A uniform exponent bound for asymptotic tensor rank

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Abstract

Over every field $\mathbb{F}$, a three-tensor with local ranks at most d has asymptotic rank at most $d^{\omega_{\mathbb{F}}/\sqrt{3}}$, where $\omega_{\mathbb{F}}$ is the matrix multiplication exponent. This uniform improvement on Strassen's bound answers a question of Alman and Li. Using rectangular matrix multiplication, we show that the asymptotic rank equals the largest local rank D whenever the other two are at most $D^{\sqrt{5}/4}$. In odd order $2m+1$ with all dimensions $d \geq 2$, the worst-case exponent in base d is $m + O(m^{-1/3})$. For any fixed tensor T, adjoining a rank-q matrix on two new parties gives asymptotic rank $q(M(T) + o(1))$ as $q \to \infty$, where $M(T)$ is the largest flattening rank of T. The proofs use spectral duality and CAT(0) geometry.

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1 Introduction

Fix a field $\mathbb{F}$. Write $\mathrm{R}(T)$ for the rank of a k-party tensor T and $\boxtimes$ for the Kronecker product, which groups factors by party. The asymptotic rank of T is

$$\widetilde{\mathrm{R}}(T) = \lim_{n\to\infty} \mathrm{R}(T^{\boxtimes n})^{1/n}.$$

The limit exists by submultiplicativity. Matrix rank is multiplicative, so each bipartition flattening bounds $\widetilde{\mathrm{R}}(T)$ from below. Single-party flattenings give the local ranks.

Let $\omega_{\mathbb{F}}$ be the exponent of square matrix multiplication over $\mathbb{F}$. For $T \in (\mathbb{F}^d)^{\otimes 3}$, Strassen's reduction gives $\widetilde{\mathrm{R}}(T) \leq d^{2\omega_{\mathbb{F}}/3}$ [12, 7]. Alman and Li improved this exponent for each fixed d and asked whether the saving could be made uniform in d [2, Problem 41].

Theorem 1.1. *Over every field $\mathbb{F}$, a three-tensor T with local ranks at most d satisfies*

$$\widetilde{\mathrm{R}}(T) \leq d^{\omega_{\mathbb{F}}/\sqrt{3}} \leq d^{3\sqrt{3}/4}. \tag{1}$$

Since $\omega_{\mathbb{F}} \geq 2$, the exponent decreases by at least $4/3 - 2/\sqrt{3}$. The numerical exponent $3\sqrt{3}/4 = 1.299038\ldots$ follows from the $9/4$ matrix multiplication bound [10] over arbitrary fields [3, 6]. For unequal local ranks, Corollary 3.4 gives exact asymptotic ranks even in formats with strictly larger generic border rank.

Let $U_{\mathbb{F}}(n_1, \ldots, n_k)$ be the supremum of the asymptotic ranks in format $(n_1, \ldots, n_k)$. Khoruzhii proved $U_{\mathbb{F}} \leq (\prod_i n_i)^{1/2}$, with equality in even cubical formats [8]. We show that equality holds exactly when the dimensions split into two groups with equal products. In odd order $2m+1$ with all dimensions $d \geq 2$,

$$m \leq \log_d U_{\mathbb{F}}(\underbrace{d, \ldots, d}_{2m+1}) \leq m + O(m^{-1/3}), \tag{2}$$

with an absolute constant. A full-rank $d^m \times d^{m+1}$ flattening gives the lower bound. Theorem 5.1 recovers the largest flattening rank of a fixed tensor by adjoining a matrix on two new parties.

2 Spectral points and auxiliary parties

Write log for the base-two logarithm. A restriction applies a linear map at each party. The tensors $S_1, \dots, S_h$ cover T if T is a sum of their restrictions. A relaxed spectral point is a nonnegative real-valued function ϕ that equals one on nonzero simple tensors, is multiplicative under $\boxtimes$, and satisfies $\phi(T) \le \sum_j \phi(S_j)$ for every cover of T [5, Definition A.1]. A nonzero tensor restricts to the unit, so its value is at least one. For ordinary spectral points, which are additive under direct sums, Strassen's duality theorem gives $\widetilde{\mathrm{R}}(T) = \max_\phi \phi(T)$ [12, 14]. Covering tensor powers by simple tensors gives $\phi(T) \le \widetilde{\mathrm{R}}(T)$ for every relaxed point.

