications [3, Theorem 4.3 and Conjecture 3.14]. Here $\mathrm{Gr}(k,n)$ is the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$.

1.3 Reachability criteria

Fix a height function $\xi : D \to \mathbb{Z}$, so that $|\xi_i - \xi_j| = 1$ for adjacent vertices, and an integer $\ell \geq 0$. We use the upper-endpoint convention

$$\mathcal{C}_{\ell}^{\leq \xi} = \big\langle L(Y_{i,\varepsilon^{\xi_i - 2r}}) : i \in D,\ 0 \leq r \leq \ell \big\rangle. \tag{1.1}$$

The brackets denote the full subcategory whose composition factors have highest ℓ-weights in the monoid generated by the displayed fundamental weights. Its original frozen variables are

the longest Kirillov–Reshetikhin modules of the window. The initial seed and its complete basis identification are specified in Section 2.

We first define the defect used in the main criterion. For simple modules U, V, let V_z be the spectral translate obtained by multiplying every spectral parameter of V by z. Normalize the rational intertwiner

$$R^{\mathrm{norm}}_{U,V_z} : U \otimes V_z \longrightarrow V_z \otimes U$$

to send the tensor product of highest ℓ-weight vectors to its reversed tensor product. Let $d_{U,V}(z)$ be the monic polynomial of least degree clearing its denominators over $\mathbb{C}[z, z^{-1}]$. Set

$$\mathfrak{d}(U, V) = \mathrm{ord}_{z=1}(d_{U,V}(z)d_{V,U}(z^{-1})). \tag{1.2}$$

Here the order is the multiplicity of the zero at $z = 1$. This is the symmetric nonnegative integer of [33, Proposition 2.14]. If one factor is real, $\mathfrak{d}(U, V) = 0$ is equivalent to simplicity of $U \otimes V$; such modules *strongly commute* [33, Corollary 2.15].

Let t be a reachable seed, with mutable index set M, frozen index set F, simple modules P_i, and principal quiver Q_t. For a real simple module S, define

$$H_t(S) = \{i \in M : \mathfrak{d}(P_i, S) \geq 2\}. \tag{1.3}$$

A quiver is mutation-acyclic if a finite sequence of mutations makes it acyclic. Its reachable g-vector fan is the collection of cones spanned by the mutable g-vectors of finitely reachable seeds, together with their faces, computed with principal coefficients and one initial seed. The fan is dense if the union of these cones is dense in the full real g-vector space.

Theorem 1.1. *Let S be a real simple module in $\mathcal{C}_\ell^{\leq \xi}$. Suppose that, for one reachable seed t, each connected component of the induced quiver $Q_t[H_t(S)]$ is either mutation-acyclic or has a dense reachable g-vector fan. Then S is reachable. Starting at t, a witnessing mutation path can be chosen to fix every P_j for which $\mathfrak{d}(P_j, S) = 0$.*

The acyclic alternative includes wild quivers. The hypotheses are tested on the induced high-defect subquiver, and the two alternatives may occur in different components. The threshold 2 results from a finite elimination of all defect-one coordinates.

Finite mutation type means that the mutation class contains finitely many quivers up to isomorphism.

Corollary 1.2. *The conclusion of Theorem 1.1 holds if every component of $Q_t[H_t(S)]$ has finite mutation type.*

The density results of Yurikusa and Plamondon–Yurikusa [46, 37] supply most of the finite-mutation cases. In Section 6 we prove density for X_6, the assertion of [37, Conjecture 1.3(1)]. Applying the support criterion to the entire mutable quiver also gives the following four windows.

Corollary 1.3. *Every real simple module in any of the following standard Hernandez–Leclerc windows is reachable:*

$$(D_4, 2), \qquad (A_3, 3), \qquad (A_2, 5), \qquad (A_5, 2).$$

The entries indicate the Dynkin type and the parameter ℓ. The standard window uses a bipartite height function $\xi_i \in \{0, 1\}$; a common spectral translation gives an equivalent statement.

The second criterion uses a hereditary representation space. Section 7 assigns to each simple module a PBW vector $\mathbf{a}$ for a specified enlarged reduced word and an acyclic quiver K. We call this vector *prehomogeneous* if the affine variety $\mathrm{rep}(K, \mathbf{a})$ has a Zariski open orbit under the product of the general linear groups at its vertices.

Theorem 1.4. *A simple module in $\mathcal{C}_\ell^{\leq \xi}$ whose separator-model PBW vector is prehomogeneous is real and reachable.*

The construction uses the full extension spaces between the reduced-word layers. It produces a finite mutation path in the enlarged category and transports the same mutable labels to the original window. The original frozen variables retain nonnegative exponents in the resulting cluster monomial.

1.4 Related results and the proofs

Hernandez Juan's recent preprint states a correspondence between real simples, after removing frozen tensor factors, and rigid objects in every finite window, together with a generic-kernel formula for real simple modules [30, Theorems 1.2–1.3]. Combined with ordinary g-fan density and the index and frozen-factor identifications, that correspondence also gives the four consequences in Corollary 1.3. We give independent proofs through the support criterion. Real-power and theta-chamber arguments also occur in [30, Sections 3.2–3.8]. Smith's recent preprint states equality of the dual canonical elements specialized at quantum parameter 1 and the dual semicanonical elements with the same crystal label attached to open-orbit components for symmetric Cartan data [45, Theorem 1.1]. Our PBW proof constructs a finite ordinary mutation path and then identifies its cluster monomial by its full degree label.

The support argument begins with the normalized quantum simple class. If its leading degree is a, every nonleading Laurent exponent has the form $a + \tilde{B}e$, where $\tilde{B}$ is the full exchange matrix and e is a nonzero nonnegative charge vector. Real powers and triangularity imply $a_M \cdot e < 0$. A finite Schur elimination excludes $a_M \cdot e \geq 0$ for every nonzero correction charge e. Valuative independence [10] transfers the resulting Newton-polytope constraint to the leading theta function. The associated wall-free open set places its degree in one reachable cone, using either dense fans or the acyclic chamber theorem [2].

The tensor product of two real simple modules with defect one has two real simple composition factors [33]. We use their fixed-seed defect inequalities to form a finite recursion. A reconstruction lemma turns the reachability of the two factors into that of their parent and preserves all zero-defect variables. For the PBW criterion, an explicit exact functor covers the entire prescribed layer stratum. An open hereditary orbit gives a rigid preprojective module. Schofield induction and reachable co-Bongartz completions give its finite mutation path. Auxiliary normalization and the comparison of common triangular bases return this path to the original Kirillov–Reshetikhin seed.

2 The original window and its bases

2.1 The based cluster structure

Throughout, $\mathbb{N} = \mathbb{Z}_{\geq 0}$. The vector $\mathbf{e}_j$ denotes the standard coordinate vector in the indicated lattice; e_j denotes the scalar jth coordinate of a vector e. For a real number r, write $[r]_+ = \max(r, 0)$; positive and negative parts of vectors are taken coordinatewise. We write $\mathcal{A}_+ = K_0(\mathcal{C}_\ell^{\leq \xi})$ and $\mathcal{A} = \mathcal{A}_+[P_f^{-1} : f \in F]$. The distinction is used whenever a denominator is removed. Choose a Coxeter word c in decreasing order of ξ, with any order

among nonadjacent equal-height vertices, and put $\mathbf{j} = c^{\ell+1}$. Occurrence (i, r) is assigned the fundamental weight $Y_{i,\varepsilon^{\xi_i-2r}}$. The word seed variable at that occurrence is

$$x_{i,r} = \left[L\left(\prod_{s=0}^{r} Y_{i,\varepsilon^{\xi_i-2s}} \right) \right]. \tag{2.1}$$

Its last occurrences, $r = \ell$, are frozen. Qin's exact seed identification and triangular-basis theorem apply to this adapted word [40, Section 7.6 and Theorem 7.10]. Thus this is an identification of the original nonlocalized based rings, with the variables (2.1) fixed.

The quantum deformation $\mathcal{A}_v$ of $\mathcal{A}$ has a positive common triangular basis $\{L_a\}$, consisting of frozen-normalized actual simple classes. In a reachable seed, let $L = \mathbb{Z}^{M \sqcup F}$ and $P = \widetilde{B} : \mathbb{Z}^M \to L$ be the full extended exchange matrix. The normalized quantum torus satisfies

$$X^u X^w = v^{\Lambda(u,w)} X^{u+w}, \qquad \Lambda(u, Pe) = -\delta\, u_M \cdot e, \qquad \delta > 0. \tag{2.2}$$

Here Λ is an integral skew-symmetric form, and v is the quantum deformation parameter. The bar involution is the anti-automorphism determined by $\overline{v} = v^{-1}$ and $\overline{X^u} = X^u$. Positivity means that basis products have coefficients in $\mathbb{N}[v^{\pm 1}]$. Compatibility implies that P is injective. We choose a common rescaling of v giving the same positive multiplier δ throughout the simply laced connected window. Every reachable seed is injective-reachable: a finite mutation word gives mutable g-vectors $-\mathbf{e}_j$, up to permutation. Its normalized injective standard elements I_a satisfy

$$L_a = I_a + \sum_{h \in \mathbb{N}^M \setminus \{0\}} b_a(h) I_{a+Ph}, \qquad b_a(h) \in v^{-1}\mathbb{Z}[v^{-1}]. \tag{2.3}$$

The dominance completion consists of formal series in nonnegative charges $h \in \mathbb{N}^M$; each equality below is read on finite componentwise intervals $[0, E]$. Injectivity of P makes these charge labels unique. The basis element L_a has a finite pointed Laurent expansion

$$L_a = X^a + \sum_{0 \neq e \in \mathbb{N}^M} c_e(v) X^{a+Pe}. \tag{2.4}$$

These statements, including the coefficient direction in (2.3), follow from the injective standard construction [41, Section 6.1, Lemma 6.2, Proposition 6.5 and Theorem 6.7(2)]. Positivity and the simple-class identification are also available in [38, Sections 7.1, 8.4 and 9.1].

The longest KR modules strongly commute with every simple in the window [38, Proposition 9.1.5]. Consequently the localization above is obtained by multiplying simple classes by inverse frozen monomials.

Lemma 2.1. *If S is real and L_a is its normalized quantum class, then*

$$L_a^r = L_{ra} \qquad (r \geq 1). \tag{2.5}$$

If S strongly commutes with P_j for $j \in M$, every correction in (2.4) *has $e_j = 0$.*

Proof. The pairwise simplicity theorem for tensor products [26] makes every power of S simple. Expand L_a^r in the positive quantum simple basis. At $v = 1$ the sum of the nonnegative coefficients is one. Thus only one basis element occurs, with a coefficient v^m. Its leading degree is ra, and its leading coefficient is one by $\Lambda(a, a) = 0$, so $m = 0$.

The same argument applied to $P_j \otimes S$ in both orders gives

$$X^{\mathbf{e}_j} L_a = v^{2\Lambda(\mathbf{e}_j, a)} L_a X^{\mathbf{e}_j}.$$

Comparing the coefficient of $X^{a+Pe+\mathbf{e}_j}$ and using the injectivity of P gives $\Lambda(\mathbf{e}_j, Pe) = 0$. Compatibility (2.2) says that this is $-\delta e_j$ when $j \in M$. Hence the jth correction coordinate is zero. □

2.2 Laurent positivity and nonnegative exponents

Lemma 2.2. *The class of every actual object has nonnegative integral Laurent coefficients in every reachable seed, with all original frozen coordinates retained.*

Proof. Let V be an object of the category. Choose a nonnegative seed monomial D clearing all Laurent denominators of $[V]$. Write $D[V] = \sum_u n_u x^u$ as a polynomial. Every x^u on the right is the class of a simple module, and distinct exponents give distinct simple classes by algebraic independence. On the left, $D[V]$ is the class of an actual tensor product. Independence of the simple classes forces each n_u to be its corresponding Jordan–Hölder multiplicity. Thus $n_u \geq 0$. □

Lemma 2.3. *If an element of $\mathcal{A}_+$ is a Laurent monomial with coefficient one in a reachable seed, all its exponents are nonnegative.*

Proof. Suppose its exponent at a mutable variable x_k is negative. In the adjacent seed, use

$$x_k x_k^* = F_k = M_+ + M_-, \qquad M_\pm = \prod_i x_i^{[\pm b_{ik}]_+}.$$

The two exchange monomials are distinct because P has no zero column. The resulting expression would require the nonunit F_k to divide a Laurent monomial in the other variables. This is impossible in their Laurent polynomial ring.

For a frozen variable z, use the z-adic valuation on the ambient rational function field. Every mutable cluster variable has valuation zero. This holds initially. At a mutation, the two exchange monomials have disjoint supports, so at least one has z-exponent zero; the other mutable factors have valuation zero by induction. Their subtraction-free expressions have positive leading coefficients, so these lowest terms cannot cancel. The new mutable variable therefore also has valuation zero. The algebra $\mathcal{A}_+$ is generated by these variables and the noninverted frozen variables, so all its elements have nonnegative z-valuation. For the monomial under consideration, that valuation is exactly its exponent of z. □

Lemma 2.4. *A real simple module strongly commuting with every variable of a reachable seed is a nonnegative monomial in that seed.*

Proof. Clear its Laurent denominators by a nonnegative seed monomial D. All factors of D strongly commute with the module and with one another, so their tensor product with it is simple by [26]. The polynomial in the proof of Lemma 2.2 therefore consists of one monomial, with coefficient one. Apply Lemma 2.3 after dividing by D. □

3 Strict support from real powers

All Laurent expansions in this section are taken in one fixed reachable seed. For charge vectors, $u \leq w$ means coordinatewise inequality, and $u < w$ additionally means $u \neq w$. We set $\deg_v 0 = -\infty$; for a nonzero rational function, $\deg_v$ is numerator degree minus denominator degree.

Theorem 3.1. *Suppose that L_a satisfies* (2.5)*. In its expansion* (2.4)*, every nonzero coefficient $c_e(v)$ with $e \neq 0$ satisfies*

$$a_M \cdot e < 0. \tag{3.1}$$

3.1 One fixed standard-element formula

Use the quantum scattering diagram of the seed, retaining the full exponent lattice L. Let H be its path-ordered element in the positive-charge completion between the positive chamber and its injective negative chamber, with orientation chosen so that the normalized injective monomial of degree $-b$ is

$$J_b = H^{-1} X^{-b} H, \qquad b \in \mathbb{N}^M, \tag{3.2}$$

where b is extended by zero on the frozen coordinates. This is the cluster-chamber description of quantum theta functions [11, Proposition 4.9]. Frozen monomials normalize the leading frozen degree to zero. We use this fixed chamber and element H for every power of L_a.

Put $Y^d = X^{Pd}$ and define

$$\tau_c(Y^d) = v^{-2\delta c\cdot d} Y^d.$$

Extend τ_c linearly and continuously to the completion. The commutation convention (2.2) gives

$$Y^d X^k = v^{2\delta k_M\cdot d} X^k Y^d, \qquad A X^k = X^k \tau_{-k_M}(A). \tag{3.3}$$

For $k \in L$, let k_+, k_- be the coordinatewise positive and negative parts of its mutable coordinates, extended by zero on F. Then $k = k_+ - k_- + z$ for a frozen vector z. The normalized standard product of $X^{k_+ + z}$ and J_{k_-} is, by (3.2)–(3.3),

$$I_k = X^k \tau_{k_-}(H^{-1}) H. \tag{3.4}$$

The scalar from the leading product is removed in the normalization, and the frozen part satisfies $\Lambda(z, Pd) = 0$.

