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NATURAL BOUNDARIES OF PRIME-PAIR DIRICHLET SERIES

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ABSTRACT. Let $G(s) = \sum_{p,q}(p+q)^{-s}$ and $T(s) = \sum_{p,q}\log p \log q\,(p+q)^{-s}$, summed over ordered pairs of primes, and let $S(s) = \sum_{p,q} p^{1-s}q^{2-s}(p^2+q^2)^{-2}$, $G_2(s) = \sum_{p,q}(p^2+q^2)^{-s}$ and $G_{\log}(s) = \sum_{p,q}\log q\,(p+q)^{-s}$. Assume that for every $\delta > 0$ only finitely many zeros of ζ have real part above $\frac{1}{2}+\delta$; besides classical results about ζ, the primes and the Gamma function, this is the only hypothesis the proofs of the boundaries use. It follows from the Riemann Hypothesis (RH) but allows zeros off the critical line, and it puts no condition on the ordinates. We prove that $\operatorname{Re} s = 1$ is a natural boundary of T, and that $\operatorname{Re} s = 1$ for G and $G_{\log}$, $\operatorname{Re} s = \frac{1}{2}$ for G_2 and $\operatorname{Re} s = 0$ for S are natural boundaries on every branch. Assuming only this hypothesis, several pairs of candidate singularities of the prime zeta function may sum to one point, and their contributions might cancel; we find points where they do not cancel, accumulating at every point of the boundary line. For the four series other than T, which under the hypothesis have no single-valued meromorphic continuation up to these lines, we also describe the branches near each point of the region of continuation where two candidate singularities meet: in a small disc about the point, off a slit from it, every branch reached along a path in the region of continuation that avoids these points (under the hypothesis, every branch based there) is a logarithm times a holomorphic coefficient plus a holomorphic function, and the coefficient is the same on all of them. The boundary theorems are checked in Lean 4 as implications from this hypothesis and named classical inputs. With Alpöge's certified Lean proof of RH, these natural boundaries hold unconditionally, by a written argument that combines Alpöge's theorem with our Lean implications, which are checked in separate Lean builds; Section 11 lists the steps taken outside Lean.

1. INTRODUCTION

1.1. **Series over pairs of primes.** Summing over ordered pairs of primes p, q, let

$$G(s) = \sum_{p,q}\frac{1}{(p+q)^s}, \qquad T(s) = \sum_{p,q}\frac{\log p \log q}{(p+q)^s}, \qquad S(s) = \sum_{p,q}\frac{p^{1-s}q^{2-s}}{(p^2+q^2)^2}. \tag{1}$$

The first two are generating series of the Goldbach problem: they count representations of an integer as a sum of two primes, without and with the logarithmic weights. In a preprint, Li and Liu [18] state that they derive the binary Goldbach conjecture for all sufficiently large even integers from the Generalized Riemann Hypothesis for Dirichlet L-functions together with either of two further spectral hypotheses, their Hypotheses S and R. Each is equivalent to a bound on the norm of an operator built from what is left of a smoothed Goldbach correlation after an exact diagonal term and a model from Euler's Beta function are subtracted [18, (2.28), p. 12; (2.30), p. 13; Theorem 6.1, p. 35]. Their results concern the number of representations of individual even integers; ours concern the continuation of the generating series.

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G and T converge absolutely for $\operatorname{Re} s > 2$. The third series, S, converges absolutely for $\operatorname{Re} s > \frac{1}{2}$. Indeed, $p^2 + q^2 \geq p^{2t} q^{2-2t}$ for $0 \leq t \leq 1$, so with $\sigma = \operatorname{Re} s$ its terms are at most $p^{1-\sigma-4t} q^{-2-\sigma+4t}$ in absolute value, and for $\sigma > \frac{1}{2}$ there is a $t \in [0,1]$ with $2 - \sigma < 4t < 1 + \sigma$, which makes both exponents less than -1.

Each of the three series is built from the prime zeta function $P(z) = \sum_p p^{-z}$ by a Mellin–Barnes integral. For G, write $(p+q)^{-s} = q^{-s}(1 + p/q)^{-s}$ and express the kernel $(1+x)^{-s}$, at $x = p/q$, as an integral of the powers x^{-z} over a vertical line, against a quotient of Gamma functions. The powers of p and of q then separate, and summing over the primes turns $G(s)$ into an integral of $P(z)P(s-z)$ against that quotient (Section 2); for T the logarithmic weights put $Q = -P'$ in place of P. The series S arises in the same way from the kernel $(1+x)^{-2}$, at $x = (p/q)^2$, in place of $(1+x)^{-s}$: its summand is $p^{1-s} q^{-s-2} \big(1 + (p/q)^2\big)^{-2}$, so the exponent of the kernel is fixed and s enters only through the powers of p and q. We include S because it tests the method on a different kind of kernel: its summand is not symmetric in p, q, and its kernel decays at half the rate of the kernel of G. It is holomorphic, where the construction needs it to be, only for s in a strip, which is the source of an additive constant C that branches of S can acquire (Section 7). Since each series is such an integral, in the domains where the contour construction applies its singularities come from those of P, and in pairs. The candidates are $a+b$ for G, T and $(a+b-1)/2$ for S, when a and b are candidate singular points of P in $\operatorname{Re} z > 0$ (the set Σ of Section 2).

Two further series test the two places where the method could depend on the particular series. Let

$$G_2(s) = \sum_{p,q} \frac{1}{(p^2+q^2)^s}, \qquad G_{\log}(s) = \sum_{p,q} \frac{\log q}{(p+q)^s},$$

absolutely convergent for $\operatorname{Re} s > 1$ and $\operatorname{Re} s > 2$. Squaring the primes in G_2 changes the kernel and leaves both factors of the integrand equal to P. The elementary factor is $(1+x)^{-s}$ again, now at $x = (p/q)^2$, and after the change of variable this gives the integral a third kernel (Section 9.1); the boundary is on a different line, $\operatorname{Re} s = \frac{1}{2}$. Each pair (a, b) at a candidate point contributes with a weight that comes from the kernel, so G_2 tests the arithmetic: whether the sum of these contributions can still be shown to be nonzero when the weights change. $G_{\log}$ carries the logarithmic weight on one prime only, so it lies between G and T. The weight changes the kind of singularity of one factor and leaves the kernel alone: one P becomes $Q = -P'$, so where G has a candidate collision of two logarithmic singularities, $G_{\log}$ has a candidate collision of a logarithmic singularity of one factor with a pole of the other. So $G_{\log}$ tests the local calculation, made where the contour of the integral is pinched between the two candidate singular points (Section 6), which for G and S has two logarithms. Their candidates are $(a+b)/2$ and $a+b$.

Landau and Walfisz [16] proved that $\operatorname{Re} z = 0$ is a natural boundary of P. Their theorem says nothing directly about these series. To reach a boundary one must find candidate points, accumulating at every point of the line, at which the contributions of all the pairs (a, b) that give the point do not cancel. We call a set of points that accumulates at every point of the line *dense*, for short; the sets we find lie among the candidate points, the catalogue defined below, which under the hypothesis (H_∞) of the next subsection is discrete in the open half-plane, so they are not dense in any open set. This is the problem we treat. The only additional hypothesis on the zeros in the proofs of the boundaries is the one below, which follows from the Riemann Hypothesis.

1.2. **The hypothesis.** Let $\mathcal{Z}$ be the set of non-trivial zeros of ζ, each taken once. Theorems 1.1–1.3 are stated and proved under the following hypothesis on the zeros:

$$(\mathrm{H}_\infty) \qquad \text{for every } \delta > 0, \text{ only finitely many } \rho \in \mathcal{Z} \text{ satisfy } \operatorname{Re} \rho > \tfrac{1}{2} + \delta.$$

Equivalently, $B_\infty = \frac{1}{2}$, where $B_\infty = \lim_{t\to\infty} \sup_{|\gamma|\ge t} \operatorname{Re}\rho$, the supremum being over the zeros $\rho = \beta + i\gamma$ (Section 2). Write $B = \sup_\rho \operatorname{Re}\rho$, and let RH and LH stand for the Riemann Hypothesis and the Lindelöf Hypothesis. Then

$$\mathrm{RH} \Longrightarrow (\mathrm{H}_\infty) \Longrightarrow \mathrm{LH}, \qquad (\mathrm{H}_\infty) \Longrightarrow B < 1.$$

The first implication is immediate, and (H_∞) follows already from the statement that only finitely many zeros lie off the critical line, so it allows exceptions to RH. The second follows from Backlund's criterion [34, Theorem 13.5, p. 331]. The third, which gives a zero-free half-plane, is shown in Section 2.[1]

RH now has a certified Lean proof, described below, after the next paragraph. So (H_∞) holds, and one may ask why the theorems still assume it. We keep (H_∞) as the hypothesis of Theorems 1.1–1.3 because it is the only additional hypothesis on the zeros that their proofs use. It allows infinitely many zeros off the critical line, so the proofs show which property of the zeros the argument depends on: that only finitely many lie to the right of each line $\operatorname{Re} s = \frac{1}{2} + \delta$, $\delta > 0$. That all of them lie on the critical line is not used. The hypothesis puts no condition on the ordinates and no rate on how the real parts approach $\frac{1}{2}$. In the notation of Bhowmik and Schlage-Puchta [2, Theorem 2(4), p. 44] it says that their abscissa σ_0 equals $\frac{1}{2}$.

The proofs use the zeros at two strengths, which is why $B < 1$ stands beside (H_∞) in the implications above. The arithmetic uses the weaker one: the only additional hypothesis on the zeros in the non-cancellation theorem of Section 3 is $B < 1$. The continuation starts without any hypothesis on the zeros. Its contour estimates use the classical inputs of Section 2 and no hypothesis on the zeros (their constants depend on where the zeros lie), and the construction works to the right of a line fixed by B_∞: in a half-plane for four of the series, and for S in a strip with that line as its left edge, where a path may leave the strip on the right and continuation along it need not exist (Section 7). What (H_∞) does, in the form $B_\infty = \frac{1}{2}$, is put that left edge on the boundary line of the theorem. We do not show that $B < 1$ alone gives continuation up to the boundary lines of Theorems 1.1–1.3. RH gives both $B < 1$ and (H_∞). Each theorem below that needs an additional hypothesis on the zeros names it: (H_∞), $B < 1$ or RH.

The evidence for RH is a formal proof. On 20 November 2026 Alpöge [1] announced a proof of RH and of the Generalized Riemann Hypothesis for Dirichlet L-functions, released as a Lean library of about 17 million lines, without an accompanying prose proof. A joint statement of the Lean and Mathlib communities, endorsed by the Lean FRO and the Mathlib Initiative and dated 21 November 2026 [17], certifies the statement of RH and its proof. It reports that two independent tools, `comparator` and `con-leche --compare`, confirm that the theorem proved is exactly Mathlib's `RiemannHypothesis`, and that the proof was checked by Lean's official kernel and accepted by the checkers `con-leche` and `con-ron` and by several independent checkers. Sutherland [32] has run independent checks. In his update of 22 November 2026, 16:20 UTC, none of the reported checks had rejected the proof, and some were still in progress; Section 11 lists the completed checks and those still running at that time. We read the headline statements of Alpöge's library and scanned its source but did not build it; for the proof we rely on the checks cited. On the basis of the certification we take RH as proved. Then (H_∞) and $B < 1$ hold as well, and these three are the only additional hypotheses on the zeros made in the theorems of this paper. Theorems 1.1–1.3 assume (H_∞), Theorem 10.1 assumes RH, and the non-cancellation theorems used to prove Theorems 1.1–1.3 assume at most $B < 1$, which follows from (H_∞); Corollary 1.4 discharges these hypotheses and states the conclusions of the four theorems unconditionally.

[1] A zero-free half-plane with $B \le 7/8$ is asserted in an unrefereed 2026 preprint of OpenAI [25], and a sharper one in an unrefereed preprint of Liu [19]. Nothing in this paper uses either.

1.3. **The theorems.** T is the simple case: it is built from $Q = -P'$, which is meromorphic, so T is meromorphic too. A boundary for T means what it usually means.

Theorem 1.1. *Assume* (H_∞). *Then T has a single-valued meromorphic continuation to* $\operatorname{Re} s > 1$. *For every real y and every $\varepsilon > 0$ the disc $|s - (1+iy)| < \varepsilon$ contains infinitely many poles of this continuation. Consequently, no function meromorphic on a disc centred on* $\operatorname{Re} s = 1$ *agrees with it on the part of the disc in* $\operatorname{Re} s > 1$.

G and S are built from P itself, which has logarithmic singularities, so they are not single-valued and the statement has to speak of branches. A function F, holomorphic near $\gamma(0)$, *continues analytically* along a path $\gamma\colon [0,1] \to \mathbb{C}$ if there are functions f_t, $0 \le t \le 1$, with f_t holomorphic near $\gamma(t)$, $f_0 = F$ near $\gamma(0)$, and $f_{t'} = f_t$ near $\gamma(t')$ for t' close to t. The germ of f_1 at $\gamma(1)$ is the *terminal germ*; we say that it is *reached* from F along γ. A *branch* of one of the series is a germ reached along some path from the series on its half-plane of convergence.

Section 2 defines a set of candidate points, the *catalogue*, closed and discrete under (H_∞), in the half-plane of each series: $\mathcal{C}_G$ in $\operatorname{Re} s > 1$ and $\mathcal{C}_S$ in $\operatorname{Re} s > 0$. Explicitly, $\mathcal{C}_G = \Pi \cap \{\operatorname{Re} s > 1\}$ and $\mathcal{C}_S = \{s : \operatorname{Re} s > 0,\ 2s+1 \in \Pi\}$, where Π is the set of sums $a+b$ of two points of the set $\Sigma = \{1/k : k \ge 1 \text{ squarefree}\} \cup \{\rho/k : \rho \in \mathcal{Z},\ k \ge 1 \text{ squarefree}\}$ of Section 2, which contains the poles of $Q = -P'$ in $\operatorname{Re} z > 0$ (Lemma 2.1) and hence the singular points of P there. A germ of G at s_1 is *admissible* if it is reached from $\operatorname{Re} s > 2$ by analytic continuation along a path in $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}_G$; a germ of S at s_1 is admissible if it is reached from $\operatorname{Re} s > 1$ along a path in $\{\operatorname{Re} s > 0\} \setminus \mathcal{C}_S$ along which continuation exists. The words "along which continuation exists" matter for S only: for G, under (H_∞), continuation exists along every path in $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}_G$ that starts in $\operatorname{Re} s > 2$ (Theorem 7.4(a)), and the proof of Theorem 7.4(b) explains the difference. The series S converges on the larger half-plane $\operatorname{Re} s > \frac{1}{2}$; the definition starts from $\operatorname{Re} s > 1$, the form in which the statement is formalized. Starting from $\operatorname{Re} s > \frac{1}{2}$ would give the same germs, since S is holomorphic there and $\mathcal{C}_S$ lies in $\operatorname{Re} s \le \frac{1}{2}$, every point of Π having real part at most 2. Such a germ, of G or of S, *extends meromorphically* to an open set $W \ni s_1$ if some function meromorphic on W agrees with it near s_1.

Theorem 1.2. *Assume* (H_∞). *Let g be an admissible germ of G at s_1. For every real y and every $r > 0$ with $|s_1 - (1+iy)| < r$, the germ g does not extend meromorphically to the disc $|s - (1+iy)| < r$. The same holds for an admissible germ of S and the disc $|s - iy| < r$.*

The statement for S is not vacuous: under (H_∞) every point of $\{\operatorname{Re} s > 0\} \setminus \mathcal{C}_S$ carries an admissible germ of S, so every disc $|s - iy| < r$ contains such base points (Section 8, before the proof of Theorem 1.2).

So $\operatorname{Re} s = 1$ is a natural boundary of G and $\operatorname{Re} s = 0$ of S, on every branch. Section 8 removes the restriction on the path, and gives the form for connected domains (Theorem 8.2).

The method is not tied to these kernels. Section 9 applies it to G_2 and $G_{\log}$, each of which changes one layer of the argument and leaves the others as they are. Put $\mathcal{C}_2 = \{s : \operatorname{Re} s > \frac{1}{2},\ 2s \in \Pi\}$, with Π as in Section 2. A germ of G_2 at s_1 is *admissible* if it is reached from $\operatorname{Re} s > 1$ by analytic continuation along a path in $\{\operatorname{Re} s > \frac{1}{2}\} \setminus \mathcal{C}_2$; a germ of $G_{\log}$ at s_1 is admissible if it is reached from $\operatorname{Re} s > 2$ along a path in $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}_G$.

Theorem 1.3. *Assume* (H_∞).

(a) *G_2 continues analytically along every path in $\{\operatorname{Re} s > \frac{1}{2}\} \setminus \mathcal{C}_2$ starting in $\operatorname{Re} s > 1$, and $G_{\log}$ along every path in $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}_G$ starting in $\operatorname{Re} s > 2$. A point carries an admissible germ of G_2 if and only if it lies in $\{\operatorname{Re} s > \frac{1}{2}\} \setminus \mathcal{C}_2$, and one of $G_{\log}$ if and only if it lies in $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}_G$.*

(b) *Let g be an admissible germ of G_2 at s_1. For every real y and every $r > 0$ with $|s_1 - (\frac{1}{2} + iy)| < r$, the germ g does not extend meromorphically to the disc $|s - (\frac{1}{2} + iy)| < r$. The same holds for an admissible germ of $G_{\log}$ and the disc $|s - (1 + iy)| < r$.*

(c) *Let γ be a path starting in $\operatorname{Re} s > 1$ along which G_2 continues analytically, with $\gamma(1) \notin \mathcal{C}_2$. Then the terminal germ is admissible. The same holds for $G_{\log}$, with start in $\operatorname{Re} s > 2$ and $\gamma(1) \notin \mathcal{C}_G$.*

By (a), every disc $|s-(\frac{1}{2}+iy)| < r$ contains base points of admissible germs of G_2, and every disc $|s - (1 + iy)| < r$ base points of admissible germs of $G_{\log}$, so (b) is not vacuous. Part (c) is the analogue of Corollary 8.3; with (b) it gives the boundary for every branch based off the catalogue. A branch based at a point of the catalogue is not covered by (b) and (c) as stated; the remark that closes the proof of (c) in Section 9.3 treats it, as Theorem 8.2(a) does for G and S, and it is checked in Lean (Section 11). So $\operatorname{Re} s = \frac{1}{2}$ is a natural boundary of G_2 and $\operatorname{Re} s = 1$ of $G_{\log}$, on every branch. For G and S the corresponding facts are stated separately. Under (H_∞), continuation of G along every path in $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}_G$ starting in $\operatorname{Re} s > 2$ is Theorem 7.4(a), the existence of admissible germs of S is shown in Section 8 before the proof of Theorem 1.2, and Corollary 8.3 says that a continuation of G or S along any path that starts in $\operatorname{Re} s > 2$ for G, or in $\operatorname{Re} s > 1$ for S, and ends off the catalogue gives an admissible germ. For G_2 and $G_{\log}$ we collect them in the theorem because Section 9 proves them for these two series.

We know of no earlier proof of Theorems 1.1, 1.2 or 1.3 under this hypothesis, nor of a boundary for G, S, G_2 or $G_{\log}$ on every branch. Under RH the arithmetic is much simpler, and Section 10 records what more can then be said: the derivative of the coefficient for G at every candidate point in $\operatorname{Re} s > 1$, and the corresponding residue of T, are nonzero.

RH implies (H_∞), so the certified Lean proof of RH discharges the hypotheses on the zeros in these theorems, as the following corollary records. If that proof were withdrawn, Theorems 1.1, 1.2 and 1.3 would stand as stated under (H_∞), and Theorem 10.1 under RH; only the unconditional statements, Corollary 1.4 and the paragraph after it, rest on that proof.

Corollary 1.4. *The conclusions of Theorems 1.1, 1.2, 1.3 and 10.1 hold unconditionally.*

Proof. RH holds, by Alpöge's proof [1] as certified in [17]; this is the hypothesis of Theorem 10.1. Under RH no zero of ζ has real part above $\frac{1}{2}$, so (H_∞) holds, and this is the hypothesis of Theorems 1.1, 1.2 and 1.3. Taken together, the written proofs of the four theorems also use the classical inputs (Z1)–(Z6) listed in Section 2 and the two facts stated after them there. These are unconditional classical results (none assumes RH or (H_∞); for the distinctness in (Z4) see Section 2), so the corollary adds no hypothesis. □

In the same way RH supplies the hypothesis on the zeros in every statement below that assumes (H_∞), $B < 1$ or RH, so that hypothesis can be dropped from each of them; the other conditions of each statement stay as they are.

What Lean checks is narrower than the corollary. Alpöge's theorem and our release, PrimePairNB (Section 11), are separate Lean developments. The release checks Theorems 1.1, 1.2 and 1.3 as implications from (H_∞) and the applicable inputs among (Z1)–(Z6), which it takes as hypotheses, and it checks Theorem 10.1 from RH. The step from RH to (H_∞) is made in the written proof above, and the inputs (Z1)–(Z6) come from the sources cited in Section 2; both are checked by hand, not in Lean. The two facts stated after (Z1)–(Z6) are not hypotheses of the Lean proofs: one comes from Mathlib, and the other is proved in the release. Section 11 says what each development covers and what joins them. The corollary rests on a formal proof released without an accompanying prose proof; the proofs of Theorems 1.1, 1.2, 1.3 and 10.1 do not depend on it and proceed from their stated hypotheses.

1.4. **The idea.** The proof has two separate jobs. The first is arithmetic: at a dense set of candidate points, the sum of the contributions of all the pairs must be shown to be nonzero. The second is analytic: a nonzero sum at such a point must be turned into an obstruction to meromorphic continuation, and for the four series other than T the obstruction must hold on every branch. For G, S and G_2 the two jobs meet in one quantity, the derivative $X'_x(x)$ of the coefficient of the logarithm (displayed below) at a collision $v = x = a + b$, with the derivative taken in v, which the arithmetic computes and the analysis shows to be the same on every branch. For T the sum is itself the residue at x, and for $G_{\log}$ it is, up to sign, the value $Y_x(x)$ of the coefficient of the logarithm; Table 1 lists the five cases.

The arithmetic. The function $Q = -P'$ is meromorphic on $\operatorname{Re} z > 0$, with at most simple poles, all lying in a set Σ containing 1 and the scaled zeros ρ/k with k squarefree; the scaling comes from the Möbius expansion $Q(z) = \sum_k \mu(k)\psi(kz)$, with $\psi = -\zeta'/\zeta$. A Mellin–Barnes integral writes T as an integral of $Q(w)Q(s-w)$ against a Beta kernel, and moving the contour across the poles shows that T has at most a simple pole at each x with $\operatorname{Re} x > 2B_\infty$, with residue

$$\sum_{a+b=x} R_a R_b \,\mathrm{B}(a,b), \qquad R_a = \operatorname{Res}_{z=a} Q,$$

over ordered pairs; we write $\Theta_G(x)$ for this sum. Under RH there are few cases. Every point of Σ other than 1 then has real part at most $\frac{1}{2}$, so a point $x = 1 + \rho/q$ has only the pairs $(1, \rho/q)$ and $(\rho/q, 1)$, any other pair having $\operatorname{Re} a + \operatorname{Re} b \le 1$, and non-vanishing reduces to the residue of Q itself at ρ/q. Without RH, zeros to the right of the critical line allow other pairs, and three things can go wrong. The residue of Q at ρ/q may be small: if ρ/q has several representations ρ'/k, their contributions carry the signs $\mu(k)$; for prime q and $\operatorname{Re}\rho \ge \frac{1}{2}$ a denominator argument still shows that the sum is nonzero when q does not divide the multiplicity $m(\rho)$ of ρ, which holds once $q > m(\rho)$, and so for the large q used below. But it no longer gives the lower bound $1/q$ that the comparison below needs. Finitely many pairs of low height may cancel the main term. And we have no uniform cutoff for the high pairs. Under (H_∞) each fixed x with $\operatorname{Re} x > 1$ has finitely many pairs, by Lemma 2.3, but the lemma gives no bound uniform as $\operatorname{Re} x$ decreases to 1, on either their number or their height, and their sum too might cancel the main term.

We deal with each in turn in Section 3. Take $x = 1 + \rho/q$ with q prime and ρ a zero on the critical line. Hardy and Littlewood's argument provides at least $A_0 t$ distinct such zeros up to height t, for sufficiently large t; the constant A_0 is that of (Z4) in Section 2. The first problem is settled by counting, which supplies points where the residue is a single term of size at least $1/q$. A second representation ρ'/k of ρ/q forces $\operatorname{Re}\rho' = k/(2q)$, so it forces a zero off the critical line whose real part has denominator q or $2q$; for $\gamma \le X$ there are at most $N(2X)$ such zeros, and each accounts for at most two pairs (ρ, q). So there are $\gg X^2/\log^2 X$ pairs (ρ, q) with $\gamma \le X$ and γ/q in a fixed window, of which only $O(X \log X)$ have a second representation. The second problem is settled the same way: an extra ordered pair (a, b) at x, other than $(1, u)$ and $(u, 1)$, with $|\operatorname{Im} a| \le H = 2\log X$ determines $u = a + b - 1$, hence (ρ, q), and there are only $\log^{O(1)} X$ such pairs (a, b), far fewer than the points available. The third is settled by the decay of the Gamma factors: the pairs above height H contribute at most $e^{-H} = X^{-2}$ in total, against a main term of size at least about $1/X$. Exact coincidences above height H are allowed in any number, since a sum smaller in absolute value than the main term cannot cancel it. The kernel of S decays at half the rate of that of G and is estimated separately; any exponential rate would do, with H a suitable multiple of $\log X$, since the near-pair count stays polylogarithmic, and $H = 2\log X$ suits both kernels.

The continuation. For G and S the same integral has P in place of Q, and P has its logarithmic singularities among the points of Σ. The integrand $P(z)P(v-z)$ has a left family of candidate singular points, the $a \in \Sigma$, and a right family, the points $v - b$, which move with v. When a right

point $v-b$ reaches a left point a, at $v=x=a+b$, the contour is pinched between them. The usual way to continue such an integral is to push the contour along a path in v while continuing the integrand, and that is hard to control near the line, where infinitely many candidate singular points accumulate. We avoid continuing along paths. Fixing a direction, we cut the half-plane along parallel rays from all candidate singular points, which gives a single-valued branch of P and a sheet D^κ of the series, defined by one contour integral with a fixed integrand (Section 5). Near a candidate point x, for a generic direction and off the cut L_x from x (Section 6),

$$D^\kappa(v)=\operatorname{Log}_\kappa(v-x)\,X_x(v)+H(v),\qquad X_x(v)=\sum_{a+b=x}c(a)c(b)\int_a^{v-b}\mathcal{K}(z,v)\,dz,$$

with H holomorphic near x; here $\operatorname{Log}_\kappa$ is the branch of the logarithm whose cut is the ray from 0 parallel to the cuts, $\mathcal{K}$ is the kernel of the integral, $c(a)=-R_a$, and every colliding pair contributes one term, by a local calculation at its pinch.