Write $E_{ij}(s) = \sum_{h=1}^{s} e_h \otimes e_h$ for the rank-s matrix, or *wire*, on parties i, j, with one-dimensional factors elsewhere. The wire distances $d_{ij} = \log \phi(E_{ij}(2))$, with $d_{ii} = 0$, form a $[0,1]$-valued pseudometric and satisfy $\phi(E_{ij}(s)) = s^{d_{ij}}$ [5, Lemma A.2].

Lemma 2.1. *Let $T \neq 0$ have local ranks r_i on a finite set of parties I. For each relaxed spectral point ϕ on I, there are points x_i in a complete* CAT(0) *space X such that $d(x_i, x_j) = d_{ij}$ and*

$$\log \phi(T) \le \inf_{y \in X} \sum_{i \in I} (\log r_i) d(x_i, y). \tag{3}$$

Proof. Apply the spectral extension theorem [5, Proposition A.7] with a new party o assigned to i or j with equal probability. Each extended value is bounded by the geometric mean of its two assigned values, with equality at one prescribed tensor on the original parties. Prescribe the product of T and all existing rank-two wires. The value of each factor is positive and cannot increase, so equality holds for each factor. Thus $d_{io}, d_{jo} \le d_{ij}/2$, and the triangle inequality forces equality.

At each stage insert midpoints between all current sites. Distances agree on overlaps, and every finite subset occurs at some stage. The resulting pseudometric is therefore a price in the sense of [5, Definition A.4]. After identifying points at distance zero, it embeds in a complete CAT(0) space [5, Proposition C.3]. Its closure X contains its midpoints by continuity, hence its geodesics by dyadic subdivision. Thus X is complete and CAT(0).

At each constructed site o, the product of the wires $E_{io}(r_i)$ over $i \neq o$ restricts to T. In bases of the local supports, contract the wire ends at o with the coefficient array of T, retaining its output at o if $o \in I$. Passing to the closure gives (3) by continuity. □

3 Three parties and rectangular matrix multiplication

Index the edges of a triangle by the opposite vertex. For $c_i \ge 0$, let $\omega_{\mathbb{F}}(c_1, c_2, c_3)$ be the rectangular matrix multiplication exponent with edge sizes $\lceil 2^{nc_i} \rceil$. The product of the three edge tensors is the rectangular matrix multiplication tensor. Spectral duality gives

$$\omega_{\mathbb{F}}(c_1, c_2, c_3) = \max_{\phi} \big(c_1 d_{23} + c_2 d_{31} + c_3 d_{12} \big), \tag{4}$$

where ϕ ranges over ordinary three-party spectral points [3, Theorem 4.2 and Lemma 4.4]. Indeed, dividing the log asymptotic rank of the edge product by n gives the right side up to a rounding error of at most $3/n$. In particular, $\omega_{\mathbb{F}}(1,1,1) = \omega_{\mathbb{F}}$.

Theorem 3.1. *Let $T \neq 0$ have three local ranks r_i. Let Q_1, Q_2, Q_3 be the vertices of a possibly degenerate Euclidean triangle with opposite sides $w_i \ge \log r_i$. Then*

$$\log \widetilde{\mathrm{R}}(T) \le \min_{z \in \mathrm{conv}\{Q_1, Q_2, Q_3\}} \omega_{\mathbb{F}}\big(\|z - Q_1\|, \|z - Q_2\|, \|z - Q_3\| \big). \tag{5}$$

In particular, one may take $w_i = \log r_i$.

For a Euclidean triangle with sides s_i opposite P_i and weights $w_i \geq 0$, write $F(w,s) = \min_y \sum_i w_i \|P_i - y\|$ for the weighted Fermat minimum [1].