Fix $E \in \mathbb{N}^M$. Uniformly for $0 \le h \le E$, once r is large,

$$(ra + Ph)_- = ra_- + \eta_h. \tag{3.5}$$

Here $(\eta_h)_i$ is 0 if $a_i > 0$, is $-(Ph)_i$ if $a_i < 0$, and is $\max\{-(Ph)_i, 0\}$ if $a_i = 0$. These are fixed offsets; η_h itself need not be nonnegative.

Let $j_r(h, e)$ be the coefficient of X^{ra+Pe} in I_{ra+Ph}. It is zero unless $h \le e$, and $j_r(e, e) = 1$. Each such coefficient is a Laurent polynomial: a standard element is a normalized product of cluster monomials, and quantum Laurentness applies in the original seed. Write, in the positive-charge completion,

$$H^{-1} = \sum_d A_d(v) Y^d, \qquad H = \sum_d B_d(v) Y^d.$$

The coefficients are rational functions independent of r, with constant coefficients one. For a fixed difference $e - h$, only the finitely many decompositions $d_1 + d_2 = e - h$ occur in (3.4). The coefficient of each decomposition is a fixed rational function times $v^{r\alpha}$, where

$$\alpha = -\delta\big(2a_- \cdot d_1 + a_M \cdot (e - h)\big) \le -\delta a_M \cdot (e - h). \tag{3.6}$$

Indeed, the first term comes from $\tau_{ra_- + \eta_h}$ and the second from normalizing $X^{ra+Ph} Y^{e-h}$. All remaining phases, including those involving Ph and η_h, are independent of r. Thus

$$j_r(h, e) = \sum_\alpha v^{r\alpha} R_{h,e,\alpha}(v), \qquad \alpha \le -\delta a_M \cdot (e - h), \tag{3.7}$$

with finitely many rational functions $R_{h,e,\alpha}$. Along a chain $h = p_0 < p_1 < \cdots < p_s = e$ the bounds telescope:

$$\alpha_1 + \cdots + \alpha_s \le -\delta \sum_{j=1}^{s} a_M \cdot (p_j - p_{j-1}) = -\delta a_M \cdot (e - h). \tag{3.8}$$

3.2 Eliminating corrections with nonnegative pairing

Proof of Theorem 3.1. Suppose a nonleading support vector E has $a_M \cdot E \geq 0$. Choose such an E of least total height $|E| = \sum_i E_i$. Every actual support vector d with $0 < d < E$ then has $a_M \cdot d < 0$.

Let $G_r(e)$ be the coefficient of X^{ra+Pe} in $L_a^r = L_{ra}$. For every proper $0 < e < E$ with $a_M \cdot e \geq 0$, one has

$$G_r(e) = 0. \tag{3.9}$$

In fact, a contribution to this coefficient would be a sum of actual nonleading support vectors strictly below E. Their pairings with a_M are negative and cannot sum to a nonnegative number. The coefficient at E can involve only one nonleading insertion, namely E itself. With $\lambda = \Lambda(a, PE) = -\delta a_M \cdot E$, direct multiplication gives

$$G_r(E) = c_E(v) \sum_{j=0}^{r-1} v^{(2j-r+1)\lambda}. \tag{3.10}$$

This is nonzero. When $a_M \cdot E = 0$, it is $r c_E(v)$.

Expand L_{ra} by (2.3). Abbreviate $b_r(h) = b_{ra}(h)$ and $b_r(0) = 1$. On the finite interval $[0, E]$,

$$G_r(e) = \sum_{h \leq e} b_r(h) j_r(h, e), \qquad \deg_v b_r(h) \leq -1 \quad (h > 0). \tag{3.11}$$

Partition the proper nonzero vectors in the interval into

$$A = \{p : 0 < p < E,\ a_M \cdot p \geq 0\}, \qquad B = \{h : 0 < h < E,\ a_M \cdot h < 0\}.$$

Eliminate the unknowns $b_r(p)$, $p \in A$, from (3.11), in increasing height order, using (3.9). The effective entry from $h \in \{0\} \cup B$ to E is

$$j_r^{\mathrm{eff}}(h, E) = \sum_{\substack{h < p_1 < \cdots < p_s < E \\ p_i \in A}} (-1)^s j_r(h, p_1) j_r(p_1, p_2) \cdots j_r(p_s, E). \tag{3.12}$$

The term with $s = 0$ means $j_r(h, E)$. This is precisely inversion of a finite upper unitriangular submatrix. Every entry in (3.12) is Laurent, and (3.8) shows that every one of its r-slopes is at most $-\delta a_M \cdot (E - h)$. The remaining equation is

$$G_r(E) = z_r(E) + b_r(E), \tag{3.13}$$

$$z_r(E) = j_r^{\mathrm{eff}}(0, E) + \sum_{h \in B} b_r(h) j_r^{\mathrm{eff}}(h, E). \tag{3.14}$$

For $h \in B$ one has $a_M \cdot (E - h) > 0$. All slopes of the corresponding effective entry are therefore strictly negative, so its upper Laurent degree tends to $-\infty$. This assertion follows also by expanding the fixed rational functions at $v = \infty$: the degree of $v^{r\alpha} R(v)$ is $r\alpha + \deg R$. Multiplication by $b_r(h)$ preserves escape to $-\infty$, because $\deg_v b_r(h) \leq -1$ uniformly in r.

If $a_M \cdot E > 0$, the same escape holds for $j_r^{\mathrm{eff}}(0, E)$. For sufficiently large r, the right side of (3.13) is supported in strictly negative powers of v. The left side is bar invariant: both L_{ra} and the normalized monomials are bar invariant. A bar-invariant Laurent polynomial with only negative powers is zero, contrary to (3.10).

It remains to treat $a_M \cdot E = 0$. The slopes in $j_r^{\mathrm{eff}}(0, E)$ are now nonpositive. Its zero-slope terms are independent of r, and all its negative-slope terms escape to $-\infty$. Consequently

$z_r(E)$ has a common upper degree bound, and above every fixed cutoff its coefficients eventually stabilize. For a Laurent polynomial $z = \sum_n z_n v^n$, put

$$\pi(z) = z_0 + \sum_{n>0} z_n(v^n + v^{-n}).$$

It is the unique bar-invariant Laurent polynomial differing from z by strictly negative powers. Equation (3.13) therefore gives $G_r(E) = \pi(z_r(E))$. The finitely many nonnegative coefficients of $z_r(E)$ eventually stabilize, so $G_r(E)$ is eventually constant. But (3.10) gives $G_r(E) = rc_E(v)$, impossible in characteristic zero with $c_E(v) \neq 0$. Both cases give the required contradiction. □

4 From strict support to a reachable cone

Theorem 4.1 (Support criterion)**.** *Let S be a real simple module, and let $I \subseteq M$ in a reachable seed t. Suppose that S strongly commutes with P_j for all $j \in J = M \setminus I$. If every component of $Q_t[I]$ is mutation-acyclic or has a dense reachable g-vector fan, a finite mutation word using only I makes S a nonnegative cluster monomial.*

4.1 The leading theta function

For a Laurent polynomial, its Newton polytope Newt is the convex hull of its exponent support. A theta function with label p is the sum of the final monomials of broken lines with initial exponent p. The middle theta algebra is spanned by the theta functions which are Laurent polynomials. Write $f = L_a|_{v=1} = [S]$. The middle theta algebra contains the cluster algebra, so f has a finite expansion

$$f = \sum_p n_p \theta_p \tag{4.1}$$

in theta functions which are Laurent polynomials [11], with integer coefficients n_p.

Lemma 4.2. *In* (4.1)*, $n_a = 1$ and*

$$\mathrm{Newt}(\theta_a) \subseteq \mathrm{Newt}(f). \tag{4.2}$$

Every nonleading term x^{a+Pe} of θ_a satisfies $a_M \cdot e < 0$ and $e_J = 0$.

Proof. Order the finite set of labels in (4.1) by $p \leq q$ if $q - p \in P\mathbb{N}^M$. A minimal label contributes its leading monomial without cancellation: another theta could contribute to that monomial only from a strictly smaller label. Therefore every minimal label belongs to $a + P\mathbb{N}^M$, the support region of f. Following a decreasing chain shows that every label belongs to that region. Only the label a can contribute to x^a, and its coefficient is one. This proves $n_a = 1$.

We check the coefficient specialization needed for valuative independence. Adjoin independent formal variables y_i, $i \in M$, and give a scattering term of charge d the coefficient y^d. A wall function has the form $1 + z\mathbb{Z}[[z]]$, with $z = x^{Pd}y^d$ and $Pd \neq 0$. Its term of zero x-degree is therefore just one; if a highest x-power exists, it occurs in a single coefficient monomial. This verifies Assumption 3.9 of [10]. A theta monomial then has the form $x^{p+Pd}y^d$. Since P is injective, its x-exponent determines d. Positivity of scattering coefficients implies that a Laurent-polynomial theta after $y = 1$ has only finitely many charges before specialization. Moreover, no nonzero coefficient monomial is killed by $y = 1$.

These are the remaining finiteness and nonvanishing conditions for coefficient specialization in [10, Theorem 3.10]. Consequently, for every full linear weight n,

$$\mathrm{val}_n(f) = \min_{n_p \neq 0} \mathrm{val}_n(\theta_p) \leq \mathrm{val}_n(\theta_a). \tag{4.3}$$

Here val_n is the minimum of the weight over the Laurent support. It suffices initially to use integral weights; rational scaling and continuity give all real weights. Separation of finite convex polytopes gives (4.2).

By Lemma 2.1 and Theorem 3.1, every nonleading exponent of f is $a + Pd$ with $d_J = 0$ and $a_M \cdot d < 0$. A nonzero correction of θ_a is a convex combination of these corrections and zero. Injectivity of P makes this a statement about the charge vectors themselves. A nonzero combination must use a nonzero correction with positive coefficient; its charge pairing is strictly negative. All its J-coordinates are zero as well. □

Freeze J, retaining its coordinates as coefficients in the full torus. The positive-charge Lie algebra has an ideal spanned by charges having a positive coordinate in J: the bracket adds charges. Quotienting by this ideal gives exactly the scattering diagram with initial walls indexed by I. A broken line ending in charge $e_J = 0$ cannot have used any positive J-charge, since charges cannot cancel. By Lemma 4.2, this applies to every term of θ_a. Thus the theta function with label a is unchanged by this freezing.

4.2 A wall-free open set

Let P_I consist of the columns of P indexed by I, and let $J_I : \mathbb{Z}^I \hookrightarrow L$ be the coordinate inclusion. On the full real exponent space, using the coordinate bases to identify a form with its matrix, set

$$\lambda = -\Lambda/\delta.$$

Compatibility gives

$$\lambda P_I = J_I, \qquad P_I^{\mathsf{T}} \lambda = -J_I^{\mathsf{T}}. \tag{4.4}$$

These are the Λ-structure identities for the scattering seed datum (P_I, J_I). An essential wall is a wall whose crossing automorphism is different from the identity.

Lemma 4.3. *Let a finite theta function in this datum have leading label a and only corrections $e > 0$ satisfying $a_I \cdot e < 0$. There is a nonempty open convex set $V \subset \mathbb{R}^I$, disjoint from the closure of the essential wall support, such that $a_I \in \overline{V}$.*

Proof. If $I = \varnothing$, take $V = \mathbb{R}^0$. If $a_I = 0$, the strict inequality rules out every nonleading correction. Take $V = (0, \epsilon)^I$ for any $\epsilon > 0$. Its closure contains zero, and all wall normals are nonzero nonnegative charge vectors, so it misses the closed wall support. We may therefore assume $a_I \neq 0$ in the argument that follows.

For each nonleading term the difference between its weight and the leading weight at λp is

$$(a + P_I e)^{\mathsf{T}} \lambda p - a^{\mathsf{T}} \lambda p = -e \cdot p_I. \tag{4.5}$$

At $p = a$ this is positive. Because there are finitely many terms, choose a small convex neighborhood U of a on which all these differences remain positive. Thus the leading monomial is the unique minimizer for $\mathrm{val}_{\lambda p}$ for every $p \in U$.

To apply the scattering results with saturated injections, replace the data by

$$\widehat{L} = L \oplus \mathbb{Z}^I, \quad \widehat{P}(e) = (P_I e, e), \quad \widehat{Q}(e) = (J_I e, 0), \quad \widehat{\lambda} = \lambda \oplus 0. \tag{4.6}$$

Both $\widehat{P}$ and $\widehat{Q}$ are saturated injections: the second-coordinate projection is a left inverse for the former, and J_I is a coordinate inclusion for the latter. The two identities (4.4) persist. Projection to L is a linear morphism of scattering seed data. Walls are cylinders in the added directions, and the theta function with label $(a, 0)$ has terms $(a + P_I e, e)$. One sees this directly by lifting a broken line with a prescribed auxiliary endpoint: its projected path and all bend choices determine its auxiliary pieces uniquely. Positivity and the injectivity of P_I preserve its finite support. Formula (4.5) is unchanged, since $\widehat{\lambda}(p, z) = (\lambda p, 0)$. We may therefore work in this enlargement and project back.

For a generic endpoint p, let $\theta_{a,p}$ be the expansion obtained from broken lines with initial exponent a and endpoint p, and let m_γ be the final exponent of such a line γ. The positive-chamber expansion is θ_a. The basepoint identity [10, Lemma 5.17] is

$$\mathrm{val}_{\lambda p}(\theta_a) = \mathrm{val}_{\lambda p}(\theta_{a,p}) = \inf_{\substack{\gamma \text{ with initial exponent } a \\ \text{and endpoint } p}} m_\gamma^{\mathsf{T}} \lambda p. \tag{4.7}$$

For $p \in U$ the left side is $a^{\mathsf{T}} \lambda p$. The straight broken line has final exponent a, so it attains this infimum. By [10, Lemmas 5.12 and 5.14], a minimizing line is transpose-taut: at a transverse crossing from a nontrivial wall's negative side to its positive side, it takes the largest available bend. Thus the straight line can make no such crossing.

Put

$$W = U \cap \{a + b : b_i > 0 \text{ for every } i \in I\}.$$

This is nonempty, open and convex, with $a \in \overline{W}$. Suppose a nontrivial conical wall with positive normal charge d contains a point $r = a + b \in W$. Its normal in the exponent space is $J_I d$, so $r_I \cdot d = 0$ and

$$a_I \cdot d = -b_I \cdot d < 0.$$

For $s > 1$ sufficiently close to one, $p = a + sb$ is still in W. The backward extension of the zero-bend line ending at p is $p + ta$, $t \geq 0$. At $t = s - 1$ it passes through sr, which belongs to the wall by conicality. Before reaching p, the forward line crosses from the negative side to the positive side: its endpoint has pairing $(s - 1)b_I \cdot d > 0$, whereas its far end has negative pairing. The crossing is transverse. A generic perturbation of r within the wall and of p avoids joints and preserves these strict inequalities. Transpose-tautness now requires a nonzero bend, contradicting the zero-bend choice. Hence W contains no essential wall. Being open, it contains no point of the closure of their union.