The sheet depends on the direction. Rotating the direction changes the sheet by an integer combination of the functions $2\pi i X_x$ of the points x swept over, and these functions are holomorphic on the whole kernel domain $\mathcal{V}$, so no local coefficient changes (Section 7).

The branches are then described in two steps, and only the second uses the hypothesis. First, a branch reached along a path that stays in the domain of the sheets and avoids the candidate points is a sheet plus such a combination (Theorem 7.4(a)). The domain of the sheets is the part of the kernel domain $\mathcal{V}$ in $\operatorname{Re}v>2B_\infty$, a half-plane for G. For S it is a strip, and a second statement is needed, because a path in $\operatorname{Re}v>2B_\infty$ that avoids the candidate points may leave the strip on the right and come back. If continuation exists along such a path and the path ends back in the strip, Section 7 follows the branch through the excursion and finds that it returns as a sheet plus such a combination plus an additive constant C (Theorem 7.4(b)). Second, (H_∞) makes $2B_\infty=1$, so that $\operatorname{Re}v>2B_\infty$ is the whole half-plane up to the boundary line, and Section 8 then shows that a path of continuation ending off the catalogue can be replaced by one that avoids the catalogue and reaches the same germ (Corollary 8.3). So under (H_∞), a branch reached along any path, at a base point off the catalogue in the domain of the sheets, is a sheet plus such a combination, up to a constant (Corollary 8.3 with Theorem 7.4(a),(b)). So, for a generic direction, every such branch based near x and off L_x has the same coefficient X_x (Theorem 7.4(c)). Since $X_x'(x)$ is the sum the arithmetic controls, up to the factor $\frac12$ for S, X_x is not identically zero at the points of Section 3. A meromorphic extension of any branch over a disc on the line would have to agree near such a point with a function that has non-trivial monodromy, which a single-valued function cannot do (Section 8).

Two more kernels. The two series of Theorem 1.3 show which layer of this argument depends on what. For G_2, in the variable $v=2s$, the integral has the same form with the kernel $\frac12\Gamma(z/2)\Gamma((v-z)/2)/\Gamma(v/2)$, holomorphic for $\operatorname{Re}v>1$ and $0<\operatorname{Re}z<\operatorname{Re}v$, as the kernel of G is. Sheets, local model, rotation and obstruction carry over unchanged; only the arithmetic sum changes, to $\sum c(a)c(b)\mathrm{B}(a/2,b/2)$, and the counting argument is run again for it. For $G_{\log}$ one factor of the integrand is Q, so at a pinch a logarithm meets a pole. The local calculation becomes a residue, the coefficient of the logarithm becomes $Y_x(s)=-\sum c(a)c(b)K_s(s-b)$, in which each ordered collision contributes a value of the kernel $K_s(w)=\Gamma(w)\Gamma(s-w)/\Gamma(s)$ of G and T at the moving right point $s-b$, weighted by $-c(a)c(b)$; at $s=x$ this is $-\Theta_G(x)$, minus the residue of T at x. So the arithmetic for $G_{\log}$ is that of T and G, unchanged. Table 1 lists, for each series, what meets at a pinch, the coefficient that must be nonzero, and where each step is proved.

1.5. **Credit.** The natural boundary of P is due to Landau and Walfisz [16]; see Titchmarsh [34, §9.5, pp. 215–216], who credits them (p. 215), and Dahlquist [5, §1.2, p. 534]. Kluyver [13]

Series	Variable	At a pinch	Must be nonzero at x	Sections
T	$v = s$	pole meets pole	residue $\Theta_G(x)$	2, 3, 4
$G_{\log}$	$v = s$	logarithm meets pole	$Y_x(x) = -\Theta_G(x)$	3, 5–8, 9.2
G	$v = s$	two logarithms	$X'_x(x) = \Theta_G(x)$	3, 5–8
G_2	$v = 2s$	two logarithms, new kernel	$X'_x(x) = \frac{1}{2}\Theta_2(x)$	3, 5–8, 9.1
S	$v = 2s + 1$	two logarithms, kernel on a strip	$X'_x(x) = \frac{1}{2}\Theta_S(x)$	3, 5–8

TABLE 1. Strategy map. Θ_G and Θ_S are the sums of Theorem 3.1, Θ_2 that of Lemma 9.2 and Theorem 9.4; derivatives are in v. The parts of Section 9 named in the last column replace only the step that changes.

had asserted it under RH, without details (see [15, p. 157]); Landau reconstructed his argument and proved the assertion under RH [15, pp. 157–158]. Dahlquist [5, §1.2, p. 534] likewise credits Kluyver, under RH, with the natural boundary of e^P. The denominator step in the non-vanishing of $R_{\rho/p}$ is Titchmarsh's proof of (9.5.7) [34, §9.5, p. 216]. For Euler products $\prod_p h(p^{-s})$, where h is analytic with $h(0) = 1$ and has no limit point of zeros or singularities in $|z| \le 1$, Dahlquist [5, Theorem I, p. 533] proved that the imaginary axis is a natural boundary except when the product is a finite product of powers $\zeta(\nu s)^{\beta_\nu}$. Estermann [9] had treated the case in which h is a polynomial with integral coefficients [5, §1.2, p. 534]; Titchmarsh cites him for the natural boundary $\operatorname{Re} s = 0$ of $\sum_n d_k(n)^l n^{-s}$, $k \ge 2$, except when $l = 1$ or $l = k = 2$ [34, §9.5, p. 216].

Egami and Matsumoto [6, pp. 11–13] isolate a main pair of zeros and bound the rest by the decay of the Gamma factors under their condition (B). The comparison of a main term with far pairs is in Bhowmik and Schlage-Puchta [2, p. 52]. Both remove near pairs by a hypothesis on additive relations among ordinates. In the form of Bhowmik and Schlage-Puchta, their Conjecture 1 [2, p. 44], it says that $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 \neq 0$ forces $\{\gamma_1, \gamma_2\} = \{\gamma_3, \gamma_4\}$; Fujii [10, p. 50] noted that the conjectured linear independence of the positive ordinates implies that a sum $\gamma + \gamma'$ has essentially one representation. In the present setting the abundance of the points ρ/q lets the near-pair count of Section 3 take the place of that hypothesis; this says nothing about the hypothesis in their setting. That zeros scaled by $1/p$, for p prime in a window, are too many for the zeros of ζ to cancel them all is the idea of Bhowmik and Schlage-Puchta [3, Lemma 1, Theorem 1], for Euler products of shifted zeta factors, where cancellation is an exact coincidence of a zero with a pole. Theorem 3.3 is that count in our setting; it must also supply the lower bound $|c(u)| \ge 1/q$ that the comparison needs.

Separating contours for the multiplicative kernel dz/z go back to Hadamard [11, pp. 56–59] and Borel [4, pp. 239–240]; cut domains, for finitely many singular points, are Hadamard's. For the local jump of a pinched integral see Pham [30, (Disc 1), p. 356]; his mixed case, a polar locus meeting a ramified one, is [30, (Disc 4), p. 357]. For Hadamard products see Pérez-Marco [27, Theorem 1.5, p. 5] and [28]. We adapt these to an additive kernel integral over an infinite contour. That a convolution can develop singularities at sums of singular points, with a monodromy that depends on the path, is standard in resurgence theory, for integrals from 0 over a discrete set of singular points [31, §1.3, p. 9], [26]; we use no theorem from there, since our contour is infinite and the singular points accumulate at the line.

The theorem of Tanigawa and Zhai [33] does not apply: their condition C1 [33, p. 468] asks for auxiliary series meromorphic on all of $\mathbb{C}$, which P and Q are not. Nor does Peter's theorem on Dirichlet series attached to a polynomial evaluated at sequences, with weights taken from auxiliary Dirichlet series [29, Theorem 1.3, pp. 246–247]. It asks each auxiliary series to be meromorphic on $\mathbb{C}$ with finitely many poles in each vertical strip of finite width; here the auxiliary series are P and Q, for which $\operatorname{Re} z = 0$ is a natural boundary [34, §9.5, pp. 215–216]. Matsumoto's

survey [21] describes recent results on multiple Dirichlet series with arithmetic coefficients in the numerators, among them those for series weighted by the von Mangoldt function Λ. For a function-field analogue of the Λ-weighted Goldbach series, Egami and Matsumoto [7] prove unconditionally that it continues meromorphically to $\mathbb{C}$ when the two function fields have the same characteristic, and has the natural boundary $\operatorname{Re} s = 2$ when they do not. The results for series weighted by Λ [6, 2] do not transfer when the sums over prime powers are replaced by sums over primes, since the poles of $-\zeta'/\zeta$ in $\operatorname{Re} z > 0$ are confined to 1 and the unscaled zeros. The Möbius expansion of Q introduces additional scaled-zero candidates; our non-cancellation argument supplies actual poles among them. The contour shift of those results does carry over to T; Section 4 opens with the credit.

For G_2 and $G_{\log}$ the Mellin–Barnes integrals come from the same formula [6, (1.4), p. 2], with p^2, q^2 in place of p, q, and with the weight $\log q$ on one prime. For the weighted analogue $\sum \Lambda(m)\Lambda(n)(m^2+n^2)^{-s}$ of G_2, Tanigawa and Zhai use the same formula [33, (3.1)–(3.2), p. 480], with $-\zeta'/\zeta(2z)$ in each factor [33, Remark 1.2, p. 472], and state that under RH it continues meromorphically only to $\operatorname{Re} s > \frac{1}{2}$ [33, Remark 1.2 and Corollary 1.4, pp. 472–473]; Egami and Matsumoto write that the proof of the natural-boundary claim of [33, Theorem 1.3] "seems incomplete" [6, p. 5]. El-Gammal [8] studies G_2 and conjectures that it continues meromorphically to at least $\operatorname{Re} s > 0$ [8, Conjecture 7.8, p. 13]. It fails without any hypothesis at $s = 1$. By the prime number theorem $\#\{(p,q) : p^2 + q^2 \le X\} \asymp X/\log^2 X$, so by partial summation the series for G_2 converges at $s = 1$ and diverges at every real $s < 1$ (the convergence at $s = 1$ is [8, Proposition 2.1, p. 4]). Its coefficients are nonnegative, so by Landau's theorem [23, Theorem 1.7, p. 16] the point $s = 1$ is a singularity of G_2; and $G_2(\sigma)$ tends to the finite limit $G_2(1)$ as $\sigma \to 1^+$, so the singularity is not a pole, and no function meromorphic on $\operatorname{Re} s > 0$ agrees with G_2 on $\operatorname{Re} s > 1$. This one singularity rules out a single meromorphic function on the half-plane of the conjecture. It does not stop continuation along paths. The point $s = 1$ belongs to the catalogue $\mathcal{C}_2$, since $2 = 1 + 1 \in \Pi$, and under (H_∞) Theorem 1.3(a) continues G_2 along every path in $\{\operatorname{Re} s > \frac{1}{2}\} \setminus \mathcal{C}_2$ that starts in $\operatorname{Re} s > 1$. Such a path can cross the line $\operatorname{Re} s = 1$ at any point off the catalogue, and every point of $\operatorname{Re} s > \frac{1}{2}$ off $\mathcal{C}_2$ is reached. Parts (b) and (c) say where the continuation must stop: what Theorem 1.3(b),(c) add under (H_∞) is that every branch of G_2 reached by continuation along a path ending off $\mathcal{C}_2$ has $\operatorname{Re} s = \frac{1}{2}$ as a natural boundary. The argument at $s = 1$ is not part of the Lean release.

1.6. **Organization.** Section 2 collects the classical inputs and the facts on P, Q, the set Σ and the two kernels. Section 3 proves non-cancellation. Section 4 treats T, as a complete first application. Sections 5–7 construct the sheets of G and S, the local model, and the description of the germs reached as in Theorem 7.4; under (H_∞), Section 8 extends the description to arbitrary arrival paths with endpoints in the sheet domain off the catalogue, and proves the boundary. Section 9 treats G_2 and $G_{\log}$ and proves Theorem 1.3, changing one layer of the argument for each; the one new local calculation it needs is Lemma 6.2, at the end of Section 6. Section 10 gives the stronger statements under RH, and Section 11 describes the formal verification, whose main statements are printed in the appendix.

2. The prime zeta function and two kernels

This section fixes the objects: the classical inputs, the function Q and the set Σ in which its poles lie, the two kernels that turn a sum over pairs of primes into a contour integral, and the coefficient X_x that the rest of the paper is about. Without RH the set Σ below is only a set of *candidates*, because a point of Σ with several representations may have coefficient zero.

2.1. **Notation and classical inputs.** Table 2 collects the notation of this section and the principal symbols of later sections.

Symbol	Meaning	Where
$\mathcal{Z}$, $m(\rho)$	non-trivial zeros of ζ, each once; multiplicity	Section 2
B, B_∞	$\sup_\rho \operatorname{Re}\rho$; $\lim_{t\to\infty}\sup_{\lvert\gamma\rvert\ge t}\operatorname{Re}\rho$	Section 2
Σ	the points $1/k$ and ρ/k, k squarefree	Section 2
$c(a)$, R_a	coefficient (2); $R_a=-c(a)=\operatorname{Res}_{z=a}Q$	(2), Lemma 2.1
Π, $\mathcal{X}$	$\Sigma+\Sigma$; $\Pi\cap\{\operatorname{Re}v>2B_\infty\}$	Section 2
F_δ	$\{a\in\Sigma:\operatorname{Re}a>B_\infty+\delta\}$, a finite set	Lemma 2.1(ii)
$\mathcal{K}$, $\mathcal{V}$, $\mathcal{V}_\infty$, D	kernel; its domain in v; $\mathcal{V}\cap\{\operatorname{Re}v>2B_\infty\}$; the series in v	Section 2
K	$\Gamma(w)\Gamma(2-w)$, from which the kernel of S is built	Section 2
K_s	$\Gamma(w)\Gamma(s-w)/\Gamma(s)$, the kernel of T and $G_{\log}$; $\mathcal{K}(w,s)$ for G	(4)
$\mathcal{C}_G$, $\mathcal{C}_S$, $\mathcal{C}_2$	catalogues: the candidate points in the variable s	Sections 1, 2
J_{ab}, X_x, M_x	pair integral; coefficient of the logarithm; $2\pi i X_x$	(5), (6)
Θ_G, Θ_S	the sums over pairs for G (and T) and for S	Theorem 3.1
D^κ, $\mathcal{M}$	sheet for the slope κ; integer combinations of the M_x	(11), Section 7
e, $\operatorname{Log}_\kappa$, P^κ	unit vector of slope κ; logarithm cut along $-e[0,\infty)$; branch of P	Section 5
ℓ_a, U_κ; L_x, Ω_κ	cuts and cut domain of P^κ; of the sheet	Section 5
Θ_2	the sum over pairs for G_2	Lemma 9.2
Y_x	coefficient of the logarithm for $G_{\log}$	Theorem 9.7
$D^\kappa_{\log}$, $\mathcal{M}_{\log}$	sheet of $G_{\log}$; integer combinations of the $2\pi i Y_x$	Section 9.2

TABLE 2. Notation.

Throughout, p,q are primes and $\rho=\beta+i\gamma$ runs over $\mathcal{Z}$. Let $m(\rho)$ be the multiplicity of ρ, and $\mathrm{B}(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$. Write $N(t)=\sum_{0<\gamma\le t}m(\rho)$, and let $N_0(t)$ be the number of distinct zeros $\frac12+i\gamma$ with $0<\gamma\le t$. We use the six classical facts (Z1)–(Z6), and two more stated after them.

(Z1) For every $\rho\in\mathcal{Z}$, also $\bar\rho, 1-\bar\rho\in\mathcal{Z}$, with multiplicity $m(\rho)$. This follows from conjugation and the functional equation [34, pp. 13, 30], whose factors have no zeros and no poles in the open strip.

(Z2) There is C with $\lvert N(t)-\{\frac{t}{2\pi}\log\frac{t}{2\pi}-\frac{t}{2\pi}\}\rvert\le C\log t$ for large t, and C_1 with $N(t+1)-N(t)\le C_1\log(t+3)$ for $t\ge0$ [34, Theorems 9.4, 9.2, pp. 214, 211].

(Z3) There is C such that for $-1\le\operatorname{Re}z\le2$, $\lvert\operatorname{Im}z\rvert\ge2$ and $\zeta(z)\ne0$,

$$\Big\lvert\frac{\zeta'}{\zeta}(z)-\sum_{\lvert\gamma-\operatorname{Im}z\rvert\le1}\frac{m(\rho)}{z-\rho}\Big\rvert\le C\log\lvert\operatorname{Im}z\rvert,$$

the sum over distinct zeros [34, Theorem 9.6(A), p. 217]. Titchmarsh states this for large positive height; bounded heights are handled by subtracting the local poles, and negative heights by conjugation.

(Z4) There are $A_0>0$, t_0 with $N_0(t)\ge A_0t$ for $t\ge t_0$ (Hardy and Littlewood [12]; [34, §§10.7–10.8, pp. 264–268]). The printed wording of these theorems does not state that the zeros are distinct. The proofs give it: they find the zeros in separate intervals, and a zero on a shared endpoint costs only a factor 2 in A_0. A count with multiplicity would not serve.

(Z5) For fixed $\theta > 0$ and $x \geq x_0(\theta)$, the open interval $(x, (1+\theta)x)$ contains at least $\theta x/(2\log x)$ primes, by the prime number theorem [34, p. 46]: their number is at least $\pi((1+\theta)x) - \pi(x) - 1 = \theta x/\log x + o(x/\log x)$.

(Z6) For $0 < x_1 \leq x_2$ there is C with $|\Gamma(x+iy)| \leq C(1+|y|)^{x_2} e^{-\pi|y|/2}$ for $x \in [x_1, x_2]$ and real y, by Stirling's formula [24, 5.11.9].

The two others are that ζ has no zeros in $\operatorname{Re} s \geq 1$, which for $\operatorname{Re} s > 1$ follows from the Euler product and for $\operatorname{Re} s = 1$ is [34, §3.2, p. 46], and that ζ has no real zero in $(0,1)$ [34, (2.12.4), p. 30]. In the Lean proofs neither is a hypothesis: the first comes from Mathlib, and the second is proved in the release from the Abel-summation formula of PrimeNumberTheoremAnd [14]. None of these facts assumes RH or (H_∞). Put

$$B = \sup_{\rho} \operatorname{Re}\rho, \qquad B_\infty = \lim_{t\to\infty} \sup_{|\gamma|\geq t} \operatorname{Re}\rho.$$

Thus (H_∞) says $B_\infty = \frac{1}{2}$. Since ρ and $1-\bar\rho$ are both zeros and there are infinitely many zeros, $\frac{1}{2} \leq B_\infty \leq B \leq 1$. Under ($\mathrm{H}_\infty$) finitely many zeros have $\operatorname{Re}\rho > \frac{3}{4}$, and each has $\operatorname{Re}\rho < 1$, so $B < 1$.

2.2. **The function Q and the candidates Σ.** Let $P(z) = \sum_p p^{-z}$ and $Q(z) = -P'(z) = \sum_p \log p\, p^{-z}$ for $\operatorname{Re} z > 1$, and $\psi = -\zeta'/\zeta$. Let

$$\Sigma = \{1/k : k \geq 1 \text{ squarefree}\} \cup \{\rho/k : \rho \in \mathcal{Z},\ k \geq 1 \text{ squarefree}\}.$$

A point of Σ may have several representations ρ/k with $\rho \in \mathcal{Z}$, k squarefree. Let $\operatorname{ord}_w \zeta$ be $m(w)$ at a zero, -1 at $w = 1$, and 0 at any other point of $\operatorname{Re} w > 0$, and for $\operatorname{Re} a > 0$ put

$$c(a) = \sum_{k\geq 1} \frac{\mu(k)}{k} \operatorname{ord}_{ka}\zeta, \qquad R_a = -c(a). \tag{2}$$

The sum is finite, since ka can be a zero or 1 only if $k \operatorname{Re} a \leq 1$. The next lemma continues Q and identifies its residues with these coefficients; $c(a)$ is then the quantity the arithmetic of Section 3 has to control, and the formula (2) shows why it can vanish: several representations of one point contribute with the signs $\mu(k)$. Since $P' = -Q$ and Q has residue $R_a = -c(a)$ at a pole a (Lemma 2.1), a branch of P near a is $c(a)\log(z-a)$ plus a holomorphic function (Lemma 5.1(a)); this is why $c(a)$, not R_a, appears in the coefficients of the series built from P.

Lemma 2.1. (i) *Q continues to a meromorphic function on $\operatorname{Re} z > 0$, with $Q(\bar z) = \overline{Q(z)}$. Its poles are simple and lie in Σ. For $a \in \Sigma$, $\operatorname{Res}_{z=a} Q = R_a$, so a is a pole if and only if $c(a) \neq 0$. Also $c(1) = -1$, and $c(a)$ is real with $c(\bar a) = c(a)$.*

(ii) *Σ is locally finite in $\operatorname{Re} z > 0$. Every point of Σ other than 1 has real part at most B, and every point of Σ that is neither 1 nor a zero has real part at most $\frac{1}{2}$. For $\delta > 0$ the set $F_\delta = \{a \in \Sigma : \operatorname{Re} a > B_\infty + \delta\}$ is finite and lies in $\{1\} \cup \mathcal{Z}$.*

(iii) *For $\varepsilon \in (0,1)$ there is $C(\varepsilon)$ with $|c(a)| \leq C(\varepsilon)\log(|\operatorname{Im} a| + 3)$ for $a \in \Sigma$ with $\operatorname{Re} a \geq \varepsilon$, and at most $C(\varepsilon)\log(|t|+3)$ such points have $t \leq \operatorname{Im} a \leq t+1$.*

Proof. (i) For $\operatorname{Re} z > 1$, $\psi(z) = \sum_{j\geq 1} Q(jz)$, and Möbius inversion gives $Q(z) = \sum_{k\geq 1} \mu(k)\psi(kz)$, the rearrangement being justified by $\sum_m d(m) Q(m \operatorname{Re} z) < \infty$. Fix $x_0 > 0$. For $k \geq 2/x_0$ and $\operatorname{Re} z \geq x_0$, $|\psi(kz)| \leq \psi(kx_0) \leq \psi(2) 2^{2-kx_0}$, as $\psi(x)2^x$ is non-increasing for $x > 1$. So the series converges normally on $\operatorname{Re} z \geq x_0$ off the poles of finitely many terms, and is meromorphic on $\operatorname{Re} z > 0$. The function $\psi(kz)$ has at $z = a$, with ka a zero or 1, a simple pole with residue $-\operatorname{ord}_{ka}\zeta/k$. Summing over k gives $\operatorname{Res}_a Q = -c(a)$. No real point is a zero, so $c(1) = -1$. Reality and the symmetry come from (Z1).

(ii) On $\operatorname{Re} z \ge x_0$ only finitely many k occur, and for each k finitely many zeros in a bounded set. For $k \ge 2$, $1/k \le \frac{1}{2}$ and $\operatorname{Re}(\rho/k) < \frac{1}{2}$. The rest follows from the definition of B_∞, as only finitely many zeros have $\operatorname{Re}\rho > B_\infty + \delta$.

(iii) Only $k \le 1/\varepsilon$ contribute. By (Z2) and (Z1), $m(\rho) \le 2C_1 \log(|\gamma|+3)$, and a unit window holds $O(\log(|t|+3))$ zeros at height about kt. □

The contour arguments below integrate along lines that run to infinite height, and on these vertical tails they use a bound on the growth of Q. For each fixed $\delta > 0$, the next lemma gives a logarithmic bound on $\operatorname{Re} w \ge B_\infty + 2\delta$ at sufficiently large $|\operatorname{Im} w|$, beyond the heights of the finitely many zeros with $\operatorname{Re}\rho > B_\infty + \delta$; Sections 4–9 use it on the tails of their contours, directly or through Lemma 5.1(b).

Lemma 2.2. *Let $\delta > 0$ and $t_\delta \ge 2$ be such that every zero with $|\gamma| \ge t_\delta - 1$ has $\operatorname{Re}\rho \le B_\infty + \delta$. Then $|Q(w)| \le C_\delta \log|\operatorname{Im} w|$ for $\operatorname{Re} w \ge B_\infty + 2\delta$, $|\operatorname{Im} w| \ge t_\delta$.*

Proof. Write $Q(w) = \psi(w) + \sum_{k\ge2} \mu(k)\psi(kw)$. Since $\operatorname{Re}(kw) \ge k(\frac{1}{2} + 2\delta)$, the sum over $k \ge 2$ is bounded in terms of δ. If $\operatorname{Re} w \ge 2$, $|\psi(w)| \le \psi(2)$. Otherwise (Z3) applies: each zero in its sum has $|\gamma| \ge t_\delta - 1$, so $|w-\rho| \ge \delta$, and the multiplicities in a unit window sum to $O(\log|\operatorname{Im} w|)$ by (Z2). □

2.3. **The kernels.** The series share one form. Let v be the variable in which the singularities pinch: for G and T, $v = s$, and for S, $v = 2s+1$. Put

$$\begin{aligned} G:&\quad \mathcal{V} = \{\operatorname{Re} v > 1\}, && \mathcal{K}(z,v) = \Gamma(z)\Gamma(v-z)/\Gamma(v),\\ S:&\quad \mathcal{V} = \{1 < \operatorname{Re} v < 3\}, && \mathcal{K}(z,v) = \tfrac{1}{2}K\big(\tfrac{2z-v+3}{4}\big), \quad K(w) = \Gamma(w)\Gamma(2-w). \end{aligned}$$

Let $D(v) = G(v)$, respectively $D(v) = S((v-1)/2)$. Under (H_∞) the half-plane $\operatorname{Re} v > 2B_\infty$ contains $\mathcal{V}$; in general we work in $\mathcal{V}_\infty = \mathcal{V} \cap \{\operatorname{Re} v > 2B_\infty\}$. Apart from the non-vanishing that Section 3 supplies, three facts about $\mathcal{K}$ are all that Sections 5–8 use, with one exception: the excursion argument for S in Theorem 7.4(b) uses the explicit formula for its kernel. This is what lets Section 9.1 treat a third kernel, that of G_2 with $v = 2s$, by checking the same three facts.

