THE DOMINATION NUMBER OF THE 48×48 QUEEN'S GRAPH

WILL HOWE, GRANT HUFFMAN, JENSEN KOHLMEYER, AND LIAM KRUER

Abstract. We prove that the domination number and the independent domination number of the 48×48 queen's graph are both 25. The upper bound is the independent dominating set of Östergård and Weakley. For the lower bound we classify hypothetical placements of 24 queens by their outermost empty rows and columns and the numbers of occupied lines. The resulting 47,580 cases are excluded by checked propositional refutations. We also derive a coordinate identity for domination by k queens on a $2k \times 2k$ board for even k. Together with Cauchy–Schwarz, the identity gives an arithmetic obstruction in 9,727 of the cases at order 48.

hexagon:2611.00974v1

1. Introduction

Let Q_n be the graph on the squares of an $n \times n$ board in which two distinct squares are adjacent if they share a row, a column, or a diagonal. A set of queens *dominates* the board if every square is occupied or adjacent to an occupied square. Let $\gamma(Q_n)$ be the least size of a dominating set, and let $i(Q_n)$ be the least size of a dominating set whose queens are pairwise nonattacking.

Östergård and Weakley [3, §5] gave an independent dominating set of size 25 on the board of order 48 and obtained $24 \le \gamma(Q_{48}) \le i(Q_{48}) \le 25$. Davies [2, §9] listed 48 as the first even order still open in the certified literature. We settle this case.

Theorem 1.1. *The queen's graph of order* 48 *satisfies*

$$\gamma(Q_{48}) = i(Q_{48}) = 25.$$

The lower bound adapts Davies's rectangle and line-deficit reduction [2, §2]. Section 2 verifies the independent placement explicitly. Section 3 proves that every hypothetical dominating set of 24 queens yields one of the enumerated cases. Section 4 derives the coordinate identity, and Section 5 proves the lower bound by checked refutations of the case formulas.

2. The independent upper bound

Here columns and rows are numbered from 1 to 48, increasing to the right and upward. The entries in the following array, read row by row, are the queen rows in columns $2, 4, \ldots, 48$, respectively.

$$\begin{array}{rrrrrrrr} 21 & 39 & 13 & 31 & 11 & 3 & 23 & 15 \\ 45 & 41 & 5 & 29 & 19 & 43 & 25 & 35 \\ 1 & 47 & 9 & 37 & 17 & 7 & 33 & 27. \end{array}$$

Add a queen at $(9, 40)$. This is the placement in [3, §5] expressed in the present coordinates.

Date: November 24, 2026.

2020 *Mathematics Subject Classification.* Primary 05C69.

Key words and phrases. Queen's graph, domination number, independent domination, computer-assisted proof.

Proposition 2.1. *The displayed placement is an independent dominating set of Q_{48}.*

Proof. The first 24 queens occupy every even column and every odd row. Write P_0 for these queens, and write D^- and D^+ for their sets of difference labels $v - u$ and sum labels $u + v$, where $(u, v) \in P_0$. For each odd x, remove from the set of even rows the elements of $(x + D^-) \cup (D^+ - x)$. The nonempty results are listed in Table 1. Each remaining square lies on one of

$$x = 9, \qquad y = 40, \qquad y - x = 31, \qquad x + y = 49,$$

so the additional queen covers it. Direct substitution for the 25 points gives 25 distinct labels in each of the four families $x, y, y - x, x + y$. Thus the placement is independent. □

TABLE 1. Squares uncovered by the queens in the even columns.

