A Proof of the Galois–McKay Conjecture

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Abstract

We prove the Galois–McKay conjecture [142, Conjecture A]. More precisely, for a nonabelian finite simple group S and a prime p, denote by $\mathrm{iMN}(S,p)$ the inductive Galois–McKay condition, and by $\mathcal{H}_p$ the subgroup of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ consisting of the automorphisms σ for which some $f \in \mathbb{Z}$ satisfies $\sigma(\zeta) = \zeta^{p^f}$ for every root of unity ζ of order prime to p. We verify $\mathrm{iMN}(S,p)$ for every nonabelian finite simple group S with $p \mid |S|$. By the reduction theorem of Navarro, Späth and Vallejo [141], it follows that for every finite group G, every prime p, and every $P \in \mathrm{Syl}_p(G)$, there is an $\mathcal{H}_p$-equivariant bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathrm{N}_G(P))$.

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This is a corollary of a candidate proof of a "Galois-Dade theorem" shared by Baoyu Zhang. See Zhang [208, §14.3] for a well-written formulation and proof of its reduction theorem to finite simple groups. It is my own decision to post this manuscript here. He believes that it would be ideal if the proofs are written by mathematicians who know the problems best. For example, at the end of July when he obtained the candidate proof of the inductive Alperin weight condition for several full families of remaining simple groups, he told me that he would like to collaborate with the mathematicians who are most likely doing Alperin's weight conjecture. He expressed deep concerns that with the emerging wide use of AI in mathematics, trust inside the mathematical community would probably experience a difficult time. Not long after starting his PhD, one day he made a post in "moments" of Wechat. He said that among the many Chinese mathematicians that messaged him, a young professor specializing in Alperin's weight conjecture was the first one who contact him and wished him a successful PhD. Baoyu said that it would be most inappropriate for him to post a full proof of Alperin's weight conjecture himself. On the same day when OpenAI announced that they had obtained hundreds of proofs of the most important problems in most areas of mathematics and would release them shortly, I also thought that he should not wait any longer. If it turned out that OpenAI didn't post a proof of Alperin's weight conjecture, leaving the proof of types D and E would still be a good chance to collaborate, let alone what he actually had obtained was already stronger than a candidate proof of a Galois version of the Character Triple Conjecture. So there would be a lot to collaborate.

I decided to post this manuscript on Hexagon because I was notified as a follower of his GitHub that Baoyu Zhang posted the formalisation of the proof of Feit's conjecture yesterday. My AI also notified me of this GitHub upload when I asked it questions on the Galois-McKay conjecture. OpenAI claimed that they had more than 4000 proofs of

major problems in different areas of mathematics. Earlier in this month, someone post about one hundred candidate proofs in representation theory on Hexagon, including Broue's abelian defect group conjecture and Brauer's $k(B)$ conjecture for $p = 2$. I know how quickly proofs of some related conjecture could be produced with these ingredients in his GitHub, and anyone like the one who post about one hundred proofs on the same day may do this again at any time. Baoyu had already obtained candidate proofs of those problems claimed by AI, but only saw them as material for self-learning. Fortunately, a group of mathematicians are working on the draft of over thousand of pages on the abelian defect group conjecture that he shared, as many people reached Baoyu after the one hundred proofs on Hexagon. So I post this manuscript that I am reading and revising. It is my own decision and I take full responsibility for any consequence of doing this.

1 Introduction

Local–global questions in character theory relate the irreducible characters of a finite group to those of its local subgroups. Let P be a Sylow p-subgroup of a finite group G, and let $M = \mathrm{N}_G(P)$ be the normalizer of P in G. A basic question is whether G and M have the same number of irreducible characters of degree prime to p. McKay [135] observed that certain finite simple groups have the same number of odd-degree irreducible characters as the normalizers of their Sylow 2-subgroups. The McKay conjecture extends this observation to every finite group G and every prime p. In the notation $\mathrm{Irr}_{p'}(X)$ for the irreducible characters of X of degree prime to p, the conjecture states that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(M)|$. Isaacs, Malle and Navarro [96] reduced the conjecture to conditions on finite simple groups, and Cabanes and Späth [35] completed its proof.

Navarro [142] asks how much of the arithmetic of the characters of G can also be read from M. If $n = |G|$, the character values lie in $\mathbb{Q}_n = \mathbb{Q}(\zeta_n)$. A Galois automorphism σ acts by $\chi^\sigma(g) = \sigma(\chi(g))$ and leaves the degree $\chi(1)$ fixed. Thus, it acts on both $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(M)$. On roots of unity of order prime to p, the Frobenius at p acts by $\zeta \mapsto \zeta^p$. Navarro considers Galois automorphisms that act on all such roots by the same power of this Frobenius.

Conjecture 1.1 (Galois–McKay conjecture, Navarro [142, Conjecture A])**.** *Let G be a finite group of order n, let p be a prime, and let P be a Sylow p-subgroup of G. Put $M = \mathrm{N}_G(P)$ and $\mathbb{Q}_n = \mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of unity. Suppose that $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ and that one integer $e \geq 0$ satisfies $\sigma(\zeta) = \zeta^{p^e}$ for every root of unity $\zeta \in \mathbb{Q}_n$ of order prime to p. For $X \in \{G, M\}$, let $\mathrm{Irr}_{p'}(X)^\sigma$ be the set of irreducible characters χ of X with $p \nmid \chi(1)$ and $\sigma(\chi(x)) = \chi(x)$ for all $x \in X$. Then $|\mathrm{Irr}_{p'}(G)^\sigma| = |\mathrm{Irr}_{p'}(M)^\sigma|$.*

Let $\mathcal{H}_p$ be the subgroup of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ whose elements act on all roots of unity of order prime to p by a single integral power of the Frobenius. Navarro [142, p. 1139] also asks whether one bijection is equivariant for the whole action of $\mathcal{H}_p$. If Ω is such a bijection, it maps the σ-fixed characters of G onto those of M in the two p'-degree sets. Moreover, $\mathrm{Stab}_{\mathcal{H}_p}(\chi) = \mathrm{Stab}_{\mathcal{H}_p}(\Omega(\chi))$. By the local description of $\mathcal{H}_p$ in Navarro, Späth and Vallejo [141, §1], the values of χ and those of $\Omega(\chi)$ generate the same field over $\mathbb{Q}_p$.

Navarro, Späth and Vallejo [141] reduce the problem for G to the inductive Galois–McKay condition $\mathrm{iMN}(S, p)$ for the nonabelian simple sections S of G with $p \mid |S|$. The condition combines equivariance with a relation between character triples. Definition 2.1 states the condition. The reduction is restated as Theorem 2.2.

Brunat and Nath [30] construct the $\mathcal{H}_p$-equivariant bijection for A_n at odd primes. Brunat and Nath [29, Corollary 5.3] obtain a spin correspondence for $2.A_n$ equivariant under the subgroup fixing all roots of unity of order prime to p. Ruhstorfer [162] and Johansson [101] establish the inductive condition for groups of Lie type at the characteristic of the field. Johansson [103] also treats the Suzuki and Ree groups. Ruhstorfer and Schaeffer Fry [159] complete the inductive condition at $p = 2$.

Together with these results, the verifications below establish $\mathrm{iMN}(S, p)$ for every finite nonabelian simple group S and every prime $p \mid |S|$. Theorem 2.2 then yields the following conclusion.

Theorem 1.2. *Let G be a finite group, let p be a prime, and let $P \in \mathrm{Syl}_p(G)$. There is an $\mathcal{H}_p$-equivariant bijection $\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(\mathrm{N}_G(P))$. In particular,* Conjecture 1.1 *holds for (G, p).*

Section 2 gives the reduction and the character-triple criteria. Sections 3–8 treat groups of Lie type and sporadic groups. Section 9 treats alternating groups, completes the verification for the simple groups, and deduces Theorem 1.2.

2 Preliminaries

Let S be a finite simple group, with $[S,S] = S \neq 1$, and fix a prime $p \mid |S|$. Let X be the universal cover of S. Choose $R \in \mathrm{Syl}_p(X)$ and put $\Gamma = \mathrm{Aut}(X)_R$. Given $\chi \in \mathrm{Irr}(X)$, we write $\Gamma_{\chi^{\mathcal{H}_p}}$ for the stabilizer of $\chi^{\mathcal{H}_p}$ in Γ, and Γ_χ for the stabilizer of χ. Moreover, for $N \lhd G$ and a G-invariant $\theta \in \mathrm{Irr}(N)$, the Brauer–Clifford group for (G,N,θ) is defined in Turull [193]. We recall $\mathrm{iMN}(S,p)$ from Navarro et al. [141, Definition 3.1].

Definition 2.1. The condition $\mathrm{iMN}(S,p)$ holds if there is a Γ-stable subgroup $\mathrm{N}_X(R) \leq M < X$ and a $\Gamma \times \mathcal{H}_p$-equivariant bijection $\Omega : \mathrm{Irr}_{p'}(X) \longrightarrow \mathrm{Irr}_{p'}(M)$ such that, for $\psi = \Omega(\chi)$ and $D = \Gamma_{\chi^{\mathcal{H}_p}}$, $(X \rtimes D, X, \chi)_{\mathcal{H}_p} \succeq_c (M \rtimes D, M, \psi)_{\mathcal{H}_p}$.

Theorem 2.2 (Reduction theorem, Navarro, Späth and Vallejo [141])**.** *Let G be a finite group, p a prime, and $P \in \mathrm{Syl}_p(G)$. Assume that $\mathrm{iMN}(S,p)$ holds for each nonabelian simple section $S \cong U/V$ of G with $p \mid |S|$, where $V \lhd U \leq G$. Then, there is an $\mathcal{H}_p$-equivariant bijection $\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(\mathrm{N}_G(P))$.*

In Navarro et al. [141, Definition 1.5(iii)–(iv)], take associated projective models $\mathcal{P}, \mathcal{P}'$ on $I = X \rtimes \Gamma_\chi$ and $I' = M \rtimes \Gamma_\chi$. Their factor sets satisfy $\alpha|_{I' \times I'} = \alpha'$, and at $z \in \mathrm{C}_{X \rtimes D}(X)$ the scalar λ_z is common. Thus, $\mathcal{P}(z) = \lambda_z I_{V_\chi}$ and $\mathcal{P}'(z) = \lambda_z I_{V_\psi}$, for the respective spaces V_χ, V_ψ. For (iv), put $B = M \rtimes D$. Under (i) $-$ (iii), Navarro et al. [141, Lemma 1.8] reduces the test $\mu_{a,\sigma}|_{I'} = \mu'_{a,\sigma}$ to representatives of $(B \times \mathcal{H}_p)_\chi / I'$.

Put $Z_p = \mathrm{Z}(X)_p$, $N = \mathrm{N}_X(R)$ and $\overline{X} = X/Z_p$. By Johansson [101, proof of Lemma 5.1], inflation from $\overline{X}$ and $\mathrm{N}_{\overline{X}}(R/Z_p) = N/Z_p$ gives the p'-degree characters of X and N, respectively.

Remark 2.3. Let $q = r^f$ with r an odd prime and f odd. Assume that $\mathrm{ord}_\ell(q) \equiv 2 \bmod 4$ for an odd prime ℓ. Given $\sigma \in \mathcal{H}_\ell$, write $\sigma(\zeta_r) = \zeta_r^a$ and choose $j^2 = (-1)^{(q+1)/2}$. By Ireland–Rosen [92, Chapter 6], the quadratic Gauss sum g satisfies $g^2 = (-1)^{(r-1)/2} r$ and $\sigma(g)/g = (a/r)$. Thus, $(jg)^2 = -r$. By Ruhstorfer et al. [156, Lemma 2.3(c)], σ fixes $\sqrt{-q}$ and hence jg. Therefore, $\sigma(j)/j = (a/r)$. More generally, put $b = v_2(\mathrm{ord}_\ell(r)) > 0$ and choose α of order 2^{b+1}. By Euler's criterion, $(r/\ell) = (-1)^{(\ell-1)/2^b}$. Thus, $\sigma(\alpha)/\alpha = (r/\ell)^u = \sigma(\sqrt{r})/\sqrt{r}$ for the ℓ^u-action on ℓ'-roots. By Ireland–Rosen [92, Chapter 5], this is (a/r) if r or ℓ is 1 mod 4. If $\mathrm{ord}_\ell(q) \equiv 2 \bmod 4$, then $b = 1 + v_2(f)$.

Let A, B be groups with normal subgroups X, L, and fix surjective homomorphisms $\pi_A : A \to J$ and $\pi_B : B \to J$ with kernels X and L, respectively. The fibre product $D = A \times_J B$ consists of the pairs (a,b) satisfying $\pi_A(a) = \pi_B(b)$. Thus, we put $D = A \times_J B$, $K = X \times L \lhd D$. The map $(a,b) \mapsto \pi_A(a) = \pi_B(b)$ has kernel K and induces $D/K \cong J$. Suppose $\chi \in \mathrm{Irr}(X)$ is A-invariant and $\beta \in \mathrm{Irr}(L)$ is B-invariant. A simultaneous transport $h = (a,b,\sigma)$ has $a \in \mathrm{Aut}(A)$, $b \in \mathrm{Aut}(B)$, and a coefficient automorphism σ. Given $j_h \in \mathrm{Aut}(J)$, we require $\pi_A \circ a = j_h \circ \pi_A$ and $\pi_B \circ b = j_h \circ \pi_B$. Moreover, the group action on D is $(u,v) \mapsto (a(u), b(v))$. Together, these actions define h on $\mathrm{Irr}(D)$. Throughout, $\chi^h = \chi$ and $\beta^h = \beta$. Moreover, we write $\vartheta = \chi \boxtimes \beta^\vee$, where $\boxtimes$ is the external tensor product.

Proposition 2.4. *Let Ξ be a virtual character of D for which $\langle \Xi|_K, \vartheta \rangle_K = 1$. Then, we have that ϑ has an ordinary extension V to D, fixed by each transport h with $\Xi^h = \Xi$. If $K \leq D_1 \leq D$, the construction with $\Xi|_{D_1}$ gives $V|_{D_1}$. If $c \in \mathrm{C}_D(K)$ acts by a common scalar λ on the two parts of Ξ, then $V(c) = \lambda I$. In particular, if Ξ is inflated from D/C_0, where $C_0 \lhd D$ centralizes K, then C_0 acts trivially on V. If $\langle \Xi|_K, \vartheta \rangle_K = -1$, the conclusion holds for $-\Xi$.*

Proof. Let $H_\pm$ afford the parts $\Xi_\pm$ of $\Xi = \Xi_+ - \Xi_-$. Choose a model (U,ρ) of ϑ and an associated projective representation T with factor set f on J. By [208, proof of Lemma 3.17, Step 3], on $E_\pm = \mathrm{Hom}_K(U, H_\pm)$ put $S_{\pm,d}(h) = H_\pm(d) h T_d^{-1}$. Since $m_\pm = \dim E_\pm$ satisfy $m_+ - m_- = 1$, $\eta(d) = \det(S_{+,d})/\det(S_{-,d})$ satisfies $\delta\eta = f^{-1}$. Thus, the extension is $V(d) = \eta(d) T_d$. The common summands cancel in η by the same argument. Moreover, $T_d \mapsto cT_d$ multiplies $\eta(d)$ by c^{-1}. Hence, $[V]$ depends only on Ξ, ϑ. For $d \in D_1$, the restrictions $H_\pm|_{D_1}$ give the same operators on $E_\pm$. Consequently, the construction for $\Xi|_{D_1}$ gives $V|_{D_1}$. Given h, the equations for $E_\pm$ give $\mathrm{Hom}_K(U^h, H_\pm^h)$. The operator $S_{\pm,d}^h$ at d^h has determinant $\sigma(\det S_{\pm,d})$, where σ is the coefficient part of h. If $\Xi^h = \Xi$, $\Xi_+^h = \Xi_+$ and $\Xi_-^h = \Xi_-$. Consequently, the extension V satisfies $V^h \cong V$. Given $c \in \mathrm{C}_D(K)$ acting by λ on $H_\pm$, take $T_c = 1$. By [208, proof of Lemma 3.17, Step 5], the quotient at c is $\lambda^{m_+ - m_-} = \lambda$. Thus, $V(c) = \lambda I$. If Ξ is inflated from D/C_0, then $\Xi_\pm$ are inflated from D/C_0. Hence, each $c \in C_0$ gives $\lambda = 1$. □

Corollary 2.5. *Let Ξ satisfy the hypotheses of* Proposition 2.4. *Let Q be an associated model of B for β, with normalized factor c inflated from J. Then, we have that A has a projective model $\mathcal{P}$ for χ with the same factor c. Given a simultaneous transport h with $\Xi^h = \Xi$, we have that the twist functions $\mu_h^{\mathcal{P}}, \mu_h^Q$ agree on J. If Ξ is inflated from D/C_0 as in* Proposition 2.4, *then for $(a,b) \in C_0$, $\mathcal{P}(a) = \lambda I$ and $Q(b) = \lambda I$ for a scalar λ.*

Proof. Let U_β be the space of Q, and take $H = (V \otimes U_\beta)^L$, with $l \in L$ acting by $V(1,l) \otimes Q(l)$. As an X-module, H has character χ, since $(U_\beta^\vee \otimes U_\beta)^L$ is the scalar endomorphism space of U_β. Given $a \in A$, choose $b \in B$ with $\pi_A(a) = \pi_B(b)$. The operator $V(a,b) \otimes Q(b)$ stabilizes H. Thus, we put $\mathcal{P}(a) = (V(a,b) \otimes Q(b))|_H$. The operator $\mathcal{P}(a)$ is independent of b. Indeed, another lift is bl for $l \in L$. Thus, the extra factor $V(1,l) \otimes Q(l)$ is trivial on H. Moreover, the normalized factor c of Q is inflated from J, so $c(b,l) = c(l,b) = 1$. Given $\pi_A(a_i) = \pi_B(b_i) = j_i$, the product lift of a_1a_2 is b_1b_2. Hence, $\mathcal{P}(a_1)\mathcal{P}(a_2) = c(j_1,j_2)\mathcal{P}(a_1a_2)$ by the product laws for V, Q. Since c is normalized on J, $\mathcal{P}$ is associated to χ.

Let h be a simultaneous transport with $\Xi^h = \Xi$. Write V^h, Q^h, H^h for the corresponding h-twists. Then, $H^h \cong (V^h \otimes U_\beta^h)^L$. By Proposition 2.4, we obtain an intertwiner $S : V^h \to V$. By [141, Lemma 1.4], we find an intertwiner $R : U_\beta^h \to U_\beta$ and a unique normalized function $\mu_h : B \to \overline{\mathbb{Q}}^\times$, constant on L-cosets, with $RQ^h(b)R^{-1} = \mu_h(b)Q(b)$, $\mu_h(1) = 1$. In particular, $\mu_h(l) = 1$ for $l \in L$. Thus, $W = (S \otimes R)|_{(V^h \otimes U_\beta^h)^L}$ is an isomorphism onto H. For $\pi_A(a) = \pi_B(b)$, the tensor product equations give $W\mathcal{P}^h(a)W^{-1} = \mu_h(b)\mathcal{P}(a)$ on H. By [141, Lemma 1.4], the correcting function for $\mathcal{P}$ is $\mu_h^A(a) = \mu_h(b)$. Thus, the functions coincide on $A/X \cong J \cong B/L$ as in [141, Definition 1.5(iv)]. Given a diagonal pair $(a,b) \in C_0$, the operator $V(a,b)$ is the identity. When b centralizes L, Schur's lemma makes $Q(b)$ scalar. Thus, write $Q(b) = \lambda_b I$. Then, $\mathcal{P}(a) = \lambda_b I_H$. Thus, $\mathcal{P}(a)$ and $Q(b)$ have the same scalar [141, Definition 1.5(iii)]. □

Let $N \lhd D$ be finite, $J = D/N$, and let $\vartheta \in \mathrm{Irr}(N)$ be D-invariant. A mixed automorphism (a,σ) acts on ξ by $x \mapsto \sigma(\xi(a^{-1}x))$. Moreover, we fix a group $\mathcal{A}$ of mixed automorphisms $h = (a,\sigma)$ with $a(N) = N$ and $\vartheta^h = \vartheta$. Put $F = \mathbb{Q}(\zeta_{|D|})$, and take the finite image of $\mathcal{A}$ on D, F. Moreover, for $U \le D$ and $\xi \in \mathrm{Irr}(U)$, we have that all values of ξ lie in F. Adjoin the inner automorphisms $\mathrm{Int}(d)$, for $d \in D$, with trivial coefficient part 1. Moreover, we have that $\mathrm{Int}(d)$ fixes ϑ and every $\xi \in \mathrm{Irr}(D)$. Below, $\mathcal{A}$ is this finite group.

Lemma 2.6. *The integers $\langle \Xi|_N, \vartheta \rangle_N$, $\Xi \in (\mathbb{Z}\,\mathrm{Irr}(D))^{\mathcal{A}}$ form an ideal $d\mathbb{Z}$, where $d > 0$ divides $|J|$. Moreover, the conditions on d, ϑ are equivalent. In* (1)*, the character ϑ has an $\mathcal{A}$-invariant ordinary extension to D. In* (2)*, $d = 1$. Finally, in* (3)*, for every prime $r \mid |J|$, some $\Xi \in (\mathbb{Z}\,\mathrm{Irr}(D))^{\mathcal{A}}$ satisfies $r \nmid \langle \Xi|_N, \vartheta \rangle_N$.*

Proof. Put $\mathcal{V} = (\mathbb{Z}\,\mathrm{Irr}(D))^{\mathcal{A}}$ and $m : \Xi \mapsto \langle \Xi|_N, \vartheta \rangle_N$. Then, $m : \mathcal{V} \to \mathbb{Z}$ is additive. Thus, $m(\mathcal{V}) = d\mathbb{Z}$ is an ideal of $\mathbb{Z}$. Put $\Xi_0 = \mathrm{Ind}_N^D \vartheta$. By [95, proof of Theorem 6.2] and D-invariance of ϑ, we have $\Xi_0|_N = |J|\vartheta$. Given $h \in \mathcal{A}$, $(\mathrm{Ind}_N^D \vartheta)^h = \mathrm{Ind}_N^D(\vartheta^h)$ and $\vartheta^h = \vartheta$. Thus, $\Xi_0 \in \mathcal{V}$ and $m(\Xi_0) = |J|$. Consequently, $d > 0$ and $d \mid |J|$. An $\mathcal{A}$-invariant extension $\widehat{\vartheta}$ to D satisfies $m(\widehat{\vartheta}) = 1$, so $d = 1$. If $d = 1$, some $\Xi \in \mathcal{V}$ has $m(\Xi) = 1$, so (3) holds. Conversely, (3) gives $r \nmid d$ for every $r \mid |J|$. Since $d \mid |J|$, we have $d = 1$. Thus, some $\Xi \in \mathcal{V}$ satisfies $m(\Xi) = 1$. Proposition 2.4 with $(A, X, B, L, \beta) = (D, N, J, 1, 1)$ gives an extension of ϑ. Each element of $\mathcal{A}$ acts on the quotient $J = D/N$, so the extension is $\mathcal{A}$-invariant. □

Theorem 2.7. *If r is a prime divisor of $|J|$, we choose $R_r \in \mathrm{Syl}_r(J)$ and let D_r be its full preimage in D. Put $\mathcal{N}_r = \{(a,\sigma) \in \mathcal{A} : a(D_r) = D_r\}$ and $Q_r \in \mathrm{Syl}_r(\mathcal{N}_r)$. Then, an $\mathcal{A}$-invariant extension of ϑ to D exists precisely when, for each $r \mid |J|$, there is a Q_r-fixed extension $\xi_r \in \mathrm{Irr}(D_r)$ of ϑ.*

Proof. Let $\widehat{\vartheta}$ be an invariant extension to D. Its restriction $\widehat{\vartheta}|_{D_r}$ is irreducible, since $\widehat{\vartheta}|_N = \vartheta$ is irreducible. It is fixed by $\mathcal{N}_r$, so Q_r fixes it. Conversely, we choose $r \mid |J|$ and a Q_r-fixed extension $\xi_r \in \mathrm{Irr}(D_r)$ of ϑ. Let $\mathcal{O}_r$ be its $\mathcal{N}_r$-orbit. Since Q_r stabilizes ξ_r, the size $|\mathcal{O}_r|$ is prime to r. Consequently, $\omega_r = \sum_{\xi \in \mathcal{O}_r} \xi$ is $\mathcal{N}_r$-invariant and $\omega_r|_N = |\mathcal{O}_r|\vartheta$. Set $\Xi_r = \mathrm{Ind}_{D_r}^D \omega_r$. Let $(a,\sigma) \in \mathcal{A}$. Moreover, we have that the groups $a(D_r)/N$ and $R_r = D_r/N$ are Sylow r-subgroups of J. Thus, we choose $x \in D$ with $(xa(D_r)x^{-1})/N = R_r$. Then, $h = \mathrm{Int}(x)(a,\sigma)$ belongs to $\mathcal{N}_r$. Since $\omega_r^h = \omega_r$, induction gives $\Xi_r^h = \Xi_r$. Since $\Xi_r^{\mathrm{Int}(x)} = \Xi_r$, $\Xi_r^{(a,\sigma)} = \Xi_r$. By [95, Chapter 5, p. 62], $\Xi_r|_N = [D : D_r]|\mathcal{O}_r|\vartheta$, since $\omega_r|_N = |\mathcal{O}_r|\vartheta$ and ϑ is D-invariant. Therefore, $\langle \Xi_r|_N, \vartheta \rangle_N = [J : R_r]\,|\mathcal{O}_r|$. Both factors $[J : R_r], |\mathcal{O}_r|$ are prime to r. Thus, Lemma 2.6 gives an $\mathcal{A}$-invariant extension of ϑ to D. □

Corollary 2.8. *Let $X \lhd A$, $A = XH$, $M = X \cap H$, and suppose $C = \mathrm{C}_A(X) \le H$. Let $\chi \in \mathrm{Irr}(X)$ and $\psi \in \mathrm{Irr}(M)$ be invariant under A and H, respectively. Suppose that $\chi(z)/\chi(1) = \psi(z)/\psi(1)$ for $z \in \mathrm{Z}(X)$. Put $\Delta C = \{(c,c) : c \in C\}$. Choose a group $\mathcal{A}$ of simultaneous mixed transports u with $X^u = X$, $M^u = M$, $\chi^u = \chi$, $\psi^u = \psi$, and $(\Delta C)^u = \Delta C$. Moreover, for an invariant extension V of $\vartheta = \chi \boxtimes \psi^\vee$, require $\Delta C \le \ker V$. By* Theorem 2.7*, such V exists precisely when the tests for $A/(XC)$ hold. By* Corollary 2.5*, the factors satisfy $\alpha|_{H \times H} = \alpha'$, the central scalars satisfy $\lambda = \lambda'$, and the mixed twist functions satisfy $\mu = \mu'$.*

Proof. The inclusion $H \hookrightarrow A$ induces an isomorphism $H/M \to A/X$, since $A = XH$ and $M = X \cap H$. Form $F = A \times_{A/X} H$ with base $K = X \times M$, and write $\Delta C = \{(c, c) : c \in C\} \leq F$. Let $(a, h) \in F$. Then, $a = hx$ for some $x \in X$. If $c \in C = \mathrm{C}_A(X)$, we have $aca^{-1} = hch^{-1}$. Thus, conjugation by (a, h) acts by $(c, c) \mapsto (hch^{-1}, hch^{-1})$. Hence, $\Delta C \lhd F$.

The intersection $\Delta C \cap K$ is $\Delta \mathrm{Z}(X)$. For $z \in \mathrm{Z}(X)$, the central characters of χ, ψ agree. Thus, $\vartheta(z, z) = \chi(1)\psi(1)$ for $\vartheta = \chi \boxtimes \psi^\vee$, and $\Delta \mathrm{Z}(X)$ acts trivially on ϑ. Moreover, C centralizes X and $M \leq X$, so ΔC centralizes K. Hence, ϑ defines an irreducible character $\overline{\vartheta}$ of $\overline{K} = K\Delta C/\Delta C$. This group $\overline{K}$ is normal in $\overline{F} = F/\Delta C$. Moreover, the character $\overline{\vartheta}$ is invariant under $\overline{F}$ and the given transports $\mathcal{A}$. The map $F \to F/K \cong A/X$ acts by $\Delta C \mapsto XC/X$. Consequently, $\overline{F}/\overline{K} \simeq A/(XC)$. Apply Theorem 2.7 to $\overline{K} \lhd \overline{F}$, adjoining inner automorphisms $\mathrm{Inn}(\overline{F})$. Inflation along $F \to \overline{F}$ bijects these extensions with extensions on F whose kernels contain ΔC. By Corollary 2.5, we obtain the $\succeq_c$ comparison for (χ, ψ). Conjugation defines $A \to \mathrm{Aut}(X)$. Given $a \in A$, the induced automorphism of X is inner if and only if $a \in XC$. Thus, $A/(XC) \to \mathrm{Out}(X)$ is injective. Only prime divisors of $|A/(XC)|$ enter the criterion. $\square$

If $\psi = \Omega(\chi)$, suppose that $\Omega(\nu^{a\sigma}) = \Omega(\nu)^{a\sigma}$ for $\nu \in \mathrm{Irr}_{p'}(X)$ and $(a, \sigma) \in \Gamma \times \mathcal{H}_p$. Suppose also that $\chi(z)/\chi(1) = \psi(z)/\psi(1)$ for $z \in \mathrm{Z}(X)$, as in Section 2. In $\mathrm{Out}(X)$, we write I_χ for the image of Γ_χ. If $\gcd(\langle \mathrm{Res}^X_M \chi, \psi\rangle_M, |I_\chi|) = 1$, the $\succeq_c$ relation holds for (χ, ψ). Indeed, we set $A = X \rtimes \Gamma_\chi$, $H = M \rtimes \Gamma_\chi$ and $C = \mathrm{C}_A(X)$. An element $(x, a) \in C$ satisfies $a = \mathrm{Int}(x^{-1}) \in \Gamma_\chi$, whence $x \in \mathrm{N}_X(R) \leq M$. Thus, we obtain that $C \leq H$, and $A/(XC) \cong I_\chi$. Let $F = A \times_{A/X} H$ and $\overline{K} \lhd \overline{F}$ be as in Corollary 2.8. Since $aX = hX$ on F, the rule $(a, h)y = ayh^{-1}$ acts on X. Since C centralizes X, this action descends to $\overline{F} = F/\Delta C$. Let Π be its permutation character. Each $u = (b, \sigma)$ in the mixed stabilizer of (χ, ψ) fixes Π, since b permutes X and σ fixes the integer values. On $K = X \times M$, the action is transitive with stabilizer ΔM at $1 \in X$. Thus, by Isaacs [95, Lemma 5.2], $\langle \Pi|_K, \chi \boxtimes \psi^\vee\rangle_K = \langle \mathrm{Res}^X_M \chi, \psi\rangle_M$. By inflation, this is also $\langle \Pi|_{\overline{K}}, \overline{\vartheta}\rangle_{\overline{K}}$. The ideal in Lemma 2.6 for $\overline{K} \lhd \overline{F}$ contains $\langle \mathrm{Res}^X_M \chi, \psi\rangle_M$ and $[\overline{F} : \overline{K}] = |I_\chi|$. Under the coprime condition, $\gcd(\langle \mathrm{Res}^X_M \chi, \psi\rangle_M, |I_\chi|) = 1$. Thus, the ideal is $\mathbb{Z}$. By Corollary 2.8, $\overline{\vartheta}$ has an invariant extension.

Given b of cyclic defect, Navarro [142, Theorem 3.4(a)] gives a bijection $\mathrm{Irr}(b) \to \mathrm{Irr}(c_b)$ with its Brauer correspondent c_b, equivariant under $(\mathcal{H}_\ell)_b$. If S is nonabelian simple, $X \to S$ its universal cover and ℓ odd, Koshitani–Späth [112, Theorem 1.1] proves that b satisfies inductive Alperin–McKay. By Kessar–Linckelmann [107, Theorem 1.10], Rouquier's $C_b^\bullet$ descends to the minimal coefficient ring $\mathcal{O}_b$ of b. The Euler character gives Ω_b with signs by Kessar–Linckelmann [107, Lemma 2.1 and Corollary 1.11].

Proposition 2.9. *Let X be a finite perfect group, ℓ an odd prime, and $P \in \mathrm{Syl}_\ell(X)$ nontrivial and cyclic. Let $P[\ell]$ be its subgroup of order ℓ. Suppose $\mathrm{N}_X(P[\ell]) = \mathrm{N}_X(P) = M$, or equivalently that P is the unique Sylow ℓ-subgroup of $\mathrm{C}_X(P[\ell])$. There is an $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant bijection $\Omega : \mathrm{Irr}_{\ell'}(X) \to \mathrm{Irr}_{\ell'}(M)$ with the $\succeq_c$ relation at $(\chi, \Omega(\chi))$. Moreover, for each $\chi \in \mathrm{Irr}_{\ell'}(X)$ some $\varepsilon_\chi \in \{1, -1\}$ gives $\chi(m) = \varepsilon_\chi \Omega(\chi)(m)$ at all ℓ-singular $m \in M$.*

Proof. Put $Z_1 = P[\ell]$ and $C_1 = \mathrm{C}_X(Z_1)$. Since $C_1 \lhd \mathrm{N}_X(Z_1)$ and $P \in \mathrm{Syl}_\ell(C_1)$, [97, Lemma 1.13] gives $\mathrm{N}_X(Z_1) = C_1 M$. Thus, $\mathrm{N}_X(Z_1) = M$ exactly when P is normal in C_1. By [95, Theorem 5.6], $P \cap \mathrm{Z}(X) = 1$, since X is perfect and P is abelian. Moreover, $\mathrm{Aut}(O_\ell(X))$ is abelian since $O_\ell(X) \leq P$ is cyclic. Thus, perfectness gives $O_\ell(X) \leq P \cap \mathrm{Z}(X) = 1$. By [97, Lemma 5.12 and Theorem 5.20], M controls fusion in P and $[P, M] = P \cap X' = P$.

With $\Delta = \mathrm{Aut}(X)_P$, write $K = (X \times M)/\operatorname{diag}\mathrm{Z}(X)$. Let A be the permutation group of X generated by left X, inverse right M and Δ. Moreover, write the epimorphism $\pi : (X \times M) \rtimes \Delta \longrightarrow A$, $(x, m; d)(t) = x\, d(t) m^{-1}$. A translation $t \mapsto xtm^{-1}$ fixes 1 precisely when $x = m$, and then it acts as $\mathrm{inn}(m)$. Hence, the translation kernel is $\operatorname{diag}\mathrm{Z}(X)$, and $A = K\Delta$, with $K \cap \Delta = \mathrm{Inn}_X(M)$. Thus, $A/K = \Delta/\mathrm{Inn}_X(M)$. Let $\mathcal{B}$ be the full-defect blocks of X. For $b \in \mathcal{B}$, let c_b be its Brauer correspondent. By [51], $\mathrm{ht}(\chi) = 0$ on $\mathrm{Irr}(b) \cup \mathrm{Irr}(c_b)$. Thus, the height formula gives $\mathrm{Irr}_{\ell'}(X) = \bigcup_{b \in \mathcal{B}} \mathrm{Irr}(b)$ and $\mathrm{Irr}_{\ell'}(M) = \bigcup_{b \in \mathcal{B}} \mathrm{Irr}(c_b)$. By [107, Theorem 1.10 and §7], Rouquier's complex $C_b^\bullet$ for (b, c_b) has modules U_b, V_b over $\mathcal{O}_b$, the minimal coefficient ring of b. Take its Brauer tree root r_b as below. Over $\mathbb{K}$, a splitting field of characteristic 0, its Euler character is Θ_b. By [107, Lemma 2.1], $\Theta_b = \sum_{\chi \in \mathrm{Irr}(b)} \varepsilon_b(\chi)\, \chi \boxtimes \Omega_b(\chi)^\vee$, with $\varepsilon_b(\chi) \in \{1, -1\}$ and a bijection $\Omega_b : \mathrm{Irr}(b) \to \mathrm{Irr}(c_b)$. By [107, Theorem 1.6 and Corollary 1.11], Ω_b is $(\mathcal{H}_\ell)_b$-equivariant, since $C_b^\bullet$ is over $\mathcal{O}_b$. Put $\Theta = \sum_{b \in \mathcal{B}} \Theta_b$, a virtual character of K.

Given $K \leq A_0 \leq A$, assume $\ell \nmid |A_0/K|$. Put $k_0 = \mathbb{F}_\ell$ and take a finite splitting field k/k_0 for every group G_0 in the construction. Enlarge k to split the endomorphism-division fields $\mathrm{End}(U_i)/J(\mathrm{End}(U_i))$ of all selected summands U_i, for every A_0 with $K \leq A_0 \leq A$. Moreover, for b a full-defect block idempotent of kX, we let c_b be its Brauer correspondent in kM, and put $A_b = kXb$, $B_b = kMc_b$. The map $e_b : kX \to kX$ given by $e_b(y) = byc_b$ is an idempotent. Moreover, put $e = \sum_{b \in \mathcal{B}} e_b$, and let $F_\ell : k \to k$ be residue Frobenius. By [107, proof of Theorem 1.10],

$c_b = \mathrm{Br}_{Z_1}(b)$ since $M = \mathrm{N}_X(Z_1)$. For $h \in \Delta \cup \{F_\ell\}$, $\mathrm{Br}_{Z_1}(h(b)) = h(\mathrm{Br}_{Z_1}(b))$. Thus, $c_{h(b)} = h(c_b)$ and $heh^{-1} = e$. For $(x, m) \in X \times M$ and $v \in kX$, $e((x, m) \cdot v) = (x, m) \cdot e(v)$. Therefore, a summand Y of the permutation k_0A_0-module k_0X satisfies $k \otimes_{k_0} Y \cong e(kX)$.

Since $O_\ell(X) = 1$ and $M = \mathrm{N}_X(Z_1)$, [107, proof of Theorem 1.10] gives a unique isomorphism class $[U_b]$ of nonprojective indecomposable summands of $b\,kX\,c_b$. Its vertex is $\Delta P = \{(p, p) : p \in P\}$. By [83], the Green correspondent U_b occurs with multiplicity 1 in $(kXb)|_{X\times M}$. Since $e(kX) = \bigoplus_{b\in\mathcal{B}} b\,kX\,c_b$, its multiplicity in $e(kX)$ is 1. Put $Z_\Delta = \mathrm{diag}\,\mathrm{Z}(X)$ and $e_Z = |Z_\Delta|^{-1}\sum_{z\in Z_\Delta} z$. Since Z_Δ has ℓ'-order and acts trivially on kX, the factor $k[X \times M]e_Z \cong kK$ gives equal multiplicities for U_b. Moreover, projective covers over $k[X \times M]$ stay in $k[X \times M]e_Z$. For $h \in \Delta$ or $h = F_\ell$, Brauer correspondence gives $h(b\,kX\,c_b) = h(b)\,kX\,c_{h(b)}$. By uniqueness, ${}^hU_b \cong U_{h(b)}$. The isomorphism $k[X \times M]e_Z \cong kK$ is h-equivariant.

Let Y_0 be a finite-dimensional k_0A_0-module and put $E = \mathrm{End}_{k_0K}(Y_0)$, $S = E/J(E)$, $H = A_0/K$. Since K centralizes E, conjugation by A_0 factors through H. Since $|H|$ is invertible in k_0, averaging makes $E^H \to S^H$ surjective with nilpotent kernel $J(E) \cap E^H$. By [64, Proposition 9.1.1], an idempotent $\overline{e} \in S^H$ lifts to $\widetilde{e} \in E^H$. If $\overline{e}$ sums identities of simple factors of S, then $\widetilde{e}Y_0$ contains each selected type L with its multiplicity in $Y_0|_K$. Indeed, lifts to E are conjugate by $1 + J(E)$ by the same proposition.

Since k_0 is finite, S is separable and $S_k = k \otimes_{k_0} S$ is the semisimple quotient of $k \otimes_{k_0} E$. By [107, Lemma 6.2(a)], each simple factor of S splits into one Frobenius orbit over k. Thus, a set $\mathcal{S}$ of split factors stable under H and Frobenius gives a sum $e \in S^H$, whose lift selects a summand of Y_0. For $Y_0 = Y$, the factors of U_b are stable since ${}^hU_b \cong U_{h(b)}$. Thus, a summand U of Y has split restriction $\bigoplus_b U_b$ to K.

Over k_0A_0, choose a projective cover $T \to U$. Put $J_K = J(k_0K)$. Since $K \lhd A_0$, the ideal $J_Kk_0A_0$ is nilpotent. The quotient $k_0A_0/J_Kk_0A_0$ is strongly A_0/K-graded with identity component k_0K/J_K. Since $\ell \nmid |A_0/K|$, [140, Corollary 3.5.3] gives semisimplicity. Thus, $J(k_0A_0) = J_Kk_0A_0$. The kernel of $T \to U$ lies in J_KT, and $T|_K$ is projective. Hence, $T|_K \to U|_K$ is a projective cover. Since k/k_0 is separable, the cover persists over k.

For (b, c_b), choose primitive-idempotent labels $i \in I_b$ for the simple A_b-modules. Moreover, in [107, §7, proof of Theorem 1.10], the projective cover T_b of U_b and the projective bimodule V_b are $T_b = \bigoplus_{i\in I_b} A_be_{\rho(i)} \otimes_k f_{\delta(i)}B_b$, $V_b = \bigoplus_{i\in I_{b,1}} A_be_{\rho(i)} \otimes_k f_{\delta(i)}B_b$. ρ permutes I_b, and δ bijects I_b with the labels of B_b. The bimodules $A_be_{\rho(i)} \otimes_k f_{\delta(i)}B_b$ have pairwise distinct simple tops for $i \in I_b$. Therefore, each summand of T_b occurs with multiplicity 1. Moreover, distinct block pairs (b, c_b) lie on distinct central idempotents. Thus, $I_{b,1}$ selects whole matrix factors of S_k for $Y_0 = T$.

In the Brauer tree $\mathcal{T}_b$, take $r_b = v$ at an exceptional vertex v. Otherwise, [107, proof of Lemma 7.2] gives $\ell - 1$ edges. At a central vertex v, set $r_b = v$. At a central edge e, take the endpoint r_b in the smaller component of $\mathcal{T}_b - e$. The component sizes satisfy $s_1 + s_2 = \ell$, so $s_1 \neq s_2$. Thus, $h(r_b) = r_{h(b)}$. In [107, proof of Theorem 1.10], $I_{b,1}$ is defined by distances from r_b and the maps obtained from $U_b^\vee \otimes_{A_b} -$ and $U_b \otimes_{B_b} -$. Since ${}^hU_b \cong U_{h(b)}$, both functors transport to the pair $(h(b), c_{h(b)})$. Hence, $h(I_{b,1}) = I_{h(b),1}$, and the formula for V_b gives ${}^hV_b \cong V_{h(b)}$.

For $\mathrm{End}_{k_0K}(T)$, apply the invariant-idempotent argument to $I_{b,1}$. Thus, the lifted idempotent gives V as a projective k_0A_0-summand whose split restriction to K is $\bigoplus_b V_b$. The modules U, V are ℓ-permutation modules of A_0: U is a summand of k_0X, and V is projective. By [107, §7, proof of Theorem 1.10], their unique ℓ-permutation lifts over $\mathbb{Z}_\ell$ are $\widehat{U}_{A_0}, \widehat{V}_{A_0}$. Over a splitting modular system $(\mathbb{F}, \mathcal{O}', k')$ extending $(\mathbb{Q}_\ell, \mathbb{Z}_\ell, k_0)$ with $k' \supseteq k$, their restrictions to K reduce to $k' \otimes_k \bigoplus_b U_b$ and $k' \otimes_k \bigoplus_b V_b$. By uniqueness over $\mathcal{O}'$, these are the modules of $\bigoplus_b C_b^\bullet$, so their difference has character Θ on K. Since $h \in \Delta$ permutes both families of summands U_b, V_b, uniqueness of the lifts gives $\Theta^h = \Theta$. Thus, $\Theta^a = \Theta$ for $a \in A = K\Delta$, and Ω_b is equivariant under Δ across block orbits. The character of $\widehat{U}_{A_0} - \widehat{V}_{A_0}$ takes values in $\mathbb{Q}_\ell$, and its restriction to K is Θ. By [107, Lemma 3.3(b)], the cyclotomic action of $\mathcal{H}_\ell$ comes from automorphisms over $\mathbb{Q}_\ell$. Thus, $\widehat{U}_{A_0} - \widehat{V}_{A_0}$ affords an $\mathcal{H}_\ell$-fixed lift of Θ for each ℓ'-subgroup A_0/K. Its restriction gives $\Theta^\sigma = \Theta$ for $\sigma \in \mathcal{H}_\ell$.

If A_1/K is an ℓ-group, put $I_1 = (A_1)_\vartheta$ for each irreducible constituent ϑ of Θ. Since the matched characters have ℓ'-degree, $\ell \nmid \vartheta(1)$. Moreover, $\det\vartheta$ is trivial on X and P in K, since $X = [X, X]$ and $[P, M] = P$. Thus, $\det\vartheta$ has ℓ'-order. Since I_1/K is an ℓ-group, [95, Corollary 8.16] gives a unique extension $\widetilde{\vartheta}$ to I_1 of determinantal ℓ'-order. Thus, uniqueness gives $\widetilde{\vartheta}^h = \widetilde{\vartheta^h}$ for the transports h below. For each A_1-orbit $\mathcal{O}$ in the support of Θ, choose $\vartheta \in \mathcal{O}$ and write $\varepsilon_\mathcal{O}$ for its coefficient in Θ. Put $\Xi_1 = \sum_\mathcal{O} \varepsilon_\mathcal{O}\,\mathrm{Ind}_{(A_1)_\vartheta}^{A_1}\widetilde{\vartheta}$. By uniqueness, induction is independent of $\vartheta \in \mathcal{O}$. On K, each restriction is $\sum_{\eta\in\mathcal{O}}\eta$ by [95, Problem 5.6]. Thus, $\Xi_1|_K = \Theta$. Since $\Theta^\sigma = \Theta$ and $\widetilde{\vartheta}^\sigma = \widetilde{\vartheta^\sigma}$, the character Ξ_1 is $\mathcal{H}_\ell$-fixed.

Write $J = A/K$. For every prime $v \mid |J|$, choose $J_v \in \mathrm{Syl}_v(J)$ and its inverse image A_v in A. Since $|J_v|$

is a prime power, choose an $\mathcal{H}_\ell$-fixed virtual lift Ξ_v of Θ on A_v. Since Θ is A-invariant, [95, Problem 5.2] gives $(\mathrm{Ind}_{A_v}^{A}\,\Xi_v)|_K = [J:J_v]\Theta$. Since $[J:J_v] \mid |J|$ and $v \nmid [J:J_v]$, the indices have gcd 1. Thus, for integers n_v with $\sum_v n_v[J:J_v] = 1$, $\widehat{\Theta} = \sum_v n_v \,\mathrm{Ind}_{A_v}^{A}\,\Xi_v$ is an $\mathcal{H}_\ell$-fixed virtual lift to A. If $J = 1$, put $\widehat{\Theta} = \Theta$.

Fix $\vartheta = \chi \boxtimes \psi^\vee$ in the support of Θ, with coefficient $\varepsilon_\chi \in \{1, -1\}$. Since Θ gives a signed equivariant bijection, $(\Delta\times\mathcal{H}_\ell)_\chi = (\Delta\times\mathcal{H}_\ell)_\psi$. Set $I = \Delta_\chi = \Delta_\psi$ and $D = \Delta_{\chi^{\mathcal{H}_\ell}} = \Delta_{\psi^{\mathcal{H}_\ell}}$. Moreover, the fibre product $B_I = (X \rtimes I)\times_I (M \rtimes I)$ is isomorphic to $(X \times M) \rtimes I$. Put $\pi_I = \pi|_{B_I}$ and $\Xi = \varepsilon_\chi \pi_I^* \widehat{\Theta}$. Since $\widehat{\Theta}|_K = \Theta$, the coefficient of ϑ in $\Xi|_{X\times M}$ is 1. If (d, σ) fixes χ, then d normalizes I, since $(\eta^d)^\sigma = (\eta^\sigma)^d$ for $\eta \in \mathrm{Irr}(X)$. Thus, (d, σ) acts on B_I and fixes Ξ, since conjugation by $(1, 1; d)$ induces inner conjugation on A and $\widehat{\Theta}^\sigma = \widehat{\Theta}$.

Put $\mathcal{G} = X \rtimes D$ and $\mathcal{H} = M \rtimes D$. Since $d \in D$ normalizes P, the condition $(x, d) \in \mathrm{C}_{\mathcal{G}}(X)$ gives $d = \mathrm{Int}(x^{-1})$ with $x \in M$. Moreover, such d lies in I. Evaluating $\pi_I(x, m; d)$ at 1 and then at all $t \in X$ gives $\ker \pi_I = \{(x, x; \mathrm{Int}(x^{-1})) : x \in M\}$. Thus, $\ker \pi_I$ is the diagonal image of $\mathrm{C}_{\mathcal{G}}(X)$ and centralizes $X \times M$. Also, Ξ is inflated through π_I. Now $\mathcal{G} = X\mathcal{H}$, $X \cap \mathcal{H} = M$ and $\mathrm{C}_{\mathcal{G}}(X) \le \mathcal{H}$. Since M fixes both χ and ψ, their mixed stabilizers in $\mathcal{H} \times \mathcal{H}_\ell$ agree. The ordinary inertia groups are $X \rtimes I$ and $M \rtimes I$. By [141, Corollary 1.2], choose an associated model $\mathcal{P}'$ on $M \rtimes I$ with root-of-unity factors on I. By Corollary 2.5 with Ξ and $C_0 = \ker \pi_I$, the upper model $\mathcal{P}$ shares the factor set of $\mathcal{P}'$ and its scalars on $\mathrm{C}_{\mathcal{G}}(X)$. Moreover, both correcting functions agree at (d, σ). Such elements meet every $M \rtimes I$-coset of $(\mathcal{H} \times \mathcal{H}_\ell)_\chi$. Thus, [141, Definition 1.5 and Lemma 1.8] gives the $\succeq_c$ relation. Finally, $M < X$ since $O_\ell(X) = 1$ and $P \ne 1$.

Write $W_b = b\,kX(1 - c_b)$. Since $kX \cong \mathrm{Ind}_{\Delta M}^{X\times M} 1$, its vertices are conjugate into ΔP. Thus, each nontrivial vertex of W_b contains a conjugate of ΔZ_1. Since $\mathrm{Br}_{Z_1}(b) = c_b$, the Brauer quotient $W_b(\Delta Z_1) = c_b\, k\mathrm{C}_X(Z_1)(1 - c_b) = 0$. Thus, [26, Theorem 3.2] gives projectivity of W_b. Moreover, $b\,kX\,c_b$ contributes only U_b and projectives. Thus, $kXb \cong U_b \oplus Q_{b,k}$ with $Q_{b,k}$ projective. By uniqueness of the ℓ-permutation lifts, $(\mathcal{O}'Xb)|_{X\times M} \cong \widehat{U}_b \oplus Q_b$, with Q_b projective. Hence, $\Theta_b - \Pi_b = -\mathrm{char}(Q_b) - \mathrm{char}(\widehat{V}_b)$ is a virtual projective character, where Π_b is the character of $(\mathcal{O}'Xb)|_{X\times M}$. By [170, Theorem 36], $\Theta_b(x, m) = \Pi_b(x, m)$ at ℓ-singular $m \in M$. Since $\Pi_b = \sum_{\chi\in\mathrm{Irr}(b)} \chi \boxtimes (\chi|_M)^\vee$, comparison of coefficients of χ gives $\chi(m) = \varepsilon_b(\chi)\Omega_b(\chi)(m)$ on those elements. □

Corollary 2.10. *Let S be nonabelian finite simple and p odd. Let $1 \ne Q \in \mathrm{Syl}_p(S)$ be cyclic. Suppose $\mathrm{N}_S(Q[p]) = \mathrm{N}_S(Q)$, where $Q[p]$ is the subgroup of order p. Then, $\mathrm{iMN}(S, p)$ holds on the universal cover X, with the Sylow normalizer $\mathrm{N}_X(P)$ for $P \in \mathrm{Syl}_p(X)$. In particular, $\mathrm{iMN}(S, p)$ holds when $|Q| = p$, for arbitrary $\mathrm{Out}(S)$.*

Proof. Put $X_0 = X/\mathrm{Z}(X)_p$ and let $\pi_0 : X_0 \to S$ be the covering map. Write $C = \ker \pi_0$ of p'-order and take $P_0 \in \mathrm{Syl}_p(X_0)$ above Q. By [97, Theorems 3.8 and 3.12], $\pi_0^{-1}(Q) = C \times P_0$ and $\pi_0^{-1}(Q[p]) = C \times P_0[p]$. Since both p-factors are characteristic, $\mathrm{N}_{X_0}(P_0) = \mathrm{N}_{X_0}(P_0[p])$ by the hypothesis on Q. Proposition 2.9 gives the comparison on $X_0, \mathrm{N}_{X_0}(P_0)$, and [101, Lemma 5.1] gives it on $X, \mathrm{N}_X(P)$. At $|Q| = p$, the hypothesis holds since $Q[p] = Q$. □

Lemma 2.11. *Let $\mathbf{G}$ be connected reductive with Frobenius F, and let ℓ be odd and nondefining. Suppose $P \in \mathrm{Syl}_\ell(\mathbf{G}^F)$ is cyclic and nontrivial, and $P \le \mathbf{T}$ for an F-stable maximal torus $\mathbf{T}$. For a generator x of $P[\ell]$, we let $W_x \le W(\mathbf{G}, \mathbf{T})$ be the stabilizer of x. If $\ell \nmid |W_x|$, $C_{\mathbf{G}}(x) = C_{\mathbf{G}}(P)$, $N_{\mathbf{G}^F}(P[\ell]) = N_{\mathbf{G}^F}(P)$. In particular, we let $\mathbf{G}$ be semisimple with Weyl group W and fundamental degrees $m_1, \dots, m_n$. For split F, we put $u = q$. For F twisted by opposition, we put $u = -q$. Suppose $\ell \nmid q|W|$ and $d = \mathrm{ord}_\ell(u) > 1$ divides exactly one degree m_i, counted with multiplicity. Then, every $P \in \mathrm{Syl}_\ell(\mathbf{G}^F)$ is cyclic. Given P, both conclusions above hold.*

Proof. Write $|P| = \ell^a$, and write $\mathbf{T}[\ell^a]$ additively. Since $x \in \mathbf{T}^F$, its Weyl stabilizer W_x is F-stable. Choose an integer c with $c|W_x| \equiv 1 \bmod \ell^a$. The operator $e = c\sum_{w\in W_x} w$ is an F-equivariant idempotent, and W_x fixes $e(\mathbf{T}[\ell^a])$. Moreover, the group $\mathbf{T}^F$ is abelian and contains the Sylow ℓ-subgroup P. Thus, $\mathbf{T}[\ell^a]^F = P$, and $e(P) \subseteq P$. Moreover, $e(x) = x$. Since P is cyclic, every endomorphism $f : P \to P$ with $f(x) = x$ is invertible. Thus, the idempotent $e|_P$ is id_P, and $w(t) = t$ for $w \in W_x$ and $t \in P$. Let α be a root of $\mathbf{G}$ relative to $\mathbf{T}$ with $\alpha(x) = 1$. Its reflection s_α lies in W_x, so the reflection formula gives, for $t \in P$, $\alpha^\vee(\alpha(t)) = 1$. The kernel of $\alpha^\vee$ has exponent dividing 2, since $\langle \alpha, \alpha^\vee \rangle = 2$. Thus, $\alpha(t) = 1$, as $\alpha(t)$ has ℓ-power order and ℓ is odd. By [133, Theorem 14.2], we obtain $C_{\mathbf{G}}(x)^\circ$ from $\mathbf{T}$ and the root groups U_α with $\alpha(x) = 1$. Consequently, $C_{\mathbf{G}}(x)^\circ \le C_{\mathbf{G}}(P)$. Moreover, [133, Theorem 14.2(b)] gives $C_{\mathbf{G}}(x)$ from $C_{\mathbf{G}}(x)^\circ$ and representatives of W_x in $N_{\mathbf{G}}(\mathbf{T})$. Since W_x fixes P pointwise, $C_{\mathbf{G}}(x) = C_{\mathbf{G}}(P)$. Thus, P is central and Sylow in $C_{\mathbf{G}^F}(P[\ell])$. Since P is characteristic there, its normalizer contains $N_{\mathbf{G}^F}(P[\ell])$. Conversely, $P[\ell]$ is characteristic in P, so $N_{\mathbf{G}^F}(P[\ell]) = N_{\mathbf{G}^F}(P)$.

We write $e = \mathrm{ord}_\ell(q)$. By [133, Corollary 24.6], the nondefining factors are $q^{m_i} - 1$ for split F and $q^{m_i} - (-1)^{m_i}$ under opposition. Indeed, in degree m_i, opposition acts on invariants by $(-1)^{m_i}$ since it acts as $-w_0$. Each factor is proportional to $u^{m_i} - 1$. The factor $\Phi_d(u)$ is proportional to $\Phi_e(q)$ since $u = \pm q$ and ℓ is odd. Thus, Φ_e has multiplicity 1 in the generic order. Since $\ell \nmid 2|W|$, [133, Theorem 25.14] gives cyclic P in a Sylow e-torus $\mathbf{S}$. For simple simply connected $\mathbf{G}$, [130, Proposition 2.2 and its proof] gives $C_{\mathbf{G}}(x) = C_{\mathbf{G}}(P) = C_{\mathbf{G}}(\mathbf{S})$ and $N_{\mathbf{G}^F}(P[\ell]) = N_{\mathbf{G}^F}(P) = N_{\mathbf{G}^F}(\mathbf{S})$. In general, take an F-stable maximal torus $\mathbf{T}$ of $C_{\mathbf{G}}(\mathbf{S})$ by [133, Corollaries 8.13(a) and 21.12]. Then $\mathbf{S} \leq \mathbf{T}$, so the first part applies since $\ell \nmid |W_x|$. □

Lemma 2.12. *Suppose X is finite, $R \in \mathrm{Syl}_p(X)$, $\Gamma \leq \mathrm{Aut}(X)_R$, and $\mathrm{N}_X(R) \leq M < X$ is Γ-stable. Given X, M, Γ, take* the $\succeq_c$ relation *and $\mathcal{H}_p$. Moreover, let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be a $\Gamma \times \mathcal{H}_p$-equivariant bijection between stable subsets of* $\mathrm{Irr}(X)$ *and* $\mathrm{Irr}(M)$*, satisfying* the $\succeq_c$ relation *at every pair. Also, let $f : \mathcal{X}_0 \longrightarrow \mathcal{Y}_0$ be another such bijection on $\Gamma \times \mathcal{H}_p$-stable subsets $\mathcal{X}_0 \subseteq \mathcal{X}$ and $\mathcal{Y}_0 \subseteq \mathcal{Y}$. Then, we have that f extends to an equivariant bijection $\mathcal{X} \longrightarrow \mathcal{Y}$ satisfying* the $\succeq_c$ relation *at every pair.*

Proof. On $\mathcal{X}_0$, take f. From $x_0 \notin \mathcal{X}_0$, set $y_0 = F(x_0)$. If $y_j \in \mathcal{Y}_0$, put $x_{j+1} = f^{-1}(y_j)$ and $y_{j+1} = F(x_{j+1})$. The partial map $f^{-1}F$ is injective with image in $\mathcal{X}_0$, so the path from x_0 cannot repeat. It ends at $y_r \in \mathcal{Y} \setminus \mathcal{Y}_0$. From $y \notin \mathcal{Y}_0$, the inverse path alternates F^{-1}, f and ends in $\mathcal{X} \setminus \mathcal{X}_0$ by injectivity of fF^{-1}. Thus, the endpoint bijection is $\Gamma \times \mathcal{H}_p$-equivariant by equivariance of F, f.

Along the path, $B \leq \Gamma \times \mathcal{H}_p$ is the common stabilizer. Set $I = \{\gamma : (\gamma, 1) \in B\}$, $D = \mathrm{pr}_\Gamma B$, and $C = \mathrm{C}_{X \rtimes D}(X)$. Thus, $C \leq M \rtimes I$, and the models are on $X \rtimes I, M \rtimes I$. At a shared vertex η, [141, Remark 1.3] gives $\mathcal{P}_2 \sim u\mathcal{P}_1$, with u on I and $u(1) = 1$. By [141, Corollary 1.2], both matrices at $i \in I$ have finite order, so $u(i)$ does too. Extend u by inflation to $X \rtimes I$. By [141, Lemma 1.6], scale the later pair $(\mathcal{Q}, \mathcal{Q}')$ by u^{-1} within $\succeq_c$. Align both bases at η. Thus, the factors on I, scalars on C and twists on B coincide along the path. Thus, (x_0, y_r) satisfies $\succeq_c$. Together with f on $\mathcal{X}_0$, this gives the comparison. □

Given $X \lhd Y$ with $|Y : X| = 2$, we let X be perfect with nonabelian simple quotient $X/\mathrm{Z}(X)$. Suppose that conjugation induces $Y/\mathrm{Z}(X) \simeq \mathrm{Aut}(X)$. The kernel of $Y \to \mathrm{Aut}(X)$ is $\mathrm{C}_Y(X)$, so $\mathrm{C}_Y(X) = \mathrm{Z}(X)$. Put $Z = \mathrm{Z}(X)$. Moreover, the automorphisms induced by Y can act nontrivially on Z. Fix an odd prime $p \mid |X/Z|$, and let $R \in \mathrm{Syl}_p(X)$. Since $|Y : X| = 2$, we deduce that the subgroup R also belongs to $\mathrm{Syl}_p(Y)$. Put $B = \mathrm{N}_Y(R)$. By the Frattini argument (Isaacs [97, Lemma 1.13]), $Y = XB$, so $B \setminus X \neq \varnothing$. Put $\Gamma = \mathrm{Aut}(X)_R = B/Z$. Let $B \leq U \leq Y$, and put $M = X \cap U$. Assume $M < X$. Since B meets both X-cosets, we deduce that $U/M \cong Y/X \cong C_2$. Thus, $M \lhd U$ and $|U : M| = 2$. Moreover, we have that B normalizes M, and $B \cap X = \mathrm{N}_X(R)$ lies in M. Consequently, we have that M is Γ-stable and contains $\mathrm{N}_X(R)$.

Write ε for the nontrivial linear character of Y/X, inflated to Y, and for its restriction to U. Choose $b \in B \setminus X$, and let c denote the action of conjugation by b on $\mathrm{Irr}(X)$ and $\mathrm{Irr}(M)$. Moreover, any other choice lies in $b(B \cap X)$, where $B \cap X = \mathrm{N}_X(R) \subseteq M$. Thus, c is independent of b, and c^2 fixes $\mathrm{Irr}(X)$ and $\mathrm{Irr}(M)$. Given $\sigma \in \mathcal{H}_p$ and $\theta \in \mathrm{Irr}(X) \cup \mathrm{Irr}(M)$, we have that ${}^c(\theta)^\sigma = {}^c(\theta^\sigma)$. Moreover, for $\eta \in \mathrm{Irr}(Y) \cup \mathrm{Irr}(U)$, we have that $(\varepsilon\eta)^\sigma = \varepsilon\eta^\sigma$. We set $\mathcal{E}_Y = \{\eta \in \mathrm{Irr}_{p'}(Y) : \varepsilon\eta \neq \eta\}$ and $\mathcal{E}_U = \{\eta' \in \mathrm{Irr}_{p'}(U) : \varepsilon\eta' \neq \eta'\}$. Next, we put $\mathcal{D}_X = \{\chi \in \mathrm{Irr}_{p'}(X) : {}^c\chi \neq \chi\}$ and $\mathcal{D}_M = \{\psi \in \mathrm{Irr}_{p'}(M) : {}^c\psi \neq \psi\}$. Thus, multiplication by ε acts freely on $\mathcal{E}_Y$ and $\mathcal{E}_U$, while c acts freely on $\mathcal{D}_X$ and $\mathcal{D}_M$. Given $\eta \in \mathcal{E}_Y$, we put $\chi = \eta|_X$. Then, we have that χ is irreducible with central character λ on Z. Call λ the central character of η on Z, even when Z is not central in Y. Follow the analogous convention on $\mathcal{E}_U$. Similarly, for $\chi \in \mathcal{D}_X$, take its central character λ_χ. Under c, λ_χ may change.

Proposition 2.13. *Let $f_E : \mathcal{E}_Y \to \mathcal{E}_U$ and $f_D : \mathcal{D}_X \to \mathcal{D}_M$ be bijections. Suppose that f_E, f_D are $\mathcal{H}_p$-equivariant. Also, suppose that $f_E(\varepsilon\eta) = \varepsilon f_E(\eta)$ for $\eta \in \mathcal{E}_Y$, and $f_D({}^c\chi) = {}^c f_D(\chi)$ for $\chi \in \mathcal{D}_X$. Moreover, suppose that η and $f_E(\eta)$ have the same central character on Z, as do χ and $f_D(\chi)$. Then, we have that f_E, f_D induce a $\Gamma \times \mathcal{H}_p$-equivariant bijection $F : \mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(M)$ satisfying* the $\succeq_c$ relation. *For orbit unions $\mathcal{O}_E, \mathcal{O}_D$, the same holds with f_E, f_D defined only on $\mathcal{O}_E, \mathcal{O}_D$, respectively.*

Proof. By [78, Theorem 4.3], restriction to X gives χ or $\chi + {}^c\chi$, and similarly on M. By [95, Corollaries 6.17 and 11.22], a fixed χ has extensions $\eta, \varepsilon\eta$. By [95, Theorem 6.11], a moving χ gives $\eta(1) = 2\chi(1)$. Since $p \neq 2$, the degrees have equal p-parts. Thus, on X, M the fixed parents correspond to $\mathcal{E}_Y/\langle\varepsilon\rangle, \mathcal{E}_U/\langle\varepsilon\rangle$, respectively. The other parents form $\mathcal{D}_X, \mathcal{D}_M$. The map f_E induces a bijection $\overline{f}_E : \mathcal{E}_Y/\langle\varepsilon\rangle \to \mathcal{E}_U/\langle\varepsilon\rangle$. Together with f_D, the map $\overline{f}_E$ gives the base bijection F. Since $B \cap X$ acts by inner automorphisms, $B \cap X$ fixes $\mathrm{Irr}(X)$ and $\mathrm{Irr}(M)$. The other B-coset

acts by c. Thus, F is $\Gamma \times \mathcal{H}_p$-equivariant. For $\chi \in \mathrm{Irr}_{p'}(X)$ and $\psi = F(\chi)$, equivariance gives $(\Gamma \times \mathcal{H}_p)_\chi = (\Gamma \times \mathcal{H}_p)_\psi$. Thus, the projection to Γ is $D = \Gamma_{\chi^{\mathcal{H}_p}} = \Gamma_{\psi^{\mathcal{H}_p}}$.

Put $B_D = \{b \in B : bZ \in D\}$. Thus, $Y_D := Y_{\chi^{\mathcal{H}_p}} = XB_D$ and $U_D := U_{\psi^{\mathcal{H}_p}} = MB_D$. Moreover, $Y_D = XU_D$, $X \cap U_D = M$ and $\mathrm{C}_{Y_D}(X) = Z \le U_D$. By the common mixed stabilizer on Γ, the stabilizers in $U_D \times \mathcal{H}_p$ of χ, ψ agree. If χ is Y-invariant, choose the matched extensions $\widetilde{\chi}, \widetilde{\psi} = f_E(\widetilde{\chi})$ on Y, U. Both factors are 1 and the scalars on Z agree. Given $\sigma \in \mathcal{H}_p$ with $\chi^\sigma = \chi$, f_E-equivariance gives $\widetilde{\chi}^\sigma = \varepsilon^e \widetilde{\chi}$ and $\widetilde{\psi}^\sigma = \varepsilon^e \widetilde{\psi}$ for the same $e \in \{0, 1\}$. Since each group part $u \in U_D$ acts by inner automorphisms on Y, U, the common twist is ε^e.

Otherwise, the ordinary inertia groups are X, M. Thus, the models share scalars at Z, with ordinary quotients $X/X = M/M = 1$. By [141, Lemma 1.4], both comparison functions are 1, including at stabilizer elements in $(U_D \setminus M) \times \mathcal{H}_p$. Thus, [141, Definition 1.5] gives $\succeq_c$ on Y_D, U_D in both cases. Take $\widehat{Y} = X \rtimes D$ and $\widehat{U} = M \rtimes D$.

The map $(x, a) \mapsto \mathrm{Int}(x)a$ gives the images $\mathrm{Inn}(X)D, \mathrm{Int}_X(M)D$. These are also the images of Y_D, U_D. Given $(x, a) \in \mathrm{C}_{\widehat{Y}}(X)$, we have $a = \mathrm{Int}(x^{-1}) \in D \le \Gamma$, so $x \in \mathrm{N}_X(R) \le M$. Hence, $\mathrm{C}_{\widehat{Y}}(X) \le \widehat{U}$. By [141, Theorem 2.9], the comparison holds on $\widehat{Y}, \widehat{U}$. □

Let A be a finite Galois image of $\mathcal{H}_p$, acting on a set $\mathcal{V}$. Let c be a free involution on $\mathcal{V}$, with $a(cv) = c(av)$ for $a \in A$ and $v \in \mathcal{V}$. If $P \subseteq \mathcal{V}$ is a c-orbit, we let A_P be its setwise stabilizer. Choosing $v \in P$ defines a homomorphism $e_P : A_P \to \mathbf{Z}/2$, $av = c^{e_P(a)}v$. The map e_P is independent of the choice of $v \in P$, since $a(cv) = c(av)$ for $a \in A_P$. For $a, b \in A_P$, the identity $abv = c^{e_P(a)+e_P(b)}v$ shows that e_P is a homomorphism. Moreover, v has stabilizer $\{(a, c^{e_P(a)}) : a \in A_P\}$ in $A \times \langle c \rangle$.

Lemma 2.14. *Let $\mathcal{O}, \mathcal{O}'$ be transitive A-orbits of pairs. Write $\mathcal{V}_\mathcal{O}, \mathcal{V}_{\mathcal{O}'}$ for their point sets. Then, an $A \times \langle c \rangle$-equivariant bijection $\mathcal{V}_\mathcal{O} \to \mathcal{V}_{\mathcal{O}'}$ exists exactly when some $P \in \mathcal{O}$ and $P' \in \mathcal{O}'$ satisfy $A_P = A_{P'}$ and $e_P = e_{P'}$. Suppose that the label maps λ, λ' are $A \times \langle c \rangle$-equivariant. For $v \in P, w \in P'$, it suffices to match $\lambda(v), \lambda(cv)$ with $\lambda'(w), \lambda'(cw)$.*

Proof. Since A is transitive on pairs, $D = A \times \langle c \rangle$ is transitive on $\mathcal{V}_\mathcal{O}, \mathcal{V}_{\mathcal{O}'}$. By [138, Proposition 4.7], $\mathcal{V}_\mathcal{O} \cong D/D_v$ and $\mathcal{V}_{\mathcal{O}'} \cong D/D_w$ for $v \in P, w \in P'$. Thus, an equivariant bijection taking v to w exists precisely when $D_v = D_w$. These stabilizers are the graphs of $e_P, e_{P'}$, with projections $A_P, A_{P'}$. Moreover, an equivariant label match at v, w extends to their D-orbits. □

Let J be finite, let $\phi \in \mathrm{Aut}(J)$ have order d, and let $\widehat{R}$ afford an ordinary extension $\widehat{\chi}$ of $\chi \in \mathrm{Irr}(J)$ to $J \rtimes \langle \phi \rangle$. If a coefficient operation σ fixes χ, we write $\widehat{\chi}^\sigma = \widehat{\chi}\tau_\sigma$, with τ_σ inflated from $\langle \phi \rangle$. With $b = h\phi^a$ and $h \in J$, we put $J_b = \langle J, b \rangle$. Since τ_σ is inflated from $\langle \phi \rangle$, we deduce that its value at b is $\tau_\sigma(b) = \tau_\sigma(\phi)^a$. Consequently, an $\mathcal{H}_p$-fixed extension remains fixed on J_b. If $am \equiv 0 \bmod d$, we have that $b^m \in J$, and the matrices of $\widehat{R}$ satisfy $\widehat{R}(b)^m = R\big(h\,\phi^a(h) \cdots \phi^{(m-1)a}(h)\big)$, $R = \widehat{R}|_J$. The formula for $\widehat{R}(b)^m$ follows from multiplication in $J \rtimes \langle \phi \rangle$. For $\alpha : K \to J_b$, compose $\widehat{\chi}, \widehat{R}$ with α. The first identity holds on K. If $k \in K$ maps to b, $(\widehat{R} \circ \alpha)(k)^m = \widehat{R}(b)^m$. If J is adjoint and χ is unipotent, the result of Ruhstorfer et al. [156, Theorem 7.5] gives $\widehat{\chi}$ for the prime-field map $\phi = F_r$.

Lemma 2.15. *Let $X \lhd U = XV$, put $M = X \cap V$, and let $X \le N \lhd U$ and $N' = N \cap V$. Let $\mathcal{X} \subseteq \mathrm{Irr}(X)$ and $\mathcal{Y} \subseteq \mathrm{Irr}(M)$ be $V \times \mathcal{H}_p$-stable. Suppose a $V \times \mathcal{H}_p$-equivariant bijection $f : \mathcal{X} \to \mathcal{Y}$ satisfies, for $\chi \in \mathcal{X}$ and $\psi = f(\chi)$, the comparison $(U_{\chi^{\mathcal{H}_p}}, X, \chi)_{\mathcal{H}_p} \succeq_c (V_{\psi^{\mathcal{H}_p}}, M, \psi)_{\mathcal{H}_p}$. Then, a $V \times \mathcal{H}_p$-equivariant bijection $F : \mathrm{Irr}(N \mid \mathcal{X}) \to \mathrm{Irr}(N' \mid \mathcal{Y})$ satisfies, for $\Theta \in \mathrm{Irr}(N \mid \mathcal{X})$ and $\Theta' = F(\Theta)$, $(U_{\Theta^{\mathcal{H}_p}}, N, \Theta)_{\mathcal{H}_p} \succeq_c (V_{\Theta'^{\mathcal{H}_p}}, N', \Theta')_{\mathcal{H}_p}$. Moreover, if $\chi \in \mathrm{Irr}(X)$ is a constituent of $\Theta|_X$, we put $\psi = f(\chi)$. Then, we obtain that $\Theta(1)/\chi(1) = \Theta'(1)/\psi(1)$, and $\Theta(z)/\Theta(1) = \Theta'(z)/\Theta'(1)$ for $z \in \mathrm{Z}(N)$.*

Proof. On each $V \times \mathcal{H}_p$-orbit, choose associated models P, P' for one pair (χ, ψ) with $\psi = f(\chi)$. By [141, Lemmas 2.2–2.3], transport P, P' on the orbit. Since $N = XN'$, the N-orbits on $\mathcal{X}$ are the N'-orbits. By [141, Lemma 1.9 and Corollary 1.11], the restricted models give $F : \mathrm{Irr}(N \mid \mathcal{X}) \to \mathrm{Irr}(N' \mid \mathcal{Y})$, equivariant for $V \times \mathcal{H}_p$. Given $\Theta \in \mathrm{Irr}(N \mid \chi)$, we put $\Theta' = F(\Theta)$, $I = N_\chi$ and $I' = I \cap V$. By [95, Theorem 6.11], let $\vartheta \in \mathrm{Irr}(I \mid \chi)$ and $\vartheta' \in \mathrm{Irr}(I' \mid \psi)$ be the Clifford correspondents of Θ, Θ'. By [141, Lemma 1.9], ϑ, ϑ' share a projective label τ on $I/X = I'/M$. Since $N = XN'$ and $X \le I$, we have $[N : I] = [N' : I']$. Thus, $\Theta(1)/\chi(1) = \Theta'(1)/\psi(1)$. Given $z \in \mathrm{Z}(N)$, we have $z \in \mathrm{C}_U(X) \le V$, so $z \in I'$. The common scalar of $P(z), P'(z)$ is $\omega(z)$. By [141, Lemma 1.9], $\vartheta(z)/\vartheta(1) = \vartheta'(z)/\vartheta'(1) = \omega(z)\tau(zX)/\tau(1)$. Since $z \in \mathrm{Z}(N)$, the same holds for Θ, Θ' by induction.

By [141, Corollary 1.2], choose a projective representation T associated to Θ on U_Θ, with a finite-order factor α inflated from U_Θ/N. Next, we put $J = U_\Theta \cap U_\chi$ and $J' = J \cap V$. By [95, Theorem 6.2], the constituents of $\Theta|_X$ and $\Theta'|_M$ are single orbits under N, N'. Therefore, $U_\Theta = NJ$ and $V_{\Theta'} = N'J'$. On the χ-isotypic space write $T(j) = P(j) \otimes W(jX)$ $(j \in J)$. Indeed, on $V_\chi \otimes L_\chi$, the operator $A_j = (P(j)^{-1} \otimes 1)T(j)$ satisfies $A_j T(x) = T(x)A_j$ for $x \in X$. Thus, A_j acts on L_χ. Moreover, W is a projective representation of $J/X = J'/M$ with factor α/α_P, where α_P is the factor of P. Thus, $W|_{I/X}$ is the projective label of ϑ. Choose a cyclotomic field E for the matrices of P, P', T, W.