(K1) $\mathcal{K}$ is holomorphic on $\{v \in \mathcal{V},\ 0 < \operatorname{Re} z < \operatorname{Re} v\}$. For S this holds because the function $K(w) = (1-w)\pi/\sin\pi w$ is holomorphic on $0 < \operatorname{Re} w < 2$ (the singularity at $w = 1$ is removable) and $w = (2z - v + 3)/4$ has $0 < \operatorname{Re} w < \frac{3}{2}$: for $0 < \operatorname{Re} z < \operatorname{Re} v$, $\operatorname{Re} w$ lies between $(3 - \operatorname{Re} v)/4$ and $(3 + \operatorname{Re} v)/4$, and both bounds lie in $(0, \frac{3}{2})$ when $1 < \operatorname{Re} v < 3$. This bound is the only use of $\operatorname{Re} v < 3$.

(K2) On compact subsets of $\mathcal{V}$, for $\operatorname{Re} z$ in a compact subset of $(0, \operatorname{Re} v)$, $\mathcal{K}(z,v) \ll (1 + |\operatorname{Im} z|)^A e^{-\pi|\operatorname{Im} z|/2}$. For G this is (Z6), with $e^{-\pi|\operatorname{Im} z|}$; for S it is elementary from the formula for K.

(K3) For $\operatorname{Re} v > 2$ (and $\operatorname{Re} v < 3$ for S) and $1 < c < \operatorname{Re} v - 1$,

$$D(v) = \frac{1}{2\pi i}\int_{(c)} \mathcal{K}(z,v)P(z)P(v-z)\,dz. \tag{3}$$

For G, Mellin–Barnes for $(p+q)^{-s}$ gives the integrand $\mathcal{K}(z,s)p^{-z}q^{-(s-z)}$; see [6, (1.4), p. 2]. For S, write its summand as $p^{1-s}q^{-s-2}(1 + (p/q)^2)^{-2}$ and use the case of exponent 2 of [6, (1.4), p. 2],

$$(1+x)^{-2} = \frac{1}{2\pi i}\int_{(d)} K(w)x^{-w}\,dw, \qquad 0 < d < 2.$$

With $x = (p/q)^2$, make the change of variable

$$z = 2w + s - 1, \qquad dw = \tfrac{1}{2}\,dz.$$

Then $p^{1-s-2w}q^{-s-2+2w} = p^{-z}q^{-(2s+1-z)}$, and $w = (2z - v + 3)/4$ with $v = 2s + 1$. This gives exactly the factor $\frac{1}{2}$ in $\mathcal{K}$ and the integral in (K3). On the chosen line $\operatorname{Re} z = c$, both $c > 1$ and $\operatorname{Re} v - c > 1$. The sum of the absolute values of the prime factors is

$$\sum_{p,q} p^{-c}q^{-(\operatorname{Re} v - c)} = P(c)P(\operatorname{Re} v - c) < \infty.$$

Multiplying this constant by the integrable kernel bound (K2) gives an integrable majorant for the summed integrand. Fubini therefore proves (K3) for both series. Likewise, for $\sigma > 2$ and $1 < c < \sigma - 1$, the majorant uses $Q(c)Q(\sigma - c)$ and gives

$$T(s) = \frac{1}{2\pi i}\int_{(c)} K_s(w)Q(w)Q(s - w)\,dw, \qquad K_s(w) = \frac{\Gamma(w)\Gamma(s - w)}{\Gamma(s)}. \tag{4}$$

This $K_s(w)$ is the kernel $\mathcal{K}(w, s)$ of G, and by (Z6) $K_s(w) \ll (1 + |\operatorname{Im} w|)^A e^{-\pi|\operatorname{Im} w|}$ uniformly for s in a compact subset of $\operatorname{Re} s > 0$ and $\operatorname{Re} w$ in a compact subset of $(0, \operatorname{Re} s)$.

2.4. **Pairs and the coefficient.** The candidate singularities of the two factors of the integrand in (3), in their right half-planes, are the points a of Σ in the first factor, the *left points*, and the points $v - b$, $b \in \Sigma$, in the second, the *right points*; a left point and a right point meet at a v that is a sum $a + b$, and within the continuations of Theorem 7.4, singularities in $\mathcal{V}_\infty$ can occur only at such sums. Let $\Pi = \Sigma + \Sigma$, and let $\mathcal{X} = \Pi \cap \{\operatorname{Re} v > 2B_\infty\}$. In the variable s these candidate points form the *catalogues* of the introduction: $\mathcal{C}_G = \Pi \cap \{\operatorname{Re} s > 1\}$ for G, and $\mathcal{C}_S = \{s : \operatorname{Re} s > 0,\ 2s + 1 \in \Pi\}$ for S. Under (H_∞), where $2B_\infty = 1$, the catalogue $\mathcal{C}_G$ is $\mathcal{X}$ itself and $\mathcal{C}_S$ is the image of $\mathcal{X}$ under $v \mapsto (v - 1)/2$, so the next lemma makes them closed and discrete in $\operatorname{Re} s > 1$ and in $\operatorname{Re} s > 0$. The *pairs at* x are the ordered pairs $(a, b) \in \Sigma^2$ with $a+b = x$, a diagonal pair occurring once. The next lemma is what keeps the local analysis finite: in the half-plane that matters, each candidate point has finitely many pairs, and the candidates do not accumulate.

Lemma 2.3. *$\mathcal{X}$ is locally finite in $\operatorname{Re} v > 2B_\infty$, and each $x \in \mathcal{X}$ has finitely many pairs.*

Proof. If $\operatorname{Re} x \geq 2B_\infty + 2\eta$ and $x = a + b$ then one of them, say a, has $\operatorname{Re} a \geq B_\infty + \eta$, so lies in the finite set $F_{\eta/2}$, while $\operatorname{Re} b \geq \operatorname{Re} x - 1 \geq 2\eta$. A compact subset of $\operatorname{Re} v > 2B_\infty$ lies in such a half-plane for some $\eta > 0$. For x in the compact set, a ranges over the finite set $F_{\eta/2}$, and $b = x - a$ has bounded height and real part at least 2η; as Σ is locally finite in $\operatorname{Re} z > 0$ (Lemma 2.1(ii)), only finitely many such b lie in Σ. So only finitely many pairs sum to points of the compact set, which therefore contains only finitely many points of $\mathcal{X}$. □

The coefficient of the logarithm at x is built from one integral per pair, taken along the segment from the left point a to the right point $v - b$ that collide at $v = x$. Why these integrals appear is shown by the local calculation of Section 6; here we record only that they are holomorphic, and compute their derivative at the collision. For $a, b \in \Sigma$ and $v \in \mathcal{V}$ let

$$J_{ab}(v) = \int_a^{v-b} \mathcal{K}(z, v)\,dz \tag{5}$$

along the straight segment. On it the real part lies between $\operatorname{Re} a \leq 1 < \operatorname{Re} v$ and $\operatorname{Re} v - \operatorname{Re} b \in [\operatorname{Re} v - 1, \operatorname{Re} v)$, so inside $(0, \operatorname{Re} v)$. Hence J_{ab} is single-valued and holomorphic on all of $\mathcal{V}$, with no hypothesis on the zeros. For $x \in \mathcal{X}$ put

$$X_x(v) = \sum_{a+b=x} c(a)c(b)J_{ab}(v), \qquad M_x = 2\pi i X_x. \tag{6}$$

Then X_x is holomorphic on $\mathcal{V}$, $X_x(x) = 0$, and since $\partial_v J_{ab}(a+b) = \mathcal{K}(a, a+b)$,

$$X'_x(x) = \sum_{a+b=x} c(a)c(b)\mathrm{B}(a,b) \quad (G), \qquad X'_x(x) = \tfrac{1}{2} \sum_{a+b=x} c(a)c(b)K\big(\tfrac{a-b+3}{4}\big) \quad (S), \tag{7}$$

the second as a derivative in v; in the variable $s = (v-1)/2$ the factor $\frac{1}{2}$ disappears. Because $X_x(x) = 0$ for every x, the value of the coefficient at x says nothing; the first quantity that can show $X_x \not\equiv 0$ is the derivative $X'_x(x)$, and in it no integral is left, only the kernel at the collision. These two sums are the quantities Section 3 controls. At $x = 1+u$ with $u \neq 1$ the pairs $(1,u)$ and $(u,1)$ contribute $-2c(u)/u$ to the first and $-\frac{1}{2}c(u)\{K(1-\frac{u}{4}) + K(\frac{1}{2}+\frac{u}{4})\}$ to the second. These are the main terms of the non-cancellation argument, and for S the argument needs the main term to be comparable to $c(u)$ on each fixed compact window used in the proof, which holds by continuity and Lemma 2.4 below.

Lemma 2.4. *For non-real u, $K(1-\frac{u}{4}) + K(\frac{1}{2}+\frac{u}{4}) \neq 0$.*

Proof. From $K(w) = (1-w)\pi/\sin\pi w$ we get $K(1-x) = \pi x/\sin\pi x$ and $K(\frac{1}{2}+x) = (\frac{1}{2}-x)\pi/\cos\pi x$. With $x = \frac{1}{4}+t$ and $t = (u-1)/4$,

$$K(1-x) + K(\tfrac{1}{2}+x) = \frac{\pi\,(\frac{1}{2}\cos\pi t - 2t\sin\pi t)/\sqrt{2}}{\sin\pi x\cos\pi x}.$$

The numerator vanishes only if $\cot\pi t = 4t$. For non-real t this is impossible, since $\operatorname{Im}\cot\pi t$ has the sign of $-\operatorname{Im} t$. □

3. Non-cancellation near the line

Density of candidate points is not enough. A point $x = 1+u$ with $u \in \Sigma$ always has the pairs $(1,u)$ and $(u,1)$, but $c(u)$ may vanish, and without RH other pairs may be present and may cancel the main term. This section proves one statement.

Theorem 3.1. *Assume $B < 1$. Let $y \in \mathbb{R}$ and $\varepsilon > 0$. The disc $|x - (1+iy)| < \varepsilon$ contains infinitely many points $x = 1+\rho/q$, with $\rho \in \mathcal{Z}$ on the critical line and q prime, at which*

$$\Theta_G(x) = \sum_{a+b=x} c(a)c(b)\mathrm{B}(a,b) \neq 0, \qquad \Theta_S(x) = \sum_{a+b=x} c(a)c(b)K\big(\tfrac{a-b+3}{4}\big) \neq 0.$$

The theorem is used for four of the five series. When $x \in \mathcal{X}$, (7) gives $\Theta_G(x) = X'_x(x)$ for G and $\Theta_S(x) = 2X'_x(x)$ for S, the derivative taken in v; Section 8 uses the theorem in this form. The same sum $\Theta_G(x)$ is the residue of T at x (Theorem 4.2), and it is $-Y_x(x)$, where Y_x is the coefficient of the logarithm for $G_{\log}$ (Lemma 9.9). For G_2 the sum has another kernel, and Section 9.1 runs the proof again (Theorem 9.4).

Under (H_∞) each x with $\operatorname{Re} x > 1$ has finitely many pairs by Lemma 2.3, but the lemma gives no bound uniform as $\operatorname{Re} x$ decreases to 1: its proof bounds the pairs on a compact set through the finite set $F_{\eta/2}$, with η fixed by the distance of the set from the line, so it gives no bound on the number of pairs, or on their heights, that holds for all x near the line. The points we need do approach the line, so the lemma supplies no cutoff for them, and the argument does not use it. All it takes from (H_∞) is $B < 1$. Under $B < 1$ alone a point x might have infinitely many pairs, so part of the statement is that both sums converge absolutely; the estimate for the high pairs (Lemma 3.4) gives that as well, and it is uniform for $x = 1+u$ with $0 < \operatorname{Re} u \le 1$ and $|\operatorname{Im} u| \le Y$, for each fixed Y. The proof has three steps: choose points whose coefficient $c(\rho/q)$ is a single term, discard the few points with a low extra pair, and show the high pairs are too small to matter.

3.1. **Single representations.** A *representation* of u is a pair (ρ', k) with $\rho' \in \mathcal{Z}$, $k \geq 1$ square-free and $\rho'/k = u$. By (2), for non-real u,

$$c(u) = \sum_{(\rho',k) \text{ representing } u} \frac{\mu(k)m(\rho')}{k}.$$

If (ρ, q) with q prime is the only representation of $u = \rho/q$, then $c(u) = -m(\rho)/q \neq 0$. So we want points ρ/q with a single representation, and we want many of them. When ρ is on the critical line, a second representation is tightly constrained: the next lemma shows that it forces a zero off the critical line whose real part has a prescribed denominator, and that this zero almost determines the pair (ρ, q). Such zeros are counted by N, and that count is what makes the pairs with a second representation rare.

Lemma 3.2. *Let $\rho = \frac{1}{2} + i\gamma \in \mathcal{Z}$ with $\gamma > 0$, q prime and $u = \rho/q$. Let $(\rho', k) \neq (\rho, q)$ represent u. Then $\gcd(k, q) = 1$, $k < 2q$, $k \neq q$ and $\operatorname{Re}\rho' = k/(2q) \neq \frac{1}{2}$. Put $\zeta^* = \rho'$ if $k > q$ and $\zeta^* = 1 - \bar{\rho}'$ if $k < q$. Then ζ^* is a zero with $\operatorname{Re}\zeta^* > \frac{1}{2}$ and $0 < \operatorname{Im}\zeta^* < 2\gamma$, and the reduced denominator d of $\operatorname{Re}\zeta^*$ lies in $\{q, 2q\}$. Conversely ζ^* determines q, and then (ρ, q) is one of at most two pairs.*

Proof. From $\rho' = k\rho/q$ we get $\operatorname{Re}\rho' = k/(2q)$ and $\operatorname{Im}\rho' = k\gamma/q$. Since $\operatorname{Re}\rho' < 1$, $k < 2q$. If $q \mid k$ then $k = q$ and $\rho' = \rho$, which is excluded; so $\gcd(k, q) = 1$. The statements on ζ^* follow from (Z1), with $\operatorname{Re}\zeta^* = j/(2q)$, $j \in \{k, 2q - k\}$, $q < j < 2q$, $\gcd(j, q) = 1$. If q is odd then $d = 2q$ for odd j and $d = q$ for even j; if $q = 2$ then $j = 3$ and $d = 4 = 2q$. So $q = d$ when d is odd and $q = d/2$ when d is even. Given ζ^* and q, we have $\rho' \in \{\zeta^*, 1 - \bar{\zeta}^*\}$, and each choice fixes $k = 2q \operatorname{Re}\rho'$ and $\rho = q\rho'/k$. □

Now count. Fix $y > 0$ and $0 < \varepsilon < y$, with $J = (y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2})$. For $X \geq 3$ put $M = \lceil 2\log X/A_0 \rceil$ with A_0 from (Z4), and let $\mathcal{P}(X)$ be the set of pairs (ρ, q) with $\rho = \frac{1}{2} + i\gamma \in \mathcal{Z}$, $X/M < \gamma \leq X$, q prime, and $\gamma/q \in J$. Call a pair *good* if (ρ, q) is the only representation of ρ/q. The points we use are $u = \rho/q$ for these pairs. For each zero the admissible primes q fill an interval of relative length about ε/y, so (Z5) supplies $\gg \gamma/\log\gamma$ of them once γ is large, with constants depending on y and ε; the zeros themselves are supplied by Hardy and Littlewood. We use only the worst case, γ about X/M, which gives $\gg X/\log^2 X$ primes per zero. The lower cutoff X/M on γ keeps every q large, so that $\operatorname{Re} u = 1/(2q)$ is small and u lies in the disc; a fixed M would do that much. M is of size $\log X$ for another reason: the only upper bound we have on the zeros below X/M is $N(X/M)$, of size $(X/M)\log X$, and it has to stay below $A_0 X$. That choice is what puts $\log^2 X$ in the count.

Theorem 3.3. *There are $c_2 > 0$ and X_0, depending on y, ε, such that for $X \geq X_0$:*

(a) *$|\mathcal{P}(X)| \geq c_2 X^2/\log^2 X$, and at most $4X\log(2X)$ pairs in $\mathcal{P}(X)$ are not good;*
(b) *for a good pair, $u = \rho/q$ satisfies $|u - iy| < \varepsilon$, $\operatorname{Re} u = 1/(2q)$, $c(u) = -m(\rho)/q$, $|c(u)/u| = m(\rho)/|\rho| \geq 1/(X+1)$, and $q \ll_{y,\varepsilon} X$;*
(c) *distinct pairs give distinct u.*

Proof. (a) The number of distinct zeros $\frac{1}{2} + i\gamma$ with $X/M < \gamma \leq X$ is at least $N_0(X) - N(X/M)$. By (Z4) and (Z2), for large X it is at least $A_0 X - (X/M)\log X \geq \frac{1}{2}A_0 X$; this is where the count of *distinct* zeros is needed. Fix such a γ. The primes with $\gamma/q \in J$ are those in $(x, (1+\theta)x)$ with $x = \gamma/(y + \frac{\varepsilon}{2})$ and $\theta = \varepsilon/(y - \frac{\varepsilon}{2})$. (In this proof x is this real number, as in (Z5), not a candidate point.) As $x \geq X/(M(y + \frac{\varepsilon}{2})) \to \infty$, (Z5) gives at least $\theta x/(2\log x) \gg X/\log^2 X$ of them. This proves the lower bound.

If a pair is not good, Lemma 3.2 attaches to it a zero ζ^* with $0 < \operatorname{Im}\zeta^* < 2X$, and each such zero arises from at most two pairs. There are at most $N(2X) \leq 2X\log(2X)$ such zeros

for large X. These zeros may lie arbitrarily close to the critical line, so (H_∞) gives no bound on their number; the count uses only the total $N(2X)$, from (Z2), and Theorem 3.3 assumes neither (H_∞) nor $B < 1$.

(b) $u = \frac{1}{2q} + i\gamma/q$, so $\operatorname{Re} u = \frac{1}{2q}$ and $|\operatorname{Im} u - y| < \frac{\varepsilon}{2}$. Since $q > x \to \infty$, $\frac{1}{2q} < \frac{\varepsilon}{2}$ for large X. And $q < \gamma/(y - \frac{\varepsilon}{2}) \ll X$. The rest is the formula for $c(u)$.

(c) $\operatorname{Re} u$ determines q, and then $\operatorname{Im} u$ determines γ. □

3.2. **High pairs.** Of the three steps, the second (discarding the points that have a low extra pair) is a count that needs no estimate of the pair contributions; it is done inside the proof of Theorem 3.1, at the end of the section. The third needs a lemma, which comes first.

For $x = 1 + u$ every pair other than $(1, u)$ and $(u, 1)$ has $a, b \neq 1$. Then $\operatorname{Re} a, \operatorname{Re} b \leq B$ by Lemma 2.1(ii), so $\operatorname{Re} a = 1 + \operatorname{Re} u - \operatorname{Re} b > 1 - B$, and likewise for b. A point of $\Sigma \setminus \{1\}$ with real part above $1 - B$ is $1/k$ or ρ/n with denominator at most $n_0 = \lfloor 1/(1 - B) \rfloor$. So Lemma 2.1(iii) applies with $1 - B$ in place of ε: the extra pairs have bounded denominators, and there are only logarithmically many of them per unit of height. The next lemma says what "too high to matter" means: above height H the extra pairs contribute at most e^{-H} in total, uniformly in u on the region where the chosen points lie. The two kernels decay at different rates, and we estimate them separately.

Lemma 3.4. *Assume $B < 1$ and let $Y > 0$. There is $H_0 \geq 2Y + 2$ such that for all u with $0 < \operatorname{Re} u \leq 1$, $|\operatorname{Im} u| \leq Y$, and all $H \geq H_0$,*

$$\sum_{\substack{a+b=1+u,\ a,b\neq 1\\ |\operatorname{Im} a|>H}} \big|c(a)c(b)\mathrm{B}(a,b)\big| \leq e^{-H}, \qquad \sum_{\substack{a+b=1+u,\ a,b\neq 1\\ |\operatorname{Im} a|>H}} \big|c(a)c(b)K\big(\tfrac{a-b+3}{4}\big)\big| \leq e^{-H}.$$

In particular both full sums at $1 + u$ converge absolutely.

Proof. Write $t = \operatorname{Im} a$, so $\operatorname{Im} b = \operatorname{Im} u - t$ and $|\operatorname{Im} b| \geq |t| - Y$. By Lemma 2.1(iii), $|c(a)c(b)| \ll \log^2(|t| + Y + 3)$, and $O(\log(j + 4))$ points a have $|\operatorname{Im} a| \in [j, j + 1]$; the pair is determined by a.

For G, (Z6) on $[1 - B, B]$ gives $|\Gamma(a)\Gamma(b)| \ll (1 + |t| + Y)^2 e^{-\pi(2|t|-Y)/2}$, and $1/\Gamma$ is bounded on the compact set where $1 + u$ lies. So the terms are $\ll (1 + |t|)^4 e^{-\pi|t|}$.

For S, let $w = (a - b + 3)/4$. Then $\operatorname{Re} w \in (\frac{1}{2}, 1)$, since $|\operatorname{Re} a - \operatorname{Re} b| \leq 2B - 1 < 1$, and $|\operatorname{Im} w| \geq \frac{1}{2}|t| - \frac{1}{4}Y$. From $K(w) = (1 - w)\pi/\sin \pi w$ and $|\sin \pi w| \geq \sinh \pi |\operatorname{Im} w|$ we get $|K(w)| \leq 7(1 + |\operatorname{Im} w|)e^{-\pi|\operatorname{Im} w|}$ for $|\operatorname{Im} w| \geq 1$, and $|t| > H \geq 2Y + 2$ gives $|\operatorname{Im} w| \geq 1$. So the terms are $\ll (1 + |t|)^3 e^{-\pi|t|/2}$. This is half the rate of G, and still faster than $e^{-|t|}$.

For both kernels, then, the terms are $\ll (1 + |t|)^4 e^{-\pi|t|/2}$. Summing over $|t| > H$, with $O(\log(j + 4))$ points per unit interval, gives $\ll (H + 3)^5 e^{-\pi H/2}$, and since $\pi/2 > 1$ this is at most e^{-H} for large H; the slower kernel of S affects only how large H_0 must be. Finitely many a have $|\operatorname{Im} a| \leq H_0$ and $\operatorname{Re} a > 1 - B$, which gives absolute convergence. □

The rate e^{-H} is chosen for convenience. Any exponential rate would serve, with H a suitable multiple of $\log X$, since the near-pair count below stays polylogarithmic. The value used below is $H = 2 \log X$. Then $e^{-H} = X^{-2}$, and since Lemma 3.4 gives the same bound e^{-H} for both kernels, the far pairs fall below the main terms, of size $\gg 1/X$, for both.

3.3. **Proof of Theorem 3.1.** By conjugation, using $c(\bar{a}) = c(a)$, it suffices to treat $y \geq 0$; and a disc about 1 contains a disc about $1 + iy'$ with $y' > 0$. So let $y > 0$ and $0 < \varepsilon < \min(y, 1)$. Put $Y = y + 1$ and let H_0 be as in Lemma 3.4 for this Y. As y is fixed, so is H_0, and $H = 2 \log X \geq H_0$ once $X \geq e^{H_0/2}$; we take X that large. For a good pair put $u = \rho/q$ and $x = 1 + u$.

The sum at x is split into the two main pairs, the near pairs of height at most H, and the far pairs above H. The main term is of size at least a constant times $1/X$; the far pairs total

at most X^{-2}; and the near pairs are not estimated at all but counted, and the points at which any occur are discarded. The comparison of the main term with the far pairs follows [2, p. 52], and isolating the main pair and bounding the rest by the decay of the Gamma factors follows [6, pp. 11–13]; here the count of near pairs replaces the condition on ordinates. We give the estimates for both kernels.

Main terms. The pairs $(1,u),(u,1)$ contribute to Θ_G the amount $-2c(u)/u$, of modulus at least $2/(X+1)$. They contribute to Θ_S the amount $-c(u)\{K(1-\frac{u}{4})+K(\frac{1}{2}+\frac{u}{4})\}$. The bracket is continuous and, by Lemma 2.4, nonzero on the closed disc $|u-iy|\le\varepsilon$, which misses the real axis, so it is at least some $\lambda>0$ in modulus. With $|c(u)|\ge 1/q$ and $q\ll X$ this main term is $\gg_{y,\varepsilon} 1/X$.

Near pairs. Call a good pair *near-bad* if x has a pair $(a,b)\ne(1,u),(u,1)$ with $|\operatorname{Im} a|\le H$. Then a,b lie in the set $\{z\in\Sigma\setminus\{1\} : \operatorname{Re} z>1-B,\ |\operatorname{Im} z|\le H+Y\}$, of size $\ll (H+Y)\log(H+Y)$ by (Z2), as the denominators are at most n_0. Since $u=a+b-1$ determines (ρ,q) by Theorem 3.3(c), there are $\ll(\log X\log\log X)^2$ near-bad pairs. In our setting it is the abundance of the points ρ/q that lets this count stand in for the hypothesis on additive relations among ordinates used in [6, 2].

Far pairs. For a good pair that is not near-bad, every pair other than the two main ones has $|\operatorname{Im} a|>H$. By Lemma 3.4 their total contribution to each sum is at most $e^{-H}=X^{-2}$, which is smaller than either main term for large X. Exact coincidences above height H are allowed, in any number and with any signs. So $\Theta_G(x)\ne 0$ and $\Theta_S(x)\ne 0$.

By Theorem 3.3(a) at least $\frac{1}{2}c_2X^2/\log^2 X$ pairs are good, and only $\ll(\log X\log\log X)^2$ of them are near-bad. As $X\to\infty$ this gives infinitely many points, distinct by Theorem 3.3(c), with $|x-(1+iy)|=|u-iy|<\varepsilon$. □

Apart from the reduction to $y>0$, which uses that the kernels commute with complex conjugation, the kernel entered this proof in two places only: the main term and its lower bound, where the main term is $-2c(u)/u$ for G and the bound for S uses Lemma 2.4, and the far-pair bound of Lemma 3.4. The choice of the points, Theorem 3.3 and the count of near pairs are independent of the kernel. Section 9.1 uses this: for the kernel of G_2 it supplies those two inputs (Lemmas 9.2 and 9.3), and the rest of the proof is repeated unchanged.

Remark 3.5. The denominator step in the non-vanishing of $c(\rho/p)$ is Titchmarsh's proof of (9.5.7) [34, §9.5, p. 216], in his account of the theorem of Landau and Walfisz [16]; Section 1 gives the attributions. The step needs no zeros on the critical line: the term $-m(\rho)/p$ is the only one with denominator divisible by p, for $\operatorname{Re}\rho\ge\frac{1}{2}$ and $p\nmid m(\rho)$, and p is chosen large, so that $p>m(\rho)$. So non-vanishing alone is classical and needs no count. The far-pair comparison needs more: the lower bound $|c(u)|\ge 1/q$. A sum of several terms could have a denominator as large as the least common multiple of many $k<2q$, which would not beat the far-pair total X^{-2}; a single representation guarantees the bound without further cancellation estimates. That is why we take the critical line: there the single-term points come with the count of Theorem 3.3(a) (compare [3, Lemma 1]).