Column x	1	3	5	9	13	17	45	47
Rows y	32, 40, 48	46	36, 44	40, 48	44	48	4	2

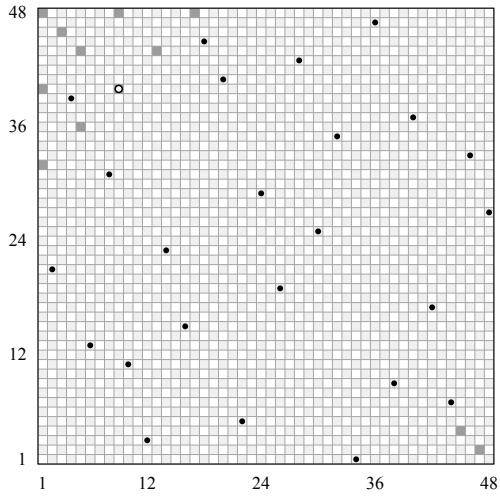

FIGURE 1. The independent dominating set. The open circle marks $(9, 40)$. The twelve shaded squares are precisely those uncovered upon its removal.

3. Reduction to finitely many cases

For the lower bound use integer coordinates $0, \ldots, 47$ on both axes. Suppose a set of 24 queens dominates the board. Let R, C, T, S be the sets of occupied row, column, difference-diagonal and sum-diagonal labels. Define the deficits

$$d_R = 24 - |R|, \quad d_C = 24 - |C|, \quad d_T = 24 - |T|, \quad d_S = 24 - |S|, \qquad \Delta = d_R + d_C + d_T + d_S.$$

At least 24 rows and 24 columns are empty. Let $a < b$ be the least and greatest empty column labels, and $c < d$ the least and greatest empty row labels. Write $w = b - a$ and $h = d - c$ for the width and height of their rectangle. Call a square *needy* if both its row and its column are empty. Such a square must be covered by an occupied diagonal.

Lemma 3.1 (Border and side bounds)**.** *Let $I = 0$ when $w \neq h$. When $w = h$, let I be the number of occupied corner-to-corner diagonals of the rectangle. Then*

$$\Delta \leq 2, \tag{3.1}$$

$$2d_R + d_T + d_S \leq I \quad (w \geq h), \qquad 2d_C + d_T + d_S \leq I \quad (h \geq w). \tag{3.2}$$

Proof. Every empty row lies between c and d, so each vertical side contains $24 + d_R$ needy squares. Each horizontal side contains $24 + d_C$. Subtracting the four corners counted twice, the border has $92 + 2d_R + 2d_C$ needy squares. Each of the $48 - d_T - d_S$ occupied diagonals meets the border at most twice. Therefore

$$0 \leq 2(48 - d_T - d_S) - (92 + 2d_R + 2d_C) = 4 - 2\Delta.$$

If $w \geq h$, a diagonal meeting both vertical sides changes its row coordinate by exactly w. This requires $w = h$ and makes it one of the two corner-to-corner diagonals. The vertical sides thus receive at most $48 - d_T - d_S + I$ diagonal incidences and require $48 + 2d_R$. Transposition gives the analogous horizontal bound. □

Define the margins $(l, r, b_0, t) = (a, 47 - b, c, 47 - d)$. All lines exterior to the rectangle are occupied in their respective row or column families, so $l + r \leq 24 - d_C$ and $b_0 + t \leq 24 - d_R$. Reflections and transposition allow the normalization

$$l \leq r, \qquad b_0 \leq t, \qquad (l, r) \leq_{\text{lex}} (b_0, t). \tag{3.3}$$

The deficit tuple is taken after normalization, since a symmetry can exchange the two diagonal families.

Lemma 3.2 (Case enumeration)**.** *There are* 47,580 *tuples of nonnegative integer margins and deficits satisfying* (3.3)*, the two exterior-line bounds,* (3.1)*, and* (3.2) *with I replaced by $\mathbf{2}_{\{w=h\}}$. Every dominating set of* 24 *queens gives one of these tuples after a symmetry.*

Proof. For each margin sum $s \in \{0, \ldots, 24\}$, there are $p_s = 1 + \lfloor s/2 \rfloor$ nondecreasing margin pairs. The total is 169. Choosing two such pairs with repetition gives $169 \cdot 170/2 = 14{,}365$ rectangles. Of these, $\sum_{s=0}^{24} p_s(p_s + 1)/2 = 819$ are square, including 91 with $s = 24$.