Lemma 3.2. *If both w and s are triples of triangle side lengths, then $F(w,s) = F(s,w)$.*

Proof. At a minimizer y the subgradient condition gives vectors f_i with $\sum_i f_i = 0$, $\|f_i\| \leq w_i$, and $f_i \cdot (P_i - y) = w_i \|P_i - y\|$. Thus $F(w,s) = \sum_i P_i \cdot f_i$. Write $f_i = V_{i+1} - V_{i+2}$ with cyclic indices, and let Q_i be opposite side w_i. The map $Q_i \mapsto V_i$ is 1-Lipschitz. For any z, Kirszbraun's theorem extends it to $z \mapsto v$ with $\|V_i - v\| \leq \|Q_i - z\|$. Collecting terms gives

$$F(w,s) = \sum_i (P_{i+2} - P_{i+1}) \cdot (V_i - v) \leq \sum_i s_i \|Q_i - z\|.$$

Minimizing over z gives $F(w,s) \leq F(s,w)$. Interchanging s and w gives equality. □

Proof of Theorem 3.1. Let P_1, P_2, P_3 be the vertices of the comparison triangle for the sites in Lemma 2.1, with sides $s_i = d_{i+1,i+2}$. By the Lang–Schroeder theorem [9], $P_i \mapsto x_i$ admits a 1-Lipschitz extension to a minimizer of $F(w,s)$. Lemmas 2.1 and 3.2 give

$$\log \phi(T) \leq F(w,s) = F(s,w) \leq \sum_i s_i \|z - Q_i\| \qquad (z \in \mathbb{R}^2). \tag{6}$$

Maximize over ordinary spectral points and apply (4). Projection onto the triangle can only decrease the three distances, giving (5). Compression to the local supports gives $r_i \leq r_{i+1} r_{i+2}$, so $\log r_i$ satisfy the triangle inequalities. □

For Theorem 1.1, take the center of an equilateral triangle of side $\log d$. Its circumradius $(\log d)/\sqrt{3}$ gives (1). The objective in (5) is convex by (4) and invariant under rotations by $2\pi/3$, so the center is a minimizer. More precisely, (6) gives $\phi(T)^{\sqrt{3}} \leq \phi(\mathrm{MM}_d)$ for every relaxed point, where MM_d denotes the $d \times d$ matrix multiplication tensor.

Corollary 3.3. *For each fixed three-tensor T with local ranks at most $d \geq 2$, there are integers h_n with $h_n/n \to 1/\sqrt{3}$ such that $T^{\boxtimes n}$ is a restriction of $\mathrm{MM}_d^{\boxtimes h_n}$.*

Proof. Assume $T \neq 0$ and fix positive integers a, b with $b/a > 1/\sqrt{3}$. The pointwise bound and cover duality [5, Theorem A.6] cover $T^{\boxtimes aN}$ by $2^{o(N)}$ restrictions of $\mathrm{MM}_d^{\boxtimes bN}$. A sum of K restrictions of S is a restriction of $S \boxtimes \Delta_K$, where the K diagonal entries of the three-party diagonal tensor Δ_K index the summands. Projecting each local index pair onto its diagonal restricts MM_d to Δ_d. Thus $o(N)$ extra copies of MM_d absorb the cover. Take $N = \lceil n/a \rceil$ and contract surplus copies of T to the unit. The least number k_n of factors satisfies $\limsup_n k_n/n \leq b/a$. Letting $b/a \downarrow 1/\sqrt{3}$, we may take $h_n = \max\{k_n, \lceil n/\sqrt{3} \rceil\}$. □

Fix a binary operation $*$ on d symbols. Coordinatewise convolution on words of length N has structure tensor $T^{\boxtimes N}$, where $T_{abc} = \mathbf{1}_{\{a*b=c\}}$. It takes $O_{*,\mathbb{F},\varepsilon}((d^{\omega_{\mathbb{F}}/\sqrt{3}} + \varepsilon)^N)$ field operations for every $\varepsilon > 0$. A rank-r formula on a block of h coordinates gives the recurrence $C_N \leq r C_{N-h} + O_{d,h}(d^{N-h})$ by applying its linear maps to the remaining coordinates. Choose h by (1) so that $r < (d^{\omega_{\mathbb{F}}/\sqrt{3}} + \varepsilon)^h$. The bound follows since $d < d^{\omega_{\mathbb{F}}/\sqrt{3}} + \varepsilon$.

Exact asymptotic ranks

Write $g(u) = \sqrt{u^3(4-3u)}$ for $0 \leq u \leq 1$.