4. The meromorphic series T

This section proves Theorem 1.1. T gives the argument in its simplest setting: since Q is single-valued, moving the contour in (4) across poles produces residues and nothing else, so T continues as a meromorphic function and every point of Π is at worst a simple pole, whose residue is the sum over pairs of Section 2. Section 3 then supplies dense points where that residue is nonzero, and poles accumulating at every point of the line are the boundary. Everything before the last proof holds without (H_∞).

The contour shift below parallels the proof of Egami and Matsumoto [6, Theorem 2.1, proved on pp. 8–10] for the series weighted by $\Lambda(k)\Lambda(m)$, under RH; Bhowmik and Schlage-Puchta state the half-plane $\operatorname{Re} s > 2\sigma_0$ for that series under their Conjecture 2 and say the proof runs parallel [2, Theorem 2(4), p. 44; p. 50]. In our reading the argument for T follows theirs with Q in place of $-\zeta'/\zeta$. What differs: Q has poles at scaled zeros, and we stop at $\operatorname{Re} w = c$, so finitely many residues appear and the right-hand poles cross the contour, which Step 3 handles.

Theorem 4.1. *T continues to a single-valued meromorphic function on $\operatorname{Re} s > 2B_\infty$, with $T(\bar{s}) = \overline{T(s)}$. Its poles are simple and lie in Π. More precisely, let $0 < \delta < \frac{1}{2}(1 - B_\infty)$, $F = F_\delta$, and $c \in (B_\infty + 2\delta, 1)$ with $c \neq \operatorname{Re} a$ for $a \in F$. Then for $\sigma = \operatorname{Re} s > c + B_\infty + 2\delta$ with $\sigma \neq c + \operatorname{Re} a$ for $a \in F$,*

$$T(s) = I_c(s) + \sum_{a \in F} \big([\operatorname{Re} a > c] + [\sigma - \operatorname{Re} a < c]\big) R_a K_s(a) Q(s-a), \tag{8}$$

where $I_c(s) = \frac{1}{2\pi i}\int_{(c)} K_s(w)Q(w)Q(s-w)\,dw$ is holomorphic on that set, and $[A]$ is 1 if A holds and 0 otherwise.

If $B_\infty = 1$ there is nothing to prove, so let $B_\infty < 1$. Write $g_s(w) = K_s(w)Q(w)Q(s-w)$. In the strip $0 < \operatorname{Re} w < \sigma$ its poles lie among the *left points* $a \in \Sigma$ and the *right points* $s - b$, $b \in \Sigma$. The idea is that a left point crossing the contour leaves a residue, and so does a right point; the two families are exchanged by $w \mapsto s - w$, and the indicators record which have crossed. The left points are fixed, and the contour at $\operatorname{Re} w = c$ is placed once, so the first indicator does not depend on s. The right points move with s, and one of them crosses the contour whenever σ passes a value $c + \operatorname{Re} b$; the second indicator switches on at that moment, and Step 3 checks that the switch is exactly compensated by the residue the contour picks up. Only the finitely many points of F can ever cross, because every other point of Σ has real part at most $B_\infty + \delta$, which keeps it, and its mirror image, away from the line.

Proof. Step 1: $\sigma > 2$. Take $c_0 \in (1, \sigma - 1)$, so that (4) holds on $\operatorname{Re} w = c_0$. In $c \le \operatorname{Re} w \le c_0$ the only poles are the left points $a \in F$ with $\operatorname{Re} a > c$, since every right point has $\operatorname{Re}(s-b) \ge \sigma - 1 > c_0$. On horizontal segments at height h, Lemma 2.2 and the decay of K_s give $\ll h^A e^{-\pi h}$. So the residue theorem, with $\operatorname{Res}_{w=a} g_s = R_a K_s(a) Q(s-a)$, gives (8), each $[\sigma - \operatorname{Re} a < c]$ being 0 here.

Step 2: I_c is holomorphic. Let s range over a compact part of the set in the statement, and $w = c + it$. No left point lies on the line. A right point on it would be $s - b$ with $\operatorname{Re} b = \sigma - c > B_\infty + 2\delta$, so $b \in F$ and $\sigma = c + \operatorname{Re} b$, which is excluded. For large $|t|$, Lemma 2.2 bounds both Q factors by $\ll \log|t|$, and with the decay of K_s the integrand has an integrable bound uniform in s.

Step 3: crossing $\sigma = c + \operatorname{Re} b_1$. Let $b_1 \in F$ with $\operatorname{Re} b_1 > B_\infty + 2\delta$, let $F_1 = \{b \in F : \operatorname{Re} b = \operatorname{Re} b_1\}$, and let s_1 be on the line $\operatorname{Re} s_1 = c + \operatorname{Re} b_1$. Take $\varepsilon > 0$ small enough that the disc $|s - s_1| < \varepsilon$ meets no line $\sigma = c + \operatorname{Re} b'$ with $b' \in F \setminus F_1$, that the following contour is at positive distance from all left points and from all right points except those it is built around, that its half-discs contain no left point and no right point other than these, and that distinct $b \in F_1$ have imaginary parts differing by more than 4ε, so that the replaced segments are disjoint. Let $\mathcal{L}'$ be $\operatorname{Re} w = c$ with, for each $b \in F_1$, the segment $|\operatorname{Im} w - (\operatorname{Im} s_1 - \operatorname{Im} b)| \le 2\varepsilon$ replaced by the semicircle of radius 2ε about $s_1 - b$ in $\operatorname{Re} w \le c$. As in Step 2, $I'(s) = \frac{1}{2\pi i}\int_{\mathcal{L}'} g_s$ is holomorphic on the disc. For $\sigma > \operatorname{Re} s_1$ the point $s - b$ lies to the right of $\operatorname{Re} w = c$, and $I' = I_c$. For $\sigma < \operatorname{Re} s_1$ it lies inside the half-disc, and since $Q(s-w)$ has residue $-R_b$ at $w = s - b$ and $K_s(s-b) = K_s(b)$, we get $I'(s) = I_c(s) + \sum_{b \in F_1} R_b K_s(b) Q(s-b)$. On the same side the indicator of b rises by one, so the right side of (8) equals one meromorphic function, I' plus the sum with the indicators on the side $\sigma > \operatorname{Re} s_1$, on both half-discs.

Step 4: assembly. Each term $R_a K_s(a) Q(s-a)$ is meromorphic on $\sigma > \operatorname{Re} a$, since $K_s(a)$ is holomorphic there, with at most simple poles, all at points $a+b$, $b \in \Sigma$. This half-plane contains the whole domain, since $\sigma > c + B_\infty + 2\delta > 2B_\infty + 4\delta \geq 1 + 4\delta > \operatorname{Re} a$, using $B_\infty \geq \frac{1}{2}$. Steps 2 and 3 give a meromorphic function on $\sigma > c + B_\infty + 2\delta$, the local expressions agreeing by the identity theorem, and it equals T on $\sigma > 2$ by Step 1. Its poles are at most simple: F is finite, each term has order at most 1 at any point, because $K_s(a)$ is holomorphic there and Q has at most simple poles, and I_c or I' is holomorphic. Several terms can have a pole at one point, as Theorem 4.2 shows, and they may cancel. Letting $c \downarrow B_\infty + 2\delta$ and $\delta \downarrow 0$ gives the half-plane $\sigma > 2B_\infty$. The symmetry holds for $\sigma > 2$ and so everywhere. □

The formula (8) depends on the choices of c and δ, and its indicators only record which points have crossed the contour. The residue they compute does not depend on those choices, and the next theorem puts it in the symmetric form over pairs announced in the introduction.

Theorem 4.2. *Let* $\operatorname{Re} s_0 > 2B_\infty$. *Then* s_0 *has finitely many pairs, and*

$$\operatorname{Res}_{s=s_0} T = \sum_{a+b=s_0} R_a R_b \mathrm{B}(a,b) = \sum_{a+b=s_0} c(a)c(b)\mathrm{B}(a,b). \tag{9}$$

Proof. If $B_\infty = 1$, then $\operatorname{Re} s_0 > 2$, where T is holomorphic; no pair sums to s_0, since $\operatorname{Re}(a+b) \leq 2$. Both sides vanish. Otherwise choose

$$0 < \delta < \min\left\{ \frac{1 - B_\infty}{2}, \frac{\operatorname{Re} s_0 - 2B_\infty}{4} \right\}$$

and c with $B_\infty + 2\delta < c < \min(1, \operatorname{Re} s_0 - B_\infty - 2\delta)$, $c \neq \operatorname{Re} a$ and $c \neq \operatorname{Re} s_0 - \operatorname{Re} a$ for $a \in F$. Near s_0, (8) holds with constant indicators $\nu(x) = [\operatorname{Re} x > c] + [\operatorname{Re} s_0 - \operatorname{Re} x < c]$, and I_c is holomorphic. The term of index $x \in F$ contributes $\nu(x) R_x R_{s_0 - x} \mathrm{B}(x, s_0 - x)$. A pair (a,b) at s_0 has at least one member in F, since $\operatorname{Re} a + \operatorname{Re} b > 2B_\infty + 4\delta$. There are three cases. If $a \neq b$ are both in F, then $\nu(a) + \nu(b) = 2$. If $a \in F$ and $b \notin F$, then $\operatorname{Re} b < c < \operatorname{Re} a$ and $\nu(a) = 2$. If $a = b$, then $\nu(a) = 1$. In each case the two ordered pairs, or the one diagonal pair, are counted exactly once, and the summand is symmetric. □

Proof of Theorem 1.1. Under (H_∞), $2B_\infty = 1$ and $B < 1$, so T is meromorphic on $\operatorname{Re} s > 1$ by Theorem 4.1. By Theorem 3.1 every disc about a point of the line contains infinitely many x with $\Theta_G(x) \neq 0$, and by Theorem 4.2 these are poles of T. A function meromorphic on a disc about $1 + iy$ agreeing with T on the right would have poles accumulating at $1 + iy$, a point of its domain. □

The boundary conclusion applies to every meromorphic continuation of T to $\operatorname{Re} s > 1$, since by the identity theorem there is only one.

5. Fixed sheets for G and S

This section builds a single-valued object that stands in for G or S below $\operatorname{Re} v = 2$, where the series no longer converge. We now replace Q by P, and the integrand becomes multivalued. The usual way to continue the integral (3) is to push the contour along a path in v while continuing the integrand, which means tracking the branches of both factors as the contour moves. That is hard to control near the line $\operatorname{Re} v = 2B_\infty$: under (H_∞) infinitely many singular points accumulate there (Section 1), and, with or without (H_∞), the room for the vertical tails of the contour, which must stay to the right of B_∞ on both sides, shrinks to nothing. We do not continue the integral along paths. We fix a direction, cut the half-plane along parallel rays from all the singular points, and so get a single-valued branch of P and one contour integral with a fixed integrand. On $\Omega_\kappa \cap \{\operatorname{Re} v > 2\}$ it is the series. The object so defined, a sheet, is

one branch of the series. Turning the direction supplies the jumps between sheets, and Section 7 shows that every germ reached along a path in $\{\operatorname{Re} v > 2B_\infty\} \setminus \Pi$, at a point of $\mathcal{V}_\infty \setminus \mathcal{X}$, is a sheet plus an integer combination of these jumps, up to a constant; under (H_∞), Corollary 8.3 removes the restriction on the path. This is the additive, infinite-contour form of Hadamard's construction [11]; Borel [4] has the contour deformation without the cuts.

Everything in Sections 5–7 holds without (H_∞), in $\mathcal{V}_\infty$. Under (H_∞), $\mathcal{V}_\infty = \mathcal{V}$.

5.1. **Directions and the branch P^κ.** All cuts will be parallel rays pointing west, in a direction chosen once. The direction is described by a slope, and the vocabulary that follows (height, abscissa, east, west) is the usual one rotated so that the cuts become horizontal. Fix a real slope κ and let $e = (1+i\kappa)/|1+i\kappa| = e^{i\theta}$, with $|\theta| < \pi/2$. For $w \in \mathbb{C}$ call $\operatorname{Im}(w/e)$ its *height* and $\operatorname{Re}(w/e)$ its *abscissa*; a *κ-line* is a set of constant height. *East* is increasing abscissa. On a κ-line,

$$w' \text{ is east of } w \iff \operatorname{Re} w' > \operatorname{Re} w, \tag{10}$$

because $\cos\theta > 0$. Call κ *generic* if no two distinct points of Σ, and no two distinct points of Π, lie on one κ-line. All but countably many κ are generic; genericity is what will let the collisions at a candidate point be treated one at a time, each on its own κ-line. The horizontal direction $\kappa = 0$ will not do: the points $1/k$ all lie on the real axis, and as soon as there is a zero ρ off the critical line, ρ and $1-\bar\rho$ have the same imaginary part. That is why a slope is needed at all. Put $\operatorname{Log}_\kappa(y) = i\theta + \operatorname{Log}(y/e)$, with cut $y \in -e[0,\infty)$; it agrees with the principal logarithm on $\operatorname{Re} y > 0$.

Let $\ell_a = (a - e[0,\infty)) \cap \{\operatorname{Re} z > 0\}$ be the westward ray from $a \in \Sigma$, and let U_κ be $\{\operatorname{Re} z > 0\}$ minus all ℓ_a. A ray ℓ_a meeting a compact subset of $\operatorname{Re} z > 0$ starts at a point with $\operatorname{Re} a \le 1$, hence at bounded distance from the set and with real part at least the set's minimum; by local finiteness only finitely many rays meet the set, and U_κ is open. It contains $\operatorname{Re} z > 1$, and each of its points flows east into $\operatorname{Re} z > 1$: the eastward ray from a point of U_κ meets no cut, since a cut it met would already contain the point. So U_κ is simply connected, and $-Q$, which is holomorphic on it, has a primitive there. Let P^κ be the primitive of $-Q$ on U_κ that equals P on $\operatorname{Re} z > 1$. Near a singular point it is a logarithm with coefficient $c(a)$ plus a holomorphic function, and at large height it grows no faster than Q does; the next lemma records both.

Lemma 5.1. (a) *If a is the only point of Σ on its κ-line, then for small R, on $|z-a| < R$ minus ℓ_a, $P^\kappa(z) = c(a)\operatorname{Log}_\kappa(z-a) + E_a(z)$, with E_a holomorphic on the whole disc.*

(b) *For $\eta > 0$ there are t_1, C with $|P^\kappa(z)| \le C\log(|\operatorname{Im} z| + 2)$ for $\operatorname{Re} z \ge B_\infty + 2\eta$, $|\operatorname{Im} z| \ge t_1$.*

Proof. (a) Q has residue $-c(a)$ at a, so $P' - c(a)/(z-a)$ is holomorphic on the disc.

(b) The cuts meeting $\operatorname{Re} z \ge B_\infty + 2\eta$ come from the finite set F_η. So for $|t|$ large the horizontal segment from $x+it$ to $2+it$, with $x \ge B_\infty + 2\eta$, avoids all cuts, and $P^\kappa(x+it) = P(2+it) + \int_x^2 Q(\xi+it)\,d\xi$. Lemma 2.2 bounds the integrand by $\ll \log|t|$. For $x \ge 2$, $|P^\kappa| \le P(2)$. □

5.2. **Separating graphs.** The sheet is to be the integral of $\mathcal{K}(z,v)P^\kappa(z)P^\kappa(v-z)$ over a contour, and the integrand is holomorphic at z when z lies on no cut ℓ_a, $v-z$ lies on no cut ℓ_b, and $0 < \operatorname{Re} z < \operatorname{Re} v$. (If $c(a) = 0$ the cut ℓ_a is artificial, since a then contributes no jump to P^κ; such a cut is harmless, and we avoid it like the others.) The second condition is a condition on z relative to the right points: $v - z \in \ell_b$ means $z = v - b + te$ with $t \ge 0$, that is, z is at or east of $v-b$ on the κ-line through $v-b$. So the contour has to pass, κ-line by κ-line, east of every left point a and west of every right point $v-b$. On a κ-line that carries both a left and a right point this is possible only if the right point is strictly east of the left one, and it fails exactly when $v-b$ is at or west of a, that is, when v lies on the westward ray from $a+b$. That is why the cuts for the sheet are rays from the points of Π, in the same direction as the cuts of P^κ.

For $x \in \mathcal{X}$ let $L_x = (x - e[0,\infty)) \cap \mathcal{V}_\infty$, and let Ω_κ be $\mathcal{V}_\infty$ minus all L_x. Since $\operatorname{Re} x \le 2$ on $\mathcal{X}$, the same argument with Lemma 2.3 shows that only finitely many L_x meet a compact set, so Ω_κ is open. It is connected: from a point with $\operatorname{Re} v \le 2$, follow the eastward ray only until $\operatorname{Re} v = \frac{5}{2}$. This segment stays in Ω_κ (a cut it met would already contain the point). The region $\operatorname{Re} v > 2$ for G, or $2 < \operatorname{Re} v < 3$ for S, contains no cuts and is connected.

A *separating κ-graph for v* is a curve Γ that is a piecewise linear graph of abscissa over height, such that its real parts lie in a compact subset of $(0, \operatorname{Re} v)$; for $|\operatorname{Im} z|$ large it is a vertical line $\operatorname{Re} z = c$ with $B_\infty < c < \operatorname{Re} v - B_\infty$; and every $a \in \Sigma$ is strictly west of Γ on its κ-line, while every right point $v - b$, $b \in \Sigma$, is strictly east. We orient Γ by increasing height, which on the tails is upward in $\operatorname{Im} z$, since $\cos\theta > 0$. Being a graph over height, Γ meets each κ-line once, so "west of Γ" and "east of Γ" make sense; the compact range of real parts keeps $\mathcal{K}$ holomorphic; and the vertical tails, placed strictly to the right of B_∞ on both sides, are where the growth bound of Lemma 5.1(b) will apply. If Γ separates and $z \in \Gamma$ then $z \in U_\kappa$ and $v - z \in U_\kappa$. Indeed $z \in \ell_a$ would put z at or west of a, and $v - z \in \ell_b$ would put z at or east of $v - b$.

The next lemma shows that the cuts L_x are the right ones: a separating graph exists exactly when v is off the cuts.

Lemma 5.2. *For $v \in \mathcal{V}_\infty$ a separating κ-graph exists if and only if $v \in \Omega_\kappa$.*

Proof. If $v = x - te$ with $t \ge 0$ and $x = a + b$, then $v - b = a - te$ is at or west of a on one κ-line, and no graph separates them.

Conversely let $v \in \Omega_\kappa$. Whenever a and $v - b$ lie on one κ-line, $v - b$ is strictly east of a: if $\operatorname{Re}(a + b) \le 2B_\infty$ then $\operatorname{Re}(v - b - a) > 0$ and (10) applies, and otherwise $a + b \in \mathcal{X}$, and $v - b$ at or west of a would put v on L_{a+b}. So on every κ-line there is room between the left points and the right points, and the graph has only to be threaded through it. All but finitely many left points lie west of the midline $\operatorname{Re} z = \operatorname{Re} v/2$ by a fixed margin, and all but finitely many right points east of it, so the midline does most of the work; the finitely many exceptions are handled by local detours. Take $c = \operatorname{Re} v/2$ and $\eta > 0$ with $4\eta < \operatorname{Re} v - 2B_\infty$. Left points with $\operatorname{Re} a \ge c$ and right points with $\operatorname{Re}(v - b) \le c$ involve a or b in the finite set F_η, so they lie on finitely many κ-lines. On each such line choose a point strictly east of every left point and strictly west of every right point on it, with real part in $(0, \operatorname{Re} v)$; the condition concerns only the points present, and such a point exists because every right point is strictly east of every left point and, Σ being locally finite in $\operatorname{Re} z > 0$, the left points on the line can accumulate only towards $\operatorname{Re} z = 0$ and the right points only towards $\operatorname{Re} z = \operatorname{Re} v$. Let the graph be $\operatorname{Re} z = c$ outside thin disjoint bands about these lines, and a tent up to the chosen point inside each band. The bands can be taken thin enough to contain no other relevant point, since only finitely many left points with real part above a small ε_1, and right points with real part below $\operatorname{Re} v - \varepsilon_1$, lie near the lines. Points with real part below ε_1 on the left, or above $\operatorname{Re} v - \varepsilon_1$ on the right, are on the correct side by (10). □

With a separating graph in hand, the sheet is the integral over it. Three things need checking: that the value does not depend on the graph, that it is holomorphic in v, and that it is the series where the series converges.

Theorem 5.3. *For $v \in \Omega_\kappa$ let*

$$D^\kappa(v) = \frac{1}{2\pi i} \int_\Gamma \mathcal{K}(z, v)\, P^\kappa(z)\, P^\kappa(v - z)\, dz, \tag{11}$$

with Γ any separating κ-graph for v. The value does not depend on Γ; D^κ is holomorphic on Ω_κ; and $D^\kappa = D$ on $\Omega_\kappa \cap \{\operatorname{Re} v > 2\}$.

Proof. The conditions on a graph are convex in the abscissa function. On a κ-line the real part is an increasing affine function of the abscissa, by (10), so each condition there (east of every left point on the line, west of every right point on it, real part in $(0, \operatorname{Re} v)$) confines the abscissa to an open interval or half-line. An intersection of such sets is again an interval, so on each κ-line the admissible abscissae form an interval. On the tails, the abscissa of the line $\operatorname{Re} z = c$ at a given height is affine in c, and the admissible c form the interval $(B_\infty, \operatorname{Re} v - B_\infty)$. A convex combination of two graphs whose real parts lie in compact subsets of $(0, \operatorname{Re} v)$ has real parts in the convex hull of their union, which is again a compact subset of $(0, \operatorname{Re} v)$. So the graph with abscissa $(1-t)\alpha_0 + t\alpha_1$ separates whenever the graphs with abscissae α_0 and α_1 do. Two separating graphs are therefore joined by a homotopy of separating graphs, inside the open set where the integrand is holomorphic, and Cauchy's theorem applies once the ends are controlled: the connecting segments at large height vanish by (K2) and Lemma 5.1(b).

Fix v_1 and a graph Γ for it, with tails on $\operatorname{Re} z = c$, and take $\eta > 0$ so small that $B_\infty + 2\eta < \min(c, \operatorname{Re} v_1 - c)$. Then Γ still separates for v near v_1: left points do not move, right points with small $\operatorname{Re} b$ stay east by (10), right points at heights where Γ is vertical have $\operatorname{Re}(v-b) > c$ unless $b \in F_\eta$, and the remaining finitely many have a positive margin. So near v_1 the sheet is one integral over a fixed contour, of an integrand holomorphic in v and dominated by (K2) and Lemma 5.1(b), since on the tails both $\operatorname{Re} z$ and $\operatorname{Re}(v-z)$ are at least $B_\infty + 2\eta$ for v near v_1; holomorphy in v follows by differentiation under the integral.

For $\operatorname{Re} v > 2$ the line $\operatorname{Re} z = c$ with $1 < c < \operatorname{Re} v - 1$ separates, and $P^\kappa = P$ on it; this is (3). □

For each fixed v the tails sit a fixed distance to the right of B_∞, and only finitely many zeros lie further east. The bounds in this section depend on v, and every later use of them is local in v.

6. One collision and the local logarithm

Near a point $x \in \mathcal{X}$, for generic κ and off the slit L_x, a sheet is a logarithm times the holomorphic coefficient X_x of (6), plus a function holomorphic at x; this is (12) below. A contour is pinched at x between a and $v-b$ for every pair at x; with a generic direction these pinches occur on different κ-lines, so they can be isolated from one another. The part of the contour away from the pinches contributes a function holomorphic in v. Each pinch is a short arc on which both factors of the integrand are explicit logarithms, and a one-variable calculation shows that the arc integral is $c(a)c(b)\operatorname{Log}_\kappa(v-x)J_{ab}(v)$ plus a holomorphic function. Summing the arcs gives the coefficient, one term per pair. The section ends with Lemma 6.2, the same calculation with one of the two logarithms replaced by a simple pole. It is used only in Section 9.2, for $G_{\log}$.

Theorem 6.1. *Let $x \in \mathcal{X} \cap \mathcal{V}$ and let κ be generic. There is $r > 0$ such that the disc $|v-x| < r$ minus L_x lies in Ω_κ, and on it*

$$D^\kappa(v) = \operatorname{Log}_\kappa(v-x)\, X_x(v) + H(v), \tag{12}$$

with H holomorphic on the whole disc.

Proof. The configuration at $v = x$. Suppose $a \in \Sigma$ and $x - b$ lie on one κ-line. Either $\operatorname{Re}(a+b) \le 2B_\infty$, and $x-b$ is east of a; or $a+b \in \mathcal{X}$ lies on the κ-line of x, and genericity gives $a+b = x$, that is $x - b = a$. So each left-right pair on a κ-line is either strictly ordered or a *collision*, and by genericity each line carries at most one left and one right point.

The outer contour. The aim is a contour for x itself that passes through each collision point and is a separating graph everywhere else, with enough margin that removing short pieces around

the collisions leaves a contour that still works for every v near x. Figure 1 shows the picture. There are three steps: the graph, the size r_0 of its flat tops, and the radius r.

As in Lemma 5.2, build a κ-graph Γ_0 for x: vertical tails on $\operatorname{Re} z = \operatorname{Re} x/2$, tents on the finitely many κ-lines that carry no collision and on which Lemma 5.2 needs a detour, and at each colliding a_i a flat top passing through a_i, namely abscissa constant for heights within r_0 of that of a_i, joined to the rest of the graph by ramps at heights between r_0 and $2r_0$ from that of a_i. Let Γ_{out} be Γ_0 without the open flat tops.