On the α-cover of $V_{\Theta'}$ with kernel C_α, induce the lift of $P'(j) \otimes W(jX)$ from the preimage of J'. Moreover, on C_α, induction gives the scalar action $\zeta \mapsto \zeta I$. Thus, [95, Corollary 11.10] gives a model T' with factor α restricted to $V_{\Theta'}$. Since $V_{\Theta'} = N'J'$, take representatives in N'. Moreover, α is trivial on N', so [95, Problem 5.2] gives $T'|_{N'}$ affording $\Theta' = \mathrm{Ind}_{I'}^{N'} \vartheta'$. Let $a \in V \times \mathcal{H}_p$ fix Θ, Θ'. Then, χ^a lies in the N'-orbit of χ. Thus, choose $n' \in N'$ with $b = (n', 1)a$ fixing χ. By equivariance of f, b also fixes ψ. By [141, Lemma 1.7], b gives the same functions as a for T, T', since $n' \in N' \leq N$. On J, the comparison functions for T, P give $W^b \sim (\mu_b^T/\mu_b^P)W$. Indeed, we restrict an intertwiner for T^b to the χ-isotypic space. Compare its factors for P^b and W^b. Since $\mu_b^{P'} = \mu_b^P$ on J', replacing P by P' gives the same μ_b^T. Induction extends μ_b^T to $V_{\Theta'}$. Moreover, the extension of μ_b^T is constant on N'-cosets. After permutation, each representative r and its chosen counterpart r' satisfy $r^b(r')^{-1} \in I'$. The N'-correction adds no factor, since α is inflated from U_Θ/N. Finally, $z \in \mathrm{C}_U(N)$ lies in J'. On the χ-isotypic space, $T(z)$ has scalar given by $P(z) \otimes W(zX)$. Both $P(z)$ and $P'(z)$ have scalar $\omega(z)$. Thus, $P'(z) \otimes W(zX)$ has the same scalar as $T(z)$. Since z centralizes N' and α is inflated from U_Θ/N, every summand of $T'(z)$ has that scalar. Moreover, equivariance of F gives the common mixed stabilizer in $V \times \mathcal{H}_p$ and $U_{\Theta^{\mathcal{H}_p}} = NV_{\Theta'^{\mathcal{H}_p}}$. Since $\mathrm{C}_U(N) \leq \mathrm{C}_U(X) \leq V$, the centralizer condition holds. Consequently, T, T' satisfy [141, Definition 1.5]. □

Lemma 2.16. *Let $\mathbf{G}$ be connected reductive, F_0 a Frobenius endomorphism, and γ an algebraic automorphism of finite order h, with $\gamma F_0 = F_0 \gamma$. Put $F = \gamma^{-1} F_0^n$ for $n \geq 1$, and $G = \mathbf{G}^F$. Moreover, we let $\mathbf{T}$ be an F_0- and γ-stable maximal torus and choose a γ-stable Borel subgroup $\mathbf{TU}$ containing it. Write $T = \mathbf{T}^F$. Let A be finite. Suppose that each $a \in A$ is an algebraic automorphism with $a(\mathbf{T}) = \mathbf{T}$, $a(\mathbf{U}) = \mathbf{U}$, $aF_0 = F_0 a$ and $a\gamma = \gamma a$. Set $J = C_{nh} \times A$, with a generator of C_{nh} acting on G, T by F_0. Given $\theta \in \mathrm{Irr}(T)^J$, we have that the virtual character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ on G is the restriction of a $\mathbb{Q}(\theta)$-valued virtual character of $G \rtimes J$. If $\chi \in \mathrm{Irr}(G)^J$ satisfies $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi \rangle \in \{1, -1\}$, we have that χ has an extension $\widehat{\chi} \in \mathrm{Irr}(G \rtimes J)$ with values in $\mathbb{Q}(\chi, \theta)$.*

Proof. Set $\tau(x_0, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, \gamma(x_0))$ and $\mathcal{F} = \tau^{-1} \mathrm{diag}(F_0)$ as in [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19]. The map $g \mapsto \widehat{g} = (g, F_0(g), \ldots, F_0^{n-1}(g))$ gives $G \cong (\mathbf{G}^n)^{\mathcal{F}}$ and $T \cong (\mathbf{T}^n)^{\mathcal{F}}$. Moreover, $\tau(\widehat{g}) = \widehat{F_0(g)}$ and $\tau^{nh} = 1$. Since τ fixes $\mathbf{U}^n$ and $\tau\mathcal{F} = \mathcal{F}\tau$, C_{nh} acts by τ on the Deligne–Lusztig variety $Y = \{x\mathbf{U}^n : x^{-1}\mathcal{F}(x) \in \mathbf{U}^n \mathcal{F}(\mathbf{U}^n)\}$. The diagonal action of $a \in A$ on $\mathbf{G}^n$ satisfies $a\tau = \tau a$. Together with left translation by $\widehat{G}$ and inverse right translation by $\widehat{T}$, it gives an action of $(G \times T) \rtimes J$ on Y. Moreover, let Θ be the alternating cohomology character on $H_c^*(Y)$. Each $s \in (G \times T) \rtimes J$ acts on Y by a finite-order automorphism. Thus, s acts algebraically, and Θ takes values in $\mathbb{Z}$ [53, Proposition 3.3].

For the right action of T, we choose the $\theta^\vee$-isotypic part of each $H_c^i(Y)$, with projector $e_\theta = |T|^{-1} \sum_{t \in T} \theta(t)(1, t)$. Since J fixes θ, the projector e_θ satisfies $he_\theta = e_\theta h$ for $h \in G \rtimes J$. For $(g, j) \in G \rtimes J$, the alternating projected character is $\Xi_\theta(g, j) = |T|^{-1} \sum_{t \in T} \theta(t)\Theta((g, t), j)$. Thus, Ξ_θ has values in $\mathbb{Q}(\theta)$. On restriction to G, [53, Corollary 4.3] allows the radical $\mathbf{V} = \prod_{i=0}^{n-1} F_0^i(\mathbf{U})$. By [208, proof of Lemma 4.19(b)], the projection p_0 gives a $G \times T$-equivariant isomorphism with the variety for $\mathbf{U}, F$. Thus, $\Xi_\theta|_G = R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. The τ-action was on the variety for $\mathbf{U}^n$. If $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi \rangle = \varepsilon \in \{1, -1\}$, we apply Proposition 2.4 to $\varepsilon\Xi_\theta$ with second base group 1. Thus, $\widehat{\chi}$ is fixed by every coefficient automorphism σ fixing χ, θ. □

Lemma 2.17. *Let $N \lhd Y$ with Y/N cyclic of order n. Let Σ act by coefficient automorphisms. Suppose that Ξ is a virtual character of Y, and that a Y-invariant $\chi \in \mathrm{Irr}(N)$ satisfies $\Xi^\sigma = \Xi$ and $\chi^\sigma = \chi$ for every $\sigma \in \Sigma$. If $m = \langle \Xi|_N, \chi \rangle$ satisfies $\gcd(m, n) = 1$, we have that χ has a Σ-fixed ordinary extension to Y.*

Proof. If $n = 1$, the extension is χ. Otherwise, take the virtual χ-part Ξ_χ, so $\Xi_\chi|_N = m\chi$ with $m \neq 0$. Since Σ fixes χ and Ξ, it fixes Ξ_χ. Thus, [208, Lemma 9.10 and its proof] gives the normalized extension at $t_\sigma = 1$. □

Corollary 2.18. *Let $J = \mathbf{J}^F$ be a finite reductive group. Let B be a Frobenius endomorphism with $BF = FB$, $B(X(w)) = X(w)$, and $B^n = F^b$, where $n > 1$ is a prime power. Also, let C_n act on J by B. Let r be the field characteristic, and $\ell \neq r$. In $\mathrm{Irr}(J)^{C_n}$, we fix ρ with values in $\mathbb{Q}_\ell$. Suppose that $m = \langle R_w, \rho\rangle$ satisfies $\gcd(m, n) = 1$, and every eigenvalue μ of $(F^b)^*$ at ρ on $H^i_c(X(w), \overline{\mathbb{Q}}_\ell)$, in every degree i, lies in $(\mathbb{Q}_\ell^\times)^n$. Then, we have that ρ has a $\mathbb{Q}_\ell$-valued extension to $J \rtimes C_n$.*

Proof. Since $\rho^B = \rho$, the central idempotent $e_\rho \in \mathbb{Q}_\ell J$ is B-invariant. On $e_\rho H^i_c(X(w), \mathbb{Q}_\ell)$, apply the filtration in [208, proof of Proposition 9.11, Step 2] at μ. Since $\mu \in \mathbb{Q}_\ell$, each kernel $\ker(((F^b)^* - \mu)^k)$ is defined over $\mathbb{Q}_\ell$. On a graded piece V, the generator of C_n acts by $t(B^*)^{-1}|_V$ with $t^n = \mu$, $t \in \mathbb{Q}_\ell$. The alternating character Ξ takes values in $\mathbb{Q}_\ell$ and restricts to $m\rho$. Thus, Lemma 2.17 gives a $\mathbb{Q}_\ell$-valued extension. □

3 Linear and unitary groups

Let S be nonabelian simple of type $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$, with universal cover X. If $n = 2$, $\mathrm{PSU}_2(q) \cong \mathrm{PSL}_2(q)$, and Tapp [186, Theorem C] gives $\mathrm{iMN}(S, \ell)$ for every $\ell \mid |S|$. At $\ell \mid q$, apply Ruhstorfer [162, Theorem 1.3] and Johansson [101, Theorem 1.1].

Theorem 3.1. *Let S be a nonabelian simple group equal to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$, where $n \geq 2$ and q is a prime power. Then $\mathrm{iMN}(S, p)$ holds for every prime $p \mid |S|$.*

For $\mathrm{PSL}_n(q)$, Proposition 3.22 gives the generic comparison. For $\mathrm{PSU}_n(q)$, apply Proposition 3.49. Moreover, for linear S with q even, Corollary 3.11 gives the comparison for S. For odd q and an odd nondefining prime ℓ, Proposition 3.14 gives descent at odd $d = \mathrm{ord}_\ell(q)$. Similarly, for even d, Proposition 3.20 normalizes the cochain for d. Moreover, in either family, a constituent χ with $\ell \nmid \chi(1)$ may lie under $\widetilde{\chi}$ with $\ell \mid \widetilde{\chi}(1)$. Theorem 3.3 describes these parents $\widetilde{\chi}$. Moreover, for unitary $\widetilde{\chi}$, Lemma 3.48 compares global and local $\widetilde{\chi}$-operators.

Suppose $n \geq 3$, with natural universal cover X and odd nondefining prime ℓ. At $n = \ell = 3$, Johansson [103, Theorem A(ii)] gives $\mathrm{iMN}(S, \ell)$ for the linear residues $q \equiv 4, 7 \bmod 9$ and unitary residues $q \equiv 2, 5 \bmod 9$. Otherwise, by Ruhstorfer et al. [156, Corollary C and its proof], choose a Sylow Φ_e-torus $\mathbf{S}$ with $e = \mathrm{ord}_\ell(q)$. For $R \in \mathrm{Syl}_\ell(X)$, we have $\mathrm{N}_X(R) \leq M = \mathrm{N}_X(\mathbf{S}) < X$, with M stable under $\mathrm{Aut}(X)_R$. The bijection $\Omega\colon \mathrm{Irr}_{\ell'}(X) \to \mathrm{Irr}_{\ell'}(M)$ is $\mathrm{Aut}(X)_R \times \mathcal{H}_p$-equivariant, with $\frac{\chi|_{\mathrm{Z}(X)}}{\chi(1)} = \frac{\psi|_{\mathrm{Z}(X)}}{\psi(1)}$ for $\psi = \Omega(\chi)$. On $\mathcal{E}(X, 1)_{\ell'}$, Ruhstorfer et al. [156, Theorem D] gives $\succeq_c$ for the pairs from Malle [132]. For unitary X, the block length is $d = \mathrm{ord}_\ell(-q)$, the order denoted by d' in Ruhstorfer et al. [156, §8, before Lemma 8.1].

Take $\mathcal{T}_X, \mathcal{T}_M$ from Ruhstorfer et al. [156, Theorems 5.8, 6.1(i) and 6.2], with $\mathcal{T}_M$ due to Petschick and Späth. Then, $\mathcal{T}_M$ is $B\widehat{M}$-stable for $\widehat{M} = \mathrm{N}_{XE}(\mathbf{S})$. By Ruhstorfer et al. [156, Corollary 5.9], B normalizes $\widehat{M}$ and stabilizes $\mathcal{T}_X$. By Ruhstorfer et al. [156, proof of Corollary C], (i)–(vi) of Theorem 3.4 hold. Write $\Phi_{\mathrm{glo}}, \Phi_{\mathrm{loc}}$ for its extension maps. Next, let $\chi \in \mathcal{T}_X \cap \mathrm{Irr}_{\ell'}(X)$ and put $\psi = \Omega(\chi)$. The ordinary extensions are $\Phi_{\mathrm{glo}}(\chi)$ on $(XE)_\chi$ and $\Phi_{\mathrm{loc}}(\psi)$ on $\widehat{M}_\psi = \widehat{M}_\chi$. By Ruhstorfer et al. [156, Definition 3.2], for α in the mixed stabilizer of the underlying character, write $\xi^\alpha = \xi[\xi, \alpha]$, where $[\xi, \alpha]$ is the quotient twist. With χ, ψ, the extension identity is $[\Phi_{\mathrm{glo}}(\chi), \alpha]\big|_{\widehat{M}_\chi} = [\Phi_{\mathrm{loc}}(\psi), \alpha]$ $\big(\alpha \in (\widehat{M}B)_\psi\big)$. If this identity holds, Ruhstorfer et al. [156, Corollary 3.5] gives $\succeq_c$ for (χ, ψ). By Ruhstorfer et al. [156, proof of Corollary 3.5], it suffices to check $\chi \in \mathcal{T}_X \cap \mathrm{Irr}_{\ell'}(X)$. Write $m = \langle \chi|_M, \psi\rangle_M$, and let $|I_\chi|$ be the outer inertia order. If $\gcd(m, |I_\chi|) = 1$, the coprime criterion gives $\succeq_c$. Only pairs with $\gcd(m, |I_\chi|) > 1$ require the extension identity.

For q prime, suppose that X is the centreless natural cover. Write $\epsilon = 1$ in the linear case and $\epsilon = -1$ in the unitary case. Then, $\gcd(n, q - \epsilon) = 1$, and $\mathrm{Out}(X)$ is generated by inverse transpose γ by Gorenstein et al. [79, Theorem 2.5.12]. By Dolfi–Guralnick–Praeger–Spiga [60, proof of Theorem 2.1(a)–(b)], x^γ and x^{-1} are conjugate in $\widetilde{X} = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$. Since $\widetilde{X} = X\mathrm{Z}(\widetilde{X})$, they are conjugate in X. Thus, $\chi^\gamma = \overline{\chi}$. For nonreal χ, we have $I_\chi = 1$, so the coprime condition $\gcd(m, |I_\chi|) = 1$ holds even if $m = 0$. For real χ and odd m, the coprime criterion holds. Thus, only χ with $\chi = \overline{\chi}$ and even m can require the $\Phi_{\mathrm{glo}}, \Phi_{\mathrm{loc}}$-extension identity.

3.1 Determinants and Jordan blocks

Let $\widetilde{X} = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$. In the unitary case, $g \in \widetilde{X}$ satisfies $(g^{(q)})^{\mathsf{T}} g = I_n$ in an orthonormal basis. Next, put $X = \ker(\det)$ and $C = \widetilde{X}/X$. Then, C is cyclic of order $q - 1$ or $q + 1$, respectively. By Schaeffer Fry–Vinroot [163, Lemma 2.4 and Theorem 3.1], the restriction $\widetilde{\chi}|_X$ determines the $\mathrm{Irr}(C)$-orbit of $\widetilde{\chi}$. Thus, σ fixes

$\widetilde{\chi}|_X$ exactly when $\widetilde{\chi}^\sigma \in \mathrm{Irr}(C)\widetilde{\chi}$. By Schaeffer Fry–Vinroot [163, Proposition 2.5], each constituent has stabilizer $D = \bigcap_{\alpha\widetilde{\chi}=\widetilde{\chi}} \ker\alpha$ in C, with $[C:D] \mid \gcd(n,|C|)$. Next, we fix $K \lhd L$ with abelian $C = L/K$, and choose $\widetilde{\chi} \in \mathrm{Irr}(L)$. Put $T = \{\alpha \in \mathrm{Irr}(C) : \alpha\widetilde{\chi} = \widetilde{\chi}\}$. By Isaacs [95, Lemma 5.2 and Problem 5.3], $\langle \widetilde{\chi}|_K, \widetilde{\chi}|_K\rangle_K = |T|$. Suppose now that C is cyclic. By Isaacs [95, Corollary 11.22], each constituent θ of $\widetilde{\chi}|_K$ extends to $I_L(\theta)$. Then, by Isaacs [95, Theorem 6.11 and Corollary 6.17], $\widetilde{\chi}|_K$ is multiplicity-free, and $\mathrm{Irr}(L \mid \theta)$ is one $\mathrm{Irr}(C)$-orbit. Thus, $\widetilde{\chi}(1) = |T|\theta(1)$. In particular, $p \nmid \theta(1)$ exactly when $\widetilde{\chi}(1)_p = |T|_p$, with b_p the p-part of b. This includes $(L,K) = (\widetilde{M}, \widetilde{M} \cap X)$ with cyclic L/K. Write $\mu : L \to C$ for the quotient map. If $C^2 \le \mu(\mathrm{Z}(L))$, the self-twist group T is trivial on C^2. Indeed, for any $\delta \in T$ and $z \in \mathrm{Z}(L)$, $\widetilde{\chi}(z) \neq 0$ gives $\delta(\mu(z)) = 1$. Hence, $|T| \mid |C : C^2|$.

For the constituent spaces of $\widetilde{\chi}|_K$ in the cyclic-quotient formulas, let δ generate T of order t, and fix a representation ρ on V affording $\widetilde{\chi}$. Choose an intertwiner A satisfying $A\rho(g)A^{-1} = \delta(g)\rho(g)$. By Isaacs [95, Corollary 1.6], we determine A up to a scalar. Since $\delta^t = 1$, we have that the operator A^t is scalar and A can be normalized so that $A^t = 1$. Since $\delta|_K = 1$, on each constituent A is scalar by Isaacs [95, Corollary 1.6]. Let x lift a generator of $C = L/K$. Then, $\rho(x)V_\lambda = V_{\delta(x)\lambda}$ with $\delta(x)$ of order t. Thus, the t constituents of $\widetilde{\chi}|_K$ give exactly the eigenspaces V_λ. Moreover, $V_\lambda(A) = V_{\zeta\lambda}(\zeta A)$ for $\zeta \in \mu_t$.

Let α act with $\widetilde{\chi}^\alpha = \nu\widetilde{\chi}$ for $\nu \in \mathrm{Irr}(C)$ and $\delta^\alpha = \delta^j$. After transfer by an isomorphism $\rho^\alpha \cong \nu\rho$, A^α intertwines ρ with $\delta^j\rho$. Thus, Isaacs [95, Corollary 1.6] gives $A^\alpha = cA^j$ for $c \in \mu_t$, since both operators have t-th power 1. On T, the action is the power j. On the eigenspaces, λ goes to the unique $\beta \in \mu_t$ with $c\beta^j = \lambda^\alpha$. By Lemma 3.4, the factor condition $\alpha_V = \alpha_W$ pairs the eigenspaces.

Put $q = r^f$ with r prime, $\epsilon = 1$ in the linear case and $\epsilon = -1$ in the unitary case. For odd $p \neq r$, put $Q = \epsilon q$ and $d = \mathrm{ord}_p(Q)$. Then, write $a = v_p(Q^d - 1)$ and $n = dw + n_0$ with $0 \le n_0 < d$. Label $\rho \in \mathrm{Irr}(\widetilde{X})$ by Q-orbits of eigenvalues of lengths e_i, with partitions λ_i and $n = \sum_i e_i|\lambda_i|$. The degree formula is Green [84, Theorem 14] for $\epsilon = 1$ and Thiem–Vinroot [190, Theorem 5.1] for $\epsilon = -1$. Thus, $v_p(\rho(1))$ comes from numerator factors $Q^j - 1$ for $1 \le j \le n$ and denominator factors $Q^{e_i h} - 1$ for $h \in \mathrm{Hook}(\lambda_i)$. Next, put $g_i = (e_i, d)$, $m_i = d/g_i$ and $b_i = e_i/g_i$. Let $\kappa_i = \mathrm{core}_{m_i}(\lambda_i)$ and let λ_{ij} be the m_i-quotient partitions of λ_i, with $w_i = \sum_j |\lambda_{ij}|$. The multiset of hook lengths is $\mathrm{Hook}(\lambda)$.

Lemma 3.2. *An ambient character ρ has p'-degree precisely when $w = \sum_i w_i$, $v_p(w!) = \sum_{i,j} \sum_{h\in\mathrm{Hook}(\lambda_{ij})} v_p(h)$. Moreover, in that case $e_i \mid d$ whenever $w_i > 0$, and $\sum_i e_i|\kappa_i| = n_0$.*

Proof. By [132, Proposition 7.2], for the Jordan label (s,u) the factors are $[\widetilde{X}^* : C_{\widetilde{X}^*}(s)]$ and $u(1)$ at p. The numerator contributes $aw + v_p(w!)$, since $v_p(Q^{dk} - 1) = a + v_p(k)$. The denominator contributes only at $m_i \mid h$, with valuation $a + v_p(b_i) + v_p(h/m_i)$. By [15, Theorem 4.7, equal-residue hooks], the multiset of h/m_i consists of the hooks of λ_{ij} over all j. Thus,

$$v_p(\rho(1)) = a\Big(w - \sum_i w_i\Big) + v_p(w!) - \sum_i w_i v_p(b_i) - \sum_{i,j} \sum_{h\in\mathrm{Hook}(\lambda_{ij})} v_p(h).$$

By $n = d\sum_i b_i w_i + \sum_i e_i|\kappa_i|$, we have $w \ge \sum_i b_i w_i \ge \sum_i w_i$. Write $H = \prod_i (C_{b_i} \wr \mathfrak{S}_{w_i}) \le \mathfrak{S}_w$ on disjoint sets of sizes $b_i w_i$. By [99, Theorem 2.3.21], the last two subtractions from $v_p(w!)$ leave $v_p([\mathfrak{S}_w : H]\prod_i D_i)$, where $D_i = \binom{w_i}{(|\lambda_{ij}|)_j} \prod_j f^{\lambda_{ij}}$. This valuation is nonnegative. Thus, $v_p(\rho(1)) = 0$ gives $w = \sum_i w_i$, whence $b_i = 1$ if $w_i > 0$ and $\sum_i e_i|\kappa_i| = n_0$. Under $w = \sum_i w_i$, the valuation is zero exactly when $v_p(w!) = \sum_{i,j}\sum_{h\in\mathrm{Hook}(\lambda_{ij})} v_p(h)$. □

If $d > 1$, we have that $p \nmid |C|$. By the cyclic-quotient formulas, every p'-degree character of X lies under a p'-degree character of $\widetilde{X}$. Similarly, the local quotient $\widetilde{M}/(\widetilde{M} \cap X)$ has the same property. The cores κ_i in Lemma 3.2 have total rank n_0. Conversely, a label of rank n_0 consists of such cores, since $e_i|\kappa_i| < d$ implies $|\kappa_i| < m_i$. If $d = 1$, we have that a constituent of $\rho|_X$ may have degree prime to p even when $p \mid \rho(1)$.

Theorem 3.3. *Let $n \ge 3$, let p be an odd prime with $p \nmid q$, and let $\rho \in \mathrm{Irr}(\widetilde{X})$ have a constituent of p'-degree on X. Then, $p \mid \rho(1)$ if and only if $p \mid q - \epsilon$ and some $\eta \in \widehat{C}$ gives $\eta\rho$ with the following label. For some $h \ge 0$, we have that $n = p^{h+1}$. The occupied labels form a subgroup $B \le \widehat{C} = \mathrm{Irr}(C)$ of order p, and all carry one hook $\nu \vdash p^h$. Conversely, each such label gives $\rho(1)_p = p$, and $\rho|_X$ has p distinct constituents of p'-degree. Moreover, we have that $|\{\eta \in \widehat{C} : \eta\rho = \rho\}| = p$.*

Proof. Let $T = \{\eta \in \widehat{C} : \eta\rho = \rho\}$, $t = |T|$, and let χ be a p'-degree constituent of $\rho|_X$. Suppose $p \mid \rho(1)$. By the quotient formulas, $v_p(t) = v_p(\rho(1)) > 0$, so $p \mid q - \epsilon$. Write (s,u) for the Jordan label of ρ. By [33, Notation 6.2 and

proof of Proposition 6.3], s centralizes a Sylow $d_p(q)$-torus $\mathbf{S}^*$ in the dual algebraic group with F-fixed points $\widetilde{X}^*$. Since $p \mid q-\epsilon$, the centralizer of $\mathbf{S}^*$ is a maximal torus $\mathbf{T}^*$. In diagonal coordinates, $s \in (\mathbf{T}^*)^F$ has eigenvalues in $\widehat{C}$.

With $b = p^{v_p(t)}$, we let $T_b \le T$ have order b. By [163, Lemma 2.3], T_b acts freely by multiplication on Λ, the occupied label set. Thus, orbit i has a common partition ν_i. Next, we put $m_i = |\nu_i| > 0$ and $L = \sum_i m_i$. Thus, $n = bL$. Write f^ν for the Specht degree of ν and $s_p(L)$ for the sum of the base-p digits of L. Since $Q \equiv 1 \bmod p$, the hook formula gives $v_p(\rho(1)) = v_p((bL)!) - b\sum_i v_p(m_i!) + b\sum_i v_p(f^{\nu_i})$. By Legendre's formula [81, (18)], we obtain $v_p(\rho(1)) = \frac{b-1}{p-1} s_p(L) + b\, v_p\binom{L}{m_1,m_2,\dots} + b\sum_i v_p(f^{\nu_i})$. Write $b = p^v$. Then, $v_p(\rho(1)) = v$, with nonnegative summands. Since $s_p(L) \ge 1$, we have $v = 1$, as $(p^v-1)/(p-1) > v$ for $v > 1$. Thus, $s_p(L) = 1$ and $L = p^h$. Both residual valuations are 0. The multinomial valuation gives $\sum_i s_p(m_i) = s_p(L) = 1$. Thus, one occupied T_b-orbit remains, and its partition ν satisfies $p \nmid f^\nu$ by the valuation formula.

By [75, Lemma 3.1], for $h > 0$ the condition $p \nmid f^\nu$ at size p^h is equivalent to $h_{11}(\nu) = p^h$. Thus, ν is a hook, including $\nu = (1)$ for $h = 0$. The occupied coset aB has $B = T_b$ of order p. The twist by a^{-1} gives the occupied subgroup B. Conversely, a hook $\nu \vdash p^h$ on B has $p \nmid f^\nu$ by that criterion. By [163, Lemma 2.3], $T = \{\eta \in \widehat{C} : \eta B = B\} = B$. Hence, the valuation gives $\rho(1)_p = p$ and the quotient formulas give p constituents of p'-degree. □

Let $L, \widetilde{M}$ be finite groups with cyclic quotients $L/X \cong C \cong \widetilde{M}/M$. Thus, the quotient maps det give the fibre product $T = L \times_C \widetilde{M}$. Let ρ, τ be irreducible representations on V, W over $\mathbb{Q}^{\mathrm{ab}}$. As in Zhang [208, §3.2 and proof of Lemma 3.17], the action on $\mathrm{Hom}(W, V)$ is $\pi(x,y)F = \rho(x)F\tau(y)^{-1}$ for $(x,y) \in T$. Suppose ρ, τ have the same self-twist group $B \le \widehat{C}$. Let $E = \mathbb{Q}^{\mathrm{ab}}$, or take a finite Galois subfield E that contains the matrices and splits X, M. Moreover, the ordinary actions on $L, \widetilde{M}$ coincide on C. Write J for the stabilizer of the pair (ρ, τ) under $\widehat{C}$-twists and simultaneous operations (a, σ). Thus, the projection $j \mapsto (a, \sigma)$ has kernel B. For $j \in J$, denote the ordinary operation by $g \mapsto {}^j g$, the coefficient operation by σ_j, and the twist by δ_j. Choose invertible σ_j-semilinear operators P_j, Q_j with $P_j \rho(x) P_j^{-1} = \delta_j(\det({}^j x))^{-1} \rho({}^j x)$ and the same identity for Q_j, τ, y. By Schur's lemma, write $P_j P_k = \alpha_V(j,k) P_{jk}$ and $Q_j Q_k = \alpha_W(j,k) Q_{jk}$.

Lemma 3.4. *Suppose $\alpha_V = \alpha_W$ on $J \times J$. Then, we find a bijection $\Omega : \mathrm{Irr}(X \mid \rho) \to \mathrm{Irr}(M \mid \tau)$. The determinant translations and J/B act equivariantly through Ω. Moreover, we have that $\sum_{\chi \in \mathrm{Irr}(X|\rho)} \chi \otimes \Omega(\chi)^\vee$ is the character of an irreducible T-representation restricted to $X \times M$, with compatible semilinear action of J/B.*

Proof. Put $U_j(F) = P_j F Q_j^{-1}$, a σ_j-semilinear operator on $\mathrm{Hom}_E(W, V)$. Since $\alpha_V = \alpha_W$, $U_j U_k = U_{jk}$. Moreover, for $(x,y) \in T$, the two twist scalars cancel since $\det({}^j x) = \det({}^j y)$. Thus, $U_j \pi(x,y) U_j^{-1} = \pi({}^j x, {}^j y)$. Since B acts trivially on T and E, its operators centralize $\pi(T)$. Thus, $H = \mathrm{Hom}_E(W, V)^B$ is a T-module. Since $B \lhd J$, the operators on H give a J/B-action.

Choose a generator $b \in B$ of order t, and normalize $P_1 = Q_1 = 1$. The common factors give $P_b^t = c\, 1_V$ and $Q_b^t = c\, 1_W$ for one scalar c. Moreover, $P_b \in \mathrm{End}_X(V)$ and $Q_b \in \mathrm{End}_M(W)$. Since E splits X, M, both spectra lie in E. Choose a generator $a \in C$, and put $\zeta = b(a)^{-1}$. A lift of a multiplies both sets of eigenvalues by ζ, a primitive t-th root. Consequently, the spectra satisfy $z^t = c$. By the cyclic-quotient formulas, each P_b-eigenspace affords a constituent of $\rho|_X$, and each Q_b-eigenspace affords a constituent of $\tau|_M$.

Given $F \in \mathrm{Hom}(W, V)$, we have $U_b(F) = F$ if and only if $P_b F = F Q_b$. Therefore, $H = \bigoplus_{\lambda^t = c} \mathrm{Hom}(W_\lambda, V_\lambda)$. These summands are distinct irreducibles of $X \times M$. Since a acts by $\lambda \mapsto \zeta\lambda$, they form one T-orbit, each with multiplicity 1. Thus, H is irreducible by [95, Theorem 6.11]. Given $j \in J$, the decomposition matches the action on $\mathrm{Irr}(X \mid \rho)$ with that on $\mathrm{Irr}(M \mid \tau)$. Thus, the pairing Ω is J-equivariant. □

For $c : J \to E^\times$, put $(\partial c)(j,k) = c(j)\sigma_j(c(k))/c(jk)$. If $\alpha_V/\alpha_W = \partial c$, then $P_j' = c(j)^{-1} P_j$ has factors α_W. Suppose $z \in J$ acts by conjugation by $a_0 \in \widetilde{M} \le L$, with $\delta_z = 1$ and $\sigma_z = 1$. Assume $P_z = \rho(a_0)$, $Q_z = \tau(a_0)$, $P_{jz} = P_j \rho(a_0)$ and $Q_{jz} = Q_j \tau(a_0)$. If $c(z) = 1$ and $c(jz) = c(j)$, then $P_z' = \rho(a_0)$ and $P_{jz}' = P_j' \rho(a_0)$. Thus, $U_z' = \pi(a_0, a_0)$ and $\pi(a_0^{-1}, a_0^{-1}) U_z' = 1$ on the new H. For an inner power $z = \phi^d$, impose the same conditions on c.

Assume $\alpha_V = \alpha_W$. For $\nu \in \mathrm{Irr}(B)$, write H_ν for the subspace where U_b acts as $\nu(b)$ for all $b \in B$. Then, $U_j(H_\nu) = H_{\nu^j}$, with $\nu^j(b) = \sigma_j(\nu(j^{-1}bj))$. If c is a semilinear 1-cocycle, $U_j' = c(j)^{-1} U_j$ again gives an action of J. Since B acts trivially on E, $c|_B \in \mathrm{Irr}(B)$ and the new fixed space is $H_{c|_B}$. For the generator b above, H_ν pairs V_λ with $W_{\lambda/\nu(b)}$, since $P_b F = \nu(b) F Q_b$. Thus, $c|_B$ determines Ω. Transport of both models by j gives $\Omega^j(\chi^j) = \Omega(\chi)^j$.

Proposition 3.5. *Let X, M, R, Γ be as in* Section 2. *Suppose χ, ψ have equal full mixed stabilizers. Put $I = \Gamma_\chi = \Gamma_\psi$, $D = \Gamma_{\chi^{\mathcal{H}_p}}$, and $K = (X \times M) \rtimes I$, with I acting simultaneously on both coordinates. Suppose $\chi \otimes \psi^\vee$ has an ordinary*

extension Θ to K. Assume $(\operatorname{tr}\Theta)^b = \operatorname{tr}\Theta$ for each b in the diagonal mixed stabilizer in $(M \rtimes D) \times \mathcal{H}_p$. Moreover, we suppose each $c = (x, a) \in \mathrm{C}_{X \rtimes D}(X)$ satisfies $\Theta(x, x, a) = 1$. Then, the $\succeq_c$ relation *holds.*

Proof. Put $G = X \rtimes D$ and $H = M \rtimes D$. Thus, $K = (X \rtimes I) \times_I (M \rtimes I)$. If $(x, a) \in \mathrm{C}_G(X)$, then $a = \mathrm{Int}(x^{-1})$. Since a fixes R, $x \in \mathrm{N}_X(R) \leq M$. Moreover, $a \in I$ since it is inner on X. Thus, the diagonal image C_0 of $\mathrm{C}_G(X)$ lies in K. Since C_0 centralizes $X \times M$ and is I-stable, $C_0 \lhd K$. By hypothesis, $\Xi = \operatorname{tr}\Theta$ is inflated from K/C_0, and $\Xi|_{X\times M} = \chi \boxtimes \psi^\vee$. By [141, Corollary 1.2], choose an associated model Q of $M \rtimes I$ for ψ with root-of-unity factors $c(j, k)$. Since $\Xi^b = \Xi$ for every b in the diagonal mixed stabilizer, Corollary 2.5 gives an associated model $\mathcal{P}$ of $X \rtimes I$. Thus, $\mathcal{P}, Q$ have the same factors on I and the same correcting functions μ_b. Moreover, $\mathcal{P}(x, a)$ and $Q(x, a)$ have equal scalars at $(x, a) \in \mathrm{C}_G(X)$, by the same corollary. Since $G = XH$, $X \cap H = M$, and the mixed stabilizers coincide, [141, Definition 1.5] gives $\succeq_c$. □

For linear or unitary L, we apply Proposition 3.5. Let $\widetilde{M} \leq L = \widetilde{X}$ map onto C, and put $M = X \cap \widetilde{M}$. Suppose that $A = L\Lambda = XH$, $H = \widetilde{M}\Lambda$, and $X \cap H = M$. Λ consists of lifts that normalize $\widetilde{M}$, and $\Lambda \cap \widetilde{M} \neq 1$ is permitted. Assume conjugation maps A onto $\mathrm{Aut}(X)$, with kernel $\mathrm{Z}(L) \leq \widetilde{M}$. Take the assumptions on M, R from Section 2. Suppose P_j, Q_j in Lemma 3.4 are normalized and give $F \mapsto \rho(k)F\tau(k)^{-1}$ on $H_{\mathrm{lin}} = \mathrm{Hom}(W, V)^B$, for every $k \in \Lambda \cap \widetilde{M}$. Assume also that ρ, τ have the same central character on $\mathrm{Z}(L)$. Let $\Lambda_0 \leq \Lambda$ stabilize the determinant orbit of (ρ, τ). Two restoring twists for H_{lin} differ by an element of B, so both act identically on H_{lin}. The map $T \rtimes \Lambda_0 \to A \times_{A/X} H$ acts by $((g, m), u) \mapsto (gu, mu)$. Its kernel consists of $((k^{-1}, k^{-1}), k)$, with $k \in \Lambda_0 \cap \widetilde{M}$. Thus, the kernel acts trivially on H_{lin} by the inner-operator hypothesis.

For $(a, h) \in A_\chi \times_{A/X} H_\psi$, write $h = mu$ with $m \in \widetilde{M}$, $u \in \Lambda$. Since $ah^{-1} \in X$, we have $a = gu$ for $g = (ah^{-1})m$, and $(g, m) \in T$. Since $(\chi, \psi)^{(a,h)} = (\chi, \psi)$, the cyclic-quotient formulas gives $u \in \Lambda_0$. Thus, the corresponding H_{lin}-summand U affords an ordinary extension Θ_F of $\chi \otimes \psi^\vee$ to $A_\chi \times_{A/X} H_\psi$. Given b in the mixed stabilizer, we have that the σ_b-semilinear action on H_{lin} stabilizes U and intertwines Θ_F^b with Θ_F. Thus, we have that Θ_F has a b-invariant character. Given $z \in \mathrm{Z}(L)$, the common central character ω gives $\rho(z) = \omega(z)1_V$ and $\tau(z) = \omega(z)1_W$. Thus, we find $\Theta_F(z, z) = 1$.

Put $Z_L = \mathrm{Z}(L)$ and $F = A_\chi \times_{A/X} H_\psi$. Since $\Delta Z_L \leq \ker\Theta_F$, the model descends to $\overline{\Theta}_F$ on $F/\Delta Z_L$. By Isaacs [97, Lemma 1.13], $H = M\mathrm{N}_H(R)$, so $A = X\mathrm{N}_H(R)$. Since $\mathrm{N}_X(R) \leq M$, it follows that $\mathrm{N}_A(R) = \mathrm{N}_H(R)$. Thus, conjugation gives $\mathrm{N}_H(R)/Z_L \cong \Gamma$. Choose a lift $s(a) \in \mathrm{N}_H(R)$ for each $a \in I = \Gamma_\chi = \Gamma_\psi$. For $K = (X \times M) \rtimes I$, put $\iota_I(x, m, a) = (xs(a), ms(a))\Delta Z_L$. Since $a \in I$, the pair $(xs(a), ms(a))$ lies in F. The cosets $s(a)Z_L$ multiply as in I, so ι_I is a homomorphism. A lift $t \in H$ gives equivariance of ι_I by conjugation. Thus, $\Xi = \overline{\Theta}_F \circ \iota_I$ satisfies $(\operatorname{tr}\Xi)^b = \operatorname{tr}\Xi$ for b in the mixed stabilizer. Moreover, if (x, a) centralizes X, then $xs(a) \in Z_L$, so $\Xi(x, x, a) = 1$. Consequently, Proposition 3.5 gives the $\succeq_c$ relation. Given a parent orbit $\mathcal{O}$, fix Ω on one parent (ρ, τ), and transport its normalized models by j. By Lemma 3.4, Ω is J-equivariant. Thus, on $(\rho, \tau)^j$ the pairing is independent of j. Next, for comparison with the $\Phi_{\mathrm{glo}}, \Phi_{\mathrm{loc}}$-extension identity, let the transversal character χ and its partner ψ be the selected constituents. In Ruhstorfer et al. [156, Theorem 3.4], put $U = (XE)_\chi$ and $V = \widehat{M}_\psi$. Then, $U = XV$, $X \cap V = M$, and $J_0 = U/X \simeq V/M$. Put $F_0 = U \times_{J_0} V$. For $(u, v) \in F_0$, we put $x = uv^{-1} \in X$. Moreover, v normalizes M. By Isaacs [97, Lemma 1.13], we obtain $\mathrm{N}_X(M) = M\mathrm{N}_{\mathrm{N}_X(M)}(R) = M$. Since $A = XH$ and H normalizes M, we have $\mathrm{N}_A(M) = H$. Thus, choose a lift $t \in H$ of the action of v on X. Since v fixes ψ, $t \in H_\psi$, and since $u = xv$ fixes χ, $xt \in A_\chi$. Put $\iota(u, v) = (xt, t)\Delta Z_L$ in $F/\Delta Z_L$. Since the action of v determines tZ_L, the map ι is well defined. Since t and v act identically on X, ι is a homomorphism. For each such b, the map ι is equivariant. Thus, $\Theta_0 = \overline{\Theta}_F \circ \iota$ extends $\chi \otimes \psi^\vee$ and satisfies $(\operatorname{tr}\Theta_0)^b = \operatorname{tr}\Theta_0$.

Put $\xi = \Phi_{\mathrm{glo}}(\chi)$ and $\zeta = \Phi_{\mathrm{loc}}(\psi)$. By Isaacs [95, Corollary 6.17], $\operatorname{tr}\Theta_0 = T\eta$ for a unique linear $\eta \in \mathrm{Irr}(J_0)$, where $T(u, v) = \xi(u)\overline{\zeta(v)}$ and η is inflated to F_0. Since $(T\eta)^\alpha = T\eta$, each α in the mixed stabilizer gives $[\xi, \alpha]|_V/[\zeta, \alpha] = \eta/\eta^\alpha$ on V/M. Thus, if $\eta^\alpha = \eta$ for each α in the mixed stabilizer, the extension identity holds for ξ, ζ. In general, the local twist $\zeta_0 = \zeta\eta^{-1}$ gives $[\zeta_0, \alpha] = [\xi, \alpha]|_V$. Suppose that $t = gz$ is as in Ruhstorfer et al. [156, Lemma 5.10], acting on V and fixing ψ. By that lemma, $[\zeta_0, t](v) = \omega([z, v]) = [\zeta, t](v)$ for $v \in V$, with ω the central character of ψ.

Given $c \in \mathrm{C}_E(X)$, the lift of its action on X lies in Z_L. Thus, $\iota(c, c) = 1$ and $(\operatorname{tr}\Theta_0)(c, c) = (\operatorname{tr}\Theta_0)(1, 1)$. Under Ruhstorfer et al. [156, Theorem 3.4(iv)], the centralizer scalars for ξ, ζ agree, so $T(c, c) = T(1, 1)$. Consequently, $\eta(cM) = 1$, and ζ_0 has the same scalar as ζ at c. Since the stabilizer V acts by inner automorphisms on $\mathrm{Irr}(V)$, the corrected choice extends along each $\widehat{M}$-orbit. The pairs on H_1 satisfy Proposition 3.5.

For $q = 2^f$, put $G_e = \mathrm{GL}_e(q)$, and let $U_e \leq G_e$ be upper unitriangular. Moreover, we let J_e be the permutation

matrix for $i \mapsto e+1-i$. Define $\gamma_e(g) = J_e g^{-T} J_e^{-1}$ and $\phi_e(g) = (g_{ij}^2)_{ij}$. With $K = \mathbb{Q}^{\mathrm{ab}}$, we choose $v = \sqrt{q}$. Moreover, for $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we put $\epsilon_\sigma = \sigma(v)/v \in \{1,-1\}$. Define $\xi_e(u) = (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\sum_{i=1}^{e-1} u_{i,i+1})}$. Given $u \in U_e$, $(u^{-1})_{i,i+1} = -u_{i,i+1}$ for $1 \le i < e$. Thus, $(\gamma_e(u))_{i,i+1} = u_{e-i,e-i+1}$, since q is even. Moreover, $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(t^2) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(t)$. Hence, we find $\xi_e \gamma_e = \xi_e \phi_e = \xi_e$.

The Gelfand–Graev module $W_e = \mathrm{Ind}_{U_e}^{G_e} \xi_e$ contains each cuspidal ρ with multiplicity one (Digne–Michel [58]) and (Deriziotis–Gotsis [54]). Let $W_{e,\rho}$ be its ρ-isotypic component. Moreover, for $f \in W_e$, we define $(D_a f)(g) = f(a^{-1}(g))$ for $a \in \{\gamma_e, \phi_e\}$, and define $(U_\sigma f)(g) = \sigma(f(g))$. Given a linear character δ of $\mathbb{F}_q^\times$, we put $(C_\delta f)(g) = \delta(\det g) f(g)$. For right translation $R(x)$, we have $R(x) C_\delta = \delta(\det x) C_\delta R(x)$. Therefore, we find $C_\delta W_{e,\rho} = W_{e,\rho(\delta \circ \det)}$. The operators D_a, U_σ, C_δ satisfy the semidirect-product relations on W_e. Moreover, $D_a W_{e,\rho} = W_{e,a\cdot\rho}$, where $(a\cdot\rho)(g) = \rho(a^{-1}(g))$, and $U_\sigma W_{e,\rho} = W_{e,\rho^\sigma}$.

Fix a Harish–Chandra label $(\rho_j, \nu_j)_j$, with distinct $\rho_j \in \mathrm{Irr}(G_{e_j})$. Put $m_j = |\nu_j|$ and $N_j = e_j m_j$, so $\sum_j N_j = n$. Order the ρ_j with the m_j occurrences of each ρ_j consecutive. Let I be Harish–Chandra induction of $\bigotimes_j W_{e_j,\rho_j}^{\otimes m_j}$. Moreover, let a stabilize $(\rho_j,\nu_j)_j$ under the operations on I. Write $\gamma_a \in \{0,1\}$ for its graph exponent, and σ_a for its coefficient component. Let F_a act before sorting. If $\gamma_a = 1$, the graph map γ_n puts both the blocks and the m_j occurrences of each ρ_j in opposite order. There δ acts by $f(g) \mapsto \delta(\det g) C_\delta f(g)$, where $C_\delta = \bigotimes_j C_{\delta,j}^{\otimes m_j}$. Consequently, the relation for $R(x)C_\delta$ gives parabolic covariance and $F_a F_b = F_{ab}$.

Lemma 3.6. *For an ordering o of the ρ_j-groups, we write I_o for its induced module. Moreover, for $o \to o'$, there is a normalized isomorphism $J_{o',o} : I_o \to I_{o'}$. These maps satisfy $J_{o'',o'} J_{o',o} = J_{o'',o}$. Moreover, on $\mathrm{End}_{G_n}(I_o)$, conjugation by $J_{o',o}$ acts by $T_s \mapsto T_{s'}$, where $s' = wsw^{-1}$ and w acts by $o \mapsto o'$. On each ρ_j-block the ordering is increasing under w.*

Proof. First, transpose neighboring factors of ranks u, w, with distinct cuspidal labels ρ, τ. In $\mathrm{GL}_{u+w}(q)$, let $P = LV$ and $P^- = LV^-$ be opposite parabolics. For induced functions f, we put $(A_- f)(g) = \sum_{y \in V^-} f(yg)$ and $(A_+ f)(g) = \sum_{x \in V} f(xg)$. Then, A_- maps the P-space to the P^--space, and A_+ maps back. Indeed, L normalizes both V, V^-, so A_-, A_+ are L-equivariant. Mackey theory gives $A_+ A_- = c1$ and $A_- A_+ = c'1$. Indeed, the transposition t can intertwine the labels only if $u = w$ and $\rho = \tau$. For b in the source module, we choose f supported on P with $f(1) = b$. Given $x \in V$ and $y \in V^-$, the condition $yx \in P$ gives $y = 1$. Thus, $(A_+ A_- f)(1) = q^{uw} b$. Similarly, $A_- A_+$ has scalar q^{uw}. Hence, the normalized maps $v^{-uw} A_-$ and $v^{-uw} A_+$ are inverse. By tensor permutation, the P^--space is isomorphic to $I_{o'}$.

On adjacent factors of ranks u, w, z, write $n_{ij}(X)$ for the block transvection. Then, $n_{12}(X) n_{23}(Y) n_{13}(Z) = n_{13}(Z + XY) n_{23}(Y) n_{12}(X)$. Thus, both braid words parametrize $\mathcal{R}$. Their tensor permutations $\pi_{\mathrm{left}}, \pi_{\mathrm{right}}$ agree. Both normalization exponents are $uw + uz + wz$. Hence, $J_{\mathrm{left}} = J_{\mathrm{right}}$. Operators at disjoint positions satisfy $J_i J_j = J_j J_i$. Induction in stages carries them to I. Moreover, a shuffle of whole ρ_j-groups crosses only distinct labels. Consequently, $J_{o'',o'} J_{o',o} = J_{o'',o}$.

Let s transpose neighboring factors of ρ_j. We choose w_0' increasing on their positions, and put $s' = w_0' s (w_0')^{-1}$. Moreover, both words in $w_0' s = s' w_0'$ are reduced. Their common radical is $V_{w_0' s}$. Hence, the shuffle $J_{w_0'} = J_{o',o}$ carries the s-supported line ℓ_s to $\ell_{s'}$. In the type A algebra, the normalized generator $T_s \in \ell_s$ satisfies $T_s^2 = (q^{e_j} - 1) T_s + q^{e_j}$. The cuspidal Harish–Chandra series of GL has label q^{e_j}. Equivalently, the Green degree ratio is q^{e_j}, with larger degree at label $(1,1)$ [84]. The supported operator T_s has $\mathrm{tr}\, T_s = 0$, since every diagonal P-coset block of T_s is zero. Normalize its $(1,1)$-eigenvalue to -1, and the displayed relation follows. If cT_s satisfies the relation, $c^2 = 1$ and $c^2 = c$. Thus, $c = 1$, so $T_{s'}$ is unique on $\ell_{s'}$. The raw operations stabilize $\ell_{s'}$, while $q^{e_j} \in \mathbb{Q}$. Moreover, a permutation matrix P has $\det P = 1$, since q is even. Therefore, $T_{s'}$ remains normalized under $\gamma_n, \phi_n, \delta, \sigma$. □

Given the transformed ordering ao, we let $w_a : ao \to o$ be the stable shuffle of the ρ_j-groups. By the choice of w_a, its restriction to each ρ_j-group is increasing. Put $L_\chi(a) = \ell(w_a)$. Moreover, a crossing of groups i, j contributes $N_i N_j$. Choose J_w from Lemma 3.6. Moreover, for $\nu \vdash m$, put $K(\nu) = \binom{m}{2} + \sum_{(i,j)\in\nu} (j - i)$, $\qquad H_\chi = \sum_j e_j K(\nu_j)$.

Proposition 3.7. *Let χ have label $(\rho_j, \nu_j)_j$, and let $\mathcal{S}_\chi$ be its joint stabilizer. Given $a, b \in \mathcal{S}_\chi$, choose intertwiners with semilinear factor $\alpha_\chi(a,b) = \epsilon_{\sigma_a}^{k_\chi(b)}$, $k_\chi(b) = L_\chi(b) + \gamma_b H_\chi \pmod 2$. In particular, we put $E_\chi = \langle \phi_n, \gamma_n \rangle_\chi$. Then, χ extends to $\mathrm{GL}_n(q) \rtimes E_\chi$.*

Proof. On I, we put $D_a = J_{w_a} F_a$. Moreover, under a, the coefficient $v^{-\ell(w_b)}$ in J_{w_b} acquires the sign $\epsilon_{\sigma_a}^{L_\chi(b)}$. The unnormalized intertwining maps are a-equivariant. Moreover, for the longest permutation w_0, we have $\ell(w_0 w_b w_0) =$

$\ell(w_b)$. By Lemma 3.6, $D_aD_b = \epsilon_{\sigma_a}^{L_\chi(b)} D_{ab}$. Given any ρ_j, we let S_j be a rational Specht module for $\mathcal{H}_j$, the Hecke algebra of $\mathfrak{S}_{m_j}$ with deformation value q^{e_j}. Let w_j^* be its longest permutation. Then, $C_j = v^{-e_j K(\nu_j)} T_{w_j^*}$ conjugates T_i to T_{m_j-i}. With $Q = q^{e_j}$, the quadratic relation is $(T_i - Q)(T_i + 1) = 0$. By [147, Lemma 2.1] and [208, Lemma 4.4, (4.1)], $T_{w_j^*}^2 = Q^{K(\nu_j)} I$ on S_j. Thus, $C_j^2 = I$.

Put $S = \bigotimes_j S_j$. Given $a \in \mathcal{S}_\chi$, we define C_a by permuting factors, applying σ_a in the rational bases, and with C_j on each factor if $\gamma_a = 1$. The Hecke matrices in C_j are rational. Hence, $C_aC_b = \epsilon_{\sigma_a}^{\gamma_b H_\chi} C_{ab}$. By Lemma 3.6, D_a and C_a induce the same action β_a on $\mathcal{A} = \bigotimes_j \mathcal{H}_j$. Thus, $D_a x = \beta_a(x) D_a$ and $C_a x = \beta_a(x) C_a$ for $x \in \mathcal{A}$. Therefore, $U_a : u \mapsto D_a u C_a^{-1}$ acts on $\mathrm{Hom}_{\mathcal{A}}(S, I)$. The factor of U_aU_b is the quotient of those for D_aD_b and C_aC_b. Since both factors are signs, $U_aU_b = \alpha_\chi(a,b) U_{ab}$. The parity of L_χ is additive. Indeed, $(-1)^{L_\chi(a)} = \mathrm{sgn}(w_a)$, and $\mathrm{sgn}(w_0 w_a w_0) = \mathrm{sgn}(w_a)$. Moreover, H_χ is invariant under label permutations. Therefore, k_χ is additive modulo two. If $\sigma_a = 1$, $\alpha_\chi(a,b) = 1$, giving the extension to $\mathrm{GL}_n(q) \rtimes E_\chi$. □

With ℓ odd, we put $d = \mathrm{ord}_\ell(q)$. With $n = Ad + r$ and $0 \le r < d$, we put $a_0 = v_\ell(q^d - 1)$. Moreover, the label ρ_j of Section 3.1 has Frobenius orbit $\mathcal{O}(\theta_j)$, where θ_j is regular on $\mathbb{F}_{q^{e_j}}^\times$ and $|\mathcal{O}(\theta_j)| = e_j$. Put $g_j = (d, e_j)$, $D_j = d/g_j$, $b_j = e_j/g_j$. Fix an abacus convention for D_j. Moreover, we let κ_j be the D_j-core of ν_j, and let $(\mu_{j,k})_{0 \le k < D_j}$ be its quotient. Put $w_j = \sum_k |\mu_{j,k}|$. In Lemma 3.2, set $(Q, p, w, n_0, m_j, a) = (q, \ell, A, r, D_j, a_0)$. Thus, χ has ℓ'-degree only if $e_j \mid d$ for $w_j > 0$ and $\sum_j e_j |\kappa_j| = r$.

Decompose $\mathbb{F}_q^n$ into A blocks of dimension d, each isomorphic with $\mathbb{F}_{q^d}$, and a residual space of dimension r. Put $T_d = \mathbb{F}_{q^d}^\times$, $N_1 = T_d \rtimes \langle r_0 \rangle$, $\widetilde{M} = G_r \times (N_1 \wr \mathfrak{S}_A)$ and $r_0(t) = t^q$. Moreover, we put $G_0 = 1$ and $\widetilde{G} = G_n$. Then, $\widetilde{M}$ is the Sylow Φ_d-torus normalizer in $\widetilde{G}$. It contains $\mathrm{N}_{\widetilde{G}}(P)$, for some $P \in \mathrm{Syl}_\ell(\widetilde{G})$, and may be larger. If $w_j > 0$, $e_j \mid d$. With $\theta = \theta_j \circ \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_{q^{e_j}}}$, we put $m = d/e_j$. Moreover, we choose $\zeta_m = \zeta_d^{e_j}$. The inertia group of θ in N_1 is $T_d \rtimes \langle r_0^{e_j} \rangle$. Thus, we find that its m extensions $\widehat{\theta}_k$ satisfy $\widehat{\theta}_k(r_0^{e_j}) = \zeta_m^k$, for $0 \le k < m$. Set $V_{\theta,k} = \mathrm{Ind}_{T_d \rtimes \langle r_0^{e_j} \rangle}^{N_1} \widehat{\theta}_k$. Hence, Clifford theory gives $\mathrm{Irr}(N_1 \mid \theta) = \{V_{\theta,k} : 0 \le k < m\}$. Conversely, we let $\eta \in \mathrm{Irr}(T_d)$ have orbit length e. Then, we obtain that $e \mid d$, and $\eta(t^{q^e - 1}) = 1$ for $t \in T_d$. Write $N_e = \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_{q^e}}$. Thus, $\ker(N_e) = \{t^{q^e-1} : t \in T_d\} \le \ker(\eta)$. Since N_e is onto, we find a unique $\eta_0 \in \mathrm{Irr}(\mathbb{F}_{q^e}^\times)$ with $\eta = \eta_0 \circ N_e$.

Proposition 3.8. *There is a bijection $\widetilde{\Omega} : \mathrm{Irr}_{\ell'}(\widetilde{G}) \to \mathrm{Irr}_{\ell'}(\widetilde{M})$. Next, the residual label of $\widetilde{\Omega}(\chi)$ is $\chi_0 \in \mathrm{Irr}(G_r)$ from the cores (κ_j). The partition at $V_{\theta,k}$ is $\mu_{j,k}$. Moreover, $\widetilde{\Omega}$ matches central characters. If $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$, $\widetilde{\Omega}$ is equivariant for the twist $\lambda \circ \det$. It is also equivariant for $\mathcal{H}_\ell$ and the lifts u, c_T below. Moreover, the factors $\alpha_\chi, \alpha_{\chi_0}$ from* Proposition 3.7 *agree on $\mathcal{S}_\chi$.*

Proof. The one-block character $V_{\theta,k}$ has degree $e_j \mid d$. Since $d \mid \ell - 1$, its degree is prime to ℓ. By [99, §4.3 and Theorem 2.3.21], $v_{\mathrm{bulk}} = v_\ell(A!) - \sum_{j,k,c \in \mu_{j,k}} v_\ell(h(c))$. Moreover, $|G_r|$ is prime to ℓ, since $r < d$. Conversely, each residual partition is a D_j-core. Indeed, a D_j-hook would require $e_j D_j = d b_j \ge d$ dimensions. Combine the bulk quotients $\mu_{j,k}$ with the residual cores κ_j. Moreover, we set $\kappa_j = \varnothing$ for absent ρ_j. The inverse map cq^{-1} gives a unique label $(\rho_j, \nu_j)_j$. Thus, $v_\ell(\chi(1)) = v_\ell(\widetilde{\Omega}(\chi)(1))$, and $\widetilde{\Omega}$ is bijective on $\mathrm{Irr}_{\ell'}(\widetilde{G})$. By Proposition 3.7, the operations act on ρ_j and leave the partition ν_j fixed. Locally they act on θ. Moreover, $\mathcal{H}_\ell$ fixes ζ_d, since $d \mid \ell - 1$. The graph lift has $c_T(t) = t^{-1}$ for $t \in T_d$ and $c_T(r_0) = r_0$. A determinant twist by λ restricts to $\lambda \circ \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_q}$. It has value one on r_0 and the block permutation matrices, since q is even. Thus, the twist fixes k. The Specht modules $\mathrm{Sp}^{\mu_{j,k}}$ are rational, so $\widetilde{\Omega}$ is equivariant under these operations. Given $z \in \mathbb{F}_q^\times$, the local label has value $\theta_j(z)^{d/e_j}$. Moreover, $|\nu_j| - |\kappa_j| = (d/e_j) w_j$. Hence, the central characters agree on z.

Take $\kappa_j = \varnothing$ for absent cores. The shuffles w_a, w_a^0 have the same label permutation. If d is even, $N_j - e_j|\kappa_j| = d w_j$ for each active label. Consequently, $L_\chi \equiv L_{\chi_0} \pmod 2$. For partitions $\mu \subset \lambda$ with $|\lambda| = s$, suppose that λ/μ is a rim hook of length D, with contents $c, c+1, \dots, c+D-1$. Thus, the change in K is $\binom{s}{2} - \binom{s-D}{2} + Dc + \binom{D}{2} = D(s + c - 1)$. For active labels, $e_j D_j = d$. Thus, if d is even, $H_\chi \equiv H_{\chi_0} \pmod 2$. If d is odd, $\mathcal{H}_\ell$ fixes $\sqrt{q}$ by [156, Lemma 2.3(d)]. Thus, $\epsilon_\sigma = 1$ for $\sigma \in \mathcal{H}_\ell$. In either parity, Proposition 3.7 gives $\alpha_\chi = \alpha_{\chi_0}$ on $\mathcal{S}_\chi$. □

Lemma 3.9. *On the bulk character spaces $\mathcal{V}$, the field and graph lifts u, c_T, coefficient action in $\mathcal{H}_\ell$, and twists by $\lambda \circ \det$ admit factor 1. Moreover, $u^f = r_0$ on each block of $\mathcal{V}$.*

Proof. We define $u(t) = t^2$ and $c(t) = t^{-1}$ on T_d. Put $A_1 = T_d \rtimes (C_{fd} \times C_2)$, with generators u, c, and put $r_0 = u^f$. Moreover, for $0 \le k < d$, we choose $\eta_k \in \mu_{\ell-1}$ with $\eta_k^f = \zeta_d^k$. Such η_k exists. Indeed, $q = 2^f \in \mathbb{F}_\ell^\times$ has order d. Thus,

the image of the f-power map on $\mathbb{F}_\ell^\times$ contains the subgroup of order d. The same holds in $\mu_{\ell-1}$, a cyclic group. Hence, $\eta_k^{fd} = 1$.

Let $\xi_k \in \mathrm{Irr}(\langle u, c\rangle)$ satisfy $\xi_k(u) = \eta_k$ and $\xi_k(c) = 1$. Put $W_k = \mathrm{Ind}_{\langle u,c\rangle}^{A_1} \xi_k$. Moreover, we find one N_1-$\langle u, c\rangle$ double coset. Thus, $W_k|_{N_1} = \mathrm{Ind}_{\langle r_0\rangle}^{N_1}(r_0 \mapsto \zeta_d^k)$. If $e = e_j$, the restriction $V_{\theta,j}|_{\langle r_0\rangle}$ is induced from $r_0^e \mapsto \zeta_{d/e}^j$. By [95, Lemma 5.2], $\langle W_k|_{N_1}, V_{\theta,j}\rangle_{N_1} = 1$ precisely when $k \equiv j \pmod{d/e}$, and is zero otherwise. We choose $k = j$, with $0 \le j < d/e$, throughout each orbit. The matrices of W_k in the T_d-indexed basis are monomial over $\mathbb{Q}(\eta_k)$, which $\mathcal{H}_\ell$ fixes.

Moreover, for a determinant twist $\lambda \circ \det$, the basis action is multiplication by $\lambda \circ \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_q}$. Write $\mathrm{N} = \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_q}$. Given $t \in T_d$, we have $\mathrm{N}(u(t)) = \mathrm{N}(t)^2$ and $\mathrm{N}(c(t)) = \mathrm{N}(t)^{-1}$. Moreover, $(\lambda \circ \mathrm{N})(1) = 1$. Hence, $u, c, \mathcal{H}_\ell$ compose ordinarily on the W_k-family. Its isotypic components $W_k[V_{\theta,j}]$ inherit this composition. Moreover, $u^f = r_0$ on W_k. For $\mathcal{V}$, tensor permutations and rational Specht factors Sp^μ satisfy these relations. On $N_1 \wr \mathfrak{S}_A$, induction gives the same relations. Thus, $\mathcal{V}$ inherits them. □

Choose a trace-orthonormal $\mathbb{F}_q$-basis $\mathcal{B}$ of $\mathbb{F}_{q^d}$. The pairing $B(x, y) = \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(xy)$ is nondegenerate. Moreover, $B(x, x) = \mathrm{Tr}(x)^2$, and Tr is onto. Thus, B is nonalternating. Since $\mathbb{F}_q$ is perfect, B is equivalent to I_d. In $\mathcal{B}$, the multiplication matrix m_t satisfies $m_t^T = m_t$ for $t \in T_d$, and r_0 is orthogonal. Write $u = g\phi$, where ϕ squares matrix entries. The identity $\mathrm{Tr}(x^2y^2) = \mathrm{Tr}(xy)^2$ gives $g^T g = 1$, so $\det g = 1$. On each bulk block $u^f = r_0$. Similarly, on the residual block, take ϕ.

With $B_0 = \mathrm{diag}(I_{Ad}, J_r)$, we set $c_T = \mathrm{Int}(B_0^{-1})(-T)$, where $(-T)$ is inverse transpose. Then, $c_T(t) = t^{-1}$ for $t \in T_d$, while c_T fixes r_0 and block permutations. Moreover, c_T satisfies $c_T\, u = u\, c_T$. Relative to γ_n, the inner correction is $B_0^{-1} J_n$, a permutation matrix over $\mathbb{F}_2$. Thus, $\det(B_0^{-1} J_n) = 1$, so $\lambda(\det(B_0^{-1} J_n)) = 1$. The inner corrections for u, c_T are multiplicative on each base representation ρ. In particular, $u^f = r_{\mathrm{all}}$, where $r_{\mathrm{all}} = \mathrm{diag}(\underbrace{r_0, \ldots, r_0}_{A}, I_r)$.

Proposition 3.10. *Let S be the simple quotient of $X = \mathrm{SL}_n(q)$. Suppose $n \ge 3$, X is the universal cover of S, and $M = \widetilde{M} \cap X$ is proper in X. Suppose $\mathrm{N}_X(R) \le M$ for $R \in \mathrm{Syl}_\ell(X)$, and put $\Gamma = \mathrm{Aut}(X)_R$. Let $\mathcal{X}$ consist of $\chi \in \mathrm{Irr}(X)$ occurring in $\widetilde{\chi}|_X$ for some $\widetilde{\chi} \in \mathrm{Irr}_{\ell'}(\widetilde{G})$. Define $\mathcal{Y}$ similarly for $\widetilde{M}, M$. There is a bijection $\Omega : \mathcal{X} \to \mathcal{Y}$ such that $(\Gamma \times \mathcal{H}_p)_\chi = (\Gamma \times \mathcal{H}_p)_{\Omega(\chi)}$ and the $\succeq_c$ relation holds for every $\chi \in \mathcal{X}$. If $d > 1$, $\mathcal{X} = \mathrm{Irr}_{\ell'}(X)$ and $\mathcal{Y} = \mathrm{Irr}_{\ell'}(M)$.*

Proof. The determinant maps $\widetilde{M}$ onto $\mathbb{F}_q^\times$. For $A > 0$, we choose the norm on T_d. Similarly, for $r > 0$, take the determinant on G_r. Hence, $\widetilde{G} = X\widetilde{M}$. Let $\widetilde{\chi} \in \mathrm{Irr}_{\ell'}(\widetilde{G})$ and put $\widetilde{\psi} = \widetilde{\Omega}(\widetilde{\chi})$. For the mixed operations on $\widetilde{G}, \widetilde{M}$, write $\mathcal{S}_{\widetilde{\chi}}, \mathcal{S}_{\widetilde{\psi}}$ for their stabilizers. Moreover, by Proposition 3.8, $\mathcal{S}_{\widetilde{\chi}} = \mathcal{S}_{\widetilde{\psi}}$, and their central characters agree on $\mathrm{Z}(\widetilde{G})$. The bulk operators for $\widetilde{\psi}$ have factor 1, by Lemma 3.9. On G_r, both parents have W_r and the Specht factors S_j. Thus, $\alpha_{\widetilde{\chi}} = \alpha_{\widetilde{\psi}}$ on $\mathcal{S}_{\widetilde{\chi}}$, including self-twists of $\widetilde{\chi}$.

Apply Lemma 3.4 to the fibre product $\widetilde{G} \times_{\mathbb{F}_q^\times} \widetilde{M}$. We write $H = \mathrm{Hom}(W, V)^B$. Moreover, the paired summands $V_\chi \otimes W_\psi^\vee$ occur with multiplicity 1 in H. If a mixed operation g fixes χ, cyclic Clifford theory gives $\widetilde{\chi}^g = (\lambda \circ \det)\widetilde{\chi}$ for some $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$. The same holds for ψ and $\widetilde{\psi}$. Since $\widetilde{\Omega}$ is injective, each paired χ, ψ satisfies $(\Gamma \times \mathcal{H}_p)_\chi = (\Gamma \times \mathcal{H}_p)_\psi$. Given any parent orbit $\mathcal{O}$, choose one pairing $\Omega_{\mathcal{O}}$. Given an operation g on $\mathcal{O}$, put $\Omega_{\mathcal{O}}(\chi^g) = \Omega_{\mathcal{O}}(\chi)^g$. Since $(\Gamma \times \mathcal{H}_p)_\chi = (\Gamma \times \mathcal{H}_p)_\psi$, the value at χ^g is independent of the choice of g. Put $\widetilde{A} = \langle \widetilde{G}, \phi, \gamma_n\rangle$. Diagonal, field and graph automorphisms of X are induced by $\widetilde{A}$, whose action on X has kernel $\mathrm{Z}(\widetilde{G})$. Moreover, $\widetilde{H} = \langle \widetilde{M}, u, c_T\rangle$ normalizes M and maps onto $\mathrm{Out}(X)$. The fibre-product quotient has $u^f = r_{\mathrm{all}}$ in both factors. Globally, $u^f = r_{\mathrm{all}}$ by Proposition 3.7 and the correction for u. Similarly, $u^f = r_{\mathrm{all}}$ holds locally by Lemma 3.9. Given $z \in \mathrm{Z}(\widetilde{G})$, $\widetilde{\chi}(z)/\widetilde{\chi}(1) = \widetilde{\psi}(z)/\widetilde{\psi}(1)$. Thus, (z, z) acts as 1.

Since $R \in \mathrm{Syl}_\ell(M)$ and $M \lhd \widetilde{H}$, [97, Lemma 1.13] gives $\widetilde{H} = M\mathrm{N}_{\widetilde{H}}(R)$. Thus, $\mathrm{N}_{\widetilde{H}}(R)$ maps onto $\mathrm{Out}(X)$. Moreover, it contains $\mathrm{N}_X(R)$, whose image is $\Gamma \cap \mathrm{Inn}(X)$. Hence, $\mathrm{N}_{\widetilde{H}}(R) \to \Gamma$ is surjective and Γ normalizes M. The descent after Proposition 3.5 gives Ξ on $(X \times M) \rtimes \Gamma_\chi$, extending $\chi \otimes \psi^\vee$. Moreover, $(\mathrm{tr}\,\Xi)^b = \mathrm{tr}\,\Xi$ for b in the mixed stabilizer, and $\Xi(x, x, a) = 1$ for $(x, a) \in \mathrm{C}_{X \rtimes \Gamma_\chi \mathcal{H}_p}(X)$. Thus, Proposition 3.5 yields $\succeq_c$. Finally, let (L, K) be $(\widetilde{G}, X)$ or $(\widetilde{M}, M)$. For $\eta \in \mathrm{Irr}(L)$ and a constituent ϑ of $\eta|_K$, the cyclic-quotient formulas give $\eta(1)/\vartheta(1) \mid q - 1$. If $d > 1$, $\ell \nmid q - 1$. Thus, $v_\ell(\eta(1)) = v_\ell(\vartheta(1))$, and the stated sets $\mathcal{X}, \mathcal{Y}$ contain all ℓ'-degree characters. □

Corollary 3.11. *Let $q = 2^f$, and let $S = \mathrm{PSL}_n(q)$ be nonabelian simple. Then,* $\mathrm{iMN}(S, \ell)$ *holds for every prime $\ell \mid |S|$.*

Proof. If $n = 2$, apply [186, Theorem C]. At $\ell = 2$, apply [162, Theorem 1.3] and [101, Theorem 1.1]. For exceptional $\mathrm{Mult}(S)$, apply [157, Lemma 4.1(2)]. If $n = \ell = 3$ and $q \equiv 4, 7 \bmod 9$, apply [103, Theorem A(ii)]. Otherwise, $X = \mathrm{SL}_n(q)$ and $\ell \nmid q$. By [132, Theorem 5.14], a Sylow Φ_d-torus $\mathbf{S}$ gives $\mathrm{N}_X(P) \le M = \mathrm{N}_X(\mathbf{S}) < X$ for $P \in \mathrm{Syl}_\ell(X)$. Thus, Proposition 3.22 gives the comparison on M. □

3.2 Linear mixed comparison

With $q = p^f$, we let p, ℓ be distinct odd primes. Moreover, take $d, A, r, e_i, \nu_i, \kappa_i, w_i$ from Section 3.1. By Lemma 3.2, $\chi(1)$ has the linear valuation. With $N_i = e_i|\nu_i|$, we write $R_i = e_i|\kappa_i|$. For an ℓ'-degree parent, these satisfy $N_i - R_i = dw_i$, since $w_i > 0$ implies $e_i \mid d$. For a determinant character δ let $t_\delta \in \mathbb{F}_2$ be defined by $\delta(-1) = (-1)^{t_\delta}$. Moreover, for a cuspidal label ρ_i, we put $s_i = \omega_{\rho_i}(-I_{e_i})$ and write $s_i = (-1)^{b_i}$.

Lemma 3.12. *Let ρ be cuspidal of odd rank e. The ρ-isotypic Gelfand–Graev space V_ρ admits simultaneous transports for field automorphisms φ, the graph automorphism γ, determinant characters δ, and coefficient automorphisms σ. Moreover, the transports for $(\varphi, \gamma, \delta, \sigma)$ satisfy their semidirect-product law.*

Proof. Given U_e, the source character is $\xi(u) = \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\sum u_{i,i+1})}$. Next, for $a \in \mathbb{F}_p^\times$, we put $t(a) = \mathrm{diag}(a^{(e-1)/2}, a^{(e-3)/2}, \dots, a^{-(e-1)/2})$. Adjacent diagonal entries of $t(a)$ have ratio a, and $\det t(a) = 1$. Moreover, $\gamma : g \mapsto J_e g^{-T} J_e^{-1}$ gives $\gamma(t(a)) = t(a)$ and $\gamma(B_e) = B_e$ for the upper triangular subgroup B_e. Given a coefficient automorphism σ with $\sigma(\zeta_p) = \zeta_p^{a_\sigma}$, we define operators on functions by $(S_\sigma h)(g) = \sigma\big(h(t(a_\sigma)^{-1} g)\big)$ and $(Ch)(g) = h(t(-1)^{-1}\gamma(g))$. The two changes to ξ cancel in each formula. Thus, S_σ and C stabilize the model. The maps $a \mapsto t(a)$ and $\sigma \mapsto a_\sigma$ are homomorphisms. Therefore, $S_\sigma S_\tau = S_{\sigma\tau}$. Since $\gamma(t(a)) = t(a)$, one also has $C^2 = 1$ and $CS_\sigma = S_\sigma C$. The matrices $t(a)$ are fixed by entrywise p-power. Moreover, multiplication by $\delta(\det g)$ contributes no scalar at $t(a)$, and the multiplier $\delta(\det g)$ has the given transform under φ, γ, σ. By [54, p. 76], V_ρ has multiplicity 1. Thus, each transport maps V_ρ to its target summand. □

Suppose that d is odd. By Proposition 3.8, for d, tensor the Specht factor at (θ_i, k) with sgn^{b_i}. Moreover, at $r_0^{e_i}$, keep the value ζ_{d/e_i}^k on the inertia group I of the character. The determinant of r_0 is $(-1)^{d-1} = 1$, while a permutation π of the d-dimensional blocks has determinant $\mathrm{sgn}(\pi)$. Every active e_i is odd, so the twist by δ changes b_i by t_δ. The sign $(-1)^{t_\delta}$ occurs on the complement under δ. Hence, the partition index ν satisfies $\nu^\delta = \nu$. The modification indexed by b_i has the original scalar values at zI. Given any Specht character ξ, $(\xi\,\mathrm{sgn}^{b_i})(1) = \xi(1)$. Given $a \in \{\varphi, \gamma, \sigma\}$, $\theta_i^a(-1) = \theta_i(-1)$, so $b_i^a = b_i$. Next, $\mathcal{S}$ admits Harish–Chandra transport. At even d, see Section 3.2. Moreover, the ranks e of cuspidal factors are arbitrary. If $2 \mid e$, take the transports of Lemma 3.17 for e. The source space V has the same scalar factors.

Lemma 3.13. *With $G = \mathrm{GL}_n(q)$, we let $L \le G$ be an ordered Levi subgroup. Let $V = \bigotimes_i V_i$ be an ordered tensor of modules V_i, cuspidal for the factors of L, grouped by label. Next, we write $I = R_L^G(V)$ and choose $v = \sqrt{q}$. Let a be a composition of determinant twists δ, field operations φ, graph operations γ, and coefficient operations σ. Moreover, for a repeated label ρ, let $T \in \mathrm{End}_G(I)$ be a normalized generator for ρ. Under a, the normalized generator is T_a on I_a. In particular, on m identical labels, γ acts by $i \mapsto m + 1 - i$ on positions. Let a stable shuffle w permute groups of labels. The transposition J_w under a has scalar $\left(\delta_a(-1)\,\frac{\sigma_a(v)}{v}\right)^{L(w)}$ relative to $J_{w,a}$ for the transported order. Moreover, $L(w)$ counts inversions of w, weighted by block dimensions.*

Proof. Let $\lambda : L \to \mathrm{GL}(V)$ be the representation on V. Let $P = LU$ be the parabolic subgroup for L. Next, on L, choose the induced-function space, with $f(ug) = f(g)$ for the unipotent radical and $f(lg) = \lambda(l)f(g)$ for $l \in L$. A source intertwiner acts on $f(g)$. Moreover, the field operation φ and graph operation γ replace the argument g. For γ, the Levi L^γ has coordinates in opposite order. Given f, apply the coefficient action σ to $f(g)$. Moreover, for a determinant twist δ, let C_δ be the tensor product of multipliers $C_{\delta,i}$. Take the map $f(g) \longmapsto \delta(\det g) C_\delta f(g)$. Indeed, on the source module, $R(l)C_\delta = \delta(\det l) C_\delta R(l)$. Thus, $\delta(\det g)C_\delta f(g)$ has left covariance at lg. Each raw map $F_a : I \to I_a$ is invertible. The compensating diagonal matrices t in the models act within V and fix support on $P\backslash G$. Consequently, $F_a F_b$ has scalar factor $\beta(a, b)$, the product of cuspidal scalars. $\beta(a, b)$ can be nontrivial.

Next, fix e and an irreducible cuspidal module V_e of rank e. Choose adjacent summands of type V_e. Moreover, choose w_e, a permutation transposing the factors of V_e. Let $K \neq 0$ be a kernel on Pw_eP. The value $K(w_e)$ lies on the intertwining line for $V_e \otimes V_e$. Moreover, the flip τ is a nonzero intertwiner. Since $V_e \otimes V_e$ is irreducible on the common Levi L_0, the intertwining line is $\langle \tau \rangle$. Induction in stages for G gives the supported kernel line $\mathcal{L}$ for the longer list. The map F_a has $F_a(\mathcal{L}) = \mathcal{L}_a$. Moreover, for $a \in \{\varphi, \gamma\}$, the images are P^a and $P^a w_e^a P^a$. The actions of σ, δ have support Pw_eP. Moreover, the scalar factor in F_a cancels in its conjugation of $\mathcal{L}$. Under γ, positions $j, j+1$ among m summands become $m-j, m-j+1$.

The rank-two generator T satisfies $(T-q^e)(T+1)=0$. Next, normalize T before extending to the longer list $\mathcal{O}$. For the cuspidal label ρ, [84, Theorem 14] gives degree ratio $1 : q^e$ for $(2), (1,1)$. Moreover, the diagonal blocks of K on parabolic cosets are zero. Thus, $\operatorname{tr}(K) = 0$. Let λ_1, λ_2 be the eigenvalues of K. Then, $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 = -q^e$. Scale K to give the larger constituent eigenvalue -1. The same normalization defines T and T' at the source and target. If $F_a T F_a^{-1} = cT'$, the rational quadratic relation gives $c^2 = 1$ from the constant coefficient and $c^2 = c$ from the coefficient of T', since $q^e > 1$. Thus, $c = 1$. Consequently, the determinant-twist sign at w_e is absent from T.