Choose the cylindrical representative of the frozen diagram, pulled back from $\mathbb{R}^I$; its wall normals depend only on the I-coordinates. The projection V of W is open and convex. If it met the closure of a reduced wall, W would meet its cylindrical preimage. Thus V has the asserted disjointness, and $a_I \in \overline{V}$. □

4.3 Two ways to obtain one reachable chamber

Lemma 4.4. *If the reachable g-vector fan of $Q_t[I]$ is dense, the vector a_I in Lemma 4.3 lies in a single reachable cluster cone.*

Proof. The open set V meets the support of the fan. After a small perturbation inside V, it meets the interior of a maximal reachable cone C. Every facet of C carries its nontrivial exchange wall [11, Proposition 4.9(3)]; hence ∂C belongs to the closed wall support. If any point of V lay outside $\operatorname{Int} C$, the segment to an interior point of $V \cap C$ would meet ∂C inside V, a contradiction. Thus $V \subseteq \operatorname{Int} C$, and $a_I \in \overline{V} \subseteq C$. □

Lemma 4.5. *If $Q_t[I]$ is acyclic, the conclusion of Lemma 4.4 holds.*

Proof. Let $A = \mathbb{C}Q_t[I]$. We use right A-modules. For a weight θ, a nonzero module is semistable if its total weight is zero and every submodule has weight at most zero; it is stable if every proper nonzero submodule has strictly negative weight. Semistability is equivalent to Asai's nonnegative quotient convention. In comparing with the left-module convention in Bridgeland's construction, we use the opposite quiver as in [11, Section 7.1], keeping the same stability weight. Thus the initial projective cone corresponds to the initial positive g-chamber. We identify its closed stability-wall support with the closed essential cluster-wall support. Here the closed stability support means the closure of the set of weights admitting a nonzero semistable A-module.

The acyclic stability/scattering comparison is the corrected form of [4, Theorem 1.5 and Section 11.7]. Its nondegeneracy hypothesis is met by the following framing. Let $B = (b_{ij})_{i,j\in I}$ be the principal matrix of $Q_t[I]$. Attach one source to each vertex, giving the extended skew matrix

$$\begin{pmatrix} B & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

The resulting quiver is still acyclic and this matrix is invertible. Quotient the positive-charge Lie algebra by all charges with a positive added coordinate. The surviving representations have dimension vector $(d, 0)$ and have exactly the subobjects of the original A-representations. Thus their semistability conditions and their wall functions are unchanged.

In the decomposition $L = \mathbb{Z}^I \oplus \mathbb{Z}^{J\sqcup F}$, write $P_I = \binom{B}{C}$. Freezing the framing vertices gives the principal-coefficient matrix $P^{\mathrm{prin}} = \binom{B}{\mathrm{Id}}$. The coefficient map

$$\Phi : \mathbb{Z}^I \oplus \mathbb{Z}^I \longrightarrow L, \qquad \Phi(u, w) = (u, Cw)$$

satisfies $\Phi P^{\mathrm{prin}} = P_I$ and preserves every wall normal $d \in \mathbb{N}^I$. It sends a charge monomial of degree d to $x^{P_I d}$. Injectivity of P_I keeps distinct charges distinct and preserves nonconstant wall functions. The walls in both diagrams are cylindrical in the frozen directions, so their essential supports have the same projection to $\mathbb{R}^I$.

For completeness, a semistable object has a stable Jordan–Hölder factor of dimension d. Its strict stability inequalities remain strict under small generic perturbations within $d^{\perp}$; only finitely many subdimension vectors are involved. At such a generic weight, the stable representation of dimension d gives a nonzero primitive cohomological Donaldson–Thomas invariant by [11, Proposition 6.14]. Choose the smallest charge e on the ray of d with this property. The same proposition makes its invariant a nonzero polynomial $f_e(v) \in \mathbb{N}[v^{\pm 1}]$, so $f_e(1) > 0$. This first charge cannot be cancelled by higher charges. Its exponent and normal are nonzero, hence its integrated wall-crossing automorphism is nontrivial. Thus the original semistable weight belongs to the closure of essential walls. The reverse inclusion follows from the stability construction. Taking closures gives the claimed equality of supports.

For a finite-dimensional algebra, a two-term complex of projectives is concentrated in degrees $-1, 0$. It is presilting if its morphisms to every positive shift vanish; it is silting if it is presilting and its shifts, triangles and direct summands generate the bounded homotopy category of projectives. It is basic if its indecomposable summands are pairwise nonisomorphic. For an object T, $\operatorname{add} T$ is the category of direct summands of finite direct sums of T. In a 2-Calabi–Yau category, a cluster-tilting object satisfies $\operatorname{add} T = \{X : \operatorname{Ext}^1(T, X) = 0\}$. Asai's chamber theorem identifies every connected component of $K_0(\operatorname{proj} A)_{\mathbb{R}}$ minus this closed wall support with the interior of a basic two-term silting cone [2, Theorem 3.17]. Since V is convex, it is contained in one such chamber. For a finite acyclic quiver, the hereditary cluster correspondence identifies these cones with cluster-tilting cones; all cluster-tilting objects are connected to the initial one by finite mutations [5, Proposition 3.5],[6, Theorem 4]. The coefficient/index correspondence retains

the mutable g-coordinates [19]. Therefore this is one reachable cluster cone C, and again $a_I \in \overline{V} \subseteq C$. □

4.4 Components, the actual simple class, and denominators

Proof of Theorem 4.1. In each mutation-acyclic component first perform a finite acyclicizing word, using only that component. Different components have no principal arrows between them; mutation in one cannot create an arrow to another. These words therefore coexist in the full seed and fix $J \cup F$.

In the resulting reachable seed, let a denote the full leading degree of $[S]$ and apply Theorem 3.1. Its strong commutation with J is unchanged, so Lemma 4.2 and the freezing argument apply again. Fix a component K of I. In the positive-charge completion retain the charges e with $e_{I\setminus K} = 0$. Charges add under brackets and at broken-line bends. The retained terms therefore form exactly the theta function with label a for the diagram whose mutable vertices are K. Its support is a subset of the finite support obtained after freezing J. A retained nonleading charge satisfies $a_K \cdot e = a_I \cdot e < 0$. Lemmas 4.3, 4.4 and 4.5 provide a reachable cone containing a_K.

The principal exchange entries between distinct components are zero. In the mutation formula an entry between them can change only through a product containing one of these zero entries. It therefore stays zero, and the component words concatenate to a word in the full seed. After freezing J, the g-matrix on the I-coordinates is block diagonal, with one block for each component. In g-matrix mutation, sign coherence makes the changed column equal to its negative plus an integral combination of the other columns; those columns stay fixed. Thus each mutation multiplies the determinant by -1. Starting from the identity matrix, every component block is unimodular. Hence the nonnegative real coordinates of the integral vector a_K in its cone are nonnegative integers. Choose the corresponding nonnegative monomial N in the terminal I-variables. Its full degree h has $h_I = a_I$ and retains the coordinates in $J \cup F$. Set

$$c = a - h; \qquad c_I = 0.$$

In the theta algebra $x^c N$ has label a after freezing J. To identify it with the actual simple, choose a nonnegative monomial $D = x^d$ in $J \cup F$ such that $d + c \geq 0$.

The product $D[S]$ is the class of a simple module, because all its tensor factors strongly commute. Its normalized quantum class has full degree $d + a$. The product $x^{d+c}N$ is a nonnegative monomial in the terminal full seed, hence another simple class of the same full degree $d + a$. Uniqueness of the common triangular basis element with that degree identifies these two classes. Cancellation in the ambient field gives

$$[S] = x^c N.$$

This equality initially permits negative exponents in $J \cup F$. Lemma 2.3, applied in the terminal full seed, makes every exponent nonnegative. The constructed path mutates vertices in I and fixes $J \cup F$. □

5 Unit defects and the high-defect subquiver

5.1 Tensor products of defect one

We recall the exact R-matrix properties required here. For simple objects, $\mathfrak{d}$ is symmetric and nonnegative. If T is a simple subquotient of $U \otimes V$, then

$$\mathfrak{d}(W, T) \leq \mathfrak{d}(W, U) + \mathfrak{d}(W, V). \tag{5.1}$$

It is additive in a tensor product which is simple. If U, V are real and $\mathfrak{d}(U, V) = 1$, their tensor product has length two, with simple factors

$$T_+ = \mathrm{hd}(U \otimes V), \qquad T_- = \mathrm{hd}(V \otimes U).$$

Here hd denotes the head, the largest semisimple quotient. Both factors are real and strongly commute with U and V. These facts are the invariant and real-head results of [33, Section 2.6 and Lemmas 2.26–2.29]; the additivity statement is also [32, Lemma 3.10].

Two consequences are useful. First, the two factors strongly commute with one another, since

$$0 \le \mathfrak{d}(T_+, T_-) \le \mathfrak{d}(T_+, U) + \mathfrak{d}(T_+, V) = 0.$$

Second, if $U = P_j$ belongs to a fixed seed, then

$$\mathfrak{d}(P_i, T_\pm) \le \mathfrak{d}(P_i, V) \quad (i \in M), \qquad \mathfrak{d}(P_j, T_\pm) = 0. \tag{5.2}$$

The first inequality uses (5.1) and $\mathfrak{d}(P_i, P_j) = 0$. The reference seed stays fixed in this formula.

Lemma 5.1 (Reconstruction)**.** *Fix a reachable seed t, let X be the simple module represented by one of its mutable variables, and let S be real with $\mathfrak{d}(X, S) = 1$. If the two factors of $X \otimes S$ are reachable, then S is reachable. If Z is a set of variables of this same seed strongly commuting with S, a witnessing path starting at t can be chosen to fix Z.*

Proof. Write S_+, S_- for the two factors. They strongly commute with each other, with X, and with every variable in Z by (5.2). Factor their reachable cluster monomials into cluster variables. Additivity and nonnegativity of $\mathfrak{d}$ show that all variables in these two factorizations, together with X and Z, pairwise strongly commute.

For two variables in this collection, let A, B be their normalized quantum classes, with full leading degrees α, β in one reachable seed. Their tensor product is simple. Positivity and comparison of leading terms give

$$AB = v^{\Lambda(\alpha,\beta)} L_{\alpha+\beta}, \qquad BA = v^{-\Lambda(\alpha,\beta)} L_{\alpha+\beta}.$$

The Poisson bracket induced by the quantum torus is consequently

$$\{A|_{v=1}, B|_{v=1}\} = \left.\frac{AB - BA}{v - v^{-1}}\right|_{v=1} = \Lambda(\alpha, \beta)(A|_{v=1})(B|_{v=1}).$$

The quotient is taken in the integral Laurent form before specialization. Thus the variables are pairwise log-canonical, meaning that their Poisson brackets are scalar multiples of their products. The pairwise and finite compatibility results [8, Theorem 4.26(i)–(ii) and Lemma 4.18] imply that any finite set of pairwise log-canonical mutable variables lies in one cluster. Thus all mutable variables in the collection belong to one reachable seed T. Frozen variables occur in every seed. The seeds t and T both contain Z; the fixed-face connectivity theorem [9, Theorem 6.2] gives a finite path between them fixing Z.

Put $x = [X]$ and write the actual length-two identity in T as

$$x[S] = x^a h + x^b s, \qquad a, b \ge 0, \tag{5.3}$$

where h, s are nonnegative monomials in the other variables of T. Let $xx^* = F = M_+ + M_-$ be the exchange relation at x in this seed. If both a, b were positive, division of (5.3) by x would express the simple class $[S]$ as a sum of two simple classes, a contradiction. If $a = 0 < b$, then in the adjacent seed

$$[S] = \frac{h}{F} x^* + s F^{b-1} (x^*)^{1-b}.$$

The two displayed powers of x^* are different. Laurentness therefore requires h/F to be a Laurent polynomial in the other variables. This is impossible because F is a nonunit binomial and h is a Laurent unit. The case $b = 0 < a$ is identical. Hence $a = b = 0$.

It follows that

$$[S] = x^* Q, \qquad FQ = h + s, \tag{5.4}$$

where Q is Laurent in the variables other than x^*. Lemma 2.2 shows that its coefficients are nonnegative integers. If A, B are finite nonempty subsets of a free abelian group, then

$$|A + B| \geq |A| + |B| - 1. \tag{5.5}$$

To see this, choose a real linear functional injective on the finite sums in question. Order the elements of both sets by this functional, and consider the strictly increasing chain of sums obtained first by running through the first set with the smallest element of the second, and then through the second with the largest element of the first. It has the asserted length.

There is no cancellation in FQ. Since F has two support points and $h + s$ has at most two, (5.5) forces Q to be a single monomial. Its coefficient is one by (5.4). The two exchange monomials M_+, M_- have coordinatewise minimum exponent zero. Thus every exponent of Q, including every frozen exponent, is

$$\min\{\exp_i h, \exp_i s\} \geq 0.$$

Therefore $[S]$ is a nonnegative monomial in $\mu_x T$. Finally, $X \notin Z$ because $\mathfrak{d}(X, S) = 1$, so the last mutation also fixes Z. □

5.2 A finite recursion with a fixed core

Proof of Theorem 1.1. Fix t and $H = H_t(S)$ for the whole argument, and put

$$Z = \{j \in M : \mathfrak{d}(P_j, S) = 0\}, \qquad U = \{j \in M : \mathfrak{d}(P_j, S) = 1\}.$$

We construct a binary recursion on real simple objects C obtained from S. At every node all the defects against this same fixed seed are bounded above by those of S.

If $\mathfrak{d}(P_j, C) = 1$ at some $j \notin H$, replace C by the two real simple factors of $P_j \otimes C$. Formula (5.2) shows that all fixed defects are nonincreasing and that the selected coordinate becomes zero in both children. A zero coordinate can never become positive later. Thus the integer

$$u(C) = \big|\{j \notin H : \mathfrak{d}(P_j, C) = 1\}\big|$$

strictly decreases along each branch. Every branch has length at most $|U|$, so the recursion has finitely many nodes.

At a leaf C, all coordinates outside H have defect zero. Apply Theorem 4.1 with $I = H$. Its hypotheses concern the fixed quiver $Q_t[H]$, and remain valid even if some defects inside H have also become zero. The leaf is reachable by a path fixing $M \setminus H$, in particular fixing Z. Apply Lemma 5.1 upward through the finite tree. Every parent becomes reachable, with Z fixed, and the root is S. This proves the theorem. □

5.3 Finite mutation type and the four windows

Proof of Corollary 1.2. A c-vector is a column of the coefficient part of a seed's extended exchange matrix with principal coefficients. A reddening sequence is a finite mutation sequence whose terminal c-vectors are all componentwise nonpositive. Injective reachability supplies such a sequence for the full original quiver. Existence of a reddening sequence is

mutation invariant and passes to induced subquivers [35, Theorem 3.1.3 and Corollary 3.2.2]. Hence every component of $Q_t[H_t(S)]$ has a reddening sequence. Its terminal g-vectors are $-\mathbf{e}_i$, up to a permutation [35, Theorem 1.3.1(2)].

The classification of dense reachable g-fans in finite mutation type [37, Corollary 1.2] leaves the following cases to consider. The m-Kronecker quiver K_m has two vertices joined by m parallel arrows. For $m \geq 3$ it is acyclic and is covered by that alternative in Theorem 1.1. A closed once-punctured surface quiver has its reachable fan in a proper half-space. This is incompatible with reddening, which puts both coordinate orthants in the reachable fan. A defining linear functional for such a half-space would have to be nonnegative on both $\mathbf{e}_i$ and $-\mathbf{e}_i$ for every i, and would therefore be zero. The half-space could not be proper. For X_7, Seven's positive radical vector can be written $u = \sum_i a_i c_i$ in every reachable c-basis, with $a_i > 0$ [44, Proposition 2]. At an all-red seed every c_i is componentwise nonpositive, which would make u nonpositive. This contradicts $u > 0$ and excludes a reddening sequence for X_7. The density case X_6 is proved in Theorem 6.1 below. Every remaining component therefore satisfies one of the two alternatives of Theorem 1.1. □

Proof of Corollary 1.3. For the standard windows, the principal H–L quivers and their mutation classes are as follows:

Window	Mutation type
$(D_4, 2)$	$E_6^{(1,1)}$
$(A_3, 3)$	$E_7^{(1,1)}$
$(A_2, 5)$	$E_8^{(1,1)}$
$(A_5, 2)$	$E_8^{(1,1)}$

The initial quivers are the square-product quivers of [27, Sections 13.1–13.2]; their displayed mutation types are identified in [16, Remark 12.9 and Figure 29]. The reachable g-fans of these three tubular mutation types are dense by [37]. Apply Theorem 4.1 with $I = M$ to obtain reachability of every real simple in each listed window. □

6 The exceptional quiver X_6

Throughout this section the cluster g-vector fan has principal coefficients and one prescribed initial seed.