Choose r_0 small enough for four things. First, the bands of heights within $2r_0$ of the collisions are disjoint from one another (take r_0 below a quarter of the least distance between collision heights, when there is more than one) and from the bands of the tents. Second, the discs $|z-a_i| < 2r_0$ are disjoint, meet Σ and $x-\Sigma$ only in a_i, and carry the expansion of Lemma 5.1(a); likewise the discs about the b_i. Third, within its band no left point lies east, and no right point west, of a flat top or its ramps. The third condition can be met because only finitely many points compete for it: the real parts of the graph in these bands lie in a compact subset of $(0, \operatorname{Re} x)$ that does not depend on r_0, so a left point a east of the graph, or a right point $x-b$ west of it, has $\operatorname{Re} a$, or $\operatorname{Re} b$, bounded below and its height bounded, which leaves finitely many candidates independent of r_0, and by genericity none but a_i itself has the height of a_i. The fourth condition looks ahead to the arc calculation at the end of the proof. There the kernel near a collision (a_i, b_i) is written in two local variables, as $f(t,\mu) = e\,\mathcal{K}(a_i + et, x + e\mu)$, with e the unit vector of Section 5: t is the position of $z = a_i + et$ measured from the left point, and μ stands for the displacement of $v = x + e\mu$ from x. The two are kept apart because the calculation differentiates the moving logarithm with the kernel's second argument held fixed, and puts μ equal to the actual displacement only at the end. For that f has to be holomorphic in the two variables jointly. So, fourth, for each collision the function f is holomorphic on $|t| < 2r_0$, $|\mu| < 2r_0$, which (K1) allows since $\operatorname{Re} a_i > 0$, $\operatorname{Re} b_i > 0$ and x lies in the open set $\mathcal{V}$.

To choose r, write $v = x + \delta$ and first fix $\eta > 0$ with $B_\infty + 2\eta < \operatorname{Re} x/2$. Beyond a fixed height all a, b have real part at most $B_\infty + \eta$, so the vertical tails have a margin of at least η on both sides at $v = x$. On the remaining compact part only finitely many left and right points come near the graph, and each non-colliding one has a positive margin, its distance in the plane from the graph. Left points do not move and right points move by δ, and the graph separates the plane, so a point keeps its side as long as $|\delta|$ is below its margin. Take r below half the least of these finitely many margins and of the tail margin η, below $r_0/4$, and small enough that v stays in $\mathcal{V}_\infty$. For $|\delta| < r$ the non-colliding points then keep their sides, and the moved colliding points $a_i + \delta$ have heights inside the removed bands. So Γ_{out} avoids all cuts. On the tails $\operatorname{Re} z = \operatorname{Re} x/2$ and $\operatorname{Re}(v - z) > \operatorname{Re} x/2 - \eta/2$ are both above $B_\infty + \eta$, so Lemma 5.1(b) with $\eta/2$ in place of η, with (K2), dominates the integrand uniformly in v on closed smaller discs. Hence

$$I_{\text{out}}(v) = \frac{1}{2\pi i} \int_{\Gamma_{\text{out}}} \mathcal{K}(z, v) P^\kappa(z) P^\kappa(v - z)\, dz$$

is holomorphic on the whole disc $|v - x| < r$.

The arcs. Each removed piece is replaced by a short arc that passes east of the left point and west of the right point, which have moved apart by δ. Fix a collision (a, b), and in its disc put $t = (z - a)/e$ and $\epsilon = \delta/e$, where e is the unit vector of Section 5, not the base of the exponential; so the left point is $t = 0$ and the right point is $t = \epsilon$, with $\epsilon \notin (-\infty, 0]$ when $v \notin L_x$; in these coordinates height is $\operatorname{Im} t$ and east is increasing $\operatorname{Re} t$. Let A_ϵ run from $-ir_0$ to ir_0, a graph over $\operatorname{Im} t$ with $\operatorname{Re} t > 0$ at height 0 and $\operatorname{Re} t < \operatorname{Re} \epsilon$ at height $\operatorname{Im} \epsilon$: if $\operatorname{Im} \epsilon \neq 0$, take a polygon through a small $\tau > 0$ and $\epsilon - \tau$, in their height order. If $\epsilon > 0$ is real, both conditions hold at the single vertex $\epsilon/2$; use the polygon through that vertex. Then Γ_{out} with these arcs

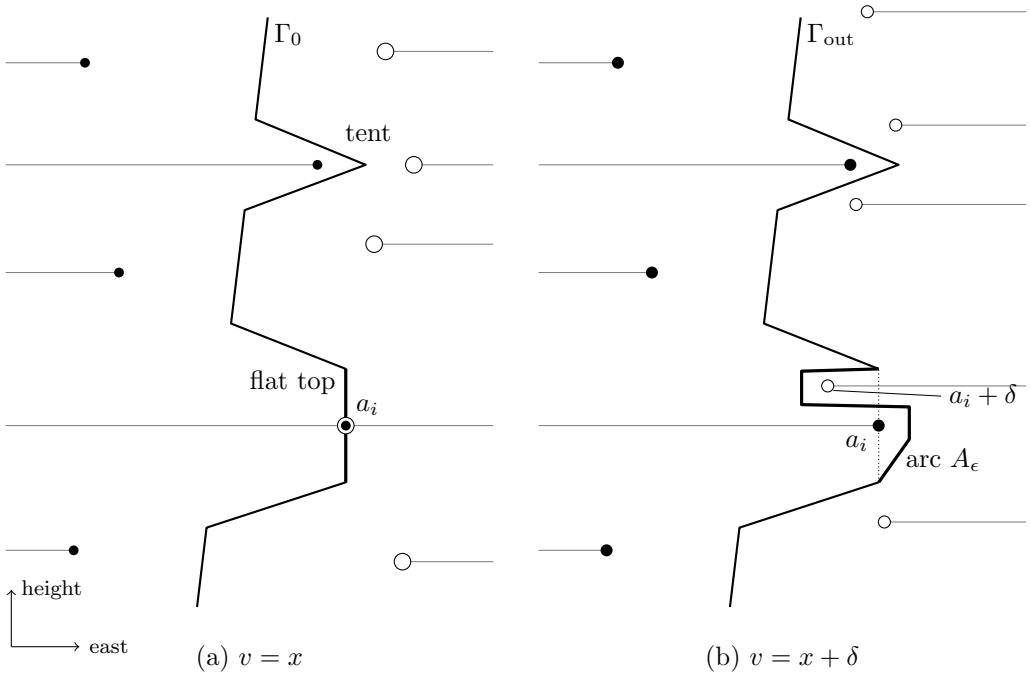


FIGURE 1. Schematic of the outer contour. The axes are abscissa and height, so κ-lines are horizontal and the vertical tails of the graph appear slightly tilted. Filled dots are left points a, whose cuts run west; open dots are right points $v-b$, whose cuts run east. (a) At $v = x$ the graph Γ_0 has a tent at a non-colliding pair and a flat top through a collision, where a left and a right point coincide. The flat top is the stretch of constant abscissa, which is vertical in these coordinates. (b) At $v = x + \delta$ the right points have moved by δ and Γ_{out} has not. The arc replacing the flat top crosses the height of a_i to its east and the height of $a_i + \delta$ to its west. Here $\operatorname{Re}(\delta/e) < 0$, the case in which the arc has to double back.

is a separating graph for v, so by Theorem 5.3, $D^\kappa = I_{\text{out}} + \sum_i I_i$ with one arc integral I_i per collision.

One arc integral. This is the local pinch calculation of [30, (Disc 1), pp. 356–357] and [27, Theorem 1.5]. We calculate the orientation below because their jump conventions differ. By Lemma 5.1(a), with constants absorbed, on A_ϵ,

$$P^\kappa(z) = c(a)\operatorname{Log} t + E_1(t), \qquad P^\kappa(v - z) = c(b)\operatorname{Log}(\epsilon - t) + E_2(\epsilon - t),$$

with E_1, E_2 holomorphic on a disc. Let $f(t, \mu) = e\,\mathcal{K}(a + et, x + e\mu)$, jointly holomorphic since $\operatorname{Re} a, \operatorname{Re} b > 0$. The product has four terms, and only the one with both logarithms can be singular in ϵ: a single logarithm has one cut, and the arc can be moved off it onto a contour that does not depend on ϵ. The term E_1E_2 has a holomorphic integrand on a fixed segment. The term in $\operatorname{Log} t \cdot E_2$ deforms to a fixed polygon east of $t = 0$, and the term in $E_1 \operatorname{Log}(\epsilon - t)$ to a fixed polygon west of the cut from ϵ; both are holomorphic in ϵ. What remains is the pinch term $c(a)c(b)\Phi(\epsilon, \epsilon)$, with

$$\Phi(\epsilon, \mu) = \frac{1}{2\pi i} \int_{A_\epsilon} f(t, \mu) \operatorname{Log} t \, \operatorname{Log}(\epsilon - t)\, dt.$$

The two logarithms are kept apart by treating μ as a separate variable and differentiating in ϵ. That turns $\operatorname{Log}(\epsilon - t)$ into a simple pole at $t = \epsilon$, which the arc can then be pushed across; the

residue is the singular part. Differentiate in ϵ, where the arc can be frozen locally:

$$\partial_\epsilon \Phi = \frac{1}{2\pi i}\int_{A_\epsilon} \frac{f(t,\mu)\operatorname{Log} t}{\epsilon - t}\,dt = f(\epsilon,\mu)\operatorname{Log}\epsilon + h(\epsilon,\mu), \qquad h = \frac{1}{2\pi i}\int_{A^E} \frac{f\operatorname{Log} t}{\epsilon - t}\,dt,$$

where A^E is a fixed polygon from $-ir_0$ to ir_0, east of every point of the disc $|\epsilon| < r$ and off the cut of $\operatorname{Log} t$, for example the one through $-ir_0$, $r_0/2 - ir_0$, $r_0/2 + ir_0$, ir_0, since $r < r_0/4$; it lies in $|t| < 2r_0$. The closed curve $A_\epsilon - A^E$ runs clockwise about $t = \epsilon$, with residue $-f(\epsilon,\mu)\operatorname{Log}\epsilon$, and h is holomorphic. The singular part $f(\epsilon,\mu)\operatorname{Log}\epsilon$ of the derivative is itself the derivative of something explicit. With $\mathcal{P}(\epsilon,\mu) = \int_0^\epsilon f(t,\mu)\,dt$ we have $\partial_\epsilon[\operatorname{Log}\epsilon\cdot\mathcal{P}] = \mathcal{P}/\epsilon + \operatorname{Log}\epsilon\cdot f(\epsilon,\mu)$, and $\mathcal{P}/\epsilon$ is holomorphic, since $\mathcal{P}$ vanishes at $\epsilon = 0$. So $\mathcal{R} = \Phi - \operatorname{Log}\epsilon\cdot\mathcal{P}$ has ϵ-derivative $h - \mathcal{P}/\epsilon$, which is holomorphic jointly in (ϵ,μ) on the whole disc in ϵ. Integrating from a fixed base point ϵ_0 gives $\mathcal{R}(\epsilon,\mu) = \mathcal{R}(\epsilon_0,\mu) + \int_{\epsilon_0}^{\epsilon}(h - \mathcal{P}/\epsilon')\,d\epsilon'$. Here $\mathcal{R}(\epsilon_0,\mu)$ is holomorphic in μ, since $\Phi(\epsilon_0,\mu)$ is an integral over a fixed arc, so $\mathcal{R}$ is holomorphic jointly in (ϵ,μ). Setting $\mu = \epsilon$ gives

$$\Phi(\epsilon,\epsilon) = \operatorname{Log}\epsilon\int_a^{a+\delta}\mathcal{K}(z,v)\,dz + \text{holomorphic} = \operatorname{Log}\epsilon\cdot J_{ab}(v) + \text{holomorphic},$$

since $a + \delta = v - b$. Finally $\operatorname{Log}\epsilon = \operatorname{Log}_\kappa\delta - i\theta$, and J_{ab} is holomorphic. Summing over the collisions gives (12): each ordered pair at x occurs once, a diagonal pair (a,a) being one collision. □

When $B_\infty < 1$, the sign can be checked at $x = 2$ for G, where the only pair is $(1,1)$ with $c(1)^2 = 1$: the leading singular term is $+\delta\log\delta$, which matches the prime number theorem heuristic $\int_{y_0}^\infty e^{-\delta y}y^{-2}\,dy$ for a fixed $y_0 > 0$ (an integer n has on average about $n/\log^2 n$ representations as an ordered sum of two primes, so $G(2+\delta)$ behaves like $\sum_n n^{-1-\delta}\log^{-2} n$; put $n = e^y$). The heuristic serves only as a check of the sign and the normalization; no proof uses it. For S, written in s, with $\alpha = (x-1)/2$, we have $v - x = 2(s-\alpha)$, so $\operatorname{Log}_\kappa(v-x) = \operatorname{Log}_\kappa(s-\alpha) + \log 2$, and $\log 2\cdot X_x(2s+1)$ goes into the holomorphic part. So the local model reads $\operatorname{Log}_\kappa(s-\alpha)X_x(2s+1)$ plus a holomorphic function, and $\frac{d}{ds}X_x(2s+1)$ at α is $\Theta_S(x)$; the factor $\frac{1}{2}$ of (7) cancels against $dv/ds = 2$.

The residue step in the proof of Theorem 6.1 is needed again in Section 9, for a series in which one factor has a pole to begin with: a pole meeting a logarithm leaves the logarithm times the value of the rest of the integrand at the pole; compare the mixed case of Pham's discontinuity formula [30, (Disc 4), p. 357]. We record the form used there.

The calculation is shorter than the one in the proof of Theorem 6.1. There both factors were logarithms: differentiating in ϵ turned the moving logarithm into a pole at $t = \epsilon$, with the auxiliary variable μ keeping the kernel out of the differentiation, and integrating back is what turned the value of the kernel at that pole into the segment integral J_{ab}. For $G_{\log}$ the right factor is Q, so the pole is present from the start: the calculation begins at the residue step, nothing is integrated back, and the coefficient of the logarithm keeps the value of the kernel at the moving pole. This is why $G_{\log}$ is governed by the value of its coefficient at the collision, where G is governed by a derivative; at the collision the coefficient, Y_x of Theorem 9.7, takes the value $Y_x(x) = -\Theta_G(x)$, where $\Theta_G(x)$ is the pair sum that is the residue of T (Theorem 4.2, Lemma 9.9). In the application (Theorem 9.7) the factor $c_1\operatorname{Log} t + E_1(t)$ is the branch P^κ near the left point, the factor $c_2/(\epsilon - t) + E_2(\epsilon - t)$ is Q near its pole at the right point, and f is the kernel in the local coordinates, times the constant e.

As in the proof, t is a local coordinate in which the left point is $t = 0$ and the right point is $t = \epsilon$, and Log is the principal logarithm, with cut $(-\infty, 0]$.

Lemma 6.2 (Pole–logarithm pinch)**.** *Let $r_0 > 0$, let $f(t,\mu)$ be holomorphic on $|t| < 2r_0$, $|\mu| < 2r_0$, let E_1, E_2 be holomorphic on $|t| < 2r_0$, and let $c_1, c_2 \in \mathbb{C}$. There exist an r with*

$0 < r < r_0/4$ and a function H holomorphic on $|\epsilon| < r$ with the following property. Let $|\epsilon| < r$ with $\epsilon \notin (-\infty, 0]$, and let A_ϵ be a piecewise linear path from $-ir_0$ to ir_0 in $|t| < \frac{3}{2}r_0$ that is a graph over $\operatorname{Im} t$, with $\operatorname{Re} t > 0$ at height 0 and $\operatorname{Re} t < \operatorname{Re} \epsilon$ at height $\operatorname{Im} \epsilon$. Then

$$\frac{1}{2\pi i}\int_{A_\epsilon} f(t,\epsilon)\big(c_1 \operatorname{Log} t + E_1(t)\big)\Big(\frac{c_2}{\epsilon - t} + E_2(\epsilon - t)\Big)\,dt = c_1 c_2\, f(\epsilon,\epsilon)\operatorname{Log}\epsilon + H(\epsilon).$$

Proof. Take any r with $0 < r < r_0/4$. Let A^E be the polygon through $-ir_0$, $r_0/2 - ir_0$, $r_0/2 + ir_0$, ir_0. It lies in $|t| < 2r_0$, off the cut of $\operatorname{Log} t$, and east of every point of the disc $|\epsilon| < r$, since $r < r_0/4$. The cut $(-\infty, 0]$ lies west of A_ϵ, because A_ϵ is a graph over height and has $\operatorname{Re} t > 0$ at height 0; and ϵ lies strictly between A_ϵ and A^E on its own height. So the closed curve $A_\epsilon - A^E$ runs clockwise about $t = \epsilon$, encloses no other singularity of the integrands below, and stays in $|t| < 2r_0$; also $|\epsilon - t| < \frac{7}{4}r_0$ for t between the two paths, so $E_2(\epsilon - t)$ is defined there.

Expand the product into four terms. The term $c_1 c_2 f(t,\epsilon)\operatorname{Log} t/(\epsilon - t)$ has a simple pole at $t = \epsilon$ with residue $-c_1 c_2 f(\epsilon,\epsilon)\operatorname{Log}\epsilon$ and no other singularity between the paths, so by the residue theorem its integral is $c_1 c_2 f(\epsilon,\epsilon)\operatorname{Log}\epsilon$ plus the same integral over A^E; the latter is holomorphic in ϵ, because A^E is fixed and stays at positive distance from the disc. The term $c_2 f(t,\epsilon)E_1(t)/(\epsilon - t)$ has a pole at $t = \epsilon$ and no other singularity; the same argument gives $c_2 f(\epsilon,\epsilon)E_1(\epsilon)$ plus an integral over A^E, both holomorphic in ϵ. In the terms $c_1 f(t,\epsilon)\operatorname{Log} t\, E_2(\epsilon - t)$ and $f(t,\epsilon)E_1(t)E_2(\epsilon - t)$ the integrand is holomorphic between the two paths, so A_ϵ may be replaced by A^E, and the integrals are holomorphic in ϵ. None of these holomorphic terms depends on A_ϵ, so their sum is the function H. □

7. Rotation and every reached branch

A sheet is one branch. This section finds which germs are reached by continuing the series along paths, and what each of them looks like at a candidate point. To reach them all we rotate the direction. Fix a point v_1 and turn κ. Call a direction *critical* for v_1 if the eastward ray from v_1 in that direction passes through a point of $\mathcal{X}$, which is to say that v_1 lies on a cut L_x for that direction. Between critical directions nothing changes, and at a critical direction the sheet jumps by the functions M_x of the points on the critical ray. Those functions are holomorphic on all of $\mathcal{V}$, so they change no local coefficient. Hence the coefficient of the logarithm at x is the same on every branch. The local jump is the pinch monodromy of [30, p. 356] and [27, 28]; here the jumps at all the points swept over are collected in one identity, Theorem 7.3.

The first lemma is the easy half: as long as no cut sweeps over v_1, the sheet at v_1 does not depend on the direction, because every sheet is the series on $\operatorname{Re} v > 2$ and v_1 is joined to that region.

Lemma 7.1. *If $v_1 \in \Omega_\kappa$ for all κ in an interval, the germ of D^κ at v_1 is constant on it.*

Proof. If $\operatorname{Re} v_1 > 2$ then $D^\kappa = D$ near v_1 by Theorem 5.3. Otherwise fix κ_0. The segment from v_1 in the direction of κ_0 to $\operatorname{Re} v = \frac{5}{2}$ contains no point of $\mathcal{X}$. Only finitely many x have a cut with direction near κ_0 meeting a neighbourhood of it, and each has positive distance from it. So there is a convex tube about the segment inside Ω_κ for κ near κ_0. It meets $\operatorname{Re} v > 2$, where all these sheets equal D, and the identity theorem finishes. □

The second lemma computes the jump when a cut does sweep over v_1. The two sheets are compared along the segment from v_1 east to the region where both are the series; they agree there, and walking west, each point x_i on the segment adds M_{x_i}, because the two logarithms of Theorem 6.1 at x_i differ by $2\pi i$ west of x_i and by nothing east of it.

Lemma 7.2. *Let $v_1 \in \mathcal{V}_\infty \setminus \mathcal{X}$, and let $x_1, \dots, x_k$ be the points of $\mathcal{X}$ on the open eastward ray from v_1 in the direction of slope κ. For generic κ^-, κ^+ with $\kappa^- < \kappa < \kappa^+$, close to κ, as germs at v_1,*

$$D^{\kappa^+} - D^{\kappa^-} = \sum_{i \le k} M_{x_i}.$$

Proof. Take a thin tube about the segment from v_1 to $\operatorname{Re} v = \frac{5}{2}$, in which only the slanted cuts of the x_i occur, meeting the segment only at the x_i. Let $D_* = D^{\kappa^+} - D^{\kappa^-}$; it is holomorphic near each open piece of the segment minus the x_i, and it is 0 near the east end. Label the x_i from east to west. At x_j, Theorem 6.1 gives $D^{\kappa^\pm} = \operatorname{Log}_{\kappa^\pm}(v - x_j)X_{x_j} + H^\pm$ on a disc. The two logarithms agree on the sector containing the east radius, and differ by exactly $2\pi i$ on the sector containing the west radius, the one for κ^+ being the larger. Hence $H^+ - H^- = D_*$ east of x_j, and D_* west of x_j is $(H^+ - H^-) + M_{x_j}$. By induction, D_* east of x_j is the sum of the M_{x_i} with $i < j$, which is holomorphic on $\mathcal{V}$, so $H^+ - H^-$ equals it on the whole disc and the step closes. □

Nothing later depends on the sign of this jump. All that is used is that the difference of two sheets is an integer combination of the M_x.

Theorem 7.3 (Rotation)**.** *If $v_1 \in \Omega_\kappa \cap \Omega_{\kappa'}$ with $\kappa < \kappa'$, then as germs at v_1,*

$$D^{\kappa'} - D^{\kappa} = \sum_{x} M_x,$$

the finite sum over $x \in \mathcal{X}$ in the open eastward sector at v_1 between the two directions.

Proof. Finitely many critical slopes occur between κ and κ'. Apply Lemma 7.1 between them and Lemma 7.2 at them. □

Let $\mathcal{M}$ be the set of finite integer combinations of the M_x; its members are holomorphic on $\mathcal{V}$. For $v_1 \in \mathcal{V}_\infty \setminus \mathcal{X}$ all but countably many κ have $v_1 \in \Omega_\kappa$, and by Theorem 7.3 the family of germs $D^\kappa + m + C$, with $m \in \mathcal{M}$ and $C \in \mathbb{C}$ a constant, does not depend on which such κ is used. This family is the candidate for the set of all branches, and the next theorem shows that continuation along paths never leaves it: the family contains every reached germ. We do not claim the converse, that every $m \in \mathcal{M}$ and every constant C is reached from the series, so the family does not classify the monodromy that occurs. The constant C is needed only for S, and only because a path for S may leave the region where the sheets are defined; for G it can be taken to be 0.

Theorem 7.4 (Every branch)**.** (a) *A germ $D^\kappa + m + C$ continues along every path in $\mathcal{V}_\infty \setminus \mathcal{X}$, staying in this family with the same C. In particular G continues along every path in $\{\operatorname{Re} v > 2B_\infty\} \setminus \Pi$ starting in $\operatorname{Re} v > 2$, and each reached germ has the form $D^\kappa + m$.*

(b) *For S, let a germ at a point of $\mathcal{V}_\infty \setminus \mathcal{X}$ be reached from $\operatorname{Re} v > 3$ along a path in $\{\operatorname{Re} v > 2B_\infty\} \setminus \Pi$ along which continuation exists. Then it has the form $D^\kappa + m + C$.*

(c) *Let $x \in \mathcal{X}$ and let κ be generic. There is a disc B_x about x, depending only on x and κ, such that every germ of this family at a point of $B_x \setminus L_x$ is $\operatorname{Log}_\kappa(v - x)X_x$ plus a function holomorphic on B_x. So every germ reached as in (a) or (b) at a point of $B_x \setminus L_x$ has the same coefficient X_x.*

Proof. (a) For p on the path choose κ with $p \in \Omega_\kappa$ and a disc about p in Ω_κ. A germ of the family at a point of the disc has this form with this κ, by Theorem 7.3, and is holomorphic on the disc. Compactness of the path finishes.

(b) A path for S may leave $\mathcal{V}$ on the right, into $\operatorname{Re} v \ge 3$, where the formula (11) is not available, and come back; these excursions are where the constant C comes from. Out there

the series itself causes no trouble: D is given by its convergent series, which is holomorphic for $\operatorname{Re} v > 2$. So the only thing to follow through an excursion is the correction $m + C$ that the germ carries when it leaves, and the way to follow it is through its derivative. The derivative of the correction extends to a single-valued meromorphic function on the whole plane, and the continued correction keeps that derivative all along the excursion. So the correction that comes back can differ from the one that left only by a constant. To see the derivative explicitly, pass to the variable $s = (v-1)/2$. The substitution $w = (2z - v + 3)/4$ in (5) gives $J_{ab}(2s+1) = \int_{(a-s+1)/2}^{(s+2-b)/2} K(w)\,dw$, in which s occurs only in the endpoints. Differentiating gives

$$\frac{d}{ds} J_{ab}(2s+1) = \tfrac{1}{2} K\big(\tfrac{s+2-b}{2}\big) + \tfrac{1}{2} K\big(\tfrac{a-s+1}{2}\big).$$

Both terms are meromorphic on the whole plane, which is the fact the argument rests on. The formula also shows how a constant can arise, and why a path for S may have no continuation at all. Its term $\frac{1}{2}K(\frac{a-s+1}{2})$ has a simple pole at $s = a+1$ with residue -1, a point with $\operatorname{Re} v > 3$, and its other term has a pole of residue -1 at $s = b+2$, also beyond $\mathcal{V}$. If the derivative of the correction $m + C$ of a germ $D + m + C$ has a pole there with nonzero residue, a loop round the point adds $2\pi i$ times that residue to the germ, and a path through the point has no continuation; if the terms with a pole there cancel, neither happens. That is why admissibility for S asks for a path "along which continuation exists", while for G, whose functions J_{ab} are holomorphic on the whole half-plane, it need not.

The constant is explicit: a term ωJ_{ab} of the correction gives $-2\pi i\omega$ on a positive loop round $s = b+2$, so, when $B_\infty < 1$, the germ $D + M_2$, where $M_2 = 2\pi i J_{11}$ is M_x for $x = 2$ (the point $s = \frac{1}{2}$), which one positive loop round $s = \frac{1}{2}$ produces, gains $4\pi^2$ on a positive loop round $s = 3$, and likewise round $s = a + 1 = 2$.

On an excursion, D is still given by its convergent series, which is holomorphic for $\operatorname{Re} v > 2$, and what we track is the continuation of the correction; the question is what has become of the germ when the path re-enters $\mathcal{V}$. The germ leaves $\mathcal{V}$ as $D + h$ with $h = m + C$, and h continues along the excursion as some holomorphic function $\tilde{h}$. We do not need its values, only its derivative: m' is a finite combination of explicit functions that are meromorphic on the whole plane, so $\tilde{h}'$ is pinned down along the entire excursion, and on re-entry $\tilde{h}$ can differ from h only by a constant.