For different adjacent labels of dimensions u, w_1, let A_{u,w_1} be the opposite-radical average for (u, w_1) in Lemma 3.6. Follow A_{u,w_1} by the block-permutation matrix w_{mat}, and take w_{mat}^{-1} for the inverse crossing on (w_1, u). Moreover, a function f, supported at P, gives $q^{uw_1}1$ for the composite $J'_{\mathrm{raw}}J_{\mathrm{raw}}$ of the two raw maps $J_{\mathrm{raw}}, J'_{\mathrm{raw}}$. The same radical entries occur when q is odd. Thus, normalization has factor v^{-uw_1}. Given a radical translate R_x, the determinant multiplier D_δ satisfies $D_\delta R_x = R_x D_\delta$. Moreover, at w_{mat}, the multiplier contributes $\delta_a(\det w_{\mathrm{mat}})^{\pm 1} = \delta_a(-1)^{uw_1}$. A coefficient operation contributes $(\sigma_a(v)/v)^{uw_1}$. Moreover, for source maps C_i, C_j and factor permutation τ_{ij}, $\tau_{ij}(C_i \otimes C_j) = (C_j \otimes C_i)\tau_{ij}$. The field operation φ permutes the finite indexing set $\mathcal{A}$. Similarly, the graph acts by $\gamma : \mathcal{A} \to \mathcal{A}^\gamma$ for the opposite crossing. Moreover, $w_{\mathrm{mat}}^{-T} = w_{\mathrm{mat}}$. Thus, the crossing scalar is $(\delta_a(-1)\sigma_a(v)/v)^{uw_1}$.

For three positions, we write $u_{ij}(X)$ for the unipotent block with entry X in position (i,j). For compatible blocks X, Y, Z, $u_{12}(X)u_{13}(Z - XY)u_{23}(Y) = u_{23}(Y)u_{13}(Z)u_{12}(X)$. Thus, both braid words have coordinates X, Y, Z. Let J_i be the normalized transposition operator at i. If $|i-j| > 1$, $J_iJ_j = J_jJ_i$. Moreover, the two-block identity gives $J'_iJ_i = 1$ for the inverse crossing J'_i. Thus, for a stable shuffle w, the normalized operators compose. Let s be an internal adjacent swap. Set $s' = wsw^{-1}$. Then, $ws = s'w$. Since w is increasing on the block of s, $ws, s'w$ have additive length. Raw products for ws and $s'w$ have the same double-coset support $\mathcal{C}$ and intertwining line $\mathcal{L}_\mathcal{C}$, by the root coordinates. Given $\mathcal{C}$, the quadratic argument gives $J_w T_s J_w^{-1} = T_{s'}$ for the normalized internal generators $T_s, T_{s'}$. Moreover, for $\eta = \delta_a(-1)\sigma_a(v)/v$, the crossing scalars give $\eta^{L(w)}$. □

On the ordered induction I, we let F_a be the raw transport and put $D_a = J_{w_a}F_a$. The shuffle w_a regroups the labels. Moreover, for a, b, the raw cuspidal factors have product $\beta(a,b)$, with multiplicities. The partition ν_i gives a Specht representation S_{ν_i} of the Hecke algebra $\mathcal{H}_i$ with deformation value q^{e_i}, defined over $\mathbb{Q}$. The graph operator on S_{ν_i} is $B_i = v^{-e_iK(\nu_i)}T_{w_0}$, where $K(\nu_i)$ is defined in Proposition 3.7 for ν_i. By Zhang [208, Lemma 4.4, (4.1)], $B_i^2 = 1$. Let $\mathcal{V} = \bigotimes_i S_{\nu_i}$. Rational factor permutations and B_i give the intertwiner C_a on $\mathcal{V}$. Moreover, the intertwining space $\mathcal{W}$ admits the map $h \mapsto D_a h C_a^{-1}$. Given any normalized generator T, Lemma 3.13 proves that the map on $\mathcal{W}$ is well defined. For a, b in the joint stabilizer J, the scalar is $\beta(a,b)\big(\delta_a(-1)\epsilon_{\sigma_a}\big)^{L(w_b)}\epsilon_{\sigma_a}^{\gamma_b \sum_i e_iK(\nu_i)}$, where $\epsilon_\sigma = \sigma(v)/v$. Indeed, the stable shuffles for a, b compose to that for ab. The rational Specht matrices for S_{ν_i} are fixed by σ. Thus, the coefficient effect concerns B_i's normalization. We fix w_{mat} in block order. Moreover, for γ, φ, inner corrections enter the factors on $\widetilde{M}$.

Proposition 3.14. *With d odd, we let (χ, ψ) be an ℓ'-degree pair with local labels (θ_i, k). Moreover, we let J be the full joint stabilizer of (χ, ψ). Given $a, b \in J$, the torus lifts and transports $\mathcal{T}$ from* Lemma 3.13 *satisfy* $\alpha_{\mathrm{glob}}(a,b) = \alpha_{\mathrm{loc}}(a,b)$.

Proof. If $2 \nmid q$, Lemma 3.6 gives the braid relations for $\mathbb{F}_q$. Moreover, the two-block composition has w^{-1} for inverse crossing. For δ, $\delta(\det(w^{-1}g)) = \delta(\det w)^{-1}\delta(\det g)$. Moreover, $\delta(\det w)^{-1} = (-1)^{t_\delta \ell(w)}$. The radical elements x have $\det x = 1$. Thus, the averaging operator A_{av} has factor $\delta(\det w)^{-1}$. For the generator T, the quadratic relation gives $F_aTF_a^{-1} = T_a$ on the transported Hecke line. Since d is odd, $q \in (\mathbb{Q}_\ell^\times)^2$. Thus, for $v = \sqrt{q}$ and $\sigma \in \mathcal{H}_\ell$, the coefficient sign $\epsilon_\sigma = \sigma(v)/v$ is 1, for Specht operators. Active ranks e_i are odd. Moreover, for active i, Lemma 3.12 gives transports. For inactive i, choose a common model for ρ_i at each of the m_i positions, with the same operators O_i on both sides. Thus, the m_i factors multiply to $\beta(a,b)$. However, individual factors may differ from 1.

For $b \in J$, let π_b permute its labels, and let γ_b be its graph exponent. Next, for π_b, we choose weights N_i, R_i in the total shuffle lengths $L_N(b), L_R(b)$. Moreover, both include the graph part γ_b on the label groups. Let $L_w(b)$ be the local block length modulo 2, with weights w_i. On each torus, γ fixes the block order. Since k is fixed, order by colour, k. On the source subgroup L, the corrected Specht sign for π accounts for $\det \pi$, for each internal block permutation π. Different orders of L admit a permutation isomorphism P_w. Moreover, the calculation at $w^{-1}g$ gives $\alpha_{\mathrm{glob}}(a,b) = \beta(a,b)(-1)^{t_{\delta_a} L_N(b)}$ and $\alpha_{\mathrm{loc}}(a,b) = \beta(a,b)(-1)^{t_{\delta_a}(L_R(b)+L_w(b))}$. The local field lift u on one block satisfies $u^f = r_0$, by Lemma 3.9 with $u(t) = t^p$. A suitable root exists because $p^f \bmod \ell$ is an f-th power of order d, and $\det r_0 = 1$. If $2 \nmid d$, choose a trace-self-dual basis (β_j) of $\mathbb{F}_{q^d}/\mathbb{F}_q$ [13, Theorem 5.6]. Its Moore matrix $M_0 = (\beta_j^{q^i})_{0 \le i < d,\, 1 \le j \le d}$ satisfies $M_0^T M_0 = 1$, so $\det M_0 = \pm 1$. Write $u = g\phi$. Then, $M_0 g = M_0^{(p)}$ and $\det g = 1$. Let $c = \mathrm{Int}(B^{-1})(-T)$, where $B = \mathrm{diag}(I_{Ad}, J_r)$. The graph γ fixes the Borel, and $B^{-1}J_n$ gives the inner correction, with determinant $(-1)^{C_0}$, where $C_0 = \binom{n}{2} + \binom{r}{2}$. Since g is orthogonal and $u = \phi$ on the residual factor, $cu = uc$. Thus, replacing the graph lift by c changes the factor by $(-1)^{t_{\delta_a}\gamma_b C_0}$.

It remains to show that $L_N(b) + L_R(b) + L_w(b) + \gamma_b C_0$ is even when $t_{\delta_a} = 1$. Given $X \in \{N, R, w\}$, we let $\mathcal{I}_X = \{i : X_i \text{ is odd}\}$. Then, the weighted sign for π_b is the ordinary sign on $\mathcal{I}_X$. Each $\mathcal{I}_X$ is π_b-stable. Since $N_i = R_i + dw_i$ and d is odd, $\mathcal{I}_N = \mathcal{I}_R \triangle \mathcal{I}_w$. Write $\varepsilon_X = \mathrm{sgn}(\pi_b|_{\mathcal{I}_X})$. Thus, $\varepsilon_N \varepsilon_R \varepsilon_w = 1$. In the opposite order, the sums $\sum_{i<j} N_i N_j$ and $\sum_{i<j} R_i R_j$ cancel against their contributions to C_0. The remaining parity is $\gamma_b \sum_i h_i$, where $h_i = \binom{N_i}{2} + \binom{R_i}{2} \pmod 2$. In particular, for inactive i, $h_i = 0$. When $t_{\delta_a} = 1$, every active rank e_i is odd, while other operations fix s_i. Thus, $s_{\pi_a(i)} = -s_i$ for active i. Each active π_a-orbit has even length, and h_i is constant on that orbit. Hence, $\sum_i h_i \equiv 0 \pmod 2$. Therefore, $\alpha_{\mathrm{glob}}(a,b) = \alpha_{\mathrm{loc}}(a,b)$ for all $a, b \in J$. □

Now, assume that p and ℓ are odd and d is even. Put $K_0 = \mathbb{F}_{q^d}$. The trace form on K_0 is $b_0(x,y) = \mathrm{Tr}_{K_0/\mathbb{F}_q}(xy)$. Next, for a basis of K_0, we write B for the Gram matrix of b_0. Since b_0 is nondegenerate, $\det B \neq 0$. Moreover, the square class of $\det B$ in $\mathbb{F}_q^\times$ enters Lemma 3.15.

Lemma 3.15. *Let T_d be the multiplication torus of K_0. Moreover, on K_0, we choose semilinear lifts u, c_*, and $v_0 = u^f$ is a linear element of $\mathrm{N}_{\mathrm{GL}_d(q)}(T_d)$. Given $t \in T_d$, one has $utu^{-1} = t^p$ and $c_*(t) = t^{-1}$. Moreover, $c_* u = u c_*$, $c_*^2 = 1$, and $v_0^d = -I$. The linear part g_u of u and v_0 satisfy $\det g_u = \det v_0 = 1$. Take graph maps $c_{\mathrm{Bor}}, c_{\mathrm{loc}}$. Moreover, c_{Bor} fixes the Borel, and c_{loc} is the local graph on A bulk blocks and the residual space V_{res}. Then, the inner correction h from c_{Bor} to c_{loc} satisfies $\det h = (-1)^{(d/2)A}$.*

Proof. Choose an $\mathbb{F}_q$-basis of K_0, with Moore matrix M_0. Next, we write $D = \det M_0$ and $B = M_0^T M_0$. Then, B is the Gram matrix of b_0. The q-power permutes the rows of M_0 in a d-cycle. Hence, $D^q = -D$ and $D^2 = \det B$. For $u_0(x) = x^p = g\phi(x)$ in these coordinates, $M_0 g = M_0^{(p)}$ and $g^T B g = B^{(p)}$. Moreover, $\det g = D^{p-1}$. In particular, $r_0 = u_0^f$ satisfies $\det r_0 = -1$. The graph $c_0 = \mathrm{Int}(B^{-1})(-T)$ satisfies $c_0(t) = t^{-1}$ for $t \in T_d$. Moreover, $c_0(r_0) = r_0$ and $c_0 u_0 = u_0 c_0$.

Let $L = (q^d - 1)/(q-1)$. In $\mathbb{F}_{q^{2d}}^\times$, we choose z with $z^{q^d} = -z$ and $z^L = D^{-1}$. Indeed, a nonzero trace-zero element z_0 satisfies $(z_0^L)^{q-1} = -1$, as does D^{-1}. Thus, $z_0^L D \in \mathbb{F}_q^\times$. By [118, Theorem 2.28(ii)], choose $z = z_0 t$ with $t \in K_0^\times$ of norm $(z_0^L D)^{-1}$. Consequently, $b = z^2$, $x = z^{p-1}$, $y = z^{q-1}$ lie in K_0. Given $\xi \in K_0$, take the multiplication map $m_\xi : k \mapsto \xi k$. Put $u = xu_0$, $v_0 = yr_0$, and $c_* = \mathrm{Int}(b^{-1})c_0$. Then, $u^j = z^{p^j - 1} u_0^j$, $u^f = v_0$, and $u^{fd} = -I$. The linear determinant of u is $\mathrm{Norm}(x) D^{p-1} = D^{1-p} D^{p-1} = 1$. Thus, $\det v_0 = 1$. Inversion on T_d gives $c_*^2 = 1$. Since $b^{p-1} = x^2$, $b^{-1} x^{-1} u_0 b = x u_0$. Hence, $c_* u = u c_*$. Relative to the standard graph $-T$, the inner representative is $b^{-1} B^{-1} J_d$. Since $\mathrm{Norm}(b^{-1}) = D^2$, $\mathrm{Norm}(b^{-1}) \det(B^{-1}) = 1$. On the residual block V_{res}, take the Borel graph c_{Bor}. Thus, the full representative h has $\det h = (-1)^{\binom{n}{2} + \binom{r}{2}} = (-1)^{(d/2)A}$. □

Lemma 3.16. *Let $e \mid d$, $m = d/e$, and let θ_e be a regular character of $\mathbb{F}_{q^e}^\times$. Moreover, we put $s = \theta_e(-1) = (-1)^b$ and $\theta = \theta_e \circ \mathrm{Norm}_{K_0/\mathbb{F}_{q^e}}$. If $0 \le j < m$, we choose an extension $\widehat{\theta}_j$ of θ to $T_d\langle v_0^e \rangle$. The value $\lambda = \widehat{\theta}_j(v_0^e)$ is ζ_m^j for odd e, and $s\zeta_m^j$ for even e. Moreover, there are roots η, κ with $\eta^f = \kappa$, $\kappa^e = \lambda$. The component indexed by η, κ has multiplicity 1 in the field-lift model. Given $\sigma \in \mathcal{H}_\ell$, $\sigma(\eta) = \eta$ when e is odd. If $2 \mid e$, $\sigma(\eta)/\eta = (\sigma(\sqrt{p})/\sqrt{p})^{mb}$.*

Proof. First, take r_0. Let λ_0 be the inertia value at r_0^e. Then, $\lambda_0 = s\zeta_m^j$ for odd e, and $\lambda_0 = \zeta_m^j$ for even e. When e is odd, m is even. A determinant twist δ has value $\delta(-1)^e$ at r_0^e, while the cuspidal twist acts by $s \mapsto s\delta(-1)^e$. Hence, $\zeta_m^{j^\delta} = \zeta_m^j$, so $j^\delta = j \pmod m$. At v_0^e, we multiply λ_0 by $\theta(z^{q^e - 1}) = s$. Indeed, $\mathrm{Norm}_{K_0/\mathbb{F}_{q^e}}(z^{q^e-1}) = z^{q^d - 1} = -1$.

Thus, $\lambda^m = s^m = \theta(-1)$, as in Lemma 3.15 for v_0. Given a block permutation P_π, $\det P_\pi = (\operatorname{sgn}\pi)^d = 1$, since d is even. Thus, ordinary Specht labels ν_i suffice.

Choose compatible roots ζ_d, ζ_{2d}. If e is odd or $s = 1$, set $\kappa = \zeta_d^j$. If e is even and $s = -1$, we set $\kappa = \zeta_{2d}^{2j+m}$. Then, $\kappa^e = \lambda$. Write $h = \operatorname{ord}_\ell(p)$, so $d = h/(h, f)$. Let κ_π be the π-part of κ for an odd prime π. By Lemma 3.9, we choose η_π with $\eta_\pi^f = \kappa_\pi$ and $|\eta_\pi|$ dividing the odd part of $\ell - 1$. For the 2-part put $a = v_2(h)$, $t = v_2(f)$, $b_0 = v_2(\ell - 1)$. Since d is even, $t < a \le b_0$ and $v_2(d) = a - t$. If e is odd or $s = 1$, an f-th root η of κ exists in $\mu_{\ell-1}$. Otherwise, if m is even, $v_2(|\kappa|) \le a - t$, and a root η has 2-part dividing 2^{b_0}. If m is odd, the 2-part of $|\kappa|$ is 2^{1+a-t}. We choose η with 2-part of order 2^{1+a}. By Section 2, the sign on this 2-part is $\sigma(\sqrt{p})/\sqrt{p}$. Thus, for even e and $\sigma \in \mathcal{H}_\ell$, $\sigma(\eta)/\eta = (\sigma(\sqrt{p})/\sqrt{p})^{mb}$.

In the external model, we put $Y = \langle T_d, u, c_* \rangle$ and $E = \langle u, c_* \rangle$. Then, $E \cap T_d = \langle -I \rangle$. Define $\xi \in \operatorname{Irr}(E)$ by $\xi(u) = \eta$ and $\xi(c_*) = 1$. In particular, at u^{fd}, $\xi(u^{fd}) = \lambda^m = \theta(-1)$. Put $H = T_d\langle v_0 \rangle$. By [95, Problem 5.2], $\operatorname{Res}_H^Y \operatorname{Ind}_E^Y \xi = \operatorname{Ind}_{\langle v_0 \rangle}^H(\xi|_{\langle v_0 \rangle})$, with $\xi(v_0) = \kappa$. Let ψ_j be the local irreducible above θ. The eigenvalues of v_0 on ψ_j are the distinct e-th roots $\kappa\zeta_e^r$, where $0 \le r < e$. Hence, the component indexed by κ has multiplicity 1. □

Lemma 3.17. *Let ρ be cuspidal of rank $e = 2r_1$, with central character ω. Put $t(a) = \operatorname{diag}(a^{r_1}, a^{r_1-1}, \dots, a^{1-r_1})$. Moreover, on the Gelfand–Graev space V_ρ, for $\sigma(\zeta_p) = \zeta_p^{a_\sigma}$, define $S_\sigma h(g) = \sigma(h(t(a_\sigma)^{-1}g))$ and $Ch(g) = h(t(-1)^{-1}\gamma(g))$. Define also $D_\delta h(g) = \delta(\det g)h(g)$. Let $\varphi(g) = g^{(p)}$ and $\Phi h(g) = h(\varphi^{-1}(g))$. For the starting label ρ, $C^2 = \omega(-1)$ and $S_\sigma C = \omega(a_\sigma)^\sigma C S_\sigma$. Moreover, $S_\sigma D_\delta = (\delta^\sigma(a_\sigma))^{-r_1} D_{\delta^\sigma} S_\sigma$ and $CD_\delta = \delta(-1)^{r_1} D_{\delta^{-1}} C$. $\Phi S_\sigma = S_\sigma \Phi$, $\Phi C = C\Phi$, and $S_\sigma S_\tau = S_{\sigma\tau}$.*

Proof. Let ξ be the source character of U_e. If $1 \le i < e$, $t(a)_{ii}/t(a)_{i+1,i+1} = a$. Thus, the substitutions for S_σ, C compensate the change in ξ, as in Lemma 3.12. Moreover, $\det t(a) = a^{r_1}$ and $\gamma(t(a)) = a^{-1}t(a)$. Thus, $t(-1)^{-1}\gamma(t(-1)^{-1}) = -I$, and $C^2 = \omega(-1)$. The arguments in $S_\sigma C$ and CS_σ differ by central multiplication by $a_\sigma I_e$ before σ. Hence, the scalar for $S_\sigma C$ is $\omega(a_\sigma)^\sigma$. The factor in $S_\sigma D_\delta$ is $(\delta^\sigma(a_\sigma))^{-r_1}$. The factor in CD_δ is $\delta(-1)^{r_1}$, since $\det\gamma(g) = \det(g)^{-1}$ and $\det t(a_\sigma) = a_\sigma^{r_1}$. Given $a, b \in \mathbb{F}_p^\times$, $t(ab) = t(a)t(b)$ and $\varphi(t(a)) = t(a)$. Thus, $\Phi S_\sigma = S_\sigma \Phi$, $\Phi C = C\Phi$, and $S_\sigma S_\tau = S_{\sigma\tau}$. □

Let $p \ne \ell$ be odd primes, $q = p^f$, and $d = \operatorname{ord}_\ell(q)$ even. Choose the field lift u and graph lift γ_* from Lemma 3.15. Next, on T_d, we put $v = u^f$ for each torus block. Then, $v^d = -I_d$, and the linear part g_u of u satisfies $\det g_u = 1$. Moreover, $\gamma_*(v) = v$ and $\gamma_*(t) = t^{-1}$ for $t \in T_d$. Let $\mathcal{B}$ index the torus blocks, with v_j on block j. Thus, put $v_{\text{all}} = \prod_{j\in\mathcal{B}} v_j$, acting trivially on the residual space V_{res}.

Fix a Harish–Chandra label Λ with cuspidal labels ρ_i. On Λ, we write e_i for the rank of ρ_i and ν_i for its partition. Moreover, write ν_i^0 for its residual partition and w_i for its total quotient weight. Call i active if $w_i > 0$. If i is active, $e_i \mid d$, and ν_i^0 is the m_i-core of ν_i, where $m_i = d/e_i$ and $|\nu_i| - |\nu_i^0| = m_i w_i$. Next, let (χ, ψ) be the pair with label Λ on global and local sides. For i inactive, the same partition ν_i and cuspidal intertwiners occur for χ and ψ. Moreover, we put $A = \sum_i w_i$. Let J be the stabilizer of Λ under determinant twists δ, field φ, graph γ and $\mathcal{H}_\ell$. Keep the external cyclic field group C_{2fd} while checking $u^f = v_{\text{all}}$ after normalization.

Choose a cyclotomic field $\mathbb{K}$ with compatible primitive roots, and enlarge $\mathbb{K}$ if necessary. With $m_0 = \operatorname{ord}_p(\ell)$, we choose $s_0 \in \mathbb{F}_{p^2}$ with $s_0^2 = \ell \bmod p$. Moreover, adjoin $\mu_{\ell^{2m_0}-1}$ to $\mathbb{K}$. Given $\sigma \in \mathcal{H}_\ell$, we write $\sigma(\zeta) = \zeta^{\ell^{k_\sigma}}$ for $\zeta \in \mu_{\ell^{2m_0}-1}$, where k_σ is defined modulo $2m_0$. Next, we set $s_\sigma = s_0^{k_\sigma}$ and $\sigma(\zeta_p) = \zeta_p^{a_\sigma}$. Since $s_0^{2m_0} = 1$ and $p \mid \ell^{2m_0} - 1$, $s_{\sigma\tau} = s_\sigma s_\tau$ and $s_\sigma^2 = a_\sigma$. Choose $i_0 \in \mathbb{F}_{p^2}$ and $\mathrm{i} \in \mathbb{C}$ with $i_0^2 = \mathrm{i}^2 = -1$.

Given an active label ρ of even rank $e = 2r$, let $\theta \in \operatorname{Irr}(\mathbb{F}_{q^e}^\times)$ be the regular character for ρ. Moreover, write $\theta(-1) = (-1)^b$ with $b \in \{0, 1\}$. Next, for $a \in \mathbb{F}_p^\times$, put $t_r(a) = \operatorname{diag}(a^r, a^{r-1}, \dots, a^{1-r})$. Let $F_{\sigma,\text{near}}, F_{\gamma,\text{near}}$ be the operators of Lemma 3.17 with $t_r(a)$. On the space with starting label θ, define $\Sigma_\sigma(\theta) = \sigma\big(\theta(s_\sigma)\big)F_{\sigma,\text{near}}$ and $\Gamma(\theta) = \mathrm{i}^b\theta(i_0)F_{\gamma,\text{near}}$. Moreover, the scalars $\theta(s_\sigma), \theta(i_0)$ have arguments in $\mathbb{F}_{p^2} \subseteq \mathbb{F}_{q^e}$, even when s_σ or i_0 lies outside $\mathbb{F}_q$. The argument matrices for $F_{\sigma,\text{near}}, F_{\gamma,\text{near}}$ lie over $\mathbb{F}_q$. Next, for the field automorphism φ, put $(\Phi h)(g) = h(\varphi^{-1}(g))$. For δ, put $(C_\delta h)(g) = \delta(\det g)h(g)$. We compute the sign exponents in $\mathbb{F}_2$. Thus, we put $P = (p-1)/2$, $L = (\ell-1)/2$, $F = f$ and $(-1)^R = \left(\frac{p}{\ell}\right)$, $(-1)^S = \left(\frac{\ell}{p}\right)$. Thus, $S = R + PL$. In sign exponents k_σ denotes reduction modulo two. Moreover, write $\delta(-1) = (-1)^{t_\delta}$.

Lemma 3.18. *Given an even-rank cuspidal label ρ, the normalized operators satisfy $\Sigma_\sigma\Phi = (-1)^{bSk_\sigma}\Phi\Sigma_\sigma$ and $\Sigma_\sigma\Gamma = (-1)^{bLk_\sigma}\Gamma\Sigma_\sigma$. Moreover, $\Gamma\Phi = (-1)^{bP}\Phi\Gamma$ and $\Sigma_\sigma C_\delta = (-1)^{rFSk_\sigma t_\delta} C_{\delta^\sigma}\Sigma_\sigma$. Finally, $\Gamma C_\delta = (-1)^{rFPt_\delta} C_{\delta^{-1}}\Gamma$. Given a composite AB, the operator A acts at the label $B(\theta)$. Moreover, $\Sigma_\sigma\Sigma_\tau = \Sigma_{\sigma\tau}$ and $\Gamma^2 = 1$. Given h, g, $(\Phi C_\delta h)(g) = \delta(\det(\varphi^{-1}(g)))(\Phi h)(g)$, and $C_\delta C_\varepsilon = C_{\delta\varepsilon}$.*

Proof. Indeed, the field signs satisfy $s_\sigma^p/s_\sigma = (-1)^{Sk_\sigma}$, $i_0^p/i_0 = (-1)^P$, and $\sigma(\mathrm{i})/\mathrm{i} = (-1)^{Lk_\sigma}$. Let $\mathrm{N} = \mathrm{Norm}_{\mathbb{F}_{q^e}/\mathbb{F}_q}$. Moreover, pairing the q-conjugates at positions $2j, 2j+1$ gives that $\mathrm{N}(s_\sigma) = a_\sigma^r(-1)^{rFSk_\sigma}$ and $\mathrm{N}(i_0) = (-1)^{r(1+FP)}$. Since $e = 2r$, $\mathrm{N}(-1) = 1$. Thus, the twist $\theta(\delta \circ \mathrm{N})$ has the same value as θ at -1. The norm factors and Lemma 3.17 give $\delta^\sigma(\mathrm{N}(s_\sigma))\delta^\sigma(a_\sigma)^{-r} = (-1)^{rFSk_\sigma t_\delta}$ and $\delta(\mathrm{N}(i_0))\delta(-1)^r = (-1)^{rFPt_\delta}$. For $\Sigma_\sigma\Gamma$, the inverted label contributes $\sigma(\theta(s_\sigma)^{-2})$. Since $s_\sigma^2 = a_\sigma$, $\sigma(\theta(s_\sigma)^{-2})\sigma(\theta(a_\sigma)) = 1$. Thus, we obtain the remaining factor $\sigma(\mathrm{i})^b/\mathrm{i}^b$. In Γ^2, $\theta(i_0)\theta(i_0^{-1}) = 1$, and $\mathrm{i}^{2b} = \theta(-1)$ cancels the raw square. Finally, let $c_\tau = \tau(\theta(s_\tau))$. The coefficient action on c_τ and $s_{\sigma\tau} = s_\sigma s_\tau$ imply $\Sigma_\sigma\Sigma_\tau = \Sigma_{\sigma\tau}$. □

Given $a \in J$, we write $T_a = C_{\delta_a}\Phi^{h_a}\Gamma^{\gamma_a}\Sigma_{\sigma_a}$ in normal order. Next, take h_a modulo $2fd$, and put $k_a = k_{\sigma_a} \bmod 2$ and $t_a = t_{\delta_a}$. The exponents satisfy $(h_{ab}, \gamma_{ab}, k_{ab}, t_{ab}) \equiv (h_a, \gamma_a, k_a, t_a) + (h_b, \gamma_b, k_b, t_b) \pmod 2$. Moreover, for a determinant character δ, $\delta^\varphi(-1) = \delta^{-1}(-1) = \delta^\sigma(-1) = \delta(-1)$. Choose a regular representative θ_i for each active even-rank label i. Let π_a be the permutation of labels under a, and define $j_i(a) \in \mathbb{Z}/e_i\mathbb{Z}$ by $\theta_i^a = \theta_{\pi_a(i)}^{q^{-j_i(a)}}$. In particular, the exponent q^{-1} denotes the inverse q-Frobenius on arguments. Given a label θ, $(\theta^{q^{-1}})^a = (\theta^a)^{q^{-1}}$. Thus, for ab, with b followed by a, $j_i(ab) = j_i(b) + j_{\pi_b(i)}(a) \pmod{e_i}$. Let $I_+ = \{i : w_i > 0,\ 2 \mid e_i\}$. Next, in $\mathbb{F}_2$, we put $B = \sum_{i\in I_+} b_i m_i w_i$ and $E = \frac{d}{2}\sum_{i\in I_+} w_i$. Set $Q_a = \sum_{i\in I_+} b_i m_i w_i j_i(a)$ and $X_a = Bh_a + FQ_a$. Finally, $K_a = X_a + EFt_a$. Given $i \in I_+$, the label action fixes (b_i, m_i, w_i), and e_i is even. Hence, Q_a, X_a, K_a are additive on J. For active i with odd e_i, the determinant-one operators U_a in the Gelfand–Graev model satisfy $U_aU_b = U_{ab}$.

If $I = R_L^G(V)$, let $\mathcal{V}_\nu$ be a multiplicity space for I. Next, Lemma 3.13 gives the transports on $\mathcal{V}_\nu$. Moreover, by the factor for a, b, a distinct-colour shuffle w has determinant sign $\delta(-1)^{\ell(w)}$ and coefficient sign $(\sigma(\sqrt{q})/\sqrt{q})^{\ell(w)}$. Given a partition ν, the Specht operator at w_0 is $q^{-eK(\nu)/2}T_{w_0}$, where $K(\nu) = \binom{|\nu|}{2} + \sum_{(i,j)\in\nu}(j-i)$. Moreover, the weighted length $\ell(w)$ counts matrix coordinates. Given w, the averaging A_w is as in Lemma 3.6. By Zhang [208, Lemma 4.4, (4.1)], $T_{w_0}^2 = q^{eK(\nu)}I$ on S_ν. Given any normalized generator T_i, the sorted maps satisfy $J_wT_i = T_i'J_w$. Thus, the Specht formula holds on S_ν.

Each crossing (i, j) contributes $e_i|\nu_i|\,e_j|\nu_j|$ to $\ell(w)$. Replacing ν_i by ν_i^0 changes the block size by dw_i, which is even. Moreover, let $\ell_G(w), \ell_M(w)$ denote global and local lengths. Thus, $\ell_G(w) \equiv \ell_M(w) \pmod 2$. Moreover, $e_i(K(\nu_i) - K(\nu_i^0)) \equiv 0 \pmod 2$. If $2 \nmid e_i$, the rim hooks in $\nu_i \to \nu_i^0$ have length m_i. Since m_i is even, the consecutive contents change K by an even integer. If $2 \mid e_i$, the factor e_i is even. Hence, square-root signs ϵ_i, ϵ_i^0 satisfy $\epsilon_i = \epsilon_i^0$. For a permutation π of local d-blocks, $\det P_\pi = (\operatorname{sgn}\pi)^d = 1$.

Lemma 3.19. *Given $a, b \in J$, the common residual scalar is β_{res}. Moreover, let $f_{\mathrm{glob}}(a, b)$ be the global cuspidal factor. Put $c_{\mathrm{glob}}(a, b) = f_{\mathrm{glob}}(a, b)/\beta_{\mathrm{res}}$. Moreover, the local factor $f_{\mathrm{loc}}(a, b)$ for torus and wreath parts gives $c_{\mathrm{loc}}(a, b) = f_{\mathrm{loc}}(a, b)/\beta_{\mathrm{res}}$. For the graph lift γ_*, $c_{\mathrm{glob}}(a, b) = (-1)^{G(a,b)}$ and $c_{\mathrm{loc}}(a, b) = (-1)^{H(a,b)}$, where $G(a, b) = (Sk_a + P\gamma_a)K_b + BLk_a\gamma_b + Et_a\gamma_b$ and $H(a, b) = Rk_aX_b$.*

Proof. First, for $a, b \in J$, order the operators on V_{ρ_i}, with each representative θ_i fixed. The multiplicity difference is m_iw_i. Thus, we raise each sign to m_iw_i by Lemma 3.18. The product exponent is $B(Sk_ah_b + Lk_a\gamma_b + P\gamma_ah_b) + EFt_b(Sk_a + P\gamma_a)$. Indeed, $r_im_i = d/2$ for $e_i = 2r_i$. Replacing θ by $\theta^{q^{-j}}$ changes Σ_σ, Γ by $(-1)^{bFSk_\sigma j}, (-1)^{bFPj}$, respectively, from scalar evaluations on $\mathbb{F}_{p^2}$. Moreover, both models have the operators Φ, C_δ. Restoring the representative after T_b and before T_a adds $FQ_b(Sk_a + P\gamma_a)$.

The inner correction h for γ_* has $\det h = (-1)^{dA/2}$ by Lemma 3.15. Thus, $(dA/2)t_a\gamma_b$ comes from the lift change. If $t_a = 1$, the determinant twist acts by $s_i = \omega_{\rho_i}(-I_{e_i}) \mapsto -s_i$ for each active odd-rank label i. The other label operations fix s_i. Thus, each active odd-rank label i lies in an even π_a-cycle with constant w_i. Let $I_- = \{i : w_i > 0,\ 2 \nmid e_i\}$. Then, $\sum_{i\in I_-} w_i \equiv 0 \pmod 2$, and $(dA/2)t_a = Et_a$ in $\mathbb{F}_2$, including when $I_- \neq \varnothing$. The linear part g_u of each field lift satisfies $\det g_u = 1$. Consequently, the exponent is $G(a, b)$.

Fix an active even-rank label i and an index $0 \le j < m = d/e$. With $T = \mathbb{F}_{q^d}^\times$, we put $\vartheta = \theta \circ \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_{q^e}}$. Next, choose a Fourier basis $\mathbf{e}_0, \dots, \mathbf{e}_{e-1}$, indexed modulo e. Given $t \in T$, $R_\vartheta(t)\mathbf{e}_l = \vartheta(t^{q^{-l}})\mathbf{e}_l$, $P_e\mathbf{e}_l = \mathbf{e}_{l+1}$, and $R_\vartheta(v) = \kappa P_e$. Moreover, we choose compatible roots ζ_d, ζ_{2d}. If $b_i = 0$, we put $\kappa = \zeta_d^j$. If $b_i = 1$, we put $\kappa = \zeta_{2d}^{2j+m}$. Thus, $\kappa^e = (-1)^{b_i}\zeta_m^j$. By Lemma 3.16, $\eta \in \mu_{2(\ell-1)}$ with $\eta^f = \kappa$, $\sigma(\eta)/\eta = (-1)^{Rk_\sigma mb_i}$, and $\sigma(\kappa)/\kappa = (-1)^{FRk_\sigma mb_i}$. Moreover, the choices of κ, η depend only on e, j, b_i and are stable under label permutations π_a.

For the field transport to $\vartheta^{p^{-1}}$, set $U = \eta I$. Let $\vartheta_\delta = \vartheta(\delta \circ \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_q})$. Similarly, take I_e for the graph action with target ϑ^{-1} and the determinant action with target ϑ_δ. For coefficient transport σ, we choose $J_\sigma\sigma$, where $J_\sigma\mathbf{e}_l = (\sigma(\kappa)/\kappa)^l\mathbf{e}_l$. Put $r_\sigma = \sigma(\kappa)/\kappa$. Then, $r_\sigma^e = 1$, $r_\sigma^2 = 1$, and $J_\sigma P_e = r_\sigma P_e J_\sigma$. Given $t \in T$, $J_\sigma\sigma(R_\vartheta(t))J_\sigma^{-1} = R_{\vartheta^\sigma}(t)$,

and the same relation holds at v. Since $r_\sigma \in \{\pm 1\}$ and $r_{\sigma\tau} = r_\sigma r_\tau$, $J_\sigma \sigma(J_\tau) = J_{\sigma\tau}$. The operator taking $\vartheta^{q^{-j_i(b)}}$ to ϑ is $P_e^{j_i(b)}$. Put $j_b = j_i(b)$. Then, $\sigma(U)U^{-1} = (\sigma(\eta)/\eta)I$ and $J_\sigma P_e^{j_b} = r_\sigma^{j_b} P_e^{j_b} J_\sigma$. On I_+, the local signs have exponent $\sum_{i\in I_+} Rk_a b_i m_i w_i (h_b + Fj_i(b)) = Rk_a(Bh_b + FQ_b) = H(a,b)$. Given an active odd-rank label i, $\sigma(\eta) = \eta$, and wreath and Specht matrices are $\mathbb{Q}$-valued, with scalar 1. Indeed, after f applications of U, the label is $\vartheta^{q^{-1}}$. The correction by P_e gives $\eta^f P_e = R_\vartheta(v)$. Moreover, $R_\vartheta(v)^d = \vartheta(-1)I = (-1)^{mb_i} I$ and $\eta^{2fd} = 1$. □

Proposition 3.20. *Next, adjoin μ_{ℓ^4-1} to the coefficient field $\mathbb{K}$. Given $a \in J$, we let $k_{4,a} \in \mathbb{Z}/4\mathbb{Z}$ satisfy $\sigma_a(\zeta) = \zeta^{\ell^{k_{4,a}}}$ for $\zeta \in \mu_{\ell^4-1}$. Moreover, for a, we put $c(a) = (-1)^{P\gamma_a K_a}\, \mathrm{i}^{B\gamma_a} (-1)^{BL\binom{k_{4,a}}{2}}$ for the transports of* Lemma 3.13. *Then, $\alpha_{\mathrm{global}}/\alpha_{\mathrm{local}} = \partial c$ on J, where $(\partial c)(a,b) = c(a)\sigma_a(c(b))/c(ab)$. In particular, for a a pure determinant self-twist, $c(a) = 1$. For $z = u^f$ and $a \in J$, $c(az) = c(a)$.*

Proof. Put $Z = \prod_{i\in I_+} \theta_i(i_0)^{m_i w_i}$. Then, $Z^2 = (-1)^B$. Given $a \in J$, put $Z_a = \prod_{i\in I_+} \theta_i^a(i_0)^{m_i w_i}$. Field, graph and coefficient actions contribute $(-1)^{B(Ph_a+\gamma_a+Lk_a)}$, and the determinant twist contributes $(-1)^{E(1+FP)t_a}$. Thus, $Z_a/Z = (-1)^{B(Ph_a+\gamma_a+Lk_a)+E(1+FP)t_a}$. The label permutation fixes Z, and the representative shifts contribute FPQ_a. Hence, $Z_a/Z = (-1)^{FPQ_a}$. The two expressions for Z_a/Z give $PK_a = B(\gamma_a + Lk_a) + Et_a$.

By Lemma 3.19, $\alpha_{\mathrm{global}}(a,b)/\alpha_{\mathrm{local}}(a,b) = (-1)^{G(a,b)+H(a,b)}$. Substitute $S = R + PL$, $X_b = K_b + EFt_b$, so $PK_x = B(\gamma_x + Lk_x) + Et_x$ for $x = a, b$. The sum is

$$P(\gamma_a K_b + K_a\gamma_b) + B\gamma_a\gamma_b + BLk_a\gamma_b + BLk_ak_b + E(L+FR)k_at_b.$$

If $4 \mid d$, $E = 0$, so $E(L+FR) = 0$. Suppose that $v_2(d) = 1$. If $\ell \equiv 3 \pmod 4$, the even order of p^f mod ℓ gives $2 \nmid f$ and $(p/\ell) = -1$. Thus, $L = F = R = 1$, and $L + FR = 0$. If $\ell \equiv 1 \pmod 4$, $L = 0$. If $R = 0$, $L + FR = 0$. If $R = 1$, $v_2(\mathrm{ord}_\ell(p)) = v_2(\ell - 1) \geq 2$. Since $d = \mathrm{ord}_\ell(p^f)$ and $v_2(d) = 1$, f is even. Hence, $F = 0$ and $L + FR = 0$. In every case, $E(L+FR) = 0$.

Write $c_1(a) = (-1)^{P\gamma_a K_a}$, $c_2(a) = \mathrm{i}^{B\gamma_a}$, and $c_3(a) = (-1)^{BL\binom{k_{4,a}}{2}}$. Then, $(\partial c_1)(a,b) = (-1)^{P(\gamma_a K_b + K_a\gamma_b)}$. Since $\sigma_a(\mathrm{i}) = (-1)^{Lk_a}\mathrm{i}$, $(\partial c_2)(a,b) = (-1)^{B\gamma_a\gamma_b + BLk_a\gamma_b}$. For integers x, y, $\binom{x+y}{2} = \binom{x}{2} + \binom{y}{2} + xy$, and $\binom{x+4}{2} \equiv \binom{x}{2} \pmod 2$. Hence, $(\partial c_3)(a,b) = (-1)^{BLk_ak_b}$. Therefore, $(\partial c)(a,b) = (-1)^{G(a,b)+H(a,b)} = \alpha_{\mathrm{global}}(a,b)/\alpha_{\mathrm{local}}(a,b)$. Given a pure determinant self-twist a, $\gamma_a = k_{4,a} = 0$, so $c(a) = 1$. Put $z = u^f$. Then, $h_z = f$, $\gamma_z = k_{4,z} = t_z = 0$, $\pi_z = 1$, and $j_i(z) = 1 \pmod{e_i}$. Hence, $Q_z = B$ and $K_z = Bf + FB = 0$. Given $a \in J$, $K_{az} = K_a$, $\gamma_{az} = \gamma_a$, and $k_{4,az} = k_{4,a}$. Thus, $c(z) = 1$ and $c(az) = c(a)$. Let $\mathrm{res}_{\mathbb{K}} : \mathcal{H}_\ell \to \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ be defined by $\mathrm{res}_{\mathbb{K}}(\sigma) = \sigma|_{\mathbb{K}}$. The actions on coefficients factor through $\mathrm{res}_{\mathbb{K}}$. Therefore, $(\partial c)(a,b) = \alpha_{\mathrm{global}}(a,b)/\alpha_{\mathrm{local}}(a,b)$ holds for the full group $\mathcal{H}_\ell$. □

Let ρ, τ be representations on V, W, respectively. Put $z = u^f$ and $v_{\mathrm{all}} = u^f$. Next, on each bulk block, choose v_0 from Lemma 3.15. On V_{res}, take the identity. If $\phi(x) = x^{(p)}$, the raw field operator Φ on the global source space satisfies $(\Phi h)(x) = h(\phi^{-1}(x))$. Thus, $\Phi^f = 1$, including after sorting labels (ρ_i). For pure field operations a, b, the composition factor is 1, and the identity shuffle is 1. With $u = g\phi$, $g\phi(g)\cdots\phi^{f-1}(g) = v_{\mathrm{all}}$. Hence, $P_z = \rho(v_{\mathrm{all}})$. Locally, f field steps act by $\vartheta \mapsto \vartheta^{q^{-1}}$. Thus, on the Fourier block W_i, the operator is $\eta^f P_e = R_\vartheta(v)$, the matrix of v_0. If $2 \nmid e$, Lemma 3.16 takes the induced model W_i for the corresponding power of v_0. Thus, the tensor of local blocks W_i and the residual field power yields $Q_z = \tau(v_{\mathrm{all}})$.

Given $a \in J$, replace P_a by $c(a)^{-1}P_a$. Thus, by Proposition 3.20, $c(z) = 1$ and $c(az) = c(a)$ on J, and $c(a) = 1$ for pure determinant self-twists a. Thus, the new pairing Ω_{new} satisfies $\Omega_{\mathrm{new}} = \Omega$. On $\mathrm{Hom}(W,V)^B$, the normalized operators at z act by $F \mapsto \rho(v_{\mathrm{all}})F\tau(v_{\mathrm{all}})^{-1}$. Moreover, by Lemma 3.15, $\gamma_* u = u\gamma_*$ and $\gamma_*^2 = 1$. Let $\bar{u}, \bar{\gamma}_*$ be the images modulo $\widetilde{M}$. Then, $|\bar{u}| = f$ and $|\bar{\gamma}_*| = 2$. Put $L = \widetilde{X}$, $C = L/X$, and $T_{\mathrm{fib}} = L \times_C \widetilde{M}$. Next, for the lift group Λ, put $A_{\mathrm{fib}} = L\Lambda$ and $H_{\mathrm{fib}} = \widetilde{M}\Lambda$. Let Λ_0 stabilize the determinant orbit of (ρ, τ). Moreover, the map $T_{\mathrm{fib}} \rtimes \Lambda_0 \to A_{\mathrm{fib}} \times_{A_{\mathrm{fib}}/X} H_{\mathrm{fib}}$ following Proposition 3.5 has kernel generated by $k = ((v_{\mathrm{all}}^{-1}, v_{\mathrm{all}}^{-1}), z)$. Given $F \in \mathrm{Hom}(W,V)^B$, $\rho(v_{\mathrm{all}}^{-1})[\rho(v_{\mathrm{all}})F\tau(v_{\mathrm{all}})^{-1}]\tau(v_{\mathrm{all}}) = F$. Thus, k acts trivially. Moreover, for k^d, both matrix coordinates are $\mathrm{diag}(-I_{Ad}, I_r)$, including when $r > 0$. Lemma 3.4 gives matched summands (χ, ψ), with mixed-invariant extensions on I_χ, I_ψ, the ordinary inertia groups of χ, ψ. Given $z_0 \in \mathrm{Z}(L)$, the common central character ω satisfies $\rho(z_0) = \omega(z_0)I_V$ and $\tau(z_0) = \omega(z_0)I_W$. Hence, $\rho(z_0)F\tau(z_0)^{-1} = F$. Consequently, for each b in the mixed stabilizer, Proposition 3.5 yields $\mu_b|_{H_0} = \mu_b'$, where $H_0 = M \rtimes I$ and $I = \Gamma_\chi = \Gamma_\psi$.

Put $X = \mathrm{SL}_n(q)$. Moreover, for a fixed prime ℓ, we choose one Sylow Φ_d-torus $\widetilde{T} \leq \mathrm{GL}_n(q)$ for all $\chi \in \mathrm{Irr}(X)$. If $d = 1$, the torus and its normalizer are $\widetilde{T} = (\mathbb{F}_q^\times)^n$, $\quad \mathrm{N}_{\mathrm{GL}_n(q)}(\widetilde{T}) = (\mathbb{F}_q^\times)^n \rtimes S_n$. For general n, q, we write $X = \ker(\det : \mathrm{GL}_n(q) \to \mathbb{F}_q^\times)$. However, a character $\chi \in \mathrm{Irr}_{\ell'}(X)$ may have no parent in $\mathrm{Irr}_{\ell'}(\mathrm{GL}_n(q))$.

Lemma 3.21. *Let $n \geq 3$, let ℓ be odd with $\ell \mid q-1$, and let $X = \mathrm{SL}_n(q)$. Next, we put $\widetilde{M} = \mathrm{N}_{\mathrm{GL}_n(q)}((\mathbb{F}_q^\times)^n)$ and $M = X \cap \widetilde{M}$. For the full set $\mathcal{P}_{\mathrm{spl}}$ of split principal-series parents in $\mathrm{Irr}(\mathrm{GL}_n(q))$, we put $\mathcal{P}_{\mathrm{mon}} = \mathrm{Irr}(\widetilde{M})$. Accordingly, for $\mathrm{Irr}_{\ell'}(X)$ and $\mathrm{Irr}_{\ell'}(M)$, it suffices to match $\mathcal{P}_{\mathrm{spl}}$ with $\mathcal{P}_{\mathrm{mon}}$ and take the matched constituents of ℓ'-degree.*

Proof. By [33, Notation 6.2 and Theorem 6.1], every parent over $\mathrm{Irr}_{\ell'}(X)$ has s in the split diagonal torus, since $\ell \mid q-1$. Thus, $\mathcal{P}_{\mathrm{spl}}$ covers $\mathrm{Irr}_{\ell'}(X)$ even when $\ell \mid \widetilde{\chi}(1)$. By [99, §4.3], $\mathcal{P}_{\mathrm{mon}}$ has the same partition labels $(\nu_\lambda)_{\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)}$. In our correspondence $\widetilde{\Omega}_{\mathrm{spl}}$, the factor at λ is $\mathrm{Sp}^{\nu_\lambda} \otimes \mathrm{sgn}^{b_\lambda}$ with $\lambda(-1) = (-1)^{b_\lambda}$, as in Section 3.2. For $\widetilde{\psi} = \widetilde{\Omega}_{\mathrm{spl}}(\widetilde{\chi})$, the linear valuation formula and [99, §4.3 and Theorem 2.3.21] give $v_\ell(\widetilde{\chi}(1)) = v_\ell(\widetilde{\psi}(1))$.

By [33, Theorem 6.1], $\widetilde{\Omega}_{\mathrm{spl}}$ is determinant equivariant. Given $\delta \in \mathrm{Irr}(\mathbb{F}_q^\times)$, put $\Delta_\delta = \delta \circ \det$. On $\widetilde{M}$, Δ_δ changes λ to $\lambda\delta$ and adds sgn^{b_δ}, since $\det P_\pi = \mathrm{sgn}(\pi)$. Write B_χ, B_ψ for the self-twist groups. Thus, $b_{\lambda\delta} \equiv b_\lambda + b_\delta \pmod 2$ gives $B_\chi = B_\psi$. By the cyclic-quotient formulas, $t = |B_\chi|$ gives constituent degrees $\widetilde{\chi}(1)/t, \widetilde{\psi}(1)/t$. Hence, the ℓ'-degree cuts agree. □

Proposition 3.22. *Let $S = \mathrm{PSL}_n(q)$ be simple with $n \geq 3$. Assume that $X = \mathrm{SL}_n(q)$ is its universal cover, that ℓ is an odd nondefining prime dividing $|S|$, and that the Sylow Φ_d-torus normalizer $M < X$ contains $\mathrm{N}_X(P)$ for a Sylow ℓ-subgroup P. Then, the correspondence on X, M satisfies the inductive comparison at ℓ.*

Proof. Let Ω_0 be the bijection of [156, Theorem D] and put $\mathcal{U} = \mathrm{Irr}_{\ell'}(X) \cap \mathrm{UCh}(X)$. Thus, Ω_0 gives the inductive comparison on $\mathcal{U}$ with local set $\Omega_0(\mathcal{U})$. If $d > 1$, [33, Proposition 6.3 and Theorem 6.1] give restriction coverage by ℓ'-degree parents, since $\ell \nmid (q-1)$. In the labels $(\kappa_j, \mu_{j,k})$ of Lemma 3.2, put $\widetilde{\psi} = \widetilde{\Omega}(\widetilde{\chi})$. If q is even, Proposition 3.8 gives $\widetilde{\Omega}$. If q is odd, Section 3.2 gives $\widetilde{\Omega}$. The two labels give the same $\widetilde{\psi}(1)$ and central character. If $d = 1$, Lemma 3.21 gives all parents. Then, $r = 0$ and each occupied orbit has length 1. Moreover, the 1-core is empty and the 1-quotient is ν_i.

When q is even, $\alpha_{\mathrm{glob}} = \alpha_{\mathrm{loc}}$ by Proposition 3.8, and Lemma 3.9 gives $u^f = r_0$ on each bulk block. When $d = 1$, $\mathcal{H}_\ell$ fixes $\sqrt{q}$ by [156, Lemma 2.3(d)]. Moreover, Proposition 3.7 gives factor 1 for every split parent $\widetilde{\chi}$. If q, d are odd, by Proposition 3.14, $\alpha_{\mathrm{glob}} = \alpha_{\mathrm{loc}}$. At $d = 1$, the argument covers every split parent $\widetilde{\chi}$. Indeed, $R_i = 0$, $N_i = w_i$, the cuspidal operators for i have rank 1. Moreover, the sign-cycle calculation follows i-label action. Each torus block has $u^f = r_{\mathrm{all}}$.

If q, d are odd, we let $B \leq \mathrm{Irr}(\mathbb{F}_q^\times)$ be the self-twist group of $\widetilde{\chi}, \widetilde{\psi}$. Next, put $H_1 = \mathrm{Hom}(V_\tau, V_\rho)^B$, as in Lemma 3.4. Moreover, choose c_T from Proposition 3.14. c_T is the symmetric graph lift. Hence, $c_T^2 = 1$ and $c_T u = u c_T$. Modulo $\widetilde{M}$, the image of u is the field generator of order f. Moreover, c_T gives the graph generator of order 2. Thus, $\langle u, c_T \rangle \cap \widetilde{M} = \langle u^f \rangle = \langle r_{\mathrm{all}} \rangle$, and $\det(r_{\mathrm{all}}) = 1$. Globally $u = g\phi$, where $\phi^f = 1$ on the matrix group and $\det(g) = 1$. Hence, $r_{\mathrm{all}} = g\phi(g) \cdots \phi^{f-1}(g)$. On each occupied torus block, u^f acts by rotation. On G_τ, u^f acts trivially. Consequently, U_{u^f} on H_1 acts by $F \mapsto \rho(r_{\mathrm{all}}) F \tau(r_{\mathrm{all}})^{-1}$. The operators U_j compose on H_1, and the action of r_{all}^{-1} on both sides cancels U_{u^f}. Given a central scalar z, the matched central characters give $\rho(z) = \tau(z)$ as scalars. Thus, the diagonal scalar (z, z) acts trivially on H_1. If q is odd and d is even, take the factor for $a, b \in J$ on each noncuspidal constituent χ in its Harish–Chandra series. Given χ, the mixed factors satisfy $\alpha_{\mathrm{global}}/\alpha_{\mathrm{local}} = \partial c$ by Proposition 3.20. Moreover, $c(a) = 1$ for every pure determinant self-twist a, and $c(au^f) = c(a)$. Thus, after normalization by c, self-twist operators stay fixed and $u^f = v_{\mathrm{all}}$ still holds.

In each case, $\det(\widetilde{M}) = \mathbb{F}_q^\times$, so $\widetilde{G} = X\widetilde{M}$, with $M = X \cap \widetilde{M}$. The matched central characters agree on $\mathrm{Z}(\widetilde{G})$. Put $T = \widetilde{G} \times_{\mathbb{F}_q^\times} \widetilde{M}$ and $\Delta Z = \{(z, z) : z \in \mathrm{Z}(\widetilde{G})\}$. By Lemma 3.4, the summand for (χ, ψ) in $\mathrm{Hom}(W, V)^B$ affords the extension Θ_F. Since $\Delta Z \leq \ker \Theta_F$, the descent after Proposition 3.5 gives Ξ on $(X \times M) \rtimes \Gamma_\chi$. The character $\mathrm{tr}\,\Xi$ is invariant under the elements b of the mixed stabilizer, and $\Xi(x, x, a) = 1$ for $(x, a) \in \mathrm{C}_{X \rtimes \Gamma_\chi \mathcal{H}_p}(X)$. Thus, Proposition 3.5 yields $\succeq_c$, including $\mu_b|_{H_0} = \mu_b'$ for each b in the mixed stabilizer. By the Frattini argument in Proposition 3.10, $\mathrm{Aut}(X)_P \leq \mathrm{Aut}(X)_M$. □

3.3 Unitary centralizers

With $q = p^f$ and p odd, we put $G = \mathrm{GU}_n(q)$. Similarly, we consider $G = \mathrm{GL}_n(q)$. The Frobenius F gives G and the rational structure on $\mathfrak{g} = \mathfrak{gl}_n(\overline{\mathbb{F}}_p)$. Fix a nonzero $J \in \mathfrak{g}^F$ with J nilpotent. Next, we let λ be the rational Jordan grading of J as in Zhang [208, Proposition 5.9, Step 1]. A string of length k has weights $k-1, k-3, \ldots, 1-k$, in the order of the J-action. A matrix in $\mathfrak{g}(i)$ raises weights by i, so $J \in \mathfrak{g}(-2)^F$. Thus, we put $U_i = \{u \in G : u - 1 \in$

$\bigoplus_{j\geq i}\mathfrak{g}(j)\big\}$ $(i\geq 1)$, $U=U_1$. Fix a primitive p-th root of unity ζ_p, and set $K=\mathbb{Q}(\zeta_p)$. Moreover, for $X\in\mathfrak{g}(2)^F$ put $L_J(X)=\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\big(\mathrm{tr}(JX)\big)$. In the unitary case, J and X are skew Hermitian. Hence, $\mathrm{tr}(JX)^q=\mathrm{tr}(XJ)=\mathrm{tr}(JX)$, so the field trace in L_J is defined. Given $u\in U$, we write $c(u)_i$ for the degree-i component of $c(u)=2(u-1)(u+1)^{-1}$. Next, we define $\theta(u)=\zeta_p^{L_J(c(u)_2)}$ for $u\in U_2$, $W=\mathfrak{g}(1)^F$, and $\beta(X,Y)=L_J([X,Y])$.

Lemma 3.23. *The form β is symplectic on W over $\mathbb{F}_p$. Moreover, we have that the map $\kappa(u)=(c(u)_1,L_J(c(u)_2))$ induces an isomorphism $U/\ker\theta\simeq H(W)=W\times\mathbb{F}_p$, where $(X,z)(Y,t)=\big(X+Y,z+t+\frac{1}{2}\beta(X,Y)\big)$. κ induces a map from $U_2/\ker\theta$ onto $\mathrm{Z}(H(W))$. If $\dim_{\mathbb{F}_p}W=2N$, the unique irreducible character η of U lying over θ has degree p^N, with Schrödinger model over K.*

Proof. By [208, Proposition 5.9, Step 2], $\mathrm{ad}(J)\colon\mathfrak{g}(1)\to\mathfrak{g}(-1)$ is injective. By trace duality, $\beta_q(X,Y)=\mathrm{tr}(J[X,Y])$ is nondegenerate on W. Thus, its field trace β is nondegenerate over $\mathbb{F}_p$. Modulo $\bigoplus_{i\geq 3}\mathfrak{g}(i)$, the Cayley transform satisfies $c(uv)\equiv c(u)+c(v)+\frac{1}{2}[c(u),c(v)]$. Indeed, $c(1+A)\equiv A-\frac{1}{2}A^2$ modulo $\bigoplus_{i\geq 3}\mathfrak{g}(i)$. Thus, $\kappa:U\to H(W)$ is a homomorphism and θ is linear. Given $X_i\in\mathfrak{g}(i)^F$, $i=1,2$, we set $X=X_1+X_2$ and $u=(2+X)(2-X)^{-1}$. Thus, $c(u)_1=X_1$ and $c(u)_2=X_2$. Since $J\neq 0$, the trace pairing between $\mathfrak{g}(-2)$ and $\mathfrak{g}(2)$ gives $L_J(\mathfrak{g}(2)^F)=\mathbb{F}_p$. Hence, κ is surjective and $\ker\kappa=\ker\theta$.

The commutator form of $H(W)$ is β, so $\mathrm{Z}(H(W))=\{(0,z):z\in\mathbb{F}_p\}$. The group $H(W)$ has order p^{2N+1} and exponent p, since $(X,z)^j=(jX,jz)$. By the Stone–von Neumann theorem, $H(W)$ has a unique irreducible representation π with central character $z\mapsto\zeta_p^z$. For example, with a Lagrangian decomposition $W=L\oplus L^\vee$, fix $L\simeq\mathbb{F}_p^N$ and $V=K^{\mathbb{F}_p^N}$ for π. Its translation operators $(T_af)(x)=f(x+a)$ and multiplication operators $(M_\xi f)(x)=\zeta_p^{\xi(x)}f(x)$ have $a\in\mathbb{F}_p^N$ and linear $\xi:\mathbb{F}_p^N\to\mathbb{F}_p$ [85, §§1–2]. Thus, inflation gives η. If $N=0$, we have $H(W)\simeq C_p$, and η is linear. □

Let $h:G\to G'$ be conjugation or a field map. $h(\mathfrak{g}(i))=\mathfrak{g}'(i)$ in each degree. Moreover, on $\mathfrak{g}^F$, the induced map h is $\mathbb{F}_p$-linear. $c(h(u))=h(c(u))$, and h acts separately on $c(u)_1,c(u)_2$. Suppose that $h(\ker\theta)=\ker\theta'$ for the character θ' on the target model. On $H(W)$, the induced map has the form $(X,z)\longmapsto(gX,mz)$, where $g:W\to W'$, $\beta'(gX,gY)=m\beta(X,Y)$ and $m\in\mathbb{F}_p^\times$. Thus, the central coordinate mz depends only on z. A coefficient automorphism $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$ matches the central character precisely when $\sigma(\zeta_p)=\zeta_p^m$. For example, we let T be unitary with $T\mathfrak{g}(i)T^{-1}=\mathfrak{g}(i)$ and $TJT^{-1}=mJ$. Then, $\mathrm{Ad}(T^{-1})$ has multiplier m, since $\mathrm{tr}(JT^{-1}XT)=m\,\mathrm{tr}(JX)$. Thus, on $\mathrm{Z}(H(W))$, we have that $\sigma(\zeta_p^z)=\zeta_p^{mz}$ for $z\in\mathbb{F}_p$.

Proposition 3.24. *Suppose that Γ acts on $H(W)$ by $\varphi_a(X,z)=(g_aX,m_az)$ and on K by σ_a, with $\sigma_a(\zeta_p)=\zeta_p^{m_a}$ for every $a\in\Gamma$. Assume that Γ is finite. On V, the Schrödinger model, put $(Pv)(x)=v(-x)$ and $V_\pm=\ker(P\mp 1)$. Then, unique σ_a-semilinear intertwiners Ω_a have equal linear part determinants on V_+ and V_-. Moreover, we have that $\Omega_a\pi(h)\Omega_a^{-1}=\pi(\varphi_a(h))$ and $\Omega_a\Omega_b=\Omega_{ab}$. For distinct models V,V', we fix rational bases of $V_\pm,V'_\pm$. Then, composable normalized maps Ω,Ω' have normalized composition $\Omega'\Omega$.*

Proof. The operator P implements $\iota(X,z)=(-X,z)$. In the delta basis (e_x), $\mathrm{tr}(P)=1$, since 0 is the unique fixed point of $x\mapsto -x$. Thus, V_+ and V_- have dimensions $(p^N+1)/2$ and $(p^N-1)/2$, respectively. We choose one x from each nonzero pair $\{x,-x\}$. Choose e_0,e_x+e_{-x} for a basis of V_+, and e_x-e_{-x} for a basis of V_-. Moreover, both bases are rational. If $\dim V_\pm=0$, the determinant is 1.

Apply σ_a entrywise to π. Moreover, the representations $\sigma_a(\pi)$ and $\pi\circ\varphi_a$ share the central character $z\mapsto\zeta_p^{m_az}$. By the Stone–von Neumann theorem, $\sigma_a(\pi)\cong\pi\circ\varphi_a$ over $\mathbb{C}$. The intertwining equations over K have solution space of dimension 1. Hence, a nonzero solution M_a exists over K. Since π is irreducible, M_a is invertible. Thus, $A_a=M_a\sigma_a$, with σ_a acting on coordinates. Thus, A_a is a semilinear intertwiner, unique modulo $K^\times$. Moreover, $\varphi_a\iota=\iota\varphi_a$, so $A_aPA_a^{-1}$ and P implement ι. Schur's lemma makes $A_aPA_a^{-1}$ a scalar multiple of P. The trace $\mathrm{tr}(P)=1$ changes to $\sigma_a(1)=1$, so $A_aPA_a^{-1}=P$. Hence, $A_a(V_\pm)=V_\pm=M_a(V_\pm)$.

Given a semilinear $A=M\sigma$, we put $d(A)=\det(M|_{V_+})/\det(M|_{V_-})$. The dimensions of V_+ and V_- differ by 1. Thus, for $t\in K^\times$, $d(tA)=t\,d(A)$, and $d(A_aA_b)=d(A_a)\,\sigma_a\big(d(A_b)\big)$. Consequently, $\Omega_a=d(A_a)^{-1}A_a$ is the unique intertwiner with $d(\Omega_a)=1$. More explicitly, if $A_aA_b=\alpha(a,b)A_{ab}$, $\alpha(a,b)=d(A_a)\,\sigma_a(d(A_b))/d(A_{ab})$. Therefore, $\Omega_a\Omega_b=\Omega_{ab}$ for $a,b\in\Gamma$. For $A:V\to V'$, the fixed rational parity bases determine $d(A)$. For $B:V'\to V''$, the codomain basis for A is the domain basis for B. Thus, $d(BA)=d(B)\sigma_B(d(A))$, where σ_B is the coefficient automorphism of B. For linear A, the parity argument is in [68, p. 160]. □

Lemma 3.25. *Let $W = W_1 \perp W_2$, with Schrödinger models V_1, V_2. If normalized transports Ω_1, Ω_2 have coefficient automorphism σ, the normalized transport on $V_1 \otimes_K V_2$ is $\Omega_1 \otimes \Omega_2$. For a permutation s of identical factors of dimension $D = p^N$, the normalized operator is $\operatorname{sgn}(s)^{(D-1)/2}\tau_s$, where τ_s permutes tensor factors.*

Proof. The operator $P_1 \otimes P_2$ is the parity operator on $V_1 \otimes_K V_2$. Its $+1$-eigenspace is $(V_{1,+} \otimes V_{2,+}) \oplus (V_{1,-} \otimes V_{2,-})$, and its -1-eigenspace is $(V_{1,+} \otimes V_{2,-}) \oplus (V_{1,-} \otimes V_{2,+})$. We put $\Delta_i = \dim V_{i,+} - \dim V_{i,-} = 1$. Moreover, in the tensor bases, $d(A_1 \otimes A_2) = d(A_1)^{\Delta_2} d(A_2)^{\Delta_1} = d(A_1)d(A_2)$. Hence, normalized A_1, A_2 give a normalized $A_1 \otimes A_2$. Let τ flip two factors of dimension D, and put $a = (D+1)/2$ and $b = (D-1)/2$. For the $+1$-eigenspace of $P_1 \otimes P_2$, we have that the determinant of τ is $(-1)^{\binom{a}{2}+\binom{b}{2}}$. However, on the -1-eigenspace, τ exchanges the mixed summands, with determinant $(-1)^{ab}$. Hence, $d(\tau) = (-1)^{\binom{D}{2}} = (-1)^{(D-1)/2}$. Given an adjacent transposition s, tensor τ with 1 on remaining factors. Next, choose s_i as adjacent transpositions, with $s = s_1 \cdots s_k$. Thus, multiplication gives the operator for s. □

Proposition 3.26. *Let $C = \mathrm{C}_G(J, \lambda)$. Thus, we find that each $c \in C$ satisfies $c\mathfrak{g}(i)c^{-1} = \mathfrak{g}(i)$ and $cJc^{-1} = J$. The parity normalization extends η to a character $\widehat{\eta}$ of $UC = U \rtimes C$. Let J have Jordan type ν. Suppose that $\rho \in \operatorname{Irr}(G)$ has wave-front class of type ν. Then, we have that C acts on the line $\mathcal{L}_\rho = \operatorname{Hom}_U(V_\eta, V_\rho)$, with linear character $\kappa_\rho(c) = \frac{1}{|U|}\sum_{u \in U} \rho(cu)\,\overline{\widehat{\eta}(cu)}$ for $c \in C$. In particular, we have that $\kappa_\rho(z) = \rho(z)/\rho(1)$ for $z \in \mathrm{Z}(G)$, and $\kappa_{\rho\delta} = \kappa_\rho\,\delta|_C$ for every determinant character δ of G.*

Proof. Each $c \in C$ has $\theta^c = \theta$ and acts symplectically on W. Next, apply Proposition 3.24 with $\sigma_a = 1$. Given $u \in U$, $c \in C$, put $\widehat{\pi}(uc) = \pi(\kappa(u))\Omega_c$. The intertwining and multiplication formulas make $\widehat{\pi}$ a representation. Since C acts within each weight space and $U - 1$ raises weights, $C \cap U = 1$. Moreover, C normalizes U, and $\widehat{\pi}$ affords an extension $\widehat{\eta}$ of η. The induced character $\operatorname{Ind}_U^G \eta$ is the generalized Gelfand–Graev character Γ_J for the Jordan type of J [208, Proposition 5.9, Step 3]. Since algebraic GL_n has connected centre and adjoint type A, [188, Lemma 6.3(ii) and Corollary 15.7] gives $\langle \operatorname{Ind}_U^G \eta, \rho\rangle_G = 1$ for both rational forms $\mathrm{GL}_n, \mathrm{GU}_n$ [156, proof of Theorem 5.8]. By [95, Lemma 5.2], $\dim \mathcal{L}_\rho = 1$.

Let R_ρ and $\widehat{\pi}$ afford ρ and $\widehat{\eta}$, respectively. Moreover, for $c \in C$, the action on $\mathcal{L}_\rho$ acts by $T \mapsto R_\rho(c)T\widehat{\pi}(c)^{-1}$. We have $\mathcal{L}_\rho \simeq (V_\rho \otimes V_{\widehat{\eta}}^\vee)^U$. Thus, averaging over U gives the projector $|U|^{-1}\sum_{u \in U} R_\rho(u) \otimes \widehat{\pi}^\vee(u)$. Consequently, the trace of c on $\mathcal{L}_\rho$ is the stated sum for $\kappa_\rho(c)$, including when $\widehat{\eta}(c) = 0$. Given $z \in \mathrm{Z}(G)$, the action on $H(W)$ is trivial, so $\Omega_z = 1$ and $\kappa_\rho(z) = \rho(z)/\rho(1)$. Finally, every determinant character δ is trivial on U, and the line action for $\rho\delta$ is multiplied by $\delta|_C$. □

Choose a cyclotomic splitting field $E \supseteq K$ for the characters. Let $\varphi : G \to G'$ be graded, and let $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ be a compensating coefficient automorphism. If $\rho'(\varphi(g)) = \sigma(\rho(g))$, the Hom action gives $\kappa_{\rho'}(\varphi(c)) = \sigma(\kappa_\rho(c))$. The normalized operators have coefficient action $\sigma|_K$. When $J = 0$, we have that $U = 1$, and the multiplicity-one condition is $\rho(1) = 1$. Thus, we have that V_ρ is scalar. Ungraded conjugation can add $t(X)$, with linear $t : W \to \mathbb{F}_p$, to the central coordinate of $H(W)$, and its induced map φ may fail $\varphi\iota = \iota\varphi$. Thus, normalization is for graded φ. The Sylow-normalizer comparison requires κ_ρ on the inner elements $x \in C$ in products of chosen lifts.

Let $q = p^f$ be odd and $\rho \in \operatorname{Irr}(\mathrm{GU}_n(q))$. At s, we write λ_s for the Jordan partition of ρ. Moreover, the partitions satisfy $\lambda_s = \lambda_{s^{-q}}$. The wave-front partition d has parts $d_r = \sum_s (\lambda_s')_r$. For strings of length k, put $m_k = \#\{r : d_r = k\}$. Next, fix the graded Jordan model (J, λ, U, η) of Section 3.3, and extend η to $\widehat{\eta}$ on UC by parity normalization. Given $c \in C$, each grade is stable and $cJ = Jc$. Moreover, on the string multiplicity spaces, $C = \prod_{k : m_k > 0} \mathrm{U}_{m_k}(q)$. The character κ_ρ gives the action of C on the line $\operatorname{Hom}_U(\widehat{\eta}, \rho)$. Next, we let ν be the quadratic character of μ_{q+1}, and put $M_k = \sum_{l \geq 1,\, l \not\equiv k \pmod 2} m_l \min(k, l)$. Given $\mathrm{GU}_{d_r}(q)$, we let R_r be a regular character, with multiplicities $(\lambda_s')_r$ and column labels for eigenvalues s. Moreover, we write $\beta_{\rho,r}$ for the central character of R_r.

Lemma 3.27. *Given any s, we choose a partition $\mu_s \subseteq \lambda_s$ with vertical strip λ_s/μ_s of size e_s. Suppose $\sum_s e_s = d_r$ and $\sum_s \mu_s'$ dominates the partition obtained from d by deleting its r-th part. Then, we find $e_s = (\lambda_s')_r$. Given λ_s, shorten each row of length $\geq r$ by one. The new partition is μ_s. If $d_r = d_t$, $\beta_{\rho,r} = \beta_{\rho,t}$.*

Proof. Given any column a, we put $b_a = \sum_s((\lambda_s')_a - (\mu_s')_a)$. Then, $b_a \geq 0$ and $\sum_a b_a = \sum_s e_s = d_r$. Let $d^{(r)}$ be d with its r-th part deleted. If $j < r$, the first j parts of $d^{(r)}$ have sum $\sum_{a \leq j} d_a$. By the hypothesis on $\sum_s \mu_s'$, $\sum_{a \leq j}(d_a - b_a) \geq \sum_{a \leq j} d_a$. Hence, whenever $a < r$, $b_a = 0$. Thus, shortened rows have length $\geq r$. Exactly $\sum_s (\lambda_s')_r = d_r$ rows have length at least r. Each row of λ_s loses at most one box. Since $\sum_s e_s = d_r$, every row of

length $\geq r$ loses one box. By Pieri, μ_s occurs with multiplicity 1. Suppose that $r < t$ and $d_r = d_t$. Each difference $(\lambda'_s)_r - (\lambda'_s)_t$ is nonnegative, and their sum is $d_r - d_t = 0$. Hence, $(\lambda'_s)_r = (\lambda'_s)_t$ for every s. Thus, the regular characters R_r, R_t have the same central character, so $\beta_{\rho,r} = \beta_{\rho,t}$. □

Proposition 3.28. *Given $c = (c_k)_k \in C$, we choose $r(k)$ with $d_{r(k)} = k$ for $m_k > 0$. Then, we find $\kappa_\rho(c) = \prod_{\substack{k \\ m_k>0}} \big(\beta_{\rho,r(k)}\nu^{M_k}\big)(\det c_k)$.*

Proof. Fix r and put $k = d_r$. Then, for $1 \neq x \in \mu_{q+1}$, we let V_r be one string of length k. Let $h_r(x)$ act by x on V_r and by 1 on $V_r^\perp$. Then, $\mathrm{C}_G(h_r(x)) = L = \mathrm{GU}_k(q) \times \mathrm{GU}_{n-k}(q)$. Let α be the trace on the θ-isotypic space of V_ρ for U_2. Thus, the isotypic projector gives $\alpha(h_r(x)) = \frac{1}{|U_2|}\sum_{y\in U_2} \theta(y^{-1})\,\rho(h_r(x)y)$. The order of $h_r(x)$ is prime to p. By [208, (5.18)], average instead over $\mathrm{C}_{U_2}(h_r(x))$. The generalized Gelfand–Graev pairing for L also has prefactor $d_L = q^{\dim_{\mathbb{F}_q} \mathfrak{g}_1(L)/2}$.

For the uniform torus formula of ρ, group rational embeddings into L by e_s. Next, assign e_s entries of label s to $\mathrm{GU}_k(q)$. By [208, Lemma 5.7], at e_s, the relative Weyl-group order w_s contributes w_s^{-1}. On $\mathrm{GU}_k(q)$, the regular pairing has labels (1^{e_s}). Consequently, Pieri's rule for Sp^{λ_s} requires a vertical strip λ_s/μ_s. Then, let $d^{(r)}$ be d with d_r deleted. On $\mathrm{GU}_{n-k}(q)$, the wave-front criterion requires $\sum_s \mu'_s$ to dominate $d^{(r)}$. By Lemma 3.27, $e_s = (\lambda'_s)_r$. By [188, Corollary 15.7], both pairings have multiplicity 1. Moreover, on the rational Pieri line $\mathcal{P}$, Frobenius acts by $\epsilon_{\mathrm{P}} \in \{\pm 1\}$. Given any factor of L, take the longest Weyl element w_0. Changing the unitary reference by w_0 gives a sign $\epsilon_{\mathrm{U}} \in \{\pm 1\}$ in the uniform characters [208, §5]. Thus, both $\epsilon_{\mathrm{P}}, \epsilon_{\mathrm{U}}$ are independent of $x \neq 1$. Indeed, the labels e_s, centralizer L, and Frobenius operators remain fixed. Hence, $e_{\rho,r} \in \{\pm 1\}$ may be chosen with $\alpha(h_r(x)) = e_{\rho,r}\, d_L\, \beta_{\rho,r}(x)$.

For strings of lengths k, l, the degree-one maps number $\min(k,l)$ per direction if $k \not\equiv l \pmod 2$, and 0 otherwise. Thus, the moving part for $h_r(x)$ comes from an M_k-dimensional Hermitian space over $\mathbb{F}_{q^2}$. On a Hermitian line $\mathcal{D}$, the unitary Heisenberg extension W_0 has torus character $\mathrm{Reg}_{\mu_{q+1}} - 1$ [91, Lemma 2.2]. Moreover, $W_0(-1) = -P$, where P is the parity operator. Let $W_{0,+}, W_{0,-}$ be its parity eigenspaces, and write $q+1 = 2m$. The odd character indices $1, 3, \ldots, 2m-1$ occur on $W_{0,+}$, and nonzero even indices on $W_{0,-}$. The respective sums are m^2 and $m(m-1)$, so the determinant quotient is ν. Thus, the normalization for $\widehat{\eta}$ gives $W_0 \otimes \nu$. Given $x \neq 1$, the normalized model on $\mathcal{D}$ has trace $-\nu(x)$. Moreover, the M_k lines and fixed factor of degree d_L give $\operatorname{tr}\widehat{\eta}(h_r(x)) = d_L(-1)^{M_k}\nu(x)^{M_k}$.