Corollary 3.4. *Suppose that T has local ranks D, r_2, r_3, with $D \geq 2$. If for some $u \in [0,1]$,*

$$r_2 \leq D^{g(1-u)}, \qquad r_3 \leq D^{g(u)},$$

then $\widetilde{\mathrm{R}}(T) = D$. In particular it suffices that $r_2, r_3 \leq D^{\sqrt{5}/4}$.

Proof. Measure lengths in units of $\log D$. Set $Q_2 = (0,0)$, $Q_3 = (1,0)$, $z = (u,0)$ and $c = u(1-u)$. Applying $\omega_{\mathbb{F}}(a,b,ab/(a+b)) = a+b$ at $(a,b) = (u,1-u)$ gives $\omega_{\mathbb{F}}(u,1-u,c) = 1$. Amanov, van den Berg and Zuiddam proved the identity over every field [3, Theorem 1.2], following Uznański's characteristic-zero result [13]. Take

$$Q_1 = \bigl(3u^2 - 2u^3,\ 2[u(1-u)]^{3/2}\bigr).$$

Then $\|Q_1 - z\| = c$, $\|Q_1 - Q_2\| = g(u)$ and $\|Q_1 - Q_3\| = g(1-u)$. Theorem 3.1 gives $\widetilde{\mathrm{R}}(T) \le D$, with equality by the first flattening. At $u = 1/2$ both exponents equal $\sqrt{5}/4$. □

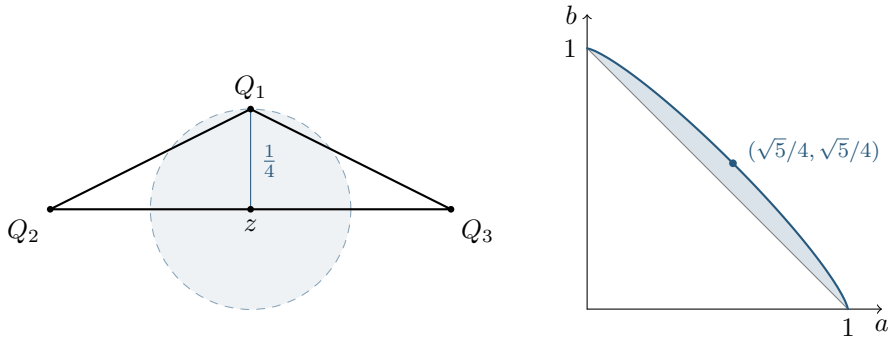


Figure 1: The construction at $u = 1/2$ (left) and the exact-rank region of Corollary 3.4 (right), with $a = \log_D r_3$ and $b = \log_D r_2$. The first flattening imposes $a + b \ge 1$.

For example, every $64 \times 10 \times 10$ tensor of first local rank 64 has asymptotic rank 64. Over an algebraically closed field a generic tensor of this format has border rank at least 79. Indeed, the affine cone of simple tensors has dimension $64 + 10 + 10 - 2 = 82$, so the closure of sums of 78 simple tensors has dimension at most $6396 < 6400$. More generally, for fixed $1/2 < t \le \sqrt{5}/4$ and $s = \lfloor D^t \rfloor$, a generic $D \times s \times s$ tensor has asymptotic rank D and border rank $(1 + o(1))s^2$ as $D \to \infty$. Its first flattening has full rank for large D, and the same dimension count gives the lower bound $Ds^2/(D + 2s - 2)$ against the basis-expansion upper bound s^2.

4 Nearly balanced formats

For $k \ge 2$ and $n_i \ge 1$, put $W = \sum_i \log n_i$. When $W > 0$, define

$$a_i = \frac{\log n_i}{W}, \qquad \delta = \min_{A \subseteq [k]} \left| \frac{1}{2} - \sum_{i \in A} a_i \right|, \qquad K = \tfrac{1}{2} - \delta.$$

The largest attainable flattening rank is 2^{WK}. When $W = 0$, every nonzero tensor is simple.