The details follow. Write $\mathsf{R} = \{\operatorname{Re} v > 2\}$ and $\mathsf{V} = \mathcal{V}_\infty$. They cover $\{\operatorname{Re} v > 2B_\infty\}$, and $\mathsf{R} \cap \mathsf{V}$ lies in every Ω_κ, since all points of $\mathcal{X}$ have $\operatorname{Re} x \le 2$. On R the series D is holomorphic. A finite subdivision of the parameter interval, subordinate to the inverse images of these two open sets, puts each closed subpath in one of them. Consecutive pieces assigned to the same set can be merged, so switches occur in $\mathsf{R} \cap \mathsf{V}$. On the first R-piece the continued germ is D, and we initialize $\tilde{h} = 0$.

The invariant on each V-piece is the family $D^\kappa + m + C$. Part (a) preserves it there. At a switch into R, all sheets equal D, so the entering germ is $D + h$ with $h = m + C$ on the overlap. Along the ensuing R-piece subtract D from the continued representatives; this gives a holomorphic continuation $\tilde{h}$ of h. The derivative h' extends meromorphically to $\mathbb{C}$. To see this explicitly, return to the variable s, with $v = 2s + 1$, and to the formula for $\frac{d}{ds} J_{ab}(2s+1)$ displayed at the start of (b). Both of its terms are meromorphic on $\mathbb{C}$, and m' is a finite integer combination of such functions times constants. So $h' = m'$, the derivative in v, has a meromorphic extension, say φ. Here $m'(v) = \frac{1}{2}\frac{d}{ds} m(2s+1)$ at $s = (v-1)/2$.

Choose an ordered finite chain of continuation discs along this R-piece, with representatives of $\tilde{h}$ agreeing on successive overlaps. On the first disc $\tilde{h}' = \varphi$ near the switch, hence on the disc by the meromorphic identity theorem. On the next disc the same equality holds on the overlap and therefore throughout that disc. Induction gives the equality on the whole chain. Any apparent

pole of φ encountered in a disc is removable, since there φ agrees meromorphically with the holomorphic function $\tilde{h}'$. On re-entry into V, both $\tilde{h}$ and h are holomorphic near the switch and have equal derivatives. Thus $\tilde{h} - h$ is a constant there, so the entering germ again has the form $D^\kappa + m + C'$ for some constant C'. Induction over the finite subdivision proves (b).

(c) Take for B_x the disc of Theorem 6.1. At a point of $B_x \setminus L_x$, which lies in Ω_κ, Theorem 7.3 writes the germ with this κ, and m and C are holomorphic on B_x. □

So the coefficient of the logarithm at a candidate point does not change from branch to branch, among the germs reached as in (a) or (b).

8. The boundary for G and S

We now combine the two jobs and prove Theorem 1.2. Theorem 3.1 gives points x near any point of the line with $X_x'(x) \neq 0$, so $X_x \not\equiv 0$. By Theorem 7.4(c) every branch near such x has non-trivial monodromy. A function meromorphic across the line would be single-valued there. The lemma is the obstruction in its local form; the proof of the theorem then consists in steering an admissible germ to such a point without leaving the domain where the proposed extension is analytic.

Lemma 8.1. *Let $x \in \mathcal{X} \cap \mathcal{V}$ with $X_x'(x) \neq 0$, κ generic, and B_x the disc of Theorem 6.1. No germ $D^\kappa + m + C$ at a point q of $B_x \setminus L_x$ agrees at q with a function f analytic on a punctured disc about x containing q.*

Proof. Replace the larger of B_x and the disc of f by the smaller; the base point q lies in both. The slit punctured disc is connected and contains q, so if f were such a function then by the identity theorem $f = \mathrm{Log}_\kappa(v - x)X_x + H_1$ on it, with H_1 holomorphic on the disc, by Theorem 7.4(c). Now f and H_1 are continuous across the slit and Log_κ jumps by $2\pi i$, so $X_x = 0$ on a segment, hence $X_x \equiv 0$ near x. That contradicts $X_x'(x) \neq 0$. □

Admissible germs of S exist. For G, continuation exists along every path in $\{\mathrm{Re}\, s > 1\} \setminus \mathcal{C}_G$ starting in $\mathrm{Re}\, s > 2$, by Theorem 7.4(a). For S, admissibility asks for a path along which continuation exists, and the statement for S in Theorem 1.2 would be vacuous if there were none. Under (H_∞), an admissible germ of S exists at every point s_1 of $\{\mathrm{Re}\, s > 0\} \setminus \mathcal{C}_S$, and by definition only there. In particular every disc $|s - iy| < r$ contains such a point, since its part in $\mathrm{Re}\, s > 0$ is open and nonempty and $\mathcal{C}_S$ is closed and discrete in $\mathrm{Re}\, s > 0$ by Lemma 2.3.

To see this, let $\gamma(t) = s_1 + t(1 + i\kappa)$ for $0 \le t \le 1$, with a slope κ chosen below; then $\mathrm{Re}\,\gamma(1) > 1$ and $\mathrm{Re}\,\gamma(t) \ge \mathrm{Re}\, s_1 > 0$. If $\mathrm{Re}\, s_1 > \frac{1}{2}$, the series S is holomorphic along γ, which lies in $\mathrm{Re}\, s > \frac{1}{2}$ and so misses $\mathcal{C}_S$, and its germ at s_1 is admissible. So let $0 < \mathrm{Re}\, s_1 \le \frac{1}{2}$ and $v_1 = 2s_1 + 1$, so that $1 < \mathrm{Re}\, v_1 \le 2$ and $v_1 \notin \Pi$. Under (H_∞), $\mathcal{V}_\infty = \mathcal{V} = \{1 < \mathrm{Re}\, v < 3\}$, so $v_1 \in \mathcal{V}_\infty \setminus \mathcal{X}$. Each $x \in \mathcal{X}$ lies on the eastward ray from v_1 for at most one slope, and $\mathcal{X}$ is countable, so all but countably many κ put v_1 on no cut L_x; fix such a κ. The points $2\gamma(t) + 1 = v_1 + 2t(1 + i\kappa)$ run east along the κ-line through v_1. Those with $\mathrm{Re}\, v \le \frac{5}{2}$ lie in Ω_κ, since a cut through one of them would contain v_1. By Theorem 5.3, $D^\kappa(2s + 1)$ is holomorphic near each point $\gamma(t)$ with $\mathrm{Re}(2\gamma(t) + 1) \le \frac{5}{2}$, and it equals $S(s)$ wherever $2 < \mathrm{Re}(2s + 1) < 3$, a strip that no cut meets because $\mathrm{Re}\, x \le 2$ on $\mathcal{X}$. The series S is holomorphic near each $\gamma(t)$ with $\mathrm{Re}\,\gamma(t) > \frac{1}{2}$. So continuing S along γ from $\gamma(1)$ back to s_1 is possible, and ends with the germ of $D^\kappa(2s + 1)$ at s_1. The path misses $\mathcal{C}_S$: its points with $\mathrm{Re}(2s + 1) \le 2$ map into Ω_κ, which contains no point of $\mathcal{X}$, and its other points map to $\mathrm{Re}\, v > 2$, which contains no point of Π. So the germ is admissible. The Lean declaration for this statement is named in Section 11.

Proof of Theorem 1.2. We treat S, which carries the change of variable $v = 2s + 1$; the changes for G are listed at the end. Under (H_∞), $\mathcal{V}_\infty = \mathcal{V}$ and $B < 1$. Suppose f is meromorphic on

$\Delta = \{|s - iy| < r\}$ with germ g at s_1, and let E be its pole set. Choose $0 < \varepsilon' < r/2$. Apply Theorem 3.1 at $1 + 2iy$ with $2\varepsilon'$ in place of ε: there is $x = 1 + u$ with $\Theta_S(x) \neq 0$ and $\alpha = u/2$ in $|s - iy| < \varepsilon'$, $\operatorname{Re}\alpha > 0$. Then $X'_x(x) = \frac{1}{2}\Theta_S(x) \neq 0$.

Fix a generic κ and the disc of Theorem 6.1 at x, which in the variable s is a disc Δ_α about α, taken inside $\Delta \cap \{0 < \operatorname{Re} s < 1\}$. Take a punctured disc about α in Δ_α free of E and of $\mathcal{C}_S$, and a point s_2 in it off the cut. $\mathcal{C}_S$ is closed and discrete in $\operatorname{Re} s > 0$ by Lemma 2.3. So there is a path from s_1 to s_2 in $(\Delta \cap \{\operatorname{Re} s > 0\}) \setminus (E \cup \mathcal{C}_S)$, along which g continues as f; hence the germ of f at s_2 is admissible. By Theorem 7.4(b), it has the form $D^\kappa + m + C$. And f is analytic on the punctured disc about α, which contradicts Lemma 8.1.

For G the argument is the same with $v = s$: the disc is $|s - (1 + iy)| < r$, Theorem 3.1 is applied at $1 + iy$ with ε' in place of ε, α is x itself, with $X'_x(x) = \Theta_G(x)$, the strip $0 < \operatorname{Re} s < 1$ and the half-plane $\operatorname{Re} s > 0$ both become $\operatorname{Re} s > 1$, $\mathcal{C}_S$ becomes $\mathcal{C}_G$, the condition $\operatorname{Re}\alpha > 0$ becomes $\operatorname{Re} x > 1$, and Theorem 7.4(a) replaces (b), since there are no excursions. □

The theorem restricts the path to avoid the catalogue. That restriction can be removed from both the boundary statement (Theorem 8.2) and the description of branches at endpoints off the catalogue (Corollary 8.3). The method is the same in both: a path along which the function continues is replaced, piece by piece inside the discs of a continuation chain, by a path that avoids the catalogue. In Theorem 8.2(a) the new path ends at a nearby point off the catalogue, carrying the germ of the same representative there, which is admissible, so that Theorem 1.2 applies to it; for the corollary the endpoint is already off the catalogue, and the new path keeps it and reaches the same terminal germ. In Theorem 8.2(b), where a meromorphic function is given on a connected open set, the path is perturbed so that it also avoids the poles of that function. The further conclusion of Theorem 10.1(a), that a path of continuation never meets the catalogue, is proved here only under RH and only for G: under (H_∞) we do not exclude a point x where $X_x \equiv 0$, on which a continuation might be holomorphic and through which a path might pass.

Theorem 8.2. *Assume* (H_∞).

(a) *Let γ be a continuous path in $\mathbb{C}$ starting in $\operatorname{Re} s > 2$, along which G continues analytically. Then γ stays in $\operatorname{Re} s > 1$. If g is the terminal germ and $\gamma(1)$ lies in a disc centred on $\operatorname{Re} s = 1$, then g does not extend meromorphically to that disc. The same holds for S, with start in $\operatorname{Re} s > 1$, staying in $\operatorname{Re} s > 0$, and discs centred on $\operatorname{Re} s = 0$. No condition is placed on the endpoint, which may be in the catalogue.*

(b) *Let g be an admissible germ at s_1, of G or of S. No connected open set containing s_1 and meeting the boundary line carries a meromorphic function with germ g at s_1.*

Proof. (a) We treat G. First let γ stay in $\operatorname{Re} s > 1$. Choose a partition $0 = t_0 < \cdots < t_N = 1$ and discs $U_j \subset \{\operatorname{Re} s > 1\}$, with holomorphic representatives f_j, such that $\gamma([t_{j-1}, t_j]) \subset U_j$ and f_j, f_{j+1} agree on a neighbourhood of $\gamma(t_j)$. These are an ordered continuation chain for the given continuation. The last representative has germ g at $\gamma(1)$. For each $j < N$, choose a small disc $U'_j \subset U_j \cap U_{j+1}$ about $\gamma(t_j)$ on which $f_j = f_{j+1}$, and a junction $q_j \in U'_j \setminus \mathcal{C}_G$. Keep $q_0 = \gamma(0)$, which is outside $\mathcal{C}_G$ since $\operatorname{Re} q_0 > 2$. Choose $q_N = s_2$ outside $\mathcal{C}_G$ in a small neighbourhood of $\gamma(1)$ contained in U_N and in the given disc about the line. It can also be taken in the neighbourhood where a proposed extension agrees with g.

For each j, join q_{j-1} to q_j inside $U_j \setminus \mathcal{C}_G$. This is possible because a disc minus a relatively closed discrete set is path-connected: a compact polygonal arc can be detoured around the finitely many removed points it meets. On this piece use f_j. At q_j the representatives agree by the choice of U'_j, so the concatenated path reproduces the specified continuation, ending with the

germ of f_N at s_2. This germ is admissible. A meromorphic extension of g to the boundary disc would agree with f_N near s_2, hence extend that admissible germ, contradicting Theorem 1.2.

Now suppose γ leaves $\operatorname{Re} s > 1$, and let t^* be the first time with $\operatorname{Re}\gamma(t^*) = 1$. The germ at $\gamma(t^*)$ is holomorphic on a disc about that point, which is centred on the line. For $t < t^*$ close to t^*, $\gamma(t)$ lies in the disc and the germ there extends to it. The first part, applied to γ up to t, forbids this.

For S nothing changes except the start, the catalogue and the line: $\gamma(0)$ has $\operatorname{Re} s > 1$, so it lies off $\mathcal{C}_S$, which is in $\operatorname{Re} s \le \frac{1}{2}$; the half-plane is $\operatorname{Re} s > 0$ and the boundary line is $\operatorname{Re} s = 0$.

(b) Let f be such a function on W. Take a path γ in W from s_1 to a point of the line, and let t^* be its first contact with the line, at s_0. Choose a disc B_0 about s_0 whose closure is in W. Stop at a parameter $t < t^*$ sufficiently close to t^* that $\gamma(t)$ lies in B_0. Choose a small disc about $\gamma(t)$ inside B_0 and the half-plane $\operatorname{Re} s > 1$ ($\operatorname{Re} s > 0$ for S), choose q in it outside the pole set of f and the catalogue, and append a segment to q. Only the compact initial subpath ending at q is now used. It is contained in the open half-plane, where the catalogue is relatively closed and discrete; the poles of f are also relatively closed and discrete in W. Cover this compact subpath by finitely many discs in their common domain and perturb it inside their union, keeping s_1 fixed and its endpoint in B_0, to avoid both discrete sets; since f is single-valued on W, no homotopy argument is needed. The germ of f at s_1 is g, so following f on the perturbed path makes its terminal germ admissible. That germ extends meromorphically to B_0 as f, contrary to Theorem 1.2. The perturbation stops strictly before contact with the line. □

Corollary 8.3. *Assume* (H_∞). *Let γ be a continuous path in $\mathbb{C}$ starting in $\operatorname{Re} s > 2$, along which G continues analytically, with $\gamma(1) \notin \mathcal{C}_G$. Then the terminal germ is admissible. The same holds for S, with start in $\operatorname{Re} s > 1$ and $\gamma(1) \notin \mathcal{C}_S$.*

Proof. By Theorem 8.2(a), γ stays in $\operatorname{Re} s > 1$, or in $\operatorname{Re} s > 0$ for S. Run the chain construction from the first part of that proof with $q_N = \gamma(1)$, which is off the catalogue. The concatenated path avoids the catalogue and ends with the terminal germ. □

So by Theorem 7.4, every germ reached along any path and ending off the catalogue is $D^\kappa + m$ for G, and $D^\kappa + m + C$ for S at endpoints with $0 < \operatorname{Re} s < 1$. Near a candidate point it has the coefficient X_x of Theorem 7.4(c).

9. Two more kernels

Sections 5–8 used three things about G and S: the kernel facts (K1)–(K3), the local calculation at a pinch of two logarithms (Theorem 6.1), and the non-vanishing of the derivative $X'_x(x)$ at dense points, which Section 3 supplies. This section proves Theorem 1.3. Each of its two series changes one of these layers and leaves the rest of the argument as it stands, so the section shows which parts of the method depend on what. Everything in this section before the proofs of Theorem 1.3 holds without (H_∞) (Lemma 9.3 and Theorem 9.4 assume $B < 1$); those proofs use it.

For G_2 the kernel is different, but Proposition 9.1 shows that it has the properties that Sections 5–8 use, on a half-plane, as for G. So the change is in the arithmetic: $X'_x(x)$ becomes $\frac{1}{2}\sum c(a)c(b)\mathrm{B}(a/2, b/2)$, and the argument of Section 3 has to be run again for this sum, with a new main term and a new far-pair bound (Section 9.1).

For $G_{\log}$ the local calculation changes. The weight $\log q$ turns one copy of P into $Q = -P'$, which has poles where P has logarithms. At a pinch a logarithm now meets a pole, and Lemma 6.2 takes the place of the log–log calculation. The coefficient of the logarithm is then a sum of values of the kernel, one for each ordered collision, weighted by $-c(a)c(b)$, not of integrals of it. At $s = x$, Lemma 9.9 gives the negative of the sum $\Theta_G(x)$ of Theorem 3.1. Under $B < 1$,

that theorem supplies dense points where the sum is nonzero, and under (H_∞) these points lie in $\mathcal{X}$. So no new arithmetic is needed (Section 9.2).

9.1. **The squares.** Put $v = 2s$ and $D(v) = G_2(v/2) = \sum_{p,q}(p^2+q^2)^{-v/2}$, absolutely convergent for $\operatorname{Re} v > 2$. The variable is $v = 2s$ because the Mellin–Barnes formula, applied to $\big(1+(p/q)^2\big)^{-s}$, splits the total exponent $2s$ of the primes between the two copies of P; so the pinch still occurs at $v = a + b$, and the line $\operatorname{Re} v = 1$ corresponds to $\operatorname{Re} s = \frac{1}{2}$. In this subsection $\mathcal{K}$, D, J_{ab}, X_x and D^κ are the objects of (5), (6) and (11) for the kernel of the next proposition.

Proposition 9.1. *Let* $\mathcal{V} = \{\operatorname{Re} v > 1\}$ *and*

$$\mathcal{K}(z,v) = \frac{\Gamma(z/2)\,\Gamma\big((v-z)/2\big)}{2\,\Gamma(v/2)}.$$

Then (K1) *and* (K2) *hold, and for* $2 < \operatorname{Re} v < 3$ *and* $1 < c < \operatorname{Re} v - 1$,

$$D(v) = \frac{1}{2\pi i}\int_{(c)} \mathcal{K}(z,v)P(z)P(v-z)\,dz.$$

Proof. (K1): $\Gamma(z/2)$ is holomorphic for $\operatorname{Re} z > 0$, $\Gamma((v-z)/2)$ for $\operatorname{Re} z < \operatorname{Re} v$, and $1/\Gamma$ is entire. (K2): let v range over a compact subset of $\mathcal{V}$ and $\operatorname{Re} z$ over a compact subset of $(0, \operatorname{Re} v)$, and write $t = \operatorname{Im} z$. By (Z6), applied to $\Gamma(z/2)$, of imaginary part $t/2$, and to $\Gamma((v-z)/2)$, of imaginary part $(\operatorname{Im} v - t)/2$, we get $|\Gamma(z/2)\Gamma((v-z)/2)| \ll (1+|t|)^A e^{-\pi|t|/2}$, while $1/\Gamma(v/2)$ is bounded on the compact set. This is the rate of S, half that of G. For the integral, the case $x = (p/q)^2$ of the Mellin–Barnes formula [6, (1.4), p. 2], $(1+x)^{-s} = \frac{1}{2\pi i}\int_{(d)} K_s(w)x^{-w}\,dw$ with $0 < d < \operatorname{Re} s$, gives

$$(p^2+q^2)^{-s} = q^{-2s}\big(1+(p/q)^2\big)^{-s} = \frac{1}{2\pi i}\int_{(d)} K_s(w)\,p^{-2w}q^{-2s+2w}\,dw.$$

The change of variable $z = 2w$, $dw = \frac{1}{2}\,dz$, with $v = 2s$, turns $p^{-2w}q^{-2s+2w}$ into $p^{-z}q^{-(v-z)}$ and $\frac{1}{2}K_{v/2}(z/2)$ into $\mathcal{K}(z,v)$; the factor $\frac{1}{2}$ comes from dw, as for S. With $c = 2d$, the majorant $P(c)P(\operatorname{Re} v - c)$ times the bound (K2) justifies Fubini as in Section 2. □

The proof gives the integral formula for every $\operatorname{Re} v > 2$; the proposition states it on the strip $2 < \operatorname{Re} v < 3$, which is what the formal proof records.

The proposition puts G_2 in the setting of Sections 5–8. The sheet, local-form, rotation and obstruction arguments of those sections use nothing about the kernel beyond (K1)–(K3), and the domain $\mathcal{V}$ is a half-plane, as for G, not a strip, as for S. The excursion argument of Theorem 7.4(b), which produced the constant C for S, was needed only because a path for S can leave its strip; a path in $\operatorname{Re} v > 2B_\infty$ never leaves this $\mathcal{V}$. On the strip, the last step of the proof of Theorem 5.3 gives $D^\kappa = D$; both sides are holomorphic on $\Omega_\kappa \cap \{\operatorname{Re} v > 2\}$, which is the whole half-plane $\operatorname{Re} v > 2$, since no cut L_x reaches it ($\operatorname{Re} x \le 2$ on $\mathcal{X}$ and L_x runs west from x); so $D^\kappa = D$ on all of it by the identity theorem.

So Theorems 5.3, 6.1, 7.3, 7.4(a),(c) and Lemma 8.1 hold for this D, with $C = 0$: D continues along every path in $\{\operatorname{Re} v > 2B_\infty\} \setminus \Pi$ starting in $\operatorname{Re} v > 2$, every germ reached that way is $D^\kappa + m$ with $m \in \mathcal{M}$, and near $x \in \mathcal{X}$, for generic κ and off the slit L_x, every such germ is $\operatorname{Log}_\kappa(v-x)X_x(v)$ plus a holomorphic function.

The candidate points in the variable v are the points of $\mathcal{X}$, as for G; in the variable s they are the points $x/2$, and under (H_∞), where $2B_\infty = 1$, they form the catalogue $\mathcal{C}_2$ of Theorem 1.3.

The candidate points and the near-pair count do not depend on the kernel. The coefficient is the one place where the kernel enters the arithmetic, so to use that count for G_2 we first identify the new sum over pairs and the contribution of its two main pairs.

Lemma 9.2. *For $x \in \mathcal{X} \cap \mathcal{V}$, with the derivative taken in v,*

$$X_x'(x) = \tfrac{1}{2}\,\Theta_2(x), \qquad \Theta_2(x) = \sum_{a+b=x} c(a)c(b)\,\mathrm{B}\big(\tfrac{a}{2},\tfrac{b}{2}\big).$$

At $x = 1+u$ with $u \neq 1$ the pairs $(1,u)$ and $(u,1)$ contribute $-2c(u)\mathrm{B}(\frac{1}{2},\frac{u}{2})$ to $\Theta_2(x)$, and $\mathrm{B}(\frac{1}{2},\frac{u}{2}) \neq 0$ for non-real u.

Proof. As in (7), $\partial_v J_{ab}(a+b) = \mathcal{K}(a,a+b) = \Gamma(a/2)\Gamma(b/2)/(2\Gamma((a+b)/2)) = \frac{1}{2}\mathrm{B}(a/2,b/2)$. The main pairs give $c(1)c(u)\{\mathrm{B}(\frac{1}{2},\frac{u}{2}) + \mathrm{B}(\frac{u}{2},\frac{1}{2})\} = -2c(u)\mathrm{B}(\frac{1}{2},\frac{u}{2})$, since $c(1) = -1$. Finally $\mathrm{B}(\frac{1}{2},\frac{u}{2}) = \Gamma(\frac{1}{2})\Gamma(\frac{u}{2})/\Gamma(\frac{1+u}{2})$; Γ has no zeros, and $u/2$ and $(1+u)/2$ are not real, so not poles of Γ. □

In the variable $s = v/2$ the derivative of $X_x(2s)$ at $x/2$ is $\Theta_2(x)$ itself: the factor $\frac{1}{2}$ cancels against $dv/ds = 2$, as for S. For S the non-vanishing of the main-term bracket needed the cotangent argument of Lemma 2.4; here it is immediate, because the main term is a single Beta value.

The far pairs are bounded as in Lemma 3.4, at the rate of S. The main term will again have size at least a constant times $1/X$, and the next lemma shows that the slower decay of this kernel still puts the sum over the far pairs below that scale at the same cutoff $H = 2\log X$.

Lemma 9.3. *Assume $B < 1$ and let $Y > 0$. There is $H_0 \geq 2Y+2$ such that for all u with $0 < \operatorname{Re} u \leq 1$, $|\operatorname{Im} u| \leq Y$, and all $H \geq H_0$,*

$$\sum_{\substack{a+b=1+u,\ a,b\neq 1\\ |\operatorname{Im} a|>H}} \big|c(a)c(b)\mathrm{B}\big(\tfrac{a}{2},\tfrac{b}{2}\big)\big| \leq 2e^{-H}.$$

Proof. As in the proof of Lemma 3.4, the extra pairs have $\operatorname{Re} a, \operatorname{Re} b \in [1-B, B]$, so $a/2$ and $b/2$ have real parts in $[\frac{1-B}{2}, \frac{B}{2}] \subset (0,1)$; with $t = \operatorname{Im} a$, $|\operatorname{Im} b| \geq |t| - Y$, and $|c(a)c(b)| \ll \log^2(|t|+Y+3)$, with $O(\log(j+4))$ points a for $|\operatorname{Im} a| \in [j, j+1]$. By (Z6), $|\Gamma(a/2)\Gamma(b/2)| \ll (1+|t|+Y)^2 e^{-\pi(2|t|-Y)/4}$. And $1/\Gamma$ is bounded on the compact set of points $(1+u)/2$, which have real part in $[\frac{1}{2}, 1]$, since Γ has no zeros. So the terms are $\ll (1+|t|)^4 e^{-\pi|t|/2}$, the rate of S. Summing over $|t| > H$ gives $\ll (H+3)^5 e^{-\pi H/2} \leq 2e^{-H}$ for large H. □

The factor 2 has no significance. The formal proof bounds by e^{-H} the sum of the absolute values of the terms $\frac{1}{2}c(a)c(b)\mathrm{B}(\frac{a}{2},\frac{b}{2})$ over the same far extra pairs, which for $x \in \mathcal{X}\cap\mathcal{V}$ are terms of $X_x'(x)$ (Lemma 9.2), and that is the same statement.

Only finitely many pairs with $a+b = 1+u$ have $|\operatorname{Im} a| \leq H$: the pairs $(1,u)$ and $(u,1)$, and pairs with $a, b \neq 1$, for which $\operatorname{Re} a > 1-B$ as in the proof of Theorem 3.1, so that a is $1/k$ or ρ/n with $n \leq \lfloor 1/(1-B)\rfloor$ and $|\operatorname{Im}\rho| \leq nH$. So Lemma 9.3 shows that $\Theta_2(1+u)$ converges absolutely. The main term of Lemma 9.2 and the far-pair bound of Lemma 9.3 are the two inputs the counting argument of Section 3 needs, so that argument applies to Θ_2 and gives a dense set of points where it does not vanish.