On the θ-isotypic space, $\alpha = \kappa_\rho \operatorname{tr}\widehat{\eta}$. Moreover, the linear characters of $\mathrm{U}_{m_k}(q)$ factor through determinant for odd q, including $(m_k, q) = (2,3)$ [91, proof of Proposition 2.5]. The determinant of $h_r(x)$ on the multiplicity space is x. Next, we define the character $\xi_k(x) = \kappa_\rho(h_r(x))/(\beta_{\rho,r}(x)\nu(x)^{M_k})$ of μ_{q+1}. Since $\nu^2 = 1$, the two trace formulas give $\xi_k(x) = e_{\rho,r}(-1)^{M_k}$ for every $x \neq 1$. Choose a generator t of μ_{q+1}. Since $q+1 \geq 4$, both t and t^2 are nonidentity, so $\xi_k(t) = \xi_k(t^2) = \xi_k(t)^2$. Thus, $\xi_k(t) = 1$ and $\xi_k = 1$. In particular, for each k, we have that the factor of κ_ρ is $\beta_{\rho,r(k)}\nu^{M_k}$. □

Let $q = p^f$ be odd and $G = \mathrm{GU}_n(q)$. Write $X^{(p)} = (x_{ij}^p)$ for a matrix $X = (x_{ij})$, and put $X^* = X^{(q)T}$. Next, we choose $J, \lambda, C, m_k, d_r, \beta_{\rho,r}, \nu$ from Proposition 3.28. For $x \in \mu_{q+1}$, we let c_x act as x on even Jordan strings and as 1 on odd Jordan strings. With $E = \frac{1}{2}\sum_{k \text{ even}} k m_k$, we put $D = \sum_{k \text{ even}, l \text{ odd}} m_k m_l \min(k,l)$ and $K_\rho = \nu^D \prod_{r:\, d_r \text{ even}} \beta_{\rho,r}$. Then, we find $D = \frac{1}{2}\dim_{\mathbb{F}_q} \mathfrak{g}(1)^F$. Given any even k, c_x has determinant x^{m_k} on the multiplicity space for k. Thus, by Proposition 3.28, we obtain $\kappa_\rho(c_x) = K_\rho(x)$ and $\det(c_x) = x^{2E}$. Moreover, we have that $K_{\rho\delta} = K_\rho\, \delta^{\,2E}$, where $\delta \in \mathrm{Irr}(\mu_{q+1})$ is inflated by determinant in $\rho\delta$.

Lemma 3.29. *Let ϕ_p act by entrywise p-th power. Moreover, the lift $u_J = g\phi_p$ exists with $g \in \mathrm{SU}_n(q)$ and stabilizes the graded model (J, λ). In Jordan coordinates, we have that $u_J^{2f} = c_{-1}$.*

Proof. In $\mathbb{F}_{q^2}^\times$, we choose h with $h^q = -h$. Next, for a length k Jordan string, we let H_k be antidiagonal with $(i, k+1-i)$-entry $(-1)^{i-1}h_k$, where $h_k = 1$ for odd k and $h_k = h$ for even k. Then, $H_k^* = H_k$ and $\det H_k = h_k^k$. On each string, J has entries 1 immediately above the diagonal and 0 elsewhere. The weights of λ on the string are $1-k, 3-k, \ldots, k-1$. Thus, we have that J has degree -2, $J^T H_k + H_k J = 0$, and opposite weights pair. Put $H = \bigoplus_k H_k$.

We choose z with $z^{q+1} = h$. Then, $z^{q^2} = -z$, so $z \in \mathbb{F}_{q^4}$ and $c = z^{p-1} \in \mathbb{F}_{q^2}$. Moreover, $\mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(c) = h^{p-1}$. Let C_0 act as c on even strings and as 1 on odd strings. Thus, the map $C_0\phi_p$ fixes J and stabilizes each grade, and $C_0^* H C_0 = H^{(p)}$. On an even string, $(C_0\phi_p)^{2f}$ acts as $z^{p^{2f}-1} = -1$. Thus, $(C_0\phi_p)^{2f} = c_{-1}$ and $\det(c_{-1}) = 1$. Moreover, $\mathrm{Ad}(c_{-1}) = -1$ on $\mathfrak{g}(1)^F$, since maps of degree 1 join strings of lengths k, l with opposite parity. We

choose B_0 with $B_0^*B_0 = H$. Now, $z^{2E} \in \mathbb{F}_{q^2}$ has norm $h^{2E} = \det H$, so $z^{2E}/\det B_0 \in \mu_{q+1}$. In $\mathrm{GU}_n(q)$, choose v with $\det v = z^{2E}/\det B_0$, and put $B = vB_0$. Then, $B^*B = H$ and $\det B = z^{2E}$. In standard coordinates, we put $g = BC_0(B^{(p)})^{-1}$. Then, g is unitary and $\det g = z^{2E}c^{2E}z^{-2pE} = 1$. Therefore, $u_J^{2f} = Bc_{-1}B^{-1}$. □

Now, fix an odd prime $\ell \neq p$, and put $d = \mathrm{ord}_\ell(-q)$. On a block V of dimension d, we write T_d for the cyclic torus with $|T_d| = q^d - (-1)^d$. Thus, $\ell \mid |T_d|$. By Cabanes–Späth [33, Notation 5.2 and proof of Proposition 5.5], the relative Weyl group on V is C_d, and the signed cycle has d-th power $(-1)^{d-1}I$. In Proposition 3.30, the lift v has this power and satisfies $uv = vu$.

Proposition 3.30. *On this block, there are lifts u, v with $uv = vu$, $utu^{-1} = t^p$ and $vtv^{-1} = t^{-q}$ for $t \in T_d$, and $u^{2f} = v^2$. Moreover, $v^d = 1$ when d is odd and $v^d = -I$ when d is even. In standard Hermitian coordinates, $v \in \mathrm{SU}_d(q)$ and the linear part of u lies in $\mathrm{SU}_d(q)$. For odd d one also has $v = u^{f(d+1)}$. For even d one has $u^{fd} = -I$.*

Proof. Suppose that d is odd. Put $K = \mathbb{F}_{q^{2d}}$ over $\mathbb{F}_{q^2}$, with Hermitian form $\mathrm{Tr}_{K/\mathbb{F}_{q^2}}(xy^{q^d})$. Let $T_d = \mu_{q^d+1}$ act by multiplication, and put $u(x) = x^p$ and $v(x) = x^{q^{d+1}}$. Then, v is $\mathbb{F}_{q^2}$-linear, $vtv^{-1} = t^{-q}$, and v has order d. On K, composition gives $uv = vu$, $u^{2f} = v^2$, and $v = u^{f(d+1)}$. In an orthonormal basis (β_j), put $M = (\beta_j^{q^{2i}})_{0\le i<d,\,1\le j\le d}$ and $D_M = \det M$. The q^2-power cycles the rows of M, with sign 1 since d is odd. Thus, v has determinant 1 and $D_M^{q^2} = D_M$. Let P shift the rows of $M^{(q)}$ by $(d-1)/2$, giving rows $(\beta_j^{q^{2i+d}})_j$. The Gram matrix is $M^{\mathsf{T}}PM^{(q)} = I$, so $D_M^{q+1} = 1$. A unitary basis change S with $\det S = D_M^{-1}$ gives $D_M = 1$. Thus, the linear part of u has determinant $\det(M^{-1}M^{(p)}) = 1$.

For even $d = 2a$, we set $K = \mathbb{F}_{q^d}$ and $K \oplus K$ over $\mathbb{F}_{q^2}$, with nondegenerate Hermitian form $\big\langle (x,y),(x',y')\big\rangle = \mathrm{Tr}_{K/\mathbb{F}_{q^2}}\big(x(y')^{q^{d-1}} + y(x')^q\big)$. Next, we define $T_t(x,y) = (tx, t^{-q}y)$ for $t \in K^\times$, $r_0(x,y) = (y, x^{q^2})$ and $u_0(x,y) = (x^p, y^p)$. r_0 is an isometry of determinant -1, with $r_0^d = 1$, $r_0T_tr_0^{-1} = T_{t^{-q}}$, and $u_0^{2f} = r_0^2$. Moreover, $u_0r_0 = r_0u_0$. By [21, Ch. III, §9.4, Proposition 6], $\det T_t = \mathrm{N}_{K/\mathbb{F}_{q^2}}(t)^{1-q}$.

Choose an $\mathbb{F}_{q^2}$-basis $\mathcal{B}$ of K, with Moore determinant D_M. Then, $D_M^{q^2} = (-1)^{a-1}D_M$, so $D_M^2 \in \mathbb{F}_{q^2}$. In the doubled basis $\mathcal{B} \oplus \mathcal{B}$, the Gram block G_{12} has determinant $D_MD_M^{q^{d-1}}$, where $D_M^{q^{d-1}} = (-1)^{a-1}D_M^q$. The other block is G_{12}^*. Thus, the Gram determinant is $-\mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(D_M^2)$. We choose $t_0 \in \mathbb{F}_{q^2}^\times$ of norm -1, and choose a standard-coordinate isometry B with $\det B = D_M^2t_0$. Thus, the linear part of u_0 has determinant t_0^{1-p} in the basis given by B. Indeed, each Moore matrix contributes D_M^{p-1}.

Choose w with $w^{1-q} = t_0$. Then, $w^{q^2} = -w$. Put $L = (q^d-1)/(q^2-1)$ and choose $z_0 \neq 0$ in $\mathbb{F}_{q^{2d}}$ with $z_0^{q^d} = -z_0$. Thus, $(z_0^L)^{q^2-1} = -1 = w^{q^2-1}$, so $w/z_0^L \in \mathbb{F}_{q^2}^\times$. By [118, Theorem 2.28(ii)], choose $t \in K^\times$ of norm w/z_0^L. Then, $z = z_0t$ has $z^{q^d} = -z$ and $z^L = w$. With $c = z^{p-1}$, we put $a_0 = z^{-(q+1)}$ and $u = T_cu_0$ and $v = T_{a_0}r_0$. Both c, a_0 belong to K. Moreover, $\det T_c = t_0^{p-1}$ and $\det T_{a_0} = w^{q^2-1} = -1$. Thus, the linear part of u has determinant 1, and $\det v = 1$. From $a_0^{p-1} = c^{-(q+1)}$, $uv = vu$. The powers satisfy $u^{2f} = T_{z^{q^2-1}}r_0^2 = v^2$. Thus, $v^d = u^{fd} = T_{z^{q^d-1}} = -I$. □

Given $e \mid d$, we put $m = d/e$. Moreover, on T_e, the signed norm and determinant are $N_{d/e}(t) = t^{\frac{(-q)^d-1}{(-q)^e-1}}$ and $\det_e(y) = y^{\frac{1-(-q)^e}{q+1}}$. The map $N_{d/e} : T_d \to T_e$ is surjective, and $\det_e \circ N_{d/e} = \det_d$. The exponent of $N_{d/e}$ is negative when d is even and e is odd. Let $\theta = \theta_e \circ N_{d/e}$, where the $(-q)$-orbit of $\theta_e \in \mathrm{Irr}(T_e)$ has length e. Next, we put $s = \theta_e(-1)$, and choose compatible primitive roots $\zeta_m, \zeta_d, \zeta_{2d}$. If $0 \le j < m$, assign ζ_m^j to power e of the complement. Also, include s when d is even and e is odd. If d is even and e is odd, a determinant twist δ changes s by $\delta(-1)$, in accordance with $\det(r_0^e)$. Moreover, for even d, the torus part $z^{(-q)^e-1}$ of v^e has signed norm $z^{q^d-1} = -1$. Thus, on the chosen θ-weight line, the operator v^e has eigenvalue $\lambda_j = s\zeta_m^j$ when d, e are even, and $\lambda_j = \zeta_m^j$ otherwise. Since $m \mid \ell - 1$, we have that the group $\mathcal{H}_\ell$ fixes the indices j. Choose the e-th root $\kappa = \zeta_d^j$, except when e is even and $s = -1$. In that case, take $\kappa = \zeta_{2d}^{2j+m}$.

Next, write η_{loc} for the scalar attached to u, and write η for the Heisenberg character. Put $K_0 = \langle u, v\rangle$, with $\tau(u) = \eta_{\mathrm{loc}}$ and $\tau(v) = \kappa$. If $2 \mid d$, choose $\eta_{\mathrm{loc}}^{2f} = \kappa^2$. Then, $K_0 \cap T_d = \langle -I\rangle$, and Proposition 3.30 gives a character τ of K_0 with $\tau(-I) = \kappa^d = \theta(-1)$. If $2 \nmid d$, we choose $\eta_{\mathrm{loc}}^f = \kappa$. Then, $\kappa^d = 1$, so $\eta_{\mathrm{loc}}^{f(d+1)} = \kappa$. Hence, we obtain that $\tau(v) = \tau(u)^{f(d+1)}$, as $v = u^{f(d+1)}$. Moreover, we have that $\mathrm{ord}_\ell(q) = 2d$, and the order formula for p^f mod ℓ permits $\eta_{\mathrm{loc}} \in \mu_{\ell-1}$.

In either case, we have that $\mathrm{Ind}_{K_0}^{T_d K_0}\tau$ restricts to $\mathrm{Ind}_{\langle v\rangle}^{T_d\langle v\rangle}(\tau|_{\langle v\rangle})$. The irreducible $W_{\theta,j}$ of $T_d\langle v\rangle$ has weight θ and the chosen inertia value. Thus, $W_{\theta,j}$ occurs with multiplicity 1. Indeed, the e eigenvalues of v on $W_{\theta,j}$ are the distinct e-th roots of λ_j. If $2 \mid e$, restriction to $\langle v^2\rangle$ merges pairs of those roots. Thus, $v \in K_0$ is necessary for the construction of $W_{\theta,j}$ with multiplicity one.

Let $\mathcal{D}$ be the determinant-twist group and Φ the field automorphism group. Next, let $\mathcal{O} \subseteq \mathrm{Irr}(G)$ be a joint orbit of $\mathcal{D}, \Phi, \mathcal{H}_\ell$. On $\mathcal{O}$, the wave-front partition $\mathbf{d}$ is fixed. Thus, choose the model (J, λ, U, η) on $\mathcal{O}$, with field lift u_J. In Jordan coordinates, we let $a_\sigma \in \mathbb{F}_p^\times$ be determined by $\sigma(\zeta_p) = \zeta_p^{a_\sigma}$ for $\sigma \in \mathcal{H}_\ell$. Choose $b_0 \in \mathbb{F}_{q^2}^\times$ of norm $(\ell \bmod p)^{-1}$ and put $m_0 = |b_0|$. Moreover, we let $\mathbb{K}$ be a finite cyclotomic field containing $\mu_{\ell^{m_0}-1}$. The ℓ-power exponent k_σ is well defined modulo m_0. Thus, $b_\sigma = b_0^{k_\sigma}$ depends multiplicatively on σ and has norm a_σ^{-1}, since $\mathrm{ord}_p(\ell) \mid m_0$.

On a string of length k, we put $T_\sigma|_k = \mathrm{diag}(a_\sigma^{(k-1)/2}, \dots, a_\sigma^{-(k-1)/2})$ for odd k and $T_\sigma|_k = b_\sigma\,\mathrm{diag}(a_\sigma^{k/2}, \dots, a_\sigma^{1-k/2})$ for even k. Write $t_i = (T_\sigma|_k)_{ii}$. Moreover, the exponents in $T_\sigma|_k$ descend by 1, and $t_i t_{k+1-i}^q = 1$. Thus, we have that T_σ is unitary. $T_\sigma c = cT_\sigma$ for $c \in C$, and T_σ stabilizes each grade. The matrices satisfy $T_\sigma J T_\sigma^{-1} = a_\sigma J$, $T_\sigma T_\tau = T_{\sigma\tau}$, $w_\sigma = a_\sigma b_\sigma^2 = b_\sigma^{1-q} \in \mu_{q+1}$, and $\det T_\sigma = w_\sigma^E$. Next, conjugate T_σ, c_x, J, C by B from Lemma 3.29. Keep the symbols T_σ, c_x, J, C in standard coordinates, with field operator u_J.

We set $I = \mathrm{Ind}_U^G \eta$ on functions f satisfying $f(xg) = \pi(x)f(g)$ for $x \in U$. Moreover, let R act by right translations. Let Ω_σ be the normalized σ-semilinear Heisenberg intertwiner for $\mathrm{Ad}(T_\sigma^{-1})$, with central multiplier a_σ. Take the linear normalized intertwiners Ω_{u_J}, Ω_c for u_J and $c \in C$. Next, we define $(I_\sigma f)(g) = \Omega_\sigma f(T_\sigma g)$, $(A_{u_J} f)(g) = \Omega_{u_J} f(u_J^{-1}(g))$, $(C_\delta f)(g) = \delta(\det g) f(g)$, and $(B_c f)(g) = \Omega_c f(c^{-1}g)$. The expression $u_J(g)$ denotes conjugation on G. Thus, the intertwiners satisfy left U-equivariance. Given $g \in G$, we have that $I_\sigma R(g) = R(g) I_\sigma$. Moreover, I_σ maps V_ρ to V_{ρ^σ}, and $I_\sigma I_\tau = I_{\sigma\tau}$ by Proposition 3.24. For determinant characters δ, ϵ, we have that $C_\delta C_\epsilon = C_{\delta\epsilon}$. On the unique summand V_ρ of I, we have that the operator B_c has scalar $\kappa_\rho(c)^{-1}$. Indeed, the map into I is $v \mapsto [g \mapsto TR_\rho(g)v]$, with $T \in \mathrm{Hom}_U(V_\rho, V_\eta)$, the dual multiplicity line. Then, $\Omega_c T R_\rho(c)^{-1} = \kappa_\rho(c)^{-1} T$. Equivalently, $B_c = A_c R(c)^{-1}$, where A_c represents inner conjugation by c.

Proposition 3.31. *Given $h \in \mathbb{Z}$, the induced operators satisfy $A_{u_J}^h I_\sigma A_{u_J}^{-h} = B^{-1}_{c_{b_\sigma^{p^h-1}}} I_\sigma$, $I_\sigma C_\delta = \delta^\sigma(w_\sigma^E)\, C_{\delta^\sigma} I_\sigma$, and $A_{u_J}^{2f} = B_{c_{-1}} R(c_{-1})$. When $h < 0$, b^{p^h-1} means b^{p^h}/b, by inverse Frobenius on $\mathbb{F}_{q^2}$. Also, we have that $A_{u_J} C_\delta = C_{\delta^{u_J}} A_{u_J}$, where $\delta^{u_J}(x) = \delta(x^{p^{-1}})$. Whenever (δ, u_J^h, σ) jointly stabilizes ρ, $K_\rho = K_\rho^{u_J^h \sigma}\, \delta^{2E}$.*

Proof. In Jordan coordinates, let $c_J I$ be the scalar part of u_J on each even string. Then, $(c_J I)T_\sigma = T_\sigma(c_J I)$, and $a_\sigma \in \mathbb{F}_p$. Thus, $u_J^h(T_\sigma) = c_{b_\sigma^{p^h-1}} T_\sigma$, with $\mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(b_\sigma^{p^h-1}) = 1$. Substitute $u_J^h(T_\sigma)$ in the arguments of f. The normalized intertwiners give $A_{u_J}^h I_\sigma A_{u_J}^{-h} = B^{-1}_{c_{b_\sigma^{p^h-1}}} I_\sigma$, see field conjugation. Moreover, on V_ρ, the operator I_σ acts before $B^{-1}_{c_{b_\sigma^{p^h-1}}}$. Thus, the latter operator acts on V_{ρ^σ} by the scalar $K_{\rho^\sigma}(b_\sigma^{p^h-1})$. The formula $\det T_\sigma = w_\sigma^E$ gives the determinant twist relation. For $g \in G$, $\det(u_J(g)) = (\det g)^p$. Thus, $A_{u_J} C_\delta = C_{\delta^{u_J}} A_{u_J}$, with $\delta^{u_J}(x) = \delta(x^{p^{-1}})$. By Lemma 3.29, $A_{u_J}^{2f} = A_{c_{-1}}$, giving the field power relation on V_ρ. $A_{u_J}^{2f} = K_\rho(-1)^{-1} R(c_{-1})$, so $A_{u_J}^{4f} = 1$. Given $c \in C$, $T_\sigma c = cT_\sigma$, and $u_J(c_x) = c_{x^p}$. Thus, the line action gives $K_{\rho^\sigma} = K_\rho^\sigma$ and $K_{\rho^{u_J}}(x) = K_\rho(x^{p^{-1}})$. The determinant twist δ and the formulas for $K_{\rho^\sigma}, K_{\rho^{u_J}}$ give the condition on K_ρ. □

Suppose that d is even. Put $b = 1$ when e is even and $s = -1$, and put $b = 0$ otherwise. With $L = \ell - 1$, we write $h_0 = \mathrm{ord}_\ell(p)$. Next, we put $F_2 = v_2(f)$ and $H_2 = v_2(h_0)$. Set $L_2 = v_2(L)$ and $(-1)^R = (\frac{p}{\ell})$. Let k_σ be the parity of the ℓ-power exponent on $\mathbb{K}_2$, a finite cyclotomic field containing μ_{ℓ^2-1}. Moreover, the odd parts of d and $\mathrm{ord}_\ell(q) = h_0/\gcd(h_0, f)$ agree. Thus, the odd-order part κ_{odd} has an f-th root in $\mu_{L_{\mathrm{odd}}}$, where L_{odd} is the odd part of L. Suppose that $4 \mid d$. Then, we obtain that $\mathrm{ord}_\ell(q) = d$ and $v_2(d) = H_2 - F_2 \geq 2$. Choose $\eta_{\mathrm{loc}}^f = \kappa$ with minimal 2-part. If mb is even, the order of η_{loc} divides L. If mb is odd, the 2-part of η_{loc} has order 2^{H_2+1}. The 2-part gives a σ-sign exactly when $H_2 = L_2$, or equivalently when $R = 1$. Hence, we obtain that $\eta_{\mathrm{loc}} \in \mu_{2L}$, $\sigma(\eta_{\mathrm{loc}})/\eta_{\mathrm{loc}} = (-1)^{Rk_\sigma mb}$, and $r_\sigma := \sigma(\kappa)/\kappa = (-1)^{fRk_\sigma mb}$.

Suppose that $d \equiv 2 \pmod 4$. Then, we have that $\mathrm{ord}_\ell(q) = d/2$ is odd and $H_2 \leq F_2$. If $b = 0$, the 2-part of κ has order at most 2. If $b = 1$, we have that the 2-part has order 4 and m is odd. Fix $\omega = \zeta_{2^{F_2+2}}$ for all labels, and put $\eta_{\mathrm{loc}} = \eta_{\mathrm{odd}}\omega^b$, where $\eta_{\mathrm{odd}} \in \mu_{L_{\mathrm{odd}}}$ and $\eta_{\mathrm{odd}}^f = \kappa_{\mathrm{odd}}$. Put $t_\sigma = \sigma(\omega)/\omega$. Now, $\omega^{2f} = -1$, so $\eta_{\mathrm{loc}}^{2f} = \kappa^2$. Moreover, we have that $\sigma(\eta_{\mathrm{loc}})/\eta_{\mathrm{loc}} = t_\sigma^b$ and $r_\sigma = (-1)^{(L/2)k_\sigma b}$. The scalars satisfy $t_{\sigma\tau} = t_\sigma \sigma(t_\tau)$ and $t_\sigma^{2f} = 1$. Moreover, the order of t_σ can exceed 2.

In the Fourier basis e_l, with $l \in \mathbb{Z}/e\mathbb{Z}$, put $P_e e_l = e_{l+1}$. Next, for $t \in T_d$, we write $R(t)e_l = \theta(t^{(-q)^{-l}})e_l$, and put

$R(v) = \kappa P_e$. The field transport $U_{\mathrm{loc}} = \eta_{\mathrm{loc}} I$ has target $\theta^{p^{-1}}$. Similarly, the determinant transport to $\theta(\delta \circ \det_d)$ is Id. Keep the same roots $\kappa, \eta_{\mathrm{loc}}$. For $\sigma \in \mathcal{H}_\ell$, we put $G_\sigma = \mathrm{diag}(r_\sigma^l)_{0 \le l < e}\, \sigma$. Since $r_\sigma \in \{\pm 1\}$ and $r_\sigma^e = 1$, for $t \in T_d$ we have $G_\sigma R(t) G_\sigma^{-1} e_l = \theta^\sigma(t^{(-q)^{-l}}) e_l$ and $G_\sigma R(v) G_\sigma^{-1} = R(v)$. Thus, $G_\sigma G_\tau = G_{\sigma\tau}$, $G_\sigma U_{\mathrm{loc}} = (\sigma(\eta_{\mathrm{loc}})/\eta_{\mathrm{loc}})\, U_{\mathrm{loc}} G_\sigma$, and $G_\sigma P_e^j = r_\sigma^j P_e^j G_\sigma$. Transport $\theta^{(-q)^{-j}}$ to θ by P_e^j. After $2f$ field steps, the transport operator is $\eta_{\mathrm{loc}}^{2f} P_e^2 = R(v)^2$. The determinant of v is 1, and the linear part of u also has determinant 1. If $2 \mid d$, permutations of the d-dimensional torus blocks have determinant 1. Hence, tensor permutations and rational Sp^ν suffice on wreath factors.

Suppose that $d \equiv 2 \pmod 4$, and let w_i be the active colour weights. Write $\mathbf{G}_\sigma$ and $\mathbf{U}_{\mathrm{loc}}$ for the corresponding tensor operators. Then, $\mathbf{G}_\sigma \mathbf{U}_{\mathrm{loc}} = t_\sigma^{B_0} \mathbf{U}_{\mathrm{loc}} \mathbf{G}_\sigma$, where $B_0 = \sum_{e_i \text{ even}} b_i w_i$ is integral. Given an action a on the labels, let $j_i(a)$ be the relative $(-q)$-orbit shift of label i. Moreover, the transports to the chosen representatives contribute $(-1)^{(L/2) k_\sigma Q_a}$, where $Q_a = \sum_{e_i \text{ even}} b_i w_i j_i(a) \pmod 2$. The shifts $j_i(a)$ add under composition with index permutation π_a. $b_{\pi_a(i)} w_{\pi_a(i)} = b_i w_i$. Since e_i is even in the sum, $Q_{ab} = Q_a + Q_b \pmod 2$.

Finally, we write $K_\rho(-1) = (-1)^{\epsilon_\rho}$ with $\epsilon_\rho \in \{0, 1\}$. The transformation rules for K_ρ and $\delta(-1)^{2E} = 1$ make ϵ_ρ constant on $\mathcal{O}$. Next, take $\omega = \zeta_{2^{F_2+2}}$, and put $\Phi_{u_J}|_{V_\rho} = \omega^{\epsilon_\rho} A_{u_J}|_{V_\rho}$. By field conjugation and field powers, $\Phi_{u_J}^{2f} = R(c_{-1})$ and $\Phi_{u_J}^h I_\sigma \Phi_{u_J}^{-h} = t_\sigma^{-h\epsilon_\rho} B^{-1}_{c_{b_\sigma^{p^h-1}}} I_\sigma$. In standard coordinates, we write $u = a u_J$. The linear parts of u, u_J have determinant 1, so $a \in \mathrm{SU}_n(q)$. Put $\Phi_u = R(a)\Phi_{u_J}$. Then, we have that Φ_u has the same scalar factors and $\Phi_u^{2f} = R(u^{2f})$, by covariance with $R(G)$. For a matched pair, we let ρ_{res} be the residual unitary character. Compare ρ_{res}, the values of K_ρ in field conjugation, and the K_ρ condition. The formulas do not imply $B_0 = \epsilon_\rho - \epsilon_{\rho_{\mathrm{res}}}$.

3.4 Unitary Gauss sums

Suppose that p, ℓ are distinct odd primes, $q = p^f$, and $d = \mathrm{ord}_\ell(-q) \equiv 2 \pmod 4$. Put $C = \mu_{q+1}$, $T = \mathbb{F}_{q^2}^\times$, $N(x) = x^{q+1}$, and $w(x) = x^{1-q}$. Next, on T_e, we choose $N_{d/e}, \det_e$ from Section 3.3. Fix matched characters $\rho \in \mathrm{Irr}(\mathrm{GU}_n(q))$ and $\rho_0 \in \mathrm{Irr}(\mathrm{GU}_{n_0}(q))$, with ρ_0 residual. A colour i is a $(-q)$-orbit of a regular character $\theta_i \in \mathrm{Irr}(T_{e_i})$, of length e_i. Moreover, we have that the partitions λ_i, μ_i are global and residual. Put $\nu_i = \theta_i|_C$. For active colours, we have that $e_i \mid d$, $m_i = d/e_i$ and μ_i is the m_i-core of λ_i, with $|\lambda_i| - |\mu_i| = m_i w_i$. Otherwise, on colour i, we have that $\lambda_i = \mu_i$. Write $\theta_i(-1) = (-1)^{b_i}$, with $b_i \in \{0, 1\}$.

Next, we let $\mathcal{P}$ be the matched labels for (ρ, ρ_0). Let J be the full joint stabilizer of $\mathcal{P}$. When $a = (\delta_a, u^{h_a}, \sigma_a) \in J$, we let π_a permute the colours, and let $j_i(a) \in \mathbb{Z}/e_i\mathbb{Z}$ satisfy $\theta_i^a = \theta_{\pi_a(i)}^{(-q)^{j_i(a)}}$. See character shift. Moreover, the permutation π_a fixes partitions, ranks, weights. If $2 \mid e_i$, we put $s_i(a) = j_i(a) \bmod 2$. Thus, $s_i(ab) = s_i(b) + s_{\pi_b(i)}(a) \pmod 2$. Put $\delta_a(-1) = (-1)^{t_a}$, $P = (p-1)/2 \bmod 2$, $F = f \bmod 2$, and $L = (\ell - 1)/2 \bmod 2$. Moreover, we let k_a be the ℓ-power exponent of σ_a. For each period m of the power actions, adjoin $\mu_{\ell^m - 1}$ to $\mathbb{K}$, and record $k_a \bmod m$. Given a torus character χ, we write $\chi^{[a]}(x) = \sigma_a(\chi(x^{p^{-h_a}}))$. Next, we let $A_a \in \mathbb{F}_p^\times$ satisfy $\sigma_a(\zeta_p) = \zeta_p^{A_a}$. Choose the compensator $y_a = b_{\sigma_a}$ from Section 3.3. Thus, we have that $N(y_a) = A_a^{-1}$ and $y_{ab} = y_a y_b$.

Next, for ρ, ρ_0, choose Jordan models from Proposition 3.31. Let $K_\rho, K_{\rho_0} \in \mathrm{Irr}(C)$ be the corresponding characters. Moreover, E_ρ, E_{ρ_0} are the integer halves of $2E_\rho, 2E_{\rho_0}$, the even-string dimensions. Put $K = K_\rho / K_{\rho_0}$ and $E = E_\rho - E_{\rho_0}$. In particular, by the condition on K_ρ, $K = K^{[a]} \delta_a^{2E}$ on J. The calculation below holds for any (K, E) satisfying $K = K^{[a]}\delta_a^{2E}$ on J. Moreover, we write $(\partial c)(a, b) = c(a)\sigma_a(c(b))/c(ab)$.

Lemma 3.32. *Let $\Lambda \in \mathrm{Irr}(T)$ extend K. Next, define $\Xi_a \in \mathrm{Irr}(\mathbb{F}_q^\times)$ by $\Lambda^{[a]}(\delta_a \circ w)^E = \Lambda(\Xi_a \circ N)$. Fix the order $C_\delta A_u^h I_\sigma$. Thus, by* field conjugation of I_σ and the C_δ-twist*, the cyclic factor $A_{K,E}(a, b)$ satisfies $A_{K,E}(a, b) = \sigma_a\Big(K(y_a^{p^{h_b}-1})^{-1} \delta_b\big(w(y_a)^E y_a^{2E(p^{h_b}-1)}\big)\Big)$. Moreover, we have that $A_{K,E}(a, b) = (\partial c_\Lambda)(a, b)\, \sigma_a\big(\Xi_b(A_a^{-1})\big)$, where $c_\Lambda(a) = \sigma_a(\Lambda(y_a))$.*

Proof. Given $z \in C$, $w(z) = z^2$. Thus, the character $\chi_a = \Lambda^{[a]}(\delta_a \circ w)^E \Lambda^{-1}$ is trivial on C, by $K = K^{[a]}\delta_a^{2E}$. Since $\ker N = C$ and $N(T) = \mathbb{F}_q^\times$, we find a unique $\Xi_a \in \mathrm{Irr}(\mathbb{F}_q^\times)$ with $\chi_a = \Xi_a \circ N$. This gives the Λ, Ξ_a identity. Moreover, the relation for $I_{\sigma_a} C_{\delta_b}$ contributes the value of $\delta_b^{\sigma_a}$ at $w(y_a)^E$. The field operator $A_u^{h_b}$ contributes the inverse character at $y_a^{p^{h_b}-1}$. Since b lies in the joint stabilizer, the target K-character is $\sigma_a(K\delta_b^{-2E})$. Thus, the δ_b-argument in the cyclic factor includes $y_a^{2E(p^{h_b}-1)}$. Apply the Λ, Ξ_a identity for b at $y^{p^{h_b}}$, where $y = y_a$. Then, $N(y) \in \mathbb{F}_p^\times$, and $\Lambda(y^{p^{h_b}-1}) = \frac{\sigma_b(\Lambda(y))}{\Lambda(y)} \frac{\delta_b(w(y)^{p^{h_b}})^E}{\Xi_b(N(y))}$. Moreover, $y^{p^{h_b}-1} \in C$, and $w(y)^{1-p^{h_b}} y^{2(p^{h_b}-1)} = N(y)^{p^{h_b}-1} = 1$. Substitution gives $\sigma_a(\Lambda(y_a))/\sigma_a\sigma_b(\Lambda(y_a))$. Since $y_{ab} = y_a y_b$, the factor is ∂c_Λ, with $c_\Lambda(a) = \sigma_a(\Lambda(y_a))$. □

The box contributions are $K_{\mathrm{b}} = \prod_{i \text{ active}} \nu_i^{m_i w_i}$ and $E_{\mathrm{b}} = (d/2) \sum_{i \text{ active}} w_i$. Next, we extend K_{b} to T by $\Lambda_{\mathrm{b}}(x) = \prod_{\substack{i \text{ active} \\ e_i \text{ even}}} \theta_i(x)^{m_i w_i} \prod_{\substack{i \text{ active} \\ e_i \text{ odd}}} \nu_i(w(x))^{m_i w_i/2}$. If $2 \mid e_i$, $T \subseteq T_{e_i}$ and $\det_{e_i}(x) = w(x)^{e_i/2}$ for $x \in T$. If $2 \nmid e_i$, we have that m_i is even, and the twist by δ changes ν_i to $\nu_i \delta^{e_i}$. Thus, the twist exponent is E_{b}. Given a, before returning to the chosen representatives, Λ_{b} satisfies $\Lambda_{\mathrm{b}}^{[a]}(x) = \sigma_a(\Lambda_{\mathrm{b}}(x^{p^{-h_a}}))$. Moreover, for even e_i and $x \in T$, we have that an odd $(-q)$-shift gives $\theta_i(x^{-q})/\theta_i(x) = \theta_i(x^{-q-1})$, and an even shift fixes $\theta_i(x)$. On even colours put $\xi_i = \theta_i|_{\mathbb{F}_q^\times}$ and $n_i = m_i w_i$. Thus, Λ_{b} gives $\Xi_a = \prod_{i:\, e_i \text{ even}} \xi_{\pi_a(i)}^{-n_i s_i(a)}$. In Ξ_a, all other weights are 0.

Lemma 3.33. *Suppose that n_i are invariant integer weights for even e_i, with norm defect Ξ_a. Put $Q_a = \sum_i n_i b_i s_i(a) \bmod 2$. Then, there is a cochain c_G such that $(\partial c_G)(a,b) = \sigma_a(\Xi_b(A_a^{-1}))\,(-1)^{Lk_a Q_b}$. If $z = u^{2f} = v_{\mathrm{all}}^2$, we have that $c_G(az) = c_G(a)$. Moreover, we have that $c_G(a) = 1$ for every pure determinant self-twist a.*

Proof. For nontrivial $\xi \in \mathrm{Irr}(\mathbb{F}_q^\times)$, we put $G(\xi) = \sum_{x \in \mathbb{F}_q^\times} \xi(x) \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)}$. Then, $G(\xi)G(\xi^{-1}) = \xi(-1)q$, $G(\xi^{p^{-h}}) = G(\xi)$ and $\sigma(G(\xi)) = \xi^\sigma(A_\sigma)^{-1} G(\xi^\sigma)$, where $\sigma(\zeta_p) = \zeta_p^{A_\sigma}$. The order of $q \bmod \ell$ is the odd integer $d/2$. Hence, $\mathcal{H}_\ell$ fixes $\sqrt{q}$ by [156, Lemma 2.3(d)]. We put $g(1) = 1$ and $g(\xi) = G(\xi)/\sqrt{q}$ for $\xi \neq 1$. Let ν_q be the quadratic character of $\mathbb{F}_q^\times$. We choose the rational sign of $g(\nu_q)$ so that $g(\nu_q) = \mathrm{i}^{b(\nu_q)}$, where $\xi(-1) = (-1)^{b(\xi)}$ and $b(\xi) \in \{0,1\}$. The field and Galois actions fix ν_q, so the sign of $g(\nu_q)$ is compatible with both actions. Put $r(\xi) = g(\xi)/\mathrm{i}^{b(\xi)}$. Then, $r(1) = r(\nu_q) = 1$, $r(\xi^{-1}) = r(\xi)^{-1}$, and $r(\xi^{p^{-h}}) = r(\xi)$, $\sigma_a(r(\xi)) = \xi^{\sigma_a}(A_a)^{-1}(-1)^{b(\xi) Lk_a} r(\xi^{\sigma_a})$. Define $c_G(a) = \prod_i r(\xi_{\pi_a(i)})^{-n_i s_i(a)}$. Moreover, the values $c_G(a)$ lie in $\mathbb{Q}^{\mathrm{ab}}$ and may have infinite order. On $\mathbb{F}_q^\times$, determinant twists are trivial and $\xi_i^{[a]} = \xi_{\pi_a(i)}^{(-1)^{s_i(a)}}$. Apply the Galois action on $r(\xi)$ to $\sigma_a(c_G(b))$. Thus, the powers of r cancel those in $c_G(a)/c_G(ab)$, since $r(\xi^{-1}) = r(\xi)^{-1}$ and $s + t \bmod 2 = t + (-1)^t s$ for $s,t \in \{0,1\}$. The factors at A_a give $\sigma_a(\Xi_b(A_a^{-1}))$, and the powers of -1 give $Lk_a Q_b$. Thus, the formula for c_G holds.

Let $a \in \mathcal{D}$ be a self-twist. Next, let $\mathcal{C}$ be an index cycle. On $\mathcal{C}$, the odd shifts give alternating factors $r(\xi)$ and $r(\xi)^{-1}$ in cyclic order. If the number $m_{\mathcal{C}}$ of odd shifts is odd, $\xi = \xi^{-1}$, so $\xi \in \{1, \nu_q\}$ and $r(\xi) = 1$. The weights n_i are constant on each cycle. Hence, $c_G(a) = 1$. If $z = u^{2f}$, the index permutation is 1 and each shift is $-2 \bmod e_i$. Thus, $s_i(az) = s_i(a)$, and the cochain c_G satisfies $c_G(az) = c_G(a)$. The cochain c_Λ depends only on σ, by the cyclic factor. Thus, $c_\Lambda(az) = c_\Lambda(a)$. If a is a pure determinant self-twist, $c_\Lambda(a) = 1$. □

Given ρ, the wave-front parts are $d_j(\rho) = \sum_i e_i(\lambda_i')_j$. Moreover, for the regular column, write $\beta_{\rho,j}$ for the central character. Put $O_\rho = \prod_{d_j(\rho) \text{ odd}} \beta_{\rho,j}$ and $o_\rho = \sum_{d_j(\rho) \text{ odd}} d_j(\rho)$. Let ω_ρ be the central character of ρ, and let ν_C be quadratic on C. Moreover, the eigenvalue multiplicities give $\prod_j \beta_{\rho,j} = \omega_\rho$. Hence, by Proposition 3.28, we obtain $K_\rho = \omega_\rho O_\rho^{-1} \nu_C^{D_\rho}$ and $2E_\rho = n - o_\rho$. Thus, $K = K_{\mathrm{b}} K_{\mathrm{o}}$ and $E = E_{\mathrm{b}} + E_{\mathrm{o}}$, where $K_{\mathrm{o}} = (O_{\rho_0}/O_\rho)\, \nu_C^{D_\rho - D_{\rho_0}}$ and $E_{\mathrm{o}} = (o_{\rho_0} - o_\rho)/2$. The integer $n - n_0 = d \sum w_i$ is even, so $E_{\mathrm{o}} \in \mathbb{Z}$. Each pair (K', E') in the formulas for K, E satisfies $K' = K'^{[a]} \delta_a^{2E'}$ for $a \in J$, as in Lemma 3.32. Next, we set $\omega = \zeta_{2^{v_2(f)+2}}$ and $t_\sigma = \sigma(\omega)/\omega$ from Section 3.3. Put $B_{\mathrm{b}} = \sum_{i:\, e_i \text{ even}} b_i w_i$. Since m_i is odd for even e_i and even for odd e_i, we deduce that $K_{\mathrm{b}}(-1) = (-1)^{B_{\mathrm{b}}}$. Set $\omega^{B_{\mathrm{b}}}$ as bulk field-power correction, giving $t_{\sigma_a}^{B_{\mathrm{b}} h_b}$. Moreover, the local sorting parity is $\sum_i n_i b_i s_i(b) \equiv \sum_{e_i \text{ even}} w_i b_i s_i(b) \pmod 2$. By the cyclic factor and the c_G identity, the bulk quotient is $\partial(c_{\Lambda_{\mathrm{b}}} c_G)$.

The remaining quotient is $A_{K_{\mathrm{o}}, E_{\mathrm{o}}}(a,b) t_{\sigma_a}^{\Delta h_b}$, where $\Delta = \epsilon_\rho - \epsilon_{\rho_0} - B_{\mathrm{b}}$ and $K_\rho(-1) = (-1)^{\epsilon_\rho}$, with $\epsilon_\rho \in \{0,1\}$. Given any even z, the root exponent may change by z, since $\partial[\omega^{z h_a}](a,b) = t_{\sigma_a}^{z h_b}$ and $\omega^{2fz} = 1$. If $d_j(\rho)$ and $d_j(\rho_0)$ are even for every j, we have that $K_{\mathrm{o}} = 1$, $E_{\mathrm{o}} = 0$, and $\Delta \in 2\mathbb{Z}$. Thus, rescaling makes the quotient 1. Given $a \in J$, we have that $b_{\pi_a(i)} = b_i + e_i t_a \bmod 2$. Suppose that $t_a = 1$ and e_i is odd. Let $\mathcal{C}_i$ be the π_a-cycle of i. Then, $|\mathcal{C}_i| \in 2\mathbb{Z}$. The partitions are constant on $\mathcal{C}_i$, so $\mathcal{C}_i$ contributes an even number to $d_j(\rho)$ and $d_j(\rho_0)$. Every even-rank colour i also contributes an even number to $d_j(\rho)$ and $d_j(\rho_0)$. Consequently, if $d_j(\rho)$ or $d_j(\rho_0)$ is odd for some j, we find $\delta_a(-1) = 1$ for all $a \in J$.

Let $n_i \in \mathbb{Z}$ be invariant weights for even e_i. Moreover, we put $B = \sum_i n_i b_i$, $Q_a = \sum_i n_i b_i s_i(a) \bmod 2$, and $E_* = \sum_i n_i e_i/2$. In T, we choose ι of order 4, and put $Z_\iota = \prod_i \theta_i(\iota)^{n_i}$. Then, we find $Z_\iota^2 = (-1)^B$. Field and coefficient actions contribute $(-1)^{B(Ph_a + Lk_a)}$. Moreover, we have that $\det_{e_i}(\iota) = (-1)^{(e_i/2)PF}$, so the determinant twist contributes $(-1)^{E_* PF t_a}$. The representatives at $\pi_a(i)$ give $(-1)^{(1+PF)Q_a}$, since $-q \equiv (-1)^{1+PF} \bmod 4$. Thus, $(1+PF)Q_a = B(Ph_a + Lk_a) + E_* PF t_a \pmod 2$. Keep J throughout. Moreover, with Harish–Chandra multiplicity 1, a coordinate correction on J is needed.

Lemma 3.34. *Let q be odd. Let $\mathcal{P}_{\mathrm{orig}}$ be the original labels. Next, let $\mathcal{S}$ be the stabilizer of $\mathcal{P}_{\mathrm{orig}}$ for determinant, field, coefficient actions. Let $J \leq \mathcal{S}$. Choose a J-stable set $\mathcal{I}$ of colours with common partition λ. Next, we choose*

a partition $\mu \subseteq \lambda$ with rim hook $H = \lambda/\mu$ of size 2. For each $i \in \mathcal{I}$, take $\mu_i = \mu$. Let l count distinct selected eigenvalues. Thus, $\mathcal{I}$ determines a rational parabolic with Levi $L = \mathrm{GL}_l(q^2) \times \mathrm{GU}_{n-2l}(q)$. Let χ_{reg} be the regular character of $\mathrm{GL}_l(q^2)$. Then, we have that χ_{reg} has compatible ordinary and semilinear J-transports. On J, we let ρ be the original unitary character and ρ' the unitary character of rank $n-2l$ after $\lambda \mapsto \mu$. Then, ρ, ρ' share the mixed factor class on J, with $u^{2f} = v_{\mathrm{all}}^2$. If every determinant twist on J has odd order, we have that J satisfies the conclusion. Also, if $q \equiv 3 \pmod 4$ and an original wave-front partition $\mathbf{d}_{\mathrm{orig}}$ has an odd part d_j, the conclusion holds.

Proof. For $\mathrm{GL}_l(q^2)$, let $\mathcal{U}_l$ be upper unitriangular. Next, we choose $\mathrm{Ind}_{\mathcal{U}_l}^{\mathrm{GL}_l(q^2)} \psi$, with $\psi(x) = \zeta_p^{\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_p}(\sum_{i=1}^{l-1} x_{i,i+1})}$ for $x = (x_{ij}) \in \mathcal{U}_l$. The unitary determinant twist restricts to $\delta_H(g) = \delta((\det g)^{1-q})$. Define the field operator by $(F_H f)(g) = f(g^{(p^{2f-1})})$, and define the twist operator by $(D_{H,\delta} f)(g) = \delta_H(g) f(g)$. Moreover, the coefficient operator satisfies $(I_{H,\sigma} f)(g) = \sigma(f(T_\sigma g))$, where $T_\sigma = \mathrm{diag}(A_\sigma^{1-l}, A_\sigma^{2-l}, \dots, 1)$. Conjugation by T_σ multiplies each simple-root coordinate by A_σ^{-1}. Moreover, $T_\sigma T_\tau = T_{\sigma\tau}$ and $T_\sigma^{(p)} = T_\sigma$. Thus, $F_H I_{H,\sigma} = I_{H,\sigma} F_H$, $F_H^{2f} = 1$, and $\delta_H(T_\sigma) = 1$. The semidirect formulas hold on $\mathrm{Ind}_{\mathcal{U}_l}^{\mathrm{GL}_l(q^2)} \psi$. The formulas also hold on the χ_{reg}-isotypic space, of multiplicity 1.

Let I be the union of orbits in $\mathcal{I}$, and let $L = \mathrm{GL}_l(q^2) \times \mathrm{GU}_{n-2l}(q)$. By [208, Proposition 5.3, Steps 1–2], the isotropic space takes each $s \in I$ with multiplicity 1, giving χ_{reg}. On the residual space, ρ' has partition μ on I and λ_i on each unselected colour i. Since I is J-stable, both labels are stable under J. By [208, Proposition 5.3, Step 3], a selected orbit of length e has hook coefficient 1 at μ, for either parity of e. The unselected orbits have coefficient 1 at λ_i. Thus, [208, (5.2)] gives $\langle \rho, R_L^G(\chi_{\mathrm{reg}} \otimes \rho') \rangle = 1$.

Let P_L be the rational parabolic with Levi L. Moreover, in a basis for P_L, the form has entries in $\mathbb{F}_p$, and P_L is defined over $\mathbb{F}_p$. The group J has $P_L^J = P_L$. Thus, inflation to P_L and induction to G give the factors of the L-representation. The unique summand of type ρ has the same factors. The $\mathrm{GL}_l(q^2)$-factor has the compatible operators $F_H, I_{H,\sigma}, D_{H,\delta}$, so only the factors of ρ' remain. Compare the field coordinates on $\mathrm{GU}_n(q)$ and $\mathrm{GU}_{n-2l}(q)$.

Next, let H_{ad} be the adapted Gram matrix. Choose B with $B^* B = H_{\mathrm{ad}}$. We put $t = \det B$. Moreover, for h field steps, the coordinate change contributes $g_h = B(B^{(p^h)})^{-1}$, with $\det g_h = t^{1-p^h}$. Let δ have odd order, and choose s with $2s \equiv 1 \bmod (q+1)_{\mathrm{odd}}$. Define $\widehat{\delta}(x) = \delta(w(x))^s$ on T. Then, $\widehat{\delta}|_C = \delta$ and $\widehat{\delta}|_{\mathbb{F}_q^\times} = 1$. Moreover, $\widehat{\delta^\sigma} = \widehat{\delta}^\sigma$ and $\widehat{\delta^{p^{-1}}} = \widehat{\delta}^{p^{-1}}$. Replace the adapted field operator A by $R(g_1)A$, and multiply the twist operator by $\widehat{\delta}(t)$. The additional factor is 1, since $\frac{\widehat{\delta}(t)}{\widehat{\delta}^{p^{-1}}(t)} \delta^{p^{-1}}(t^{1-p}) = 1$. After $2f$ steps, $\prod_{j=0}^{2f-1} g_1^{(p^j)} = B(B^{(q^2)})^{-1} = 1$. Hence, an original inner power x, when present, has the ordinary operator $R(x)$ in either coordinate system. If $q \equiv 1 \pmod 4$, the 2-part of C has order 2. Thus, $\delta_a(-1) = 1$ implies that δ_a has odd order.

Suppose that $q \equiv 3 \pmod 4$. Then, f is odd, $p \equiv 3 \pmod 4$, and $H = v_2(q+1) = v_2(p+1) \geq 2$. Moreover, put $M = 2^{H+1}$, and fix an odd part d_j of either *original* wave front. Let β_j be the column central character. For $a \in J$, $\beta_j = \beta_j^{[a]} \delta_a^{d_j}$. Indeed, π_a fixes column multiplicities, and $x^{-q} = x$ for $x \in C$. In $\mathbb{F}_{p^2}^\times$, we choose t of order M, put $z = t^2$, and fix a primitive M-th root ζ_M in $\mathbb{C}$. Then, $t^{q+1} = t^{p+1} = -1$, and $z \in C$ has order 2^H. Write $\delta_a(z) = \zeta_M^{2D_a}$ and $\beta_j(z) = \zeta_M^{2c_j}$. Next, put $x_0 = c_j/d_j \pmod{2^H}$ and $r_a = p^{-h_a} \ell^{k_a} \pmod M$. The formula for β_j gives $c_j \equiv r_a c_j + d_j D_a \pmod{2^H}$. Since d_j is odd, $D_a = (1 - r_a) x_0 \pmod{2^H}$. The same x_0 satisfies the congruence for J at every later step $\lambda \mapsto \mu$.

The matrix H_{ad} has a hyperbolic block of dimension $2l$ and a residual identity block. Thus, $\det H_{\mathrm{ad}} = (-1)^l$. By unitary determinant surjectivity, we choose B with $\det B = 1$ for even l, and $\det B = t$ for odd l. If $2 \nmid l$, replacing $T_a = C_{\delta_a} A^{h_a} I_{\sigma_a}$ by $R(g_{h_a}) T_a$ gives $\tau(a,b) = \delta_a\big(t^{p^{h_a}(p^{h_b}-1)}\big)$. Indeed, the standard field action ϕ satisfies $g_h \phi^h(g_k) = g_{h+k}$. Moreover, $T_a R(x) = \delta_a(\det \phi^{h_a}(x))^{-1} R(\phi^{h_a}(x)) T_a$. Put $Z = \zeta_M^{2x_0}$ and $c_B(a) = Z^{-(p^{h_a}-1)/2}$. Since $v_2(q^2-1) = H+1$, the value $c_B(a)$ depends only on $h_a \bmod 2f$. By the congruence for D_a, $(\partial c_B)(a,b) = Z^{(p^{h_a} - \ell^{k_a})(p^{h_b}-1)/2} = \tau(a,b)$. Moreover, for $z_0 = u^{2f} = v_{\mathrm{all}}^2$, $c_B(z_0) = 1$ and $c_B(a z_0) = c_B(a)$.

On an irreducible space V, intertwiners S_1, S_2 for the same transport satisfy $S_1 = \lambda_0 S_2$, with $\lambda_0 \in \mathbb{C}^\times$. The relation $u^{2f} = v_{\mathrm{all}}^2$ fixes the scalar convention on V. Moreover, the common group X consists of determinant self-twists. On X, twist operators multiply exactly. A cochain c for the factor quotient satisfies $(\partial c)|_{X \times X} = 1$, and $c|_X$ is a character of X. Moreover, in Lemma 3.4, the character $c|_X$ reindexes eigenspaces for the pairing Ω chosen by descent. □

For $\mathcal{I}$, we have that the step $\lambda \mapsto \mu$ has central-character quotient $\chi_{\mathcal{I}} = \prod_{i \text{ selected}} \nu_i^2$. The half-rank changes by $\sum_{i \text{ selected}} e_i$. Next, we put $\mathcal{I}_{\mathrm{e}} = \{i \in \mathcal{I} : 2 \mid e_i\}$ and $\mathcal{I}_{\mathrm{o}} = \mathcal{I} \setminus \mathcal{I}_{\mathrm{e}}$. In Λ_{b}, take weight 2 on $\mathcal{I}_{\mathrm{e}}$ and $\nu_i \circ w$ on $\mathcal{I}_{\mathrm{o}}$. The sorting sign in the c_G identity is 1, and $\chi_{\mathcal{I}}(-1) = 1$. Thus, choose field-power root 1. The central factor on J

is a coboundary. Thus, by Lemma 3.34, $(K_{\rm o}, E_{\rm o})$ has the same class on J. Both models have field-period relation $u^{2f} = v_{\rm all}^2$. Consider J separately on ρ, ρ_0. Moreover, at each step the character χ has unrestricted degree $\chi(1)$.

Iterate to the 2-core of every partition. For odd e_i and active i, $2 \mid m_i$. Indeed, on the abacus with 2 runners, an m_i-hook moves a bead by m_i places. Thus, runner charges for (λ_i, μ_i) coincide, and $\mathrm{core}_2(\lambda_i) = \mathrm{core}_2(\mu_i)$. For inactive i, $\lambda_i = \mu_i$. Let t_i, t_i^0 be the staircase heights of the 2-cores for λ_i, μ_i. Thus, $t_i = t_i^0$ for odd e_i. Both reduced wave fronts set $I_{\rm o} = \{j : \sum_{i:\, e_i \text{ odd}} e_i (t_i + 1 - j)_+ \text{ is odd}\}$, $x_+ = \max(x, 0)$ for odd positions. If $r < s$, we have that $d_r(\rho) \geq d_s(\rho)$, so the minimum is $d_s(\rho)$. Thus, $D_\rho \bmod 2$ counts pairs (r, s) with $d_r(\rho)$ even and $d_s(\rho)$ odd. Consequently, we find $D_\rho \equiv D_{\rho_0} \pmod 2$. For each even colour put $n_i = \sum_{j \in I_{\rm o}} \big((t_i^0 + 1 - j)_+ - (t_i + 1 - j)_+\big)$. Given $a \in J$, we have that the weights satisfy $n_{\pi_a(i)} = n_i$. In particular, for $\lambda_i = \mu_i$, we have that $n_i = 0$. The formulas for K, E become $K_{\rm o} = \prod_i \nu_i^{n_i}$ and $E_{\rm o} = \frac{1}{2} \sum_i e_i n_i$. Thus, we apply Lemmas 3.32 and 3.33 with n_i and $\Lambda(x) = \prod_i \theta_i(x)^{n_i}$. Put $B = \sum_i n_i b_i \in \mathbb{Z}$ and $Q_a = \sum_i n_i b_i s_i(a) \bmod 2$. Since $K_{\rm o}(-1) = (-1)^B$, adjust the root exponent Δ by an even integer to B. The remaining factor is $\mathcal{B}(a, b) = t_{\sigma_a}^{Bh_b} (-1)^{Lk_a Q_b}$.

Proposition 3.35. *For the matched unitary labels above, $\mathcal{B}$ is a semilinear coboundary on J. Next, choose c with $\mathcal{B} = \partial c$. For every pure determinant self-twist a, we have that $c(a) = 1$. Moreover, for $a \in J$ and $z_0 = u^{2f} = v_{\rm all}^2$, we have that $c(az_0) = c(a)$.*

Proof. First, let $q \equiv 1 \pmod 4$. Then, $PF = 0$. Apply the congruence for Q_a with n_i. Thus, $Q_a = B(Ph_a + Lk_a) \pmod 2$. If B is even, $Q_a = 0$ and $\mathcal{B} = \partial[\omega^{Bh_a}]$, with $2f$-periodic exponent as above. Suppose B is odd. With $s = v_2(f)$, we write $H = v_2(q-1)$ and $M = 2^{H+1}$. Every active even e_i has $v_2(e_i) = 1$, so $v_2(q^{e_i} - 1) = H + 1$. The J-invariant set $\mathcal{C} = \{i : n_i \text{ odd},\ b_i = 1\}$ has odd cardinality. Thus, for $a \in J$, we choose an odd π_a-cycle $\mathcal{C}_a \subseteq \mathcal{C}$ of length l. Given $i \in \mathcal{C}_a$, θ_i is faithful on the subgroup $T_{e_i}[M]$ of order M. Every determinant twist δ_a has odd order. Thus, along $\mathcal{C}_a$, the relation gives $(p^{-h_a} \ell^{k_a})^l \in \langle -q \rangle \pmod M$. Since l is odd, exponentiation by l is invertible on the abelian 2-group $(\mathbb{Z}/M\mathbb{Z})^\times / \langle -q \rangle$. Thus, $p^{-h_a} \ell^{k_a} \in \langle -q \rangle \pmod M$.

We have $H \geq s + 2$ by $v_2(p^f - 1) \geq v_2(f) + 2$. Next, put $\mathfrak{U}_4 = \{x \in (\mathbb{Z}/M\mathbb{Z})^\times : x \equiv 1 \pmod 4\}$. Modulo M, $q = 1 + 2^H$ is the unique involution in $\mathfrak{U}_4$. Thus, $(\mathbb{Z}/M\mathbb{Z})^\times / \langle -q \rangle$ is cyclic of order 2^{H-1}, generated by $[5]$, and $[p]^f = [q] = [-1]$ is its involution. Hence, $[p]$ has order 2^{s+1}. Write $[p] = [5]^{gu_0}$, $g = 2^{H-s-2}$, with u_0 odd, and put $[\ell] = [5]^v$ and $r = g/\gcd(g, v)$. In particular, $p^{-h_a} \ell^{k_a} \in \langle -q \rangle \pmod M$ implies $r \mid k_a$ and $h_a \equiv c\, k_a / r \bmod 2^{s+1}$, where $c = u_0^{-1} v r / g \bmod 2^{s+1}$. Put $\Omega = \omega \mathrm{i}^P$, still primitive of order $N_0 = 2^{s+2}$. Indeed, if $s = 0$, $P = 0$, and otherwise i has smaller order than ω. The congruence for Q_a and the formula for $\mathcal{B}$ now yield $\mathcal{B}(a, b) = \big(\sigma_a(\Omega)/\Omega\big)^{Bh_b} (-1)^{BLk_a k_b}$. Next, put $R_0 = 2M$, record $k_a \bmod rR_0$, and let $x_a = k_a / r \bmod R_0$. This is additive on J. For $b_0 = \ell^r$ put $g_0(x) = \sum_{j=0}^{x-1} b_0^j - x$, with $g_0(0) = 0$. Moreover, the cochains $c_2(a) = \Omega^{-Bc\, g_0(x_a)}$ and $c_1(a) = (-1)^{BL\binom{k_a \bmod 4}{2}}$ satisfy $\partial(c_1 c_2) = \mathcal{B}$. For c_2, the geometric sum satisfies $\sum_{j=0}^{R_0 - 1} b_0^j \equiv 0 \pmod{R_0}$, by its factors $1 + b_0^{2^j}$. Since $N_0 \mid R_0$, $g_0(x + R_0) \equiv g_0(x) \pmod{N_0}$. From $g_0(x) + b_0^x g_0(y) - g_0(x + y) = -(b_0^x - 1)y$, $\partial c_2 = (\sigma_a(\Omega)/\Omega)^{Bcx_b}$. Since $(\sigma_a(\Omega)/\Omega)^{2^{s+1}} = 1$, the congruence for h_b gives $\partial c_2 = (\sigma_a(\Omega)/\Omega)^{Bh_b}$. The binomial identity gives $\partial c_1 = (-1)^{BLk_a k_b}$. Thus, both $c_1(a), c_2(a)$ depend only on σ_a.

Now, let $q \equiv 3 \pmod 4$. Moreover, keep H, M, t, x_0, D_a, r_a as above, with $D_a = (1 - r_a)x_0 \pmod{2^H}$. In this case, $\omega = \mathrm{i}$ and $\mathcal{B}(a, b) = (-1)^{Lk_a(Bh_b + Q_b)}$. Write $\theta_i(t) = \zeta_M^{u_i}$. Every relevant $e_i = 2a_i$ has a_i odd and $v_2(q^{e_i} - 1) = H + 1$, so $u_i \bmod 2 = b_i$. The determinant exponent on t is

$$\frac{1 - q^{e_i}}{q + 1} = (1 - q)(1 + q^2 + \cdots + q^{2(a_i - 1)}) \equiv a_i(2 - 2^H) \pmod M.$$

By the congruence for D_a, $D_a \equiv 0 \pmod 2$. Hence, the determinant twist at t has additive exponent $e_i D_a \bmod M$. Thus, the equation at t is $r_a u_i + e_i D_a \equiv u_{\pi_a(i)}(1 - 2^H j_i(a)) \pmod M$. Put $v_i = u_i - e_i x_0 \bmod M$, well defined since e_i is even. Applying the congruence for D_a gives $r_a v_i \equiv v_{\pi_a(i)}(1 - 2^H j_i(a)) \pmod M$. Next, put $\mathcal{C}_1 = \{i : n_i \neq 0,\ b_i = 1\}$. Given $i \in \mathcal{C}_1$, v_i is a unit modulo M. Moreover, $n_{\pi_a(i)} = n_i$, so $\prod_{i \in \mathcal{C}_1} v_{\pi_a(i)}^{n_i} = \prod_{i \in \mathcal{C}_1} v_i^{n_i} \pmod M$. Thus, the equations yield $r_a^B = (1 - 2^H)^{Q_a} \pmod M$.

Since f is odd, $p \equiv q \equiv -1 + 2^H \bmod M$. With $Y_a = Bh_a + Q_a \bmod 2$ the last identity gives $\ell^{Bk_a} \equiv (-1)^{Bh_a}(1 - 2^H)^{Y_a} \pmod M$. Modulo 4, this gives $Bh_a \equiv BLk_a \bmod 2$. Thus, with $g = \big((-1)^L \ell\big)^B$, we obtain $g^{k_a} \equiv (1 - 2^H)^{Y_a} \pmod M$. The group $\mathfrak{U}_4 = \{x \in (\mathbb{Z}/M\mathbb{Z})^\times : x \equiv 1 \pmod 4\}$ is cyclic, with unique involution $1 - 2^H$. If $g = 1$, it follows that $Y_a = 0$ for every a. If $|g| \geq 4$, $g^{k_a} \in \{1, 1 - 2^H\}$ implies $2 \mid k_a$. In both cases $\mathcal{B} = 1$. If g has order two, $Y_a = k_a \bmod 2$, and the remaining factor is $(-1)^{Lk_a k_b} = \partial\big[(-1)^{L\binom{k_a \bmod 4}{2}}\big](a, b)$. Put $c_{\rm s}(a) = (-1)^{L\binom{k_a \bmod 4}{2}}$. Since

$c_{\mathrm{s}}(a)$ depends only on σ_a, it is 1 on pure determinant twists and $c_{\mathrm{s}}(az_0) = c_{\mathrm{s}}(a)$. If B is even, put $c_{\mathrm{f}}(a) = \omega^{Bh_a}$. Since $\omega^{2fB} = 1$, $c_{\mathrm{f}}(az_0) = c_{\mathrm{f}}(a)$. For pure determinant twists a, $c_{\mathrm{f}}(a) = 1$. □

Next, let α_{glob} be the global Jordan factor on J, and α_{loc} the local Fourier factor. For $d \equiv 2 \pmod 4$, choose the cyclic and Gauss cochains $c_{\mathrm{cyc}}, c_{\mathrm{gauss}}$ and the coordinate and field cochains $c_{\mathrm{coord}}, c_{\mathrm{field}}$. Thus, $\alpha_{\mathrm{glob}} = \alpha_{\mathrm{loc}}$ after rescaling. For u, apply the field conjugation relation. For δ_a, also take the relation for δ. Similarly, for u^{2f}, take the field power relation. For K_ρ, apply the condition on K_ρ. Moreover, for active i, the matching assigns μ_i as the m_i-core of λ_i. The residual label is ρ_0. Lemma 3.34 gives rational induction of multiplicity 1. Moreover, by Lemma 3.4, a character $\xi \in \mathrm{Irr}(X)$ can change the constituent pairing Ω in descent. Next, we choose Θ as an invariant extension of $\chi \otimes \psi^\vee$. Moreover, we choose D from Proposition 3.5. For $c = (x, a) \in \mathrm{C}_{\mathrm{SU}_n(q)\rtimes D}(\mathrm{SU}_n(q))$, require $\Theta(x, x, a) = 1$.

Let $q = p^f$ be odd, let $\ell \neq p$ be odd, and put $d = \mathrm{ord}_\ell(-q)$, $C = \mu_{q+1}$. Apply Proposition 3.46 for partitions (λ_i, μ_i), and Proposition 3.30 for lifts u, v. Thus, we obtain that $n = Ad + r$, $0 \leq r < d$, and A counts torus blocks. A colour i of degree e_i is a $(-q)$-orbit represented by θ_i on the cyclic torus T_{e_i} of order $q^{e_i} - (-1)^{e_i}$. Moreover, its partition is λ_i, its residual partition is μ_i. Given an active colour $e_i \mid d$, we set $m_i = d/e_i$ and $w_i = (|\lambda_i| - |\mu_i|)/m_i$. Thus, we find $A = \sum_i w_i$. Write $\nu_i = \theta_i|_C$ and $\theta_i(-1) = (-1)^{b_i}$.

The common stabilizer J is as in §3.4. Next, we write $a = (\delta_a, h_a, \sigma_a)$ in determinant, field, coefficient order. Put $\delta_a(-1) = (-1)^{t_a}$, and define $a_\sigma \in \mathbb{F}_p^\times$ by $\sigma(\zeta_p) = \zeta_p^{a_\sigma}$. Moreover, on prime-to-$\ell$ roots write k_a for the ℓ-power exponent of σ_a. Choose $B \in 4\mathbb{Z}$ divisible by the orders m of the power actions below. Adjoin $\mu_{\ell^B - 1}$ to determine the residues of k_a. Thus, the colour action is σ_a. For each a write $\theta_i^{p^{-h_a}\sigma_a}(\delta_a \circ \det_{e_i}) = \theta_{\pi_a(i)}^{(-q)^{j_i(a)}}$. Moreover, shifts add modulo e_i with permutation of the subscripts. When e_i is even, we let $s_i(a) \in \{0, 1\}$ be the parity of $j_i(a)$. In particular, for $z = (1, 2f, 1)$ the shift is -2, and the operator represents $u^{2f} = v_{\mathrm{all}}^2$. For global and local factors $\alpha_{\mathrm{glob}}, \alpha_{\mathrm{loc}}$, we choose $\alpha_{\mathrm{glob}}/\alpha_{\mathrm{loc}}$. Moreover, for a cochain c, we put $\partial c(a, b) = c(a)\sigma_a(c(b))/c(ab)$. Rescale the global operators by c^{-1}.

For a unitary source ρ and $x \in C$, we let c_x act by x on even strings and by 1 on odd strings. Next, we put $K_\rho(x) = \kappa_\rho(c_x)$, and let $2E_\rho$ be the total dimension. For global ρ_g and residual ρ_0, follow §3.3 and put $K = K_g/K_0$, $E = E_g - E_0$. Set $T = \mathbb{F}_{q^2}^\times$, $N(y) = y^{q+1}$, $w(y) = y^{1-q}$ and the multiplicative compensators $b_a = b_{\sigma_a}$ with $N(b_a) = a_{\sigma_a}^{-1}$. Thus, by Lemma 3.32, $K = K^{p^{-h_a}\sigma_a}\delta_a^{2E}$ permits cyclic normalization. Namely, if $\Lambda|_C = K$ and Ξ_a is defined by the Λ, Ξ_a identity, the raw factor is $\partial c_\Lambda(a, b)\sigma_a(\Xi_b(a_{\sigma_a}^{-1}))$, where $c_\Lambda(a) = \sigma_a(\Lambda(b_a))$. Take the unnormalized cyclic factor.

Lemma 3.36. *Assume $4 \mid d$. Write $F = f \bmod 2$ and $(p/\ell) = (-1)^R$. Moreover, the field-period correcting root $\omega = \zeta_{2^{v_2(f)+2}}$ is fixed by $\mathcal{H}_\ell$. For the contribution of the A torus blocks, we put $n_i = m_i w_i$. For active colours with even e_i, we write $\xi_i = \theta_i|_{\mathbb{F}_q^\times}$. Next, we put $B_0 = \sum_i n_i b_i$ and $Q_a = \sum_i n_i b_i s_i(a)$. Set $T_a = \sum_i n_i \mathbf{1}_{\xi_i \neq 1} s_i(a)$. After c_Λ, c_G, the global torus factor $\alpha_{\mathrm{t,g}}$ and local torus factor $\alpha_{\mathrm{t,loc}}$ satisfy $\alpha_{\mathrm{t,g}}/\alpha_{\mathrm{t,loc}} = (-1)^{Rk_a B_0 h_b + FRk_a(T_b + Q_b)}$.*

Proof. Put $D = v_2(d)$ and $s = v_2(f)$. Since $4 \mid d$, $-1 = (-q)^{d/2}$ modulo ℓ and $1 + d/2$ is coprime to d. Thus, $q = (-q)^{1+d/2}$ also has order d, $\ell \equiv 1 \bmod 4$, and $v_2(\mathrm{ord}_\ell(p)) = s + D \leq v_2(\ell - 1)$. Thus, $\sigma(\omega) = \omega$ and $\sigma(i) = i$ for every $\sigma \in \mathcal{H}_\ell$. If $2 \nmid f$, the action on $\sqrt{q}$ is the one on $\sqrt{p}$. The quadratic Gauss sum for $\mathbb{F}_p$ has square $(-1)^{(p-1)/2}p$, and its Galois ratio is $(\ell/p)^{k_a}$. Since i is fixed and $\ell \equiv 1 \bmod 4$, reciprocity gives $\sigma_a(\sqrt{q})/\sqrt{q} = (-1)^{FRk_a}$, also for even f.

By §3.3, the local root ratios are $(-1)^{Rk_a m_i b_i}$ on the field root and $(-1)^{FRk_a m_i b_i}$ on the rotation root for even e_i. Next, on each block, take the w_i-fold tensor power. The representative shifts $j_i(b)$ give the local factor $(-1)^{Rk_a(B_0 h_b + FQ_b)}$. If $2 \nmid e_i$, the ratios are 1. Moreover, for a permutation τ_{blk} of even-dimensional blocks, $\det(\tau_{\mathrm{blk}}) = 1$. The rational Specht factor is 1.

Similarly, for the global pair, put $K_{\mathrm{t}} = \prod_i \nu_i^{m_i w_i}$, $E_{\mathrm{t}} = dA/2$. Moreover, we put $\Lambda_{\mathrm{t}}(y) = \prod_{e_i \text{ even}} \theta_i(y)^{m_i w_i} \prod_{e_i \text{ odd}} \nu_i(w(y))^{m_i w_i/2}$. $y \in T$, $T \subset T_{e_i}$ for even e_i, and m_i is even for odd e_i. The determinant on T in an even block is $w(y)^{e_i/2}$. Because $-q$ acts on T by an involution and $y^{-q}/y = N(y)^{-1}$, the shifts $s_i(a)$ in the character-shift identity give $\Xi_a = \prod_{e_i \text{ even}} \xi_{\pi_a(i)}^{-n_i s_i(a)}$. For weights $n_i \in \mathbb{Z}$ with $n_{\pi_a(i)} = n_i$ on even e_i, the formula for Ξ_a also holds. The central formulas give $K = \prod_i \nu_i^{n_i}$ and $2E = \sum_i e_i n_i$.

Moreover, for nontrivial $\xi \in \mathrm{Irr}(\mathbb{F}_q^\times)$, put $g(\xi) = q^{-1/2} \sum_{x \in \mathbb{F}_q^\times} \xi(x)\zeta_p^{\mathrm{Tr}(x)}$ and $r(\xi) = g(\xi)/i^{b(\xi)}$, where $\xi(-1) = (-1)^{b(\xi)}$ and $b(\xi) \in \{0, 1\}$. Set $g(1) = r(1) = 1$. On $\mathbb{F}_q^\times$, we write ξ_2 for its quadratic character. If $r(\xi_2) \neq 1$, replace $g(\xi_2)$ by $-g(\xi_2)$. Thus, $r(\xi_2) = 1$. The product formula gives $r(\xi^{-1}) = r(\xi)^{-1}$. Given $x \in \mathbb{F}_q$, $\mathrm{Tr}(x^p) = \mathrm{Tr}(x)$. Thus, $r(\xi^p) = r(\xi)$. With $y = a_{\sigma_a} x$ and $\sigma_a(\sqrt{q})/\sqrt{q} = (-1)^{FRk_a}$, $\sigma_a(r(\xi)) = \xi^{\sigma_a}(a_{\sigma_a})^{-1}(-1)^{FRk_a \mathbf{1}_{\xi \neq 1}} r(\xi^{\sigma_a})$. Since $\xi_2^{\sigma_a} = \xi_2$, changing the sign of $r(\xi_2)$ does not affect the formula. Set $c_G(a) = \prod_i r(\xi_{\pi_a(i)})^{-n_i s_i(a)}$ on even-degree colours.

Moreover, in $\partial c_G(a,b)$, transport of $r(\xi_{\pi_b(i)})$ replaces it by $r(\xi_{\pi_a\pi_b(i)})^{(-1)^{s_{\pi_b(i)}(a)}}$ times $\xi_{\pi_b(i)}^{\sigma_a}(a_{\sigma_a})^{-1}(-1)^{FRk_a\mathbf{1}_{\xi_{\pi_b(i)}\neq 1}}$. With $s = s_{\pi_b(i)}(a)$, we put $t = s_i(b)$. Then, $(s+t \bmod 2) = t+(-1)^t s$, so the r-factors contribute 1. Thus, $\partial c_G(a,b) = \sigma_a\big(\Xi_b(a_{\sigma_a}^{-1})\big)(-1)^{FRk_aT_b}$. Put $L(a,b) = (-1)^{Rk_a(B_0h_b+FQ_b)}$. Next, divide the identity by $L(a,b)\partial(c_\Lambda c_G)(a,b)$. The result is $(-1)^{Rk_aB_0h_b+FRk_a(T_b+Q_b)}$. □

Proposition 3.37. *Given $c \in \{0,1\}$, we put $\mathcal{B}_c(a) = (-1)^{c\binom{k_a \bmod 4}{2}}$ on J. Then, $\partial\mathcal{B}_c(a,b) = (-1)^{ck_ak_b}$. Moreover, the sign $(-1)^{Rk_aB_0h_b+FRk_a(T_b+Q_b)}$ is a product of differentials $\partial\mathcal{B}_c$.*

Proof. On μ_{ℓ^B-1}, the ℓ-power map has order B. Thus, $k_{ab} \equiv k_a + k_b \pmod 4$ since $4 \mid B$. Moreover, $\binom{k+4}{2} \equiv \binom{k}{2} \pmod 2$ for $k \in \mathbb{Z}$. The identity $\binom{x+y}{2} = \binom{x}{2} + \binom{y}{2} + xy$ gives the differential since $\sigma_a(-1) = -1$. Next, assume $FR = 1$, so f is odd. Moreover, i contributes to $T_a + Q_a$ only if $\xi_i \neq 1$ and $\xi_i(-1) = 1$. Let $\mathcal{I}_{\mathrm{ns}}$ be a J-invariant set of indices i with $\xi_i \neq \xi_i^{-1}$. We put $S_a = \sum_{i\in\mathcal{I}_{\mathrm{ns}}} n_i s_i(a) \bmod 2$. Thus, the permutation rule gives $S_{ab} = S_a + S_b \pmod 2$. Since $\ell \nmid q-1$, we put $\upsilon_a = p^{-h_a}\ell^{k_a} \in (\mathbb{Z}/(q-1)\mathbb{Z})^\times$. If υ_a fixes every ξ_i, the inversion count N_{inv} on each π_a-cycle is even, since $\xi_i \neq \xi_i^{-1}$. Thus, $S_a = 0$. The p-power image has odd order dividing f, so S is zero there. Let A_ℓ be the cyclic ℓ-power image modulo the p-power subgroup. Consequently, S factors through A_ℓ. Record $k_a \bmod m$, where $4 \mid m$ and $|A_\ell| \mid m$. If all k_a are even, $k_aS_b \equiv 0 \pmod 2$. Otherwise, recorded exponents form $\mathcal{K} \leq \mathbb{Z}/m\mathbb{Z}$ of odd index. The nonzero homomorphism $\mathcal{K} \to \mathbb{F}_2$ is unique and is $k \mapsto k \bmod 2$. Hence, $S_b = ck_b \bmod 2$ for a fixed bit c, and $(-1)^{k_aS_b} = \partial\mathcal{B}_c(a,b)$.

For $q \equiv 1 \bmod 4$, let ξ_2 be the quadratic character of $\mathbb{F}_q^\times$. Then, the remaining ξ_i is ξ_2. Since f is odd, $p \equiv 1 \bmod 4$. With $H = v_2(q-1)$, we choose $y \in T$ of order 2^{H+1}. Choose a generator t of T, and write $\theta_i(t) = \zeta_{q^2-1}^{u_i}$. Since $\theta_i|_{\mathbb{F}_q^\times} = \xi_2$, $u_i \equiv (q-1)/2 \pmod{q-1}$ and $v_2(u_i) = H-1$. Thus, $\theta_i(y)^2 = -1$, and $\theta_i(y)^p = \theta_i(y)^\ell = \theta_i(y)$. For odd $j_i(a)$, $\theta_i(y)^{(-q)^{j_i(a)}} = \theta_i(y)^{-1}$. Also, $w(y) = -1$. Apply the character-shift identity at y, with weights n_i. Since $n_ie_i/2 = (d/2)w_i$ is even, the determinant factors are 1. We put $\mathcal{I}_2 = \{i : \xi_i = \xi_2\}$. On $\mathcal{I}_2$, the remaining factor is $(-1)^{\sum_{i\in\mathcal{I}_2} n_is_i(a)}$. Since π_a permutes $\mathcal{I}_2$, $\sum_{i\in\mathcal{I}_2} n_is_i(a) \equiv 0 \pmod 2$. The $T+Q$ sign is a coboundary. Moreover, for $FR = 0$, the sign is 1. For the field sign $Rk_aB_0h_b$, it remains to consider $R = 1$ and B_0 odd. For $q \equiv 3 \bmod 4$, choose $\iota \in T$ with $\iota^2 = -1$. Thus, on even colours i, we apply the shift identity at ι. Then, $-q \equiv 1 \bmod 4$, $\ell \equiv 1 \bmod 4$, and $p \equiv -1 \bmod 4$. The determinant exponent is $\sum_{e_i \text{ even}} n_ie_i/2 = (d/2)\sum_{e_i \text{ even}} w_i \in 2\mathbb{Z}$. Thus, field transport gives $(-1)^{B_0h_b} = 1$.