Theorem 6.1. *If B is a skew-symmetric exchange matrix of mutation type X_6, the union of its reachable cluster g-vector cones is dense in $\mathbb{R}^6$.*

The finite polyhedral calculation is proved below and implemented in full in Appendix A.

6.1 The root and its negative leaf half-space

Use the label order

$$(a_1, a_2, b_1, b_2, s, t) = (0, 1, 2, 3, 4, 5)$$

and the convention that $b_{ij} > 0$ counts arrows $i \to j$. The centered matrix is

$$B_0 = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & -1 \\ -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}. \tag{6.1}$$

Thus the arrows are $a_i \rightrightarrows b_i \to t \to a_i$ and $t \to s$. This is obtained from the quiver in [20, Section 9.4.1] by mutation at its vertex 5; the numbered vertices $(1,2,3,4,5,6)$ there correspond to $(a_1, a_2, b_1, b_2, s, t)$ here. In particular it is the exceptional quiver, and s is a sink. Here the arrows record morphisms between the additive projectives; when a Jacobian or endomorphism algebra is written with the opposite Gabriel-quiver convention, take the corresponding opposite algebra and left/right module convention. The index transformation (6.3) below fixes the convention used in this section. The opposite arrow convention gives the same mutation class: reverse B_0, interchange a_i with b_i for both i, and mutate the leaf s to recover B_0.

Let $\mathscr{F}$ be the union of the cluster cones rooted at (6.1), and put $Z = \overline{\mathscr{F}}$. The nondegenerate potential of [20, Section 9.4.1] is Jacobi-finite, meaning that its Jacobian algebra is finite-dimensional, by [20, Lemma 9.7 and its proof]. Its generalized cluster category gives a Hom-finite 2-Calabi–Yau model. Write $T = \bigoplus_i T_i$ for the canonical cluster-tilting object and Σ for suspension. The arrows above record irreducible maps $T_i \to T_j$. For an approximation triangle $T^1 \to T^0 \to X \to \Sigma T^1$, where $T^0, T^1 \in \operatorname{add} T$ and $T^0 \to X$ is a right $\operatorname{add} T$-approximation, the index is

$$\operatorname{ind}_T(X) = [T^0] - [T^1] \in \mathbb{Z}^6$$

in the basis $[T_i]$. Here a right approximation means that every map from a summand of a finite sum of T factors through it. Reachable cluster-tilting objects correspond to ordinary clusters, with indices equal to g-vectors [46, Theorems 4.2–4.3].

We make the coordinate changes explicit. At a mutation k, the matrix changes by

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & i, j \neq k. \end{cases} \tag{6.2}$$

On the region $\sigma g_k \geq 0$, where $\sigma \in \{1, -1\}$, the coordinate change for an index is

$$g'_k = -g_k, \qquad g'_i = g_i + [-\sigma b_{ki}]_+ g_k \quad (i \neq k). \tag{6.3}$$

The formulas agree when $g_k = 0$. The index mutation theorem [12, Section 3], together with sign coherence for summands of a rigid object, shows that this map sends each reachable cluster cone onto the corresponding cone in the new root coordinates. Its inverse is mutation at k with the new matrix. Thus it is an integral piecewise-linear homeomorphism of the whole real vector space.

A *return word* is a mutation sequence followed by a permutation $p = (p_0, \dots, p_5)$ for which the readout matrix $(b_{p_i p_j})_{i,j}$ is again B_0. Its readout on vectors is $(g_{p_i})_i$. Each return word preserves $\mathscr{F}$ and Z: the two roots are connected by the word, and their ordinary cluster patterns have the same labeled matrix after readout.

Lemma 6.2. *The half-space $\{g_s \leq 0\}$ is contained in Z.*

Proof. Let P_i be the projectives of the Jacobian model corresponding to T_i, with the convention $\operatorname{Hom}(P_i, P_j) = \operatorname{Hom}(T_i, T_j)$. Since s is a sink,

$$\operatorname{Hom}(P_s, P_j) = 0 \qquad (j \neq s). \tag{6.4}$$

Indeed, in a finite-dimensional basic algebra the radical is nilpotent and is generated by representatives of irreducible maps. A nonzero radical map starting at P_s would therefore begin with an arrow leaving s.

Let $\mathcal{P}_{\mathrm{old}}$ be the full additive subcategory on the five projectives P_j, $j \neq s$. Its endomorphism algebra is the Jacobian algebra of the quiver obtained by deleting s. No path

between old vertices can pass through s, and the potential has no cycle through s. By [20, Section 9.4.1], this is the quiver with potential of an annulus with one puncture and one marked point on each boundary component.

Let U be a basic two-term silting complex over this old algebra, viewed as a complex in the full projective category. The embedding of homotopy categories on these complexes is fully faithful: all terms and all chain homotopies use the same old projectives. Hence $\mathrm{Hom}(U, U[1]) = 0$. Furthermore,

$$\mathrm{Hom}(P_s[1], U[1]) = \mathrm{Hom}(P_s, U) = 0, \qquad \mathrm{Hom}(U, P_s[2]) = 0.$$

The first vanishing follows from (6.4), and the second from the degrees $-1, 0$ of U. Higher positive shifts also give zero. Thus $P_s[1] \oplus U$ is a two-term presilting complex with six nonisomorphic indecomposable summands. It is silting by [1, Proposition 3.3(b)].

This construction preserves actual paths. The old identity complex plus $P_s[1]$ corresponds to the leaf neighbor $\bigoplus_{j \neq s} T_j \oplus \Sigma T_s$: the left approximation of T_s by the other summands is zero. Two adjacent old silting objects have four summands in common, so their lifts have five and are adjacent full silting objects [1, Corollary 3.8(b), Theorem 4.7 and Corollary 4.8]. Consequently every old reachable cone, together with the negative s-ray, lies in $\mathscr{F}$.

The index of each lifted old summand has s-coordinate zero, since both projective terms are old, while $\mathrm{ind}(P_s[1]) = -\mathbf{e}_s$. The ordinary rooted fan of the punctured annulus is dense in $\mathbb{R}^5$ by [46, Theorem 1.2]. Taking its product with $\mathbb{R}_{\leq 0}\mathbf{e}_s$ and then its closure proves the assertion. □

6.2 Return words and normalization

The following table specifies all return words used below. Mutations within a word, and concatenated words in the finite comparison, are executed from left to right. The last column gives the readout order.

Word	Mutation sequence	Readout
α_1	0	$2,1,0,3,4,5$
α_1^{-1}	2	$2,1,0,3,4,5$
α_2	1	$0,3,2,1,4,5$
α_2^{-1}	3	$0,3,2,1,4,5$
β_1	$5,2,1,0,1$	$3,2,4,5,0,1$
β_1^{-1}	$5,4,5,1,3$	$4,5,1,0,2,3$
β_2	$5,3,0,1,0$	$3,2,5,4,1,0$
β_2^{-1}	$5,4,5,0,2$	$5,4,1,0,3,2$
F	$4,2,5,0,3,1,2,0,5,3$	$1,5,0,2,4,3$
F^{-1}	$5,1,2,3,0,5,2,1,3,4$	$2,0,3,5,4,1$

Applying (6.2) verifies $B = B_0$ after each readout. An inverse word can also be checked by reversing the mutation sequence and applying the inverse readout permutation.

Write a vector as $g = (x, z, y, w, r, c)$. The word α_1 preserves z, w, r and acts by

$$(x, y) \longmapsto (y + 2\min(x, 0), -x), \qquad c \longmapsto c + \max(x, 0). \tag{6.5}$$

Its inverse sends (x, y, c) to $(-y, x + 2\max(y, 0), c + \min(y, 0))$; the second pair has the same formulas. Each pair can be brought, in finitely many steps, either to the closed nonnegative quadrant or to the exceptional ray $(-u, u)$ with $u > 0$. To see this in the only potentially persistent region $x < 0 < y$, put $h = x + y$. While the region persists, a forward step replaces x by $x + h$ and an inverse step replaces y by $y + h$. If $h > 0$, forward iteration leaves the

region for the nonnegative quadrant; if $h < 0$, inverse iteration leaves it for the nonpositive quadrant. Two inverse steps take a pair in the nonpositive quadrant to the nonnegative quadrant. The other mixed quadrant enters the nonpositive quadrant after one forward step. The same formulas handle the axes. If $h = 0$ and $x < 0 < y$, the pair is exactly the fixed exceptional ray.

Suppose $Z \neq \mathbb{R}^6$. Its open complement contains a rational point. Since Z is invariant under positive scaling, choose an integral point $\theta \notin Z$. Every return word sends it to an integral point outside Z. Lemma 6.2 implies that its leaf coordinate is always a positive integer. Choose a return image for which this integer r is least. Normalize its two pairs by (6.5); the leaf coordinate stays minimal.

The simultaneous interchange of the two pairs preserves B_0 and the rooted fan. Up to this interchange, the normalized point belongs to one of five sectors. We describe these using parameters $q = (q_0, \dots, q_5)$, with $q_j \geq 0$ for $j \in J$ and $q_j = 0$ otherwise. In the table $g = Aq$ and the last column lists the parameters that must be strictly positive.

Sector	J	$Aq = (x, z, y, w, r, c)$	Strict
NN	$0, 1, 2, 3, 4, 5$	$(q_0, q_1, q_2, q_3, q_4, -q_5)$	q_4, q_5
LN^-	$0, 1, 3, 4, 5$	$(-q_0, q_1, q_0, q_3, q_4, -q_5)$	q_0, q_4
LN^+	$0, 1, 3, 4, 5$	$(-q_0, q_1, q_0, q_3, q_4, q_5)$	q_0, q_4
LL^-	$0, 1, 4, 5$	$(-q_0, -q_1, q_0, q_1, q_4, -q_5)$	q_0, q_1, q_4
LL^+	$0, 1, 4, 5$	$(-q_0, -q_1, q_0, q_1, q_4, q_5)$	q_0, q_1, q_4

In NN a nonnegative center would already put g in the initial cluster cone, so $c < 0$. The common boundary $c = 0$ of the other sectors is deliberately included in both signs.

6.3 The closed polyhedral calculation

For vectors $v_1, \dots, v_k$, write $\operatorname{cone}(v_1, \dots, v_k) = \{\sum_i a_i v_i : a_i \in \mathbb{R}_{\geq 0}\}$. Set

$$D_0 = (0, 0, 0, 0, 2, -1), \quad K_1 = (-1, 0, 1, 0, 0, 0), \quad K_2 = (0, -1, 0, 1, 0, 0). \tag{6.6}$$

For a word v, write $r_v(g)$ for the leaf coordinate after executing v. We use the following six comparisons, in the displayed order:

$$\begin{gathered} \alpha_1\beta_1^{-1}, \quad \alpha_2\beta_1, \quad \alpha_2\beta_2^{-1}, \quad \alpha_1\beta_2, \\ \alpha_2^{-1}F, \qquad \alpha_2^{-1}F^{-1}. \end{gathered} \tag{6.7}$$

Lemma 6.3. *If a point in one of the five sectors satisfies $r_v(g) \geq r$ for every word in* (6.7)*, then it belongs to* $\operatorname{cone}(D_0, K_1, K_2)$.

Proof. Here is a finite calculation over exact rational numbers that proves the assertion on the entire real parameter cones. Its cells are pointed cones in $V_J = \{q : q_j = 0 \text{ for } j \notin J\}$, together with the fixed strict coordinate conditions in the preceding table.

Represent a closed cell by a pair (H, R): H is a list of row vectors defining inequalities $h \cdot q \geq 0$, and R is its set of primitive integer extreme rays. Initially H consists of the six coordinate rows and $R = \{\mathbf{e}_j : j \in J\}$. A state during a word also contains a matrix G whose rows are the six current linear coordinate forms, so the current vector is Gq. At the start of each comparison set $G = A$ and $B = B_0$.

For clarity, first specify the clipping operation $\mathcal{H}((H, R), l)$, which intersects the cell with $l \cdot q \geq 0$. If all $l \cdot u$, $u \in R$, are nonnegative, the cell is unchanged. Otherwise append the row l to H and form the following finite set of candidate rays:

$$\begin{aligned} R_{\text{raw}} = &\{u \in R : l \cdot u \geq 0\} \\ &\cup \{(-l \cdot v)u + (l \cdot u)v : u, v \in R,\ l \cdot u > 0 > l \cdot v\}. \end{aligned} \tag{6.8}$$

Normalize each nonzero candidate by the positive greatest common divisor of its coordinates. Keep a candidate u precisely when the rows of H vanishing on u, restricted to the columns in J, have rank $|J| - 1$. Compute this rank by Gaussian elimination over $\mathbb{Q}$. If no ray remains, the cell contributes no point satisfying the strict conditions. In particular, when all evaluations are nonpositive and some are zero, the zero-evaluation face is retained.

This operation is exact. Indeed, write a point of the original cone as a nonnegative combination of rays. If its l-value is nonnegative, the negative l-contributions can be paired with portions of the positive contributions until they are exhausted. Each pairing gives a nonnegative multiple of a crossing vector in (6.8); the unused parts have nonnegative l-value. This proves that R_{raw} generates the intersection. For a nonzero vector in a pointed polyhedral cone, the dimension of its smallest face is $|J|$ minus the rank of its tight inequalities. Indeed, at a relative interior point of that face the tight rows vanish and every other defining inequality is strict. A sufficiently small neighborhood in their common kernel still satisfies the strict inequalities, so this kernel is the face's linear span. This proves the dimension formula. The rank test therefore retains exactly the extreme rays, including when the cone is contained in a proper subspace.

Discard a cell if, for one of the marked strict coordinate rows h, every retained ray satisfies $h \cdot u = 0$. Conversely, if each marked row is positive on some retained ray, summing one such ray for each row gives a point satisfying all strict conditions: all those coordinate rows remain nonnegative throughout. Thus the strict-feasibility test is exact.

At a pivot k put $l = G_k$. Split a state into the two clips $\mathcal{H}((H, R), \sigma l)$, $\sigma = 1, -1$, and apply the strict-feasibility test. On each surviving branch replace the rows by

$$G'_k = -l, \qquad G'_i = G_i + [-\sigma b_{ki}]_+ l \quad (i \neq k), \tag{6.9}$$

and update B by (6.2). Readouts permute the rows of G and both indices of B. After the whole comparison word, clip once more by the row $G_4 - A_4$, encoding $r_v - r \geq 0$. Retain this closed cell, reset $G = A$, and process the next comparison. At every pivot the two closed branches agree on their common wall, so no sign wall is omitted.

To remove repetitions, states may be identified when their ray sets are identical and their maps agree on every generating ray. At the end of a comparison only identical ray sets are identified. Thus every operation preserves exactly the required union, with the original strict conditions. The result of applying it is

Sector	Numbers of cells after the six comparisons	Final extreme rays
NN	$16, 30, 18, 1, 1, 1$	D_0
LN^-	$10, 17, 1, 1, 1, 1$	D_0, K_1
LN^+	$4, 0, 0, 0, 0, 0$	none
LL^-	$4, 4, 4, 4, 2, 1$	D_0, K_1, K_2
LL^+	$2, 2, 2, 2, 2, 0$	none

The final column lists the physical rays Aq. All entries are obtained from (6.2), (6.8) and (6.9) with the two displayed input tables; Appendix A carries out exactly this recurrence. The possible rays in every sector are contained in $\{D_0, K_1, K_2\}$, proving the lemma. □

6.4 Simultaneous limits and completion of the proof

Lemma 6.4. *The cone* $\mathrm{cone}(D_0, K_1, K_2)$ *is contained in* Z.