Theorem 9.4. *Assume $B < 1$. Let $y \in \mathbb{R}$ and $\varepsilon > 0$. The disc $|x - (1+iy)| < \varepsilon$ contains infinitely many points $x = 1 + \rho/q$, with $\rho \in \mathcal{Z}$ on the critical line and q prime, at which $\Theta_2(x) \neq 0$.*

The theorem does not require the points to be chosen together with those of Theorem 3.1; the two families are found separately.

Proof. Follow the proof of Theorem 3.1 with Θ_2 in place of Θ_G. The reduction to $y > 0$ and $0 < \varepsilon < \min(y, 1)$ is the same, since $c(\bar{a}) = c(a)$ and $\overline{\mathrm{B}(\bar{a}/2, \bar{b}/2)} = \mathrm{B}(a/2, b/2)$. The good pairs

and the near-bad pairs depend only on the points, not on the kernel, so Theorem 3.3 and the near-pair count apply unchanged. By Lemma 9.2 the main term is $-2c(u)\mathrm{B}(\frac{1}{2},\frac{u}{2})$. The function $\mathrm{B}(\frac{1}{2},\frac{u}{2})$ is continuous and nonzero on the closed disc $|u-iy|\le\varepsilon$, which misses the real axis, so its modulus there is at least some $\lambda>0$, and with $|c(u)|\ge 1/q$ and $q\ll X$ the main term is $\gg_{y,\varepsilon} 1/X$. By Lemma 9.3 with $H=2\log X$ the far pairs contribute at most $2X^{-2}$. The count of points is as before. □

Proof of Theorem 1.3(a) *and* (b) *for* G_2. Assume (H_∞), so $2B_\infty=1$ and $B<1$. A point s with $\operatorname{Re}s>\frac{1}{2}$ lies in $\mathcal{C}_2$ exactly when $2s\in\Pi$, so $s\mapsto 2s$ carries paths in $\{\operatorname{Re}s>\frac{1}{2}\}\setminus\mathcal{C}_2$ starting in $\operatorname{Re}s>1$ to paths in $\{\operatorname{Re}v>1\}\setminus\Pi$ starting in $\operatorname{Re}v>2$, and $G_2(s)=D(2s)$. Under this map an admissible germ g of G_2 at s_1 becomes a germ $g(v/2)$ of D at $2s_1$ that is admissible in the sense of Theorem 1.2 with G replaced by D: reached from $\operatorname{Re}v>2$ along a path in $\{\operatorname{Re}v>1\}\setminus\mathcal{C}_G$, with $\mathcal{C}_G$ read in the variable v.

(a) Continuation along every such path is Theorem 7.4(a) for this D. An admissible germ sits at the end of a path in $\{\operatorname{Re}s>\frac{1}{2}\}\setminus\mathcal{C}_2$, so only points of that set carry one. Conversely let s_1 be such a point and $v_1=2s_1$, so $\operatorname{Re}v_1>1$ and $v_1\notin\Pi$. As in the existence argument for S in Section 8, all but countably many slopes κ put v_1 on no cut L_x; fix one. The segment $v_1+t(1+i\kappa)$, $0\le t\le 2$, has real parts at least $\operatorname{Re}v_1>1$, ends in $\operatorname{Re}v>2$, and meets no point $x\in\Pi$ with $\operatorname{Re}x>1$, since such an x on it would put v_1 on L_x. Continuing D along it from its east end to v_1, which is possible by Theorem 7.4(a), gives an admissible germ at v_1, hence one of G_2 at s_1. Since $\mathcal{C}_2$ is closed and discrete in $\operatorname{Re}s>\frac{1}{2}$ (Lemma 2.3), every disc $|s-(\frac{1}{2}+iy)|<r$ contains such points.

(b) Let g be an admissible germ at s_1 with $|s_1-(\frac{1}{2}+iy)|<r$, and suppose f is meromorphic on that disc with germ g at s_1. Then $f(v/2)$ is meromorphic on $|v-(1+2iy)|<2r$ and extends the admissible germ $g(v/2)$ at $2s_1$. Run the proof of Theorem 1.2 for G with D in place of G and this disc. The only change is the source of the point x: choose $0<\varepsilon'<r$ and apply Theorem 9.4 at $1+2iy$ with ε', which gives $x\in\mathcal{X}$ with $\operatorname{Re}x>1$, $|x-(1+2iy)|<\varepsilon'$ and, by Lemma 9.2, $X'_x(x)=\frac{1}{2}\Theta_2(x)\neq 0$. Theorem 7.4(a) and Lemma 8.1, which hold for this D, then give the contradiction exactly as there. □

9.2. **One logarithm.** The series $G_{\log}$ sits between G and T: one of its two factors is P, the other is Q. We write the integral with the weighted factor Q on the right, so that its poles lie among the right points $s-b$, which move with s; the next lemma gives the integral from which the sheets of $G_{\log}$ are built.

Lemma 9.5. *For* $\operatorname{Re}s>2$ *and* $1<c<\operatorname{Re}s-1$,

$$G_{\log}(s)=\frac{1}{2\pi i}\int_{(c)}K_s(z)P(z)Q(s-z)\,dz.$$

Proof. As for (4): the Mellin–Barnes integrand for $(p+q)^{-s}$ is $K_s(z)p^{-z}q^{-(s-z)}$, and multiplying by $\log q$ and summing gives $P(z)Q(s-z)$. The majorant is $P(c)Q(\operatorname{Re}s-c)$ times the bound for K_s. □

Matsumoto, Nawashiro and Tsumura [22] continue the double series

$$\sum_{m_1,m_2\ge 1}\Lambda(m_2)m_1^{-s_1}(m_1+m_2)^{-s_2}$$

meromorphically to $\mathbb{C}^2$. That series also carries a weight on one variable only, the von Mangoldt function, but its other variable runs over all positive integers with coefficient 1; in $G_{\log}$ the weight $\log q$ is on one prime and both variables run over primes.

In this subsection $v = s$, $\mathcal{V} = \{\operatorname{Re} s > 1\}$, $\mathcal{V}_\infty = \{\operatorname{Re} s > 2B_\infty\}$ and $\mathcal{X}$, L_x, Ω_κ are as for G. The sheet is built as in Section 5. The same contours serve. The left factor $P^\kappa(z)$ has a cut west of every left point a, so the contour must pass east of each a, as before. The right factor $Q(s-z)$ is single-valued, with poles only among the right points $s-b$. A contour could pass a pole on either side, but the two choices differ by a residue, which contains $P^\kappa(s-b)$, a function of s that can jump whenever $s-b$ crosses a cut ℓ_a. The residue can change the jump across the existing cuts L_{a+b}. To define one function we keep every right point east of the contour, as in Section 5. The condition for such a contour to exist is then the same as before: on a κ-line holding a left point a and a right point $s-b$, the right point must be east of the left one. So separating graphs, Lemma 5.2 and the cuts L_x carry over unchanged, and a cut L_x is now where a right candidate point $s-b$ meets the cut of a left candidate point a. Figure 1 applies with one change: the open dots are now right-hand pole candidates and carry no cuts, so the eastward rays drawn from them are to be omitted.

Theorem 9.6. *For $s \in \Omega_\kappa$ let*

$$D^\kappa_{\log}(s) = \frac{1}{2\pi i}\int_\Gamma K_s(z)\,P^\kappa(z)\,Q(s-z)\,dz,$$

with Γ any separating κ-graph for s. The value does not depend on Γ; $D^\kappa_{\log}$ is holomorphic on Ω_κ; and $D^\kappa_{\log} = G_{\log}$ on $\Omega_\kappa \cap \{\operatorname{Re} s > 2\}$.

Proof. As for Theorem 5.3. On a separating graph z lies on no cut ℓ_a, and $s-z$ is not a pole of Q, since every right point is strictly east of the graph; so the integrand is holomorphic near Γ, and the homotopy of separating graphs stays in the set where it is holomorphic. On the vertical tails both $\operatorname{Re} z$ and $\operatorname{Re}(s-z)$ are at least $B_\infty + 2\eta$, with η as in the proof of Theorem 5.3, so Lemma 5.1(b) bounds $P^\kappa(z)$ and Lemma 2.2 bounds $Q(s-z)$ by $\ll \log|\operatorname{Im} z|$; with the decay of K_s this dominates the integrand, locally uniformly in s. For $\operatorname{Re} s > 2$ the line $\operatorname{Re} z = c$ with $1 < c < \operatorname{Re} s - 1$ separates, $P^\kappa = P$ on it, and Lemma 9.5 applies. □

At a pinch the calculation is now a residue, and the coefficient of the logarithm is a value of the kernel.

Theorem 9.7. *Let $x \in \mathcal{X} \cap \mathcal{V}$ and let κ be generic. Put*

$$Y_x(s) = -\sum_{a+b=x} c(a)c(b)\,K_s(s-b) = -\sum_{a+b=x} c(a)c(b)\,\frac{\Gamma(s-b)\Gamma(b)}{\Gamma(s)}.$$

Then Y_x is holomorphic on $\operatorname{Re} s > 1$, and there is $r > 0$ such that the disc $|s-x| < r$ minus L_x lies in Ω_κ, and on it

$$D^\kappa_{\log}(s) = \operatorname{Log}_\kappa(s-x)\,Y_x(s) + H(s),$$

with H holomorphic on the whole disc.

Proof. The sum is finite by Lemma 2.3. Each $b \in \Sigma$ has $0 < \operatorname{Re} b \le 1$, so $\Gamma(s-b)$ is holomorphic on $\operatorname{Re} s > 1$, and $1/\Gamma$ is entire.

For the local form, follow the proof of Theorem 6.1. The configuration at $s = x$, the outer contour Γ_{out}, the radius r and the arcs depend only on the positions of the left and right points, which are the same as for G, so they are chosen as there, with the same four conditions on r_0; the radius r is shrunk once more below, to the radius that Lemma 6.2 provides. The fourth is now asked of the kernel K_s in the local variables: for each collision (a,b) the function $f(t,\mu) = e\,K_{x+e\mu}(a+et)$ used below is holomorphic on $|t| < 2r_0$, $|\mu| < 2r_0$. Here it serves the residue calculation of Lemma 6.2, which takes the place of the two-logarithm calculation of Theorem 6.1; it is that lemma's hypothesis on f. In the bound that makes I_{out} holomorphic, the right factor is bounded on the tails by Lemma 2.2 in place of Lemma 5.1(b). It remains to

compute one arc integral. Fix a collision (a,b), and put $t=(z-a)/e$, $\delta=s-x$ and $\epsilon=\delta/e$, as there. By Lemma 5.1(a), with the constant $i\theta$ absorbed, $P^\kappa(z)=c(a)\operatorname{Log}t+E_1(t)$. Since Q has residue $R_b=-c(b)$ at b, and $s-z-b=e(\epsilon-t)$,

$$Q(s-z)=\frac{-c(b)e^{-1}}{\epsilon-t}+E_2(\epsilon-t),$$

with E_2 holomorphic on a disc. With $f(t,\mu)=e\,K_{x+e\mu}(a+et)$, which the fourth condition on r_0 allows since $\operatorname{Re}a>0$ and $\operatorname{Re}(x-a)=\operatorname{Re}b>0$, the arc integral is

$$\frac{1}{2\pi i}\int_{A_\epsilon} f(t,\epsilon)\big(c(a)\operatorname{Log}t+E_1(t)\big)\Big(\frac{-c(b)e^{-1}}{\epsilon-t}+E_2(\epsilon-t)\Big)\,dt.$$

Lemma 6.2, with $c_1=c(a)$ and $c_2=-c(b)e^{-1}$ (here $e^{-1}=1/e$ is the reciprocal of the unit vector e of Section 5, not an exponential), gives $-c(a)c(b)e^{-1}f(\epsilon,\epsilon)\operatorname{Log}\epsilon$ plus a function holomorphic in ϵ, on the disc of the radius the lemma provides; shrink r so that $|\epsilon|$ stays below that radius for every collision. Here $e^{-1}f(\epsilon,\epsilon)=K_s(a+\delta)=K_s(s-b)$, since $a+\delta=x-b+\delta=s-b$, and $\operatorname{Log}\epsilon=\operatorname{Log}_\kappa\delta-i\theta$. Summing over the collisions, each ordered pair once, gives the formula. □

For G the arc gave $c(a)c(b)\operatorname{Log}_\kappa(v-x)J_{ab}(v)$, whose coefficient is an integral from a to $v-b$ that vanishes at $v=x$: there the singular part of the sheet is of the order of $(v-x)\log(v-x)$ at most, and exactly of that order when $X_x'(x)\neq0$. Here a logarithm meets a pole, each ordered collision contributes the kernel evaluated at the moving right point $s-b$, weighted by $-c(a)c(b)$, and $D^\kappa_{\log}$ behaves like $Y_x(x)\log(s-x)$ when $Y_x(x)\neq0$, and $Y_x(x)$ need not vanish.

When $B_\infty<1$, the sign can be checked heuristically at $x=2$, where the only pair is $(1,1)$ with $c(1)^2=1$. There $Y_2(s)=-\Gamma(s-1)/\Gamma(s)=-1/(s-1)$, so $Y_2(2)=-1$, and with $s=2+\delta$ the leading singular term is $-\log\delta$. This matches the prime number theorem heuristic. Let $A(X)=\sum_{p+q\le X}\log q$. For a fixed p the sum over q is $\vartheta(X-p)$, where $\vartheta(y)=\sum_{q\le y}\log q$ is the weighted count of the primes up to y, and $\vartheta(y)=y+o(y)$ by the prime number theorem. So $A(X)$ is asymptotic to $\sum_{p\le X}(X-p)=\int_2^X\pi(t)\,dt$, where $\pi(t)$ is the number of primes up to t; the terms with $X-p$ bounded, where the estimate for ϑ says nothing, number at most $\pi(X)$ and contribute $O(\pi(X))$, which is of lower order. Since $\pi(t)\sim t/\log t$, the integral is asymptotic to $X^2/(2\log X)$. So $A(X)\sim X^2/(2\log X)$, and partial summation gives

$$G_{\log}(2+\delta)=(2+\delta)\int_4^\infty A(t)t^{-3-\delta}\,dt.$$

With $t=e^y$ the right side is about $\int_{\log 4}^\infty e^{-\delta y}y^{-1}\,dy$, which grows like $-\log\delta$ as $\delta\to0^+$. As for G, the heuristic serves only as a check of the sign and the normalization; no proof uses it.

The rotation argument uses only that the jump functions are holomorphic on the whole domain, and Y_x is. Let $\mathcal{M}_{\log}$ be the set of finite integer combinations of the functions $2\pi iY_x$, $x\in\mathcal{X}$; its members are holomorphic on $\operatorname{Re}s>1$.

Theorem 9.8. (a) *If $s_1\in\Omega_\kappa\cap\Omega_{\kappa'}$ with $\kappa<\kappa'$, then as germs at s_1*

$$D^{\kappa'}_{\log}-D^\kappa_{\log}=\sum_x 2\pi iY_x,$$

the finite sum over $x\in\mathcal{X}$ in the open eastward sector at s_1 between the two directions.

(b) *A germ $D^\kappa_{\log}+m$, $m\in\mathcal{M}_{\log}$, continues along every path in $\mathcal{V}_\infty\setminus\mathcal{X}$ and stays in this family. In particular $G_{\log}$ continues along every path in $\{\operatorname{Re}s>2B_\infty\}\setminus\Pi$ starting in $\operatorname{Re}s>2$, and each germ reached is $D^\kappa_{\log}+m$.*

(c) *Let $x\in\mathcal{X}\cap\mathcal{V}$ and κ be generic. There is a disc B_x about x such that every germ of this family at a point of $B_x\setminus L_x$ is $\operatorname{Log}_\kappa(s-x)Y_x$ plus a function holomorphic on B_x.*

Proof. (a) Lemma 7.1 holds as stated, with the same proof, since every sheet equals $G_{\log}$ on $\operatorname{Re} s > 2$ by Theorem 9.6. In the proof of Lemma 7.2, Theorem 9.7 replaces Theorem 6.1: at each point x_j on the ray the two logarithms differ by $2\pi i$ on the west sector, so the jump there is $2\pi i Y_{x_j}$, and the induction needs only that the earlier jumps are holomorphic on $\mathcal{V}$. The proof of Theorem 7.3 then applies. (b) and (c) follow from (a) and Theorem 9.7 as Theorem 7.4(a),(c) follow from Theorems 7.3 and 6.1. The paths considered in (b) lie in $\mathcal{V}_\infty \subseteq \mathcal{V}$, so there are no excursions and no constant. □

So, for generic κ, every germ of $G_{\log}$ reached along a path in $\{\operatorname{Re} s > 2B_\infty\} \setminus \Pi$, at a base point in the disc B_x of Theorem 9.8(c) off L_x, has the same coefficient Y_x. Its value at x is a sum the paper has already controlled.

Lemma 9.9. *For $x \in \mathcal{X}$, $Y_x(x) = -\Theta_G(x)$. At $x = 1+u$ with $u \neq 1$ the pairs $(1,u)$ and $(u,1)$ contribute $2c(u)/u$ to $Y_x(x)$.*

Proof. For a pair (a,b) at x, $x - b = a$, so $K_x(x-b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) = \mathrm{B}(a,b)$. For the main pairs, $\mathrm{B}(1,u) + \mathrm{B}(u,1) = 2\Gamma(u)/\Gamma(1+u) = 2/u$ and $c(1) = -1$. □

The main term $2c(u)/u$ is the negative of the main term of Θ_G in Section 3, as Lemma 9.9 requires. Under $B < 1$, Theorem 3.1 gives, near every point of the line, infinitely many x with $\Theta_G(x) \neq 0$; under (H_∞) these lie in $\mathcal{X}$, and there $Y_x(x) = -\Theta_G(x) \neq 0$ by Lemma 9.9. The obstruction is then simpler than Lemma 8.1, because the coefficient itself, not its derivative, is nonzero at x.

Lemma 9.10. *Let $x \in \mathcal{X} \cap \mathcal{V}$ with $Y_x(x) \neq 0$, κ generic, and B_x the disc of Theorem 9.8(c). No germ $D^\kappa_{\log} + m$ at a point q of $B_x \setminus L_x$ agrees at q with a function f analytic on a punctured disc about x containing q.*

Proof. As in the proof of Lemma 8.1, the identity theorem and Theorem 9.8(c) give $f = \operatorname{Log}_\kappa(s-x)Y_x + H_1$ on the slit punctured disc, with H_1 holomorphic on the disc. Since f and H_1 are continuous across the slit and $\operatorname{Log}_\kappa$ jumps by $2\pi i$, $Y_x = 0$ on the slit, and $Y_x(x) = 0$ by continuity. □

Proof of Theorem 1.3(a) *and* (b) *for $G_{\log}$.* Assume (H_∞), so $2B_\infty = 1$ and $B < 1$. (a) Continuation is Theorem 9.8(b). The existence of admissible germs exactly at the points of $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}_G$ follows as for G_2, with the segment $s_1 + t(1+i\kappa)$, $0 \le t \le 2$.

(b) Run the proof of Theorem 1.2 for G, with three replacements. Theorem 3.1 supplies $x = 1+u$ in the disc with $\Theta_G(x) \neq 0$, hence $Y_x(x) \neq 0$ by Lemma 9.9; Theorem 9.8(b) replaces Theorem 7.4(a), so the germ of f at the point s_2 near x is $D^\kappa_{\log} + m$; and Lemma 9.10 replaces Lemma 8.1. □

9.3. **Paths with an endpoint off the catalogue.** Part (c) of Theorem 1.3 is the analogue of Corollary 8.3. We give a direct proof, which uses part (b), Lemma 2.3 and the holomorphy of each series on its half-plane of absolute convergence. Recall from Section 1 that a continuation of a function F along a path γ is a family of functions f_t, $0 \le t \le 1$, with f_t holomorphic near $\gamma(t)$, $f_0 = F$ near $\gamma(0)$, and $f_{t'} = f_t$ near $\gamma(t')$ for t' close to t, and that the terminal germ is that of f_1 at $\gamma(1)$.

The proof is an open-and-closed argument on the parameter interval. Call a time t good if the representative f_t is holomorphic on a disc about $\gamma(t)$ in the open half-plane and its germs at the points of that disc off the catalogue are admissible. The property is asked of a whole disc, not of the point $\gamma(t)$ alone, so that it survives when the path passes through a catalogue point: the germs at the nearby points off the catalogue carry the admissibility, and the same holomorphic representative serves. Time 0 is good, and goodness passes to nearby times because

nearby representatives agree. The main step is that a limit of good times is good: there the path might have arrived at the boundary line, and part (b) forbids it, since the representative at the limit time would extend a nearby admissible germ holomorphically to a disc centred on the line. So the path stays in the open half-plane, and off the catalogue the germ it carries is admissible. The chain construction of Theorem 8.2 and Corollary 8.3 proves the same statement for G and S; the argument here does both of its steps, confinement to the half-plane and admissibility, at once.

Proof of Theorem 1.3(c). We treat G_2; for $G_{\log}$ replace the line $\operatorname{Re} s = \frac{1}{2}$ by $\operatorname{Re} s = 1$, the half-plane $\operatorname{Re} s > \frac{1}{2}$ by $\operatorname{Re} s > 1$, the half-plane of absolute convergence $\operatorname{Re} s > 1$ by $\operatorname{Re} s > 2$, and $\mathcal{C}_2$ by $\mathcal{C}_G$. The set $\mathcal{C}_2$ lies in $\operatorname{Re} s \le 1$, since $\operatorname{Re} x \le 2$ for $x \in \Pi$, and $\mathcal{C}_G$ lies in $\operatorname{Re} s \le 2$. Let (f_t) be a continuation of G_2 along γ, with $\operatorname{Re}\gamma(0) > 1$, terminal germ g, and $\gamma(1) \notin \mathcal{C}_2$. Call t *good* if there is a disc U about $\gamma(t)$ in $\operatorname{Re} s > \frac{1}{2}$ on which f_t is holomorphic and whose germ at every point of $U \setminus \mathcal{C}_2$ is admissible.

The time 0 is good: take U in $\operatorname{Re} s > 1$, where G_2 is holomorphic and every germ of it is admissible, by the constant path. The good times form an open set: if t is good with disc U, then for t' near t the point $\gamma(t')$ lies in U and $f_{t'} = f_t$ near it, so a small disc about $\gamma(t')$ inside U, on which $f_{t'} = f_t$, shows that t' is good.

They also form a closed set. Let t_n be good, with discs U_n, and $t_n \to t^*$. There is a disc U^* about $\gamma(t^*)$ on which f_{t^*} is holomorphic, and for large n, $\gamma(t_n) \in U^*$ and $f_{t_n} = f_{t^*}$ near $\gamma(t_n)$. For such an n, every point $w \in U_n \cap U^*$ off $\mathcal{C}_2$ near $\gamma(t_n)$ is a point where the germ of f_{t^*} is that of f_{t_n} and so is admissible. Now $\operatorname{Re}\gamma(t^*) \ge \frac{1}{2}$. If $\operatorname{Re}\gamma(t^*) = \frac{1}{2}$, then U^* is a disc centred on the line $\operatorname{Re} s = \frac{1}{2}$, it contains such a w, and f_{t^*} is holomorphic on it, which contradicts (b). So $\operatorname{Re}\gamma(t^*) > \frac{1}{2}$. Shrink U^* into $\operatorname{Re} s > \frac{1}{2}$, and then fix n and a point w as above for the smaller disc. Any point of $U^* \setminus \mathcal{C}_2$ is joined to w by a path in $U^* \setminus \mathcal{C}_2$, since $\mathcal{C}_2$ is closed and discrete in $\operatorname{Re} s > \frac{1}{2}$ (Lemma 2.3); following f_{t^*} along it continues an admissible germ along a path in $\{\operatorname{Re} s > \frac{1}{2}\} \setminus \mathcal{C}_2$, and the result is admissible. So t^* is good.

Hence 1 is good. Its disc lies in $\operatorname{Re} s > \frac{1}{2}$ and $\gamma(1) \notin \mathcal{C}_2$, so the germ of f_1 at $\gamma(1)$, which is g, is admissible.

The hypothesis $\gamma(1) \notin \mathcal{C}_2$ was used only in the last sentence; without it, 1 is still good, with some disc U. So a terminal germ g based at a point of $\mathcal{C}_2$ is obstructed in the same way, by the step that closes the first part of the proof of Theorem 8.2(a): a function meromorphic on a disc centred on $\operatorname{Re} s = \frac{1}{2}$ that contains $\gamma(1)$, with germ g at $\gamma(1)$, would agree with f_1 on a neighbourhood of $\gamma(1)$ inside U and that disc, hence near some point w of the neighbourhood off $\mathcal{C}_2$, which is discrete, and so would extend the admissible germ of f_1 at w meromorphically to that disc, contradicting (b). For $G_{\log}$ the same holds with the replacements made at the start of the proof. This remark is not part of the statement of Theorem 1.3. It is checked in Lean, as `HInf.no_extension_any_path_Sq` for G_2 and `HInf.no_extension_any_path_mix` for $G_{\log}$, the counterparts of `HInf.no_extension_any_path_G` and `HInf.no_extension_any_path_S` (Section 11). □

Together the five series give three kinds of pinch. For T two poles meet, and the test used is a residue, $\Theta_G(x)$. For $G_{\log}$ a logarithm meets a pole, and the test used is the value $Y_x(x) = -\Theta_G(x)$. For G, S and G_2 two logarithms meet, and the test used is the derivative $X'_x(x)$, taken in v, which is $\Theta_G(x)$ for G, $\frac{1}{2}\Theta_S(x)$ for S and $\frac{1}{2}\Theta_2(x)$ for G_2. In every case the residue, the coefficient's value or the coefficient's derivative at a candidate point is a sum over pairs of the same shape. Under $B < 1$, Theorem 3.1 and, for Θ_2, Theorem 9.4 give nonzero pair sums at points accumulating at every point of the line $\operatorname{Re} v = 1$; under (H_∞) these points lie in the regions of continuation, where the identities above apply.

10. What more holds under RH

Under RH the same machinery proves more, and the comparison shows what the non-cancellation argument of Section 3 is needed for. The boundary proofs needed a nonzero pair sum only at a dense family of points. The counting and decay estimates of Section 3 supply such a family under $B < 1$ (for G_2, with the far-pair bound of its own kernel, Lemma 9.3, and its own dense family, Theorem 9.4), and (H_∞) places its points in the region of continuation. RH gives non-vanishing at every point of the catalogue of G, which lies in $\operatorname{Re} s > 1$, because no extra pairs exist there and the main coefficient $c(u)$ at $x = 1 + u$ is nonzero, as u has only one representation as ρ/n or $1/n$ with n squarefree. So under RH the pair sum at each point of that catalogue can be written down, and its points are all genuine singularities of G and of T.