Suppose $q \equiv 1 \bmod 4$. We put $H = v_2(q-1)$, $M = 2^{H+D}$, and $\mathcal{I}_* = \{i : n_i \equiv b_i \equiv 1 \pmod 2\}$. Moreover, the J-invariant set $\mathcal{I}_*$ has odd cardinality. Given $i \in \mathcal{I}_*$, $v_2(e_i) = D$ and $v_2(q^{e_i}-1) = H+D$. In μ_M, we choose t_M of order M, and write $\theta_i(t_M) = \zeta_M^{u_i^*}$. Since $\theta_i(-1) = -1$, u_i^* is a unit modulo M. A determinant twist adds $\epsilon_{a,i}M/2$, with $\epsilon_{a,i} \in \{0,1\}$, since $v_2(q+1) = 1$. If $2 \nmid u$, $u + \epsilon_{a,i}M/2 \equiv u(1+M/2)^{\epsilon_{a,i}} \pmod M$. Moreover, $q^{2^{D-1}} \equiv 1 + M/2 \pmod M$, by squaring q^{2^j} for $0 \leq j < D-1$, starting from $v_2(q-1) = H \geq 2$. Since $D \geq 2$, $1 + M/2 \in \langle -q\rangle \pmod M$.

We put $U = (\mathbb{Z}/M)^\times/\langle -q\rangle$. Given $i \in \mathcal{I}_*$, the equation gives $[p]^{-h_a}[\ell]^{k_a}[u_i^*] = [u_{\pi_a(i)}^*]$ in U. Since π_a permutes $\mathcal{I}_*$, $([p]^{-h_a}[\ell]^{k_a})^{|\mathcal{I}_*|} = 1$. Odd powers are injective on the 2-group U, so $[p]^{h_a} = [\ell]^{k_a}$. If $(-q)^j \equiv -1 \pmod M$, reduction modulo 4 makes j odd and gives $q^j \equiv 1 \pmod M$. But q has order 2^D modulo M, so $[-1] \neq 1$ in U. Since $[p]^f = [q] = [-1]$, $[p]$ has even order. If some k_a is odd, $[\ell] = [p]^c$ for an integer c. Thus, $h_b \equiv ck_b \bmod 2$ for every b, and the field sign is $\partial\mathcal{B}_{RB_0c}$. If all k_a are even, the sign is 1. □

Proposition 3.38. *If $4 \mid d$, a cochain c on J satisfies $\mathcal{A}_{\mathrm{o}} = \partial c$. The coboundary comparison has field period $2f$.*

Proof. The wavefront parts in column position j on the two sides are $d_j = \sum_i e_i\lambda_{i,j}^{\mathsf{t}}$ and $d_j^0 = \sum_i e_i\mu_{i,j}^{\mathsf{t}}$. If d_j, d_j^0 are even for every j, $K = K_{\mathsf{t}}$ and $E = E_{\mathsf{t}}$ by the formulas for K, E. Proposition 3.37 suffices for d_j, d_j^0. Otherwise, choose j_0 with d_{j_0} or $d_{j_0}^0$ odd. By the character-shift identity, $b_{\pi_a(i)} = b_i + e_it_a \bmod 2$. If $t_a = 1$, each π_a-cycle $\mathcal{O}$ with odd e_i has even length. On $\mathcal{O}$, the partitions λ_i, μ_i are constant. Hence, $d_{j_0} \equiv d_{j_0}^0 \equiv 0 \pmod 2$, contrary to the choice of j_0. Thus, $t_a = 0$ for every $a \in J$.

If $q \equiv 1 \bmod 4$, $v_2(q+1) = 1$, so $|\delta_a|$ is odd for $a \in J$. If $q \equiv 3 \bmod 4$, we have the identity $\partial c_B = \tau$ for the original odd column j_0. The cochain c_B is valid for every even d. Next, for a J-orbit $\mathcal{I}_{\mathrm{rem}}$, choose partitions λ_i^- with common domino $\lambda_i/\lambda_i^- = H$, for $i \in \mathcal{I}_{\mathrm{rem}}$. Apply the same construction to μ_i. We put $l = \sum_{i\in\mathcal{I}_{\mathrm{rem}}} e_i$. Moreover, Lemma 3.34 gives ordinary and semilinear J-transports for the regular character $\chi_{\mathrm{reg}} \in \mathrm{Irr}(\mathrm{GL}_l(q^2))$. On the corresponding Levi L, the chosen constituent has multiplicity 1. Consequently, the comparison has field period $2f$.

For $\lambda_i \mapsto \lambda_i^-$ on $\mathcal{I}_{\rm rem}$, the central-character quotient is $\prod_{i\in\mathcal{I}_{\rm rem}} \nu_i^2$, and the half-dimension changes by $\sum_{i\in\mathcal{I}_{\rm rem}} e_i$. On selected even colours i, take weight 2. Similarly, on selected odd colours i, take $\nu_i \circ w$ in cyclic and Gauss normalization. Thus, $T_a \equiv 0 \pmod 2$, the central sign is 1, and both comparisons have period root ω. The central factor on J is a coboundary. Thus, the factor $\mathcal{A}'_{\rm o}$ satisfies $[\mathcal{A}'_{\rm o}] = [\mathcal{A}_{\rm o}]$. Iterate to the two 2-cores. For active i with odd e_i, $2 \mid m_i = d/e_i$. A bead moved by m_i remains on its runner, so $\mathrm{core}_2(\lambda_i) = \mathrm{core}_2(\mu_i)$. For inactive i, $\lambda_i = \mu_i$. Let t_i, t_i^0 be the staircase heights of the 2-cores of λ_i, μ_i. Thus, $t_i = t_i^0$ for odd e_i. Put $(x)_+ = \max(x, 0)$. Moreover, both reduced wave fronts set $\mathcal{I} = \{j : \sum_{e_i \text{ odd}} e_i(t_i + 1 - j)_+ \equiv 1 \pmod 2\}$ for odd positions. The parity of D_ρ counts pairs $r < s$ with d_r even and d_s odd. Thus, $D_\rho \equiv D_{\rho_0} \pmod 2$. Now, the formulas for K, E give $a_i = \sum_{j\in\mathcal{I}}\big((t_i^0 + 1 - j)_+ - (t_i + 1 - j)_+\big)$, $K_{\rm o} = \prod_{e_i \text{ even}} \nu_i^{a_i}$, and $E_{\rm o} = \frac{1}{2}\sum_{e_i \text{ even}} e_i a_i$. Given $g \in J$, $a_{\pi_g(i)} = a_i$. Moreover, the local rotation factor for $\mathcal{A}_{\rm o}$ is 1. After normalization, Lemma 3.36 leaves $(-1)^{FRk_aT_b}$, with $T_b = \sum_i a_i \mathbf{1}_{\xi_i\neq 1} s_i(b)$. If $FR = 0$, this sign is 1. If $FR = 1$, let $\mathcal{A}_{\rm ns}$ be the contribution from $\xi_i \neq \xi_i^{-1}$. By Proposition 3.37, $\mathcal{A}_{\rm ns}$ is a coboundary. For ξ_2 quadratic on $\mathbb{F}_q^\times$ and $q \equiv 1 \bmod 4$, $\sum_{i:\xi_i=\xi_2} a_i s_i(a) \equiv 0 \pmod 2$, since $\delta_a(-1) = 1$.

It remains to treat quadratic ξ_i at $q \equiv 3 \bmod 4$. In this case, f is odd and $H = v_2(q+1) = v_2(p+1) \geq 2$. Then, we put $M = 2^{H+1}$, choose $y \in \mathbb{F}_{p^2}^\times$ of order M, and write $\zeta = \zeta_M$. Thus, $p \equiv q \equiv 2^H - 1 \bmod M$, and y^2 generates the 2-part of C. Write $\delta_a(y^2) = \zeta^{2D_a}$. Moreover, we let l_0 be the dimension of odd column j_0, with central character $\beta \in \mathrm{Irr}(C)$. Write $\beta(y^2) = \zeta^{2c_0}$. Since $\beta = \beta^{[a]}\delta_a^{l_0}$, $x_0 = c_0/l_0 \bmod 2^H$, $D_a = (1 - r_a)x_0 \bmod 2^H$, and $r_a = p^{-h_a}\ell^{k_a} \bmod M$. In particular, D_a is even. If $2 \mid e_i$, write $\theta_i(y) = \zeta^{u_i}$ and $v_i = u_i - e_i x_0 \bmod M$. Since $\det_{e_i}(y) = w(y)^{e_i/2}$ and $w(y) = y^{2-2^H}$, $\delta_a(\det_{e_i}(y)) = \zeta^{e_i D_a}$. The exponent error $2^{H-1}e_i D_a$ is divisible by M, since $2 \mid e_i, D_a$. By the character-shift identity, $r_a u_i + e_i D_a \equiv u_{\pi_a(i)}(1 - 2^H s_i(a)) \pmod M$. Thus, $r_a v_i = v_{\pi_a(i)}(1 - 2^H s_i(a)) \pmod M$.

Let $\mathcal{I}_{\rm quad} = \{i : \xi_i = \xi_2\}$, with ξ_2 quadratic on $\mathbb{F}_q^\times$. Since $\xi_i(-1) = -1$, $u_i \equiv v_i \equiv 1 \pmod 2$ for $i \in \mathcal{I}_{\rm quad}$. We put $B_* = \sum_{i\in\mathcal{I}_{\rm quad}} a_i$ and $T_{*,a} = \sum_{i\in\mathcal{I}_{\rm quad}} a_i s_i(a)$. As $a_{\pi_a(i)} = a_i$ and $\pi_a(\mathcal{I}_{\rm quad}) = \mathcal{I}_{\rm quad}$, $\prod_{i\in\mathcal{I}_{\rm quad}} v_{\pi_a(i)}^{a_i} = \prod_{i\in\mathcal{I}_{\rm quad}} v_i^{a_i} \pmod M$. Hence, $r_a^{B_*} = (1 - 2^H)^{T_{*,a}} \pmod M$. Modulo 4, $B_* h_a \equiv 0 \pmod 2$. Since $p^2 \equiv 1 \pmod M$, $p^{-B_* h_a} \equiv 1 \pmod M$. Put $g_* = \ell^{B_*} \bmod M$ and $\mathfrak{U}_4 = \{x \in (\mathbb{Z}/M\mathbb{Z})^\times : x \equiv 1 \pmod 4\}$. Moreover, we have that the group $\mathfrak{U}_4$ is cyclic. Write $|g_*| = 2^v$. Thus, $g_*^{k_a} = (1 - 2^H)^{T_{*,a}} \pmod M$. If $v = 0$, $T_{*,a} \equiv 0 \pmod 2$. If $v \geq 2$, $2 \mid k_a$, since $g_*^{k_a}$ has order at most 2. In either case, $(-1)^{FRk_aT_{*,b}} = 1$. If $v = 1$, $g_* = 1 - 2^H \pmod M$, so $T_{*,a} \equiv k_a \pmod 2$. Thus, we obtain the remaining factor $\partial\mathcal{B}_{FR}$. □

Proposition 3.39. *Suppose d is odd. On A torus blocks of dimension d, we set $N_A = \binom{A}{2}$ and $L = (\ell - 1)/2 \bmod 2$. Given a base colour i, we let V_i be the base module, of multiplicity A_i, and write $\theta_i(-1) = (-1)^{b_i}$. Next, we let λ be the quotient partition. On $V_i^{\otimes A_i}$, we choose P, the permutation extension, with Specht character $[\lambda] \otimes \mathrm{sgn}^{b_i}$. Moreover, the labels (λ, b_i) are determinant-equivariant. Let V be wreath-induced, with local factor $\alpha_{\rm loc}$. Then, $\alpha_{\rm loc}(a, b) = (-1)^{N_A L k_a t_b}$. The field-period matrix represents $v_{\rm all}^2$.*

Proof. All active e_i, m_i are odd. Hence, the signed norm $N_{d/e_i} : T_d \to T_{e_i}$ gives $(\theta_i \circ N_{d/e_i})(-1) = \theta_i(-1) = (-1)^{b_i}$. Let $\delta(-1) = (-1)^t$. Thus, the twist δ changes b_i to $b_i + t \bmod 2$. For a transposition s of two d-blocks, $\delta(\det s) = (-1)^t$, and the local rotation v satisfies $\det(v) = 1$. Thus, δ changes the Specht sign and fixes j, the rotation-extension index. If $b_i = 1$, the ordinary label is λ', with $[\lambda'](1) = [\lambda](1)$ and equal hook multisets.

Next, let $\mathfrak{X}$ be the colour assignments of $\{0, \ldots, A-1\}$ with fixed multiplicities. Given $F \in \mathfrak{X}$, we put $b_r = b_{F(r)}$ and $A_i = |F^{-1}(i)|$. Next, on the summand V_F, we put $\widetilde{s}_r = (-1)^{b_r b_{r+1}}\tau_r$, where τ_r is the adjacent tensor flip. Given $\pi \in S_A$, we put $\mathrm{Inv}(\pi) = \{(r, s) : r < s,\ \pi(r) > \pi(s)\}$. Thus, we obtain the sign $\epsilon_\pi(\mathbf{b}) = \prod_{(r,s)\in\mathrm{Inv}(\pi)}(-1)^{b_r b_s}$. The operators satisfy $\widetilde{s}_r^2 = 1$, the braid relations, and $\widetilde{s}_r\widetilde{s}_s = \widetilde{s}_s\widetilde{s}_r$ for $|r - s| > 1$. Thus, S_A acts on $\bigoplus_{F\in\mathfrak{X}} V_F$. On the equal-colour Young subgroup S_{A_i}, the sign is sgn^{b_i}. Moreover, the rational Specht operators P_π satisfy $P_\pi P_\sigma = P_{\pi\sigma}$.

If $\mathbf{b} = (b_0, \ldots, b_{A-1})$, put $S(\mathbf{b}) = \sum_{r=0}^{A-1} r b_r \bmod 2$. Let D_δ be ordinary base transport. Then, on V_F, put $\widetilde{D}_\delta = i^{N_A t}(-1)^{tS(\mathbf{b})} D_\delta$. Thus, an adjacent swap changes $S(\mathbf{b})$ by $b_r + b_{r+1} \pmod 2$. For $t \in \{0, 1\}$, $(b_r + t)(b_{r+1} + t) - b_r b_{r+1} \equiv t(b_r + b_{r+1} + 1) \pmod 2$. The extra t accounts for $\delta(\det s_r) = (-1)^t$. Thus, $\widetilde{D}_\delta$ satisfies the twist relation on every $\widetilde{s}_r$, including swaps of colours $i \neq j$. For bits t, t', $S(\mathbf{b} + t'\mathbf{1}) - S(\mathbf{b}) = N_A t' \pmod 2$. The two signs are $(-1)^{N_A tt'}$ and $i^{N_A(t+t'-(t+t' \bmod 2))} = (-1)^{N_A tt'}$. Hence, $\widetilde{D}_\delta\widetilde{D}_{\delta'} = \widetilde{D}_{\delta\delta'}$.

The field fixes the bits b_i. Moreover, for odd d, Proposition 3.30 and the Fourier construction have $\eta_{\rm loc}, \kappa \in \mu_{\ell-1}$, with $\eta_{\rm loc}^f = \kappa$. Given $a \in J$, we have that $\sigma_a(\eta_{\rm loc}) = \eta_{\rm loc}$ and $\sigma_a(\kappa) = \kappa$. Permutation and Specht matrices lie in $\mathbb{Q}$. Moreover, the coefficient scalar for δ_b is $(\sigma_a(i)/i)^{N_A t_b} = (-1)^{N_A L k_a t_b}$. Thus, the local factor is $(-1)^{N_A L k_a t_b}$. Fourier

rotation relies on $P_{e_i}^{e_i}=1$, and the power relations $u^{2f}=v^2$, $v=u^{f(d+1)}$ hold on each base space. Let $U_{\mathrm{blk}}, V_{\mathrm{blk}}$ be tensor products of matrices for u, v, and let $\widetilde{P}_\pi$ be the block permutation operator. Then, $\widetilde{P}_\pi U_{\mathrm{blk}}\widetilde{P}_\pi^{-1}=U_{\mathrm{blk}}$ and $\widetilde{P}_\pi V_{\mathrm{blk}}\widetilde{P}_\pi^{-1}=V_{\mathrm{blk}}$. Hence, u^{2f} acts by v_{all}^2. Tensor V with the residual source V_{ρ_0}. □

Proposition 3.40. *Let d be odd and (ρ_g,ρ_0) a matched pair. Next, we put $K=K_g/K_0$ and $K(-1)=(-1)^\epsilon$. Moreover, we let α_{gr} be the corrected factor for (ρ_g,ρ_0). For $q\equiv 1 \bmod 4$, we have that $\alpha_{\mathrm{gr}}(a,b)$ equals $(-1)^{N_A L k_a t_b}$. For $q\equiv 3 \bmod 4$, we have that the quotient of $\alpha_{\mathrm{gr}}(a,b)$ by $(-1)^{N_A L k_a t_b}$ is $\partial\mathcal{B}_{\epsilon L}(a,b)$.*

Proof. First, $\mathrm{ord}_\ell(q)=2d$. Since $\mathrm{ord}_\ell(-q)=d$ is odd, $(-q/\ell)=1$. Thus, $(q/\ell)=(-1)^L$ and $FR=L \bmod 2$. If $s=v_2(f)$, $v_2(\mathrm{ord}_\ell(p))=s+1$. Put $P_0=(p-1)/2 \bmod 2$. Equivalently, the condition $(p/\ell)=-1$ is equivalent to $v_2(\mathrm{ord}_\ell(p))=v_2(\ell-1)$. If $\omega=\zeta_{2^{s+2}}$, $\sigma_a(\omega)/\omega=(-1)^{Rk_a}$ and $(\ell/p)=(-1)^{R+P_0L}$. Moreover, for the Jordan model of ρ, $A_{u_J}^{2f}=K_\rho(-1)^{-1}R_\rho(c_{-1})$. Put $\widehat{A}_{u_J}=\omega^{\epsilon_\rho}A_{u_J}$, where $K_\rho(-1)=(-1)^{\epsilon_\rho}$. Since $\omega^{2f}=-1$, $\widehat{A}_{u_J}^{2f}=R_\rho(c_{-1})$. For ρ_g,ρ_0, the correction contributes $(-1)^{R\epsilon k_a h_b}$.

For the remaining factors, put $b_a=b_0^{k_a}$. If $q\equiv 3 \bmod 4$, $P_0=F=1$, $R=L$, and $(\ell/p)=1$. If $q\equiv 1 \bmod 4$ and $R=0$, $L=0$ and $(\ell/p)=1$. In either case choose $b_0\in\mathbb{F}_p^\times$ with $b_0^2=(\ell \bmod p)^{-1}$. Then, $N(b_a)=a_{\sigma_a}^{-1}$, $b_a^{p^{h_b}-1}=1$, and $w(b_a)=1$. For $q\equiv 1 \bmod 4$, $R=1$, $P_0F=0$ and $L=F$. Put $\eta_{\mathrm{ns}}=(-1)^F(\ell \bmod p)^{-1}\in\mathbb{F}_p^\times$. Moreover, its quadratic character is $(\eta_{\mathrm{ns}}/p)=(-1)^{P_0F+R+P_0L}=-1$. In $\mathbb{F}_{p^2}^\times$, we choose b_0 with $b_0^2=\eta_{\mathrm{ns}}$. Then, $b_0^p=-b_0$, $b_0^q=(-1)^F b_0$, and $N(b_a)=a_{\sigma_a}^{-1}$. Moreover, $b_a^{p^{h_b}-1}=(-1)^{k_a h_b}$ and $w(b_a)=(-1)^{Fk_a}$. The finite coefficient quotient records $k_a \bmod |b_0|$.

Given $a\in J$, $(E+N_A)t_a\equiv 0 \pmod 2$. Indeed, if $t_a=1$, each π_a-cycle $\mathcal{O}$ with odd e_i satisfies $|\mathcal{O}|\in 2\mathbb{Z}$, by Proposition 3.38. Thus, $d_j, d_j^0\in 2\mathbb{Z}$ for every j, and $n,r\in 2\mathbb{Z}$. Hence, $E=(n-r)/2=dA/2$. Since d is odd, A is even and $E\equiv A/2\equiv N_A \bmod 2$. If $t_a=0$, the relation is immediate. For $q\equiv 1 \bmod 4$, $R=0$, the unnormalized cyclic factor α_{cyc} and field correction α_{f} both equal 1. Moreover, $(-1)^{N_A L k_a t_b}$ equals 1. For $q\equiv 1 \bmod 4$, $R=1$, the unnormalized cyclic factor equals $(-1)^{\epsilon k_a h_b+EFk_a t_b}$ after substituting b_a. In particular, $b_a^{2E(p^{h_b}-1)}=1$. The field correction $(-1)^{R\epsilon k_a h_b}$ cancels $(-1)^{\epsilon k_a h_b}$. Since $L=F$ and $(E+N_A)t_b=0 \bmod 2$, $(-1)^{EFk_a t_b}$ equals $(-1)^{N_A L k_a t_b}$.

Finally, we let $q\equiv 3 \bmod 4$. Since $b_a^{p^{h_b}-1}=w(b_a)=1$, the raw cyclic factor $A_{K,E}(a,b)$ is 1. Thus, we obtain the quotient $(-1)^{Lk_a(\epsilon h_b+N_A t_b)}$. In C, we choose ι with $\iota^2=-1$. Evaluate $K=K^{p^{-h_b}\sigma_b}\delta_b^{2E}$ at ι. Then, $K(\iota)^2=(-1)^\epsilon$, $p^{-h_b}\equiv(-1)^{h_b} \pmod 4$, and $\ell^{k_b}\equiv(-1)^{Lk_b} \pmod 4$. Consequently, $\epsilon(h_b+Lk_b)+Et_b\equiv 0 \pmod 2$. Since $(E+N_A)t_b\equiv 0 \pmod 2$, $\epsilon h_b+N_A t_b\equiv \epsilon L k_b \pmod 2$. Hence, $(-1)^{\epsilon L k_a k_b}=\partial\mathcal{B}_{\epsilon L}(a,b)$. □

Corollary 3.41. *For odd q and each generic pair (ρ,τ) in* Proposition 3.46*, we let J be the joint stabilizer. Let α_g,α_l be Jordan and Fourier factors on J, with residual unitary factor in α_l. Moreover, for $z=(1,2f,1)$, there is a cochain c on J with $\alpha_g/\alpha_l=\partial c$, $c(z)=1$ and $c(az)=c(a)$ for $a\in J$.*

Proof. For $d\equiv 2 \bmod 4$, the cyclic-factor identity and the c_G identity give $c_{\mathrm{tor}}=c_{\Lambda_{\mathrm{b}}}c_G$. The full torus-block quotient is $\partial c_{\mathrm{tor}}$. If $d_j(\rho), d_j(\rho_0)$ are even for every j, put $c_{\mathrm{even}}(a)=\omega^{\Delta h_a}$. By the formulas for K,E, $\Delta\in 2\mathbb{Z}$. Thus, $\omega^{2f\Delta}=1$, and $c_{\mathrm{even}}(z)=1$, $c_{\mathrm{even}}(az)=c_{\mathrm{even}}(a)$. Otherwise, keep J for the original pair (ρ,τ). For each J-orbit, Lemma 3.34 compares factor classes on J with multiplicity 1. Moreover, the coordinate cochain c_{coord} has the period $2f$. If the wave front $\mathbf{d}_{\mathrm{int}}$ has only even parts, c_{coord} remains valid at that step. Write $c_{\mathrm{g},j}$ and $c_{0,k}$ for global and residual cochains. Put $c_{\mathrm{chain}}=(\prod_j c_{\mathrm{g},j})(\prod_k c_{0,k}^{-1})$. Thus, the endpoint factor is $\mathcal{B}$, a coboundary by Proposition 3.35.

If $4\mid d$, take $c_\Lambda c_G$. By Propositions 3.37 and 3.38, take $\mathcal{B}_c$, with coordinate and multiplicity-one cochains $c_{\mathrm{coord}}, c_{\mathrm{mult}}$ for two-hook comparison. If $2\nmid d$, Proposition 3.40 gives the comparison. Let c_{bin} and c_{comp} be binomial and compensator cochains. Given $a\in J$, both depend only on σ_a. In particular, for $\sigma_a=1$, $c_{\mathrm{bin}}(a)=c_{\mathrm{comp}}(a)=1$. For $z=(1,2f,1)$ and even e_i, $j_i(az)=j_i(a)-2 \pmod{e_i}$, so $s_i(az)=s_i(a)$. Hence, $c_G(z)=1$ and $c_G(az)=c_G(a)$. Given a pure determinant self-twist a, a colour cycle $\mathcal{O}$ contributes alternating $r(\xi)$ and $r(\xi)^{-1}$. If the inversion count $N_{\mathrm{inv}}(\mathcal{O})$ is odd, we have that $\xi=\xi^{-1}$ and $r(\xi)=1$. Thus, $c_G(a)=1$. The coordinate cochains of §3.4 have period $2f$. The multiplicity-one restriction φ_{mult} has the same ordinary matrix for v_{all}^2. Thus, the product satisfies $c(z)=1$ and $c(az)=c(a)$. Given $h_a+h_b=h_0+2fk$ with $0\le h_0<2f$, the local carry is $\tau(v_{\mathrm{all}}^2)^k$. Thus, after the chosen lift change, $\rho(v_{\mathrm{all}}^2)^k$ is the global carry. Let X denote determinant self-twists. In $\mathrm{Irr}(X)$, we choose ξ. Moreover, by Proposition 3.49, ξ can reindex the pairing Ω. □

Let $q=2^f$, let ℓ be odd and set $d=\mathrm{ord}_\ell(-q)$. With $G=\mathrm{GU}_n(q)$, we put $C=\mu_{q+1}$. Next, we let ϕ be entrywise squaring over $\mathbb{F}_{q^2}$. Keep $\phi^{2f}=1$ on G. However, the field period on $H\le G$ or a representation V may be smaller than

$2f$. Let $\delta \in \mathrm{Irr}(C)$, $\sigma \in \mathcal{H}_\ell$, and $h \in \mathbb{Z}$. Put $a = (\delta, \phi^h, \sigma)$ with determinant twist $\delta \circ \det$. Moreover, let J_χ stabilize $\chi \in \mathrm{Irr}(G)$ under the joint action. Over $E = \mathbb{Q}^{\mathrm{ab}}$, we let P_a be a σ_a-semilinear intertwiner for $a \in J_\chi$. Then, we define $\alpha(a,b) \in E^\times$ by $P_a P_b = \alpha(a,b) P_{ab}$. The mixed factor class is $[\alpha]$. Thus, we find $\alpha(a,b)\alpha(ab,c) = \sigma_a(\alpha(b,c))\alpha(a,bc)$.

Let F be the unitary Frobenius on $\mathfrak{g} = \mathfrak{gl}_n(\overline{\mathbb{F}}_2)$. Fix the grading $\mathfrak{g} = \bigoplus_j \mathfrak{g}(j)$ for the wave-front partition ω. Next, we choose $J \in \mathfrak{g}(-2)$ with Jordan-string coefficients 1. Moreover, give the Hermitian Gram matrix H antidiagonal entries 1. Put $U_j = \{u \in G : u - 1 \in \bigoplus_{k \geq j} \mathfrak{g}(k)\}$. Given $\beta \in \mathbb{F}_{q^2}$ with $\beta + \beta^q = 1$, the Springer map is $\mathcal{S}(u) = (u-1)(\beta u + \beta^q)^{-1}$. Given $u \in U_2$, we have that $\mathcal{S}(u)_2 = (u-1)_2$, since $\beta + \beta^q = 1$. Put $\theta(u) = (-1)^{\mathrm{Tr}_{q/2}(\mathrm{tr}(J(u-1)_2))}$. Thus, we obtain that $\theta(\phi(u)) = \theta(u)$ and $\sigma(\theta(u)) = \theta(u)$. Given $X, Y \in \mathfrak{g}(1)^F$, the commutator form $B(X,Y) = \mathrm{Tr}_{q/2}(\mathrm{tr}(J[X,Y]))$ is nondegenerate (Zhang [208, pp. 81–83]). For ω, the generalized Gelfand–Graev character is Γ_ω. Thus, the comparison requires $\langle \Gamma_\omega, \chi \rangle_G = 1$ for the ambient character χ.

Lemma 3.42. *Given $a \geq 1$, we let E_a consist of $(x_1, \ldots, x_a, z) \in \mathbb{F}_{q^2}^{a+1}$ with $z + z^q = x_1^{q+1}$, modulo translation of z by $\ker(\mathrm{Tr}_{q/2} : \mathbb{F}_q \to \mathbb{F}_2)$. Put $t(b) = b + b^q$, $\mathbf{x} = (x_1, \ldots, x_a)$, and $\mathbf{y} = (y_1, \ldots, y_a)$. Then, the multiplication is $(\mathbf{x}, z)(\mathbf{y}, z') = (\mathbf{x} + \mathbf{y}, z + z' + x_1 y_1^q + \sum_{j=1}^{a-1} t(x_{j+1} y_j))$. Let $\omega = (w_1, w_2, \ldots)$ be decreasing. Next, we put $\mathcal{P}_\omega = \{(i,j) : i < j,\ w_j > 0,\ w_i \not\equiv w_j \pmod 2\}$ and $a_{ij} = \min(w_i, w_j)$. For $\mathfrak{g}(1)^F \neq 0$, we have that $U_1/\ker\theta$ is the central product of $E_{a_{ij}}$ for $(i,j) \in \mathcal{P}_\omega$, with entrywise squaring. Moreover, let η_1 be the faithful irreducible character of E_1. Put $b = [\alpha_{\eta_1}]$ for the mixed factor class on $\langle \phi \rangle \times \mathcal{H}_\ell$. Thus, the source class is $b^{\epsilon(\omega)}$, where $b^2 = 1$ and $\epsilon(\omega) = \sum_{i<j}(1 - (w_i \bmod 2))(w_j \bmod 2) \pmod 2$. If $\mathfrak{g}(1)^F = 0$, the source class is 1.*

Proof. We write $u = 1 + X + Y$ modulo degree three, with $X \in \mathfrak{g}(1)$ and $Y \in \mathfrak{g}(2)$. Thus, unitarity gives $X^* = X$ and $Y + Y^* = X^2$. For a Jordan string s, let $Y_s(t)$ be the entry of Y from weight t to $t+2$. If either weight is absent, put $Y_s(t) = 0$. We define $z(Y) = \sum_s Y_s(-1) + \sum_s \sum_{t \geq 0} \big(Y_s(t) + Y_s(t)^q\big)$. When $X = 0$, $Y_s(-t-2) = Y_s(t)^q$, so $z(Y) = \mathrm{tr}(JY)$. Given $v \in \ker\theta$, $z((uv-1)_2) - z((u-1)_2) \in \ker \mathrm{Tr}_{q/2}$.

Choose $(i,j) \in \mathcal{P}_\omega$, and put $a = \min(w_i, w_j)$. Next, we restrict X to the strings s_i, s_j. If $0 \leq t < a$, let x_{t+1} be the entry from weight t to $t+1$. Moreover, the entry from $-t-1$ to $-t$ is x_{t+1}^q. These entries give X on s_i, s_j. Moreover, the middle-weight part of $Y + Y^*$ gives $z(Y) + z(Y)^q = x_1^{q+1}$. Let $u' = 1 + X' + Y'$ modulo degree three. Next, give X' coordinates y_j. The degree-two part of uu' is $Y + Y' + XX'$. Moreover, the path from -1 to 1 contributes $x_1 y_1^q$ to $z(XX')$. If $0 \leq t < a-1$, the path from t to $t+2$ contributes $x_{t+2} y_{t+1} + (x_{t+2} y_{t+1})^q$. Thus, multiplication gives the law of E_a. The Springer map makes $\pi_1 : U_1/\ker\theta \to \mathfrak{g}(1)^F$, $u \ker\theta \mapsto (u-1)_1$, surjective. Moreover, $\ker \pi_1 = U_2/\ker\theta \cong C_2$, detected by $z(Y)$ modulo $\ker \mathrm{Tr}_{q/2}$. Let $W_{ij} \leq \mathfrak{g}(1)^F$ be the coordinates on s_i, s_j, and put $P_{ij} = \pi_1^{-1}(W_{ij})$. Then, $u \ker\theta \mapsto (x_1, \ldots, x_a, z(Y) + \ker \mathrm{Tr}_{q/2})$ gives $P_{ij} \simeq E_a$. The spaces W_{ij} form a direct sum. For distinct $(i,j), (k,l) \in \mathcal{P}_\omega$, we choose $X \in W_{ij}$ and $X' \in W_{kl}$. Then, $\mathrm{tr}(JXX') = \mathrm{tr}(JX'X) = 0$, so $\mathrm{tr}(J[X,X']) = 0$. Put $Z_0 = U_2/\ker\theta$. Then, $[P_{ij}, P_{kl}] = 1$ and $P_{ij} \cap P_{kl} = Z_0$. The subgroups P_{ij} generate $U_1/\ker\theta$, with common center Z_0. Moreover, $\phi(P_{ij}) = P_{ij}$, with $\phi(x_t) = x_t^2$ and $\phi(z) = z^2$.

The group E_a has order $2q^{2a}$. Next, choose $Z(E_a) \simeq \mathbb{F}_2$. The square of $(\mathbf{x}, z) \in E_a$ is $Q_a(\mathbf{x}) = \mathrm{Tr}_{q/2}(x_1^{q+1}) + \mathrm{Tr}_{q^2/2}\left(\sum_{j=1}^{a-1} x_{j+1} x_j\right)$. Next, we put $Q_0 = 0$ on the zero space, and set $B_a(\mathbf{x}, \mathbf{y}) = Q_a(\mathbf{x} + \mathbf{y}) - Q_a(\mathbf{x}) - Q_a(\mathbf{y})$. If $a = 1$, B_1 is nondegenerate. Given $a \geq 2$, let $e_a = (0, \ldots, 0, 1)$ and $W_a = \mathbb{F}_{q^2} e_a$. Then, $B_a(te_a, \mathbf{y}) = \mathrm{Tr}_{q^2/2}(t y_{a-1})$. Moreover, $W_a^\perp = \{\mathbf{y} : y_{a-1} = 0\}$, and Q_a induces Q_{a-2} on $W_a^\perp / W_a$. Thus, B_a is nondegenerate and E_a is extraspecial. By [174, (4)], the nonlinear character η_a has $\eta_a(1) = q^a$ and $\eta_a(g) = 0$ for $g \notin Z(E_a)$. With $E_0 = C_2$, let η_0 be the nontrivial character of E_0.

Given $a \geq 2$, we put $A_a = \{(0, \ldots, 0, x_a, 0) : x_a \in \mathbb{F}_{q^2}\}$. Then, $|A_a| = q^2$, $A_a \cap Z(E_a) = 1$, and $\phi(A_a) = A_a$. Moreover, $\mathrm{C}_{E_a}(A_a)$ consists of the elements with $x_{a-1} = 0$. Projection gives $\mathrm{C}_{E_a}(A_a)/A_a = E_{a-2}$, with the same ϕ-action. Let V_a afford η_a. Thus, averaging η_a over A_a gives $\dim V_a^{A_a} = q^a/q^2$. Thus, $V_a^{A_a}$ affords η_{a-2}. Given any field or coefficient operation t, we let P_t be its semilinear intertwiner on V_a. Then, $P_t(V_a^{A_a}) = V_a^{A_a}$. If $P_t P_u = \alpha_a(t,u) P_{tu}$, the restrictions to $V_a^{A_a}$ have the same factor $\alpha_a(t,u)$. Moreover, $(P_\phi|_{V_a^{A_a}})^{2f} = (P_\phi^{2f})|_{V_a^{A_a}}$. Consequently, the factor class for E_a is b when a is odd and 1 when a is even.

With $\mathcal{E} = E_1 * E_1$, we let $\Delta \leq \mathcal{E}$ be the image of $\{(g,g) : g \in E_1\}$. Then, $|\Delta| = q^2$, $\Delta \cap Z(\mathcal{E}) = 1$, and $\phi(\Delta) = \Delta$. Let λ be the rational linear character of $\Delta Z(\mathcal{E})$ with $\lambda|_\Delta = 1$ and $\lambda|_{Z(\mathcal{E})} \neq 1$. Then, $\mathrm{Ind}_{\Delta Z(\mathcal{E})}^{\mathcal{E}} \lambda = \eta_1 \otimes \eta_1$, of degree q^2. On an induced function v, we define $(F_\phi v)(g) = v(\phi^{-1}(g))$ and $(I_\sigma v)(g) = \sigma(v(g))$. Since $\phi^{2f} = 1$ on $\mathcal{E}$, $F_\phi^{2f} = 1$ and $F_\phi I_\sigma = I_\sigma F_\phi$. Therefore, $b^2 = 1$. For $(i,j) \in \mathcal{P}_\omega$, $a_{ij} = w_j$. Since $w_i \not\equiv w_j \pmod 2$, w_j is odd exactly when w_i is even. Thus, $\sum_{(i,j) \in \mathcal{P}_\omega} a_{ij} \equiv \epsilon(\omega) \pmod 2$. For m zero parts, we put $\omega^+ = (\omega, 0^m)$. Then, $\epsilon(\omega^+) = \epsilon(\omega)$. □

Proposition 3.43. *Write $s = v_2(f)$, choose $\zeta_{2^{s+2}} = \exp(2\pi i/2^{s+2})$, and set $\alpha = \sqrt{2}\,\zeta_{2^{s+2}}$, with positive $\sqrt{2}$. The representation η_1 has a model over $\mathbb{Q}(\alpha)$ with an operator U for ϕ and $U^{2f} = 1$. If d is odd, we have that every $\sigma \in \mathcal{H}_\ell$ fixes $\mathbb{Q}(\alpha)$ pointwise.*

Proof. Let C be the smooth projective model of $y^2 + y = x^{q+1}$. By [98, Definition 2.1 and (2.6)], for $R(x) = x^q$ the map $(x_1, z) \mapsto (x_1^q, \sum_{i=0}^{f-1} z^{2^i})$ gives $E_1 \simeq H_R^{\mathrm{op}}$. Thus, (2.7) gives the left action $g \mapsto t_g$ on C. The formula is over $\mathbb{F}_2$, giving $F_C t_g = t_{\phi(g)} F_C$ for $F_C(x, y) = (x^2, y^2)$. Given an odd prime v, put $V = H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_2}, \mathbb{Q}_v)$ and $T = F_C^*$. By [98, Lemma 2.13], V affords η_1, and T^{2f} is scalar since H_R is defined over $\mathbb{F}_{q^2}$. By [47, Theorem 8.12], C is maximal over $\mathbb{F}_{q^2}$. Thus, $\operatorname{tr}(T^{2f}) = -q^2$, giving $T^{2f} = -qI_V$. On V, the action $\rho(g) = (t_g^{-1})^*$ satisfies $T^{-1}\rho(g)T = \rho(\phi(g))$.

Let $\mathcal{A}$ be the rational simple component for η_1. Then, $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}_v = \operatorname{End}_{\mathbb{Q}_v}(V)$. Given $g \in E_1$, we let N_g count the fixed points of $g^{-1}F_C$. Thus, the fixed-point formula gives $\operatorname{tr}(T\rho(g)) = 1 + 2 - N_g \in \mathbb{Z}$. Choose a rational basis $\rho(g_1), \ldots, \rho(g_m)$ of $\mathcal{A}$. The matrix $(\operatorname{tr}(\rho(g_i)\rho(g_j)))_{i,j}$ is nonsingular over $\mathbb{Q}$. Thus, the traces $\operatorname{tr}(T\rho(g_i)) \in \mathbb{Z}$ give $T \in \mathcal{A}$.

By [196, Theorem 4.1], E_1 is a central product of Q_8 and $f-1$ factors D_8. Hence, $\mathcal{A} = M_{q/2}((-1,-1)_{\mathbb{Q}})$. If $s = 0$, $\alpha = \sqrt{-2}$. For quaternion generators i_0, j_0, $(i_0 + j_0)^2 = -2$, so $\alpha \mapsto i_0 + j_0$ embeds $\mathbb{Q}(\alpha)$ in $(-1,-1)_{\mathbb{Q}}$. If $s \geq 1$, $\alpha^2/2 = \zeta_{2^{s+1}}$, so $i \in \mathbb{Q}(\alpha)$. In either case $\mathbb{Q}(\alpha)$ splits $\mathcal{A}$. Thus, we have that η_1 and T have matrices over $\mathbb{Q}(\alpha)$. Since $\alpha^{2f} = -q$, $U = \alpha T^{-1}$ satisfies $U^{2f} = 1$. If $d = \operatorname{ord}_\ell(-q)$ is odd, $\operatorname{ord}_\ell(q) = 2d$. Thus, Lemma 7.291 with $p = 2$ gives $\alpha \in \mathbb{Q}_\ell$, fixed by $\mathcal{H}_\ell$. For $\sigma \in \mathcal{H}_\ell$ and $w \in \mathbb{Q}(\alpha)^q$, entrywise σ satisfies $\sigma(Uw) = U\sigma(w)$. □

Lemma 3.44. *Let $\mathfrak{p}$ be a unitary label, and let J be its stabilizer under determinant twists, ϕ, and $\mathcal{H}_\ell$. Next, we choose a union of J-orbits of colours with common partition λ. Choose a partition $\mu \subseteq \lambda$ with domino λ/μ. Then, let $\mathfrak{p}'$ be the new label, with characters χ, χ' for $\mathfrak{p}, \mathfrak{p}'$. On J, the mixed factors of χ, χ' agree up to intertwiners. The ϕ-operators $\Phi_\chi, \Phi_{\chi'}$ have powers $\Phi_\chi^{2f} = c_\chi I$ and $\Phi_{\chi'}^{2f} = c_{\chi'} I$, with $c_\chi = c_{\chi'}$. This holds on $J_0 \leq J$, even if $\mathfrak{p}'$ has additional stabilizers.*

Proof. Let $\mathcal{O}$ be the selected eigenvalues in $\overline{\mathbb{F}}_2$, including each selected $(-q)$-orbit. We put $l = |\mathcal{O}|$. Moreover, every eigenvalue in $\mathcal{O}$ has multiplicity at least 2. On an odd orbit $\mathcal{C} \subseteq \mathcal{O}$, we choose a hyperbolic plane $H_{\mathcal{C}}$ over its Hermitian extension field $k_{\mathcal{C}}$. Next, order its isotropic lines $D_{\mathcal{C}}, D'_{\mathcal{C}}$. Given an even orbit $\mathcal{C} \subseteq \mathcal{O}$, the primary spaces are $V, V^\vee$. Choose dual bases e_i, f_i, and take $Y = \langle e_1, f_2 \rangle$ and $Y' = \langle f_1, e_2 \rangle$ over the primary field. Moreover, both Y, Y' are isotropic, and $Y' \cong Y^\vee$. The trace forms for $\mathcal{O}$ give isotropic spaces X, X' with $\dim X = \dim X' = l$. Moreover, $V_0 = (X \oplus X')^\perp$ is nondegenerate of dimension $m = n - 2l$. Let P stabilize X. A rational Levi of P is $L = \mathrm{GL}_l(q^2) \times \mathrm{GU}_m(q)$. If $m = 0$, only $\mathrm{GL}_l(q^2)$ occurs.

Place one occurrence of each $\xi \in \mathcal{O}$ on X, and take μ for selected colours in $\mathrm{GU}_m(q)$. The dual element for $\mathrm{GL}_l(q^2)$ is regular semisimple. Since $\mathcal{O}$ is J-stable, both Levi labels are J-stable. On an odd orbit $\mathcal{C}$, unipotent restriction has coefficient ± 1 at μ [62, Lemma 4.10]. On an even orbit $\mathcal{C}$, the two boxes of λ/μ have one admissible order. If λ/μ is a row, the order is (b, b'), with b rightmost. If λ/μ is a column, the order is (b, b'), with b bottommost. For unselected $\mathcal{C}$, the coefficient is 1. Next, let ϑ_1 be the regular character with label $\mathcal{O}$ on $\mathrm{GL}_l(q^2)$, and put $\vartheta = \vartheta_1 \otimes \chi'$. Let R_L^G be Harish–Chandra induction. Thus, for χ, Jordan decomposition yields $\langle R_L^G(\vartheta), \chi \rangle_G = 1$, including $q = 2$ [72, Theorems 3.3.7 and 4.7.2] as in [208, (5.2)].

With $L_1 = \mathrm{GL}_l(q^2)$, let $U_{\mathrm{GL}} \leq L_1$ be upper unitriangular. Next, for $u \in U_{\mathrm{GL}}$, we put $\psi(u) = (-1)^{\mathrm{Tr}_{q^2/2}(\sum_i u_{i,i+1})}$. The character ψ defines the Gelfand–Graev representation Γ_ψ of L_1. Moreover, the character ϑ_1 occurs in Γ_ψ with multiplicity 1. On its induced functions v, put $(M_\delta v)(g) = \delta((\det g)^{1-q}) v(g)$, $(\mathsf{F} v)(g) = v(\phi^{-1}(g))$, and $(I_\sigma v)(g) = \sigma(v(g))$. Both ϕ and σ fix ψ, so F and I_σ act on the source model. Moreover, $\mathsf{F} M_\delta \mathsf{F}^{-1} = M_{\delta \circ (\cdot)^{2^{2f-1}}}$ and $I_\sigma M_\delta I_\sigma^{-1} = M_{\sigma \circ \delta}$. Finally, $\mathsf{F}^{2f} = 1$.

Let W be induced from P, with inflated models of $\vartheta_1 \otimes \chi'$. On $X \oplus V_0 \oplus X'$, choose the Gram matrix $J_H(x, v, x') = (x', v, x)$. Then, J_H is defined over $\mathbb{F}_2$, and $\det J_H = 1$. Let $B : (\mathbb{F}_{q^2}^n, J_H) \to (\mathbb{F}_{q^2}^n, H)$ be an isometry. Then, $\det B \in \mu_{q+1}$. In $\mathrm{GU}_n(q)$, we choose A with $\det A = (\det B)^{-1}$, and replace B by AB. Thus, $\det B = 1$. Put $g_h = B(B^{(2^h)})^{-1}$. Then, $\det g_h = 1$, $g_h \phi^h(g_k) = g_{h+k}$, and $g_{2f} = 1$. In particular, $\delta(\det g_h) = 1$ for each determinant twist δ. Thus, the operators on W give the mixed factors of χ'. Each $j \in J$ acts on the χ-isotypic subspace of W, where χ has multiplicity 1. Thus, the operators on χ have the factors of χ'.

Let P_j and P'_j be intertwiners on models of χ. Put $z = (1, 2f, 1)$. If $P'_j = c(j) P_j$, the common period operator gives $P'_{jz} = c(j) P_{jz}$. In the standard field group, $P_z = P'_z = 1$. Given a carried field lift $u = t\phi$ with $t \in \mathrm{SU}$, covariance

with t gives the ordinary matrix of $t\phi(t)\cdots\phi^{2f-1}(t)=u^{2f}$, with the same $c(j)$. Thus, the claim holds on J at each step $\lambda\to\mu$. □

Proposition 3.45. *Let $\widetilde{M}$ be the generic torus normalizer at ℓ. Next, match global labels λ_i with residual labels μ_i by core and quotient. Let (χ,τ) be a matched pair. Let J be its joint stabilizer, and let $\alpha_{\mathrm{glob}},\alpha_{\mathrm{loc}}$ be the semilinear factors on J. Then, $[\alpha_{\mathrm{glob}}]=[\alpha_{\mathrm{loc}}]$. If $2\nmid d$, we have that both classes are 1. Moreover, we have that u^{2f} acts as before.*

Proof. For ω, let S_ω be the source and Γ_ω the generalized Gelfand–Graev representation. Induce ϕ- and $\mathcal{H}_\ell$-intertwiners from S_ω to Γ_ω. Moreover, for $\delta\in\mathrm{Irr}(\mu_{q+1})$, let D_δ act on an induced function v by $(D_\delta v)(g)=\delta(\det g)v(g)$. Then, $D_\delta D_{\delta'}=D_{\delta\delta'}$. Let F_h and I_σ be the operators for ϕ^h and $\sigma\in\mathcal{H}_\ell$. Thus, their conjugation gives $F_hD_\delta F_h^{-1}=D_{\delta\circ(\cdot)^{2^{-h}}}$ and $I_\sigma D_\delta I_\sigma^{-1}=D_{\sigma\circ\delta}$, where 2^{-h} is taken modulo $q+1$. The source is ϕ-stable, and its central character θ is rational. The centre of GL_m is connected. Moreover, in type A, all primes are acceptable [188, Lemma 6.3(ii) and Corollary 15.7]. Given $\chi\in\mathrm{Irr}(G)$ with wave front ω, $\langle\Gamma_\omega,\chi\rangle=1$. Thus, the χ-summand has the same class on J. Define $\pi:J\to\langle\phi\rangle\times\mathcal{H}_\ell$ by $\pi(\delta,\phi^h,\sigma)=(\phi^h,\sigma)$. Consequently, by Lemma 3.42, $[\alpha_{\mathrm{glob}}]=\pi^*(b^{\epsilon(\omega)})$.

Let T be a local block torus. Then, $|T|=q^d-(-1)^d$. Given $t\in T$, we let T_t act on the block. Choose v,u with $vT_tv^{-1}=T_{t^{-q}}$ and $uT_tu^{-1}=T_{t^2}$. Moreover, $uv=vu$, $v^d=1$, and $u^{2f}=v^2$. If $2\nmid d$, we let $N(x)=x^{q^d+1}$ on $\mathbb{F}_{q^{2d}}^\times$, and take $T=\ker N$. Moreover, let T act by $T_t(x)=tx$ on the Hermitian trace space $\mathbb{F}_{q^{2d}}$. Set $v(x)=x^{q^{d+1}}$ and $u(x)=x^2$. If $2\mid d$, set $T=\mathbb{F}_{q^d}^\times$, acting on the paired space $\mathbb{F}_{q^d}\oplus\mathbb{F}_{q^d}$ with its Hermitian trace form by $T_t(x,y)=(tx,t^{-q}y)$. Next, we set $v(x,y)=(y,x^{q^2})$ and $u(x,y)=(x^2,y^2)$. Thus, T_t,u,v satisfy the displayed relations. In characteristic 2, the normal-basis permutations for v and block permutations have determinant 1. Next, in standard Hermitian coordinates, write $u=A_u\phi$ and $a_u=\det A_u\in\mu_{q+1}$. A unitary basis change of determinant c replaces a_u by $a_uc^{-1}c^2$. We choose $c=a_u^{-1}$. Thus, the new linear part of u has determinant 1, and the conjugates satisfy $(u^g)^{2f}=(v^g)^2$ for a common conjugator g.

For a colour, let $\kappa=\zeta_d^j$ be the Fourier rotation scalar. Choose γ with $\gamma^{2f}=\kappa^2$. If d is odd or $4\mid d$, we choose $\gamma\in\mu_{\ell-1}$ with $\gamma^f=\kappa$. Put $h=\mathrm{ord}_\ell(2)$. Given any prime $s\mid d$, we have that $v_s(\mathrm{ord}_\ell(q))=v_s(d)$. Since $\mathrm{ord}_\ell(q)=h/\gcd(h,f)$, $v_s(\ell-1)\geq v_s(d)+v_s(f)$. Thus, κ has an f-th root in $\mu_{\ell-1}$. If $d=2d_0$ with d_0 odd, $\mathrm{ord}_\ell(q)=d_0$, and the order of κ^2 divides d_0. The same valuation argument gives $\gamma\in\mu_{\ell-1}$ of odd order with $\gamma^{2f}=\kappa^2$. Moreover, for every $\sigma\in\mathcal{H}_\ell$, $\sigma(\gamma)=\gamma$ and $\sigma(\kappa)=\kappa$. The Fourier cycle permutations P_{cyc}, tensor permutations P_{tens}, and Specht matrices S_λ are rational. Their factor class is 1. Moreover, the determinant twists have operators D_δ. Let ω_0 be the residual wave front. Then, $[\alpha_{\mathrm{loc}}]=\pi^*(b^{\epsilon(\omega_0)})$. If $2\nmid d$, Proposition 3.43 gives $b=1$.

If $2\mid d$, we apply Lemma 3.44 on J to each orbit $\mathcal{C}$ with one partition. At each step $\lambda_i\to\lambda_i^-$, $|\lambda_i^-|=|\lambda_i|-2$, so the process ends at $\mathrm{core}_2(\lambda_i)$. For active odd $e_i\mid d$, λ_i loses d/e_i-hooks on passing to μ_i. Since d/e_i is even, each bead keeps its 2-abacus runner. Thus, $\mathrm{core}_2(\lambda_i)=\mathrm{core}_2(\mu_i)$. For inactive i, we have that $\lambda_i=\mu_i$. Let $\bar\lambda_i=\mathrm{core}_2(\lambda_i)$ and $\bar\mu_i=\mathrm{core}_2(\mu_i)$. Thus, $\bar\lambda_i=\bar\mu_i$ for odd e_i. Let $\bar\omega,\bar\omega_0$ be the reduced wave fronts. Given $k\geq1$, pad partitions by zeros and set $\bar\omega_k=\sum_i e_i(\bar\lambda_i^{\mathrm{t}})_k$ and $\bar\omega_{0,k}=\sum_i e_i(\bar\mu_i^{\mathrm{t}})_k$. Hence, $\bar\omega_k\equiv\bar\omega_{0,k}\pmod 2$ for every k, so $\epsilon(\bar\omega)=\epsilon(\bar\omega_0)$. Hence, by Lemma 3.42, $[\alpha_{\mathrm{glob}}]=[\alpha_{\mathrm{loc}}]$. Let ρ_χ,ρ_τ afford χ,τ, and let U_χ,U_τ be field operators. Moreover, $U_\chi^{2f}=\rho_\chi(u^{2f})$ and $U_\tau^{2f}=\rho_\tau(u^{2f})$. □

With $q=p^f$ and p prime, we let $\ell\neq p$ be odd. Write $\widetilde{G}=\mathrm{GU}_n(q)$, $G=\mathrm{SU}_n(q)$ and $C=\mu_{q+1}$. Moreover, put $d=\mathrm{ord}_\ell(-q)$, $\quad n=Ad+r$, $\quad 0\leq r<d$. Let e_i,λ_i record lengths and partitions for $(-q)$-orbits. If $g_i=(d,e_i)$, we put $m_i=d/g_i$, $b_i=e_i/g_i$. Let κ_i be the m_i-core and $(\mu_{ij})_j$ the m_i-quotient of λ_i, of total size w_i. Let $\widetilde{M}$ be the generic torus normalizer.

Proposition 3.46. *If $d>1$, the core and quotient labels index $\mathrm{Irr}_{\ell'}(\widetilde{G})$ and $\mathrm{Irr}_{\ell'}(\widetilde{M})$. Their restrictions cover $\mathrm{Irr}_{\ell'}(G)$ and $\mathrm{Irr}_{\ell'}(M)$, where $M=\widetilde{M}\cap G$. Suppose that $d=1$ and $n\geq2$. Let $\mathcal{P}_1$ be parents with occupied label degree 1. Thus, their restrictions cover $\mathrm{Irr}_{\ell'}(G)$. All parents on $C\wr\mathfrak{S}_n$ give the corresponding coverage on M. Matched parents ρ,τ have equal central characters on $\mathrm{Z}(\widetilde{G})$.*

Proof. For $d>1$, [33, Proposition 6.3 and Theorem 6.1] give the ℓ'-degree parametrizations and restriction coverage, since $\ell\nmid(q+1)$. Lemma 3.2 gives the degree cut in (κ_i,μ_{ij}). A residual hook of length m_i requires rank $e_im_i=db_i>r$, so residual partitions are m_i-cores. On a full torus block T, the determinant is the signed norm $a\mapsto a^{\sum_{j=0}^{d-1}(-q)^j}$. Thus,

$\det(T) = C$ and $\widetilde{G}/G \cong \widetilde{M}/M \cong C$. Write $D_\rho, D_\tau \le \mathrm{Irr}(C)$ for self-twist groups. Thus, $D_\rho = D_\tau$ by determinant equivariance. Hence, for $t = |D_\rho|$, the constituent degrees are $\rho(1)/t, \tau(1)/t$ by the cyclic-quotient formulas.

If $d = 1$ and $n \ge 2$, [33, Notation 6.2 and Theorem 6.1] give ambient coverage with s in the dual diagonal torus C^n. Thus, the occupied orbits have length 1. The wreath product $C \wr \mathfrak{S}_n$ gives all local parents τ. Moreover, λ_i and λ_i^{t} have the same hook lengths. With $t = |D_\rho| = |D_\tau|$, the degree formulas give $v_\ell(\rho(1)/t) = v_\ell(\tau(1)/t)$. By [33, Theorem 6.1], $\mathrm{Z}(\widetilde{G})$ acts on ρ, τ by the common torus character. Each $z \in C$ has signed norm z^{d/e_i} on each active orbit i. Thus, the local scalar at z is $\beta_i(z)^{|\kappa_i|+(d/e_i)w_i} = \beta_i(z)^{|\lambda_i|}$, where β_i is the orbit character on C. For inactive i, $\lambda_i = \kappa_i$. Hence, the two scalars agree on $\mathrm{Z}(\widetilde{G})$. □

Fix a parent pair (ρ, τ) in Proposition 3.46. Next, we let ϕ_p be entrywise p-power, and let u be a lift of ϕ_p that normalizes $\widetilde{M}$. The operator ϕ_p^f is the graph operation, and $|\phi_p| = 2f$. Put $N = |u|$. Next, we let v_{all} be the product of rotations v_i on the bulk. Put $a_0 = u^{2f} = v_{\mathrm{all}}^2 \in M$. Similarly, for $h \in \mathbb{Z}$, take powers u^h. Let $\delta \in \mathrm{Irr}(C)$ and $\sigma \in \mathcal{H}_\ell$. The triple (δ, h, σ) acts on labels by p^{-h}, coefficient action, and the twist δ. Thus, multiplication is $(\delta, h, \sigma)(\delta', h', \sigma') = (\delta\,(\delta')^{p^{-h}\sigma}, h + h', \sigma\sigma')$. Let K be a finite cyclotomic field. Moreover, K contains coefficients and intertwiner roots for (ρ, τ). Choose B divisible by the exponent periods. Set $K' = K(\mu_{\ell^B - 1})$. Moreover, for $\sigma \in \mathcal{H}_\ell$, we write $\sigma(\zeta) = \zeta^{\ell^{a_\sigma}}$ on μ_{ℓ^B-1}, where $a_\sigma \in \mathbb{Z}/B\mathbb{Z}$. Take the finite quotient $\mathcal{H}_\ell|_{K'}$ for the coefficient actions, and let J stabilize both labels. Next, choose representatives θ_i of the $(-q)$-orbits. Given $j \in J$, we define π_j and $s_i(j) \bmod e_i$ by $\theta_i^{p^{-h_j}\sigma_j}(\delta_j \circ \det_{e_i}) = \theta_{\pi_j(i)}^{(-q)^{s_i(j)}}$. Since $-q$ fixes determinant labels, $s_i(jj') = s_i(j') + s_{\pi_{j'}(i)}(j) \bmod e_i$. In particular, we have that $z = (1, 2f, 1)$ is central in the label group, $\pi_z = 1$, and $s_i(z) = -2 \bmod e_i$. On both modules, we have that z acts through a_0.

Lemma 3.47. *Let P_j, Q_j be normalized σ_j-semilinear intertwiners for ρ, τ on J, with common group action and* det*-covariance. Suppose that $P_z = \rho(a_0)$, $P_{jz} = P_j\rho(a_0)$, $Q_z = \tau(a_0)$, and $Q_{jz} = Q_j\tau(a_0)$. Let α_P, α_Q be their semilinear factors. Suppose that $\alpha_P(j, j')/\alpha_Q(j, j') = c(j)\sigma_j(c(j'))/c(jj')$, with $c(1) = c(z) = 1$ and $c(jz) = c(j)$. Then, the factors agree on J, with the same identities on every $j\langle z\rangle$.*

Proof. Put $P'_j = c(j)^{-1}P_j$. Since P_j is σ_j-semilinear, $P'_jP'_{j'} = \alpha_Q(j, j')P'_{jj'}$ and $P'_z = \rho(a_0)$, $P'_{jz} = P'_j\rho(a_0)$. Let V, W afford ρ, τ. Then, $\Psi_j(F) = P'_jFQ_j^{-1}$ defines a σ_j-semilinear map on $\mathrm{Hom}(W, V)$, with $\Psi_j\Psi_{j'} = \Psi_{jj'}$. The determinant scalars cancel on $\widetilde{G} \times_C \widetilde{M}$. Put $k = ((a_0^{-1}, a_0^{-1}), z)$. Then, k acts trivially, so $(kj) \cdot F = j \cdot F$ for $j \in J$. □

Let $c = \prod_i c_i$, where $c_i : J \to K'^\times$ come from the scalar corrections. Then, $c(z) = 1$ and $c(jz) = c(j)$. For the cochain $c_\Lambda(j) = \sigma_j(\Lambda(b_{\sigma_j}))$, the value depends only on σ_j, and $c_\Lambda(1) = 1$. Thus, we obtain that $c_\Lambda(z) = 1$ and $c_\Lambda(jz) = c_\Lambda(j)$. The binomial signs $c_{\mathrm{bin}}(j)$ depend on $h_j \bmod 4$, while the geometric-sum signs $c_{\mathrm{geo}}(j)$ depend on periods dividing B. Alternatively, for odd p, put $c_{\mathrm{G}}(j) = \prod_{i:e_i \text{ even}} r(\xi_{\pi_j(i)})^{-n_i(s_i(j) \bmod 2)}$. The character ξ_i is restricted to $\mathbb{F}_q^\times$, and n_i is invariant. If $2 \mid e_i$, we have that the parity of $s_i(j) \bmod e_i$ is well defined. Since $s_i(z) = -2 \bmod e_i$, we deduce that $c_{\mathrm{G}}(z) = 1$ and $c_{\mathrm{G}}(jz) = c_{\mathrm{G}}(j)$.

Given an integer t, we put $c_{\mathrm{root}}(j) = \omega^{2th_j}$, where $|\omega| = 2^{v_2(f)+2}$. Then, we obtain that $\omega^{2f} = -1$, so $c_{\mathrm{root}}(z) = 1$ and $c_{\mathrm{root}}(jz) = c_{\mathrm{root}}(j)$. Indeed, the exponent is $2t$. The scalar enforcing $P_z = \rho(a_0)$ is already part of P_z. The odd-order correction $c_{\det}(j)$ depends only on δ_j. Suppose that $q \equiv 3 \bmod 4$, and choose $h_j \ge 0$ modulo N. The other coordinate factor is $\eta_j = Z^{-(p^{h_j}-1)/2}$, where $|Z| \mid 2^{v_2(q+1)}$. Since $v_2(q^2-1) = 1 + v_2(q+1)$, we deduce that $\eta_{jz}/\eta_j = Z^{-p^{h_j}(q^2-1)/2} = 1$, and $\eta_z = 1$. Thus, the indicated product c satisfies $c(z) = 1$ and $c(jz) = c(j)$. Let V_1, V_2 be ordinary induced models. Let $\iota : V_1 \to V_2$ satisfy $\iota\rho_1(a_0) = \rho_2(a_0)\iota$. If P_j, Q_j satisfy the inner-power identities at a_0 in Lemma 3.47, their comparison scalar c_ι satisfies $c_\iota(z) = 1$ and $c_\iota(jz) = c_\iota(j)$.

Lemma 3.48. *Let (ρ, τ) be as in* Proposition 3.46. *Moreover, for $j = (\delta, h, \sigma) \in J$, the modules V_ρ, V_τ admit σ-semilinear intertwiners P_j, Q_j. For $x \in \widetilde{G}$, we have that $P_j\rho(x)P_j^{-1} = \delta(\det(u^hxu^{-h}))^{-1}\rho(u^hxu^{-h})$. Given $y \in \widetilde{M}$, $Q_j\tau(y)Q_j^{-1} = \delta(\det(u^hyu^{-h}))^{-1}\tau(u^hyu^{-h})$. Thus, we find that the factors α_P, α_Q satisfy $\alpha_P = \alpha_Q$ on $J \times J$. Moreover, $P_z = \rho(a_0)$ and $P_{jz} = P_j\rho(a_0)$. Moreover, we have that $Q_z = \tau(a_0)$ and $Q_{jz} = Q_j\tau(a_0)$, as in* Lemma 3.47. *Both depend on $h \bmod N$, where $|u| = N$.*

Proof. Choose the finite field K' and ordered composition $C_\delta A_u^h I_\sigma$ on (ρ, τ). Given an active colour i, we write $N_i = N_{d/e_i} : T_d \to T_{e_i}$. Then, $N_i(t^{p^h}) = N_i(t)^{p^h}$ and $\det_{e_i} \circ N_i = \det_d$. Given $\theta_i \in \mathrm{Irr}(T_{e_i})$, $\sigma(\theta_i(N_i(t))) = (\sigma\theta_i)(N_i(t))$. Next, put $\Theta_i = \theta_i \circ N_i$. Given $j \in J$, the transported weight Θ_i^j is $\theta_i^j \circ N_{\pi_j(i)}$, with λ_i and its ordered quotient

$(\mu_{ij})_j$. The correction $\mathcal{R}_i$ in Section 3.3 and the sign ε_i in Proposition 3.39 give the local δ-action. Moreover, when $p=2$, we have that $\mathcal{R}_i=\varepsilon_i=1$. The core κ_i and quotient $(\mu_{ij})_j$ determine λ_i. Hence, the stabilizers J_ρ, J_τ satisfy $J_\rho=J_\tau=J$. A residual label ρ_0 or label $\mathfrak{p}'$ in the reduction may have stabilizer $J_0>J$. Moreover, the intertwiners P_j, Q_j are indexed by $j\in J$.

Let ω be the wave-front class of ρ. On the generalized Gelfand–Graev model Γ_ω, let $R(x)$ be right translation by x. Then, $I_\sigma R(x) I_\sigma^{-1}=R(x)$ and $A_u^h R(x) A_u^{-h}=R(u^h x u^{-h})$. The determinant multiplier C_δ satisfies $C_\delta R(x) C_\delta^{-1}=\delta(\det x)^{-1}R(x)$. Since GL_n has connected centre and adjoint type A, [188, Corollary 15.7] gives $\langle\Gamma_\omega,\rho\rangle=1$. Put $T_j=C_\delta A_u^h I_\sigma$. Then, $T_j\Gamma_\omega[\rho]=\Gamma_\omega[\rho^j]$. On the local torus T_d and its rotation v_i, Fourier matrices F_i have equal covariance. Moreover, for blocks V_i, V_k of equal dimension, the tensor permutation Π_{ik} gives the relation when $\dim V_i$ is even or $p=2$. For odd p,d, the sign of the transposition $\varepsilon_{\mathrm{swap}}$ and assignment factor a_{swap} in Proposition 3.39 give covariance on s_i^{blk}, the adjacent transpositions. The torus factors T_i, rotations v_i, transpositions s_i^{blk}, and residual group M_0 generate $\widetilde{M}$.

Next, we write $u=bu_J$, and let L_u, L_J be their linear parts. Since $\det L_u=\det L_J=1$, $b\in\mathrm{SU}_n(q)$. If $2\nmid p$, set $A_J=\omega^{\epsilon_\rho}A_{u_J}$ on the J-orbit in Section 3.3, with $A_J^{2f}=R(u_J^{2f})$ and $u_J^{2f}=c_{-1}$ in a Jordan basis $\mathcal{B}_J$. Moreover, for $p=2$, the graded field operator u_J has order $2f$. Let $H=U_1/\ker\theta$, and let the u_J-invariant source $S\in\mathrm{Irr}(H)$ be as above. Next, let E be a cyclotomic splitting field. Extend S to its cyclic overgroup $H\langle u_J\rangle$ over E. Moreover, the operator A_J satisfies $A_J^{2f}=1$. By Proposition 3.43, we obtain A_J on the basic source S_1, affording η_1.

In either case, we set $A_u=R(b)A_J$. Moreover, for $k\geq 1$, covariance with R gives $A_u^k=R\Big(\prod_{t=0}^{k-1}u_J^t b u_J^{-t}\Big)A_J^k=R(u^k u_J^{-k})A_J^k$, with factors $u_J^t b u_J^{-t}$ ordered by increasing t. At $k=2f$, $A_J^{2f}=R(u_J^{2f})$. Next, we restrict A_u^{2f} to $\Gamma_\omega[\rho]$. Thus, $P_z=\rho(u^{2f})=\rho(a_0)$. Put $g_t=u_J^t b u_J^{-t}$. Then, $\det g_t=1$, and $C_\delta R(g_t)=R(g_t)C_\delta$. The Fourier space $\mathcal{F}_i$ has dimension e_i. Let P_e cycle the weights $\mathcal{W}_i$, and let U_i be its field operator. Then, $U_i^{2f}=\eta_{\mathrm{loc}}^{2f}P_e^2=\kappa^2P_e^2=R(v)^2$. Let W_0 be the residual module. Moreover, on W_{wr}, tensor the operators U_i. On W_0, take its field operator A_0. Thus, $Q_z=\tau(a_0)$. If $j=(\delta,h,\sigma)$, we write $h=h_0+2fk$ with $0\leq h_0<2f$ and set $j_0=(\delta,h_0,\sigma)$. Since $ua_0=a_0u$ and $\det a_0=1$, covariance gives $P_j\rho(a_0)=\rho(a_0)P_j$ and $Q_j\tau(a_0)=\tau(a_0)Q_j$. Hence, $P_j=P_{j_0}\rho(a_0)^k$ and $Q_j=Q_{j_0}\tau(a_0)^k$. The identities hold for negative k by inversion. Since $a_0^{N/(2f)}=1$, both formulas depend only on h mod N.

If $2\nmid q$, apply Corollary 3.41 on J, including the residual character τ_0. Furthermore, the corollary gives $\alpha_P/\alpha_Q=\partial c$. Replace P_j by $c(j)^{-1}P_j$. Then, $\alpha_P=\alpha_Q$, while $c(z)=1$ and $c(jz)=c(j)$ give the same operators. If $2\mid q$, we apply Proposition 3.45. Moreover, for $a\geq 2$, choose V_a, A_a from Lemma 3.42. Let the line $\mathcal{I}_a(j)$ consist of j-intertwiners on V_a. Let the line $\mathcal{I}_{a-2}(j)$ consist of j-intertwiners on $V_a^{A_a}$. Next, for $P\in\mathcal{I}_a(j)$, put $r_j(P)=P|_{V_a^{A_a}}$. The map $r_j:\mathcal{I}_a(j)\to\mathcal{I}_{a-2}(j)$ is bijective. Indeed, $\dim\mathcal{I}_a(j)=\dim\mathcal{I}_{a-2}(j)=1$. Moreover, nonzero P gives invertible $r_j(P)$. Given $P_j\in\mathcal{I}_a(j)$ and $P_k\in\mathcal{I}_a(k)$, $r_{jk}(P_jP_k)=r_j(P_j)r_k(P_k)$. Thus, the factors α_a, α_{a-2} on $V_a, V_a^{A_a}$ agree. For $\mathcal{E}=E_1*E_1$, we let the diagonal subgroup Δ and character λ be as in Lemma 3.42. Then, $\mathrm{Ind}_{\Delta Z(\mathcal{E})}^{\mathcal{E}}\lambda=\eta_1\otimes\eta_1$. The field operator F_ϕ permutes cosets, and $F_\phi^{2f}=1$. At odd d, the unpaired source S_1 has field operator U and coefficient operators I_σ over $\mathbb{Q}(\alpha)$, fixed by $\mathcal{H}_\ell$, by Proposition 3.43.

If $2\mid d$, Lemma 3.44 takes both labels to their 2-cores on J. Let $m_{\mathrm{g}}, m_{\mathrm{loc}}$ count the unpaired factors E_1. Then, $m_{\mathrm{g}}\equiv m_{\mathrm{loc}}\pmod 2$. Thus, both sides have E_1 with multiplicity $\epsilon\in\{0,1\}$. Choose identical E_1-intertwiners when $\epsilon=1$. Given any ordinary model isomorphism ι, the period operators A_z, A_z' satisfy $\iota A_z=A_z'\iota$. Thus, the scalar cochain c satisfies $c(z)=1$ and $c(jz)=c(j)$. Hence, the relative factor quotient is ∂c, even when the coefficient part $\alpha_{\eta_1}|_{\mathcal{H}_\ell\times\mathcal{H}_\ell}\neq 1$.

If $d=1$, every occupied label has degree 1, $r=0$, and the 1-quotients are the original partitions λ_i. Moreover, the odd-d comparison holds for (ρ,τ) without a degree cut. If $p=2$, the construction covers (ρ,τ) without a degree cut. Hence, parents (ρ,τ) of Proposition 3.46 are included, even if $\ell\mid\rho(1)$. Lemma 3.47 gives the same normalization for $p=2$. Next, let $\mathfrak{p}$ be the label of ρ. The construction takes $\sigma(\mathfrak{p})$ and the exponent of σ on roots of order prime to ℓ. The field K' records $\sigma(\mathfrak{p})$ and the exponent on μ_{ℓ^B-1}. Thus, each $\sigma\in\mathcal{H}_\ell$ acts by the comparison above. □

Proposition 3.49. *Let $n\geq 2$, and let $S=\mathrm{PSU}_n(q)$ be simple. For the universal cover $\widehat{S}$, $\mathrm{iMN}(S,\ell)$ holds for each prime $\ell\mid|S|$.*

Proof. If $n=2$, then $\mathrm{PSU}_2(q)\cong\mathrm{PSL}_2(q)$, and [186, Theorem C] gives $\mathrm{iMN}(S,\ell)$. At $\ell=p$, apply [162, Theorem 1.3] and [101, Theorem 1.1]. For exceptional $\mathrm{Mult}(S)$, apply [157, Lemma 4.1(2)]. If $\ell=2$ and q is odd, apply [159, Theorem A(3)]. If $n=\ell=3$ and $q\equiv 2,5$ mod 9, apply [103, Theorem A(ii)]. In the remaining cases, $\widehat{S}=G=\mathrm{SU}_n(q)$

and $\ell \nmid 2q$. By [132, Theorem 5.14], choose $\mathbf{S}$ and $P \in \mathrm{Syl}_\ell(G)$ with $\mathrm{N}_G(P) \le M = \mathrm{N}_G(\mathbf{S}) < G$. For $\Gamma = \mathrm{Aut}(G)_P$, [33, Lemma 7.1] gives $\Gamma \le \mathrm{Aut}(G)_M$. In the ambient group, $\widetilde{M} = \mathrm{N}_{\widetilde{G}}(\mathbf{S})$ gives $M = G \cap \widetilde{M}$. Moreover, [156, Corollary C] gives equivariance on $\mathrm{Irr}_{\ell'}(G)$. On $\mathcal{U} = \mathrm{Irr}_{\ell'}(G) \cap \mathrm{UCh}(G)$, [156, Theorem D] gives $\succeq_c$.

For Proposition 3.46, fix matched parents (ρ, τ). By Lemma 3.48, $\alpha_P = \alpha_Q$ on J, with the given values at $z = u^{2f}$. Thus, Lemma 3.4 gives the irreducible T-module $H_{\mathrm{lin}} = \mathrm{Hom}(V_\tau, V_\rho)^{D_0}$, where $T = \widetilde{G} \times_C \widetilde{M}$ and $D_0 = D_\rho = D_\tau$. Its restriction to $G \times M$ gives the pairing Ω. By Lemma 3.4, Ω is equivariant under determinant translations and J/D_0. If b fixes χ, the cyclic-quotient formulas give $\rho^b = \rho\delta$ for $\delta \in \mathrm{Irr}(C)$. Thus, $\tau^b = \tau\delta$ by parent equivariance, and b fixes ψ by the action through J/D_0. Conversely, if b fixes ψ, then b fixes χ. Thus, Ω extends equivariantly over the orbit of (ρ, τ).

Put $\mathcal{A} = \widetilde{G} \rtimes \langle \phi_p \rangle$ and $\mathcal{B} = \widetilde{M}\langle u \rangle$. Then, $\mathcal{A} = G\mathcal{B}$, $G \cap \mathcal{B} = M$, and $\ker(\mathcal{A} \to \mathrm{Aut}(G)) = \mathrm{Z}(\widetilde{G})$. By Lemma 3.47, the element $((a_0^{-1}, a_0^{-1}), z)$ acts trivially on H_{lin}. By Proposition 3.46, ρ, τ have equal central characters, so each (z_0, z_0) with $z_0 \in \mathrm{Z}(\widetilde{G})$ acts as 1. By the descent after Proposition 3.5, Ξ extends $\chi \otimes \psi^\vee$ to $(G \times M) \rtimes \Gamma_\chi$. Thus, $(\mathrm{tr}\,\Xi)^b = \mathrm{tr}\,\Xi$ for each b in the mixed stabilizer, and $\Xi(g, g, a) = 1$ for $(g, a) \in \mathrm{C}_{G \rtimes \Gamma_\chi \mathcal{H}_p}(G)$. Hence, Proposition 3.5 gives $\succeq_c$. □

3.5 Proof of Theorem 3.1

Proof. Suppose that $S = \mathrm{PSL}_n(q)$. If $n = 2$, [186, Theorem C] gives $\mathrm{iMN}(S, p)$. At $p = \mathrm{char}\,\mathbb{F}_q$, apply [162, Theorem 1.3] and [101, Theorem 1.1]. For exceptional $\mathrm{Mult}(S)$, apply [157, Lemma 4.1(2)]. If $p = 2$ and q is odd, apply [159, Theorem A(3)]. If $n = p = 3$ and $q \equiv 4, 7 \bmod 9$, [103, Theorem A(ii)] gives $\mathrm{iMN}(S, p)$. Otherwise, the universal cover is $X = \mathrm{SL}_n(q)$. By [132, Theorem 5.14], a Sylow Φ_d-torus $\mathbf{S}$ gives $M = \mathrm{N}_X(\mathbf{S}) < X$ with $\mathrm{N}_X(P) \le M$ for $P \in \mathrm{Syl}_p(X)$. Moreover, [33, Lemma 7.1] gives $\mathrm{Aut}(X)_P \le \mathrm{Aut}(X)_M$. Proposition 3.22 gives the inductive comparison on M. For $S = \mathrm{PSU}_n(q)$, Proposition 3.49 gives $\mathrm{iMN}(S, p)$. □

4 Odd orthogonal groups

Theorem 4.1. *Let $S = \Omega_{2n+1}(q)$ be a nonabelian simple group of type B_n, where $n \ge 3$ and q is a prime power. Then* $\mathrm{iMN}(S, p)$ *holds for every prime $p \mid |S|$.*

Given $2 \nmid q$, we fix a regular inclusion $X = \mathrm{Spin}_{2n+1}(q) \hookrightarrow \widetilde{X}$. Above a spin series of X, we keep full rational series of $\widetilde{X}$. Moreover, for a proper Levi subgroup L, its residual spin factor is L_{spin}. We compare induction from L to X through the characters of L_{spin}. Moreover, $M = \mathrm{N}_X(Q)$ is the local comparison group.