Proof. The center mutation has index $-\mathbf{e}_t + \mathbf{e}_{a_1} + \mathbf{e}_{a_2} + \mathbf{e}_s$: these are the three outgoing approximation summands at t. Adding $\mathbf{e}_{b_1} + \mathbf{e}_{b_2}$, whose objects remain in that same adjacent cluster, gives

$$g_0 = (1, 1, 1, 1, 1, -1) \in \mathscr{F}.$$

The return word F satisfies, for every integer $m \geq 0$,

$$F(g_0 + 3mD_0) = g_0 + 3(m+1)D_0. \tag{6.10}$$

To check every sign, the successive pivot coordinates along the ten mutations of F, starting at $g_0 + 3mD_0$, are

$$\begin{aligned} &1+6m, \quad 1, \quad 3m, \quad 3+3m, \quad 1+3m, \\ &2+3m, \quad 2+3m, \quad -1, \quad 3+3m, \quad 4+3m. \end{aligned}$$

They have the indicated fixed signs for $m \geq 0$; the zero at $m = 0$ is allowed in either branch. Applying (6.3) with these signs and the stated readout gives $(1, 1, 1, 1, 7 + 6m, -4 - 3m)$, proving (6.10). Since F preserves the rooted fan,

$$F^m(g_0) = g_0 + 3mD_0 \in \mathscr{F}.$$

For either pair, its first two α_i steps give $(1, 1) \mapsto (1, -1) \mapsto (-1, -1)$ and add 2 to the center. Each further step adds $2K_i$, with no change to the center or the leaf. Therefore, executing F^m first, then n_1 copies of α_1, then n_2 copies of α_2, gives

$$\begin{aligned} g_{m,n_1,n_2} = (-1,-1,-1,-1,1,3) + 3mD_0 \\ + 2(n_1-2)K_1 + 2(n_2-2)K_2 \in \mathscr{F} \end{aligned} \tag{6.11}$$

for $m \geq 0$ and $n_1, n_2 \geq 2$.

To obtain every nonnegative combination at once, fix $a, b, c \geq 0$ and choose

$$m_N = \lfloor Na/3 \rfloor, \qquad n_{1,N} = 2 + \lfloor Nb/2 \rfloor, \qquad n_{2,N} = 2 + \lfloor Nc/2 \rfloor.$$

The fan support is invariant under positive scaling, and (6.11) gives

$$N^{-1} g_{m_N, n_{1,N}, n_{2,N}} \longrightarrow aD_0 + bK_1 + cK_2.$$

This proves the full cone assertion, including zero coefficients. □

Proof of Theorem 6.1. If Z were a proper subset of $\mathbb{R}^6$, the integral minimum and normalization above would give $g \notin Z$ in one of the five sectors. Minimality of its positive leaf coordinate implies $r_v(g) \geq r$ for each of the six return comparisons. By Lemma 6.3, g belongs to $\mathrm{cone}(D_0, K_1, K_2)$; by Lemma 6.4, it belongs to Z, a contradiction. Thus the rooted fan for B_0 is dense. A mutation sequence and its integral piecewise-linear coordinate homeomorphism transport this density to every matrix in its mutation class. □

7 The separator model

We now specify the PBW parameters in Theorem 1.4. All vector spaces and representation varieties in this and the next section are over $\mathbb{C}$. Put

$$\mathbf{j} = c^{\ell+1} = (j_1, \ldots, j_N), \qquad N = |D|(\ell+1),$$

where c is the decreasing-height Coxeter word fixed in Section 2. If k is the $(r(k)+1)$st occurrence of j_k, write a simple module uniquely as

$$S(w) = L\left(\prod_{k=1}^{N} Y_{j_k, \varepsilon^{\xi_{j_k} - 2r(k)}}^{w_k} \right), \qquad w \in \mathbb{N}^N. \tag{7.1}$$

For positions in a word, k^- and k^+ denote the previous and next positions of the same color; a missing predecessor is 0 and a missing successor is $+\infty$. Set $\mathbf{e}_0 = 0$. The full initial degree of the simple class in (7.1) is

$$Uw, \qquad U\mathbf{e}_k = \mathbf{e}_k - \mathbf{e}_{k^-}. \tag{7.2}$$

Indeed the initial variable at k has highest monomial equal to the product of the fundamental monomials at all occurrences of j_k up to k, by (2.1). Inverting this cumulative change of coordinates gives (7.2), including its frozen rows. This is also the degree formula in [40, Theorem 7.10].

7.1 A reduced word and its hereditary quiver

Adjoin colors $h_1, \dots, h_{N-1}$ to the original Cartan matrix. Keep its old entries, give every new diagonal entry the value 2, and put -2 between a separator and every other distinct color, including the other separators. This defines a symmetric generalized Cartan matrix, denoted by C. Consider the word

$$\mathbf{i} = (j_1, h_1, j_2, h_2, \dots, h_{N-1}, j_N), \qquad R = 2N - 1. \tag{7.3}$$

It is reduced. In fact, each separator has infinite Coxeter exponent with every other color. The enlarged Coxeter group is the free product of the original Coxeter group and the order-two groups generated by the separators. The displayed expression is in reduced free-product normal form, with each old factor a single simple reflection. Its Coxeter length is therefore R.

Choose an acyclic orientation Q of the enlarged graph. Its double $\overline{Q}$ adds an arrow a^* opposite to each arrow a of Q. The preprojective algebra is

$$\Pi = \mathbb{C}\overline{Q} \Big/ \left\langle \sum_{a \in Q_1} (aa^* - a^*a) \right\rangle,$$

where Q_1 is the arrow set. We use finite-dimensional nilpotent left modules: all sufficiently long paths act by zero. A module X is rigid if $\operatorname{Ext}^1_\Pi(X, X) = 0$, and a brick if $\operatorname{End}_\Pi(X) = \mathbb{C}$. Use the GLS Frobenius category $\mathcal{C}_{\widetilde{w}}$ and its reduced-word cluster-tilting object $V = \bigoplus_{p=1}^{R} V_p$. Positions are numbered in the order of (7.3); in the reversed display of a reduced word used in [22], writing i_p for the pth letter gives $\widetilde{w} = s_{i_R} \cdots s_{i_1}$. Explicitly, let $\widehat{I}_i$ be the injective envelope of the vertex simple S_i. For a list $(j_1, \dots, j_t)$, form $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t \subseteq X$ by requiring that X_r/X_{r-1} is the maximal direct sum of copies of S_{j_r} in the socle of X/X_{r-1}, and set $\operatorname{soc}_{(j_1,\dots,j_t)} X = X_t$. Then

$$V_p = \operatorname{soc}_{(i_p,\dots,i_1)} \widehat{I}_{i_p}, \qquad \mathcal{C}_{\widetilde{w}} = \operatorname{Fac}(V),$$

where $\operatorname{Fac}(V)$ consists of quotients of finite direct sums of V [22, Section 2.4]. In particular the successive socles in V_p are taken in the order $i_p, i_{p-1}, \dots, i_1$. The layer modules are defined by the canonical exact sequences

$$0 \longrightarrow V_{p^-} \xrightarrow{I_p} V_p \longrightarrow B_p \longrightarrow 0, \qquad V_0 = 0. \tag{7.4}$$

Each B_p is a rigid brick. Every object of $\mathcal{C}_{\widetilde{w}}$ has a filtration whose successive quotients are $B_p^{a_p}$, in increasing order of p, with uniquely determined multiplicities [22, Lemma 9.8, Corollary 9.9 and Lemmas 10.1–10.2].

Define an acyclic quiver K on $\{1,\dots,R\}$ by taking a basis of the entire vector space $\mathrm{Ext}^1_\Pi(B_q,B_p)$ as arrows $q\to p$, for every $p<q$. Define

$$\mathbf{a}=(w_1,0,w_2,0,\dots,0,w_N)\in\mathbb{N}^R. \tag{7.5}$$

These data give the *separator model* used in the theorem. Its prehomogeneity condition means that the affine variety

$$\mathrm{rep}(K,\mathbf{a})=\bigoplus_{q\to p}\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{a_q},\mathbb{C}^{a_p})$$

has a Zariski open orbit under $\prod_p \mathrm{GL}_{a_p}(\mathbb{C})$, acting on an arrow $\alpha:q\to p$ by $y_\alpha\mapsto g_p y_\alpha g_q^{-1}$. Since this variety is irreducible, such an orbit is dense. The choices of bases of the Ext spaces do not affect the condition.

7.2 The original ice matrix and auxiliary normalization

We use the positive-word convention of [40, Lemma 6.3]. For a word $\mathbf{i}$, its exchange matrix has all positions as rows and the nonlast positions as columns. Its entries are

$$b_{pk}=\begin{cases}1, & p=k^-,\\ -1, & p=k^+,\\ \mathsf{C}_{i_p,i_k}, & p<k<p^+<k^+,\\ -\mathsf{C}_{i_p,i_k}, & k<p<k^+<p^+,\\ 0, & \text{otherwise.}\end{cases} \tag{7.6}$$

The same convention is used by GLS: their endomorphism quiver has horizontal arrows $k^+\to k$ and their matrix entry is the number of arrows $k\to p$ minus the number of arrows $p\to k$ [24, Section 9.4]. This fixes the sign in both the degree calculations and the cluster-character formula below.

Let $\mathscr{A}_v^+$ be the integral enlarged quantum unipotent algebra over $\mathbb{Z}[v^{\pm1}]$ with this reduced-word seed. Its frozen variables are the last occurrences of all colors. In particular, each separator variable H_p, at position $2p$, is frozen, while the old mutable and frozen positions are exactly those of $\mathbf{j}$. Inserting separators preserves every successor relation and every interlacing inequality between old positions in (7.6). Consequently, after placing the old rows first, its extended matrix has the form

$$\widehat{B}=\begin{pmatrix}B\\ C\end{pmatrix}, \tag{7.7}$$

where B is the entire original ice matrix, including the original frozen rows.

Project the root-lattice grading of $\mathscr{A}_v^+$ to the coordinates of the separator simple roots. Write O for the matrix of the degrees of its old initial variables and E for the matrix of the degrees of $H_1,\dots,H_{N-1}$. The weight of H_p is

$$s_{i_1}\cdots s_{i_{2p-1}}\alpha_{h_p}.$$

Reflections at colors preceding h_p can change only the coordinates of those colors. Thus its h_p coefficient is 1 and its later separator coefficients vanish. The matrix E is upper triangular with diagonal 1, hence is unimodular. Localize only these variables:

$$\mathscr{L}=\mathscr{A}_v^+[H_1^{-1},\dots,H_{N-1}^{-1}],\qquad D_v^+=\mathscr{L}_0. \tag{7.8}$$

Every homogeneous element has a unique integral power of the H_p which brings its auxiliary degree to zero. Products are normalized to have leading coefficient 1 in the torus.

Put

$$\eta = -E^{-1}O, \qquad \iota = \begin{pmatrix} I_N \\ \eta \end{pmatrix}, \qquad M_D(g) = \widehat{M}(\iota g). \tag{7.9}$$

Here $\widehat{M}$ is the normalized enlarged toric frame. Homogeneity of the exchange relations gives $OB + EC = 0$, so that

$$C = \eta B, \qquad \widehat{B} = \iota B. \tag{7.10}$$

Use the common quantum normalization with compatibility multiplier 2 [38, Lemma 7.3.2][24, Proposition 10.1]. If $\widehat{\Lambda}$ is the enlarged compatible form, then

$$\Lambda_D = \iota^t \widehat{\Lambda} \iota, \qquad \Lambda_D B = -2J_M,$$

where J_M inserts the mutable coordinates into all old coordinates. Thus (7.9) is a quantum seed with the original full ice matrix.

Lemma 7.1. *Every finite sequence of mutable labels of the enlarged seed induces the same sequence of mutations of the normalized old seed. Moreover, D_v^+ is precisely the algebra generated by these normalized cluster variables and the noninverted original frozen variables.*

Proof. At any current seed use the notation η, B for its normalization matrix and old exchange matrix. For a mutation at k, set $b = B\mathbf{e}_k$, $c = \eta b$, and let n_j be column j of η. The normalizing auxiliary exponent of the new variable is

$$n'_k = -n_k + \eta[b]_+ - [c]_+. \tag{7.11}$$

The enlarged exchange terms have exponent vectors

$$(-\mathbf{e}_k + [b]_+, [c]_+), \qquad (-\mathbf{e}_k + [-b]_+, [-c]_+).$$

Adding $(0, n'_k)$ changes them respectively into $\iota(-\mathbf{e}_k + [b]_+)$ and $\iota(-\mathbf{e}_k + [-b]_+)$. For the second identity use $c = [c]_+ - [-c]_+ = \eta b$. The quantum scalar caused by multiplication by the normalizing auxiliary monomial is the same for both terms: their difference is $\widehat{B}\mathbf{e}_k$, and every frozen vector pairs to zero with that column. After leading-coefficient normalization we obtain exactly the quantum exchange relation for (B, Λ_D).

Matrix mutation restricted to the old rows is the usual mutation of B. The grading remains homogeneous, and (7.10) is reestablished with the new η. This proves the first statement by induction on the length of the sequence, starting with the actual old variables at the initial seed.

The enlarged unipotent algebra is generated by its ordinary quantum cluster variables and its polynomial frozen variables [24, Theorem 12.3][40, Theorem 8.10(3)]. An element of $\mathscr{L}$ is therefore a finite linear combination of products of these generators and auxiliary frozen inverses. All factors are homogeneous. Project an expression for an element of $\mathscr{L}_0$ to degree zero, keeping only its homogeneous products of degree zero. Replace every old cluster-variable factor by its normalization. An auxiliary monomial remains, with degree zero; its exponent is zero because E is invertible. The result is a product of normalized old cluster variables, in which every original frozen exponent is nonnegative. Conversely, these generators belong to $\mathscr{L}_0$. This proves the assertion about the whole algebra D_v^+. □

7.3 The basis comparison

Proposition 7.2. *At $v = 1$, auxiliary normalization gives a based ring isomorphism*

$$D_v^+\big|_{v=1} \;\simeq\; K_0(\mathcal{C}_\ell^{\leq \xi})$$

which fixes the original KR initial variables. If $b(\mathbf{a})$ is the upper global basis element with PBW vector (7.5)*, its auxiliary normalization maps to $[S(w)]$.*

Proof. First invert the original frozen variables as well, and denote the resulting degree-zero algebra by $D_v^\times$. The enlarged upper global basis after frozen localization is its common triangular basis [38, 40]. Its elements are homogeneous, normalized frozen multiplication permutes them, and their full leading degrees run bijectively through $\mathbb{Z}^R$. A homogeneous pointed element has the degree of its leading monomial. Its auxiliary degree is zero exactly when its leading exponent belongs to

$$\ker(O\ E) = \{(g, \eta g) : g \in \mathbb{Z}^N\} = \iota\mathbb{Z}^N.$$

The degree-zero basis therefore has one element for every full old degree $g \in \mathbb{Z}^N$.

Dominance orders agree under this correspondence, because $\widehat{B} = \iota B$ and ι is injective. Normalizing a triangular multiplication relation by an auxiliary monomial preserves its coefficients: the scalar is common to all its terms, since their exponent differences lie in the image of $\widehat{B}$. Bar invariance, the triangular coefficient condition, and cluster-monomial containment are consequently preserved. Lemma 7.1 makes this true at every seed. Thus the resulting basis, denoted $\mathscr{B}_D^\times$, is the common triangular basis of $D_v^\times$. This is the degree-zero instance of the basis-transport argument in [40, Lemma 4.17 and Proposition 4.18]; here both the complete degree set and its dominance order have been checked explicitly.