For a nonsquare rectangle, (3.2) forces both diagonal deficits to be zero and allows a positive deficit only in the column family when $w > h$, or in the row family when $h > w$. Its corresponding margin sum is the smaller of the two sums, so a deficit of one is always allowed. A deficit of two fails precisely when that sum is 23 and the other sum is 24. This occurs for $p_{23}p_{24} = 156$ rectangles.

For a square rectangle, each single deficit is allowed except for a row or column deficit at $s = 24$. For $\Delta = 2$, the allowed tuples are $(0, 0, 2, 0)$, $(0, 0, 1, 1)$, $(0, 0, 0, 2)$ and, when $s \leq 23$, $(1, 1, 0, 0)$. The resulting counts appear in Table 2. The border and side bounds and the normalization show that the list covers all dominating placements of size 24. □

TABLE 2. Case counts by rectangle shape and total line deficit.

Shape	$\Delta = 0$	$\Delta = 1$	$\Delta = 2$	Total
Nonsquare	13,546	13,546	13,390	40,482
Square	819	3,094	3,185	7,098
Total	14,365	16,640	16,575	47,580

Lemma 3.3 (A saturated border)**.** *When $\Delta = 2$, a queen at $(x, d + u)$, with $u > 0$, must satisfy*

$$a + u + 1 \le x \le b - u - 1.$$

Proof. Equality in the border bound forces every occupied diagonal to pass through two distinct needy border squares. The two diagonals through $(x, d + u)$ cross the line $y = d$ at $x - u$ and $x + u$. If $x - u \le a$, the leftward descending ray misses the rectangle or touches only its upper left corner. The same argument at the right gives $x + u < b$. □

For example, the tuple with margins $(2, 12, 7, 16)$ and deficits $(0, 2, 0, 0)$ has $a = 2$, $b = 35$ and $d = 31$. Row 47 is exterior and must contain a queen. Taking $u = 16$ in Lemma 3.3 would give $19 \le x \le 18$, excluding the case. Figure 2 shows the two ray intersections used in the argument.

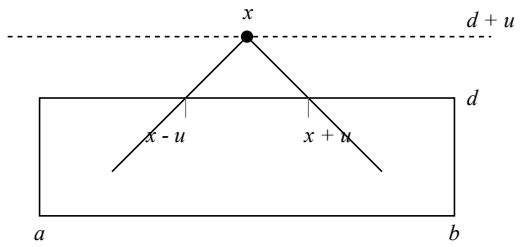

FIGURE 2. The rays from a queen u rows above the rectangle. The intersection abscissae $x - u$ and $x + u$ must both lie strictly between a and b in a saturated case.

4. A polynomial obstruction

We use the grid coefficient functional underlying the Combinatorial Nullstellensatz [1, Theorem 1.2]. For a finite set A of distinct real numbers, put

$$f_A(Z) = \prod_{a \in A} (Z - a), \qquad \Lambda_A(F) = \sum_{a \in A} \frac{F(a)}{f_A'(a)}.$$

Writing $N = |A| - 1$, Lagrange interpolation gives $\Lambda_A(Z^j) = 0$ for $0 \le j < N$ and $\Lambda_A(Z^N) = 1$. Reducing Z^{N+1} modulo f_A also gives $\Lambda_A(Z^{N+1}) = \sum_{a \in A} a$.