Theorem 4.2. *Let $X = \mathrm{Spin}_{2n+1}(q)$. Suppose that $n \ge 3$ and that q is odd. Let p be an odd prime dividing $|X|$, with $p \nmid q$, and let $P \in \mathrm{Syl}_p(X)$. Write Q for the largest abelian subgroup of P, which is unique, and $M = \mathrm{N}_X(Q)$. Then, we find an $\mathrm{Aut}(X)_P \times \mathcal{H}_p$-equivariant bijection $\Omega_X : \mathrm{Irr}_{p'}(X) \longrightarrow \mathrm{Irr}_{p'}(M)$ satisfying $\succeq_c$ in* Definition 2.1. *If the Lusztig series $\mathcal{E}(X, s)$ is rational with index s, the image of its p'-characters is the ordinary local packet $Y_s^X(M)$ of* Proposition 4.39.

4.1 Spin covers and normalizers

Put $d = \mathrm{ord}_p(q)$ and $\mathcal{U} = \{\chi \in \mathcal{E}(X, 1) : p \nmid \chi(1)\}$. By Proposition 4.4, the local subgroup $M = \mathrm{N}_X(Q)$ is $\mathrm{N}_X(\mathbf{S}_d)$ for the Sylow d-torus $\mathbf{S}_d$ containing Q. At $d = 1$, Schaeffer Fry [166, Theorem 6.2] gives the equivariant McKay bijection to M, with each pair above a common $\omega \in \mathrm{Irr}(\mathrm{Z}(X))$. Moreover, for $\chi \in \mathcal{U}$, Ruhstorfer et al. [156, Theorem D] gives $\succeq_c$ on the pairs of Malle [132, Theorem 7.8]. If $2 \nmid q$, the natural map from X to S has central kernel $\langle z \rangle$ of order two. Since z is central, any matched pair (χ, ψ) has scalar operators on z, namely $\chi(z)/\chi(1), \psi(z)/\psi(1)$. The signs on z agree by the $\succeq_c$ relation. If the sign is $+1$, put $\pi : X \to S = X/\langle z \rangle$. The central p'-kernel argument in Corollary 2.10 gives $\pi(\mathrm{N}_X(R)) = \mathrm{N}_S(\pi(R))$ for $R = P, Q$. If the sign is -1, the comparison is on X.

Suppose that q is odd and that p is an odd divisor of $|X|$ prime to q. The group X is the rational spin group for S, whether or not the multiplier of S is generic. Set $d = \mathrm{ord}_p(q)$, $e = \mathrm{ord}_p(q^2)$, and $n = ew + r$, where $0 \le r < e$. Thus, we find $e = d/\gcd(d, 2)$. Write $\epsilon = 1$ if d is odd and $\epsilon = -1$ otherwise, and put $a = v_p(q^e - \epsilon)$. Thus, $a \ge 1$ and $w \ge 1$. The Frobenius of $\mathbf{X}$ is F, and algebraic subgroups $\mathbf{H}$ appear in bold letters.

Lemma 4.3. *Suppose that A is cyclic of order p^a, with p odd and $a \geq 1$. If V is a p-subgroup of $\mathfrak{S}_w$ acting by permutation of the factors, A^w is the largest abelian subgroup of $A^w \rtimes V$, and is unique. Moreover, A^w contains every abelian normal subgroup of $A^w \rtimes V$.*

Proof. Put $E = A^w$. Choose $U \in \mathrm{Syl}_p(\mathfrak{S}_w)$ containing V. If $J \lhd E \rtimes U$ is abelian, then $[E, x] \leq J \cap E$ and $[E, x, x] = 1$ for $x \in J$. Any nontrivial cycle of x has length $\geq p$. Choose $u \neq 1$ on a moved coordinate. Then, $[u, x, x] = u(u^x)^{-2}u^{x^2} \neq 1$. Thus, $J \leq E$. By Dirichlet's theorem, choose a prime $r \equiv 1 + p^a \pmod{p^{a+1}}$. By [207, §2], $E \rtimes U$ is Sylow in $\mathrm{GL}_w(r)$. By [2], $E \rtimes U$ has a unique abelian subgroup B of maximal order. Since B is normal, $B \leq E$. Hence, $B = E \leq E \rtimes V$. The same commutator calculation gives $J \leq E$ also for abelian $J \lhd E \rtimes V$. □

Proposition 4.4. *Let $X = \mathrm{Spin}_{2n+1}(q)$ and keep p as above. Given $P \in \mathrm{Syl}_p(X)$, its unique largest abelian subgroup Q lies in a Sylow d-torus $\mathbf{S}$ and satisfies $\mathrm{N}_X(Q) = \mathrm{N}_X(\mathbf{S}) < X$. Moreover, $\mathrm{N}_X(Q)$ contains $\mathrm{N}_X(P)$ and is $\mathrm{Aut}(X)_P$-stable. Given any $P' \in \mathrm{Syl}_p(X)$, $P' = P^x$ for some $x \in X$, with $Q' = Q^x$ and $\mathbf{S}' = \mathbf{S}^x$.*

Proof. On the natural space W, [32, Proposition 4.1] gives $(e, \epsilon)^w(1, +)^r$ for the maximal torus $\overline{\mathbf{T}}$ of $\mathrm{SO}(W)$. Its active factors are $C \simeq C_{q^e - \epsilon}$ on isometric blocks W_i of dimension $2e$. By [207], with $A \in \mathrm{Syl}_p(C)$ and $V \in \mathrm{Syl}_p(\mathfrak{S}_w)$, a Sylow p-subgroup has the form $A^w \rtimes V$. The group V permutes the W_i with identical coordinates and fixes $W_0 = (\perp_i W_i)^\perp$. The covering map $\pi : X \to \mathrm{SO}(W)$ has kernel and image index 2. Thus, $\pi|_P$ is an isomorphism onto a Sylow p-subgroup of $\mathrm{SO}(W)$. By Sylow conjugacy, take $\pi(P) = A^w \rtimes V$. Lemma 4.3 gives $\pi(Q) = A^w$.

The algebraic preimage $\mathbf{T} = \pi^{-1}(\overline{\mathbf{T}})$ is a maximal torus with $Q \in \mathrm{Syl}_p(\mathbf{T}^F)$. Since each toral p-subgroup is conjugate into P, Q has maximal order among them. By [132, Proposition 5.3(b)], $Q \in \mathrm{Syl}_p(\mathbf{S}^F)$ for a Sylow Φ_d-torus $\mathbf{S}$. By [132, Propositions 5.6–5.7], $E = Q[p]$ has maximal toral rank. Moreover, $\mathbf{S}$ is unique through E by [132, Proposition 5.13]. Since E is characteristic in Q, $\mathrm{N}_X(Q) \leq \mathrm{N}_X(\mathbf{S})$. Conversely, Q is characteristic in $\mathbf{S}^F$, so $\mathrm{N}_X(Q) = \mathrm{N}_X(\mathbf{S})$. Since Q is characteristic in P, $\mathrm{N}_X(P) \leq \mathrm{N}_X(Q)$ and $\mathrm{Aut}(X)_P$ normalizes $\mathrm{N}_X(Q)$. If $Q \lhd X$, its image in S would be abelian and normal. As $|\ker \pi| = 2$ and $Q \neq 1$, the image is nontrivial. Thus, $\mathrm{N}_X(Q) < X$. If $w < p$, then $V = 1$, so $P = Q$ and $\mathrm{N}_X(Q) = \mathrm{N}_X(P)$. If $P' = P^x$, uniqueness gives $Q' = Q^x$ and $\mathbf{S}' = \mathbf{S}^x$. □

Let $\widetilde{X} = \mathrm{GSpin}_{2n+1}(q)$, and write ν for its Clifford norm. The norm ν has kernel X, and the value on a scalar c is c^2. Accordingly, for the normalizer comparison, put $M = \mathrm{N}_X(\mathbf{S})$ and $\widetilde{M} = \mathrm{N}_{\widetilde{X}}(\mathbf{S})$.

Lemma 4.5. *On $\widetilde{M}$, the norm ν is onto $\mathbb{F}_q^\times$. If $(L, K) = (\widetilde{X}, X)$ or $(\widetilde{M}, M)$, then $\Theta|_K$ has $t \in \{1, 2\}$ distinct constituents of degree $\Theta(1)/t$, for every $\Theta \in \mathrm{Irr}(L)$. Each $\theta \in \mathrm{Irr}(K)$ occurs under some $\Theta \in \mathrm{Irr}(L)$. If p is odd and $p \nmid \theta(1)$, then $p \nmid \Theta(1)$.*

Proof. In $\widetilde{\mathbf{X}} = \mathbf{X}\mathrm{Z}(\widetilde{\mathbf{X}})$, we have $C_{\widetilde{\mathbf{X}}}(\mathbf{S}) = C_{\mathbf{X}}(\mathbf{S})\mathrm{Z}(\widetilde{\mathbf{X}})$. On the centre, $\nu(c) = c^2$ is onto $\mathbb{G}_m$. Thus, the restricted norm has kernel $C_{\mathbf{X}}(\mathbf{S})$, a connected Levi. By [114, Theorem 1], $\nu(C_{\widetilde{X}}(\mathbf{S})) = \mathbb{F}_q^\times$. Moreover, [38, proof of Theorem 4.3] gives $\widetilde{M} = C_{\widetilde{X}}(\mathbf{S})M$. For either pair (L, K), the quotient map ν has $(\mathbb{F}_q^\times)^2 \leq \nu(\mathrm{Z}(L))$. Hence, the cyclic-quotient formulas give $t \mid 2$, degree $\Theta(1)/t$, and multiplicity 1. Thus, $v_p(\Theta(1)) = v_p(\theta(1))$ at every odd p and every constituent θ. By [95, Lemma 5.2], each constituent of $\mathrm{Ind}_K^L \theta$ gives a parent Θ of θ. □

Given semisimple $s \in (\mathbf{X}^*)^F$, put $I_s = [(\mathbf{X}^*)^F : \mathrm{C}_{\mathbf{X}^*}(s)^F]$. Choose $y \in \mathrm{Sp}_{2n}$ above s. By Steinberg's theorem (Malle–Testerman [133, Theorem 14.16]), $C_{\mathrm{Sp}_{2n}}(y)$ is connected. By Zhang [208, proof of Lemma 6.12], the action on $\{y, -y\}$ embeds the component group A_s in C_2. Thus, $p \nmid 2|A_s|$. By Malle [132, Proposition 7.2], $p \nmid \chi(1)$ in $\mathcal{E}(X, s)$ exactly when $p \nmid I_s \prod_i \lambda_i(1)$, where the connected factor L_i has unipotent label λ_i. For the centralizers in Zhang [208, Lemma 6.12], put $h(t) = at + v_p(t!)$ and $e' = e/\gcd(e, 2)$. By Malle–Testerman [133, Theorem 24.1], $|B^i - 1|$ are the relevant order factors for $B = q^2, q^4, \eta q$, where $\eta = 1$ for $\mathrm{GL}_c(q)$ and $\eta = -1$ for $\mathrm{GU}_c(q)$. Put $m = \mathrm{ord}_p(B)$. Thus, $m = e, e', j$, respectively, with $j = \mathrm{ord}_p(\eta q)$. Put $u = \epsilon q^e$, so $u \equiv 1 \pmod p$ and $v_p(u - 1) = a$. Since $B^m = u^k$ with $k \in \{1, 2, 4\}$, Enguehard–Michel [63, Lemma 4.1] gives $v_p(B^{mi} - 1) = a + v_p(i)$. Thus, the contributions are $h(\lfloor b/e \rfloor)$ for $\mathrm{Sp}_{2b}(q)$, $h(\lfloor b/e' \rfloor)$ for $\mathrm{Sp}_{2b}(q^2)$, and $h(\lfloor c/j \rfloor)$ for the linear or unitary factor L. Each floor w_i gives ambient rank at least ew_i. In particular, $\mathrm{Sp}_{2b}(q^2)$ has ambient rank $2b$. Thus, $\sum_i w_i \leq w = \lfloor n/e \rfloor$ and $v_p(I_s) = a(w - \sum_i w_i) + v_p\big(w!/\prod_i w_i!\big)$. By Kummer's theorem (Granville [81, §1]), $v_p(I_s) = 0$ exactly when $\sum_i w_i = w$ without base-p carries.

Suppose that s is quasi-isolated and F^* exchanges its symplectic factors of rank b. Then, the rational centralizer of s has a factor $\mathrm{Sp}_{2b}(q^2)$ and $n = 2b$ (Zhang [208, Lemma 6.12]). The formula above gives $v_p(I_s) = h(\lfloor 2b/e \rfloor) - h(\lfloor b/e' \rfloor)$.

If e is even, the two arguments of h are equal. If e is odd and $b \geq e$, $\lfloor 2b/e \rfloor \geq 2\lfloor b/e \rfloor > \lfloor b/e \rfloor$. If e is odd and $b < e$, then $\lfloor b/e \rfloor = 0 < \lfloor 2b/e \rfloor$, since $p \mid |X|$. Thus, $v_p(I_s) = 0$ exactly when $4 \mid d$. The central isogeny for X, the component quotient for s, and the parent restriction for χ contribute only powers of 2, by Lemma 4.5 and Zhang [208, Lemma 6.12]. Consequently, for $4 \mid d$, a character χ in the series of s has p'-degree exactly when its unipotent label $\lambda \in \mathrm{Irr}(\mathrm{Sp}_{2b}(q^2))$ has p'-degree.

Given s in the mixed row of Zhang [208, Lemma 6.12(ii)], write $n = 2a_0 + b,\ b > 0,\ \epsilon = (-1)^{(q-1)/2}$, and let $\delta = \pm 1$ be the scalar of z for the rational series of s. Then, the linear or unitary L has period $j = \mathrm{ord}_p(\delta\epsilon q)$. Set $a_0 = eu + x,\ b = ev + y$, with $0 \leq x, y < e$. If $\delta = +1$, both symplectic factors lie over $\mathbb{F}_q$. We have that $p \nmid I_s$ exactly under the following conditions for j.

$$j = e:\ 2x + y < e,\ p \nmid (2u+v)!/\big((u!)^2 v!\big),\ j = 2e:\ b < e,\ 2x + b < e,\ p \nmid \binom{2u}{u}.$$

Indeed, for $j = e$, the contributions are u, u, v. Their shortfall $\Delta = w - \sum_i w_i$ is $\lfloor (2x+y)/e \rfloor$. If $j = 2e$, we write $b = 2ev' + y'$ with $0 \leq y' < 2e$. Then, Δ is $v' + \lfloor (2x + y')/e \rfloor$. Equivalently, $\Delta = 0$ if and only if $v' = 0$ and $2x + b < e$. Then, u, u are the only contributions, with coefficient $\binom{2u}{u}$.

If $\delta = -1$, the symplectic factor lies over $\mathbb{F}_{q^2}$. Moreover, if e is odd and $j = 2e$, the index is divisible by p. If e is odd and $j = e$, the conditions are $a_0 < e$ and $2a_0 + y < e$. When $2 \mid e$, they are $j = 2e$, $b < e$, and $(2a_0 \bmod e) + b < e$. If e is odd and $j = 2e$, the sum is $\lfloor a_0/e \rfloor + \lfloor b/(2e) \rfloor \leq \lfloor n/(2e) \rfloor < \lfloor n/e \rfloor$, since $p \mid |X|$ implies $n \geq e$. For odd e and $j = e$, the shortfall Δ is $u + \lfloor (2x+y)/e \rfloor$. If $\Delta = 0$, only L contributes positively. If $2 \mid e$, both q and $-q$ have order $2e$ modulo p, because $-q = q^{e+1} \bmod p$ and $(e+1, 2e) = 1$. The symplectic factor contributes $\lfloor 2a_0/e \rfloor$. With $b = 2ev' + y'$, the shortfall Δ is $v' + \lfloor ((2a_0 \bmod e) + y')/e \rfloor$. If $\Delta = 0$, only the symplectic contribution remains. Thus, $w!/\prod_i w_i! = 1$ in both nonempty cases. Each λ_i, a unipotent Jordan factor, must satisfy $p \nmid \lambda_i(1)$ under the index conditions for $\delta = 1$ and the index conditions for $\delta = -1$.

Let $\pi : \mathbf{X} \longrightarrow \mathrm{SO}(V)$ be the covering isogeny over $\overline{\mathbb{F}}_q$, with kernel $\langle z \rangle$. Suppose that $\mathbf{D}$ is a rational Levi subgroup of $\mathrm{SO}(V)$ on $V = \perp_i V_i$, with rational nondegenerate spaces V_i and group of rational points $D = D_1 \times \cdots \times D_t$. Accordingly, each D_i acts on V_i by proper isometries. The linear factors $D_i = \mathrm{GL}_{a_i}(q^{e_i})$ act on $U_i \oplus U_i^\vee$, where U_i is their natural space. If $D_i = \mathrm{GU}_{a_i}(q^{e_i})$, the natural space is over $\mathbb{F}_{q^{2e_i}}$. The residual factor is $\mathrm{SO}(V_0)$, with $\dim V_0$ odd. The geometric inverse images are $\widehat{D} = \pi^{-1}(D)$ and $\widehat{D}_i = \pi^{-1}(D_i)$. Since $\ker \pi = \langle z \rangle$, both groups are finite. For $\widehat{D}$, Frobenius may move $h \in \widehat{D}$ while fixing its image in D.

Proposition 4.6. *Each $\widehat{D}_i$ has a sign homomorphism δ_i defined by $F(h) = h$ for sign $+1$, and $F(h) = zh$ for sign -1. Consequently, we find that the product δ of the signs δ_i is a homomorphism on $\widehat{D}$. With $L = \widehat{D}^F$, we have that $\widehat{D} \simeq (\prod_i \widehat{D}_i)/\{(z^{u_i})_i : \sum_i u_i = 0 \pmod 2\}$ and $L = \ker \delta$. Let Q_i be any p-subgroup of D_i, and let Q denote the unique odd lift of $\prod_i Q_i$. Then, the inverse image of $\mathrm{N}_D(\prod_i Q_i)$ is $\mathrm{N}_{\widehat{D}}(Q)$. Write $\widetilde{Q}_i$ for the odd lifts. The normalizers $\mathrm{N}_{\widehat{D}_i}(\widetilde{Q}_i)$ have central product $\mathrm{N}_{\widehat{D}}(Q)$, with kernel $\{(z^{u_i})_i : \sum_i u_i = 0 \pmod 2\}$. Moreover, $\mathrm{N}_L(Q) = \ker(\delta|_{\mathrm{N}_{\widehat{D}}(Q)})$.*

Proof. By the Clifford relations in [137, Theorem 24.56], $[\widehat{D}_i, \widehat{D}_j] = 1$ for $i \neq j$. Thus, multiplication gives $m : \prod_i \widehat{D}_i \to \widehat{D}$. Its image maps onto D and contains z, so m is onto. If $m((h_i)_i) = 1$, then $\pi(h_i) = 1$ for each i, by the direct product decomposition of D. Thus, $h_i = z^{u_i}$ and $\ker m = \{(z^{u_i})_i : \sum_i u_i = 0 \pmod 2\}$. By [67, Remark 5.2.1], the spinor norm sn is the connecting map for π. Evaluation at F gives $H^1(\mathbb{F}_q, \mu_2) \cong \langle z \rangle$. Thus, $\delta_i = \mathrm{sn} \circ \pi$ on $\widehat{D}_i$, with $\delta_i(z) = 1$. By [32, proof of Lemma 4.2], the restriction of sn to D_i is its spinor norm on V_i. Consequently, on $\widehat{D}$ the product $\delta = \prod_i \delta_i$ is $\mathrm{sn} \circ \pi$, and $L = \ker \delta$. If two factor signs are -1 and all others are $+1$, their product lift belongs to L and lies outside $\prod_i \widehat{D}_i^F$.

Put $Q_D = \prod_i Q_i$. As in Corollary 2.10, Q is the unique Sylow p-subgroup of $\pi^{-1}(Q_D)$. Thus, F fixes Q pointwise, since $\pi|_Q$ is injective and $Q_D \leq D$. The same normalizer formula gives $\mathrm{N}_{\widehat{D}}(Q) = \pi^{-1}(\mathrm{N}_D(Q_D))$ and $\mathrm{N}_{\widehat{D}_i}(\widetilde{Q}_i) = \pi^{-1}(\mathrm{N}_{D_i}(Q_i))$. Since $\mathrm{N}_D(Q_D) = \prod_i \mathrm{N}_{D_i}(Q_i)$, multiplication maps the lifted normalizers onto $\mathrm{N}_{\widehat{D}}(Q)$ with kernel $\ker m$. Finally, $L = \ker \delta$ gives $\mathrm{N}_L(Q) = \ker(\delta|_{\mathrm{N}_{\widehat{D}}(Q)})$. □

Fix irreducible ρ_i with common scalar $\rho_i(z) = \varepsilon I$, where $\varepsilon \in \{1, -1\}$. Then $\rho = \bigotimes_i \rho_i$ is trivial on $\ker m$ and descends to $\widehat{D}$. Write Θ for its character and suppose $\delta \neq 1$. By the quotient formulas, if $\Theta\delta \neq \Theta$, then $\Theta|_L$ is irreducible with parents $\Theta, \Theta\delta$. If $\Theta\delta = \Theta$, the tensor factors satisfy $\rho_i \simeq \rho_i \delta_i$. Choose S_i with $S_i \rho_i(h) S_i^{-1} = \delta_i(h)\rho_i(h)$. By

Isaacs [95, Corollary 1.6], S_i^2 is scalar, so normalize $S_i^2 = 1$, with $S_i = 1$ if $\delta_i = 1$. Thus, $S = \bigotimes_i S_i$ is the involution for δ on ρ. The spaces $V_\pm = \ker(S \mp 1)$ afford the constituents of $\Theta|_L$ by the quotient eigenspace description.

Suppose $\mathcal{A}$ consists of automorphism and coefficient operations a. The group $\mathcal{A}$ permutes the factors with signs δ_i and fixes Θ. If $a \in \mathcal{A}$, write π_a for its permutation of factors and S_i^a for the image of S_i under a. If $j = \pi_a(i)$, the factor intertwiner $T_{a,i}$ satisfies $T_{a,i} S_i^a T_{a,i}^{-1} = c_i S_j$. Since $(S_i^a)^2 = S_j^2 = 1$, uniqueness up to scalar gives $c_i \in \{1, -1\}$. With the tensor permutation, conjugation gives $S \mapsto (\prod_i c_i) S$. Put $c(a) = \prod_i c_i$. The product also tests $a \in \mathcal{A}$ even if a moves ρ_i or its coefficient field K_i. Let f_0 pair parents of central sign ε and suppose that f_0 is equivariant for the twists δ, δ'. With $\Theta' = f_0(\Theta)$, we write $L' = \ker \delta'$. Thus, we obtain that $f_0(\Theta\delta) = f_0(\Theta)\delta'$, and the irreducible restrictions on L and L' are paired. For split restrictions, let $c'(a)$ denote the local product. Suppose that $\prod_i c_i$ equals $c'(a)$ on the joint stabilizer $B = \mathcal{A}_{\Theta,\Theta'}$. Write $\theta_\pm$ and $\theta'_\pm$ for the constituents. At a representative θ, we choose the pairing f. Then, we find $f(\theta^a) = f(\theta)^a$ on the $\mathcal{A}$-orbit. Since $c(a) = c'(a)$ on B, the value $f(\theta^a)$ is independent of the choice of a. Conversely, for (Θ, Θ'), we write f_1, f_2 for the two constituent pairings. Since $\mathfrak{S}_2$ is abelian, for $b \in B$ we have that $f_1 b f_1^{-1} = f_2 b f_2^{-1}$ on $\{\theta'_+, \theta'_-\}$. By the quotient formulas, for θ on L or L', its parent degree is $\theta(1)$ or $2\theta(1)$. Thus, at odd p, $p \mid \theta(1)$ exactly when p divides the parent degree. If $\delta = 1$, then $L = \widehat{D}$. Matching permutations of $\{\theta_+, \theta_-\}$ alone does not determine $\alpha : J \times J \to \mathbb{C}^\times$, a factor set on inertia quotient J. For $\widetilde{X}$, suppose that $q = s^f$ with s prime. With $b = v_2(q-1)$, we let C be the geometric scalar group $\mu_{2^{b+1}}$, and put $\kappa(c) = c^{2^b}$. Write $\widehat{K}$ for the preimage of $\mathrm{SO}(V)^F$ in **Spin**. If $h \in \widehat{K}$, we write $\delta(h)$ for its Frobenius sign. Let C_{rat} be the rational scalar center of $\widetilde{X}$, and Z_o its odd-order part.

Lemma 4.7. *In* $\mathbf{GSpin} = (\mathbf{Spin} \times \mathbb{G}_m)/\langle (z, -1) \rangle$*, multiplication induces an isomorphism* $\{(h, c) \in \widehat{K} \times C : \delta(h) = \kappa(c)\}/\langle (z, -1) \rangle \simeq \widetilde{X}/Z_o$*. Similarly, with* $\widehat{D}$ *or* $\mathrm{N}_{\widehat{D}}(Q)$ *in place of* $\widehat{K}$*, the isomorphism describes the Levi or* Q*-normalizer preimage, respectively. Let* φ *be the prime-field map. Its* f*-th power acts by* $(h, c) \mapsto (h, c)(z, -1)^u$*, where* $(-1)^u = \delta(h)$*.*

Proof. Since $(q-1)/2^b$ is odd, $c^{q-1} = \kappa(c)$ for $c \in C$. Thus, $\delta(h) = \kappa(c)$ is exactly the condition for $[h, c]$ to lie in $\widetilde{X}$. Conversely, every rational element of $\widetilde{X}$ has a representative $[h, x]$ with $h \in \widehat{K}$ and $x^{q-1} = \delta(h)$. Then, $x^{2(q-1)} = 1$, so the odd-order part of x is rational. Modulo Z_o, the scalar is $c \in C$. If the pair maps to Z_o, its spin coordinate has $\pi(h) = 1$. The remaining scalar ζ has odd order and 2-power order. Thus, $\zeta = 1$, and the kernel is $\langle (z, -1) \rangle$. The orthogonal image of $[h, c]$ is $\pi(h)$, the image of h. Thus, the description holds on $\widehat{D}$ and $\mathrm{N}_{\widehat{D}}(Q)$. The map φ acts on both coordinates of (h, c). Its f-th power follows from $F(h) = z^u h$ and $c^q = (-1)^u c$. Thus, on $\widetilde{X}/Z_o$, the field relation holds. □

Put $J = \{(h, c) \in \widehat{K} \times C : \delta(h) = \kappa(c)\}$. Thus, J has index two in $\widehat{K} \times C$. Under its nontrivial quotient character, $\tau \in \mathrm{Irr}(C)$ becomes $\tau\kappa \neq \tau$. Consequently, by the quotient formulas, $\Theta \otimes \tau$ restricts irreducibly for every $\Theta \in \mathrm{Irr}(\widehat{K})$. The fibres are the orbits of $(\Theta, \tau) \mapsto (\Theta\delta, \tau\kappa)$. The quotient by $\langle (z, -1) \rangle$ imposes $\Theta(z)/\Theta(1) = \tau(-1)$. Thus, any parent bijection f_0 matching central signs and δ-twists yields the pullback bijection $(\Theta, \tau) \mapsto (f_0(\Theta), \tau)$. Since $\nu([h, c]) = c^2$, a norm twist γ replaces $\tau(c)$ by $\tau(c)\gamma(c^2)$ on both sides. Moreover, $(\Theta, \tau)^a$ represents the action of a on the twist orbit. Let $C_0 \lhd K$ and $C_1 \lhd N$ be finite groups, with $K \leq N$, $K \cap C_1 = C_0$. In the application, $N = \mathrm{N}_X(Q)$, $K = \mathrm{N}_L(Q)$, and the C_i are the finite centralizers of Q in L and X. In particular, for $\chi \in \mathrm{Irr}_{p'}(N)$, the restriction $\chi|_{C_1}$ may have several constituents ξ. Moreover, a semisimple element s may occur in several rational tori $\mathbf{T}$.

Lemma 4.8. *Suppose a* K*-orbit of characters of* C_0 *and an* N*-orbit of characters of* C_1 *have representatives* β *and* ξ *with* $I_N(\xi) = C_1 I_K(\beta)$*. With* $I_0 = I_K(\beta)$*, we put* $I_1 = I_N(\xi)$*. Suppose their associated projective representations* $\mathcal{P}_0, \mathcal{P}_1$ *on* I_0, I_1 *have a common factor set* α *on the quotients* $I_K(\beta)/C_0 = I_N(\xi)/C_1$*, denoted by* U*. The characters of* K *and* N *above the orbits admit a bijection* f*. Given* $\chi \in \mathrm{Irr}(K \mid \beta)$*, we put* $\chi' = f(\chi)$*. Then, we obtain that* $\chi'(1)/\chi(1) = r$*, where* r *is* $[N : C_1]\xi(1)/([K : C_0]\beta(1))$*. On degrees prime to* p*, the bijection restricts if* r *has* p*-valuation zero. Let the mixed stabilizer* $\mathcal{A}$ *fix both orbits. Suppose each* $a \in \mathcal{A}$ *can be corrected by conjugation in* K, N *to stabilize* β, ξ*, respectively, with the same action on* U *and the same comparison function for* $\mathcal{P}_0, \mathcal{P}_1$*. Then,* f *is* $\mathcal{A}$*-equivariant.*

Proof. Since $I_1 = C_1 I_0$ and $I_0 \cap C_1 = C_0$, we obtain $U \cong I_0/C_0 \cong I_1/C_1$. By [141, proof of Lemma 1.9(a)], the quotient label E of factor α^{-1} pairs inertia characters. By [95, Theorem 6.11], induction gives χ, χ' with degree ratio

$[N : I_1]\xi(1)/([K : I_0]\beta(1)) = r$. For both corrected actions, write $\eta_a \mathcal{P}_i^a \simeq \mathcal{P}_i$. Thus, [141, proof of Lemma 1.9(a)] gives $\eta_a^{-1} E^a$ on both sides. Conjugation in K, N fixes the induced characters. Hence, f is $\mathcal{A}$-equivariant. □

Given non-quasi-isolated s, Proposition 4.39 gives both inertia quotients as U, for the full rational dual element s. Moreover, Proposition 4.40 gives the common factor α through the centralizer induction variety Y. Propositions 4.41 and 4.42 give mixed comparison for $E_{\xi,\beta}$ on the larger inertia groups of β, ξ. Consequently, the comparison on (χ, χ') follows by induction to K, N.

Let $q = r^f$ be odd, with r prime, and $d = \mathrm{ord}_p(q) = 2e$, and $n = ew + h$, $0 \le h < e,\ w > 0$. The faithful cases in **GSpin** occur for even e, or for odd e and $q \equiv 1 \bmod 4$. Thus, $q^e + 1 \equiv 2 \bmod 4$ in both cases. Write $K_s = s(s+1)/2$, $f_2 = 2^{v_2(f)}$, let η_q denote the quadratic character of $\mathbb{F}_q^\times$, and put $\epsilon_\sigma = (c_\sigma/r)$ where $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$. We construct the operators in **GSpin**. Consequently, the ordinary characters are in $\mathrm{Irr}(\mathbf{GSpin}^F)$.

Choose a basis $e_1, \dots, e_n$ of the maximal isotropic space U. On $\bigwedge U$, we let m_i be exterior multiplication, so $m_i(\omega) = e_i \wedge \omega$. Let d_i be contraction by the dual vector $e_i^\vee$, with $d_i = \iota_{e_i^\vee}$. If $\omega \in \bigwedge^k U$, the parity operator P_0 has value $P_0(\omega) = (-1)^k \omega$. With $L(A) = \bigwedge A$ and $z = -1$, we put $T_i(a) = 1 + (a-1) m_i d_i$ and $R_i = (m_i + d_i) P_0$. For the unit vector v_i of hyperbolic pair i and anisotropic vector v_0, we have that $R_i = v_i v_0$ in the even Clifford algebra $\mathrm{Cl}^0(W_{\mathrm{nat}})$ of the natural quadratic space W_{nat} (Milne [137, §24.h]). Hence, the norm of R_i is $\nu(R_i) = 1$. Moreover, we have that $\nu(T_i(a)) = a$ and $\nu(L(A)) = \det A$. The identities on $\bigwedge U$ give $R_i^2 = z$, $R_i R_j = z R_j R_i$ for $i \ne j$, and $R_i T_i(a) R_i^{-1} = a T_i(a^{-1})$. Given T_j with $j \ne i$, we have that $T_j(a) R_i = R_i T_j(a)$. On the subscripts i, the permutation matrices $L(\pi)$ act by $i \mapsto \pi(i)$. Moreover, we have that F_r fixes $R_i, L(\pi)$ and satisfies $F_r(T_i(a)) = T_i(a^r)$.

On the residual coordinates, we put $E_R = \prod_{\mathrm{res}} T_i(-1)$. If x is an active signed permutation, let $s(x)$ count sign changes modulo 2. For the residual quadratic space W_R, we put $\mathcal{C}_R = \mathrm{Cl}^0(W_R)$ (Milne [137, §24.h]). On W_R, an active R_i acts as E_R by Milne [137, §24.i]. Active $L(\pi), T_i(a)$ fix W_R pointwise. Thus, x acts on $\mathcal{C}_R$ by conjugation by $E_R^{s(x)}$. Since E_R satisfies $x E_R = E_R x$, we have that $j(x) = x E_R^{s(x)}$ is a homomorphism centralizing $\mathbf{GSpin}(W_R)$. Moreover, the norm of $j(x)$ is $\nu_h(x) = \nu_A(x)(-1)^{hs(x)}$. The two embedded factors $\mathbf{A}, \mathbf{R}$ intersect in Z and generate the regular normalizer $\widetilde{\mathbf{N}}$. The centralizer of $\mathbf{S}$ has active torus $\mathbf{T}_A$ and residual factor $\mathbf{R}$. By Späth [176, §§6 and 10], the active Weyl quotient $W = C_{2e} \wr \mathfrak{S}_w$ rotates and permutes the blocks $1, \dots, w$. Multiplication $\mu : \mathbf{A} \times \mathbf{R} \to \widetilde{\mathbf{N}}$ has kernel consisting of (c, c^{-1}) for $c \in Z$. Since Z is connected, Lang [114, corollary to Theorem 1] gives $\mathbf{A}^F \times \mathbf{R}^F \twoheadrightarrow \widetilde{\mathbf{N}}^F$.

Lemma 4.9. *On block l, we write i_l for its first coordinate. If π_l cycles the e coordinates of l, we put $u_l = T_{i_l}((-1)^{h+e-1}) R_{i_l} L(\pi_l)$, $v_A = u_1 \cdots u_w$, and $v = j(v_A)$. Then, $F' = \mathrm{Int}(v) F_q$ acts on $\mathbf{A}$ by $\mathrm{Int}(v_A) F_q$ and on $\mathbf{R}$ by F_q. The active norm kernel A_h has a normal torus T and a subgroup V with $A_h = TV$, $T \cap V = H = T_2$, and $V/H = C_{2e} \wr \mathfrak{S}_w$. Moreover, we have that T_2 is elementary abelian of order 2^w. Take the quadratic character θ defined below, with $\theta(z) = -1$. Put $I_\theta = I_{A_h}(\theta)$ and $\overline{I}_\theta = I_\theta / T$. A wreath element $(a_1, \dots, a_w; \pi)$ belongs to $\overline{I}_\theta$ if and only if $a_l \equiv a_m \pmod 2$ for all l, m. Below, $\widetilde{\theta}$ extends θ linearly to I_θ.*

Proof. We calculate $\det(\pi_l) = (-1)^{e-1}$ for the e-cycle π_l. Hence, $\nu_A(u_l) = (-1)^h$ and $\nu_h(u_l) = 1$. Thus, v satisfies $\nu(v) = 1$, $F_r(v) = v$, and $[v, \mathbf{R}] = 1$. In the twisted fixed group $\mathbf{Spin}^{F'}$, $F_r^f = \mathrm{Int}(v^{-1})$. By [114, corollary to Theorem 1] on **Spin**, take x with $x^{-1} F_q(x) = v$. After conjugation by x, we obtain the field lift $x F_r(x)^{-1} F_r$ in the split model. Consequently, its f-th power is $x v^{-1} x^{-1}$. Before conjugation by x, the two carries are v_A^{-1} and 1 on $\mathbf{A}, \mathbf{R}$.

Put $E_l = \prod_{i \in l} T_i(-1)$. Moreover, we choose a fourth root ι, with $\iota^2 = -1$, in the algebraic base field. Put $h_l = (-\iota)^e E_l$. Then, $h_l^2 = z^e$. We calculate $\mathcal{R}_l = (R_{i_l} L(\pi_l))^e$, the ordered reflection product on block l, with $\mathcal{R}_l^2 = z^{K_e}$. For the $2e$-th power of u_l, we pair the torus entries at i before and after its reflection R_i. Then, the two entries have product $(-1)^{h+e-1}$. Thus, $u_l^{2e} = z^{K_e + he}$, $[u_l, u_m] = z$ for $l \ne m$, and $u_l h_l u_l^{-1} = z h_l$. Given $m \ne l$, $u_m h_l = h_l u_m$. Since $q^e + 1 \equiv 2 \bmod 4$ in the stated range of e, q, we have that $F'(h_l) = z h_l$ and $F'(u_l) = z^{w-1} u_l$. Write $b_0 = (w-1) \bmod 2$. Then, $k_l = h_l^{b_0} u_l$ satisfies $F'(k_l) = k_l$. Its $2e$-th power is that of u_l, since $(h_l u_l)^2 = z^{e+1} u_l^2$. In particular, for even e, the extra factor is z.

Let $\Pi_l = L(\pi)$ exchange blocks $l, l+1$ at matching coordinate positions. Put $p_l = E_l \Pi_l$ for odd e, and $p_l = h_l \Pi_l$ for even e. Moreover, the two factors of p_l have equal norm under ν_h. Under F', the reordering of u_l, u_{l+1} in v_A gives the factor z on Π_l. The prefactor in p_l also has Frobenius sign z. Thus, $F'(p_l) = p_l$. We calculate $p_l^2 = z^e h_l h_{l+1}$, $p_l k_l p_l^{-1} = k_{l+1}$, and $p_l k_{l+1} p_l^{-1} = z k_l$. Indeed, $d_l = p_l \Pi_l^{-1}$ is supported on block l, with $d_l u_l = z u_l d_l$. Thus, $p_l^2 = d_l \Pi_l d_l \Pi_l^{-1}$. If $j \notin \{l, l+1\}$, $p_l k_j p_l^{-1} = k_j$. The adjacent lifts satisfy $p_l p_{l+1} p_l = p_{l+1} p_l p_{l+1}$. Indeed, at matching positions, $d_l d_{l+1} = d_{l+1} d_l$. If $|l - j| > 1$, $p_l p_j = p_j p_l$. Put $H = \langle z, h_1 h_2, \dots, h_{w-1} h_w \rangle$. The images of $h_l h_{l+1}$ modulo z are independent adjacent pairs in $\{\pm 1\}^w$. Since $(h_l h_{l+1})^2 = 1$, $|H| = 2^w$, including $w = 1$. We calculate

$|T| = (q^e+1)^w$. Indeed, on block l, the cyclic norm kernel has order q^e+1. The isogeny $\mathbf{T}_1 \to \mathbf{T}_2$ gives isomorphic rational lattices $X(\mathbf{T}_i)\otimes\mathbb{Q}$ as F-modules. Thus, [133, Proposition 25.2] gives $|\mathbf{T}_1^F| = |\mathbf{T}_2^F|$. Since $q^e+1 \equiv 2 \bmod 4$, the full Sylow subgroup T_2 is H.

Put $V = \langle H, k_l, p_j\rangle$. Since $k_l^{2e} \in H$, we order the rotations k_l and reduce exponents modulo $2e$. We fix a reduced permutation word p_π, with corrections in H. Modulo H, the words $k_1^{a_1}\cdots k_w^{a_w}p_\pi$ represent distinct elements of $C_{2e}\wr\mathfrak{S}_w$. Hence, the leading factor in H is unique. Thus, $V\cap T = H$. Given $g \in A_h$, we choose $v_g \in V$ with the same signed action. Then, $gv_g^{-1} \in T$ is rational, so $A_h = TV$.

On $\widehat{H} = \langle z, h_1, \dots, h_w\rangle$, let $\lambda_H(z) = -1$. Take $\lambda_H(h_l) = i$ for odd e and $\lambda_H(h_l) = 1$ for even e. Since $T = H\times T_{2'}$, the restriction $\lambda_H|_H$ extends to θ by $\theta(T_{2'}) = 1$. The rotation k_l at block l gives $\lambda_H^{k_l}(h_l) = -\lambda_H(h_l)$. Thus, the exponent vectors stabilizing θ satisfy $a_l \equiv a_m \pmod 2$ for all l, m. Consequently, $[A_h : I_\theta] = 2^{w-1}$. On $V_\theta = V\cap I_\theta$, generated by H, k_l^2, p_l and $k_0 = k_1\cdots k_w$, take $\Lambda|_H = \theta|_H$ and $\Lambda(p_l) = 1$. If e is odd, put $(\Lambda(k_l^2), \Lambda(k_0)) = ((-1)^{(e+1)/2+h}, i^{w(w+e)/2+hw})$. If e is even, the pair is $(i^{e+1}, \zeta_8^{w(w+e)})$. Write β, κ for the values at k_l^2, k_0, respectively. Then, $\beta^e = (-1)^{K_e+he}$ and $\kappa^2 = (-1)^{\binom{w}{2}}\beta^w$. Each square p_l^2 has value 1. In $p_lk_0p_l^{-1}$, conjugation contributes z. Moreover, the order change between k_l, k_{l+1} contributes another z. Thus, $p_lk_0p_l^{-1} = k_0$. The relations and normal form in V_θ define a linear character Λ. Finally, we put $\widetilde{\theta}(tx) = \theta(t)\Lambda(x)$ for $t\in T$, $x\in V_\theta$. The formula is independent of (t,x), since $T\cap V = H$ and $\Lambda|_H = \theta|_H$. Since V_θ fixes θ, $\widetilde{\theta}$ is multiplicative on TV_θ. □

Proposition 4.10. *Suppose that e is odd and $q\equiv 1 \bmod 4$. With $B = K_{ew} + hw$, we put $W_0 = C_e\wr\mathfrak{S}_w$. Then, for $\eta\in\mathrm{Irr}(W_0)$, we have that the active group A_h has two characters χ_η^+, χ_η^- above the orbit of θ, each of degree $2^{w-1}\eta(1)$. If $\nu_h(g)$ is nonsquare, g exchanges χ_η^+ and χ_η^- with η fixed. Their coefficient exchange sign is $(\sigma(i)/i)^B$. Moreover, on the stabilizer of $\chi_\eta^\pm$, an extension to the field group $\langle a\rangle$ with $a^f = v_A^{-1}$ can have ratio ϵ_σ^B.*

Proof. In $\overline{I}_\theta$ from Lemma 4.9, the subgroup W_0 consists of elements $(2a_1, \dots, 2a_w; \pi)$. Write $\bar{k}_0$ for the image of k_0. Then, $\bar{k}_0$ is central of order $2e$, and $\bar{k}_0^2$ generates the diagonal subgroup of C_e^w. Since e is odd, $\bar{k}_0^e$ is a central involution outside W_0. Thus, $\overline{I}_\theta = W_0\times\langle\bar{k}_0^e\rangle$. Extend η to $\eta^\pm$ on $\overline{I}_\theta$, with opposite signs at $\bar{k}_0^e$. By [95, Theorem 6.11 and Corollary 6.17], the characters $\chi_\eta^\pm = \mathrm{Ind}_{I_\theta}^{A_h}(\widetilde{\theta}\eta^\pm)$ lie over θ and have degree $2^{w-1}\eta(1)$. By [99, Theorem 4.3.34], the model for η is induced from C_e-characters and Specht modules over $\mathbb{Q}$. Thus, both extensions have models over $\mathbb{Q}(\mu_{2e})$, fixed by $\mathcal{H}_p$ since $2e = d \mid p-1$.

The toral and permutation values of $\widetilde{\theta}$ are rational apart from κ. The exponent of i in κ is congruent to B modulo 2, since $K_{ew} - w(w+e)/2 = (e^2-1)w^2/2$ is even. Thus, coefficient action at k_0 has ratio $(\sigma(i)/i)^B$. To conjugate diagonally, choose a scalar c with $c^{q-1} = -1$, and put $t = cE_1$. The two Frobenius signs at c, E_1 cancel under F'. Moreover, $\nu_h(t) = -c^2$ is nonsquare, since $\eta_q(-1) = 1$. Conjugation by t contributes z at k_0 and fixes each k_l^2. If $w>1$, the extra factor at p_1 is E_1E_2, with $\theta(E_1E_2) = 1$. The toral character θ is fixed. Thus, t exchanges χ_η^+ and χ_η^-, independently of their value field $\mathbb{Q}(\chi_\eta^\pm)$.

p_l, k_l^2, and k_0 are fixed by F_r. Indeed, $F_r(k_l) = z^{b_0(r-1)/2}k_l$, and b_0w is even. The field map F_r fixes θ, since it fixes H pointwise. Hence, F_r fixes the source matrices for $\widetilde{\theta}$, including the matrices at toral elements $t\in T$ permuted by F_r. We write ν_η for the quotient scalar at k_0. The scalar at v_A in the inertia representation is $\gamma = i^{-b_0w}\kappa\nu_\eta$. Indeed, $v_A = (\prod_l h_l^{-b_0})k_0$. When $b_0 = 1$, the first factor is in H, since w is even. We assign to a the scalar α on the source space, with $\alpha^f = \gamma^{-1}$. Thus, after induction from I_θ, the relation is $a^f = v_A^{-1}$. If σ fixes $\chi_\eta^\pm$, we compare the character $\widetilde{\theta}\eta^\pm$. The quotient intertwiner J satisfies $J(\alpha I) = (\alpha I)J$. Thus, the field ratio is $\sigma(\alpha)/\alpha$.

If the prime s is odd and divides e, then $v_s(\mathrm{ord}_p(r)) = v_s(f) + v_s(e)$. Thus, choose the odd part $\alpha_{2'}$ of α in μ_{p-1}. Write γ_2 for the two-power part of γ. Its order divides 2 for even B, and equals 4 for odd B. Indeed, b_0w is even and $(\nu_\eta)_2$, the two-power part of ν_η, has order at most 2. By Section 2, $v_2(\mathrm{ord}_p(r)) = 1 + v_2(f)$. If B is even, choose $\alpha_2 \in \mu_{2f_2}$ fixed by $\mathcal{H}_p$. If B is odd, choose α_2 of order $4f_2$. Since $p\equiv 1 \bmod 4$ for even f and $r\equiv 1 \bmod 4$ for odd f, Section 2 gives $\sigma(\alpha)/\alpha = \epsilon_\sigma^B$. □

If $h>0$, write N_{full} for the full spin normalizer in $\mathrm{Spin}_{2n+1}(q)$. Let $\bar{\nu}_A : N_{\mathrm{full}} \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ be the active norm class. Then, $\ker\bar{\nu}_A = L_\square = \big(A_h\times\mathrm{Spin}_{2h+1}(q)\big)/\langle(z,z)\rangle$. Indeed, for reciprocal square norms $\nu_h(a), \nu_R(b)$, choose $c\in\mathbb{F}_q^\times$ with $c^2 = \nu_h(a)^{-1}$. Then, (a,b) has image equal to that of $(ca, c^{-1}b)$. Since the norms of $(ca, c^{-1}b)$ are 1, the remaining common scalars are $1, z$. Write $\chi_{\mathrm{res}}^+, \chi_{\mathrm{res}}^-$ for the residual pair. Since z acts by -1 on both pairs, the four products $\chi_\eta^\epsilon\otimes\chi_{\mathrm{res}}^\delta$ belong to $\mathrm{Irr}(L_\square)$ by Isaacs [95, Problem 4.4]. The nonsquare coset in $N_{\mathrm{full}}/L_\square$ acts by $(\epsilon,\delta)\mapsto(-\epsilon,-\delta)$. By Isaacs [95, Theorem 6.11], each orbit yields $\psi\in\mathrm{Irr}(N_{\mathrm{full}})$ of degree $2^w\eta(1)\chi_{\mathrm{res}}(1)$, with

$\chi_{\rm res} = \chi^+_{\rm res}$. The exchange sign is $s_A(\sigma)s_R(\sigma)$, with s_A, s_R the signs on the two pairs $(\chi^+_\eta, \chi^-_\eta)$ and $(\chi^+_{\rm res}, \chi^-_{\rm res})$. If $h = 0$, the pair is $(\chi^+_\eta, \chi^-_\eta)$ from A_h. On both tensor factors, we choose a to act by F_r. Then, before induction a^f has components v_A^{-1} and 1, with product v^{-1}. If σ fixes $(\chi^\epsilon_\eta, \chi^\delta_{\rm res})$, we write the field ratios as ρ_A, ρ_R. Then, the tensor ratio is $\rho_A\rho_R$. Given $(\chi^\epsilon_\eta, \chi^\delta_{\rm res})$, we write B_0 for its common stabilizer. Suppose that f is odd and $\rho_A\rho_R = 1$ on B_0. If b exchanges both characters, Lemma 4.34 gives $\widehat{\psi}$ as an extension fixed by b. If $b\sigma$ is the mixed action, on $N_{\rm full}$ we have that $\alpha_{\rm full} = \alpha_{\rm spin}$ by Lemma 4.11. The root $\alpha_{\rm spin}$ is for $\mathrm{Spin}_{2n+1}(q)$.

4.2 Mixed unitary series

Suppose that $q = r^f$ with r an odd prime. Put $X = \mathrm{Spin}_{2n+1}(q)$ and suppose that $p \neq r$ is an odd divisor of $|X|$. With $d = \mathrm{ord}_p(q)$, we put $e = \mathrm{ord}_p(q^2)$ and $f_2 = 2^{v_2(f)}$ and $K_m = m(m+1)/2$. Write $\Delta_m = (m, m-1, \ldots, 1)$. If $\sigma \in \mathcal{H}_p$, we define c_σ by $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$. The Legendre symbol is $\epsilon_\sigma = (c_\sigma/r)$. Thus, the quadratic character η_q of $\mathbb{F}_q^\times$ has value ϵ_σ^f at c_σ. Given a diagonal pair $\{\chi_+, \chi_-\}$, we write $s_\chi(\sigma) \in \{1, -1\}$ for the sign of π_σ, its permutation on $\{\chi_+, \chi_-\}$. Choose a field generator φ and an extension $\widetilde{\chi}$ of χ to the full field group. If σ stabilizes χ, we write $\widetilde{\chi}^\sigma = \widetilde{\chi}\lambda_\sigma$, with linear λ_σ. Then, the field ratio is $\lambda_\sigma(\varphi)$. The source comparison for χ determines $s_\chi(\sigma)$ and $\lambda_\sigma(\varphi)$.

Suppose that $q \equiv 1 \pmod 4$ and $d = 2e$, with e odd. The faithful mixed X-series in Zhang [208, Lemma 6.12] has labels (ρ, λ) for $\mathrm{Sp}_{2a}(q^2)$ and $\mathrm{GU}_b(q)$, respectively, with $n = 2a + b$. At $b = 0$, the symplectic labels form (ρ, ρ). Then, we find $\eta_q(c_\sigma) = \sigma(i)/i$. Indeed, for even f, we have that both signs are 1. If $2 \nmid f$, $r \equiv 1 \pmod 4$ and $v_2(\mathrm{ord}_p(r)) = 1$. Quadratic reciprocity gives that $(p/r) = (r/p) = (-1)^{(p-1)/2}$. Thus, we find that the common p-power action on ζ_r, i gives that $\eta_q(c_\sigma) = \sigma(i)/i$.

Lemma 4.11. *Let H be the spin group X or its normalizer $N_X(Q)$, in the twisted model $F' = \mathrm{Int}(v)F_q$, with $F_r(v) = v$ and $\nu(v) = 1$. In particular, the field generator a acts by F_r, with $a^f = v^{-1}$. Let t be rational in the regular group $\widetilde{H}$, that normalizes H, with nonsquare norm $b \in \mathbb{F}_q^\times$. With $c = b^{(1-r)/2}$, we put $h = c^{-1}tF_r(t)^{-1}$. Suppose that $h \in H$. The condition $h \in H$ holds for the full spin group H, and for a diagonal representative t in the marked rational torus of H when H is a normalizer. Suppose that (ρ, A) extends a constituent to $H\langle a\rangle$. Then, an extension of $\rho^t(x) = \rho(txt^{-1})$ has field operator $\alpha\rho(h)A$, with $\alpha^f = \omega_\rho(z)^{-1}$. Write $\omega_\rho(z)$ for the scalar at $z = -1$. Moreover, on a joint coefficient stabilizer B, the ratio for ρ^t differs from that for (ρ, A) by $\sigma(\alpha)/\alpha$.*

Proof. We calculate $\nu(tF_r(t)^{-1}) = b^{1-r} = c^2$. Hence, $\nu(h) = 1$. On the indicated torus of H, the element h also normalizes the marked subgroup of H. Since $hF_r(t) = c^{-1}t$, the operator $\rho(h)A$ implements F_r on ρ^t. The ordered norm of c is $b^{-(q-1)/2} = -1$, and $F_q(t) = v^{-1}tv$. Thus, $\prod_{j=0}^{f-1} F_r^j(h) = z\,tv^{-1}t^{-1}v$ and $(\rho(h)A)^f = \omega_\rho(z)\rho^t(v^{-1})$. Hence, $(\alpha\rho(h)A)^f = \rho^t(v^{-1})$. For σ in the coefficient stabilizer, one intertwiner compares ρ with ρ^σ, and ρ^t with $(\rho^t)^\sigma$. Since h has ordinary matrix $\rho(h)$ in both comparisons, only α contributes the extra ratio $\sigma(\alpha)/\alpha$. In the faithful sector $\omega_\rho(z) = -1$, we choose α of order $2f_2$, with $\alpha^f = -1$. Otherwise, choose $\alpha = 1$. In particular, the choices of α cover both $v = 1$ and local v. □

If f is odd and $\omega_\rho(z) = -1$, we can set $\alpha = -1$. The diagonal and coefficient actions satisfy $(\rho^t)^\sigma = (\rho^\sigma)^t$. Moreover, we have that t^2 acts by an inner automorphism on $H\langle a\rangle$, including Ha. Indeed, we have that $s = t^2/b$ is in the spin group H, and lies in the local torus T_H. The correcting elements satisfy $tht^{-1}h = sF_r(s)^{-1}$. Suppose that σ_0 exchanges the two branches. The element $\tau = t\sigma_0$ fixes one branch χ. Suppose that χ has an extension $\widehat{\chi}$ fixed by the pure coefficient stabilizer Σ_χ. The fixed extensions form a torsor $\mathcal{T}$ under an odd-order subgroup $C \leq \mathrm{Irr}(H\langle a\rangle/H)$. The square τ^2 fixes $\mathcal{T}$ pointwise, since $\sigma_0^2 \in \Sigma_\chi$. If $\tau(u) = u\mu$, with $u \in \mathcal{T}$, its linear action satisfies $\tau(\mu) = \mu^{-1}$. Thus, the unique square root $\sqrt{\mu}$ in C gives that $\tau(u\sqrt{\mu}) = u\sqrt{\mu}$. Thus, we have that τ fixes an extension of χ.

Lemma 4.12. *Suppose that $q = r^f$ is odd. Define $c_\sigma \in \mathbb{F}_r^\times$ by $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$, and put $\epsilon_\sigma = (c_\sigma/r)$. For either exceptional pair of $\mathrm{SL}_2(q)$, the exchange sign is $\eta_q(c_\sigma)$. More precisely, the degrees are $(q+1)/2$ for the principal pair and $(q-1)/2$ for the cuspidal pair. Let χ correspond to $\psi(x) = \zeta_r^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x)}$, with full field extension $\widetilde{\chi}$ to $\langle F_r\rangle$. If f is even, the ratio on $\mathrm{Stab}_{\mathcal{H}_p}(\chi)$ is 1 in the principal case, and ϵ_σ in the cuspidal case. If f is odd, the ratio is 1 for both pairs.*

Proof. We take ω_ψ, the Weil representation for $\psi(x) = \zeta_r^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x)}$ on functions on $\mathbb{F}_q$. In particular, V_δ consists of functions satisfying $g(-x) = \delta g(x)$, for $\delta = 1, -1$. On V_δ, the two square classes $u(\mathbb{F}_q^\times)^2, u'(\mathbb{F}_q^\times)^2$, with $u = 1$, give the irreducible pair $\{\chi_u, \chi_{u'}\}$ [119, §2.2, (2.1)]. Given an additive label u, the nonzero upper-unipotent weights are

$u(\mathbb{F}_q^\times)^2$. Since a coefficient operation σ replaces u by $c_\sigma u$, the square class of c_σ gives the exchange sign $\eta_q(c_\sigma)$. The operator at $-I$ is $g(x) \mapsto \eta_q(-1)g(-x)$. Hence, the two central scalars are $\delta\eta_q(-1)$.

The field operator $Pg(x) = g(F_r^{-1}x)$ permutes the function basis, with $P^f = 1$. Since $\mathrm{Tr}_{\mathbb{F}_{q^f}/\mathbb{F}_r}$ is invariant under F_r, for unipotent and torus elements g_0 $P\omega_\psi(g_0)P^{-1} = \omega_\psi(F_r(g_0))$. Similarly, the Fourier formula for ω_ψ gives the identity when g_0 is the Weyl representative. Thus, P implements entrywise F_r. If σ fixes the constituent, choose $s \in \mathbb{F}_q^\times$ with $s^2 = c_\sigma$. The dilation $D_s g(x) = g(sx)$ satisfies $D_s\omega_\psi(g_0)D_s^{-1} = \omega_\psi(g_0)^\sigma$ for $g_0 \in \mathrm{SL}_2(q)$. Indeed, for unipotent g_0, the quadratic argument gains the factor s^2. Whenever $a \in \mathbb{F}_q^\times$, put $t_a = \mathrm{diag}(a, a^{-1})$. Then, $D_s\omega_\psi(t_a) = \omega_\psi(t_a)D_s$. In ω_ψ, the Fourier substitution by s gives the same kernel factor s^2. The quadratic Gauss sum $G(\psi)$ in the normalization is fixed, since $\eta_q(c_\sigma) = 1$. Now, $s^r/s = (c_\sigma/r)$. On V_δ, $D_{-s} = \delta D_s$. Hence, conjugation of P by D_s has ratio 1 for $s^r = s$, and ratio δ otherwise. If $2 \nmid f$ and σ fixes the constituent, then $c_\sigma \in (\mathbb{F}_r^\times)^2$. Thus, the full extension ratios are $(1, \epsilon_\sigma)$ for even f, and $(1, 1)$ for odd f, on the principal and cuspidal branches $(\chi_{\mathrm{pr}}, \chi_{\mathrm{cusp}})$.

For the other additive square class $u'(\mathbb{F}_q^\times)^2$, take Lemma 4.11 in $\mathrm{GL}_2(q)$, with determinant as norm and $v = 1$. On the faithful negative blocks, $\mathrm{ord}_p(q)$ is even. Then, $v_2(\mathrm{ord}_p(r)) = v_2(f) + v_2(\mathrm{ord}_p(q))$, so $2f_2 \mid p-1$. Consequently, the common diagonal root α is fixed by $\mathcal{H}_p$. If $2 \nmid f$, we can take $\alpha \in \{1, -1\}$. Hence, the additional ratio is $\sigma(\alpha)/\alpha = 1$ on $\mathrm{Stab}_{\mathcal{H}_p}(\chi)$ for either member χ. If f is even and $\omega_\chi(z) = 1$ for a principal constituent χ, the diagonal root is $\alpha = 1$, independently of $\mathrm{ord}_p(q)$. □

Lemma 4.13. *At $a = 1$, $b = 0$, the symplectic labels are* 1 *and* St *of* $\mathrm{Sp}_2(q^2)$. *In the faithful series of* $\mathrm{Spin}_5(q)$, *both diagonal pairs have exchange sign $\eta_q(c_\sigma)^3$. Moreover, for the standard branch $\chi \in \{\chi_+, \chi_-\}$, the full field extension has ratio ϵ_σ^3 on its stabilizer* $\mathrm{Stab}_{\mathcal{H}_p}(\chi)$. *On the other branch χ', take the correction in* Lemma 4.11 *for* $\mathrm{Spin}_5(q)$.

Proof. The connected centralizer C° in the exchanged series has factor $\mathrm{Sp}_2(q^2)$. Since $\mathrm{Sp}_2(q^2)$ has unipotent characters 1 and St, the semisimple and regular pairs $(\mathcal{P}_{\mathrm{ss}}, \mathcal{P}_{\mathrm{reg}})$ exhaust the starting series for $\mathrm{Spin}_5(q)$. In Proposition 4.38, we set $n = 2$. Then, the grading element is $T_1(c_\sigma^2)T_2(c_\sigma)$, of norm c_σ^3. Both pairs are faithful at z. Thus, from P and its semisimple dual $P^\vee$, the ratio is $\epsilon_\sigma^{K_2}$, where $K_2 = 3$. Moreover, for the other member χ, the correction $\sigma(\alpha)/\alpha$ is from Lemma 4.11. □

Lemma 4.14. *Let κ be the e-core of λ, with size $s < e$. With $b = ew + s$, we write $h = 2a + s < e$. Suppose that the e-quotient has degree prime to p. The Harish–Chandra comparison has multiplicity* 1 *on the unitary two-quotient of λ, by* Lemma 4.19. *Each rank-one step $\lambda_j \to \lambda_{j+1}$ has exchange sign $\eta_q(c_\sigma)$. If f is even, the field ratio is ϵ_σ. If λ and κ have a common two-core* $\mathrm{core}_2(\lambda) = \mathrm{core}_2(\kappa)$, *their exchange-sign ratio is $(\sigma(i)/i)^A$, with $A = K_{ew} + hw$. If f is even, the field quotient is ϵ_σ^A.*

Proof. The common base is (ρ, Δ_{t_0}), with $|\Delta_{t_0}| = b_0$. We choose one extension $\widetilde{\chi}_0$ at the base. Thus, for λ, κ, the lengths are $(b - b_0)/2$ and $(s - b_0)/2$. At each step of multiplicity 1, the extensions are determined from $\widetilde{\chi}_0$. The rank-one factors are $\eta_q(c_\sigma)$ for the sign. If f is even, the ratio factor is ϵ_σ. Hence, the base factors from $\widetilde{\chi}_0$ cancel in the quotients for λ, κ. Their length difference is $m = ew/2$, so w is even and $K_{2m} + hw \equiv m \pmod 2$. Thus, the exchange-sign quotient is $\eta_q(c_\sigma)^m$. In even field degree f, the field-ratio quotient is ϵ_σ^m. Since $A \equiv m \pmod 2$, both formulas follow for λ, κ. □

If the two-cores of λ, κ differ, the joint base is $(a, b) = (k(k+1), K_t)$. With $a = k(k+1)$ and $b = K_t$, we put $m = \lfloor t/2 \rfloor$, $R = \max(k, m)$ and $h = R - k$. The wave-front partition Λ_{wf} of the regular parent is $(2\alpha_1 + 1, 2\beta_1 - 1, \ldots, 2\alpha_R + 1, 2\beta_R - 1, 2\alpha_{R+1} + 1)$, where $\alpha_j = 2(k - j + 1)_+ + (t - 2j + 2)_+$, $\beta_j = 2(k - j + 1)_+ + (t - 2j + 1)_+$, and $x_+ = \max(x, 0)$. The symplectic bipartition is (Δ_k, Δ_k). Thus, truncated induction with Δ_t gives the componentwise sum (α, β). Thus, the unique dominance maximum is (α, β). Each row has coefficient 1 in Littlewood–Richardson induction. If (μ, ν) occurs in the induction, the type B b-function satisfies $b_B(\alpha, \beta) \le b_B(\mu, \nu)$. The partition Λ_{wf} has $2k + 1$ odd sizes of multiplicity 1 and h of multiplicity 2. Accordingly, write D_j for the sizes of multiplicity 1, with $1 \le j \le 2k + 1$ and E_l for the sizes of multiplicity 2, with $1 \le l \le h$.

Lemma 4.15. *Let $\mathcal{D}$ be the multiset of parts of Λ_{wf}. Put $L = \sum_{D \in \mathcal{D}} K_{(D-1)/2}$. Then, modulo* 2 *we have that $L \equiv K_t$ and $L + K_n \equiv K_m \pmod 2$. Assume that the paired multiplicity spaces M_l are nonsplit. In* $\mathrm{Cl}(W_{\mathrm{nat}})$, *choose equal-norm volumes v_j for the single blocks D_j, and v'_l for one block of each size E_l. Moreover, let δ be the common norm. If $S \subseteq \{1, \ldots, 2k + 1\}$ indexes single blocks with $|S| \equiv h \pmod 2$, we have that the product $w(S)$*

is allowed. Its square satisfies $w(S)^2 = z^{Q(S)}$*, with* $Q(S) = \frac{1}{2}\big(\sum_{j\in S} D_j + \sum_l E_l\big) \pmod 2$*. Moreover, we have that* $\sum_{|S|\equiv h \ (2)} (-1)^{Q(S)+K_m} = 2^k$*.*

Proof. We calculate the grading sum L as $\sum_{j\le R}(K_{\alpha_j} + K_{\beta_j - 1}) + K_{\alpha_{R+1}}$. While $t - 2j + 2 > 0$, $\alpha_j = \beta_j + 1$. Thus, the summand is $\beta_j^2 + \beta_j + 1$, which is odd. After Δ_t ends, equal rows give α_j^2, which is even. If t is odd and $R = m$, the final $\alpha_{R+1} = 1$ contributes 1 more odd summand. Thus, there are $\lceil t/2 \rceil$ odd summands, and $\lceil t/2 \rceil \equiv K_t \pmod 2$. Since $n = 2k(k+1) + K_t \equiv K_t \pmod 4$, substitute $t = 2m, 2m+1$ in $K_t + K_{K_t} \equiv K_m \pmod 2$. This gives that $L + K_n \equiv K_m \pmod 2$.

By [137, Theorem 24.56 and its following volume-square formula], $v_a v_b = -v_b v_a$ for distinct odd blocks and $v_D^2 = (-1)^{(D-1)/2}\delta$. With $u = |S| + h$ even, the normalized product $w(S)$ has square $(-1)^{\binom{u}{2}}\delta^{-u}\prod_D v_D^2$, for the selected sizes D. Thus, its sign exponent is $\sum (D-1)/2 + \binom{u}{2} \equiv \sum D/2 \pmod 2$, giving $Q(S)$. If S, T are even, the polar form is $(S,T) \mapsto |S \cap T| \bmod 2$. Write $V_0 \subseteq \mathbb{F}_2^{2k+1}$ for the even-subset space. Then, $V_0^\perp = \langle \mathbf{1} \rangle$, where $\mathbf{1}$ has odd weight. Thus, $V_0 \cap V_0^\perp = 0$, so the polar form on V_0 is nondegenerate. We define $G_Q = \sum_{|S|\equiv h \ (2)} (-1)^{Q(S)}$. The Gauss sum G_Q is $\mathrm{Re}(i^{\sum_l E_l}\prod_j (1 + i^{D_j}))$. Indeed, the summand for S is real exactly when $|S| \equiv h \pmod 2$. If $t \le 2k$, no paired blocks occur. Let N_1, N_3 count the single sizes congruent to 1 and 3 mod 4. Their difference is $N_1 - N_3 = (-1)^t(2m+1)$. If $t > 2k$, all block sizes have residue $2t+1 \bmod 4$. The paired blocks E_l contribute angle $(-1)^t h\pi/2$ modulo 2π, and $h + k = m$. Thus, the real part is $2^{(2k+1)/2}\cos((2m+1)\pi/4) = 2^k(-1)^{K_m}$. □

Lemma 4.16. *Assume that* $q \equiv 1 \pmod 4$ *on the wave front* Λ_{wf}*. Let* Θ *be a faithful parent of* $\widetilde{X}$*, regular with quadratic norm self-twist* $\Theta(\eta_q \circ \nu) = \Theta$*. Then, each paired multiplicity space* $\mathcal{M}_i$ *at the selected rational class* $\mathcal{C}$ *is nonsplit. An irreducible character* θ_{src} *of the spin source* C *selects a branch* χ *with multiplicity* 1*. Moreover, the component module* W_{comp} *has a model over* $\mathbb{Q}(i)$*. The field intertwiner* F_{comp} *can be chosen over* $\mathbb{Q}(i)$*, with* $F_{\mathrm{comp}}^f = 1$*.*

Proof. We write $\widetilde{T}$ for the rational source torus, including Z, the scalar torus. In particular, the h two-dimensional spaces $\mathcal{M}_i$ have independent reflections s_i. If a factor of $\widetilde{T}$ is split, the reflection exchanges exponents $a_1, a_2 \bmod (q-1)$. The character is faithful on $\langle z \rangle$ exactly when $a_1 + a_2$ is odd. If $T_i \le \widetilde{T}$ is nonsplit, the exponent $b \bmod (q^2-1)$ is odd. The reflection acts by $b \mapsto qb$. If $qb \equiv b \pmod{q^2-1}$, $q+1 \mid b$, contrary to odd b. Thus, each faithful weight τ of $\widetilde{T}$ has reflection orbit of size 2^h, independently of $\mathcal{C}$. The remaining group E consists of even subsets of $\{1, \dots, 2k+1\}$. Its commutator on τ is $(-1)^{|S\cap T|}$, with zero radical on E. Thus, adjoin scalars $U(1)$ to the lifts. By [151, Theorem 2.3], A_{tw} has one simple complex module. Hence, $A_{\mathrm{tw}} \simeq \mathrm{Mat}_{2^k}(K)$ over a splitting field $K \subseteq \mathbb{C}$, since $\dim_K A_{\mathrm{tw}} = 2^{2k}$. Moreover, the lifts for $\mathcal{C}$, after scalar corrections, have commutator $(-1)^{|S\cap T|}$.

Thus, by Clifford induction, an irreducible V faithful at z has degree 2^{k+h}. Given a family label ρ, its component group has order 2^{2k+h}, and its generic-degree denominator is 2^k. The symbol ρ gives the denominator 2^k. Since Frobenius exchanges the geometric factors (ρ, ρ), the finite label is over $\mathbb{F}_{q^2}$. The unitary staircase Δ_t has denominator 1. Consequently, the sum of class multiplicities $m_{\mathcal{C}}$ is $\sum_{\mathcal{C}} m_{\mathcal{C}} = 2^{k+h}$, by [188, Proposition 15.4 and (11.15)]. Each $m_{\mathcal{C}}$ lies in $2^{k+h}\mathbb{Z}_{\ge 0}$. Hence, exactly one class $\mathcal{C}$ contributes one irreducible module V.

On the regular Gelfand–Graev space $\mathcal{H}_{\mathrm{ind}}$, define $J_\nu \phi(g) = \eta_q(\nu(g))\phi(g)$. The norm ν is 1 on the source unipotent group U, so J_ν implements the norm twist. On the Θ-isotypic part, J_ν splits into the parent intertwiner J_Θ and an intertwiner J_V on V. Translation by a source element y contributes $\eta_q(\nu(y))$. Thus, $V \simeq V \otimes (\eta_q \circ \nu)$. Given a torus weight τ, its twist lies in $\langle s_1, \dots, s_h \rangle \cdot \tau$. The twist is nontrivial on each $\mathcal{M}_i$, so each s_i occurs. A split T_i would require $a_1 - a_2 = (q-1)/2 \bmod (q-1)$ and odd $a_1 + a_2$. Since $4 \mid q-1$, the split case is impossible. For T_i nonsplit, $(q-1)b = (q^2-1)/2 \bmod (q^2-1)$, equivalently $b = (q+1)/2 \bmod (q+1)$. Hence, all paired spaces $\mathcal{M}_i$, $1 \le i \le h$, are nonsplit. We fix $\tau|_Z$. The two admissible exponents b_l differ by $(q^2-1)/2$. Thus, both exponents are congruent modulo $q-1$ and modulo $q+1$. Moreover, $q+1 \mid 2b_l$. Since the factors l have a common scalar character and $b_l \equiv (q+1)/2 \pmod{q+1}$, the quotients $2b_l/(q+1)$ are congruent modulo $q-1$. Thus, the squares τ_l^2 are a common norm pullback. On Z, the square is the pullback along $c \mapsto c^2$. Consequently, the joint weight τ has order at most 2 on the common norm kernel $\widetilde{T} \cap \ker \nu$.

We write $\widetilde{C}$ for the rational source centralizer and C for its spin source. Suppose that $\nu(\widetilde{C}) = \mathbb{F}_q^\times$. The quotient $\widetilde{C}/C$ is cyclic. On scalars Z, the norm ν is squaring. Therefore, a self-twist $\lambda \ne 1$ of V is quadratic. Thus, $\dim_{\mathbb{C}} \mathrm{End}_C(V) = 2$ by [95, Lemma 5.2 and Problem 5.3]. By [95, Theorem 6.2], $V|_C = V_+ \oplus V_-$ with $\dim V_+ = \dim V_-$. A nonsquare-norm element $y \in \widetilde{C}$ exchanges the spin branches χ_+, χ_- and fixes the character ϑ. Thus, the branch multiplicities satisfy $m_+ = m_-$. Their spaces are V_+, V_-. Selecting its character θ_{src} gives that branch with multiplicity 1. Suppose

instead that $\nu(\widetilde{C}) \neq \mathbb{F}_q^\times$. A nonsplit torus T_i has surjective norm $\mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$, so $h = 0$. If a single-block class $\bar{x}$ is nonidentity, we choose v and x with $vx = -xv$, where v is a volume product and x represents $\bar{x}$. Choose c with $c^{q-1} = -1$ to correct the Frobenius sign. Then, $y = cx$ has nonsquare norm, a contradiction. Thus, $\bar{x} = 1$. On C, the faithful twisted algebra of E has one module of degree 2^k, including the central-character line at $k = 0$. Each branch multiplicity is divisible by 2^k, and their sum is 2^k. Therefore, the selected character θ_{src} gives the unique occurring branch χ with multiplicity 1. The other branch χ' is the diagonal conjugate of χ. If the spin pair $\{\chi_+, \chi_-\}$ is absent, we exclude rank 0.

Choose models over $\mathbb{F}_r$ for the quadratic blocks D_j, E_l. In particular, we let (B_l, B_l') be the paired blocks of size E_l. Assign volume norm δ_{vol} to the single blocks D_j and to B_l. If the norm on B_l is δ_{vol}, we set the norm on B_l' to $-\delta_{\mathrm{vol}}$. Changing δ_{vol} changes the total discriminant on $2k+1$ single blocks. Thus, δ_{vol} can give the chosen odd space W_{nat}. The equal-norm volumes are over $\mathbb{F}_r$, so each normalized even product w satisfies $F_r(w) = w$. Since each $\mathcal{M}_l$ is nonsplit, Lemma 4.15 gives $w^2 \in \{1, z\}$. Accordingly, set $F' = \mathrm{Int}(w)F_q$. On the spin-source quotient for τ, the torus image $\overline{T}$ has exponent 2. The corrected volume lifts u_i have squares in $\{1, z\}$, after fourth-root corrections on $\mathcal{M}_i$. Moreover, we have that the commutators $[u_i, u_j]$ are central of order at most 2. Indeed, a reflection u_i inverts its torus coordinate modulo Z. The corresponding commutator c_i has order at most 2, since $\overline{T}$ has exponent 2. The reflection u_i inverts c_i, so it fixes c_i. Other reflections u_j fix c_i, since its support is in factor i. Thus, c_i is central. Distinct lifts u_i, u_j satisfy $u_i u_j = z u_j u_i$, with central involution z. Thus, the finite group G_{comp} has class at most 2 and exponent dividing 4. Write J for the image on W_{comp}. Then, its center $Z(J)$ embeds in μ_4 by Schur's lemma. Let $A \leq G_{\mathrm{comp}}$ lie over a maximal abelian $M \leq J$. Moreover, the scalar character of $Z(J)$ extends to M, with inflation ϑ_A to A. Thus, with scalars $U(1)$ adjoined, [151, Theorem 2.3] gives $W_{\mathrm{comp}} \simeq \mathrm{Ind}_A^{G_{\mathrm{comp}}} \vartheta_A$ over $\mathbb{C}$. Its induced matrices lie in $\mathbb{Q}(i)$. The field map F_r acts by $u_i \mapsto u_i$ or $z u_i$, through its fourth-root correction. Given $r \equiv 1 \pmod 4$, all lifts u_i are fixed, so take $F_{\mathrm{comp}} = I$. Otherwise, f is even. By [208, Lemma 6.12], the spin branch χ is field-invariant. Since the source θ_{src} has multiplicity 1, the selected component module W_{comp} is field-invariant. Hence, ϵ_F is trivial on the radical $\mathcal{R}$ of the represented commutator pairing. On $G_{\mathrm{comp}}/\mathcal{R}$, write $\eta_F = (-1)^{\epsilon_F}$. By [151, Lemma 3.1], the η_F-eigenspace in $\mathrm{End}_{\mathbb{C}}(W_{\mathrm{comp}})$ is spanned by the matrix Y of a lift $y \in G_{\mathrm{comp}}$. Thus, Y implements the field action, with $Y^2 \in \{I, -I\}$. If $F_{\mathrm{comp}} = Y$ or $F_{\mathrm{comp}} = iY$ according to the sign of Y^2, $F_{\mathrm{comp}}^2 = 1$. Thus, $F_{\mathrm{comp}}^f = 1$ for even f, with F_{comp} defined over $\mathbb{Q}(i)$. On the quotient $G_{\mathrm{comp}}/\mathcal{R}$, the map F_q is the identity. Thus, in the F'-model, the scalar of w on the component is ξ^{-1}, of order dividing 4. The full field operator P_{src} on the selected source space has $P_{\mathrm{src}}^f = \xi^{-1} R(w^{-1})$. Choose $\alpha^f = \xi$, with $\alpha \in \mu_{4f_2}$. Then, $(\alpha P_{\mathrm{src}})^f = R(w^{-1})$, with the original inner carry $R(w^{-1})$. □

We choose symplectic family labels ρ with multiset $\{0, \ldots, 2k\}$, as in Lemma 4.16. Let $\mathcal{O}$ be the wave-front class of ρ. If ι is a principal Springer pair on $\mathcal{O}$, we write $\mathcal{Y}_\iota$ for its local-system function on $\mathcal{O}$. Choose representatives $u \in \mathcal{U} \subseteq \mathcal{O}^F$. Their generalized Gelfand–Graev characters are Γ_u. Accordingly, define Γ_ι as the weighted sum of Γ_u, with weights $\mathcal{Y}_\iota(u)$ for $u \in \mathcal{U}$, normalized by Taylor [188, (11.15)]. Consequently, the projection for Γ_ι distinguishes the selected classes in $\mathcal{O}$.