Theorem 4.1. *For $W > 0$, one has $U_{\mathbb{F}}(n_1, \dots, n_k) = 2^{W/2}$ if and only if $\delta = 0$. If $0 < \delta \le 1/2$, then for an absolute constant C,*

$$K \le \frac{\log U_{\mathbb{F}}(n_1, \dots, n_k)}{W} \le \min\Bigl\{ \tfrac{1}{2} - \tfrac{\delta}{7},\ K + C\delta^{4/3} \Bigr\}. \tag{7}$$

For cubical tensors of order $2m+1$, this gives (2) and the explicit upper exponent $m + 3/7$.

The wire distances satisfy, for any nonnegative b_i,

$$\sum_{i<j} b_i b_j d_{ij} \le \tfrac{1}{4} \Bigl(\sum_i b_i \Bigr)^2. \tag{8}$$

This is the wire inequality for weighted graph tensors [15, Section 5.1]. For integer b_i, group b_i vertices at site i in the complete graph tensor with binary edges. Grouping preserves simple tensors, and internal wires become local vectors. Apply the complete graph upper bound $2^{(\sum b_i)^2/4}$ [8, Theorem 3] to obtain (8). Homogeneity and continuity give the general case.

For sites A, B with $D = d(A, B) = 1 - \eta$, write $p_i = d(A, x_i)$ and $q_i = d(B, x_i)$. Using weights $1, 1, 2v$ at A, B, x_i in (8) gives $D + 2v(p_i + q_i) \le (1 + v)^2$. Taking $v = \sqrt{\eta}$, or letting $v \downarrow 0$ when $\eta = 0$, yields

$$D \le p_i + q_i \le 1 + \sqrt{\eta}. \tag{9}$$

Lemma 4.2. *Suppose that A, B attain the diameter $D = 1 - \eta \ge 1/2$ of the finite set of sites. In the Euclidean comparison triangle for A, B, x_i, take the base to be $[0, D]$ and let (s_i, y_i) be the third vertex, with $y_i \ge 0$. Set $t_i = \min(p_i, q_i)$. Then $0 \le s_i \le D$ and*

$$d(x_i, h)^2 \le (h - s_i)^2 + y_i^2 \qquad (h \in [0, D]), \tag{10}$$

$$y_i^2 \le Ct_i(\eta + \sqrt{\eta t_i}), \qquad t_i + t_j \le C(1 - d_{ij} + \eta), \tag{11}$$

where h on the left denotes the corresponding geodesic point and C is absolute.

Proof. Since both p_i and q_i are at most D, the foot $s_i = (D^2 + p_i^2 - q_i^2)/(2D)$ lies in the base segment. The first inequality is triangle comparison. To bound the altitude write $v = p_i - q_i$ and $b = 1 - v^2/D$. Since $D - |v| \le 2t_i$, we have $\eta \le b \le \eta + 4t_i$. For $\eta > 0$ put $t = \sqrt{\eta/b} \le 1$ and $z = tv/D$. Apply (8) with masses $(1 + z)/2, (1 - z)/2, t$ at A, B, x_i. Expansion gives

$$2t(p_i + q_i - 1) \le \eta + t^2 + Dz^2 - 2tzv = \eta + bt^2.$$

Thus $p_i + q_i \le 1 + \sqrt{\eta b}$. Heron's identity gives

$$y_i^2 = \frac{((p_i + q_i)^2 - D^2)(D^2 - v^2)}{4D^2} \le Ct_i(\eta + \sqrt{\eta t_i}).$$

When $\eta = 0$, (9) gives $y_i = 0$. In either case, Young's inequality gives $2y_i \le t_i/2 + C\eta$. The smaller horizontal distance $\min(s_i, D - s_i)$ is at least $t_i - y_i$. Applying (10) at $h = s_i, s_j$ for the respective sites gives

$$d_{ij} \le |s_i - s_j| + y_i + y_j \le D - t_i - t_j + 2(y_i + y_j).$$

This proves the second bound in (11). □

Proof of Theorem 4.1. Choose a cut realizing K and reshape a full-rank rectangular matrix across it. This proves the lower bound. Fix a nonzero tensor in the format and a spectral point. Use the space in Lemma 2.1 and set

$$H = \inf_{y \in X} \sum_i a_i d(x_i, y), \qquad Q = \sum_{i<j} a_i a_j d_{ij}^2.$$

Then $\log \phi(T) \le WH$ and $H^2 \le Q$. For the latter, let b be the barycenter minimizing $V(y) = \sum_i a_i d(x_i, y)^2$. The CAT(0) variance inequality $V(x) \ge V(b) + d(x, b)^2$, averaged at the x_i, gives $V(b) \le Q$. Cauchy–Schwarz gives $H^2 \le V(b)$. The midpoint inequality makes minimizing sequences Cauchy. Strong convexity on $[b, x]$ gives the variance bound.