Before inverting the old frozen variables, localizing the upper basis only at the H_p gives a homogeneous basis of $\mathscr{L}$. Its degree-zero part is a basis $\mathscr{B}_D^+$ of D_v^+ and is a subset of $\mathscr{B}_D^\times$. Linear independence then proves

$$\mathscr{B}_D^+ = \mathscr{B}_D^\times \cap D_v^+. \tag{7.12}$$

Let Λ_H be the actual HL form, with the same matrix B and compatibility multiplier 2. Write $\mathscr{B}_H^\times$ for its localized common triangular simple basis and $\mathscr{B}_H^+$ for the subset of actual simple classes in the original nonlocalized ring. Set $\Omega = \Lambda_D - \Lambda_H$. Then $\Omega B = 0$ and $B^t\Omega = 0$, so Ω descends to the quotient group $G = \mathbb{Z}^N/\operatorname{im} B$. Define the linear torus map $\mathcal{T}$ by $M_D(g) \mapsto M_H(g)$ and $v \mapsto v$. Here M_H is the normalized toric frame of the original quantum affine seed. Write $*_D, *_H$ for the two quantum products and $\deg_G M(g)$ for the image of g in G. For G-homogeneous elements f, h it satisfies

$$\mathcal{T}(f *_D h) = v^{\Omega(\deg_G f, \deg_G h)}\mathcal{T}(f) *_H \mathcal{T}(h). \tag{7.13}$$

The exponent is independent of its representatives. If f, h are pointed at g, g', divide their product by its leading scalar $v^{\Lambda_D(g,g')}$; (7.13) becomes the product normalized by $v^{\Lambda_H(g,g')}$ on the right. Hence normalized products, exchange relations, and triangular multiplication coefficients are unchanged. All cluster variables and pointed basis elements are G-homogeneous. Induction on mutations shows that $\mathcal{T}$ maps each cluster variable to the variable at the same label and the same mutation word for Λ_H.

The map preserves all full leading degrees, bar invariance and dominance orders. Uniqueness of the common triangular basis, or [40, Lemma 4.17], therefore gives $\mathcal{T}(\mathscr{B}_D^\times) = \mathscr{B}_H^\times$. Equation (7.13) also shows that it maps the ordinary generated algebras onto one another:

every product of generators changes by a unit power of v, and the inverse map has the same property. Intersecting with those algebras in (7.12) gives

$$\mathcal{T}(\mathscr{B}_D^+) = \mathscr{B}_H^+.$$

By [40, Section 7.6 and Theorem 7.10], the basis on the right is the actual simple basis in the original window. At $v = 1$ the scalar in (7.13) is 1, so this linear correspondence is a ring isomorphism. To see independence after specialization, consider a finite putative relation and choose a leading label g which belongs to none of the sets $g' + B\mathbb{N}^M$ for the other labels g' in that relation. Such a label exists because injectivity of B makes the order $g' \leq g$ defined by $g - g' \in B\mathbb{N}^M$ acyclic. The coefficient of x^g in the relation is exactly the coefficient of the basis element labeled by g, so it is zero. Repeating this argument removes every term. This proves the asserted based classical identification.

Finally put $\widetilde{U}\mathbf{e}_p = \mathbf{e}_p - \mathbf{e}_{p^-}$ for the enlarged word. The upper basis element $b(\mathbf{a})$ is pointed at

$$\widetilde{U}\mathbf{a}. \tag{7.14}$$

Indeed the dual PBW generator for B_p is the fundamental word variable, whose degree is $\mathbf{e}_p - \mathbf{e}_{p^-}$ [24, Corollary 12.4][40, Lemma 8.4 and Theorem 8.10]. The standard monomials have these additive leading degrees, and the upper basis has the same labels. Normalizing $b(\mathbf{a})$ changes only separator coordinates. Its old coordinates remain Uw, so its image is exactly the simple in (7.1), by (7.2) and uniqueness of the full degree label. □

At separator position $2p$, $(2p)^- = 0$ and $B_{2p} = V_{2p}$ is projective-injective in the full GLS category. Its vertex is isolated in K. Its zero entry in (7.5) contributes a point to the representation variety. When $N = 1$, there are no separators, all matrices involving auxiliary coordinates are empty, and the normalization is the identity.

8 Prehomogeneous PBW parameters

We prove the hereditary statement needed for the separator model. The arguments in the next three subsections work for any fixed reduced word of a symmetric Cartan matrix. We keep its actual GLS category, root V, layer sequences (7.4), and the full Ext-quiver K; its length is denoted by R. In these subsections $M = \{p : p^+ < \infty\}$ is the set of its nonlast positions and $\widehat{B}$ is its full ice matrix.

8.1 An exact realization of the whole stratum

Let $\mathscr{S}_{\mathbf{a}}$ denote the GLS stratum of modules with layer multiplicities $\mathbf{a}$. We need to realize every isomorphism class in this stratum, including classes outside its generic orbit.

Lemma 8.1. *For $p < q$ there is an identification*

$$\operatorname{Ext}^1_\Pi(B_q, B_p) = \frac{\operatorname{Hom}_\Pi(V_{q^-}, V_p)}{I_p \operatorname{Hom}_\Pi(V_{q^-}, V_{p^-}) + \operatorname{Hom}_\Pi(V_q, V_p) I_q}. \tag{8.1}$$

Proof. The module V_q is projective-injective in the prefix category associated with $(i_q, \dots, i_1)$; this extension-closed category contains B_p. Consequently $\operatorname{Ext}^1_\Pi(V_q, B_p) = 0$. Apply $\operatorname{Hom}_\Pi(-, B_p)$ to the layer sequence at q. Its connecting map identifies $\operatorname{Ext}^1_\Pi(B_q, B_p)$ with the cokernel of

$$\operatorname{Hom}_\Pi(V_q, B_p) \longrightarrow \operatorname{Hom}_\Pi(V_{q^-}, B_p).$$

Every map in these two spaces lifts to V_p, because all the V_s are mutually rigid, in particular $\operatorname{Ext}^1(V_q, V_{p^-}) = \operatorname{Ext}^1(V_{q^-}, V_{p^-}) = 0$. The ambiguity in a lift to V_p is a map through I_p; the ambiguity in the cokernel is a map extending across I_q. These are precisely the two denominator terms in (8.1). □

For each $p < q$, choose a linear section σ_{pq} of the quotient in (8.1), and a basis $\epsilon_{pq,\alpha}$ of that entire Ext space. If y is a representation of K, with vertex spaces Y_q and arrow maps $y_{pq,\alpha} : Y_q \to Y_p$, define

$$D(y) : \bigoplus_q V_{q^-} \otimes Y_q \longrightarrow \bigoplus_q V_q \otimes Y_q \tag{8.2}$$

by the blocks

$$D(y)_{qq} = I_q \otimes 1, \qquad D(y)_{pq} = \sum_\alpha \sigma_{pq}(\epsilon_{pq,\alpha}) \otimes y_{pq,\alpha} \quad (p < q), \qquad D(y)_{pq} = 0 \quad (p > q).$$

Set $F(y) = \operatorname{coker} D(y)$. Write $\operatorname{rep} K$ for the category of finite-dimensional representations of K.

Proposition 8.2. *The construction F is an additive exact functor $\operatorname{rep} K \to \mathcal{C}_{\widetilde{w}}$ with $F(E_q) = B_q$, where E_q has space $\mathbb{C}$ at vertex q, zero at the other vertices, and zero arrow maps. At any dimension vector $\mathbf{a}$, its values cover every isomorphism class in $\mathscr{S}_{\mathbf{a}}$. It can be realized as a polynomial family of Π-module matrices on one fixed vector space.*

Proof. The map $D(y)$ is injective: if a vector is in its kernel, its last nonzero block would lie in the kernel of the corresponding injective diagonal block $I_q \otimes 1$. Descending induction eliminates every block. Intersecting its image with the first s target blocks gives exactly the image of the first s source blocks, by the same argument on the later blocks. The images of the first s target blocks in its cokernel therefore provide a filtration with quotients $B_s \otimes Y_s$. This puts $F(y)$ in the stated stratum and in the extension-closed category $\mathcal{C}_{\widetilde{w}}$.

A morphism of K-representations consists of vertex maps intertwining all arrow maps. Tensoring these vertex maps with the identity on the V_q and V_{q^-} produces a commuting diagram of the presentations (8.2). Passing to cokernels defines F on morphisms, respects composition, and is additive. A short exact sequence of representations is exact at each vertex. The corresponding two sequences of source and target sums are exact, and all three presentation maps are injective. The snake lemma gives a short exact sequence of their cokernels. This proves exactness.

For surjectivity on isomorphism classes, take an object X with the specified filtration and construct its presentation inductively. Suppose the preceding filtered object X_{q-1} has an upper triangular presentation with diagonal blocks $I_p^{a_p}$ for $p < q$. In an extension

$$0 \longrightarrow X_{q-1} \longrightarrow X_q \longrightarrow B_q^{a_q} \longrightarrow 0,$$

the map $V_q^{a_q} \to B_q^{a_q}$ lifts to X_q. Indeed V_q is projective in the prefix category containing the extension. The restriction of this lift to $V_{q^-}^{a_q}$ lands in X_{q-1}. It lifts through the target of the preceding presentation, because that presentation has kernel $\bigoplus_{p<q} V_{p^-}^{a_p}$, which has vanishing Ext^1 from V_{q^-}. These lifts produce an upper triangular presentation of X_q with the required diagonal. This is also the block construction underlying [22, Proposition 10.5].

It remains to reduce its upper blocks to the chosen sections. Process the block (p, q) in increasing order of $q - p$. Modulo the section, the unwanted part of this block has the form

$$(I_p \otimes 1)A + C(I_q \otimes 1)$$

by (8.1), tensored with $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{a_q}, \mathbb{C}^{a_p})$. Subtracting the appropriate multiple of source block p from source block q removes the first term. Subtracting the appropriate multiple of target block q from target block p removes the second. These are actual invertible unit upper triangular module automorphisms. The first operation can additionally change only blocks (r, q) with $r < p$; the second can additionally change only blocks (p, s) with $s > q$. Those blocks have larger distance. Thus neither the diagonal nor any already treated block changes. After finitely many operations every block belongs to its chosen section, giving precisely $D(y)$ for some representation y. Its cokernel is still X.

To check polynomial dependence, choose at each vertex of Q fixed vector-space complements W_q to $I_q(V_{q^-})$ in V_q. Every coset modulo the image of $D(y)$ has a unique representative in $\bigoplus_q W_q \otimes \mathbb{C}^{a_q}$. Find it by eliminating the image part of the last block, then the preceding blocks in descending order. Only the fixed inverses of I_q on their images are used. All other operations are sums and products of the off-diagonal blocks, hence are polynomial in the arrow matrices of y. Apply this reduction after each Π-arrow map on the target sum. The induced arrow matrices on the fixed complement space are polynomial and represent $F(y)$. □

8.2 Open orbits give actual rigidity

Proposition 8.3. *If y is a rigid representation of K, then $F(y)$ is rigid as an actual Π-module. In particular this holds for rigid direct sums, and transfers both cross-*Ext^1 *vanishings of every compatible pair.*

Proof. Let Y_q be the space of y at vertex q. The length-one projective resolution for an acyclic quiver gives the exact sequence

$$0 \longrightarrow \mathrm{End}_K(y) \longrightarrow \bigoplus_q \mathrm{End}_{\mathbb{C}}(Y_q) \xrightarrow{\partial_y} \bigoplus_{\alpha: q \to p} \mathrm{Hom}_{\mathbb{C}}(Y_q, Y_p) \longrightarrow \mathrm{Ext}^1_K(y, y) \longrightarrow 0,$$

$$\partial_y((f_q)_q)_\alpha = f_p y_\alpha - y_\alpha f_q.$$

The middle map is the derivative of vertex base change. Its kernel is the stabilizer's Lie algebra, so its image has the dimension of the orbit. Taking dimensions gives

$$\dim \mathrm{rep}(K, \mathbf{a}) - \dim\big((\textstyle\prod_q \mathrm{GL}_{a_q}) \cdot y\big) = \dim \mathrm{Ext}^1_K(y, y).$$

If y is rigid, its orbit is open dense in the irreducible affine space $\mathrm{rep}(K, \mathbf{a})$; conversely an open orbit is rigid.

Put $X = F(y)$ and $d = \sum_q a_q \dim B_q$, a dimension vector on Q. The polynomial family in Proposition 8.2 gives a morphism

$$\mathrm{GL}_d \times \mathrm{rep}(K, \mathbf{a}) \longrightarrow \mathscr{S}_{\mathbf{a}}$$

which is surjective, since every isomorphism class occurs. Here GL_d is the product of general linear groups at the vertices of Q. Base changes of y induce actual isomorphisms of the presentations, so the image of GL_d times its open orbit is the single orbit $\mathrm{GL}_d \cdot X$. This orbit is dense in the closure of $\mathscr{S}_{\mathbf{a}}$.

By [22, Proposition 14.6], this closure is an irreducible component of the nilpotent preprojective variety, of dimension

$$\dim \mathrm{GL}_d - \langle d, d\rangle_Q, \qquad \langle d, d\rangle_Q = \sum_i d_i^2 - \sum_{i \to j \text{ in } Q} d_i d_j.$$

The orbit dimension is $\dim \mathrm{GL}_d - \dim \mathrm{End}_\Pi(X)$. Density therefore gives $\dim \mathrm{End}_\Pi(X) = \langle d, d\rangle_Q$. The preprojective Euler identity [22, Lemma 2.1] now yields

$$\dim \mathrm{Ext}^1_\Pi(X, X) = 2 \dim \mathrm{End}_\Pi(X) - 2\langle d, d\rangle_Q = 0.$$

Apply the same argument to a rigid sum $y \oplus z$. Since F is additive, the four summands of $\mathrm{Ext}^1(F(y) \oplus F(z), F(y) \oplus F(z))$ all vanish. This proves the cross-extension assertion. □

8.3 Finite paths from the prescribed GLS root

In this subsection an object is called reachable if all its indecomposable summands occur together in a cluster-tilting object obtained from V by finitely many mutations. Multiplicities are allowed. Work first in the stable 2-Calabi–Yau category $\underline{\mathcal{C}}_{\widetilde{w}}$. Restoring its fixed projective-injective summands lifts every path to the Frobenius category, with its frozen variables unchanged. The stable category has the same objects and quotients morphisms by those factoring through projective-injective objects. Its suspension [1] is the cosyzygy: a sequence $0 \to X \to I \to X[1] \to 0$ with I projective-injective defines it.

For each color i, write its positions as $p_{i,1} < \cdots < p_{i,t_i}$. Process the positions $k = 1, \ldots, R$ in order. At $k = p_{i,a}$ append the mutation labels

$$p_{i,1}, p_{i,2}, \ldots, p_{i,t_i - a};$$

the block is empty for $a = t_i$. The resulting word has length $\sum_i t_i(t_i - 1)/2$. The generalized determinantal exchange identities [22, Section 13, Theorem 13.1] show that, just before the block for $p_{i,a}$, the first i-label carries $V_{p_{i,a}}/V_{p_{i,a-1}} = B_{p_{i,a}}$, with $V_{p_{i,0}} = 0$. The word ends at $I_{\widetilde{w}} \oplus \Omega^{-1}_{\widetilde{w}} V$, where $I_{\widetilde{w}}$ is the sum of the indecomposable projective-injectives and $\Omega^{-1}_{\widetilde{w}}$ is cosyzygy [22, Sections 9.8 and 15.2]. Its stable image is $V[1]$. Thus every layer is reachable. Moreover the rooted component is invariant under both shifts. Shift a known finite path $V \rightsquigarrow T$ and prepend the fixed path $V \rightsquigarrow V[1]$ to obtain a path to $T[1]$. Shifting the fixed path by $[-1]$ and reversing it gives a path $V \rightsquigarrow V[-1]$, and the same reasoning gives negative shift invariance.