RH now has a certified Lean proof: Alpöge's proof [1] was certified on 21 November 2026 [17], and Sections 1 and 11 say what has been checked. So Theorem 10.1 below describes the series G and T as they are (Corollary 1.4), and we state it with RH as its hypothesis, since that is what its proof uses.

RH implies (H_∞), and every point of Σ other than 1 then has real part at most $\frac{1}{2}$. So the catalogue $\mathcal{C}_G$ of G, which lies in $\operatorname{Re} s > 1$ and which we write $\mathcal{C}$ in this section, is

$$\mathcal{C} = \{2\} \cup \{1 + 1/k : k \geq 2 \text{ squarefree}\} \cup \{1 + \rho/n : \rho \in \mathcal{Z},\ n \geq 1 \text{ squarefree}\},$$

and a pair (a, b) with $\operatorname{Re}(a+b) > 1$ must contain 1. Each $u = \rho/n$ also has a single representation, since $\operatorname{Re} u = 1/(2n)$ determines n. So $x = 1 + u$ has exactly the pairs $(1, u), (u, 1)$, or the single pair $(1, 1)$ at $x = 2$, and (7) becomes

$$X_2'(2) = 1, \qquad X_{1+1/k}'(1 + 1/k) = 2\mu(k), \qquad X_{1+\rho/n}'(1 + \rho/n) = -\frac{2\mu(n)m(\rho)}{\rho} \neq 0. \tag{13}$$

Theorem 10.1. *Assume RH.*

(a) *Every point of $\mathcal{C}$ is a genuine logarithmic branch point of G on every branch: every branch near $\alpha \in \mathcal{C}$ is $X_\alpha(s)\log(s-\alpha)$ plus a holomorphic function, with X_α holomorphic on $\operatorname{Re} s > 1$, $X_\alpha(\alpha) = 0$, and $X_\alpha'(\alpha)$ given by (13). G continues along every path in $\{\operatorname{Re} s > 1\} \setminus \mathcal{C}$ from $\operatorname{Re} s > 2$, and a path along which G continues analytically never leaves this set.*

(b) *T has a meromorphic continuation to $\operatorname{Re} s > 1$, with a simple pole at every point of $\mathcal{C}$, of residue equal to the value in (13), and no other pole there.*

Proof. (b) is Theorem 4.2. For (a), the local form and continuation along paths avoiding $\mathcal{C}$ are Theorems 6.1 and 7.4. Let γ be any path along which G continues from $\operatorname{Re} s > 2$. It stays in $\operatorname{Re} s > 1$ by Theorem 8.2. Suppose it meets $\mathcal{C}$. The catalogue is relatively closed in this half-plane, so its inverse image under γ is a nonempty closed subset of $[0, 1]$. It has a first element $t^* > 0$, since the starting point has real part greater than 2. Put $\alpha = \gamma(t^*)$. Every restriction ending at $t < t^*$ avoids $\mathcal{C}$, and its terminal germ is admissible.

Take a holomorphic representative f of the continued germ at α. Shrink its disc of holomorphy so that it lies in $\operatorname{Re} s > 1$, contains no catalogue point except α, and is contained in the local-model disc at α for a generic κ. For $t < t^*$ sufficiently close to t^*, the continued germ at $\gamma(t)$ agrees with f. Join $\gamma(t)$ inside this disc minus $\{\alpha\}$ to a point q off the slit. Continuing as f along this short path preserves admissibility. By Theorem 7.4, its germ at q is $D^\kappa + m$, while f is analytic on the whole disc about α. Since (13) gives $X_\alpha'(\alpha) \neq 0$, this contradicts Lemma 8.1 at the off-slit base point q. Thus no first catalogue contact exists. □

This is stronger than the (H_∞) theorem for G, whose proof gives non-vanishing only on a dense family. We do not state such a theorem for S.

We now make the comparison. Under RH, at $x = 1 + \rho/q$ with q prime, only the main terms exist. For G the sum is $-2c(u)/u$. For S, in the variable s, it is $-c(u)\{K(1 - \frac{u}{4}) + K(\frac{1}{2} + \frac{u}{4})\}$, with a bracket nonzero off the real axis by Lemma 2.4. So under RH the sum is an explicit nonzero factor times $c(\rho/q) = -m(\rho)/q$, the one-term case of Titchmarsh's (9.5.7) [34, §9.5, p. 216], and the accumulation at the line is his (9.5.4)–(9.5.6) [34, §9.5, pp. 215–216] shifted by 1, followed by $s = (v - 1)/2$ for S. The near-pair count and the far-pair estimates exist to handle collisions that (H_∞) permits and RH does not.

11. Formal verification

Theorems 1.1, 1.2, 1.3, 8.2 and 10.1, Corollary 8.3 and the remark that closes the proof of Theorem 1.3(c) have been checked in the Lean 4 proof assistant (Lean v4.34.0, Mathlib [20] at tag v4.34.0, PrimeNumberTheoremAnd [14]). The formal theorems are implications. The (H_∞) results take (H_∞), as the proposition `Hinf`, and the applicable classical inputs among (Z1)–(Z6) of Section 2, each formulated with Mathlib's definitions and taken as a hypothesis, not as an axiom. The theorem for S, `HInf.boundary_S`, does not use (Z6); the conjunction `HInf.boundary_GS` printed in the appendix carries (Z6) for G. For G_2 and $G_{\log}$, parts (a) of Theorem 1.3 use (H_∞), (Z1)–(Z3) and (Z6), and parts (b) and (c) use (H_∞) and all of (Z1)–(Z6); unlike S, whose kernel is elementary, these two series need (Z6) already for continuation, as G does, because their kernels are bounded through Stirling's formula. For the kernels of Sections 5–9, the formal proofs record the decay (K2) only in the weaker form $\mathcal{K}(z, v) \ll (1 + |\operatorname{Im} z|)^{-2}$, which is all they use. Theorem 10.1 is checked from RH alone. What is machine-checked is that these imply the stated results. The Lean kernel checks the implication; whether the formal definitions say what the prose says is a matter of reading them, and the appendix gives the main statements for that purpose.

The development is a standalone release, PrimePairNB version 1.2.1, Apache-2.0, available at `https://github.com/lpkralv/prime-pair-natural-boundaries` under the tag `v1.2.1`, commit `f0d9d5fb1444e495af2cb4ea7d1e31d3fb01b2df`. Its module root is `PrimePairNB`; its declarations are in the namespace `GoldbachBoundaries`, and the names in this section are relative to it: for example, `HInf.boundary_G` means `GoldbachBoundaries.HInf.boundary_G`. The appendix shows the namespace blocks explicitly.

The main declarations are `Paper2.thm_boundary` for Theorem 1.1; `HInf.boundary_GS`, the conjunction of `HInf.boundary_G` and `HInf.boundary_S`, for Theorem 1.2; `HInf.boundary_Sq` and `HInf.boundary_mix` for Theorem 1.3(b); and `thm_A` and `rem_every_path` for G and `cor_T_poles` for T in Theorem 10.1. The appendix prints the statements for Theorems 1.1, 1.2 and 1.3(b). The existence of admissible germs of S shown in Section 8 is `HInf.exists_admS_iff`, which takes only `Hinf` and (Z1)–(Z3). Theorem 8.2 is `HInf.every_path_stays_G`, `HInf.every_path_stays_S`, `HInf.no_extension_any_path_G`, `HInf.no_extension_any_path_S`, `HInf.cor_domain_G` and `HInf.cor_domain_S`; Corollary 8.3 is `HInf.every_path_G` and `HInf.every_path_S`. For Theorem 1.3, part (a) is `HInf.continuesAlong_Sq`, `HInf.continuesAlong_mix`, `HInf.exists_admSq_iff` and `HInf.exists_admMix_iff`, and part (c) is `HInf.every_path_Sq` and `HInf.every_path_mix`. The remark that closes the proof of part (c), which gives for G_2 and $G_{\log}$ what Theorem 8.2(a) gives for G and S at an endpoint in the catalogue, is `HInf.no_extension_any_path_Sq` and `HInf.no_extension_any_path_mix`; both take `Hinf` and all of (Z1)–(Z6). The declarations for the lemmas and intermediate theorems are listed, with their hypotheses, in `CORRESPONDENCE.md` in the release. In Lean, G_2 and $G_{\log}$ are `SqSer` and `MixSer`. The formal proofs for G_2 instantiate the generic sheet, rotation and obstruction theorems with the kernel of Proposition 9.1, as the paper does; those for $G_{\log}$ repeat that development with Q as the right factor and the

pole–logarithm arc of Lemma 6.2. The formal proof of Theorem 10.1 goes by a different route, from RH alone, not via the sheets of Section 5. The passage from Titchmarsh's statements to the forms in Section 2 is checked by hand.

The release builds from a clean clone with the pinned versions (`lake exe cache get`, then `lake build`), and `lake env lean CheckAxioms.lean`, which runs `#print axioms` on every PrimePairNB theorem named in this section, lists only `propext`, `Classical.choice` and `Quot.sound`.

Corollary 1.4 is a written deduction, and its unconditional statements are not checked in a single Lean build. It joins two Lean developments, Alpöge's library and our release, by three steps taken outside Lean: the comparison of the statement of RH in two versions of Mathlib, which is a comparison of source text and not a Lean check, the implication from RH to (H_∞), and the classical inputs (Z1)–(Z6). So the unconditional statements combine four things. The first is Alpöge's Lean theorem [1], which according to the joint statement [17] two independent tools confirm to be exactly Mathlib's `RiemannHypothesis`. Alpöge's library is built against Lean v4.33.1 and Mathlib v4.33.1 [32], the release against Lean v4.34.0 and Mathlib v4.34.0; the source text of the definition of `RiemannHypothesis` is the same in both Mathlib versions; we did not check in Lean that `RiemannHypothesis` in Mathlib v4.33.1 and in v4.34.0 are the same proposition. Identical source text of that definition does not by itself show that the definitions it depends on agree between the two versions. The second is the step from RH to (H_∞): under RH no zero has real part above $\frac{1}{2}$. It is one line, but the release has no declaration deriving `Hinf` from `RiemannHypothesis`. The third is the classical results (Z1)–(Z6), which remain hypotheses of the formal statements; they come from the sources cited in Section 2 and are checked by hand, not in Lean. The fourth is the release, which proves Theorems 1.1, 1.2 and 1.3 from `Hinf` and the applicable inputs among (Z1)–(Z6), and Theorem 10.1 from Mathlib's `RiemannHypothesis` (`thm_A`, `cor_T_poles` and `rem_every_path`). The two libraries are built and checked separately. We read the headline statements of Alpöge's library and scanned its source but did not build it; for its proof we rely on the checks cited in Section 1, whose details follow.

Those checks are the certification [17] described in Section 1 and the independent runs of Sutherland [32]. In his update of 22 November 2026, 16:20 UTC, none of the reported checks had rejected the proof. Completed checks included a compile of all 175,187 modules; `#print axioms` on the five headline theorems, which lists only `propext`, `Classical.choice` and `Quot.sound`; a `leanchecker` replay of every module; the checkers `nanoda` and `con-leche`, which accepted the full export; `con-ron` in its `--compare` mode; `lean4lean` on every compiled module; a successful `con-leche --compare` run against all five headline statements; and an independent sandbox rebuild that reproduced the export byte for byte. The versions of `con-leche` and `con-ron` bundled with Lean v4.36.0 declined (exit 2, not a rejection). In Sutherland's runs, the replays of the full export by Lean's default and paranoid kernels, and the end-to-end `lake comparator` run, whose statement-comparison and axiom stages had passed in one run, were still in progress at that time; his completed `leanchecker` replay checked the compiled modules one by one.

Acknowledgement and statement on the use of AI

The author is a computer scientist, not a mathematician. Starting this work at all was the author's idea: he directed an AI research agent through a sequence of questions on natural boundaries of prime-only Goldbach series, chose at each step the direction to pursue, and decided to attack these series without the Riemann Hypothesis. He designed the verification: independent adversarial reviews of the results before they were recorded as proved, with re-derivation and small-case checks where those apply, and a formalization in Lean. The mathematics is

largely the work of AI systems under his direction: Anthropic Claude models and OpenAI models found and reviewed the arguments; an Anthropic Claude model drafted the manuscript, and another edited it; AI agents wrote the Lean development. The author did not write or check the Lean development line by line; he relies on the Lean kernel and the checks above. No human mathematician has checked the proof line by line. Queries about the proof will be answered by the author, with the same AI assistance. No AI system is an author. The author takes full responsibility for the content.

Appendix A. The Lean statements

The definitions below and the boundary statements of Theorems 1.1, 1.2 and 1.3(b) are copied verbatim from the PrimePairNB release, with docstrings and proofs omitted and line breaks adjusted; the other declarations cited in Section 11 are in `CORRESPONDENCE.md` in the release. The `namespace` and `open` lines are condensed, not copied; the declarations themselves are verbatim, and the namespace blocks give their full names. `Hinf` is (H_∞), and `Z1`–`Z6` are the classical inputs of Section 2. `PoleSetQ` is Σ, `Pinch` is Π, and `catG`, `catS`, `catSq` are the catalogues $\mathcal{C}_G$, $\mathcal{C}_S$, $\mathcal{C}_2$. The series `G`, `T` and `Sser`, the last via `WS`, are (1); `SqSer` and `MixSer`, via `WSq` and `Wlogq`, are G_2 and $G_{\log}$ of Section 1. For an affine map $L(s) = \ell s + \ell_0$, `Aff.HL` is $\operatorname{Re} L(s) > 1$ and `Aff.RL` is $\operatorname{Re} L(s) > 2$; so `Aff.twoL.HL` is $\operatorname{Re} s > \frac{1}{2}$, `Aff.twoL.RL` is $\operatorname{Re} s > 1$, `Aff.idL.HL` is $\operatorname{Re} s > 1$, `Aff.idL.RL` is $\operatorname{Re} s > 2$, and `catalogueL Aff.idL` is $\mathcal{C}_G$. `IsTerminalGerm` records continuation along a continuous path; `AdmG`, `AdmS`, `AdmSq`, `AdmMix` record admissibility. The filters $\mathcal{N}\, s$ and $\mathcal{N}[\neq]\, s$ mean a neighbourhood and a punctured neighbourhood; $f =^{\mathrm{f}} [\mathcal{F}] g$ means equality eventually in the filter $\mathcal{F}$. For `codiscreteWithin U`, this is equality in U away from a set with no accumulation point there. `MeromorphicOn`, `AnalyticAt`, `riemannZeta`, `Gamma` and `meromorphicOrderAt` are Mathlib's.

```
noncomputable section

open Complex Filter Topology Set Metric ComplexConjugate MeasureTheory

namespace GoldbachBoundaries

def halfPlane (a : ℝ) : Set ℂ := {s | a < s.re}

abbrev HH : Set ℂ := halfPlane 1

def lineOne : Set ℂ := {s | s.re = 1}

def G (s : ℂ) : ℂ :=
  ∑' pq : Nat.Primes × Nat.Primes, (((pq.1 : ℕ) + (pq.2 : ℕ) : ℕ) : ℂ) ^ (-s)

def T (s : ℂ) : ℂ :=
  ∑' pq : Nat.Primes × Nat.Primes,
    (Real.log (pq.1 : ℕ) : ℂ) * (Real.log (pq.2 : ℕ) : ℂ) *
      (((pq.1 : ℕ) + (pq.2 : ℕ) : ℕ) : ℂ) ^ (-s)

def NontrivialZeros : Set ℂ := {ρ | riemannZeta ρ = 0 ∧ 0 < ρ.re ∧ ρ.re < 1}

def zetaMult (ρ : ℂ) : ℕ := ((meromorphicOrderAt riemannZeta ρ).untopD 0).toNat

def PoleSetQ : Set ℂ :=
  {1} ∪ {z | ∃ k : ℕ, 2 ≤ k ∧ Squarefree k ∧ z = 1 / (k : ℂ)} ∪
    {z | ∃ n : ℕ, 1 ≤ n ∧ Squarefree n ∧ ∃ ρ ∈ NontrivialZeros, z = ρ / (n : ℂ)}

structure IsContinuationAlong (γ : C(unitInterval, ℂ))
```

```
    (f : unitInterval → ℂ → ℂ) : Prop where
  analyticAt : ∀ t, AnalyticAt ℂ (f t) (γ t)
  locallyEq : ∀ t, ∀ᶠ t' in 𝒩 t, f t' =ᶠ[𝒩 (γ t')] f t

def IsTerminalGerm (F : ℂ → ℂ) (γ : C(unitInterval, ℂ)) (g : ℂ → ℂ) : Prop :=
  ∃ f : unitInterval → ℂ → ℂ, IsContinuationAlong γ f ∧ f 0 =ᶠ[𝒩 (γ 0)] F ∧
    g =ᶠ[𝒩 (γ 1)] f 1

def NoMeromorphicExtensionAt (F : ℂ → ℂ) (U : Set ℂ) (z₀ : ℂ) : Prop :=
  ∀ r : ℝ, 0 < r → ¬ ∃ f : ℂ → ℂ, MeromorphicOn f (ball z₀ r) ∧
    f =ᶠ[codiscreteWithin (ball z₀ r ∩ U)] F

def IsNaturalBoundary (F : ℂ → ℂ) (U L : Set ℂ) : Prop :=
  ∀ z₀ ∈ L, NoMeromorphicExtensionAt F U z₀

def GermExtendsMeromorphically (g : ℂ → ℂ) (s₁ : ℂ) (W : Set ℂ) : Prop :=
  s₁ ∈ W ∧ ∃ f : ℂ → ℂ, MeromorphicOn f W ∧ f =ᶠ[𝒩[≠] s₁] g

end GoldbachBoundaries

namespace GoldbachBoundaries.Paper2

open GoldbachBoundaries

def Pinch : Set ℂ := {s | ∃ a ∈ PoleSetQ, ∃ b ∈ PoleSetQ, s = a + b}

def Hinf : Prop := ∀ δ : ℝ, 0 < δ → {ρ | ρ ∈ NontrivialZeros ∧ 1 / 2 + δ < ρ.re}.Finite

def Z1 : Prop :=
  ∀ ρ ∈ NontrivialZeros,
    (conj ρ ∈ NontrivialZeros ∧ zetaMult (conj ρ) = zetaMult ρ) ∧
      (1 - conj ρ ∈ NontrivialZeros ∧ zetaMult (1 - conj ρ) = zetaMult ρ)

def Nzero (t : ℝ) : ℝ :=
  ∑ᶠ ρ ∈ {ρ : ℂ | ρ ∈ NontrivialZeros ∧ 0 < ρ.im ∧ ρ.im ≤ t}, (zetaMult ρ : ℝ)

def Z2 : Prop :=
  (∃ C : ℝ, ∀ᶠ t in atTop,
      |Nzero t - (t / (2 * Real.pi) * Real.log (t / (2 * Real.pi)) - t / (2 * Real.pi))| ≤
        C * Real.log t) ∧
    ∃ C₁ : ℝ, ∀ t : ℝ, 0 ≤ t → Nzero (t + 1) - Nzero t ≤ C₁ * Real.log (t + 3)

def Z3 : Prop :=
  ∃ C : ℝ, ∀ z : ℂ, -1 ≤ z.re → z.re ≤ 2 → 2 ≤ |z.im| → riemannZeta z ≠ 0 →
    ‖deriv riemannZeta z / riemannZeta z -
        ∑ᶠ ρ ∈ {ρ : ℂ | ρ ∈ NontrivialZeros ∧ |ρ.im - z.im| ≤ 1},
            (zetaMult ρ : ℂ) / (z - ρ)‖ ≤
      C * Real.log |z.im|

def N0 (t : ℝ) : ℕ :=
  Set.ncard {ρ : ℂ | ρ ∈ NontrivialZeros ∧ ρ.re = 1 / 2 ∧ 0 < ρ.im ∧ ρ.im ≤ t}

def Z4 : Prop := ∃ A₀ : ℝ, 0 < A₀ ∧ ∃ t₀ : ℝ, ∀ t ≥ t₀, A₀ * t ≤ (N0 t : ℝ)

def Z5 : Prop :=
  ∀ θ : ℝ, 0 < θ → ∃ x₀ : ℝ, ∀ x ≥ x₀,
    θ * x / (2 * Real.log x) ≤ ({p : ℕ | p.Prime ∧ x < p ∧ (p : ℝ) < (1 + θ) * x}.ncard : ℝ)
```

```
def Z6 : Prop :=
  ∀ x₁ x₂ : ℝ, 0 < x₁ → x₁ ≤ x₂ → ∃ C : ℝ, ∀ x ∈ Icc x₁ x₂, ∀ y : ℝ,
    ‖Gamma (x + y * I)‖ ≤ C * (1 + |y|) ^ x₂ * Real.exp (-(Real.pi * |y| / 2))

def IsAdmissibleGermOn (F : ℂ → ℂ) (Ω S : Set ℂ) (g : ℂ → ℂ) (s₁ : ℂ) : Prop :=
  ∃ γ : C(unitInterval, ℂ), (∀ t, γ t ∈ Ω) ∧ γ 0 ∈ S ∧ γ 1 = s₁ ∧ IsTerminalGerm F γ g

structure Aff where
  ell : ℝ
  ell0 : ℝ
  ell_pos : 0 < ell

namespace Aff

def app (A : Aff) (s : ℂ) : ℂ := (A.ell : ℂ) * s + A.ell0

def HL (A : Aff) : Set ℂ := {s | 1 < (A.app s).re}

def RL (A : Aff) : Set ℂ := {s | 2 < (A.app s).re}

def idL : Aff := ⟨1, 0, one_pos⟩

def twoL : Aff := ⟨2, 0, two_pos⟩

def twoOneL : Aff := ⟨2, 1, two_pos⟩

end Aff

def Dser (W : ℂ → ℕ → ℕ → ℂ) (s : ℂ) : ℂ :=
  ∑' pq : Nat.Primes × Nat.Primes, W s (pq.1 : ℕ) (pq.2 : ℕ)

def catalogueL (A : Aff) : Set ℂ := {s | s ∈ A.HL ∧ A.app s ∈ Pinch}

def WS (s : ℂ) (p q : ℕ) : ℂ :=
  (p : ℂ) ^ (1 - s) * (q : ℂ) ^ (2 - s) * ((((p ^ 2 + q ^ 2 : ℕ) : ℂ)) ^ 2)⁻¹

def WG (s : ℂ) (p q : ℕ) : ℂ := (((p + q : ℕ) : ℂ)) ^ (-s)

def WSq (s : ℂ) (p q : ℕ) : ℂ := (((p ^ 2 + q ^ 2 : ℕ) : ℂ)) ^ (-s)

theorem thm_boundary (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hZ4 : Z4) (hZ5 : Z5) (hZ6 : Z6) :
    (∃ F : ℂ → ℂ, MeromorphicOn F HH ∧ EqOn F T (halfPlane 2)) ∧
    ∀ F : ℂ → ℂ, MeromorphicOn F HH → F =ᶠ[codiscreteWithin (halfPlane 2)] T →
      (∀ y ε : ℝ, 0 < ε →
        {s | s ∈ ball (1 + (y : ℂ) * I) ε ∧ s ∈ HH ∧ meromorphicOrderAt F s < 0}.Infinite) ∧
      IsNaturalBoundary F HH lineOne :=

end GoldbachBoundaries.Paper2

namespace GoldbachBoundaries.HInf

open GoldbachBoundaries GoldbachBoundaries.Paper2

abbrev Sser : ℂ → ℂ := Dser WS
```

```
abbrev catG : Set ℂ := catalogueL Aff.idL

abbrev catS : Set ℂ := catalogueL Aff.twoOneL

def AdmG (g : ℂ → ℂ) (s₁ : ℂ) : Prop := IsAdmissibleGermOn G (HH \ catG) (halfPlane 2) g s₁

def AdmS (g : ℂ → ℂ) (s₁ : ℂ) : Prop :=
    IsAdmissibleGermOn Sser (halfPlane 0 \ catS) (halfPlane 1) g s₁

abbrev SqSer : ℂ → ℂ := Dser WSq

abbrev catSq : Set ℂ := catalogueL Aff.twoL

def AdmSq (g : ℂ → ℂ) (s₁ : ℂ) : Prop :=
  IsAdmissibleGermOn SqSer (Aff.twoL.HL \ catSq) Aff.twoL.RL g s₁

def Wlogq (s : ℂ) (p q : ℕ) : ℂ := (Real.log q : ℂ) * WG s p q

abbrev MixSer : ℂ → ℂ := Dser Wlogq

def AdmMix (g : ℂ → ℂ) (s₁ : ℂ) : Prop :=
  IsAdmissibleGermOn MixSer
    (Aff.idL.HL \ catalogueL Aff.idL) Aff.idL.RL g s₁

theorem boundary_GS (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hZ4 : Z4) (hZ5 : Z5) (hZ6 : Z6) :
    (∀ (g : ℂ → ℂ) (s₁ : ℂ), AdmG g s₁ → ∀ y r : ℝ, 0 < r →
        s₁ ∈ ball (1 + (y : ℂ) * I) r →
      ¬ GermExtendsMeromorphically g s₁ (ball (1 + (y : ℂ) * I) r)) ∧
    (∀ (g : ℂ → ℂ) (s₁ : ℂ), AdmS g s₁ → ∀ y r : ℝ, 0 < r → s₁ ∈ ball ((y : ℂ) * I) r →
      ¬ GermExtendsMeromorphically g s₁ (ball ((y : ℂ) * I) r)) :=

theorem boundary_Sq (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hZ4 : Z4) (hZ5 : Z5) (hZ6 : Z6) :
    ∀ (g : ℂ → ℂ) (s₁ : ℂ), AdmSq g s₁ → ∀ y r : ℝ, 0 < r →
      s₁ ∈ ball ((1/2 : ℂ) + (y : ℂ) * I) r →
      ¬ GermExtendsMeromorphically g s₁ (ball ((1/2 : ℂ) + (y : ℂ) * I) r) :=

theorem boundary_mix (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hZ4 : Z4) (hZ5 : Z5) (hZ6 : Z6) :
    ∀ (g : ℂ → ℂ) (s₁ : ℂ), AdmMix g s₁ →
      ∀ y r : ℝ, 0 < r →
        s₁ ∈ ball (1 + (y : ℂ) * I) r →
        ¬ GermExtendsMeromorphically g s₁
          (ball (1 + (y : ℂ) * I) r) :=

end GoldbachBoundaries.HInf
```

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