Set $G = 1/4 - Q \ge 0$, using $d_{ij} \le 1$ and (8). Then $H \le \sqrt{Q} \le 1/2 - G$. Also $\sum_{i<j} a_i a_j d_{ij}(1 - d_{ij}) \le G$. Averaging gives a site A with $e = \sum_i a_i p_i (1 - p_i) \le 2G$. Let $S = \{i : p_i > 2/3\}$ have mass β. If $\beta \le 1/2$, then $\beta \le K$ and the star at A costs at most $K + 3e$. If $\beta > 1/2$, set $\mu_i = a_i/\beta$ on S. In (8) put half the mass at A and distribute half

according to μ. This gives $\sum_{i<j\in S}\mu_i\mu_j d_{ij} \le \sum_{i\in S}\mu_i(1-p_i)$. The average cost of a star centered according to μ is consequently at most $1-\beta+2\sum_{i\in S}a_i(1-p_i) \le K+3e$. Hence

$$H \le \min\{1/2-G, K+6G\} \le 1/2-\delta/7.$$

Together with the lower bound, this proves the equality criterion.

For the second estimate take $\delta > 0$ sufficiently small, since $H \le 1/2$ suffices when δ is bounded below. Assume $H > K$ and choose a diametral pair A, B of old sites. By (8), $K^2 < Q \le D/4$, so $\eta = 1-D \le 4\delta$ and $G \le \delta$. For each i, the sites j with $d_{ij} > 1/4$ have mass at least $K-1/4$, because the star centered at i costs more than K. On these pairs $1-d_{ij} \le 4d_{ij}(1-d_{ij})$. Multiply the last inequality of (11) by a_ia_j and sum over these pairs. Since $K-1/4$ is bounded below, we get

$$\sum_i a_i t_i = O(G+\eta) = O(\delta). \tag{12}$$

Let s_* be a weighted median of the s_i and write $B(h) = \sum_i a_i|s_i-h|$. Integrating the smaller mass on either side of each cut of $[0,D]$ gives $B(s_*) \le KD$, since this mass is at most K. If $\eta = 0$, all altitudes vanish and $H \le B(s_*) \le K$. Otherwise put $\lambda = \eta^{1/4}$. Orient the base so that $s_* \le D/2$ and take

$$J = [\max(s_*,\lambda),\ \max(s_*,\lambda)+\lambda] \subseteq [\lambda, D-\lambda].$$

For $h \in J$ let v be the mass strictly to the right of h. Then $v \le K$, and the integral for $B(s_*)$ saves at least $(K-v)(D-h)$ on $[h,D]$ compared with KD. Moving from s_* to h increases the cost by at most $(1-2v)(h-s_*)$. Since $1-2v = 2\delta+2(K-v)$, $h-s_* \le 2\lambda$, and $2(h-s_*) \le D-h$ for small δ, we obtain

$$B(h) \le KD + O(\delta\lambda) \qquad (h \in J). \tag{13}$$

Average (10) over $h \in J$ after taking square roots. For $t_i \le \lambda/2$, the foot is within $\lambda/2$ of an endpoint, so $|h-s_i| \ge \lambda/2$. The excess of $\sqrt{(h-s_i)^2+y_i^2}$ over $|h-s_i|$ is at most $2y_i^2/\lambda = O(\lambda t_i)$. For $t_i > \lambda/2$, (11) gives $y_i \le b_i = C_1\lambda t_i^{3/4}$. The excess is at most $\min(b_i, b_i^2/|h-s_i|)$. Integrating this even decreasing function of $|h-s_i|$ over an interval of length λ gives an average at most

$$C_2\frac{b_i^2}{\lambda}(1+\log_+(\lambda/b_i)) \le C_3\lambda t_i^{3/2}(1+|\log t_i|) \le C_4\lambda t_i,$$

where $\log_+ u = \max(0, \log u)$. Summing with weights a_i and using (12) and (13) gives

$$H \le KD + C_5\delta\lambda = K - K\eta + C_5\delta\eta^{1/4} \le K + C_6\delta^{4/3}.$$