We shall use the following precise form of reachable co-Bongartz completion. If T is cluster-tilting and a rigid U is reachable from T, choose a minimal left $\mathrm{add}\, U$-approximation and complete it to a triangle

$$T[-1] \longrightarrow U' \longrightarrow C \longrightarrow T.$$

An approximation means that every map from $T[-1]$ to an object of $\mathrm{add}\, U$ factors through the displayed map $f : T[-1] \to U'$. Minimality means that an endomorphism h of U' with $hf = f$ is invertible. The basic object with additive hull $\mathrm{add}(U \oplus C)$ is cluster-tilting and reachable from T [7, Definition 2.5.1 and Theorem 4.2.3]. Finiteness is part of the result: apply the completion to each seed on a finite path from T to a cluster containing U. Successive completed seeds are equal or differ by one mutation; reverse this finite path of completed seeds to reach the completion relative to T.

Lemma 8.4. *The following operations take place in the component of V.*

(i) *If A, B are reachable and $A \oplus B$ is rigid, then $A \oplus B$ is reachable.*

(ii) *In a triangle*

$$A \xrightarrow{f} B \longrightarrow D \longrightarrow A[1], \tag{8.3}$$

if A, B are reachable and $\mathrm{Ext}^1(D, B) = 0$, then D is reachable.

(iii) *In* (8.3)*, if* B, D *are reachable and* $\operatorname{Ext}^1(B, A) = 0$*, then* A *is reachable.*

Proof. For (i), choose a reachable cluster-tilting object T containing A. The object B is reachable from T, since both lie in the component of V. Complete B relative to T. For any indecomposable summand A_i of A,

$$\operatorname{Hom}(A_i[-1], B) = \operatorname{Ext}^1(A_i, B) = 0.$$

Its minimal left approximation is zero, whose cone is A_i. The completion thus contains both A and B. Repeating this argument combines any finite compatible collection already known individually to be reachable.

For (ii), apply $\operatorname{Hom}(-, B)$ to the triangle. The vanishing of $\operatorname{Ext}^1(D, B)$ makes $\operatorname{End}(B) \to \operatorname{Hom}(A, B)$ surjective, so f is a left $\operatorname{add} B$-approximation. Choose a reachable cluster-tilting object containing the basic part of $A[1]$, using the shift invariance proved above, and complete B relative to it. Minimal left approximations of each indecomposable summand of A contribute their cones to this completion. Their direct sums do the same for arbitrary multiplicities. Every left approximation is, after a codomain isomorphism, its minimal part together with a zero map into a redundant summand of $\operatorname{add} B$. Accordingly its cone is the minimal cone plus that redundant summand. The entire D therefore belongs to the additive hull of the reachable completion. This argument also proves its rigidity; in our applications rigidity is independently known from Proposition 8.3. For (iii), the opposite triangulated category has suspension $[1]_{\mathrm{op}} = [-1]$ and triangle

$$D \longrightarrow B \longrightarrow A \longrightarrow D[1]_{\mathrm{op}}.$$

Its required vanishing in (ii) is $\operatorname{Ext}^1_{\mathrm{op}}(A, B) = \operatorname{Ext}^1(B, A) = 0$. Both starting objects D, B are reachable, and the shift invariance proved above applies in either category. Part (ii) therefore gives reachability of A. □

Proposition 8.5. *For every rigid representation* y *of* K*, the actual module* $F(y)$ *is reachable from the GLS root* V*.*

Proof. Call an indecomposable rigid representation exceptional. Such a representation E has $\operatorname{End}_K(E) = \mathbb{C}$. To prove this, take a nonzero endomorphism f and put $J = \operatorname{im} f$, $K_0 = \ker f$. The extension $0 \to K_0 \to E \to J \to 0$ lifts along $J \hookrightarrow E$ to an extension $0 \to K_0 \to Z \to E \to 0$: the obstruction belongs to $\operatorname{Ext}^2(E/J, K_0) = 0$. The middle term of the original extension is the pullback $Z \times_E J$. If $i : E \to Z$, $p : E \to J$ and $q : Z \to E$, $j : J \to E$ are the pullback maps, then

$$0 \longrightarrow E \xrightarrow{(i,-p)} Z \oplus J \xrightarrow{(q,j)} E \longrightarrow 0$$

is exact. It splits by rigidity, so $Z \oplus J \simeq E \oplus E$. Krull–Schmidt and $0 < \dim J \leq \dim E$ give $J \simeq E$. Hence f is invertible. The finite-dimensional algebra $\operatorname{End}_K(E)$ is therefore a division algebra. Every endomorphism has an eigenvalue $\lambda \in \mathbb{C}$; the noninvertible endomorphism $f - \lambda 1$ must be zero. This proves the assertion, the hereditary argument of [25, Lemma 4.1].

Induct on total dimension. At total dimension 1 it is a vertex simple, whose image is a reachable layer B_q. For a nonsimple exceptional representation E, Schofield induction [42, Theorem 3.1] gives an exact sequence

$$0 \longrightarrow U^u \longrightarrow E \longrightarrow W^v \longrightarrow 0, \qquad u, v > 0, \tag{8.4}$$

where U, W are exceptional and

$$\operatorname{Hom}(U, W) = \operatorname{Hom}(W, U) = \operatorname{Ext}^1(U, W) = 0.$$

Both have strictly smaller total dimension than E. Their extension-closed subcategory is the representation category of the r-Kronecker quiver with sink U, source W, and $r = \dim \operatorname{Ext}^1(W, U)$ arrows. This is an exact equivalence; all extensions between its members are still computed in $\operatorname{rep} K$.

We spell out the finite rank-two construction, including its first steps. If $r = 0$ every representation is a sum of vertex simples. For $r = 1$, putting its single arrow matrix in rank normal form decomposes the representation into vertex simples and copies of the one-dimensional identity arrow. Hence the sole mixed indecomposable is the middle term of $0 \to U \to E \to W \to 0$. Suppose $r \geq 2$. Put $P_0 = U$, $J_0 = W$. The two chains start with

$$\begin{aligned} 0 &\longrightarrow U^r \longrightarrow P_1 \longrightarrow W \longrightarrow 0, \\ 0 &\longrightarrow U \longrightarrow J_1 \longrightarrow W^r \longrightarrow 0. \end{aligned} \tag{8.5}$$

Here (P_0, P_1) are the two indecomposable projectives and (J_0, J_1) the two indecomposable injectives. Subsequent terms fit in the exact sequences

$$\begin{aligned} &0 \longrightarrow P_{m-1} \longrightarrow P_m^r \longrightarrow P_{m+1} \longrightarrow 0, \\ &0 \longrightarrow J_{m+1} \longrightarrow J_m^r \longrightarrow J_{m-1} \longrightarrow 0 \qquad (m \geq 1). \end{aligned} \tag{8.6}$$

For completeness, these recursions can be obtained directly from reflection functors. Source reflection replaces the source space by the cokernel of its map to the direct sum of the r sink spaces; after interchanging the vertex names it takes P_m to P_{m+1}. For a representation with no source-simple summand, the combined arrow map from the source to the direct sum of the sink spaces is injective: its kernel is a sum of source simples, which split off because that simple is injective. Dually, on the reflected side the combined map onto the sink is surjective whenever the sink simple, which is projective, is absent as a summand. The cokernel and kernel constructions are inverse on these subcategories. The snake lemma shows that they preserve short exact sequences. The injectivity and surjectivity conditions are closed under extensions, so the same equivalence preserves the ambient Ext^1 spaces. For the first recursion, set $A = \mathbb{C}^r$ with basis $\mathbf{e}_1, \dots, \mathbf{e}_r$. The projective P_1 has sink space A, source space $\mathbb{C}$, and jth arrow $1 \mapsto \mathbf{e}_j$. Thus P_1^r has sink $A \otimes A$, source A, and arrows $v \mapsto \mathbf{e}_j \otimes v$. The representation P_0 embeds with zero source and sink spanned by $\sum_j \mathbf{e}_j \otimes \mathbf{e}_j$. Its quotient has sink $(A \otimes A)/\mathbb{C}(\sum_j \mathbf{e}_j \otimes \mathbf{e}_j)$, source A, and the induced arrows. This is the source-reflection construction of P_2, giving $0 \to P_0 \to P_1^r \to P_2 \to 0$. Repeated source reflection gives all the first sequences in (8.6); the dual kernel construction gives the second sequences. It also proves that consecutive terms of each chain are mutually Ext^1-orthogonal, starting with the projective or injective pair.

These chains exhaust the exceptional representations of the rank-two category. If its sink and source dimensions are s, t, the hereditary Euler formula and the Schur property just proved give

$$s^2 + t^2 - rst = \dim \operatorname{End}(E) - \dim \operatorname{Ext}^1(E, E) = 1.$$

When $s \geq t > 0$, sink reflection followed by interchanging vertices sends (s, t) to $(t, rt - s)$, where $0 \leq rt - s < t$. The inequality follows by viewing the quadratic as a polynomial in s: its value at $(r-1)t$ is $(2-r)t^2 \leq 0$, while $s \geq t$ lies on the large-root branch. The identity $s(s - rt) = 1 - t^2$ gives nonnegativity. The case $t \geq s$ is dual. Thus repeated reflection reduces to a vertex simple in finitely many steps, giving one of the displayed chains. At each step an indecomposable other than the excluded simple remains in the reflection equivalence. Reversing these equivalences from the final simple identifies the representation up to isomorphism.

By induction $F(U)$ and $F(W)$ are reachable. Apply the exact functor F to (8.5) and (8.6). Proposition 8.3 transfers the rigidity of each term and of every consecutive pair. For the first preprojective term rotate the triangle to

$$F(W)[-1] \longrightarrow F(U)^r \longrightarrow F(P_1) \longrightarrow F(W).$$

Its first two objects are reachable, and $\operatorname{Ext}^1(F(P_1), F(U)) = 0$. Lemma 8.4(ii) therefore gives $F(P_1)$. For the first preinjective term use

$$F(J_1) \longrightarrow F(W)^r \longrightarrow F(U)[1] \longrightarrow F(J_1)[1]$$

and $\operatorname{Ext}^1(F(W), F(J_1)) = 0$ from the rigid injective pair. Part (iii) gives $F(J_1)$. The same operations on (8.6) successively give all $F(P_m)$ and $F(J_m)$. For $r = 1$ the first preprojective argument already gives the sole mixed term. Every specified E needs finitely many steps; the outer induction terminates by the strict decrease of total hereditary dimension in (8.4).

Finally a rigid representation is a finite direct sum of pairwise Ext^1-orthogonal exceptional representations, with possible multiplicities. Their images are individually reachable by the preceding argument and are mutually rigid by Proposition 8.3. Lemma 8.4(i) places them in one reachable cluster-tilting object. Restoring the projective-injective summands gives the assertion in the actual Frobenius category. □

8.4 Full degree labels and the return to the original window

We compute the index in the full lattice, including the projective-injective coordinates.

Lemma 8.6. *If $y \in \operatorname{rep}(K, \mathbf{a})$ is rigid, the cluster monomial of $F(y)$ is the specialization at $v = 1$ of the upper basis element $b(\mathbf{a})$ for this reduced word.*

Proof. Put $X = F(y)$. Its actual presentation is

$$0 \longrightarrow \bigoplus_q V_{q^-}^{a_q} \longrightarrow \bigoplus_q V_q^{a_q} \longrightarrow X \longrightarrow 0. \tag{8.7}$$

Applying $\operatorname{Hom}_\Pi(V, -)$ preserves exactness, since the source is in $\operatorname{add} V$ and V is rigid. The resulting sequence is a projective resolution of $\operatorname{Hom}_\Pi(V, X)$ over $\operatorname{End}_\Pi(V)^{\mathrm{op}}$. Its full index is consequently

$$g_V(X) = \sum_q a_q(\mathbf{e}_q - \mathbf{e}_{q^-}) = \widetilde{U}\mathbf{a} \quad \text{in } \mathbb{Z}^R. \tag{8.8}$$

Cancelling common projective summands in the presentation leaves this difference unchanged. All frozen coordinates remain in (8.8).

Here is the relation with the full GLS cluster-character formula. Let $C_V = (\dim \operatorname{Hom}_\Pi(V_p, V_q))_{p,q}$ and let $h = (\dim \operatorname{Hom}_\Pi(V_p, X))_p$. Exactness of the projective resolution gives $h = C_V g_V(X)$. The matrix C_V is integrally invertible in the GLS character formula. In the row-vector notation of [23, (1.3) and (6.7)–(6.9)], the extended exponent is $h^t C_V^{-t} = g_V(X)^t$. Thus it is exactly (8.8), including the projective-injective rows. The mutable columns of C_V^{-t} are the exchange matrix with the sign convention (7.6). The full character formula [23, Theorem 4], equivalently the Frobenius cluster character of [19, Sections 3–4], is therefore

$$\varphi_X = x^{g_V(X)} \sum_{d \in \mathbb{N}^M} \chi\Big(\operatorname{Gr}_d \operatorname{Ext}^1_\Pi(V, X)\Big)\, x^{\widehat{B}d}. \tag{8.9}$$

Here χ is topological Euler characteristic. Precomposition makes $\operatorname{Ext}^1_\Pi(V, X)$ a left $\operatorname{End}_\Pi(V)^{\mathrm{op}}$-module. Maps through projective-injectives act by zero, so the action factors

through the stable endomorphism algebra, whose vertex set is M. The variety Gr_d parametrizes its submodules of dimension vector d. The leading monomial, however, uses every initial variable. The $d = 0$ term has coefficient 1. Compatibility implies that $\widehat{B}$ has full column rank, so no other term can have its exponent. Thus (8.9) is pointed at $\widetilde{U}\mathbf{a}$ for the same dominance order as the upper common triangular basis.

By Proposition 8.5, X is reachable, so φ_X is an actual cluster monomial. Its quantum lift belongs to the upper common triangular basis. Specializing a pointed basis element preserves its leading term. A Laurent polynomial cannot be pointed at two different degrees for this order: each of those degrees would differ from the other by an element of $\widehat{B}\mathbb{N}^M$, and injectivity forces both differences to be zero. Its quantum lift therefore has degree $\widetilde{U}\mathbf{a}$. On the other hand $b(\mathbf{a})$ has that same full degree by (7.14). Uniqueness of the degree label in the upper basis proves

$$\varphi_X = b(\mathbf{a})\big|_{v=1}.$$

□

Proof of Theorem 1.4. Use the precise separator word (7.3) and vector (7.5) of $S(w)$. Choose a representation y in the open orbit. The hereditary orbit formula shows that y is rigid. Propositions 8.3 and 8.5 produce an actual rigid module $X = F(y)$ and a finite mutation path from the specified GLS root to a cluster containing every indecomposable summand of X. Lemma 8.6 identifies its cluster monomial with $b(\mathbf{a})|_{v=1}$.

Every mutable label on this path is an old label, since all separators are frozen. Lemma 7.1 transports exactly this finite list of labels, one step at a time, to the normalized old cluster algebra. Proposition 7.2 identifies its initial variables with the prescribed KR variables (2.1), and its normalized target with $[S(w)]$. Thus the transported word starts at the required original seed.