Theorem 4.1 (Mixed coordinate identity)**.** *Let k be a positive even integer. Suppose k queens at positions (u_i, v_i), $1 \le i \le k$, dominate a $2k \times 2k$ board. Measure both coordinates from the center of the board. Then*

$$k \sum_{i=1}^{k} u_i v_i = (4k - 3) \left(\sum_{i=1}^{k} u_i \right) \left(\sum_{i=1}^{k} v_i \right). \tag{4.1}$$

Proof. Put $H(X, Y) = XY(Y^2 - X^2)$ and

$$G(X, Y) = \prod_{i=1}^{k} H(X - u_i, Y - v_i).$$

The four linear factors contributed by a queen vanish on its four lines. Thus G vanishes on $A \times A$, where A is the set of the $2k$ centered coordinates. Set $N = 2k - 1$. The product functional $\Lambda_A \otimes \Lambda_A$ annihilates G. It vanishes on any monomial with an exponent below N or equal to $N + 1$, the latter because $\sum_{a \in A} a = 0$.

Since $\deg G = 2N + 2$, the only other exponents that could contribute beyond (N, N) are $(N + 2, N)$ and $(N, N + 2)$. Both are absent from the highest homogeneous part H^k, which has even exponents in both variables. Consequently $[X^N Y^N] G = 0$.

For a polynomial F write $c(F) = [X^N Y^N]F$. In the degree $2N$ part of G, terms involving two X-derivatives or two Y-derivatives have even exponents in both variables and vanish under c. The second-order terms of the translated factors give

$$0 = C\sum_i u_i v_i + D\left(\left(\sum_i u_i\right)\left(\sum_i v_i\right) - \sum_i u_i v_i\right), \qquad C = c(H^{k-1}H_{XY}), \quad D = c(H^{k-2}H_X H_Y).$$

Let $B = (-1)^{k/2}\binom{k}{k/2} = [X^{2k}Y^{2k}]H^k$. As $H_{XY} = 3(Y^2 - X^2)$, we have $C = 3B$. Differentiating H^k once in each variable gives $kC + k(k-1)D = 4k^2 B$. Thus $D = (4k-3)B/(k-1)$ and $C - D = -kB/(k-1)$. Substitution yields (4.1). □

Proposition 4.2. *The identity* (4.1) *and Cauchy–Schwarz exclude* 9,727 *of the tuples in Lemma 3.2.*

Proof. Use doubled centered coordinates, so that integer coordinate x from Section 3 has centered value $2x - 47$. Equation (4.1) is homogeneous of degree two. For either axis let E be the set of compulsory exterior coordinates, J the set of coordinates strictly between the two extreme empty lines, and e the deficit for that family. Put $m = 24 - e - |E|$. There must be m distinct occupied elements of J, and e additional queen occurrences among the occupied coordinates.

Let A_- and A_+ be E together with the m least and greatest elements of J, respectively. The least and greatest possible coordinate sums under these restrictions are

$$s_- = \sum_{z\in A_-} z + e\min A_-, \qquad s_+ = \sum_{z\in A_+} z + e\max A_+.$$

Replacing an optional coordinate by a smaller one cannot increase the first expression, and the reversed argument gives the second. Thus $M = \max(0, s_-, -s_+)$ is a lower bound on the absolute coordinate sum. Let A_2 be E together with the m elements of J with largest squares. An upper bound for the sum of squared coordinates is $U = \sum_{z\in A_2} z^2 + e\max_{z\in A_2} z^2$, by the same extremal argument applied to the squares.

At $k = 24$, equation (4.1) is $8\sum_i u_i v_i = 31(\sum_i u_i)(\sum_i v_i)$. Apply the bounds to the two axes to obtain M_x, M_y, U_x, U_y. By Cauchy–Schwarz,

$$(31M_x M_y)^2 \le 8^2 U_x U_y. \tag{4.2}$$

For instance, the tuple with margins $(0, 12, 0, 19)$ and zero deficits gives $(M_x, U_x) = (24, 30{,}568)$ and $(M_y, U_y) = (346, 26{,}704)$; the left side of (4.2) exceeds the right side. Over the nonsquare tuples the numbers failing (4.2) at $\Delta = 0, 1, 2$ are 2,581, 2,386, and 1,999. For square tuples the corresponding counts are 337, 1,177, and 1,247. Their sum is 9,727. □