Lemma 4.17. *Write W for the ambient Weyl group and $W(s)w_1F$ for the rational Weyl coset of s. Let $\widetilde{\varphi}_\iota$ be the extended Springer character, and $\widetilde{\epsilon}$ the Weyl sign. If Θ is regular with wave front* $\operatorname{supp}\iota$*, we have that $\langle \Gamma_\iota, \Theta \rangle = a_\iota \langle R_{\mathrm{Res}^{WF}_{W(s)w_1F}(\widetilde{\varphi}_\iota \widetilde{\epsilon})}, \Theta \rangle$, with normalization* (Digne et al. [59, Corollary 3.14(i)])*. In particular, $a_\iota = |A_\iota|$ for the algebraic component group A_ι on* $\operatorname{supp}\iota$*. The uniform function R is the torus-induction average on $W(s)w_1F$.*

Proof. The regular group has type B, so the odd characteristic r is very good. Thus, r is acceptable by [188, Lemma 6.3(i)]. Since r is acceptable, we expand Γ_ι by [188, Theorems 11.10 and 11.13, Lemma 11.16]. The expansion indices γ lie in the block of ι. The principal cuspidal Levi L_0 is a torus. In particular, at $y \in WF$, the Weyl average has factor $|T_y^F|$. Since the identity indicator on T_y^F is $|T_y^F|^{-1} \sum_{\theta \in \mathrm{Irr}(T_y^F)} \theta$, the factor $|T_y^F|$ cancels $|T_y^F|^{-1}$. We expand the torus induction functions $R_{T_y}^{\widetilde{X}}(\theta)$ and project to $\mathcal{E}(\widetilde{X}, s)$. Thus, (y, θ) contributes only if s is conjugate to its dual. Given $y_0 \in W(s)w_1F$, the pair (y_0, s) has $|W|/|C_{W(s)}(y_0)|$ ambient conjugates, and $|W(s)|/|C_{W(s)}(y_0)|$ conjugates with fixed s. Consequently, the average has denominator $|W(s)|$. With preferred extensions $\widetilde{\varphi}_\gamma$, the principal-block projection is $a_\iota \sum_\gamma \widetilde{P}^*_{\iota,\gamma} R_{\mathrm{Res}^{WF}_{W(s)w_1F}(\widetilde{\varphi}_\gamma \widetilde{\epsilon})}$ [59, Proposition 3.11]. For local intersection cohomology, $\widetilde{P}^*$ is the normalized matrix.

If γ contributes against Θ, $\dim \operatorname{supp}\gamma \leq \dim \mathcal{O}$ by [59, Proposition 3.13]. If the dimensions are equal, $\operatorname{supp}\gamma =$

$\mathcal{O}$. The Springer fibre dimension d_γ is compared with Weyl-family invariant a_Θ. The wave-front description in characteristic r is [188, Theorem 14.10], with connected-center proof for Θ. If $\widetilde{P}^*_{\iota,\gamma} \neq 0$, $\operatorname{supp}\iota \subseteq \overline{\operatorname{supp}\gamma}$. On a single support $\mathcal{O}$, the matrix $\widetilde{P}^*$ is the identity. Thus, only $\gamma = \iota$ contributes. The weighted formula in characteristic r precedes projection to $\mathcal{E}(\widetilde{X}, s)$. Thus, compare supports with $\mathcal{O}$ after projection. A normalization change for $\mathcal{Y}_\iota$ rescales row ι of $\widetilde{P}^*$ by a nonzero scalar c_ι. □

Lemma 4.18. *Let θ_{src} be as in* Lemma 4.16. *Then, $\langle \Gamma_\iota, \Theta\rangle$ from* Lemma 4.17 *distinguishes classes for 2^{2k} labels.*

Proof. The Weyl family $\mathcal{F}$ indexes its columns by complementary sets A, B in $\{0, \dots, 2k\}$, of sizes $k+1, k$. We index both rows from 0. Then, the increasing entries of α, β are $A_i - i$ and $B_j - j$, respectively. The Schur functions s_α, s_β take decreasing rows. Moreover, the lengths $R+1, R$ have beta-sets $A^h = \{0, \dots, h-1\} \cup (h + A)$ and $B^h = \{0, \dots, h-1\} \cup (h + B)$. The Springer symbol of row (A, B) is U, V. If $t \leq 2k$, we apply $x \mapsto 2x + (t - 2k + x)_+$ to the entries of A, B. If $t \geq 2k$, we write $t = 2m + \varepsilon$ and take $U = \{4j + \varepsilon : 0 \leq j < h\} \cup (4h + \varepsilon + 3A)$ and $V = \{4j + 1 + \varepsilon : 0 \leq j < h\} \cup (4h + \varepsilon + 3B)$. The formulas coincide at $t = 2k$. Moreover, the gaps within U, V are at least 2. Their combined entries give the distinguished u-symbol for Λ_{wf}. Thus, (U, V) gives a principal Springer pair on $\mathcal{O}$. The associated Weyl bipartition $\Phi_{A,B}$ has increasing rows $U_i - 2i, V_j - 2j$, by the step-two convention [104, §§5.2, 7.2–7.3].

Thus, X, Y are alphabets of $R+1, R$ variables, respectively. For the column (A', B'), with bipartition (α', β'), the entry $\mathsf{M}_{(A,B),(A',B')}$ is the coefficient of $s_{\Phi_{A,B}}$ in $s_{\alpha'}(X^2) s_{\beta'}(Y^2) s_{\Delta_t}(X, -Y)$. Equivalently, in $-Y$, each variable y becomes $-y$. For the flip $\tau(v \otimes w) = w \otimes v$ on $E \otimes E$, the trace of $(x, y)\tau$ is $\chi_E(xy)$. Thus, a cycle c of xy with length l gives a cycle of length $2l$, with signed product $\epsilon(c)$. Hence, the variables are X^2, Y^2. The unitary coset acts by -1 on its coordinates, giving $s_{\Delta_t}(X, -Y)$. Preferred extensions and Weyl sign change M to $D_1 \mathsf{M} P D_2$, with invertible diagonal D_1, D_2 and a column permutation P. Also, $\Delta_t^{\mathsf{t}} = \Delta_t$.

We prove that M is signed diagonal. The entry $\mathsf{M}_{(A,B),(A',B')}$ is the coefficient at exponents $U_i - i, V_j - j$ in the alternant expression $\frac{\det(x_i^{2(A')_j^h}) \det(y_i^{2(B')_j^h})}{\prod_{i<i'}(x_i + x_{i'}) \prod_{j<j'}(y_j + y_{j'})} s_{\Delta_t}(X, -Y)$. Since the exponents are even, $D_X \mid \Delta_X$ and $D_Y \mid \Delta_Y$. If $t \geq 2k$, $t = 2R + \varepsilon$. In $2R+1$ variables z_i, the staircase is $(\prod z_i)^\varepsilon \prod_{i<j}(z_i + z_j)$, by the alternant formula. Thus, D_X, D_Y cancel, as does the common ε-monomial. The cross factors are $\prod_{i,j}(x_i - y_j)$, up to sign. We order X, Y as $(z_1, \dots, z_{2R+1})$, by $U - \varepsilon, V - \varepsilon$. The first entries are $0, 1, 4, 5, \dots, 4h-4, 4h-3$. Moreover, the remaining entries are $4h + 3l$, for $0 \leq l \leq 2k$. We write $d_1 \leq \cdots \leq d_{2R+1}$ for sorted exponents, each of $0, 2, \dots, 2h-2$ twice, followed by $2h, 2h+2, \dots, 2h+4k$. Take a prefix $\mathcal{P}$, containing u variables of X and v of Y. Then, the target degree on $\mathcal{P}$ is $\sum_{j=1}^{u+v} d_j + uv$. Indeed, a target exponent a_j is $d_j + o_j$, with o_j counting earlier variables from Y when $z_j \in X$, and from X when $z_j \in Y$. Indeed, the row indices in U, V give that $a_j = d_j + o_j$. In a contributing determinant monomial m, the degree on $\mathcal{P}$ is at least $\sum_{j=1}^{u+v} d_j$. Moreover, each of the uv internal cross factors contributes degree 1 to $\mathcal{P}$. The target degree on $\mathcal{P}$ attains both bounds. Hence, the exponents d_j occur in order, and each factor $x_i - y_j$ contributes to its later endpoint z_ℓ. Each single exponent d_j belongs to its assigned alphabet X or Y. Each repeated exponent $d_j = d_{j+1}$ has one occurrence in each alphabet X, Y. Thus, only the increasing determinant monomials m contribute with one choice for each $x_i - y_j$. The labels must satisfy $(A, B) = (A', B')$, and the diagonal coefficient is ± 1.

If $t < 2k$, allow row lengths with total N. We put $g_N(x) = 2x + (t - N + 1 + x)_+$, initially with $N = 2k+1$. Suppose that $N > t+1$. The variable z_* assigned entry 0 has target exponent 0. If $z_* = 0$, the determinant of its alphabet $Z_* \in \{X, Y\}$ is 0 unless its assigned row contains 0. Consequently, in (A, B) and (A', B'), the assignments at z_* are absent. For each other position z, replace its entry a by $a - 1$. The even alternants contribute degree 2 at each remaining variable z. Then, division by $\prod_{z \in Z_* \setminus \{z_*\}} z$ leaves degree 1 in $Z_* \setminus \{z_*\}$. The target becomes g_{N-1} minus the new row index i, since $g_N(x+1) - 2 = g_{N-1}(x)$. At $N = t+1$, the exponents d_j are distinct. The staircase formula for s_{Δ_t} and the prefix bounds for $\mathcal{P}$ are valid for arbitrary row lengths $(|X|, |Y|)$. Thus, M is signed diagonal for all t.

For principal ι, Lemma 4.17 expresses $\langle \Gamma_\iota, \Theta\rangle$ by the weighted class multiplicities on $\mathcal{U}$ and the scalar product with $R_{\mathrm{Res}^{WF}_{W(s)w_1F}(\widetilde{\varphi_\iota \epsilon})}$, up to fixed row scalar $c_\iota \neq 0$. Thus, M separates the coefficient vector $\mathbf{c}_\rho$ for the symplectic uniform projection on $\mathcal{F}$. The coefficients outside $\mathcal{F}$ are 0. Suppose that two labels ρ, ρ' select the same rational class $\mathcal{C}$. Their class multiplicities are both 2^{k+h}, so weighted multiplicities coincide for every ι. Hence, $\mathbf{c}_\rho = \mathbf{c}_{\rho'}$, and their uniform projections $\rho_{\mathrm{unif}}, \rho'_{\mathrm{unif}}$ coincide. Symplectic unipotent labels ρ are distinguished by uniform projection ρ_{unif},

by [149, Proposition 3.12 and Lemma 4.1]. Thus, $\rho = \rho'$. The selection is injective, with 2^{2k} labels and 2^{2k} allowed classes. Hence, selection $\rho \mapsto \mathcal{C}$ is bijective. □

Lemma 4.19. *We choose the series $\mathcal{E}$ of faithful mixed unitary type with $q \equiv 1 \pmod 4$, and we write ρ for its symplectic label of rank a, and λ for its unitary label of size b, with $n = 2a+b$. Moreover, we write $\boldsymbol{\lambda}_\rho$ for the ordinary relative bipartition of ρ. We choose either a predecessor $\boldsymbol{\mu} \subset \boldsymbol{\lambda}_\rho$, with $|\boldsymbol{\lambda}_\rho/\boldsymbol{\mu}| = 1$, or a predecessor $\boldsymbol{\mu} \subset \mathrm{quot}_2(\lambda)$, with $|\mathrm{quot}_2(\lambda)/\boldsymbol{\mu}| = 1$. We write $\mathrm{quot}_2(\lambda)$ for the unitary two-quotient of λ. Moreover, we suppose that the predecessor $\boldsymbol{\mu}$ has spin branches θ'_+, θ'_-. We write $s_{\mathrm{pred}}(\sigma)$ for their exchange sign under σ, and $r_{\mathrm{pred}}(\sigma)$ for the field ratio on θ'_δ, with $\delta \in \{+,-\}$. By* Lemma 4.12*, we have sign $s_1(\sigma) = \eta_q(c_\sigma)$. If f is even, $r_1(\sigma) = \epsilon_\sigma$. We take source characters ψ_+, ψ_- and target characters χ_+, χ_- with multiplicity matrix $M = (m_{ij})$. Moreover, we have that M is a permutation matrix. Thus, $s(\sigma) = s_1(\sigma)s_{\mathrm{pred}}(\sigma)$. On each stabilizer of χ_i, we have that $r(\sigma) = r_1(\sigma)r_{\mathrm{pred}}(\sigma)$ when f is even.*

Proof. First, we put $X' = \mathrm{Spin}_{2(n-2)+1}(q)$ and $\widetilde{X}' = \mathrm{GSpin}_{2(n-2)+1}(q)$. We choose the regular Levi subgroup $\widetilde{L} = \mathrm{GL}_2(q) \times \widetilde{X}'$, common to both steps, with norm $\nu_L(A,y) = \det(A)\nu(y)$. Moreover, we have residual ranks $(a-1,b)$ in the symplectic step and $(a,b-2)$ in the unitary step. We write Θ' for the residual regular parent and ω for the scalar character on $\mathbb{F}_q^\times$. Moreover, in a dual lift ty, with $t^q = -t$, we have the conjugate pair $(t,-t)$ in the symplectic step and $(it,-it)$ in the unitary case. We fix $\beta \in \mathrm{Irr}(\mathrm{GL}_2(q))$ cuspidal for the pair $(t,-t)$ or $(it,-it)$. Consequently, $t(-t) = -t^2$ and $(it)(-it) = t^2$. We write ω_β for the scalar character of β. Consequently, $\omega_\beta = \omega\eta_q$ in the first step and $\omega_\beta = \omega$ in the second.

We put $L = \ker(\det\cdot\nu)$ and $L_+ = \{(A,y) \in L : \det(A) \in (\mathbb{F}_q^\times)^2\}$. We have a surjective homomorphism $\pi : \mathrm{SL}_2(q) \times X' \times \mathbb{F}_q^\times \longrightarrow L_+$, with $\pi(x,y,c) = (cx, c^{-1}y)$ and $\ker\pi = \langle(-I,z,-1)\rangle$. Indeed, we set $c^2 = \det A$ for $(A,y') \in L_+$, which gives the preimage $(c^{-1}A, cy', c)$. We write τ_+, τ_- for the rank-one branches of β, and θ'_+, θ'_- for the branches of Θ'. Moreover, we write ϑ for the scalar character on $c \in \mathbb{F}_q^\times$ in $(\beta \boxtimes \Theta') \circ \pi$. $\vartheta = \eta_q$ in the symplectic step and $\vartheta = 1$ in the unitary step $(a,b-2)$. $\tau_\epsilon, \theta'_\delta$ have central scalar -1 at $-I, z$, whereas $\eta_q(-1) = \vartheta(-1) = 1$. Thus, $\ker\pi \le \ker(\tau_\epsilon \boxtimes \theta'_\delta \boxtimes \vartheta)$ for $\epsilon,\delta \in \{+,-\}$. We write $\alpha_{\epsilon,\delta}$ for the descended character on L_+, and the quotient $L/L_+ \simeq C_2$ acts by $(\epsilon,\delta) \mapsto (-\epsilon,-\delta)$. Hence, we have two orbits $\mathcal{O}_s$, each of size 2, with $s = \epsilon\delta$. Whenever $(\epsilon,\delta) \in \mathcal{O}_s$, we put $\psi_s = \mathrm{Ind}_{L_+}^L \alpha_{\epsilon,\delta}$. By [43], $\psi_+, \psi_- \in \mathrm{Irr}(L)$. Moreover, $s_\psi(\sigma) = s_1(\sigma)s_{\mathrm{pred}}(\sigma)$ on $\{\psi_+, \psi_-\}$.

We extend the rational character ϑ to the field group with $\widehat{\vartheta}(F_r) = 1$. Choose the full field operators Φ_1, Φ' on the factors $V_{\tau_\epsilon}, V_{\theta'_\delta}$. Moreover, we keep the original powers $\Phi_1^f, (\Phi')^f$. The tensor operator $\Phi_1 \otimes \Phi'$, followed by induction from L_+, has ratio $r_1(\sigma)r_{\mathrm{pred}}(\sigma)$ whenever σ fixes both factors. With even f, we have $\eta_q(c_\sigma) = 1$ on $\{\tau_+, \tau_-\}$. Thus, $\mathrm{Stab}_{\mathcal{H}_p}(\psi_s) \le \mathrm{Stab}_{\mathcal{H}_p}(\tau_\epsilon, \theta'_\delta)$. If f is odd, choose field extensions of θ'_δ with value field $\mathbb{Q}(\theta'_\delta)$ by [208, Lemma 6.13]. Choose the corresponding extension $\widehat{\tau}_\epsilon$ from Lemma 4.12. We write $\mathcal{S}$ for the coefficient stabilizer of ψ_s, and $\mathcal{S}_0$ for its subgroup fixing both factors $\tau_\epsilon, \theta'_\delta$. Consequently, $[\mathcal{S} : \mathcal{S}_0] \le 2$. We choose the induced extension $\widehat{\psi}_s^0$, with $(\widehat{\psi}_s^0)^\sigma = \widehat{\psi}_s^0$ for $\sigma \in \mathcal{S}_0$. Since the field quotient C has odd order dividing f, by Lemma 4.34, we obtain the extension $\widehat{\psi}_s$ fixed by $\mathcal{S}$, including a simultaneous exchange of $\tau_\epsilon, \theta'_\delta$. Thus, $r_{\widehat{\psi}_s}(\sigma) = 1$ for $\sigma \in \mathcal{S}$.

Write Θ for the target regular parent on $\widetilde{X}$, with connected algebraic center $\mathrm{Z}(\widetilde{\mathbf{X}})$, and we write $\widetilde{s} \in \mathrm{GSp}_{2n}(q)$ for the dual label of Θ, with connected centralizer $\mathbf{C}^*$. Moreover, we induce from $\mathrm{GL}_1(q^2)$ and a smaller factor H' of symplectic or unitary type. On $\mathrm{GL}_1(q^2)$, we take label $1_{\mathrm{GL}_1(q^2)}$. Moreover, write $W_m = C_2 \wr \mathfrak{S}_m$ as relative Weyl group. We write $b_{\boldsymbol{\mu},\boldsymbol{\lambda}}$ for the branching coefficient along $W_{m-1} \le W_m$, and $b_{\boldsymbol{\mu},\boldsymbol{\lambda}} = 1$ for $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$ with $|\boldsymbol{\lambda}/\boldsymbol{\mu}| = 1$. By Jordan comparison in $\widetilde{X}$, $\langle R_{\widetilde{L}}^{\widetilde{X}}(\beta \boxtimes \Theta'), \Theta\rangle = 1$. We have a corner predecessor $\boldsymbol{\mu}$ for every noncuspidal $\boldsymbol{\lambda}$.

We put $I = R_{\widetilde{L}}^{\widetilde{X}}(\beta \boxtimes \Theta')$ and $m_{ij} = \langle \chi_i, R_L^X(\psi_j)\rangle$, with $i,j \in \{+,-\}$. Moreover, $\nu(\widetilde{P}) = \mathbb{F}_q^\times$ for the regular parabolic $\widetilde{P}$. Hence, $I|_X = R_L^X((\beta \boxtimes \Theta')|_L)$. Since $\widetilde{X}/X \simeq \mathbb{F}_q^\times$ is cyclic, its regular character is $\sum_{\delta \in \mathrm{Irr}(\mathbb{F}_q^\times)} \delta$. By [95, Lemma 5.2] on X, $\sum_{i,j} m_{ij} = \sum_{\delta \in \mathrm{Irr}(\mathbb{F}_q^\times)} \langle \Theta(\delta\circ\nu), R_{\widetilde{L}}^{\widetilde{X}}(\beta \boxtimes \Theta')\rangle = 2$. We choose the nondegenerate alternating form B on $\mathsf{V} = \mathbb{F}_q^{2n}$ for $\mathrm{GSp}_{2n}(q)$. If $g \in \mathrm{GSp}_{2n}(q)$ and $v, w \in \mathsf{V}$, the multiplier $\mu : \mathrm{GSp}_{2n}(q) \to \mathbb{F}_q^\times$ satisfies $B(gv, gw) = \mu(g)B(v,w)$. Indeed, $\mu(\widetilde{s}) \ne 0$. We find $\mu(c\widetilde{s}) = c^2\mu(\widetilde{s})$. Consequently, we have conjugacy to $\widetilde{s}$ only for $c^2 = 1$. We choose the twists for $c = 1, -1$, which fix both Θ and $\beta \boxtimes \Theta'$. Therefore, $\mathcal{E}(\widetilde{X}, c\widetilde{s}) \ne \mathcal{E}(\widetilde{X}, \widetilde{s})$ for $c^2 \ne 1$. By diagonal conjugation, $m_{ij} = m_{-i,-j}$. Consequently, $\sum_j m_{ij} = 1$ and $\sum_i m_{ij} = 1$. M is a permutation matrix, since $m_{ij} \in \mathbb{Z}_{\ge 0}$.

With a coefficient power k, $(it)^k/(it^k) = i^{k-1} \in \{1,-1\}$. Consequently, the sign i^{k-1} selects the common quadratic self-twist δ_{quad} when $i^{k-1} = -1$, and the trivial twist when $i^{k-1} = 1$. Write $P_X(\sigma), P_L(\sigma)$ for the branch permutation matrices on χ_i, ψ_j. By conjugation and coefficient action on $I|_X$, $P_X(\sigma)M = MP_L(\sigma)$. Thus, we have the same sign

$s(\sigma)$ on (χ_i, ψ_j) with $m_{ij} = 1$.

Finally, we induce the full extension $\widehat{\psi}_j$ on the field-stable parabolic P. We write the induced representation as $\widehat{I}_j$. Consequently, we have multiplicity 1 for χ_i in $\widehat{I}_j|_X$ when $m_{ij} = 1$. By [208, Lemma 6.12], χ_i is field-invariant. Thus, choose $\widehat{\chi}_i$ on the isotypic subspace V_i, with $\widehat{\chi}_i|_X = \chi_i$. We have the same action of σ on induction and on the χ_i-isotypic projection. A field-quotient character ξ gives the representation $\widehat{I}_j \otimes \xi$, with isotypic subspace $V_i \otimes \xi$. Thus, $r_{\chi_i}(\sigma) = r_{\psi_j}(\sigma)$ on $\mathrm{Stab}_{\mathcal{H}_p}(\chi_i) = \mathrm{Stab}_{\mathcal{H}_p}(\psi_j)$. $K_n - K_{n-2} = 2n-1$ is odd. Hence, the sign $\eta_q(c_\sigma)$ gives one parity change for $n \mapsto n-2$ in both steps. □

Lemma 4.20. *We choose $k > 0$, $a = k(k+1)$, and $q = r^f$ with odd prime r. With $D = \mathrm{SO}_{4a+1}(q)$, we put $S = \Omega_{4a+1}(q)$ and $X = \mathrm{Spin}_{4a+1}(q)$. Moreover, we write ρ_q for the symplectic cuspidal label on $\mathrm{Sp}_{2a}(q)$. We take $\Theta \in \mathcal{E}(D, s) \subseteq \mathrm{Irr}(D)$, with $|s| = 2$ and ordered labels (ρ_q, ρ_q), and write $\Theta|_S = \chi_+ + \chi_-$. We have rational extensions $\widehat{\chi}_+, \widehat{\chi}_-$ to the ordinary field group $\widehat{S}$ of S, with $\widehat{\chi}_\pm|_S = \chi_\pm$. Moreover, when $q \equiv 3 \bmod 4$, we choose a faithful involution series $\mathcal{E}_X \subseteq \mathrm{Irr}(X)$ with cuspidal label ρ_{q^2} on $\mathrm{Sp}_{2a}(q^2)$. We write $\theta_+, \theta_- \in \mathrm{Irr}(X)$ for its two spin branches. If $4 \mid \mathrm{ord}_p(q)$, we have full field extensions $\widehat{\theta}_+, \widehat{\theta}_-$ on the ordinary field group $\widehat{X}$, with $\theta_\pm^\sigma = \theta_\pm$ and $\widehat{\theta}_\pm^\sigma = \widehat{\theta}_\pm$ for $\sigma \in \mathcal{H}_p$.*

Proof. First, choose the partition $\Lambda = (4k+1, 4k-1, \dots, 5, 3, 1)$, with $2k+1$ distinct odd parts. If $t = 0$, the truncated induction of (Δ_k, Δ_k) with itself has partition Λ, by Section 4.2. Take Jordan blocks V_j of Λ, with $\dim V_j = 2j+1$ for $0 \le j \le 2k$. We choose their volumes $u_j \in \mathrm{Cl}(V_j)$ [137, §24.h]. Moreover, we choose models over $\mathbb{F}_r$, with equal volume norms on V_j, by Lemma 4.15. Write E for the group of normalized products U_v and central involution z. Moreover, $E/\langle z \rangle \simeq V = \{v \in \mathbb{F}_2^{2k+1} : \sum_j v_j = 0\}$. We write the quotient map as $\pi_E : E \to V$, with $\pi_E(U_v) = v$. Moreover, U_v satisfy $F_r(U_v) = U_v$ and $U_v^2 \in \langle z \rangle$, with $\nu(U_v) = 1$. Given $i \neq j$, $u_i u_j = z u_j u_i$ by [137, Theorem 24.56(b)]. Thus, the even products satisfy $U_v U_{v'} = z^{v \cdot v'} U_{v'} U_v$. Since $V^\perp = \langle (1, \dots, 1) \rangle$ and $2k+1$ is odd, the form on V is nondegenerate. We choose $w \in E$ for the component of rational class $\mathcal{C}$, and put $F' = \mathrm{Int}(w) F_q$.

Moreover, write $C_{\mathrm{src}} \le D$ for the source centralizer for Λ, and $\vartheta_{\mathrm{sp}} : D \to \{1, -1\}$ for the spinor character, with $\ker \vartheta_{\mathrm{sp}} = S$. We have component order $|V| = 2^{2k}$ in D, and we have denominators $(2^k, 2^k)$ for (ρ_q, ρ_q). Consequently, $\sum_{\mathcal{C}} m_{\mathcal{C}} = 1$ by [188, Proposition 15.4 and (11.15)]. Write $m_{\mathcal{C}}$ for the source contribution at $\mathcal{C}$, and we have one contributing class $\mathcal{C}$, with a multiplicity line $\mathcal{L}$ for C_{src}. Moreover, $\Theta \vartheta_{\mathrm{sp}} = \Theta$. On the source space $\mathrm{Ind}_U^D \theta_U$, we choose $(J_{\mathrm{sp}} f)(g) = \vartheta_{\mathrm{sp}}(g) f(g)$. $\vartheta_{\mathrm{sp}}|_U = 1$, since U is unipotent. Whenever $c \in C_{\mathrm{src}}$, we write $(L_c f)(g) = f(c^{-1} g)$. Thus, $J_{\mathrm{sp}} L_c J_{\mathrm{sp}}^{-1} = \vartheta_{\mathrm{sp}}(c) L_c$. We write $\lambda_{\mathcal{L}}$ for the linear character on $\mathcal{L}$. By the unique occurrence of $\mathcal{L}$, $\lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}(\vartheta_{\mathrm{sp}}|_{C_{\mathrm{src}}})$. Consequently, $\vartheta_{\mathrm{sp}}|_{C_{\mathrm{src}}} = 1$.

We suppose that $\bar{w} = w\langle z \rangle \neq 0$ in V. Fix $y \in E$ with $[w, y] = z$ by the nondegenerate form on V, and $F'(y) = zy$ and $F'(zy) = y$. Thus, the natural image $\bar{y} \in C_{\mathrm{src}}$ is rational, with $\vartheta_{\mathrm{sp}}(\bar{y}) = -1$. We contradict $\vartheta_{\mathrm{sp}}|_{C_{\mathrm{src}}} = 1$. Hence, we choose $\bar{w} = 0$ and $w = 1$. We have a sign character $\lambda_{\mathcal{L}}$ on V, with $\mathbb{Q}(\lambda_{\mathcal{L}}) = \mathbb{Q}$. We write $D_j(c)$ for the grading on V_j, with $\nu(D_j(c)) = c^{K_j}$ and $K_j = j(j+1)/2$. Consequently, we have norm $\nu(D(c)) = c^L$, with $L = \sum_{v=1}^k (K_{2v} + K_{2v-1}) = 4 \sum_{v=1}^k v^2$. Write $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. When $\sigma \in \mathcal{G}$, we write $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$, with $c_\sigma \in \mathbb{F}_r^\times$ and primitive ζ_r of order r. Moreover, when $c = c_\sigma$, we choose $D_0(c) = c^{-L/2} D(c)$, with $\nu(D_0(c)) = 1$. $F_r(c^{-L/2}) = c^{-L/2}$ and $[D_0(c), U_v] = 1$. $\Phi^f = 1$ on the source, since $w = 1$.

We fix $d \in D \backslash S$. Since $C_{\mathrm{src}} \le S$, we have two conjugate source pairs (U, θ_U) and (U^d, θ_U^d) on S. By [95, Lemma 5.2], $\langle \mathrm{Ind}_U^S \theta_U, \chi_+ + \chi_- \rangle_S = \langle \mathrm{Ind}_U^D \theta_U, \Theta \rangle_D = 1$. Hence, the chosen source θ_U selects one branch χ_+ with multiplicity 1. We take its isotypic space V_{χ_+}. The coefficient operator T_σ from $D_0(c_\sigma)$ satisfies $T_\sigma(V_{\chi_+}) = V_{\chi_+}$ and $\Phi T_\sigma = T_\sigma \Phi$. Thus, we obtain a full extension $\widehat{\chi}_+$, with $\widehat{\chi}_+^\sigma = \widehat{\chi}_+$ for $\sigma \in \mathcal{G}$. We have rational values on $\widehat{\chi}_+$, and we obtain the other branch χ_- from the conjugate source θ_U^d. Moreover, $F_r(d) d^{-1} \in S$, since ϑ_{sp} is field-invariant. Thus, d normalizes the field group $\widehat{S}$, and we choose $\widehat{\chi}_- = \widehat{\chi}_+^d$. We have rational extensions $\widehat{\chi}_+, \widehat{\chi}_-$.

Moreover, we choose $\widetilde{X} = \mathrm{GSpin}_{4a+1}(q)$ in the faithful case. We fix ρ_{q^2} with denominator 2^k. Thus, $\sum_{\mathcal{C}} m_{\mathcal{C}} = 2^k$. Choose a contributing rational class $\mathcal{C}$ and choose its component $w \in E$, with $F' = \mathrm{Int}(w) F_q$. Moreover, choose a regular parent $\widetilde{\Theta} \in \mathrm{Irr}(\widetilde{X})$, with $\widetilde{\Theta}|_X = \theta_+ + \theta_-$. Write $\omega : \mathbb{F}_q^\times \to \mathbb{C}^\times$ for its scalar character, with $\omega(-1) = -1$. Moreover, write $\widetilde{C}_{\mathrm{src}} \le \widetilde{X}$ for the regular source centralizer. When $y \in E$, we have $F'(y) = z^{\epsilon(y)} y$, with $\epsilon(y) \in \{0, 1\}$. Moreover, we choose $c_y \in \overline{\mathbb{F}}_q^\times$ with $c_y^{q-1} = (-1)^{\epsilon(y)}$. Thus, $F'(c_y y) = c_y y$. We have the same commutator $[c_y y, c_{y'} y'] = [y, y']$. The component map $\pi_V : \widetilde{C}_{\mathrm{src}} \to V$ satisfies $\pi_V(c_y y) = \pi_E(y)$, so $\pi_V(\widetilde{C}_{\mathrm{src}}) = V$. Moreover, put $\widetilde{U}_v = c_{U_v} U_v$, with $\widetilde{U}_0 = 1$. Since $\widetilde{U}_v \widetilde{U}_{v'} \widetilde{U}_{v+v'}^{-1} \in \mathbb{F}_q^\times$, we obtain the scalar function $\gamma(v, v') = \omega(\widetilde{U}_v \widetilde{U}_{v'} \widetilde{U}_{v+v'}^{-1})$, with $\gamma(v, v')/\gamma(v', v) =$

$(-1)^{v\cdot v'}$. Thus, γ gives a nondegenerate commutator on V. By [151, Theorem 2.3], $\mathcal{A} = \mathbb{C}_\gamma[V] \simeq \mathrm{Mat}_{2^k}(\mathbb{C})$. Moreover, $2^k \mid m_{\mathcal{C}}$. Consequently, we have one class $\mathcal{C}$ and one source module W, with $\dim W = 2^k$.

On the spin group X, we have rational sign group $H = C_E(w)$. If $w \notin \mathrm{Z}(E)$, $[E : H] = 2$. Moreover, the form on $\bar{w}^\perp/\langle \bar{w}\rangle$ is nondegenerate. Since $[w, y] = z$ for $y \in E \setminus H$, the action of y exchanges the two eigenspaces of w. By [95, Theorem 6.2 and Corollary 11.22], these are the two constituents $W_\pm$ of $W|_H$, with $\dim W_\pm = 2^{k-1}$ and $\lambda_- = -\lambda_+$ on w. The rational lift $c_y y$ has $c_y^{q-1} = -1$, $\nu(c_y y) = c_y^2$ and $\eta_q(\nu(c_y y)) = -1$. Thus, $c_y y$ also exchanges θ_+, θ_-. We write m_+, m_- for the multiplicities. $(m_+ + m_-)2^{k-1} = 2^k$ and $m_+ = m_-$. Thus, $m_+ = m_- = 1$. If $w \in \mathrm{Z}(E)$, we instead have $H = E$ and $\dim W = 2^k$. We obtain multiplicity $m = 1$ from $m2^k = 2^k$. Moreover, we choose $\delta \in \widetilde{X}$ with $\eta_q(\nu(\delta)) = -1$. $\theta_- = \theta_+^\delta$ from the chosen branch θ_+.

Choose a maximal abelian subgroup $\widehat{A}_H \le H$. Since $U_v^4 = 1$, extend each central character to α on $\widehat{A}_H$ over $\mathbb{Q}(i)$. Adjoin $U(1)$ via the central character $\alpha|_{\mathrm{Z}(H)}$. By [151, Theorem 2.3], the model is $\rho_{H,\alpha} = \mathrm{Ind}_{\widehat{A}_H}^H \alpha$, with matrices over $\mathbb{Q}(i)$. If $H = E$, take $\alpha(z) = -1$ for W. If $H < E$, take $(\alpha_\pm(z), \alpha_\pm(w)) = (-1, \lambda_\pm)$ for $W_\pm$. Since $4 \mid p-1$, $\sigma(i) = i$ for $\sigma \in \mathcal{H}_p$. Thus, $\sigma(\rho_{H,\alpha}(h)) = \rho_{H,\alpha}(h)$ for $\sigma \in \mathcal{H}_p$. We have scalars $\lambda_w \in \mu_4$ on w, with $\sigma(\lambda_w) = \lambda_w$. Moreover, we fix $D_0(c)$ over $\mathbb{F}_r$, since $2 \mid L$. Thus, $\theta_\pm^\sigma = \theta_\pm$ for $\sigma \in \mathcal{H}_p$. We have odd f, since $q \equiv 3 \bmod 4$. By [208, Lemma 6.13], we choose full field extensions $\widehat{\theta}_\pm$ with $\mathbb{Q}(\widehat{\theta}_\pm) = \mathbb{Q}(\theta_\pm)$. Hence, $\widehat{\theta}_\pm^\sigma = \widehat{\theta}_\pm$ for $\sigma \in \mathcal{H}_p$. □

Lemma 4.21. *We choose the involution series $\mathcal{E}$ of rank $2a$, with $a \ge 1$, and we put $X = \mathrm{Spin}_{4a+1}(q)$ and $\widetilde{X} = \mathrm{GSpin}_{4a+1}(q)$. Moreover, we take spin branches $\chi_+, \chi_- \in \mathrm{Irr}(X)$, with equal labels (ρ, ρ) and central sign $+1$. We have coefficient sign $s(\sigma) = \eta_q(c_\sigma)^a$ on $\{\chi_+, \chi_-\}$. Moreover, we take field extensions $\widehat{\chi}_+, \widehat{\chi}_-$, with $r_{\widehat{\chi}_\pm}(\sigma) = 1$ on $\mathrm{Stab}_{\mathcal{H}_p}(\chi_\pm)$. When $q \equiv 3 \bmod 4$ and $4 \mid \mathrm{ord}_p(q)$, we have the same sign $s(\sigma) = \eta_q(c_\sigma)^a$ for the faithful pair θ_+, θ_- with exchanged eigenspaces. We choose extensions $\widehat{\theta}_+, \widehat{\theta}_-$, with $r_{\widehat{\theta}_\pm}(\sigma) = 1$ on $\mathrm{Stab}_{\mathcal{H}_p}(\theta_\pm)$.*

Proof. First, we take central sign $+1$. When $a > 1$ and noncuspidal ρ, we choose a predecessor ρ'. We have relative bipartitions $\boldsymbol{\lambda}_{\rho'} \subset \boldsymbol{\lambda}_\rho$, with $|\boldsymbol{\lambda}_\rho / \boldsymbol{\lambda}_{\rho'}| = 1$. Moreover, we choose ρ' on both eigenspaces. We choose $\widetilde{L} = \mathrm{GL}_2(q) \times \mathrm{GSpin}_{4a-3}(q)$, with linear eigenvalues $t, -t$, with $t \in \mathbb{F}_q^\times$. We fix $\beta \in \mathrm{Irr}(\mathrm{GL}_2(q))$ in the split principal series for $(t, -t)$, and the residual parent Θ', with scalar character ω. Moreover, we write $X' = \mathrm{Spin}_{4a-3}(q)$ and $\Theta'|_{X'} = \theta'_+ + \theta'_-$. Thus, $\omega_\beta = \omega\eta_q$, with quadratic η_q. A central change of t gives a common character $\xi \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Moreover, we have twists $\xi \circ \det$ on β and $\xi \circ \nu$ on Θ', with $((\xi \circ \det) \boxtimes (\xi \circ \nu))|_L = 1$.

We choose the square-determinant subgroup L_+ and map π of Lemma 4.19. We write τ_+, τ_- for the split exceptional pair on $\mathrm{SL}_2(q)$. Moreover, at $(-I, z, -1)$, we have scalars $\eta_q(-1), 1, \eta_q(-1)$. Their product is $\eta_q(-1)^2 = 1$. Thus, all four products $\tau_\epsilon \boxtimes \theta'_\delta \boxtimes \eta_q$ have $\ker \pi$ in their kernels. We have action $(\epsilon, \delta) \mapsto (-\epsilon, -\delta)$ for $L \setminus L_+$, and we write $\alpha_{\epsilon,\delta} \in \mathrm{Irr}(L_+)$ for the descended tensor and $\psi_{\epsilon\delta} = \mathrm{Ind}_{L_+}^L \alpha_{\epsilon,\delta}$.

Moreover, we have relative Weyl coefficients $b_1 = b_2 = 1$. By Jordan decomposition in $\widetilde{X}$, $\langle R_{\widetilde{L}}^{\widetilde{X}}(\beta \boxtimes \Theta'), \Theta\rangle = b_1 b_2 = 1$. Put $m_{ij} = \langle \chi_i, R_L^X(\psi_j)\rangle$, with $i, j \in \{+, -\}$. Since the quotient is cyclic with $\widetilde{X}/X \simeq \mathbb{F}_q^\times$, the proof of Lemma 4.19 yields $\sum_{i,j} m_{ij} = 2$. Only scalar twists with $c^2 = 1$ fix the regular series of Θ, and both fix Θ. Moreover, simultaneous diagonal conjugation gives that $m_{ij} = m_{-i,-j}$. Thus, we have a permutation matrix $M = (m_{ij})$. If $m_{ij} = 1$, the comparison on χ_i is from ψ_j. Write $s_\rho(\sigma)$ for the sign on χ_+, χ_-. Then, by Lemma 4.12 $s_\rho(\sigma) = \eta_q(c_\sigma) s_{\rho'}(\sigma)$. We choose the upper Borel $B_1 \le \mathrm{SL}_2(q)$ and $U_1 = \mathrm{O}_r(B_1)$ and write $\vartheta_1(\mathrm{diag}(c, c^{-1})u) = \eta_q(c)$ for $c \in \mathbb{F}_q^\times$ and $u \in U_1$. Moreover, we choose $\mathcal{V}_{\mathrm{flag}} = \mathrm{Ind}_{B_1}^{\mathrm{SL}_2(q)} \vartheta_1$. With even f, we take its operator T_{flag}, with $T_{\mathrm{flag}}^2 = qI$ and eigenvalues $\pm\sqrt{q} \in \mathbb{Q}$. We have the field permutation P_F, with $P_F T_{\mathrm{flag}} = T_{\mathrm{flag}} P_F$ by the flag sum for T_{flag}. Thus, we obtain projectors $\Pi_\pm = (I \pm q^{-1/2} T_{\mathrm{flag}})/2$, with $\Pi_\pm \in \mathrm{End}_{\mathbb{Q}}(\mathcal{V}_{\mathrm{flag}})$ and $\Pi_\pm P_F = P_F \Pi_\pm$. Take rational extensions on $\Pi_\pm \mathcal{V}_{\mathrm{flag}}$. The proof of Lemma 4.19 gives that $r_{\widehat{\chi}_i}(\sigma) = r_{\mathrm{src}}(\sigma)$ after induction and projection on χ_i. If f is odd, we choose $\widehat{\chi}_i$ with $\mathbb{Q}(\widehat{\chi}_i) = \mathbb{Q}(\chi_i)$ by [208, Lemma 6.13].

If the cuspidal rank is $a_0 = k(k+1) > 0$, there are $a - a_0$ steps from Lemma 4.20, with even a_0 and base sign 1. Consequently, the final sign is $\eta_q(c_\sigma)^{a-a_0} = \eta_q(c_\sigma)^a$. With principal support of ρ, we instead take the base $a = 1$ in the Siegel Levi of $\mathrm{Spin}_5(q)$. We have Jordan pairs $(1, 1)$ and $(\mathrm{St}, \mathrm{St})$. Moreover, on each $\mathrm{Sp}_2(q)$, we have the flag character $\mathrm{Ind}_{B_1}^{\mathrm{Sp}_2(q)} 1 = 1 + \mathrm{St}$. Thus, we have multiplicity 1 at $(1, 1), (\mathrm{St}, \mathrm{St})$. We choose the Siegel Levi $L_{\mathrm{S}} \le \mathrm{Spin}_5(q)$ and its determinant kernel $L_{\mathrm{S},0}$. Moreover, choose the split pair τ_+, τ_- on $L_{\mathrm{S},0}$, with source columns ψ_+, ψ_-. We again obtain a permutation matrix M, with base exponent 1. The remaining $a - 1$ steps give $\eta_q(c_\sigma)\eta_q(c_\sigma)^{a-1} = \eta_q(c_\sigma)^a$.

Moreover, take central sign -1, with $q \equiv 3 \bmod 4$ and $4 \mid \mathrm{ord}_p(q)$. Choose the cuspidal character $\beta \in \mathrm{Irr}(\mathrm{GL}_2(q))$ for the conjugate pair $(t, -t)$. At $-I$, we have scalar $-\eta_q(-1) = 1$ on the rank-one branches τ_+, τ_-. Moreover, we

have scalars $(-1,\eta_q(-1))=(-1,-1)$ on $(z,-1)$. Consequently, we have kernel scalar 1. We obtain four characters $\alpha_{\epsilon,\delta}\in\mathrm{Irr}(L_+)$. We have branching coefficient $b=1$ in the exchanged factor $\mathrm{Sp}_{2a}(q^2)$. Thus, we obtain sign $\eta_q(c_\sigma)$ at every step $a\mapsto a-1$. When $a_0>0$, choose the cuspidal source θ_{src} of Lemma 4.20. When $a_0=0$, we take regular and semisimple pairs $(\chi_{\mathrm{reg},\pm},\chi_{\mathrm{ss},\pm})$ from Lemma 4.13. Moreover, at $a=1$, we have grading $D(c)=T_1(c^2)T_2(c)$, with $\nu(D(c))=c^3$ and sign $\eta_q(c_\sigma)$. Choose D_X from rational parabolics P_I, including when $q\equiv 3 \bmod 4$. $\sigma(\mathsf{D}_X\chi)=\mathsf{D}_X(\chi^\sigma)$, so we have the same sign $\eta_q(c_\sigma)$ on $\chi_{\mathrm{reg},\pm},\chi_{\mathrm{ss},\pm}$. We again obtain exponent a by the rank steps $a\mapsto a-1$. Since $q\equiv 3 \bmod 4$, we have odd f. By [208, Lemma 6.13], $r_{\widehat{\theta}_\pm}(\sigma)=1$ on $\mathrm{Stab}_{\mathcal{H}_p}(\theta_\pm)$. □

Proposition 4.22. *Suppose that $q\equiv 1 \bmod 4$ and that $d=2e$, with e odd. Let $\mathcal{E}$ be the faithful series with labels (ρ,λ). We write $k(k+1)$ for the cuspidal rank of ρ. In particular, θ_{src} has multiplicity 1 by* Lemmas 4.16 and 4.18. *On θ_{src}, we have that the field operator P_{src} is from* Lemma 4.16. *Moreover, the steps $a\mapsto a+1$ and $b\mapsto b+2$ are from* Lemma 4.19. *The pair (τ_+,τ_-) is exceptional by* Lemma 4.12. *In particular, $-I\in\mathrm{SL}_2(q)$ acts by -1 on both τ_+,τ_-. Moreover, on the scalar coordinate $c\in\mathbb{F}_q^\times$, we have character $\eta_q(c)$ for $a\mapsto a+1$ and 1 for $b\mapsto b+2$. We have exchange sign $s_{\mathcal{E}}(\sigma)=(\sigma(i)/i)^{K_n+K_k}$. On $\mathrm{Stab}_{\mathcal{H}_p}(\chi_\pm)$, we choose full field extensions $\widehat{\chi}_\pm$ with ratio $r_{\widehat{\chi}_\pm}(\sigma)=\epsilon_\sigma^{K_n+K_k}$. Moreover, we set $b=ew+s$ and $h=2a+s<e$, with $s<e$. By* Lemma 4.14, *we write $\kappa=\mathrm{core}_e(\lambda)$. Then, (ρ,κ) is the residual label and η is the active character of p'-degree. By* Proposition 4.10, *we have that $s_{\mathcal{E}}(\sigma)=s_{\mathrm{act}}(\sigma)s_{\mathrm{res}}(\sigma)$ and $r_{\widehat{\chi}_\pm}(\sigma)=r_{\mathrm{act}}(\sigma)r_{\mathrm{res}}(\sigma)$. Moreover, we have that $\mathrm{core}_2(\lambda)\neq\mathrm{core}_2(\kappa)$ is allowed.*

Proof. At $(a,b)=(k(k+1),K_t)$, we choose the joint base class $\mathcal{C}$ and component representative w, with $w^2=z^Q$ and $Q\in\{0,1\}$. By Lemma 4.16, we choose $\widetilde{C}$ as rational source centralizer and $C=\ker(\nu|_{\widetilde{C}})$ as spin source. Suppose that $\nu(\widetilde{C})=\mathbb{F}_q^\times$. On the two irreducible source modules V_+,V_-, we have scalar operators $w|_{V_\pm}=\omega_\pm I$. Moreover, when $h>0$, $w|_{\widetilde{T}}=s_1\cdots s_h$. We choose torus weight τ, with free orbit $\langle s_1,\dots,s_h\rangle\cdot\tau$ of size 2^h. When $h=0$, we have single-block vector $\bar{w}\neq 0$. Since $V_0\cap V_0^\perp=0$, fix y with $[w,y]=z$. Thus, we have noncentral w and opposite scalars $\omega_-=-\omega_+$. We find $\omega_\pm^2=(-1)^Q$. Consequently, we have coefficient sign $(\sigma(i)/i)^Q$ on ω_+,ω_-. We choose the grading operator $D(c_\sigma)$, with $\nu(D(c_\sigma))=c_\sigma^L$, and $[D(c_\sigma),C]=1$. Thus, the correction to $\ker\nu$ contributes sign $\eta_q(c_\sigma)^L$. We obtain $s_{\mathcal{E}}(\sigma)=(\sigma(i)/i)^Q\eta_q(c_\sigma)^L$. When $\nu(\widetilde{C})\neq\mathbb{F}_q^\times$, $h=0$ and trivial component $w=1$. Moreover, $Q=0$. Choose $\delta\in\widetilde{X}$ with $\eta_q(\nu(\delta))=-1$, and $\chi_-=\chi_+^\delta$. Thus, we again have $s_{\mathcal{E}}(\sigma)=(\sigma(i)/i)^Q\eta_q(c_\sigma)^L$.

With even f, we choose correction $u^{-1}D(c_\sigma)$, where $u^2=c_\sigma^L$ in $\mathbb{F}_q$. We have $D(c_\sigma)$ over $\mathbb{F}_r$. Since $c_\sigma\in\mathbb{F}_r^\times$, $F_r(u)/u=c_\sigma^{L(r-1)/2}=\epsilon_\sigma^L$. Thus, the field commutator for $u^{-1}D(c_\sigma)$ has sign ϵ_σ^L. By Lemma 4.16, $P_{\mathrm{src}}^f=\xi^{-1}R(w^{-1})$. Fix αP_{src}, with $\alpha^f=\xi$, $\xi^2=(-1)^Q$ and $\alpha\in\mu_{4f_2}$. Thus, $(\alpha P_{\mathrm{src}})^f=R(w^{-1})$. When $v_2(p-1)=v_2(f)+1$, we have $p\equiv 1 \bmod 4$ and nonsquare r modulo p. Moreover, we fix $\alpha\in\mu_{2f_2}$ when $Q=0$. When $Q=1$, $|\alpha|=4f_2$. By quadratic reciprocity, $\sigma(\alpha)/\alpha=\epsilon_\sigma^Q$. Thus, we have complete field ratio ϵ_σ^{L+Q}. With odd f, we choose extensions $\widehat{\chi}_\pm$ with $\mathbb{Q}(\widehat{\chi}_\pm)=\mathbb{Q}(\chi_\pm)$ by [208, Lemma 6.13]. Moreover, in the complementary domain $4f_2\mid p-1$, we have square r modulo p. $\sigma(\alpha)=\alpha$, $\sigma(i)=i$ and $F_{\mathrm{comp}}^\sigma=F_{\mathrm{comp}}$ for $\sigma\in\mathcal{H}_p$. Also, we have square c_σ in $\mathbb{F}_r^\times$. Thus, we have coefficient factor 1 and field ratio 1 there.

Moreover, we choose $v_2(p-1)=v_2(f)+1$. With odd f, $\sigma\in\mathcal{H}_p$ with $\sigma(i)/i=-1$ on the pair χ_+,χ_-. With even f, we have σ stabilizing $\chi_\pm$, with $\epsilon_\sigma=-1$. Moreover, write $\lambda_F=\lambda_{F,2}\lambda_{F,\mathrm{odd}}$ for a field-quotient character λ_F. We have that $\lambda_{F,2}^\sigma=\lambda_{F,2}$ and odd order $|\lambda_{F,\mathrm{odd}}^\sigma/\lambda_{F,\mathrm{odd}}|$. Thus, $\lambda_F^\sigma/\lambda_F\neq -1$. Consequently, the sign after the λ_F-twist is ϵ_σ^{L+Q}. We write $E_\rho\in\mathbb{F}_2$ for the exponent after subtraction of K_n. By Lemma 4.15, $E_\rho=Q+K_m$ at $(a,b)=(k(k+1),K_t)$. Moreover, $(a,n)\mapsto(a+1,n+2)$ for symplectic label ρ. $(b,n)\mapsto(b+2,n+2)$ for unitary label λ. Thus, in both steps $K_{n+2}-K_n=2n+3\equiv 1 \bmod 2$. By Lemma 4.19, we have exponent 1 per step. Hence, we have constant E_ρ along $a\mapsto a+1$ and $b\mapsto b+2$.

In family $\mathcal{F}$, take absolute defect $2j+1$. Moreover, we have cuspidal rank $j(j+1)$ and $\binom{2k+1}{k-j}$ labels of defect $2j+1$. With $j<k$, we assume inductively that $E_\rho=K_j$. Thus, we have known exponents E_ρ except for the cuspidal label ρ_{cusp}. Moreover, by Lemma 4.18, each allowed class $\mathcal{C}$ corresponds to one label ρ. By Lemma 4.15, $\sum_{\mathcal{C}}(-1)^{Q(\mathcal{C})+K_m}=2^k$. We find $\sum_{j=0}^k\binom{2k+1}{k-j}(-1)^{K_j}=2^k$. Consequently, $E_{\rho_{\mathrm{cusp}}}=K_k$. To verify the sum 2^k, we write $(-1)^{K_j}=\mathrm{Re}((1+i)i^j)$. We pair indices $j,-j-1$. Moreover, $\frac{1}{2}\mathrm{Re}((1+i)i^k(1-i)^{2k+1})=2^k$. At $k=0$, we have one allowed class $\mathcal{C}$, with $E_\rho=0$. When $t>0$, we choose the pure unitary staircase Δ_t as source base. Moreover, when $t=0$, we choose pairs $(\chi_{\mathrm{reg},\pm},\chi_{\mathrm{ss},\pm})$ of Lemma 4.13. We have total exponent K_n+K_k. Thus, $s_{\mathcal{E}}(\sigma)=(\sigma(i)/i)^{K_n+K_k}$. With even f, $r_{\widehat{\chi}_\pm}(\sigma)=\epsilon_\sigma^{K_n+K_k}$. With odd f, $\eta_q(c_\sigma)=\epsilon_\sigma=\sigma(i)/i$. Thus, on $\mathrm{Stab}_{\mathcal{H}_p}(\chi_\pm)$, $\epsilon_\sigma^{K_n+K_k}=1$. By [208, Lemma 6.13], $r_{\widehat{\chi}_\pm}(\sigma)=1$ there.

Moreover, we choose (ρ,κ) with the same ρ. With $h=2a+s$, we write $b=ew+s$ and $A=K_{ew}+hw$. $K_n-K_h=K_{ew}+hew\equiv A \bmod 2$, since $e\equiv 1 \bmod 2$. By Proposition 4.10, $s_{\mathrm{act}}(\sigma)=(\sigma(i)/i)^A$ and field ratio $r_{\mathrm{act}}(\sigma)=\epsilon_\sigma^A$. The active field power is v_A^{-1}. Accordingly, choose extensions $\widehat{\psi}_{\mathrm{act}},\widehat{\psi}_{\mathrm{res}}$. By simultaneous diagonal conjugation on $V_{\mathrm{act}}\otimes V_{\mathrm{res}}$, we choose induced extensions $\widehat{\psi}_\pm\in\mathrm{Irr}(I_\pm)$, with $I_\pm=(M\langle\varphi\rangle)_{\psi_\pm}$. Thus, the factors are $s_{\mathrm{act}}s_{\mathrm{res}}, r_{\mathrm{act}}r_{\mathrm{res}}$. If f is odd, consider the torsor $\mathcal{T}$ of extensions $\widehat{\psi}_\pm$ fixed by the pure stabilizer S_0. We find $S_0=\mathrm{Stab}_{\mathcal{H}_p}(\psi_{\mathrm{act}})\cap\mathrm{Stab}_{\mathcal{H}_p}(\psi_{\mathrm{res}})$. Moreover, on the pair $\{\psi_+,\psi_-\}$, we have an affine exchange $\tau:\mathcal{T}\to\mathcal{T}$, with $\tau^2=1$. Since the translation group A_F of $\mathcal{T}$ has odd order, $a\mapsto 2a$ is bijective on A_F. Then, for $x\in\mathcal{T}$, the midpoint $x+\frac{1}{2}(\tau(x)-x)$ is fixed by τ. Thus, we choose $\widehat{\psi}_\pm$ fixed by τ. By Lemma 4.11, we choose normalized operators $D_{\mathrm{glo}}(\delta), D_{\mathrm{loc}}(\delta)$. Choose ordinary inertia quotients $I_{\mathrm{glo}}/X\simeq I_{\mathrm{loc}}/M$. Hence, for (δ,σ), $\mu^{\mathrm{glo}}_{\delta,\sigma}=\mu^{\mathrm{loc}}_{\delta,\sigma}$ on I_{loc}/M. □

We set $d=2e$, with $e=2j$ and odd q. We write $Q_0=q^2$ and $D=\mathrm{Sp}_{2a}(Q_0)$. Moreover, we choose the faithful X-series with other factor $\mathrm{GU}_b(q)$ when $q\equiv 1 \bmod 4$. When $q\equiv 3 \bmod 4$, we instead have factor $\mathrm{GL}_b(q)$. Thus, the ranks satisfy $n=2a+b$, $\quad b=0$ in the equal-symplectic series $\mathcal{E}$.

Lemma 4.23. *We choose an odd-defect symbol Σ of rank a and write Σ_0 for its j-cocore, with rank c. Moreover, we write $\boldsymbol{\lambda}$ for its $2j$-quotient, of size t. With $W=\lfloor a/j\rfloor$, we put $A=v_p(Q_0^j+1)$. We fix $\eta\in\mathrm{Irr}(C_{2j}\wr\mathfrak{S}_t)$ with label $\boldsymbol{\lambda}$. Thus, we find $v_p(\rho_\Sigma(1))=(W-t)A+v_p(W!/t!)+v_p(\eta(1))$. Moreover, we have eligible $\mathcal{E}$ precisely when $b<e$ and $(2a \bmod e)+b<e$. We have a bijection on its p'-labels with triples (Σ_0,λ,η), where $c=a \bmod j$, $\lambda\vdash b$ and $\eta\in\mathrm{Irr}_{p'}(C_e\wr\mathfrak{S}_w)$, and we have that $w=(a-c)/j$ and residual rank $h=2c+b<e$.*

Proof. First, we choose rows $\Sigma^{(0)},\Sigma^{(1)}$ of Σ. We choose runners $B_{u,\varepsilon}$, with $0\le u<j$ and $\varepsilon\in\{0,1\}$. Moreover, we put $v\in B_{u,\varepsilon}$ when $u+jv\in\Sigma^{(\varepsilon+v \bmod 2)}$. We have cohook $u+jv\mapsto u+j(v-1)$ across the two rows of Σ. Equivalently, on $B_{u,\varepsilon}$ the bead moves by $v\mapsto v-1$ into a vacancy. By packing $B_{u,\varepsilon}$, we choose cocore Σ_0. Then, $a-c=jt$, and the displacements give partitions $\lambda_{u,\varepsilon}$ in $\boldsymbol{\lambda}$. Moreover, a simultaneous shift of Σ permutes the runners $B_{u,\varepsilon}$. We have beta-set shifts in $\mathcal{B}(\Sigma)$ at each wrap $u=j-1\mapsto 0$. We orient Σ_0 and order its $2j$ colours. Since $\mathrm{def}(\Sigma)$ is odd, row exchange of Σ exchanges the orientation of Σ_0. We have the same orientation for Σ_0 under shifts of Σ. Choose a common shift of Σ_0 with sufficiently many beads $N_{u,\varepsilon}$. Moreover, we pad each $\lambda_{u,\varepsilon}$ to length $N_{u,\varepsilon}$. We have packed positions $N_{u,\varepsilon}-i$, for $1\le i\le N_{u,\varepsilon}$. We reconstruct positions $N_{u,\varepsilon}-i+\lambda_{u,\varepsilon,i}$ on $B_{u,\varepsilon}$. Thus, we recover Σ up to equivalence from $(\Sigma_0,\boldsymbol{\lambda})$. Moreover, a larger shift of Σ_0 gives an equivalent Σ.

By [115, §3.3], we have numerator valuation $WA+v_p(W!)$ in $\rho_\Sigma(1)$. We have denominator hooks of length $2jm$ within rows of Σ. Moreover, we take cohooks of length $j(2m+1)$ across rows of Σ. In particular, both kinds occur on $B_{u,\varepsilon}$. By the valuation formula at p, we have contribution $A+v_p(l)$ for each quotient hook l. Since there are t boxes, $\eta(1)=t!/\prod l$. Thus, $v_p(\rho_\Sigma(1))=(W-t)A+v_p(W!/t!)+v_p(\eta(1))$. $t\le W$ and $A>0$. Consequently, we have $p\nmid\rho_\Sigma(1)$ precisely when $t=W$ and $p\nmid\eta(1)$. $t=W$ precisely when $c=a \bmod j$. Moreover, any symbol of rank $c<j$ is already a j-cocore Σ_0.

We put $N=\lfloor n/e\rfloor$ and $A_0=v_p(q^{2e}-1)$, and $v_p(|X|_{r'})=A_0N+v_p(N!)$. Moreover, for $D=\mathrm{Sp}_{2a}(Q_0)$, $v_p(|D|_{r'})=A_0W+v_p(W!)$. Since e is even, $\mathrm{ord}_p(q)=\mathrm{ord}_p(-q)=2e$. Set $B=\lfloor b/(2e)\rfloor$. With factor $H_b=\mathrm{GL}_b(q)$ or $\mathrm{GU}_b(q)$, $v_p(|H_b|_{r'})=A_0B+v_p(B!)$. Moreover, when $b\ge e$, we have $\lfloor b/e\rfloor>B$ and $N>W+B$. We have $v_p(N!)-v_p(W!)-v_p(B!)\ge 0$. Thus, the centralizer index for $D\times H_b$ has positive valuation at p. When $b<e$, $B=0$. Put $c_*=a \bmod j$. Thus, $N=W$ precisely when $2c_*+b<e$. Moreover, in that case $N=W$, we have matching factorial valuations $v_p(N!)=v_p(W!)$. Thus, we have eligible index domain $b<e$ and $(2a \bmod e)+b<e$. We have p'-order for H_b in that domain. At $c=c_*$, we also have p'-order for the residual group R_0, of rank $h=2c+b<e$. Consequently, the degree condition for X is exactly $p\nmid\rho_\Sigma(1)$. □

Lemma 4.24. *We fix $w\ge 1$ active blocks and residual rank h. We write θ for the faithful quadratic character on T. Moreover, we take its inertia quotient I_θ, with index-two subgroup $W_0=C_e\wr\mathfrak{S}_w$. We have that $I_\theta=\langle W_0,c_0\rangle$, with $[c_0,W_0]=1$ and $c_0^2=d_0$. In particular, d_0 generates the diagonal of C_e^w. By the Clifford lifts for θ, we have two active characters χ_η^+,χ_η^- for each $\eta\in\mathrm{Irr}(W_0)$. We have degrees $\chi_\eta^\pm(1)=2^{w-1}\eta(1)$. Moreover, we have exchange sign $\eta_q(c_\sigma)^{jw}$ and field-extension ratio ϵ_σ^{jw}.*

Proof. By Lemma 4.9, choose $k_l^{2e}=z^{e/2}$, with $k_l=h_l^{b_0}u_l$ and $v_A=u_1\cdots u_w$. Choose extension $\widehat{\theta}$ on TV_θ, with $\widehat{\theta}|_T=\theta$, and $\widehat{\theta}(k_l^2)=\beta=i^{e+1}$. Moreover, $\widehat{\theta}(p_l)=1$ and $\widehat{\theta}(k_0)=\kappa=\zeta_8^{w(w+e)}$. We have rotation vector $c_0=(1,\dots,1)$ in I_θ, and $c_0^2=d_0$, the diagonal generator of C_e^w. Thus, $[c_0,W_0]=1$. By common parity, $I_\theta=\langle W_0,c_0\rangle$. We choose

a representation ρ_η for η on W_0. Then, we choose ν_η with $\nu_\eta^2 = \eta(d_0)/\eta(1)$. Moreover, we have two extensions $\widehat{\rho}_\eta^\pm$, with $\widehat{\rho}_\eta^\pm(c_0) = \pm\nu_\eta I$. Since $[c_0, W_0] = 1$, we have all extensions above η in the pair $\widehat{\rho}_\eta^\pm$. Take matrices for $\rho_\eta, \widehat{\rho}_\eta^\pm$ over $\mathbb{Q}(\zeta_{2e})$. $2e \mid p-1$, so $\mathcal{H}_p$ fixes $\mathbb{Q}(\zeta_{2e})$. Thus, we choose tensors $\widehat{\theta} \otimes \widehat{\rho}_\eta^\pm$ and induce to active characters χ_η^+, χ_η^-. We find $\chi_\eta^\pm(1) = 2^{w-1}\eta(1)$. Choose a lift cE_1, with $c^{q-1} = -1$ and $\eta_q(\nu(cE_1)) = -1$. Moreover, $\widehat{\theta}(k_0^{cE_1}) = -\widehat{\theta}(k_0)$. $(k_l^2)^{cE_1} = k_l^2$ and $\widehat{\theta}(p_l^{cE_1}) = \widehat{\theta}(p_l)$. When the permutation lifts p_l occur, we have correction h_1h_2, with $\theta(h_1h_2) = 1$. Thus, $\nu_\eta \mapsto -\nu_\eta$, with η fixed under cE_1.

We find $v_2(\mathrm{ord}_p(r)) = v_2(f) + v_2(2e)$. Consequently, $4f_2 \mid p-1$. $\sigma(\beta) = \beta$ and $\sigma(\nu_\eta) = \nu_\eta$. Thus, we have coefficient sign $\sigma(\kappa)/\kappa$. When $4 \mid e$, $8 \mid p-1$ and even j. Moreover, $\sigma(\kappa) = \kappa$ in that case $4 \mid e$. When $e \equiv 2 \bmod 4$, $|\kappa| = 8$ precisely when w is odd. When $e \equiv 2 \bmod 4$ and w is even, $\sigma(\kappa) = \kappa$. If $|\kappa| = 8$, $\kappa^p/\kappa = -1$ exactly when $p \equiv 5 \bmod 8$. By the order of q modulo p, we have the equivalent condition $(q/p) = -1$. We have $p \equiv 1 \bmod 4$. Thus, quadratic reciprocity gives that $\sigma(\kappa)/\kappa = \eta_q(c_\sigma)^{jw}$.

We choose one branch $\chi_\eta^\pm$ and its root ν_η, and we have scalar $\gamma = \kappa\nu_\eta$ for v_A. Moreover, $\theta(\prod_l h_l^{b_0}) = 1$. Choose field permutation operator P_F, scaled by α with $\alpha^f = \gamma^{-1}$. With odd prime $s \mid 2e$, $v_s(\mathrm{ord}_p(r)) = v_s(f) + v_s(2e)$. Thus, we fix α_{odd} fixed by $\mathcal{H}_p$. Moreover, write $\alpha = \alpha_2\alpha_{\mathrm{odd}}$ and $\gamma = \gamma_2\gamma_{\mathrm{odd}}$ with primary factors. $|\alpha_2| = 2^{v_2(|\alpha|)}$ and $|\gamma_2| = 2^{v_2(|\gamma|)}$, and we put $t = v_2(e)$ and $u = v_2(f)$. When $t \geq 2$, we have two-primary order $|\alpha_2| \mid 2^{u+t+1} \mid p-1$. When $t = 1$ and w is even, $|\alpha_2| \mid 2^{u+2} \mid p-1$. Thus, $\sigma(\alpha) = \alpha$ in both cases for $\sigma \in \mathcal{H}_p$. In the remaining case $t = 1$ with odd w, $|\gamma_2| = 8$. Moreover, $|\alpha_2| = 2^{u+3}$. Since $v_2(\mathrm{ord}_p(r)) = u+2$, Section 2 gives $\sigma(\alpha)/\alpha = \epsilon_\sigma^{jw}$ on $\mathrm{Stab}_{\mathcal{H}_p}(\chi_\eta^\pm)$. □

Proposition 4.25. *We choose the faithful series $\mathcal{E}$ of* Lemma 4.23*, with $e = 2j$. We write Σ for its symplectic label over $\mathbb{F}_{q^2}$, of rank a. Moreover, we choose partition $\lambda \vdash b$ and put $n = 2a + b$. We put $m_\lambda = \binom{b}{2} + \sum_i \binom{\lambda_i}{2} - \sum_i \binom{\lambda'_i}{2}$. When $q \equiv 1 \bmod 4$, we write $\mathrm{core}_2(\lambda) = \Delta_t$ and $u \equiv a + (b + K_t)/2 \bmod 2$. Moreover, when $q \equiv 3 \bmod 4$, we put $u \equiv a + K_b + m_\lambda \bmod 2$. At $b = 0$, $u \equiv a \bmod 2$ in both cases. We fix $\chi_+, \chi_- \in \mathcal{E}$ and local pair ψ_+, ψ_- above the orbit of θ. Thus, we have sign $s_\chi(\sigma) = \eta_q(c_\sigma)^u = s_\psi(\sigma)$. On the common stabilizer $\mathrm{Stab}_{\mathcal{H}_p}(\chi_\pm) = \mathrm{Stab}_{\mathcal{H}_p}(\psi_\pm)$, we choose full field extensions $\widehat{\chi}_\pm, \widehat{\psi}_\pm$ with ratios $r_{\widehat{\chi}_\pm}(\sigma) = \epsilon_\sigma^u = r_{\widehat{\psi}_\pm}(\sigma)$. Moreover, at $h = 0$, we have the same factors $\eta_q(c_\sigma)^u, \epsilon_\sigma^u$.*

Proof. First, we fix $q \equiv 1 \bmod 4$. $4f_2 \mid p-1$, so $\sigma(i) = i$ for $\sigma \in \mathcal{H}_p$. By Lemma 4.16, we choose module V_{comp} and its field operator S_{comp}. We take its root for normalization $\gamma_{\mathrm{comp}} \in \mu_{4f_2}$. Moreover, $V_{\mathrm{comp}}^\sigma \simeq V_{\mathrm{comp}}$, $S_{\mathrm{comp}}^\sigma = S_{\mathrm{comp}}$ and $\sigma(\gamma_{\mathrm{comp}}) = \gamma_{\mathrm{comp}}$. At the joint base for (Σ, λ), we have grading factors $\eta_q(c_\sigma)^L, \epsilon_\sigma^L$. By Lemma 4.15, $L \equiv K_t \bmod 2$. Then, there are $a - k(k+1)$ symplectic steps $a \mapsto a+1$ and $(b - K_t)/2$ unitary steps $b \mapsto b+2$. Moreover, each step contributes exponent 1 by Lemma 4.19. Since $k, k+1$ have opposite parity, $k(k+1) \equiv 0 \bmod 2$. Thus, $L + a - k(k+1) + (b - K_t)/2 \equiv a + (b + K_t)/2 \bmod 2$. We have global exponent u in the unitary case $q \equiv 1 \bmod 4$.

Moreover, fix $q \equiv 3 \bmod 4$, with odd f and $4 \mid d$. At $b = 0$, by Lemma 4.21, we have exponent a, including principal support of Σ. With $b > 0$, choose operator T for opposite parabolics P, P^- from Section 4.3. Thus, $T^2 = (-1)^{K_b} q^{4ab+K_b} I$. We take its radical coordinates $\mathcal{U}_T$, with $|\mathcal{U}_T| = q^{4ab+K_b}$. Moreover, we have scalar $(-1)^{K_b}$ from w_*^2. $T^2 = (-1)^{K_b} q^{4ab+K_b} I$ for every $p \neq r$. Given ρ_λ, we choose a rational model V_λ. By [61, Theorem 3], we choose graph operator S_λ with $S_\lambda^2 = I$ and coefficient factor $(\sigma(\sqrt{q})/\sqrt{q})^{m_\lambda}$. In particular, $\sigma(i) = i$ and $\sigma(\sqrt{q})/\sqrt{q} = \eta_q(c_\sigma)$. Moreover, we put $T_0 = i^{K_b} q^{-(4ab+K_b)/2} T$, with $T_0^2 = I$. We have coefficient factor $\eta_q(c_\sigma)^{K_b}$ for the normalization of T. Together with S_λ, we have additional exponent $K_b + m_\lambda$. Thus, $u \equiv a + K_b + m_\lambda \bmod 2$. If f is odd, we choose extensions to $X \rtimes \langle F_r \rangle$ by [208, Lemma 6.13]. Then, $\eta_q(c_\sigma) = \epsilon_\sigma$. On $\mathrm{Stab}_{\mathcal{H}_p}(\chi_\pm)$, $\epsilon_\sigma^u = \eta_q(c_\sigma)^u = 1$. Hence, we have ratio $r_{\widehat{\chi}_\pm}(\sigma) = 1 = \epsilon_\sigma^u$ there.

Choose cocore Σ_0 of rank c, with the same λ, and $a - c = jw$ for every cuspidal support of Σ. Moreover, we write u_{res} for the exponent at (Σ_0, λ). We find $u - u_{\mathrm{res}} \equiv a - c = jw \bmod 2$. If $h > 0$, the full spin normalizer is N_{full}. By Lemma 4.9, $A_h = TV$. Then, the active norm class is $\bar{\nu}_A : N_{\mathrm{full}} \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. Thus, we have index-two subgroup $L_\square = \ker \bar{\nu}_A = (A_h \times \mathrm{Spin}_{2h+1}(q))/\langle (z,z) \rangle$. Choose residual pair ξ_+, ξ_-, with $\xi_\pm(z) = -\xi_\pm(1)$. By Lemma 4.24, choose active pair χ_η^+, χ_η^-, with $\chi_\eta^\pm(z) = -\chi_\eta^\pm(1)$. Then, on $L_\square$ the four characters are $\alpha_{s,t} = \chi_\eta^s \boxtimes \xi_t$, for $s, t \in \{+,-\}$. Moreover, we have trivial scalar 1 on (z,z) in each product $\alpha_{s,t}$. We fix $x \in N_{\mathrm{full}} \setminus L_\square$, and $\alpha_{s,t}^x = \alpha_{-s,-t}$. Thus, we have two orbits $\{\alpha_{+,+}, \alpha_{-,-}\}$ and $\{\alpha_{+,-}, \alpha_{-,+}\}$. We choose $\psi_+ = \mathrm{Ind}_{L_\square}^{N_{\mathrm{full}}} \alpha_{+,+}$ and $\psi_- = \mathrm{Ind}_{L_\square}^{N_{\mathrm{full}}} \alpha_{+,-}$. Thus, ψ_+, ψ_- are distinct irreducibles of degree $2^w \eta(1) \chi_{\mathrm{res}}(1)$, with $\chi_{\mathrm{res}} = \xi_+$. By restriction to $L_\square$, we have orbit sums $\alpha_{+,+} + \alpha_{-,-}$ and $\alpha_{+,-} + \alpha_{-,+}$. Write s_A, s_R for signs on $(\chi_\eta^+, \chi_\eta^-)$ and (ξ_+, ξ_-). Consequently, $s_\psi = s_A s_R$. On both factors of $L_\square$,

we choose the field lift a_F. Then, $a_F^f = (v_A^{-1}, 1)$, with product v^{-1} in N_{full}. If f is even, $\mathcal{H}_p$ fixes both pairs. Thus, $r_A(\sigma) r_R(\sigma)$ is the tensor ratio on $L_\square$ and after induction to N_{full}. If f is odd, the compatible extensions $\widehat{\psi}_\pm$ form a torsor $\mathcal{T}$. By Proposition 4.22, fix $y \in \mathcal{T}$ fixed by the remaining exchange τ. $\tau^2 = 1$ and odd-order translation group A_F on $\mathcal{T}$. Moreover, any diagonal root γ_{diag} lies in μ_{2f_2}. $\sigma(\gamma_{\mathrm{diag}}) = \gamma_{\mathrm{diag}}$ for $\sigma \in \mathcal{H}_p$. By Lemma 4.24, we have active exponent $jw = a - c$. Thus, $s_\chi = s_\psi$ and $r_{\widehat{\chi}_\pm} = r_{\widehat{\psi}_\pm}$. If $h = 0$, $b = c = 0$ and $a = jw$. We choose active characters χ_η^+, χ_η^- directly, of degree $2^{w-1}\eta(1)$. By Lemma 4.24, we again have factors $\eta_q(c_\sigma)^{jw}, \epsilon_\sigma^{jw}$. Thus, $u \equiv jw \bmod 2$, with the same factors at $h = 0$. □

Lemma 4.26. *We choose an involution $s \in G^* = \mathrm{Sp}_{2n_0}(q)$ and write $\Theta \in \mathcal{E}(D_0, s)$, with $D_0 = \mathrm{SO}_{2n_0+1}(q)$ and ordered labels Σ_1, Σ_2. Moreover, we choose their cuspidal ranks $k(k+1), l(l+1)$. We have selected multiplicity character $\vartheta_{k,l}$ of degree 1. We take its field-relation element w, with $\vartheta_{k,l}(w) = \xi_{k,l} \in \{1, -1\}$. Thus, with $\alpha_{k,l}^f = \xi_{k,l}^{-1}$ and $\alpha_{k,l} \in \mu_{2f_2}$, we have full field ratio $r_{\widehat{\Theta}}(\sigma) = \sigma(\alpha_{k,l})/\alpha_{k,l}$. At equal supports $k = l$, we choose $\xi_{k,k} = 1$. Moreover, in the mixed linear series $q \equiv 1 \bmod 4$ with even f, we take doubled construction $\mathcal{C}$ for opposite parabolic P^-. We take residual involution parent Θ_{res}. We choose the full field group $F_{\mathrm{ord}} = \langle F_r \rangle$. Moreover, on F_{ord}, we have ratio $r_{\mathcal{C}}(\sigma) = r_{\widehat{\Theta}_{\mathrm{res}}}(\sigma)$.*

Proof. First, we order the supports k, l with $k \geq l$. We have row pair (α, α), with $\alpha_i = (k-i+1)_+ + (l-i+1)_+$. We write Λ_{wf} for the support partition of Θ. Moreover, we have $2l+1$ single odd blocks and $k-l$ paired sizes in Λ_{wf}. We take cuspidal denominators d_k, d_l, with $d_k d_l = 2^{k+l}$. The component group has order 2^{k+l}. By [188, Proposition 15.4 and (11.15)], we have total multiplicity 1. Thus, the unique contributing class is $\mathcal{O}$, with linear source $\varrho = \vartheta_{k,l}$. Set $R_i = \mathrm{O}(U_i)$, with $\dim U_i \in \{1, 2\}$. Thus, we have reductive source $C_{\mathrm{red}} = \{(x_i) \in \prod_i R_i : \prod_i \det(x_i) = 1\}$. Moreover, we take rotation torus $T_i \leq R_i$ and character $\tau \in \mathrm{Irr}(T_i)$. Choose an available reflection r_τ for τ. Thus, $\tau^{r_\tau} = \tau = \tau^{-1}$, so $\tau^2 = 1$. Choose source model M, with reflections over $\mathbb{F}_r$ and block signs ± 1. Moreover, the grading operator $D(c_\sigma)$ lies in D_0 over $\mathbb{F}_r$. By the block construction of M, $D(c_\sigma)w = wD(c_\sigma)$. Choose source field scalar $\alpha_{k,l}$, with $\alpha_{k,l}^f = \varrho(w^{-1}) = \xi_{k,l}^{-1}$. Moreover, we have coefficient ratio $\sigma(\alpha_{k,l})/\alpha_{k,l}$. We take ordinary induction steps with $\mathrm{GL}_1(q)$-characters $1, \eta_q$. In particular, F_r has value 1 on both extensions $\widehat{1}, \widehat{\eta}_q$. By multiplicity 1 in each step, we have ratio $\sigma(\alpha_{k,l})/\alpha_{k,l}$ for the ordered labels Σ_1, Σ_2.