The last inequality follows by maximizing $C_5\delta\eta^{1/4} - K\eta$ over $\eta \ge 0$, using $K \ge 1/4$. Spectral duality proves (7). For $2m+1$ equal weights one has $\delta = 1/(4m+2)$, giving (2). □

5 Matrix stabilization

For T on $k \geq 2$ parties, let $M(T)$ be its largest bipartition flattening rank. Let E_q be a rank-q matrix on two new parties, so the external product $T \otimes E_q$ has $k+2$ parties. Christandl, Jensen and Zuiddam [4] asked whether asymptotic rank is multiplicative under this product.

Theorem 5.1. *For every field and every fixed tensor T,*

$$\lim_{q\to\infty} \frac{\widetilde{\mathrm{R}}(T \otimes E_q)}{q} = M(T). \tag{14}$$

For $q \geq 2$ and $M = M(T) \geq 1$, the logarithm of $\widetilde{\mathrm{R}}(T \otimes E_q)/(qM)$ lies between zero and $C(k \log M)^{4/3}(\log q)^{-1/3}$ for an absolute constant C.

Proof. Placing the two matrix legs on opposite sides of a cut attaining M gives the lower bound qM. For the upper bound take $T \neq 0$ and a spectral point on the $k+2$ parties. In the CAT(0) realization of its wire metric [5, Proposition C.3], let A, B be the matrix sites and $D = d(A,B) = 1-\eta$. For an original site x at distances p, q' from A, B, let z lie at fraction $p/(p+q')$ along $[A,B]$, taking $z = A$ when $p + q' = 0$. Triangle comparison and (9) give

$$d(x,z)^2 \leq \frac{pq'}{(p+q')^2}\bigl((p+q')^2 - D^2\bigr) \leq \tfrac{5}{4}\sqrt{\eta} \qquad (p+q' > 0).$$

Ordering the original sites by their points z gives a path of length at most $D + \sqrt{5}(k-1)\eta^{1/4}$.

In this order, let L_j be the column space of the flattening onto the first j factors. Then $L_j \subseteq L_{j-1} \otimes V_j$, with $L_0 = \mathbb{F}$ and $L_k = \mathbb{F}T$. Expanding a basis at each inclusion gives a tensor train with bond dimensions $\dim L_j$ [11]. Its coefficient arrays restrict the product of the path wires to T. Each bond has dimension at most M, and the matrix wire has value $q^{1-\eta}$. Hence

$$\log \frac{\phi(T \otimes E_q)}{qM} \leq \sqrt{5}(k-1)(\log M)\eta^{1/4} - \eta \log q.$$

Maximizing the right side over $\eta \geq 0$ gives the stated rate. Spectral duality proves (14). For the zero tensor both sides vanish. □

By flattening and submultiplicativity, $qM(T) \leq \widetilde{\mathrm{R}}(T \otimes E_q) \leq q\widetilde{\mathrm{R}}(T)$. Together with (14), this gives $\widetilde{\mathrm{R}}(T \otimes E_q) = q\widetilde{\mathrm{R}}(T)$ for every q exactly when $\widetilde{\mathrm{R}}(T) = M(T)$. For tensors on disjoint parties the cuts can be chosen independently, so $M(S \otimes T) = M(S)M(T)$. Thus asymptotic rank is multiplicative under all external products over $\mathbb{F}$ if and only if $\widetilde{\mathrm{R}}(T) = M(T)$ for every T over $\mathbb{F}$. For MM_d with $d \geq 2$, the largest flattening rank is d^2 and the asymptotic rank is $d^{\omega_{\mathbb{F}}}$. Hence $\widetilde{\mathrm{R}}(\mathrm{MM}_d \otimes E_q) = q\widetilde{\mathrm{R}}(\mathrm{MM}_d)$ for every q if and only if $\omega_{\mathbb{F}} = 2$.

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