5. Propositional refutations and the lower bound

Fix a tuple from Lemma 3.2. Introduce a Boolean variable q_v for each board square v and o_L for each board line L. The line variables satisfy

$$o_L \longleftrightarrow \bigvee_{v\in L} q_v, \qquad \bigvee_{L\ni v} o_L \quad \text{for every square } v. \tag{5.1}$$

The first condition is expressed by the clauses $(\neg q_v \vee o_L)$ for $v \in L$ and $\neg o_L \vee \bigvee_{v\in L} q_v$. In each family the number of true line variables is 24 minus its deficit. In every family with deficit zero, add $\neg q_v \vee \neg q_{v'}$ for distinct squares on each line of that family. Since at least one deficit is zero, the line counts and these clauses force exactly 24 queens. The four extreme lines are prescribed empty and all exterior lines in their respective families are prescribed occupied by unit clauses.

5.1. **Incidence constraints.** We impose additional consequences of (5.1). Let X be the whole border, the vertical sides when $w \geq h$, or the horizontal sides when $h \geq w$. For a diagonal L put $\beta_L = |L \cap X|$. Set $\kappa_L = 2$ at the whole border. At a pair of sides use $\kappa_L = 1$, with $\kappa_L = 2$ for the two corner-to-corner diagonals of a square rectangle. In all cases $\beta_L \leq \kappa_L$. Let $B_X \subseteq X$ be the needy squares and let $m(v)$ count occupied diagonals through v. Then

$$\sum_{L \text{ occupied}} (\kappa_L - \beta_L) + \sum_{v \in X \setminus B_X} m(v) + \big|\{v \in B_X : m(v) = 2\}\big| = \sum_{L \text{ occupied}} \kappa_L - |B_X|. \tag{5.2}$$

Indeed, $\sum_{L \text{ occupied}} \beta_L = \sum_{v \in X} m(v)$, and $m(v) \in \{1, 2\}$ at every needy square. The right side of (5.2) equals $4 - 2\Delta$ for the whole border, $I - d_T - d_S - 2d_R$ for the vertical sides, and $I - d_T - d_S - 2d_C$ for the horizontal sides.

To encode these bounds, use o_L with multiplicity $\kappa_L - \beta_L$ in the first sum. For the second and third sums, introduce a fresh variable for each counted conjunction and add the clause requiring the variable to be true whenever the conjunction is true. In the second sum the conjunction states that a diagonal through v is occupied and v is nonneedy. In the third it states that both diagonals are occupied and v is needy. At a noncorner square of a vertical side, nonneediness is the row variable; at a noncorner square of a horizontal side, it is the column variable. The corners are always needy. For a pair of sides of a square rectangle, move I to the left by adding the two negated corner-diagonal variables to the list and adding 2 to the right-hand constant with I removed.

5.2. **Cardinality clauses and correctness.** An at-most constraint on a list of M literals $\ell_1, \ldots, \ell_M$ with bound b uses the sequential cardinality counter [4]. If $0 < b < M$, introduce p_{ij} for $1 \leq i < M$ and $1 \leq j \leq b$, interpreted as at least j true occurrences in the first i positions. Use the clauses corresponding to

$$\begin{aligned}
\ell_i &\Rightarrow p_{i,1} && (1 \leq i < M),\\
\neg p_{1,j} &&& (2 \leq j \leq b),\\
p_{i-1,j} &\Rightarrow p_{i,j} && (2 \leq i < M,\ 1 \leq j \leq b),\\
\ell_i \wedge p_{i-1,j-1} &\Rightarrow p_{i,j} && (2 \leq i < M,\ 2 \leq j \leq b),\\
\neg \ell_i \vee \neg p_{i-1,b} &&& (2 \leq i \leq M).
\end{aligned}$$

The first four lines inductively force p_{ij} whenever its prefix contains at least j true occurrences, so the last line forbids a $(b + 1)$-st true occurrence. If the bound holds, assigning each p_{ij} its prefix value satisfies all the clauses. For $b = 0$ each literal is prescribed false, for $b \geq M$ no clause is required, and for $b < 0$ the constraint is the empty clause. An exact count is expressed by an upper bound on the list and a complementary upper bound on its negation.