To verify nonnegativity before concluding, write the enlarged cluster monomial in its terminal seed as

$$\prod_{j \text{ old}} \widehat{x}_{j;t}^{m_j} \prod_{p=1}^{N-1} H_p^{c_p}, \qquad m_j, c_p \in \mathbb{N},$$

at $v = 1$. Normalize it to auxiliary degree zero, allowing inverses of the H_p only. Replace each old variable by its normalized terminal variable. The remaining auxiliary factor has degree zero; since E is invertible, it is 1. The resulting expression is

$$[S(w)] = \prod_{j \text{ old}} x_{j;t}^{m_j}, \qquad m_j \in \mathbb{N}.$$

These are the original terminal variables at the transported seed. In particular the original frozen exponents are still the same nonnegative integers m_j. This proves reachability in the ordinary unlocalized ring. Monoidal categorification makes this simple real.

If $w = 0$, the representation space at $\mathbf{a} = 0$ is a point, $F(0) = 0$, and the conclusion is the empty cluster monomial 1. If $N = 1$, the quiver K has one vertex and no arrows; the conclusion is the nonnegative power $x_1^{w_1}$, with the empty mutation word. □

Here *nonfrozen* means that it is not isomorphic to a module represented by one of the original frozen variables.

Corollary 8.7. *A prime nonfrozen simple satisfying Theorem 1.4 is a cluster variable in the original Hernandez–Leclerc cluster structure.*

Proof. The theorem writes its class as a nonnegative monomial in a reachable seed. Monoidal categorification identifies this with the simple tensor product of the corresponding factors.

Two nontrivial factors, including a repeated factor or a frozen factor accompanying a mutable one, would contradict primeness. The sole factor is therefore a mutable cluster variable, with exponent one. □

A An exact certificate for the X_6 comparison

The following Python 3.9 program implements Lemma 6.3 using the standard library. The names `alpha1`, `alpha2`, `beta1`, `beta2`, `F` denote the corresponding return words; `_inv` denotes the inverse. All matrices, readouts, sector maps, strict conditions and terminal rays are included. The program checks every return matrix and every terminal ray set, and prints the intermediate cell counts.

```
"""Exact closed-cone certificate for the X6 finite-comparison lemma.

Python 3.9 or later; standard library only. All arithmetic is exact.
"""
from fractions import Fraction
from functools import lru_cache
from math import gcd

N = 6
I = tuple(tuple(int(i == j) for j in range(N)) for i in range(N))
B0 = (
    (0, 0, 2, 0, 0, -1),
    (0, 0, 0, 2, 0, -1),
    (-2, 0, 0, 0, 0, 1),
    (0, -2, 0, 0, 0, 1),
    (0, 0, 0, 0, 0, -1),
    (1, 1, -1, -1, 1, 0),
)
LOOPS = {
    "alpha1": ((0,), (2, 1, 0, 3, 4, 5)),
    "alpha1_inv": ((2,), (2, 1, 0, 3, 4, 5)),
    "alpha2": ((1,), (0, 3, 2, 1, 4, 5)),
    "alpha2_inv": ((3,), (0, 3, 2, 1, 4, 5)),
    "beta1": ((5, 2, 1, 0, 1), (3, 2, 4, 5, 0, 1)),
    "beta1_inv": ((5, 4, 5, 1, 3), (4, 5, 1, 0, 2, 3)),
    "beta2": ((5, 3, 0, 1, 0), (3, 2, 5, 4, 1, 0)),
    "beta2_inv": ((5, 4, 5, 0, 2), (5, 4, 1, 0, 3, 2)),
    "F": ((4, 2, 5, 0, 3, 1, 2, 0, 5, 3), (1, 5, 0, 2, 4, 3)),
    "F_inv": ((5, 1, 2, 3, 0, 5, 2, 1, 3, 4), (2, 0, 3, 5, 4, 1)),
}
WORDS = (
    ("alpha1", "beta1_inv"), ("alpha2", "beta1"),
    ("alpha2", "beta2_inv"), ("alpha1", "beta2"),
    ("alpha2_inv", "F"), ("alpha2_inv", "F_inv"),
)


def dot(a, b):
    return sum(x * y for x, y in zip(a, b))


def scale(k, a):
    return tuple(k * x for x in a)


def add(a, b, k=1):
    return tuple(x + k * y for x, y in zip(a, b))


def evaluate(A, x):
    return tuple(dot(row, x) for row in A)


def primitive(v):
    d = gcd(*v)
    assert d > 0
    return tuple(x // d for x in v)
```

```
def mutate(B, k):
    def entry(i, j):
        if i == k or j == k:
            return -B[i][j]
        x, y = B[i][k], B[k][j]
        return B[i][j] + max(x, 0) * max(y, 0) \
            - min(x, 0) * min(y, 0)
    return tuple(tuple(entry(i, j) for j in range(N)) for i in range(N))


@lru_cache(None)
def exact_rank(rows):
    pivots = {}
    for row in rows:
        r = list(map(Fraction, row))
        for j in range(len(r)):
            if r[j] == 0:
                continue
            if j not in pivots:
                d = r[j]
                pivots[j] = [x / d for x in r]
                break
            d = r[j]
            r = [x - d * y for x, y in zip(r, pivots[j])]
    return len(pivots)


def clip(cell, row, J):
    """Exact intersection with row >= 0, including all proper faces."""
    H, rays = cell
    if not any(row):
        return cell
    values = {u: dot(row, u) for u in rays}
    if min(values.values()) >= 0:
        return cell
    Hnew = H + (primitive(row),)
    candidates = {u for u in rays if values[u] >= 0}
    for u in rays:
        if values[u] <= 0:
            continue
        for v in rays:
            if values[v] < 0:
                crossing = add(scale(-values[v], u), v, values[u])
                candidates.add(primitive(crossing))
    extreme = []
    for u in candidates:
        tight = tuple(
            tuple(h[j] for j in J) for h in Hnew if dot(h, u) == 0
        )
        if exact_rank(tight) == len(J) - 1:
            extreme.append(u)
    return (Hnew, tuple(sorted(extreme))) if extreme else None


def strict_feasible(cell, marks):
    if cell is None:
        return False
    for h in marks:
        values = [dot(h, u) for u in cell[1]]
        assert min(values) >= 0
        if max(values) == 0:
            return False
    return True


def compare(cells, names, A, J, marks):
    states = [(cell, A) for cell in cells]
    B = B0
    for name in names:
        word, p = LOOPS[name]
        for k in word:
            next_states = {}
            for cell, G in states:
```

```
                pivot = G[k]
                for sign in (1, -1):
                    part = clip(cell, scale(sign, pivot), J)
                    if not strict_feasible(part, marks):
                        continue
                    Gnew = tuple(
                        scale(-1, pivot) if i == k else
                        add(G[i], pivot, max(0, -sign * B[k][i]))
                        for i in range(N)
                    )
                    key = (
                        part[1], tuple(evaluate(Gnew, u) for u in part[1])
                    )
                    next_states[key] = (part, Gnew)
            states = list(next_states.values())
            B = mutate(B, k)
        B = tuple(tuple(B[i][j] for j in p) for i in p)
        assert B == B0, name
        states = [(cell, tuple(G[i] for i in p)) for cell, G in states]
    output = {}
    for cell, G in states:
        part = clip(cell, add(G[4], A[4], -1), J)
        if strict_feasible(part, marks):
            output[part[1]] = part
    return list(output.values())


D0 = (0, 0, 0, 0, 2, -1)
K1 = (-1, 0, 1, 0, 0, 0)
K2 = (0, -1, 0, 1, 0, 0)


def neg(j):
    return scale(-1, I[j])


# Each entry is (parameter coordinates, initial map, strict rows, final rays).
SECTORS = {
    "NN": (
        tuple(range(N)), I[:5] + (neg(5),),
        (I[4], I[5]), (D0,),
    ),
    "LN-": (
        (0, 1, 3, 4, 5), (neg(0), I[1], I[0], I[3], I[4], neg(5)),
        (I[4], I[0]), (D0, K1),
    ),
    "LN+": (
        (0, 1, 3, 4, 5), (neg(0), I[1], I[0], I[3], I[4], I[5]),
        (I[4], I[0]), (),
    ),
    "LL-": (
        (0, 1, 4, 5), (neg(0), neg(1), I[0], I[1], I[4], neg(5)),
        (I[4], I[0], I[1]), (D0, K1, K2),
    ),
    "LL+": (
        (0, 1, 4, 5), (neg(0), neg(1), I[0], I[1], I[4], I[5]),
        (I[4], I[0], I[1]), (),
    ),
}


def main():
    for name, (word, p) in LOOPS.items():
        B = B0
        for k in word:
            B = mutate(B, k)
        assert tuple(tuple(B[i][j] for j in p) for i in p) == B0, name
    for name, (J, A, marks, expected) in SECTORS.items():
        cells = [(I, tuple(I[j] for j in J))]
        counts = []
        for word in WORDS:
            cells = compare(cells, word, A, J, marks)
            counts.append(len(cells))
        actual = {evaluate(A, u) for cell in cells for u in cell[1]}
        assert actual == set(expected), (name, actual, expected)
```

```
        print(name, tuple(counts), tuple(sorted(actual)))
    print("All five exact comparisons verified.")


if __name__ == "__main__":
    main()
```

References

[1] T. Adachi, O. Iyama and I. Reiten, *τ-tilting theory*, Compos. Math. **150** (2014), 415–452; arXiv:1210.1036v4.

[2] S. Asai, *The wall-chamber structures of the real Grothendieck groups*, arXiv:1905.02180v2, 2020.

[3] K. Baur, C. Fu and J.-r. Li, *A correspondence between additive and monoidal categorifications with application to Grassmannian cluster categories*, arXiv:2410.04401v3, 2026; doi:10.1016/j.jpaa.2026.108379.

[4] T. Bridgeland, *Scattering diagrams, Hall algebras and stability conditions*, arXiv:1603.00416v4, 2020 (including the correction in Section 11.7).

[5] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, arXiv:math/0402054.

[6] P. Caldero and B. Keller, *From triangulated categories to cluster algebras II*, arXiv:math/0510251.

[7] P. Cao, *$\mathcal{G}$-systems*, Adv. Math. **377** (2021), 107500; arXiv:1902.09218v3.

[8] P. Cao, *F-invariant in cluster algebras*, arXiv:2306.11438v4, 2025.

[9] P. Cao and F. Li, *The enough g-pairs property and denominator vectors of cluster algebras*, Math. Ann. **377** (2020), 1547–1572; arXiv:1803.05281v2.

[10] M.-W. Cheung, T. Magee, T. Mandel and G. Muller, *Valuative independence and cluster theta reciprocity*, arXiv:2505.09585v2, 2026.

[11] B. Davison and T. Mandel, *Strong positivity for quantum theta bases of quantum cluster algebras*, arXiv:1910.12915v3, 2021.

[12] R. Dehy and B. Keller, *On the combinatorics of rigid objects in 2-Calabi–Yau categories*, arXiv:0709.0882v3, 2008.

[13] B. Duan, J.-R. Li and Y.-F. Luo, *Cluster algebras and snake modules*, J. Algebra **519** (2019), 325–377; arXiv:1508.03467v4.

[14] B. Duan and R. Schiffler, *Real simple modules over simply-laced quantum affine algebras and categorifications of cluster algebras*, J. Algebra **685** (2026), 608–672; arXiv:2305.08715v2.

[15] B. Duan and R. Schiffler, *Classification of real modules in monoidal categorifications of cluster algebras*, arXiv:2512.16107v2, 2026.

[16] S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, arXiv:math/0608367v3.

[17] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529; arXiv:math/0104151.

[18] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of W-algebras*, Contemp. Math. **248** (1999), 163–205; arXiv:math/9810055.

[19] C. Fu and B. Keller, *On cluster algebras with coefficients and 2-Calabi–Yau categories*, arXiv:0710.3152.

[20] C. Geiss, D. Labardini-Fragoso and J. Schröer, *The representation type of Jacobian algebras*, Adv. Math. **290** (2016), 394–452; arXiv:1308.0478v3.

[21] C. Geiss, B. Leclerc and J. Schröer, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), 589–632; arXiv:math/0503324.

[22] C. Geiss, B. Leclerc and J. Schröer, *Kac–Moody groups and cluster algebras*, Adv. Math. **228** (2011), 329–433; arXiv:1001.3545v3.

[23] C. Geiss, B. Leclerc and J. Schröer, *Generic bases for cluster algebras and the Chamber Ansatz*, J. Amer. Math. Soc. **25** (2012), 21–76; arXiv:1004.2781v3.

[24] C. Geiss, B. Leclerc and J. Schröer, *Cluster structures on quantum coordinate rings*, Selecta Math. (N.S.) **19** (2013), 337–397; arXiv:1104.0531v3.

[25] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), 399–443; author's text.

[26] D. Hernandez, *Simple tensor products*, Invent. Math. **181** (2010), 649–675; arXiv:0907.3002v2.

[27] D. Hernandez and B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), 265–341; arXiv:0903.1452.

[28] D. Hernandez and B. Leclerc, *A cluster algebra approach to q-characters of Kirillov–Reshetikhin modules*, J. Eur. Math. Soc. **18** (2016), 1113–1159; arXiv:1303.0744.

[29] D. Hernandez and B. Leclerc, *Quantum affine algebras and cluster algebras*, arXiv:1902.01432v2, 2019.

[30] Hernandez Juan, *Quantum affine representations and cluster characters*, Hexagon:2611.00712v1, 20 November 2026.

[31] S.-J. Kang, M. Kashiwara, M. Kim and S.-j. Oh, *Monoidal categorification of cluster algebras*, J. Amer. Math. Soc. **31** (2018), 349–426; arXiv:1801.05145.

[32] M. Kashiwara, M. Kim, S.-j. Oh and E. Park, *Monoidal categorification and quantum affine algebras*, Compos. Math. **156** (2020), 1039–1077; arXiv:1910.08307v2.

[33] M. Kashiwara, M. Kim, S.-j. Oh and E. Park, *PBW theory for quantum affine algebras*, J. Eur. Math. Soc. **26** (2024), 2679–2743; publisher's text.

[34] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$*, Transform. Groups **8** (2003), 95–104; arXiv:math/0202148v2.

[35] G. Muller, *The existence of a maximal green sequence is not invariant under quiver mutation*, arXiv:1503.04675v4.

[36] H. Nakajima, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51** (2011), 71–126; arXiv:0905.0002v5.

[37] P.-G. Plamondon and T. Yurikusa, with an appendix by B. Keller, *Tame algebras have dense g-vector fans*, Int. Math. Res. Not. **2023** (2023), 2701–2747; arXiv:2007.04215.

[38] F. Qin, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*, Duke Math. J. **166** (2017), 2337–2442; arXiv:1501.04085v5.

[39] F. Qin, *An analog of Leclerc's conjecture for bases of quantum cluster algebras*, SIGMA **16** (2020), 122, 22 pp.; arXiv:2004.12466v4.

[40] F. Qin, *Analogs of the dual canonical bases for cluster algebras from Lie theory*, arXiv:2407.02480v4, 2025.

[41] F. Qin, *An introduction to representation-theoretic canonical bases of cluster algebras*, arXiv:2508.12761v2, 2025.

[42] C. M. Ringel, *Exceptional objects in hereditary categories*, author's text, 1996.

[43] H. Sakamoto, *Classification of real and imaginary modules of quantum affine algebras in monoidal categorifications of affine cluster algebras*, arXiv:2512.09631v2, 2026.

[44] A. I. Seven, *Maximal green sequences of exceptional finite mutation type quivers*, SIGMA **10** (2014), 089; arXiv:1406.1072v3.

[45] B. Smith, *A proof of the open orbit conjecture*, Hexagon:2611.00923v1, 23 November 2026.

[46] T. Yurikusa, *Density of g-vector cones from triangulated surfaces*, Int. Math. Res. Not. IMRN **2020** (2020), 8081–8119; arXiv:1904.12479v3.