A dominating placement of the given tuple assigns the square and line variables according to its occupied squares and lines. It satisfies (5.1), the line counts, the clauses for the families of zero deficit, and the exterior and extreme line clauses. Assign each incidence variable the value of its conjunction, then assign the prefix counts. Identity (5.2) ensures that the incidence clauses and their bounds are satisfied. This gives an extension of every such placement to a satisfying assignment of the case formula.

5.3. **Finite verification.**

Lemma 5.1 (Checked finite computation)**.** *The case formulas for all* 47,580 *tuples in Table 2 are unsatisfiable.*

Verification. For each tuple the full formula received a checked DRAT refutation [6]. A second check used an LRAT refutation on a subformula formed by selecting clauses of the same input formula. Each of these 47,580 refutations was accepted by the formally verified checker of Tan, Heule and Myreen [5]. Unsatisfiability of a subformula implies unsatisfiability of the full conjunction. The list of checked tuples agreed with the list obtained by enumerating the outermost empty intervals, applying all eight board symmetries and testing the inequalities of Lemma 3.2. The checks covered every tuple, including those excluded by Proposition 4.2. □

Proof of Theorem 1.1. If there were a dominating set of size 24, a symmetry would place it in one of the cases of Lemma 3.2. The assignment just constructed would satisfy that case formula, contradicting Lemma 5.1. A smaller dominating set could be extended to size 24 by adding queens, since domination is preserved by supersets. Thus $\gamma(Q_{48}) \geq 25$. Proposition 2.1 gives $i(Q_{48}) \leq 25$, completing the proof. □

Acknowledgment and AI-use disclosure. OpenAI's Codex assisted with the mathematical investigation, the finite computations and their verification, and the drafting and typesetting of the manuscript.

References

[1] N. Alon, *Combinatorial Nullstellensatz*, Combin. Probab. Comput. **8** (1999), 7–29. doi:10.1017/S0963548398003411.
[2] J. Davies, $\gamma(Q_{35}) = 19$ *and certified queen domination numbers up to* $n = 46$, preprint, 2026. Hexagon 2611.00830v1.
[3] P. R. J. Östergård and W. D. Weakley, *Values of domination numbers of the queen's graph*, Electron. J. Combin. **8** (2001), #R29. Article.
[4] C. Sinz, *Towards an optimal CNF encoding of Boolean cardinality constraints*, in Principles and Practice of Constraint Programming (CP 2005), Lecture Notes in Comput. Sci. **3709**, Springer, 2005, 827–831. doi:10.1007/11564751_73.
[5] Y. K. Tan, M. J. H. Heule and M. O. Myreen, *Verified propagation redundancy and compositional UNSAT checking in CakeML*, Int. J. Softw. Tools Technol. Transf. **25** (2023), 167–184. doi:10.1007/s10009-022-00690-y.
[6] N. Wetzler, M. J. H. Heule and W. A. Hunt Jr., *DRAT-trim: Efficient checking and trimming using expressive clausal proofs*, in Theory and Applications of Satisfiability Testing (SAT 2014), Lecture Notes in Comput. Sci. **8561**, Springer, 2014, 422–429. doi:10.1007/978-3-319-09284-3_31.

Will Howe · howew@purdue.edu
Grant Huffman · huffma97@purdue.edu
Jensen Kohlmeyer · jenwin@purdue.edu
Liam Kruer · kruerl@purdue.edu