If $k = l > 0$, the cuspidal source θ_{cusp} is from Lemma 4.20. We find $\xi_{k,k} = 1$. Moreover, with principal support $k = l = 0$, we choose torus characters ν_1, ν_2. We have field extensions $\widehat{\nu}_i$ with $\widehat{\nu}_i(F_r) = 1$. At $\Sigma_1 = \Sigma_2$, by Lemma 4.21, we have sign $\eta_q(c_\sigma)^a$ for symplectic rank a, also at principal support $k = l = 0$. Moreover, in the mixed linear case $\Sigma_1 = \Sigma_2$, we choose opposite operator T. $T^2 = q^{4ab+K_b} I$, with graph contribution m_λ, and we choose $\mathrm{core}_2(\lambda) = \Delta_t$. Thus, we have exchange exponent $a + (b + K_t)/2 \bmod 2$. On Δ_t, $K_b + m_\lambda \equiv (b + K_t)/2 \bmod 2$. Moreover, with rim domino λ/μ, $(K_b + m_\lambda) - (K_{b-2} + m_\mu) \equiv 1 \bmod 2$. $(b + K_t)/2 - (b - 2 + K_t)/2 = 1$. Choose either base branch β. Consequently, with even f, $\beta^\sigma = \beta$.

Moreover, we choose quartic character ξ, with $\xi^2 = \eta_q$. We choose ambient special orthogonal group D and parabolic $P = LU_P$. Moreover, we have Levi $L \simeq \mathrm{GL}_b(q) \times D_0$, with residual character $\Sigma = \Theta_{\mathrm{res}}$. On L, choose $\mu_\pm = ((\xi^{\pm 1} \circ \det)\rho_\lambda) \boxtimes \Sigma$. Choose modules $V_\pm$ with characters $\mu_\pm$. Moreover, fix $\rho_\pm : P \to \mathrm{GL}(V_\pm)$, with $\rho_\pm(lu) = \rho_\pm(l)$ for $l \in L, u \in U_P$. Write $I_\pm = \mathrm{Ind}_P^D V_\pm$, with characters $\mathrm{Ind}_P^D \mu_\pm$. Since $C_{D^*}(\widetilde{s}) \leq L^*$, both I_+, I_- afford the irreducible character χ. We choose opposite isometry g and the corresponding double coset $\mathcal{D}$. After correction by g, the series is still $\mathcal{E}(D, \widetilde{s})$, with labels (ρ_λ, Σ). Moreover, choose graph operator T_{gr} on ρ_λ and identity operator I_Σ on the residual module V_Σ. We have kernels on $\mathcal{D}$ with value $T_{\mathrm{gr}} \otimes I_\Sigma$, and we choose maps $A_+ : I_+ \to I_-$ and $A_- : I_- \to I_+$. Since χ is irreducible, we have scalar composites $A_- A_+, A_+ A_-$. We have q^{4ab+K_b} radical entries in the coefficient at 1. Moreover, on $\mathcal{D}$, we have opposite component $u^- = 1$. Choose the block exchange $J : E_1 \to E_2$, with its map $E_2 \to E_1$, and we have contribution $J^2 T_{\mathrm{gr}}^2$ from each radical entry u. Indeed, in the orthogonal model D, $J^2 = T_{\mathrm{gr}}^2 = I$. Thus, $A_- A_+ = q^{4ab+K_b} I_{I_+}$ and $A_+ A_- = q^{4ab+K_b} I_{I_-}$. Choose doubled space $\mathcal{C} = I_+ \oplus I_-$ and put $B = q^{-(4ab+K_b)/2} \left(\begin{smallmatrix} 0 & A_- \\ A_+ & 0 \end{smallmatrix} \right)$, with $B^2 = I$. Moreover, for even f, we have rational scalar $q^{-(4ab+K_b)/2}$.

By [61, Theorem 3], we take rational T_{gr} with $T_{\mathrm{gr}}^2 = I$. Moreover, the rational flag module V_λ affords ρ_λ. We write R_λ for its representation of $\mathrm{GL}_b(q)$. Since ρ_λ is absolutely irreducible, $\mathrm{span}_{\mathbb{Q}} R_\lambda(\mathrm{GL}_b(q)) = \mathrm{End}_{\mathbb{Q}}(V_\lambda)$. We have rational traces $\mathrm{Tr}(R_\lambda(g) T_{\mathrm{gr}})$ on graph coset $\mathcal{C}_{\mathrm{gr}}$. Moreover, we have nondegenerate pairing $(A, B) \mapsto \mathrm{Tr}(AB)$ on $\mathrm{End}_{\mathbb{Q}}(V_\lambda)$. Thus, we recover T_{gr} over $\mathbb{Q}$ from its graph traces $\mathrm{Tr}(R_\lambda(g) T_{\mathrm{gr}})$. On $\mathrm{GL}_b(q)$, the graph map is $\gamma(g) = g^{-T}$. Since $[\gamma, F_r] = 1$, we choose the extension $\widehat{R}_\lambda$ over $\mathrm{GL}_b(q) \rtimes \langle \gamma, F_r \rangle$ by [156, proof of Proposition 8.5]. Moreover, $\widehat{R}_\lambda(\gamma) = T_{\mathrm{gr}}$ and $\widehat{R}_\lambda(F_r) = P_{\mathrm{gr}}$. Thus, $P_{\mathrm{gr}} T_{\mathrm{gr}} = T_{\mathrm{gr}} P_{\mathrm{gr}}$. We have the same relation for $cP_{\mathrm{gr}}, dT_{\mathrm{gr}}$, with $c, d \in \mathbb{C}^\times$.

We choose residual field operator $P_{\rm res}$ and coefficient operator C_σ. Moreover, choose opposite representatives $\mathcal{R}$ and unipotent indices $\mathcal{U}$ over $\mathbb{F}_r$. We write P for the full field operator on $\mathcal{C}$. Thus, $PB = BP$ and $C_\sigma B = BC_\sigma$. Also, $C_\sigma P = \frac{\sigma(\alpha_{k,l})}{\alpha_{k,l}} PC_\sigma$. By the quartic characters $\xi \circ \det, \xi^{-1} \circ \det$, we have that $P(I_+) = I_-$ if and only if $r \equiv 3 \bmod 4$. $C_\sigma(I_+) = I_-$ if and only if $\sigma(i) = -i$. Fix $t_F, t_\sigma \in \{0, 1\}$, with $t_F \equiv (r-1)/2 \bmod 2$ and $(-1)^{t_\sigma} = \sigma(i)/i$. Moreover, on I_+, put $\Phi = B^{t_F} P$ and $\widehat{C}_\sigma = B^{t_\sigma} C_\sigma$. We find $\widehat{C}_\sigma \Phi = \frac{\sigma(\alpha_{k,l})}{\alpha_{k,l}} \Phi \widehat{C}_\sigma$. Since f is even, $\Phi^f = B^{t_F f} P^f = I$. We fix $R_D : D \to \mathrm{GL}(I_+)$ with character χ. We take spinor character $\vartheta_{\rm sp} : D \to \{1, -1\}$, with $H = \ker \vartheta_{\rm sp}$. Moreover, at $\Sigma_1 = \Sigma_2$, $\chi \vartheta_{\rm sp} = \chi$. Thus, we fix $J_{\rm sp}$ with $J_{\rm sp}^2 = I$ and $J_{\rm sp} R_D(g) = \vartheta_{\rm sp}(g) R_D(g) J_{\rm sp}$ for $g \in D$. We take invariant eigenspaces $W_\pm = \ker(J_{\rm sp} \mp I)$. Moreover, we have ratio $\sigma(\alpha_{k,l})/\alpha_{k,l}$ on both $W_\pm$. At $\Sigma_1 \neq \Sigma_2$, $\mathrm{Res}_H^D \chi \in \mathrm{Irr}(H)$. Therefore, we have ratio $R_{\rm irr} = \sigma(\alpha_{k,l})/\alpha_{k,l}$. With equal labels $\Sigma_1 = \Sigma_2$, $R_\beta = \sigma(\alpha_{k,l})/\alpha_{k,l}$. If $D_{\rm res}$ is absent, choose $R_{\rm res} = 1$. □

Moreover, we fix $q = r^f \equiv 3 \bmod 4$, with prime r and odd $\ell \neq r$. With central character $\omega = 1$, we write $\widetilde{G} = \mathrm{Spin}_{2n+1}(q)$. Moreover, we choose the dual lift $\widetilde{s} \in \mathrm{Sp}_{2n}(q)$ with eigenvalues $1, -1, i, -i$ of multiplicities $2a, 2a, b, b$. $b > 0$ and $n = 2a + b$. In $\mathrm{SO}_{2n+1}(q)$, we write H for the image of $\widetilde{G}$. Thus, we write $\widetilde{G} \to H$, with central kernel $\langle z \rangle$. On H, we write symplectic labels ρ_1, ρ_2 and characters $\chi_+, \chi_- \in \mathrm{Irr}(H)$. Moreover, we allow $\rho_1 = \rho_2$ with $\chi_+ \neq \chi_-$. On the marked normalizer N, we write the partners $\psi_\pm = \Omega(\chi_\pm) \in \mathrm{Irr}(N)$.

Lemma 4.27. *We choose a splitting ℓ-modular system $(\mathcal{K}, \mathcal{O}, \mathbb{F})$. Thus, $\mathcal{K} = \mathrm{Frac}(\mathcal{O})$, with $\mathrm{char}(\mathcal{K}) = 0$. Moreover, we write discrete valuation ring $\mathcal{O}$, complete for v with residue field $\mathbb{F}$. On $\mathcal{K}^\times$, we write $v : \mathcal{K}^\times \to \mathbb{Z}$, with $v(\mathcal{K}^\times) = \mathbb{Z}$ and $\mathrm{char}(\mathbb{F}) = \ell$. Moreover, we write finite groups $H \lhd D$, with $[D : H] = 2$. On D/H, we write $\varepsilon : D \to \{1, -1\}$, with $\ker \varepsilon = H$. Thus, we write the $\mathcal{O}$-algebra automorphism $\tau : \mathcal{O}D \to \mathcal{O}D$, with $\tau(g) = \varepsilon(g) g$. We choose a block idempotent B of $\mathcal{O}D$, with $\tau(B) \neq B$, and have that $b_0 = B + \tau(B)$ is a block idempotent in $\mathcal{O}H$. Moreover, $\varphi : \mathcal{O}Hb_0 \to \mathcal{O}DB$ and $\psi : \mathcal{O}DB \to \mathcal{O}Hb_0$ are inverse algebra maps, with $\varphi(y) = yB$ and $\psi(x) = x + \tau(x)$. We have that $\mathrm{Res}_H^D : \mathrm{Irr}(D, B) \to \mathrm{Irr}(H, b_0)$ is bijective. With a D-stable block b' of H, we write one covering block B', or two blocks $B', \tau(B')$.*

Proof. Since $2 \in \mathcal{O}^\times$, $(\mathcal{O}D)^\tau = \mathcal{O}H$ by the signs on $D \setminus H$. Thus, $b_0 = B + \tau(B)$ belongs to $\mathcal{O}H$. Since $B\tau(B) = 0$, the maps φ, ψ are multiplicative, with $\psi\varphi(y) = yb_0 = y$ and $\varphi\psi(x) = xB = x$. Hence, b_0 is primitive in $Z(\mathcal{O}H)$ because B is primitive in $Z(\mathcal{O}D)$. On any simple $\mathcal{K}DB$-module, hB acts as h for $h \in H$, so the isomorphism gives $\mathrm{Irr}(D, B) \cong \mathrm{Irr}(H, b_0)$ by restriction. By [65, Lemma 2.1.9], the blocks over b' form a $\langle\tau\rangle$-orbit. Thus, the blocks are B' alone or $B', \tau(B')$. □

Suppose that $H \lhd D$ and that $[D : H] = 2$. In $\mathrm{Syl}_\ell(H)$, we choose P. We consider $\mathrm{N}_D(P) \leq U \leq D$ and $H_1 = U \cap H$. By Isaacs [97, Lemma 1.13], we obtain $D = HU$ and $U/H_1 \simeq D/H$. We fix orbits $\mathcal{O}_H, \mathcal{O}_{H_1}$ of ℓ'-degree characters. Moreover, we choose bijection $f : \mathcal{O}_H \to \mathcal{O}_{H_1}$, with $f(\vartheta) = \vartheta'$. We fix Brauer correspondents b, b_1 of full defect P, with $\vartheta \in \mathrm{Irr}(H, b)$ and $\vartheta' \in \mathrm{Irr}(H_1, b_1)$. Moreover, we assume that $f(\vartheta^u) = f(\vartheta)^u$ for $u \in U$, and $f(\vartheta^\gamma) = f(\vartheta)^\gamma$ for mixed actions $\gamma \in \Gamma$. We have that $P \in \mathrm{Syl}_\ell(D) \cap \mathrm{Syl}_\ell(U)$, since $\ell \neq 2$. By Cabanes [37, Theorem 1.17], we write the Brauer correspondence $\mathfrak{b}$ between full-defect blocks of D, U. Since $\mathrm{N}_D(P) \leq U$, $C_D(P) \leq U$. By Koshitani–Späth [109, Theorem B], $B_1 b_1 \neq 0$ when $Bb \neq 0$, with $B_1 = \mathfrak{b}(B)$. We write $\overline{a} \in \mathbb{F}$ for the image of $a \in \mathcal{O}$ and the Brauer map $\mathrm{Br}_P : (\mathcal{O}D)^P \to \mathbb{F}C_D(P)$, with $\mathrm{Br}_P(\sum_g a_g g) = \sum_{g \in C_D(P)} \overline{a}_g g$. Moreover, we write $\varepsilon : D \to \{1, -1\}$, with $\ker \varepsilon = H$. We have that $\ker(\varepsilon|_U) = H_1$. Indeed, $\mathrm{Br}_P(\tau(x)) = \overline{\tau}(\mathrm{Br}_P(x))$, where $\overline{\tau}(g) = \overline{\varepsilon(g)} g$. With the actions in Γ on the chosen pointed subgroups at P, $\mathrm{Br}_{P^\gamma}(x^\gamma) = \mathrm{Br}_P(x)^\gamma$. Thus, $\mathfrak{b}(\tau(B)) = \tau(\mathfrak{b}(B))$ and $\mathfrak{b}(B^\gamma) = \mathfrak{b}(B)^\gamma$.

We choose $\dot{d} \in U \setminus H_1$. By Clifford [43], $\Theta = \mathrm{Ind}_H^D \vartheta \in \mathrm{Irr}(D)$ when $\vartheta^{\dot{d}} \neq \vartheta$. Then, we find $\mathrm{Res}_H^D \Theta = \vartheta + \vartheta^{\dot{d}}$. Moreover, Θ is independent of the orbit member $\vartheta, \vartheta^{\dot{d}}$. We write $B = \mathrm{bl}(\Theta)$. If $b^{\dot{d}} \neq b$, we write its idempotent $e \in Z(\mathcal{O}H)$. Thus, $ee^{\dot{d}} = 0$ and $e\mathcal{O}De = \mathcal{O}He$. We put orbit algebra $A = \mathcal{O}D(e + e^{\dot{d}})$. Moreover, we write matrix units $E_{11} = e$ and $E_{22} = e^{\dot{d}}$, $E_{12} = e\dot{d}^{-1}$ and $E_{21} = \dot{d}e$. We have that $E_{12}E_{21} = e$ and $E_{21}E_{12} = e^{\dot{d}}$. Since $\dot{d}^2 \in H$, $A \simeq \mathrm{Mat}_2(\mathcal{O}He)$. We have that $B = e + e^{\dot{d}}$ is the unique block over $b, b^{\dot{d}}$. If $b^{\dot{d}} = b$, $\Theta\varepsilon = \Theta$. Thus, we find $\tau(B) = B$. By Lemma 4.27, B is unique. $\Theta' = \mathrm{Ind}_{H_1}^U \vartheta' \in \mathrm{Irr}(U)$, with $\mathrm{Res}_{H_1}^U \Theta' = \vartheta' + \vartheta'^{\dot{d}}$ and block $B_1 = \mathfrak{b}(B)$.

If $\vartheta^{\dot{d}} = \vartheta$, we choose extensions $\widehat{\vartheta}_+, \widehat{\vartheta}_- \in \mathrm{Irr}(D)$. By Clifford [43], $\widehat{\vartheta}_- = \widehat{\vartheta}_+ \varepsilon$ and $\mathrm{Res}_H^D \widehat{\vartheta}_\pm = \vartheta$. If two blocks $B, \tau(B)$ cover b, we consider the corresponding pair $B_1, \tau(B_1)$. By Lemma 4.27, we choose $\widehat{\vartheta}_B \in \mathrm{Irr}(D, B)$ with $\mathrm{Res}_H^D \widehat{\vartheta}_B = \vartheta$. We have that the extension $\widehat{\vartheta}'_{B_1} \in \mathrm{Irr}(U, B_1)$ is unique with $\mathrm{Res}_{H_1}^U \widehat{\vartheta}'_{B_1} = \vartheta'$. Moreover, $\widehat{\vartheta}_{\tau(B)} = \widehat{\vartheta}_B \varepsilon$ and $\widehat{\vartheta}'_{\tau(B_1)} = \widehat{\vartheta}'_{B_1} \varepsilon|_U$. We put $\widehat{f}(\widehat{\vartheta}_B) = \widehat{\vartheta}'_{B_1}$. Thus, $\widehat{f}(\chi^\gamma) = \widehat{f}(\chi)^\gamma$ and $\widehat{f}(\chi\varepsilon) = \widehat{f}(\chi)\varepsilon|_U$. With unique block B, we

choose $o_\vartheta : \{\widehat{\vartheta}_+, \widehat{\vartheta}_-\} \to \{\widehat{\vartheta}'_+, \widehat{\vartheta}'_-\}$. We write $\epsilon^D_\vartheta, \epsilon^U_{\vartheta'} : \Gamma_\vartheta \to \{1, -1\}$ for the two exchange characters, with $\Gamma_\vartheta = \Gamma_{\vartheta'}$. Moreover, o_ϑ is equivariant precisely when $\epsilon^D_\vartheta = \epsilon^U_{\vartheta'}$. Given $\gamma \in \Gamma$, we write $o_{\vartheta^\gamma}(\widehat{\vartheta}^\gamma) = o_\vartheta(\widehat{\vartheta})^\gamma$. If $\vartheta^{\gamma_1} = \vartheta^{\gamma_2}$, $\gamma_1\gamma_2^{-1} \in \Gamma_\vartheta$. By $\epsilon^D_\vartheta = \epsilon^U_{\vartheta'}$, $o_{\vartheta^{\gamma_1}} = o_{\vartheta^{\gamma_2}}$. Thus, we write $\widehat{f} : \mathrm{Irr}(D \mid \mathcal{O}_H) \to \mathrm{Irr}(U \mid \mathcal{O}_{H_1})$, with $\widehat{f}(\chi\varepsilon) = \widehat{f}(\chi)\varepsilon|_U$ and $\mathrm{bl}(\widehat{f}(\chi)) = \mathfrak{b}(\mathrm{bl}(\chi))$. We have that $\Theta(1) = 2\vartheta(1)$ and $\widehat{\vartheta}_\pm(1) = \vartheta(1)$. Since $\ell \neq 2$, $\ell \nmid \Theta(1) \iff \ell \nmid \vartheta(1)$. Also, $\ell \nmid \widehat{\vartheta}_\pm(1) \iff \ell \nmid \vartheta(1)$.

Proposition 4.28. *With $K = \mathrm{GU}_b(q)$ and $R = \mathrm{SO}_{4a+1}(q)$, we put $D_0 = K \times R$. Moreover, we write ε_K for the quadratic determinant character on K, and ε_R for the spinor character on R. Put $L_0 = \ker(\varepsilon_K \boxtimes \varepsilon_R)$. On K, we choose determinant character ξ with $\xi^2 = \varepsilon_K$. With equal labels $\rho_1 = \rho_2$, we consider unipotent $\rho_\lambda \in \mathrm{Irr}(K)$ and involution-series character $\Sigma \in \mathrm{Irr}(R)$. Thus, we find $\mu = ((\xi\rho_\lambda) \boxtimes \Sigma)|_{L_0} \in \mathrm{Irr}(L_0)$. On L_0, we write $N_0 = L_0 \rtimes \langle\gamma_h\rangle$, with $\gamma_h^2 = 1$ and spinor norm $\mathrm{sn}(\gamma_h) = 1$. On K, we write $\gamma_h(k) = h\overline{k}h^{-1}$, with $h \in K$ and $\overline{k} = (k_{ij}^q)$. Moreover, we choose extensions $\widehat{\mu}_+, \widehat{\mu}_- \in \mathrm{Irr}(N_0)$. Whenever $\epsilon \in \{1, -1\}$, $\widehat{\mu}^\sigma_\epsilon = \widehat{\mu}_{\epsilon s_\lambda(\sigma) s_R(\sigma)}$. Moreover, we write s_λ as the graph sign on ρ_λ. With $a > 0$, we write s_R on the pair $\Sigma|_{H_R} = \Sigma_+ + \Sigma_-$, with $H_R = \ker \varepsilon_R$. Thus, at $a = 0$, we consider $s_R = 1$.*

Proof. First, we write dual scalar $z_{\ell'}$ for ξ, with $|z_{\ell'}| = 4$. Since the characteristic is odd, $z_{\ell'} \neq -z_{\ell'}$. By [37, Theorem 4.8], consider $\mathcal{E}_\ell(K, z_{\ell'})$ and $\mathcal{E}_\ell(K, -z_{\ell'})$ as block unions. Consequently, $\mathcal{E}_\ell(K, z_{\ell'}) \cap \mathcal{E}_\ell(K, -z_{\ell'}) = \varnothing$. Thus, $\varepsilon_K B_K \neq B_K$ for the block B_K of $\xi\rho_\lambda$. On D_0, write product block B_0 of $(\xi\rho_\lambda) \boxtimes \Sigma$. Thus, $\tau_0(B_0) \neq B_0$, where τ_0 is the twist by $\varepsilon_K \boxtimes \varepsilon_R$. By Lemma 4.27, $b_L = B_0 + \tau_0(B_0)$ is a block of L_0, with $\mu \in \mathrm{Irr}(L_0, b_L)$. We write Hermitian norm space V_K for K, and odd orthogonal space V_R for R. Moreover, we choose unitary conjugation c on V_K, with $\det(c) = (-1)^b$. With c, we put $t = c \oplus (-1)^b I_{V_R}$, with $t^2 = I$ and $\det(t) = 1$. On R, $trt^{-1} = r$ for $r \in R$. Moreover, we write spinor map $\mathrm{sn} : \mathrm{SO}(V_K \perp V_R) \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. If $\mathrm{sn}(t) = 1$, consider $h = 1$. Otherwise, choose $u \in C_{q+1} \setminus C_{q+1}^2$, where $C_{q+1} = \{u \in \mathbb{F}_{q^2}^\times : u^{q+1} = 1\}$. We have that $-1 \in C_{q+1}^2$ and $u \neq -1$, since $4 \mid q+1$. On plane $W = \mathbb{F}_{q^2}$, we put $h = \mathrm{diag}(u, 1, \ldots, 1)$. Moreover, $Q_W(x) = x^{q+1}$. On W, we write the rotation $m_u : x \mapsto ux$, with $m_u = r_{1+u}r_1$, where r_v is reflection in $v \in W$ [137, §24.h]. Thus, $\mathrm{sn}(m_u) = N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(1+u)(\mathbb{F}_q^\times)^2$. We find $\mathrm{sn}(h) = \mathrm{sn}(m_u)$. Since $(1+u)^{q-1} = u^{-1}$, $\eta_q(N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(1+u)) = u^{(q+1)/2} = -1$. Thus, $\mathrm{sn}(ht) = 1$. Also, $tht^{-1} = h^{-1}$, so $(ht)^2 = I$. Thus, we put $\gamma_h = ht$, with $\gamma_h(k, v) = (h\overline{k}h^{-1}, v)$ on L_0. Since $\det(\gamma_h(k)) = \det(k)^{-1}$, $\xi(\gamma_h(k)) = \xi(k)^{-1}$ for $k \in K$.

On $K \times R$, choose $R_\lambda : K \to \mathrm{GL}(V_\lambda)$ for ρ_λ, and $R_\Sigma : R \to \mathrm{GL}(V_\Sigma)$ for Σ. By [61, Theorem 3], we choose graph extension on $K^+ = K \rtimes \langle c\rangle$, with operator T_0 satisfying $T_0^2 = I$. Then, $T_0 R_\lambda(h) T_0^{-1} = R_\lambda(h^{-1})$. Thus, put $T_K = R_\lambda(h)T_0$, with $T_K^2 = I$. We find $T_K R_\lambda(k) T_K^{-1} = R_\lambda(\gamma_h(k))$. At $\rho_1 = \rho_2$, $\Sigma\varepsilon_R = \Sigma$. Moreover, we choose T_R with $T_R^2 = I$ and $T_R R_\Sigma(u) T_R^{-1} = \varepsilon_R(u) R_\Sigma(u)$. On L_0, we write $M : L_0 \to \mathrm{GL}(V_\lambda \otimes V_\Sigma)$, with $M(k, u) = \xi(k) R_\lambda(k) \otimes R_\Sigma(u)$. We find $\mu(k, u) = \mathrm{tr}\, M(k, u)$. Since $(k, u) \in L_0$, $\varepsilon_R(u) = \varepsilon_K(k) = \xi(k)^2$. Thus, $\xi(k)\varepsilon_R(u) = \xi(k)^{-1}$. On $V_\lambda \otimes V_\Sigma$, put $T = T_K \otimes T_R$, with $T^2 = I$. $TM(k, u)T^{-1} = M(\gamma_h(k), u)$. Moreover, we choose the two extensions $\widehat{\mu}_\pm$, with $\gamma_h \mapsto \pm T$.

With rational ρ_λ, Σ, we choose coefficient operators $C_{\lambda,\sigma}, C_{\Sigma,\sigma}$. $C_{\lambda,\sigma} R_\lambda(k)^\sigma C_{\lambda,\sigma}^{-1} = R_\lambda(k)$. Moreover, $C_{\Sigma,\sigma} R_\Sigma(u)^\sigma C_{\Sigma,\sigma}^{-1} = R_\Sigma(u)$. $C_{\lambda,\sigma} T_K^\sigma C_{\lambda,\sigma}^{-1} = s_\lambda(\sigma) T_K$. $C_{\Sigma,\sigma} T_R^\sigma C_{\Sigma,\sigma}^{-1} = s_R(\sigma) T_R$. With $a > 0$, write constituents Σ_+, Σ_- on $\ker(T_R - I), \ker(T_R + I)$ for H_R. When $\xi^\sigma = \xi$, we put $C_\sigma = C_{\lambda,\sigma} \otimes C_{\Sigma,\sigma}$. Alternatively, when $\xi^\sigma = \xi^{-1}$, we put $C_\sigma = (I \otimes T_R)(C_{\lambda,\sigma} \otimes C_{\Sigma,\sigma})$. On L_0, write $M_{\xi^{-1}}(k, u) = \xi(k)^{-1} R_\lambda(k) \otimes R_\Sigma(u)$. On L_0, we choose $I \otimes T_R$ for $M_{\xi^{-1}}$ and M. Indeed, $(I \otimes T_R) M_{\xi^{-1}}(k, u)(I \otimes T_R)^{-1} = M(k, u)$. We find $(I \otimes T_R)T = T(I \otimes T_R)$. Thus, $C_\sigma T^\sigma C_\sigma^{-1} = s_\lambda(\sigma) s_R(\sigma) T$. At $a = 0$, $R = 1$ and $\xi|_{L_0}^2 = 1$. Moreover, we consider $T_R = I$ and $s_R = 1$. At $a = 0$, $\widehat{\mu}^\sigma_\epsilon = \widehat{\mu}_{\epsilon s_\lambda(\sigma)}$, with $\mathrm{Res}^{N_0}_{L_0} \widehat{\mu}_\epsilon = \mu$. □

Lemma 4.29. *Moreover, in* Proposition 4.28*, we consider $q = r^f \equiv 3 \bmod 4$. At rank a, we consider equal symplectic labels $\rho_1 = \rho_2$. On λ, we write the two-core Δ_t, with $m = |\Delta_t|$. Thus, we put $g_\lambda = \binom{m}{2} \bmod 2$. With $\sigma \in \mathcal{H}_\ell$, we write $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$. On N_0, $\widehat{\mu}^\sigma_\epsilon = \widehat{\mu}_{\epsilon\eta_q(c_\sigma)^{a+g_\lambda}}$ for $\epsilon \in \{1, -1\}$. Moreover, $\widehat{\mu}^\sigma_{\mathrm{loc},\epsilon} = \widehat{\mu}_{\mathrm{loc},\epsilon\eta_q(c_\sigma)^{a+g_\lambda}}$ on N_{tor}. We write N_{tor} as the marked torus normalizer, with $I_{\mathrm{loc}} = N_{\mathrm{tor}} \cap L_0$. Moreover, $\mathrm{Res}^{N_{\mathrm{tor}}}_{I_{\mathrm{loc}}} \widehat{\mu}_{\mathrm{loc},\pm} = \mu_{\mathrm{loc}} \in \mathrm{Irr}(I_{\mathrm{loc}})$. With unequal labels $\rho_1 \neq \rho_2$, we choose distinct conjugate sources θ, θ' on L_0. Thus, we find $\Theta_{\mathrm{glob}} = \mathrm{Ind}^{N_0}_{L_0} \theta \in \mathrm{Irr}(N_0)$. On I_{loc}, we choose $\theta_{\mathrm{loc}} \in \mathrm{Irr}(I_{\mathrm{loc}})$. We have that $\Theta_{\mathrm{loc}} = \mathrm{Ind}^{N_{\mathrm{tor}}}_{I_{\mathrm{loc}}} \theta_{\mathrm{loc}} \in \mathrm{Irr}(N_{\mathrm{tor}})$.*

Proof. First, we choose graph extensions $\widetilde{\rho}^\pm_\lambda$ of $\rho_\lambda \in \mathcal{E} \subseteq \mathrm{Irr}(K)$. With two-core Δ_t, write the Harish–Chandra series $\mathcal{E}$. If $m \equiv 0, 1 \bmod 4$, $\mathbb{Q}(\widetilde{\rho}^\pm_\lambda) = \mathbb{Q}$ by [61, Theorem 3]. If $m \equiv 2, 3 \bmod 4$, $\mathbb{Q}(\widetilde{\rho}^\pm_\lambda) = \mathbb{Q}(\sqrt{-q})$. Moreover, $Z(K) \leq \ker \rho_\lambda$. Write adjoint quotient $\overline{K} = K/Z(K)$, with graph action c. Moreover, write $\pi_K : K\langle c\rangle \to \overline{K}\langle c\rangle$. We

have that $\widetilde{\rho}_\lambda^\pm = \overline{\rho}_\lambda^\pm \circ \pi_K$, with $\overline{\rho}_\lambda^\pm \in \mathrm{Irr}(\overline{K}\langle c\rangle)$. Thus, $\widetilde{\rho}_\lambda^\pm(kc) = \overline{\rho}_\lambda^\pm(kZ(K)c)$. Thus, $s_\lambda(\sigma) = (\sigma(\sqrt{-q})/\sqrt{-q})^{g_\lambda}$ on T_K. With $g_\lambda = 1$, choose $\gamma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{-q})/\mathbb{Q})$, with $\gamma(\sqrt{-q}) = -\sqrt{-q}$. We find $(\widetilde{\rho}_\lambda^+)^\gamma = \widetilde{\rho}_\lambda^-$. Since $q = r^f \equiv 3 \bmod 4$, f is odd and $r \equiv 3 \bmod 4$. Choose primitive root ζ_r and $c_\sigma \in \mathbb{F}_r^\times$, with $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$. On $\mathbb{F}_r, \mathbb{F}_q$, write quadratic characters η_r, η_q, with $\eta_r(0) = 0$. Moreover, we put Gauss sum $G_r = \sum_{x\in\mathbb{F}_r} \eta_r(x)\zeta_r^x$, with $G_r^2 = -r$. We find $\sigma(G_r) = \eta_r(c_\sigma)G_r$. Since f is odd, $\eta_q(c_\sigma) = \eta_r(c_\sigma)$. Thus, choose $\sqrt{-q} = r^{(f-1)/2}G_r$. Thus, $s_\lambda(\sigma) = \eta_q(c_\sigma)^{g_\lambda}$. If $K = \mathrm{GU}_1(q)$ or $K = 1$, $\lambda = (1)$ or $\lambda = \varnothing$, respectively. $g_\lambda = 0$ and $s_\lambda(\sigma) = 1$. Moreover, by Lemma 4.21, $s_R(\sigma) = \eta_q(c_\sigma)^a$. From Proposition 4.28, we write tensor operator $T = T_K \otimes T_R$. Thus, $s_\lambda(\sigma)s_R(\sigma) = \eta_q(c_\sigma)^{a+g_\lambda}$ on $\widehat{\mu}_+, \widehat{\mu}_-$.

With $e = \mathrm{ord}_\ell(q^2)$, we write $a = eu + c$, with $0 \le c < e$. On R, we consider local semisimple element s_{loc}. Moreover, we write the two sets $\mathcal{B}_1, \mathcal{B}_2$ of active blocks, with $|\mathcal{B}_1| = |\mathcal{B}_2| = u$ and residual cores κ_1, κ_2 of rank c. By Proposition 4.45, we choose $C_{\mathrm{act},\sigma}$ on V_{act}, with $C_{\mathrm{act},\sigma}T_{\mathrm{act}}^\sigma C_{\mathrm{act},\sigma}^{-1} = \eta_q(c_\sigma)^{eu}T_{\mathrm{act}}$. In particular, on $R_c = \mathrm{SO}_{4c+1}(q)$, we write residual involution character Θ_c, with module V_c. For Θ_c, Lemma 4.21 gives the self-twist T_c, with sign $\eta_q(c_\sigma)^c$. If $c = 0$, put $T_c = I$ on $R_c = 1$. Write active parent D_{act}, with spinor character $\varepsilon_{\mathrm{act}} : D_{\mathrm{act}} \to \{1, -1\}$. Moreover, we write ε_c on R_c, with $H_{\mathrm{act}} = \ker\varepsilon_{\mathrm{act}}$ and $H_c = \ker\varepsilon_c$. In $D_{\mathrm{act}} \times R_c$, we put $H_{\mathrm{tot}} = \ker(\varepsilon_{\mathrm{act}} \boxtimes \varepsilon_c)$. We put $T_{R,\mathrm{loc}} = T_{\mathrm{act}} \otimes T_c$. Thus, $T_{R,\mathrm{loc}}^2 = I$ on $V = V_{\mathrm{act}} \otimes V_c$. On V, we write eigenspaces $V_\pm = \ker(T_{R,\mathrm{loc}} \mp I)$. Write $\delta_1, \delta_2 \in \{1, -1\}$ for eigenvalues of T_{act}, T_c. Moreover, put $V_\delta^{\mathrm{act}} = \ker(T_{\mathrm{act}} - \delta I)$ and $V_\delta^c = \ker(T_c - \delta I)$ for $\delta \in \{1, -1\}$. On $V_{\delta_1}^{\mathrm{act}} \otimes V_{\delta_2}^c$, $T_{R,\mathrm{loc}} = \delta_1\delta_2 I$. If T_{act}, T_c are both present, $[H_{\mathrm{tot}} : H_{\mathrm{act}} \times H_c] = 2$. Choose constituent characters $\chi_{\mathrm{act},\pm}, \chi_{c,\pm}$ on H_{act}, H_c. By [43], $\chi_1 = \mathrm{Ind}_{H_{\mathrm{act}}\times H_c}^{H_{\mathrm{tot}}}(\chi_{\mathrm{act},+} \boxtimes \chi_{c,+})$. $\chi_2 = \mathrm{Ind}_{H_{\mathrm{act}}\times H_c}^{H_{\mathrm{tot}}}(\chi_{\mathrm{act},+} \boxtimes \chi_{c,-})$. Thus, $(+,+), (-,-)$ are the branches of χ_1, and $(+,-), (-,+)$ are the branches of χ_2. $\chi_i^\sigma = \chi_i$ when both factor signs are -1, and $\eta_q(c_\sigma)^{eu+c} = \eta_q(c_\sigma)^a$. Moreover, at $u = 0$, $T_{R,\mathrm{loc}} = T_c$. At $a = 0$, $T_{R,\mathrm{loc}} = I$ and $s_{R,\mathrm{loc}}(\sigma) = 1$.

Moreover, we choose marked toral ℓ-subgroup $Q_K \le K$, with $K_{\mathrm{loc}} = \mathrm{N}_K(Q_K)$. Write paired character $\rho_{\lambda,\mathrm{loc}} = \Omega_K(\rho_\lambda) \in \mathrm{Irr}(K_{\mathrm{loc}})$. Moreover, choose graph extension sets $\mathcal{T}_{\mathrm{glob}}, \mathcal{T}_{\mathrm{loc}}$ over $\rho_\lambda, \rho_{\lambda,\mathrm{loc}}$. By [156, Proposition 8.5 and §8.1], we choose a bijection $j : \mathcal{T}_{\mathrm{glob}} \to \mathcal{T}_{\mathrm{loc}}$. We write quotient character $\varepsilon_2 \in \mathrm{Irr}(C_2)$, with $\varepsilon_2 \ne 1$. Then, $j(\tau\varepsilon_2) = j(\tau)\varepsilon_2$ and $j(\tau^\sigma) = j(\tau)^\sigma$. Moreover, we put finite set $\mathcal{X} = \mathcal{T}_{\mathrm{glob}} \sqcup \mathcal{T}_{\mathrm{loc}}$. We choose finite cyclotomic field E, with $\mathbb{Q}(\tau), \mathbb{Q}(\zeta_r) \subseteq E$ for $\tau \in \mathcal{X}$, and we fix $E \hookrightarrow \overline{\mathbb{Q}}_\ell$ and put $\mathbb{K}_\ell = \mathbb{Q}_\ell E$. Consequently, $\mathcal{H}_\ell|_E \simeq \mathrm{Gal}(\mathbb{K}_\ell/\mathbb{Q}_\ell)$. Thus, on $\mathcal{T}_{\mathrm{glob}}, \mathcal{T}_{\mathrm{loc}}$, $s_{\lambda,\mathrm{glob}} = s_{\lambda,\mathrm{loc}} = s_\lambda$. With active K, $d = 2e$ and e is odd. We consider $W = \mathbb{F}_{q^{2e}}$, with $\iota(x) = x^{q^e}$. Moreover, ι is antiunitary on W, with $\iota^2 = I$. Choose unitary conjugation ι_0 on the residual space V_0. We put $d_R = (-1)^b I_{V_R}$ on the odd orthogonal space V_R. Thus, put $g_{\mathrm{loc}} = (\bigoplus_W \iota) \oplus \iota_0 \oplus d_R$, with $g_{\mathrm{loc}}^2 = I$. If $\mathrm{sn}(g_{\mathrm{loc}}) \ne 1$, choose $\alpha \in C_{q+1} \setminus C_{q+1}^2$. $\alpha^{q+1} = 1$ and $\det_W(\alpha I) = \alpha^e$. Since e is odd, $\alpha^e \notin C_{q+1}^2$. We choose $h = \alpha I$ on W, with identity on $W^\perp$. By Proposition 4.28 on W, $\mathrm{sn}(hg_{\mathrm{loc}}) = 1$. $\iota h\iota^{-1} = h^{-1}$, since $\alpha^{q^e} = \alpha^{-1}$. Thus, we put $g'_{\mathrm{loc}} = hg_{\mathrm{loc}}$, with $(g'_{\mathrm{loc}})^2 = I$. With wholly residual space $V_0 \ne 0$, consider h on one unitary coordinate w. $\mathrm{sn}(hg_{\mathrm{loc}}) = 1$, with local comparison $c_0 = I$. Moreover, we choose representation $R_{\lambda,\mathrm{loc}} : K_{\mathrm{loc}} \to \mathrm{GL}(V_{\lambda,\mathrm{loc}})$. We choose the coefficient operator $C_{\mathrm{loc},\sigma}$, with $C_{\mathrm{loc},\sigma}R_{\lambda,\mathrm{loc}}(h)^\sigma C_{\mathrm{loc},\sigma}^{-1} = R_{\lambda,\mathrm{loc}}(h)$, and we choose $T_{K,\mathrm{loc}}$, with $C_{\mathrm{loc},\sigma}T_{K,\mathrm{loc}}^\sigma C_{\mathrm{loc},\sigma}^{-1} = s_\lambda(\sigma)T_{K,\mathrm{loc}}$. Moreover, put $T'_K = R_{\lambda,\mathrm{loc}}(h)T_{K,\mathrm{loc}}$. Thus, $C_{\mathrm{loc},\sigma}(T'_K)^\sigma C_{\mathrm{loc},\sigma}^{-1} = s_\lambda(\sigma)T'_K$.

With $\xi^\sigma = \xi^{-1}$, we write orthogonal self-twist $A = I \otimes T_R$ from Proposition 4.28. On $V_{\lambda,\mathrm{loc}} \otimes V$, we put $A_{\mathrm{loc}} = I_{V_{\lambda,\mathrm{loc}}} \otimes T_{R,\mathrm{loc}}$. Moreover, we put $T_{\mathrm{loc}} = T'_K \otimes T_{R,\mathrm{loc}}$. We find $A_{\mathrm{loc}}T_{\mathrm{loc}} = T_{\mathrm{loc}}A_{\mathrm{loc}}$. Thus, we write the exchange sign $s_\lambda(\sigma)\eta_q(c_\sigma)^a$ at N_{tor}. With active K, $\mathrm{ord}_\ell(-q) = e$ is odd. By [156, Lemma 2.3(c)], $\eta_q(c_\sigma) = \sigma(\sqrt{-q})/\sqrt{-q} = 1$ for $\sigma \in \mathcal{H}_\ell$. Thus, $s_\lambda(\sigma) = 1$ and $\eta_q(c_\sigma)^{a+g_\lambda} = 1$. On N_0, $\widehat{\mu}_\epsilon^\sigma = \widehat{\mu}_\epsilon$ for $\epsilon \in \{1, -1\}$. We consider ambient packet $\mathcal{P}$, with quotient twist $\nu \in \{1, \varepsilon\}$. Moreover, write quotient twists $\nu_I = \nu_{I_Q} = \nu$ for I, I_Q on $\mathcal{P}$. With $\rho_1 \ne \rho_2$, $\Sigma\varepsilon_R$ has labels (ρ_2, ρ_1). Thus, $\mathrm{Res}_{L_0}^{N_0}\Theta_{\mathrm{glob}} = \mu_1 + \mu_2$, with $\mu_1 \ne \mu_2$. Consider $\theta = \mu_1$ and $\theta' = \mu_2$, with $\theta^{\gamma_h} = \theta' \ne \theta$. Then, by [43] $\mathrm{Ind}_{L_0}^{N_0}\theta \in \mathrm{Irr}(N_0)$. On R, write unique quotient/core labels (ρ_1, ρ_2). Moreover, choose distinct local sources $\theta_{\mathrm{loc}}, \theta'_{\mathrm{loc}}$ on I_{loc}. We obtain $\Theta_{\mathrm{loc}} = \mathrm{Ind}_{I_{\mathrm{loc}}}^{N_{\mathrm{tor}}}\theta_{\mathrm{loc}} \in \mathrm{Irr}(N_{\mathrm{tor}})$. □

Lemma 4.30. *Moreover, we consider ℓ'-degree packet $\mathcal{P}$ and core/quotient bijection Ω_{D_0} on D_0. With toral ℓ-subgroups Q_K, Q_R, we put $Q_{D_0} = Q_K \times Q_R$. Moreover, we write Brauer correspondence $\mathfrak{b}_{D_0}$ with $\mathrm{N}_{D_0}(Q_{D_0})$, for defect group $P_{D_0} \in \mathrm{Syl}_\ell(D_0)$. With $\chi \in \mathcal{P}$, $\mathrm{bl}(\Omega_{D_0}(\chi)) = \mathfrak{b}_{D_0}(\mathrm{bl}(\chi))$. We put $\mathcal{P}_{L_0} = \{\theta \in \mathrm{Irr}(L_0) \mid \exists \chi \in \mathcal{P},\ \langle \mathrm{Res}_{L_0}^{D_0}\chi, \theta\rangle \ne 0\}$. Thus, $\mathrm{bl}(\Omega_{L_0}(\theta)) = \mathfrak{b}_{L_0}(\mathrm{bl}(\theta))$ for $\theta \in \mathcal{P}_{L_0}$. With defect group P_{D_0}, we write $\mathfrak{b}_{L_0}$ with $\mathrm{N}_{L_0}(Q_{D_0})$.*

Proof. First, we write $d = \mathrm{ord}_\ell(q)$ and $e = \mathrm{ord}_\ell(q^2)$. We write $2a = ew + c$, with $0 \le c < e$ and write torus centralizer $L_R = T^w \times \mathrm{SO}_{2c+1}(q)$. Moreover, $Q_R = (T_\ell)^w$ and $|T| = q^e - (-1)^{d+1}$. Consider $\lambda_R \in \mathrm{Irr}(L_R)$, with colours $1_T, \varepsilon_T$ on the w torus coordinates and choose residual character $\psi_c \in \mathrm{Irr}(\mathrm{SO}_{2c+1}(q))$, with cores κ_1, κ_2. Thus, $\lambda_R = (\boxtimes_{i=1}^w \lambda_i) \boxtimes \psi_c$, with $\lambda_i \in \{1_T, \varepsilon_T\}$. With active unitary $d = 2e$, e is odd. With $0 \le j < e$, write $b = ev + j$. Moreover, we write $L_K = T_U^v \times \mathrm{GU}_j(q)$, with $|T_U| = q^e + 1$. We find $Q_K = ((T_U)_\ell)^v$. We consider residual e-core

character $\psi_j \in \mathrm{Irr}(\mathrm{GU}_j(q))$. Thus, we put $\lambda_K = (1_{T_U^v} \boxtimes \psi_j)\xi|_{L_K}$. With inactive K, $\ell \nmid |K|$. Consider $Q_K = 1$ and $L_K = K$. Moreover, for $X \in \{R, K\}$, $Q_X = Z(L_X)_\ell$ and $Q_X \le \ker \lambda_X$. We have that $\ell \nmid |L_X/Q_X|$. Thus, consider direct product $L_X = Q_X \times C_X$, with $\ell \nmid |C_X|$. We consider $b_{L_X} \in \mathrm{Bl}(L_X)$. Thus, $|\mathrm{Irr}(L_X, b_{L_X}) \cap \mathrm{Irr}(L_X/Q_X)| = 1$.

Moreover, write ambient algebraic group $\mathbf{G}$ for X, with torus Levi subgroup $\mathbf{L}$. We put $L = \mathbf{L}^F = L_X$, $Q = Q_X$ and $\lambda = \lambda_X$ and consider $\chi_X \in \mathrm{Irr}_{\ell'}(X)$, with $\langle \chi_X, R_{\mathbf{L}}^{\mathbf{G}}\lambda_X\rangle \ne 0$. Moreover, $\mathrm{C}_{\mathbf{G}}(Q) = \mathbf{L}$. Over $\overline{\mathbb{F}}_q$, we write the natural module V, with residual space V^Q. Indeed, write active Q-weight spaces V_β, with $\beta \ne 1$. We find $\mathsf{V}_\beta = \{v \in \mathsf{V} : xv = \beta(x)v \ (x \in Q)\}$. With $A \le Q$, we put $\mathbf{H} = \mathrm{C}_{\mathbf{G}}(A)$. Since $|A|$ is odd, we write weights α, α^{-1} with $\alpha \ne 1$ in orthogonal type. $\alpha \ne \alpha^{-1}$. Thus, $\mathbf{H} = \mathbf{H}_0 \times \prod_i \mathbf{H}_i$, with connected odd orthogonal $\mathbf{H}_0$. Moreover, we write the geometric factors $\mathbf{H}_i = \mathrm{GL}_{n_i}$. With unitary $X = K$, write factors $\mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)$ over $\overline{\mathbb{F}}_q$. Moreover, consider $\chi_A \in \mathrm{Irr}(\mathbf{H}^F)$, with $m_A = \langle \chi_A, R_{\mathbf{L}}^{\mathbf{H}}\lambda\rangle \ne 0$. We find ${}^*R_{\mathbf{L}}^{\mathbf{H}}\chi_A = m_A \sum_{\lambda' \in \mathrm{N}_{\mathbf{H}}(\mathbf{L})^F \cdot \lambda} \lambda'$. We verify the orbit of λ in ${}^*R_{\mathbf{L}}^{\mathbf{H}}\chi_A$. Next, on K, we write $\xi^{-1}\lambda_K$ in place of λ_K. Write dual eigenvalues $1, -1$ on R and write dual multiplicities m_i^+, m_i^- on each factor $\mathbf{H}_i^*$. Moreover, write colour vector $\mathbf{c}_i \in \{1, -1\}^{\mathcal{A}_i}$ on its torus coordinates $\mathcal{A}_i$. With fixed m_i^+, m_i^-, write the single orbit $\mathfrak{S}_{\mathcal{A}_i} \cdot \mathbf{c}_i$. On R, $c_i = \mathrm{rk}(\kappa_i)$ in $\mathbf{H}_0$, with $c_1 + c_2 = c < e$. Thus, $c_i \equiv \mathrm{rk}(\rho_i) \pmod e$ for $i \in \{1, 2\}$. Moreover, write $s_\lambda \in (\mathbf{L}^*)^{F^*}$, with $\lambda \in \mathcal{E}(L, s_\lambda)$. Write the unique normalizer orbit $\mathcal{O}_s = \mathrm{N}_{\mathbf{H}^*}(\mathbf{L}^*)^{F^*} \cdot s_\lambda$ and put dual centralizer $\mathbf{C}^* = \mathrm{C}_{\mathbf{H}^*}(s_\lambda)$ and $\mathbf{L}_s^* = \mathrm{C}_{\mathbf{L}^*}(s_\lambda)$. Next, take $\mathbf{J}$ in an F-orbit of length m. The orbit product has fixed points $\mathbf{J}^{F^m}$ via projection. On a classical factor $\mathbf{J}$, $[\mathbf{J}, \mathbf{J}]$ is simple and $\mathrm{Z}(\mathbf{J})$ is connected. Thus, $J_{\mathbf{H}, s_\lambda}, J_{\mathbf{L}, s_\lambda}$ are products of the factor maps $J_{\mathbf{J}, s_\lambda}$, with $\theta_s \mapsto 1$ on each torus. By [154, Theorem 3.2(ii)], $J_{\mathbf{H}, s_\lambda} R_{\mathbf{L}}^{\mathbf{H}} = R_{\mathbf{L}_s^*}^{\mathbf{C}^*} J_{\mathbf{L}, s_\lambda}$. Thus, $J_{\mathbf{L}, s_\lambda} {}^*R_{\mathbf{L}}^{\mathbf{H}} = {}^*R_{\mathbf{L}_s^*}^{\mathbf{C}^*} J_{\mathbf{H}, s_\lambda}$. On $\mathbf{C}^*$, we write residual d-cuspidal labels κ. By [37, Theorem 6.7(ii)], we write the unipotent restriction ${}^*R_{\mathrm{HC}}$, with the orbit sum on $\mathbf{C}^*$.

If $A \le Q$, put $\mathbf{V} = \mathrm{C}_{\mathbf{G}}(A)$ and $V = \mathbf{V}^F$. Let χ occur in $R_{\mathbf{L}}^{\mathbf{V}}\lambda$. By the restriction formula, λ meets condition (b) of [37, Theorem 7.4], and $\mathrm{C}_V(Q) = L$ gives (a). Its proof needs $Q \le \ker \lambda$, which holds for our λ. Since $Q = \mathrm{Z}(L)_\ell$, the same holds on $\mathrm{N}_V(L) \cdot \lambda$. For $z \in Q$, $\mathrm{C}_{\mathbf{V}}(z)$ is connected by the decomposition of V. Thus, [37, proof of Theorem 7.4] gives $(1, \mathrm{bl}_V(\chi)) \le (Q, \mathrm{bl}_L(\lambda))$.

Moreover, for $X \in \{R, K\}$, we put $N_X = \mathrm{N}_X(Q_X)$. $Q_X \lhd N_X$ and $\mathrm{C}_{N_X}(Q_X) = L_X$. Let $k = \mathbb{F}$ in characteristic ℓ, and put $b_X = \mathrm{bl}_{L_X}(\lambda_X)$. By [14, Proposition 6.2.2], all block idempotents of kN_X lie in $Z(kL_X)^{N_X}$. Conversely, $Z(kL_X)^{N_X}$ is central in kN_X. Thus, the orbit sum $\sum_{b \in b_X^{N_X}} b$ defines the unique block B_X above b_X. Moreover, we choose extension $\widehat{\lambda}_X$ to $(N_X)_{\lambda_X}$. Choose $\eta \in \mathrm{Irr}((N_X)_{\lambda_X}/L_X)$ and have that $\theta_\eta = \mathrm{Ind}_{(N_X)_{\lambda_X}}^{N_X}(\widehat{\lambda}_X\eta) \in \mathrm{Irr}(N_X, B_X)$. By [37, Theorem 5.2(ii)], extend (Q_X, b_X) to a maximal pair (P_X, c_X) for $\mathrm{bl}_X(\chi_X)$. Since $\ell \nmid \chi_X(1)$, $P_X \in \mathrm{Syl}_\ell(X)$. In block coordinates, $P_X \simeq Q_X \rtimes U$ with permutation action by U. Thus, Lemma 4.3 makes Q_X the unique largest abelian subgroup of P_X, also trivially at $P_X = 1$. In particular, $\mathrm{N}_X(P_X) \le N_X$. For $Q_X \le A \le P_X$, we have $\mathrm{C}_X(A) = \mathrm{C}_{N_X}(A)$, so the inclusion $(Q_X, b_X) \le (P_X, c_X)$ holds in N_X. Hence, the block of N_X under (P_X, c_X) covers b_X. By uniqueness, it is B_X, and [37, Theorem 1.17(i)] gives $B_X = \mathfrak{b}_X(\mathrm{bl}_X(\chi_X))$. At $\mathrm{N}_{D_0}(Q_{D_0}) = N_K \times N_R$, we write products $B_K \boxtimes B_R$. We obtain $\mathrm{bl}(\Omega_{D_0}(\chi)) = \mathfrak{b}_{D_0}(\mathrm{bl}(\chi))$. Moreover, $Q_{D_0} \le L_0$, since $\ell \ne 2$. We put quadratic twist $\tau_0(g) = (\varepsilon_K \boxtimes \varepsilon_R)(g)g$ and have that $\mathrm{Br}_{Q_{D_0}} \tau_0 = \overline{\tau}_0 \mathrm{Br}_{Q_{D_0}}$. By Lemma 4.27, on L_0, we write the blocks under $\mathrm{Res}_{L_0}^{D_0}$. Consequently, for $\theta \in \mathcal{P}_{L_0}$, $\mathrm{bl}(\Omega_{L_0}(\theta)) = \mathfrak{b}_{L_0}(\mathrm{bl}(\theta))$. □

Moreover, we consider $G = \Omega_{2n+1}(q)$ and $P \in \mathrm{Syl}_\ell(G)$. By Proposition 4.4, we write toral $Q \le P$, characteristic in P. We write companion $H \le G$, with $P \le H$ and put $M = \mathrm{N}_G(Q)$ and $U = \mathrm{N}_H(Q)$. Thus, $P \in \mathrm{Syl}_\ell(M) \cap \mathrm{Syl}_\ell(U)$. With rational Levi normalizer H, we consider semisimple ℓ'-element s. Moreover, we consider companion H for quasi-isolated reduction at s. We choose splitting system $(\mathcal{K}, \mathcal{O}, k)$, with $\mathcal{K} = \mathrm{Frac}(\mathcal{O})$ and $\mathrm{char}(k) = \ell$. Moreover, we write series idempotents $e_s \in Z(\mathcal{O}G)$ and $e_s^H \in Z(\mathcal{O}H)$. We write $\bar{e}_s \in Z(kG)$ and $\bar{e}_s^H \in Z(kH)$ modulo $J(\mathcal{O})$. Moreover, we write block sums on $\mathcal{E}_\ell(G, s), \mathcal{E}_\ell(H, s)$. On kG, we write the conjugation action $u : x \mapsto uxu^{-1}$ of Q. Thus, we write the Brauer map $\mathrm{Br}_Q : (kG)^Q \to k\mathrm{C}_G(Q)$. We have that $\mathrm{Br}_Q(\sum_{g \in G} a_g g) = \sum_{g \in \mathrm{C}_G(Q)} a_g g$. We put $\overline{e}_M = \mathrm{Br}_Q(\bar{e}_s) \in k\mathrm{C}_G(Q)$. Since $\overline{e}_M$ is M-invariant, $\overline{e}_M \in Z(kM)$. We choose its unique central lift $e_M \in Z(\mathcal{O}M)$ and put $\overline{e}_U = \mathrm{Br}_Q(\bar{e}_s^H) \in Z(kU)$, with central lift $e_U \in Z(\mathcal{O}U)$. Moreover, we define $Y_{s,Q}(M) = \{\psi \in \mathrm{Irr}_{\ell'}(M, e_M) \mid Q \le \ker \psi\}$. We put $Y_{s,Q}(U) = \{\varphi \in \mathrm{Irr}_{\ell'}(U, e_U) \mid Q \le \ker \varphi\}$.

Moreover, we write $\Delta Q = \{(u, u) : u \in Q\}$ and $\Delta R = \{(u, u) : u \in R\}$ for $R \le Q$. We put bimodule action $(g, h)v = gvh^{-1}$ for $(g, h) \in G \times H$ and put $S_Q = \mathrm{N}_{G \times H}(\Delta Q) \le M \times U$. By Ruhstorfer [160, Proposition 1.12] and Ruhstorfer [161, §1.6], we write global complex $\mathcal{C}$. Then, we write $\overline{\mathcal{C}}^j = k \otimes_{\mathcal{O}} \mathcal{C}^j$ in degree j and put $\mathcal{C}_Q^j = (\overline{\mathcal{C}}^j)^{\Delta Q} / \sum_{R < Q} \mathrm{Tr}_{\Delta R}^{\Delta Q}((\overline{\mathcal{C}}^j)^{\Delta R})$. Moreover, $\mathrm{Tr}_{\Delta R}^{\Delta Q}(m) = \sum_{u \in [\Delta Q / \Delta R]} um$ for $m \in (\overline{\mathcal{C}}^j)^{\Delta R}$. We choose coset

representatives $[\Delta Q/\Delta R]$ in ΔQ. Thus, we put $d_Q^j([m]) = [\overline{d}^j(m)]$ on $\mathcal{C}_Q$.

Lemma 4.31. *We write corresponding full-defect blocks B_M, B_U of M, U and write Morita correspondence $\mathfrak{M}_Q : \mathrm{Irr}(U, B_U) \to \mathrm{Irr}(M, B_M)$ from $\mathcal{C}_Q$. Thus, $Q \le \ker\varphi \iff Q \le \ker \mathfrak{M}_Q(\varphi)$ for $\varphi \in \mathrm{Irr}(U, B_U)$. With corresponding idempotents e_M, e_U, we have that $\mathfrak{M}_Q(Y_{s,Q}(U) \cap \mathrm{Irr}(U, B_U)) = Y_{s,Q}(M) \cap \mathrm{Irr}(M, B_M)$.*

Proof. First, we choose permutation lift P_Q of $\mathcal{C}_Q$ [160, Proposition 1.12] and [161, §1.6]. $k \otimes_{\mathcal{O}} P_Q \simeq \mathcal{C}_Q$, and we write centralizer bimodule $V_Q = H^{d_Q}(P_Q)$ in its cohomology degree d_Q. Since ΔQ acts trivially on $\mathcal{C}_Q$, we write P_Q from the quotient by ΔQ. With $u \in Q$ and $v \in V_Q$, $uvu^{-1} = v$. Thus, $uv = vu$. Since Q is toral, Q is abelian, with $Q \le \mathrm{C}_M(Q) \cap \mathrm{C}_U(Q)$. Over $\mathcal{K}$, we put $e_Q = |Q|^{-1}\sum_{u\in Q} u$. Moreover, $e_Q \in Z(\mathcal{K}M) \cap Z(\mathcal{K}U)$, since $Q \lhd M, U$. We put normalizer bimodule $W_Q = B_M \mathrm{Ind}_{S_Q}^{M\times U}(V_Q) B_U$ and put $T_Q = \mathcal{K}M \otimes_{\mathcal{K}\mathrm{C}_M(Q)} \mathcal{K}V_Q \otimes_{\mathcal{K}\mathrm{C}_U(Q)} \mathcal{K}U$. $T_Q \twoheadrightarrow \mathrm{Ind}_{S_Q}^{M\times U}(\mathcal{K}V_Q)$, with $a\otimes v\otimes b \mapsto (a, b^{-1})\otimes v$ for $a \in M, b \in U$. Thus, $e_Q(a\otimes v\otimes b) = a\otimes e_Q v\otimes b = a\otimes ve_Q\otimes b = (a\otimes v\otimes b)e_Q$. Since $e_Q B_M = B_M e_Q$ and $e_Q B_U = B_U e_Q$, we consider $\mathcal{K}W_Q$. With $w \in \mathcal{K}W_Q$, $e_Q w = we_Q$. Moreover, we choose a simple $\mathcal{K}UB_U$-module V_φ, with character φ. $e_Q V_\varphi = V_\varphi^Q$ and $e_Q V_\varphi = V_\varphi \iff Q \le \ker\varphi$. Thus, $e_Q(w\otimes v) = w\otimes e_Q v$ on $\mathcal{K}W_Q \otimes_{\mathcal{K}UB_U} V_\varphi$. On the inverse bimodule $W_Q^\vee$, $e_Q w = we_Q$ for $w \in \mathcal{K}W_Q^\vee$. Thus, $Q \le \ker\varphi \iff Q \le \ker\mathfrak{M}_Q(\varphi)$. By [107, §1], choose a perfect isometry $\mathfrak{M}_Q$. In particular, $h_M(\mathfrak{M}_Q(\varphi)) = h_U(\varphi)$. P is a defect group of B_M, B_U. Thus, $\varphi(1)_\ell = \ell^{h_U(\varphi)}$ and $\mathfrak{M}_Q(\varphi)(1)_\ell = \ell^{h_M(\mathfrak{M}_Q(\varphi))}$. $\ell \nmid \varphi(1) \iff \ell \nmid \mathfrak{M}_Q(\varphi)(1)$. Since $e_M W_Q = W_Q e_U$, $\mathfrak{M}_Q(Y_{s,Q}(U) \cap \mathrm{Irr}(U, B_U)) = Y_{s,Q}(M) \cap \mathrm{Irr}(M, B_M)$. □

We write $I : \mathrm{Irr}(H, e_s^H) \to \mathrm{Irr}(G, e_s)$ from $\mathcal{C}$. Moreover, we write $I_Q : \mathrm{Irr}(U, e_U) \to \mathrm{Irr}(M, e_M)$ from the permutation lift of $\mathcal{C}_Q$. On $\mathcal{O}$, we consider coefficient operation σ, with $\sigma|_{\mathbb{Z}_\ell} = 1$. Moreover, we write $\widetilde{\mathbf{G}} = \mathrm{Spin}_{2n+1}$, with $\widetilde{G} = \widetilde{\mathbf{G}}^F$. We write quotient $\pi : \widetilde{G} \to G$, with $\ker\pi = \langle z\rangle$ and central character $+1$. Thus, we choose projective dual element s, with lift $\widetilde{s} \in \mathrm{Sp}_{2n}(q)$. We write $\mathbf{G}^* = \widetilde{\mathbf{G}}^*$ and minimal dual Levi $\mathbf{L}_s^*$ containing $\mathrm{C}_{\mathbf{G}^*}(s)^\circ$ and write dual Levi $\widetilde{\mathbf{L}}_s$ and put $L = \pi(\widetilde{\mathbf{L}}_s^F)$. In particular, by Zhang [208, Lemma 6.12(ii)], $A_s = H/L$ has order 2. With $\widetilde{s}$, we consider V_1, V_{-1} of dimension $2a$, and unitary space $V_{\{\pm i\}}$. On $\mathbf{L}_s^*$, we write the symplectic space $V_1 \perp V_{-1}$. Moreover, $s^m \in \{s, s^{-1}\}$ for odd m. On $V_{\{\pm i\}}$, $(-\widetilde{s}^{-1})|_{V_{\{\pm i\}}} = \widetilde{s}|_{V_{\{\pm i\}}}$. Since $\dim V_1 = \dim V_{-1} = 2a$, we choose $g \in \mathrm{Sp}(V_1 \perp V_{-1})(q)$, with $g(V_1) = V_{-1}$ and $g(V_{-1}) = V_1$. We write the projective image $\overline{g} \in (\mathbf{L}_s^*)^{F^*}$ of $g \oplus I_{V_{\{\pm i\}}}$. Thus, $g\widetilde{s}g^{-1} = -\widetilde{s}^{-1}$ and $\overline{g}s\overline{g}^{-1} = s^{-1}$. We put series projector $\pi_s = e_s^L$ on $\mathcal{O}L$ and have that $\pi_s^\sigma = \pi_s$. Moreover, we put central projector $e_+ = (1+z)/2$ on $\widetilde{G}$. On $G \times L$, we put $\mathcal{P}_Y = e_+(\mathcal{O}\otimes_{\mathbb{Z}_\ell} \mathcal{P}_{Y,0})e_+$. By Ruhstorfer [161, §1.3], we choose bounded permutation model $\mathcal{P}_{Y,0} = \mathrm{G}\Gamma_c(Y, \mathbb{Z}_\ell)$, with $Y = Y_{\widetilde{\mathbf{U}}}^{\widetilde{\mathbf{G}}}$. Since $F(\widetilde{\mathbf{L}}_s) = \widetilde{\mathbf{L}}_s$, we choose parabolic $\widetilde{\mathbf{P}}_s = \widetilde{\mathbf{L}}_s \ltimes \widetilde{\mathbf{U}}$. Moreover, we put $\mathcal{E}_s = \mathcal{P}_Y \otimes_{\mathcal{O}L} \mathcal{O}L\pi_s$. Over $\mathcal{O}$, $\mathcal{E}_s^\sigma \simeq \mathcal{E}_s$. By Ruhstorfer [161, Theorem 1.1 and Lemma 1.3], we choose extension $\mathcal{C}$ over H. We have that $|H^1(F, Z(\widetilde{\mathbf{G}}))| \mid 2$, so $\ell \nmid |H^1(F, Z(\widetilde{\mathbf{G}}))|$.

We put $D = G \times H$ and $D_1 = G \times L$ and write bounded permutation homotopy category $\mathcal{H}_{\mathrm{perm}}$ over $\mathcal{O}$. Moreover, we consider ambient block b', with $A_{b'} = \mathcal{O}Gb'$ and corresponding algebra $B_{b'} = \mathcal{O}Hc'$. In block b', we put $\mathcal{C}_{b'} = b'\mathcal{C}c'$, with inverse $\mathcal{C}_{b'}^{-1}$. Thus, $\mathcal{C}_{b'} \otimes_{B_{b'}} \mathcal{C}_{b'}^{-1} \simeq A_{b'}$. We have that $\mathrm{End}_{\mathcal{H}_{\mathrm{perm}}}(\mathcal{C}_{b'}) \simeq Z(A_{b'})$. Since $Z(A_{b'})$ is local, $\mathcal{C}_{b'}$ is indecomposable. We have that $[D : D_1] = 2$ and $2 \in \mathcal{O}^\times$. On H/L, we write $\varepsilon \in \mathrm{Irr}(H/L)$, with $\varepsilon \ne 1$. Thus, $\mathcal{O}[H/L] \simeq \mathcal{O} \oplus \mathcal{O}_\varepsilon$. We put $\mathcal{C}_1 = (\mathcal{C}_{b'})^\sigma$ and $\mathcal{C}_2 = \mathcal{C}_{(b')^\sigma}$ and have that $\mathrm{Res}_{D_1}^D \mathcal{C}_1 \simeq \mathrm{Res}_{D_1}^D \mathcal{C}_2$. Moreover, we write $\mathcal{C}_i(\nu) = \mathcal{C}_i \otimes (1_G \boxtimes \nu)$ for $\nu \in \{1_H, \varepsilon\}$. We have that $\mathrm{Ind}_{D_1}^D \mathrm{Res}_{D_1}^D \mathcal{C}_i \simeq \mathcal{C}_i \oplus \mathcal{C}_i(\varepsilon)$. By Krull–Schmidt in $\mathcal{H}_{\mathrm{perm}}$, $\mathcal{C}_1 \simeq \mathcal{C}_2(\nu)$ for one $\nu \in \{1_H, \varepsilon\}$. On the entire block b', we write $\nu = \nu_{\sigma,b'}$. Since $\ell \ne 2$, $\nu|_Q = 1$. We consider $m \in ((\overline{\mathcal{C}}_i)^j)^{\Delta Q}$. Thus, $(m\otimes 1)^\sigma = m^\sigma \otimes 1$ with twist ν. We have that $\sigma\,\mathrm{Tr}_{\Delta R}^{\Delta Q}(m) = \mathrm{Tr}_{\Delta R}^{\Delta Q}(\sigma(m))$. We obtain that $\mathrm{Br}_{\Delta Q}(\mathcal{C}_i(\nu)) \simeq \mathrm{Br}_{\Delta Q}(\mathcal{C}_i) \otimes (1_M \boxtimes \nu|_U)|_{S_Q}$. Thus, we write local twist $\nu_Q = \nu|_U$ for I_Q from $\mathcal{C}$.

With $\mathcal{X} \subseteq \mathrm{Irr}_{\ell'}(H, e_s^H)$, we write middle bijection $J : \mathcal{X} \to \mathcal{Y}$. Moreover, by Lemma 4.30, we write $\mathfrak{b}_{D_0}$ for J on D_0. We write extension pairs $\mathcal{E}_{\mathrm{glob}}, \mathcal{E}_{\mathrm{loc}}$ above μ. If $\rho_1 = \rho_2$, $\epsilon_{\mathrm{glob}} = \epsilon_{\mathrm{loc}}$ by Lemma 4.29 on $\Gamma_\mu = (\mathcal{H}_\ell)_\mu$. If B is unique, we choose orientation $o : \mathcal{E}_{\mathrm{glob}} \to \mathcal{E}_{\mathrm{loc}}$. Thus, $o(\alpha^\sigma) = o(\alpha)^\sigma$ for $\alpha \in \mathcal{E}_{\mathrm{glob}}$. With two blocks $B, \varepsilon B$, we choose unique $\widehat{\mu}_B \in \mathrm{Irr}(H, B)$ over μ. By Lemma 4.27, we choose the unique $\widehat{\mu}'_{B_1} \in \mathrm{Irr}(U, B_1)$ over μ_{loc}. Moreover, we write reconstruction $\mathcal{R}((\kappa_i, \eta_i)_{i=1,2}) = (\rho_1, \rho_2)$, with $\rho_1 \ne \rho_2$. We have that $\tau((\kappa_1, \eta_1), (\kappa_2, \eta_2)) = ((\kappa_2, \eta_2), (\kappa_1, \eta_1))$. We put $\tau(\rho_1, \rho_2) = (\rho_2, \rho_1)$. Thus, we find $\mathcal{R}\tau = \tau\mathcal{R}$. We obtain that $J(\Theta_{\mathrm{glob}}) = \Theta_{\mathrm{loc}}$. Thus, $J(\vartheta^\sigma) = J(\vartheta)^\sigma$ and $J(\vartheta\varepsilon) = J(\vartheta)\varepsilon|_U$.

We put $C = \mathrm{C}_H(Q)$ and $\overline{C} = C/Q$ and write residual ranks $r_i < e_i$, with torus periods e_i. Moreover, we write the component index $c_{\mathrm{comp}} \in \{1, 2\}$ in $\overline{C}$. Since $\ell \ne 2$, $\ell \nmid |\overline{C}|$. We have that $Q \in \mathrm{Syl}_\ell(C)$ and $Q \le Z(C)$. Thus, we choose the unique $\psi_{b_C} \in \mathrm{Irr}(C, b_C)$ with $Q \le \ker\psi_{b_C}$ in each block b_C. With $\overline{\psi}_{b_C} \in \mathrm{Irr}(\overline{C})$, $\psi_{b_C} = \mathrm{Inf}_{\overline{C}}^C \overline{\psi}_{b_C}$. Since the defect group is 1, we choose distinct $\overline{b}_C \in \mathcal{B}_0(\overline{C})$. By Lemma 4.30, we write characters λ_R, λ_K on L_R, L_K. We

have that $[D_0 : L_0] = [H : L] = 2$. Moreover, we write normalizer orbits $\mathcal{O}_N = \{\psi_{b_C}^U : b_C\bar{e}_U = b_C\}$. In the active case, we write $a = eu + c_0$ and $b = ev + j$, with $2c_0 + j < e$. Thus, we write active multiplicities (u, u, v) for $\widetilde{s}$. On T, we write colours $(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_4) = (1_T, \varepsilon_T, \xi_T)$, with $\xi_T^2 = \varepsilon_T$ and $|T| = q^e + 1$. Moreover, we put $\mathbf{c} = (c_0, c_0, j)$. With inactive K, we write residual unitary space V_K. Moreover, we write relative labels $\eta \in \mathcal{D}_J$ over $\mathcal{O}_N$. We put multiplicity $e_{\theta_\eta} = \langle \mathrm{Res}_C^U \theta_\eta, \psi_{b_C}\rangle_C$. Then, by Clifford [43] we have that $\mathrm{Res}_C^U \theta_\eta = e_{\theta_\eta} \sum_{\psi' \in \psi_{b_C}^U} \psi'$. We have that $\ell \nmid [U : U_{\psi_{b_C}}] e_{\theta_\eta} \psi_{b_C}(1)$. Thus, we find $J(\mathcal{X}) = \mathcal{Y} = Y_{s,Q}(U)$. By Ruhstorfer [160, §2.8], we write the series idempotents e_M, e_U. Moreover, by Lemma 4.31, $I_Q(Y_{s,Q}(U)) = Y_{s,Q}(M)$. We put $F = I_Q J I^{-1}$ on $I(\mathcal{X})$ and consider $\vartheta \in \mathcal{X}$, with $b' = \mathrm{bl}_G(I(\vartheta))$ and $\nu = \nu_{\sigma,b'}$. Moreover, $I(\vartheta)^\sigma = I(\vartheta^\sigma \nu)$. We obtain that $F(I(\vartheta)^\sigma) = I_Q J(\vartheta^\sigma \nu)$. We have that $I_Q J(\vartheta^\sigma \nu) = I_Q(J(\vartheta)^\sigma \nu|_U)$. Since $\nu_Q = \nu|_U$, $I_Q(J(\vartheta)^\sigma \nu|_U) = I_Q(J(\vartheta))^\sigma$. Thus, we find $F(I(\vartheta)^\sigma) = F(I(\vartheta))^\sigma$. By Lemma 4.31, $F(I(\mathcal{X})) = Y_{s,Q}(M)$. With $\psi \in Y_{s,Q}(M)$, $Q \le \ker\psi$ and $\psi \in \mathrm{Irr}(M, e_M)$. Moreover, with an allowed orientation o of $\mathcal{E}_{\mathrm{glob}}, \mathcal{E}_{\mathrm{loc}}$, we write J_o for the middle map J. We have that $I_Q J_o I^{-1}(I(\mathcal{X})) = Y_{s,Q}(M)$.

Lemma 4.32. *Moreover, we consider $H \lhd Y$, with cyclic Y/H of odd order m. We consider $\vartheta \in \mathrm{Irr}(H)$, with $\vartheta^y = \vartheta$ for $y \in Y$ and consider $\delta \in \mathrm{Aut}(Y)$, with $\delta(H) = H$ and $\delta(y)H = yH$ for $y \in Y$. Moreover, we assume that $\delta^2 = \mathrm{Int}(y_0)$ for $y_0 \in Y$. We consider $t \in \{0, 1\}$, with $\overline{\vartheta} = \vartheta^{\delta^t}$ and choose the unique extension $\widehat{\vartheta} \in \mathrm{Irr}(Y)$, with $\overline{\widehat{\vartheta}}^{\delta^t} = \widehat{\vartheta}$. Thus, we find $\mathbb{Q}(\widehat{\vartheta}) = \mathbb{Q}(\vartheta)$.*

Moreover, we put $B = DA$, with $D, A \lhd B$. We put $X = D \cap A$, with $|D/X| = 2$ and cyclic A/X of odd order and choose $\Theta \in \mathrm{Irr}(D)$ and extension $\widehat{\Theta} \in \mathrm{Irr}(B)$. Moreover, we write $\Gamma \le \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, with $\widehat{\Theta}^\sigma = \widehat{\Theta}$ for $\sigma \in \Gamma$. Thus, with $\chi \in \mathrm{Irr}(X)$ and $\langle \mathrm{Res}_X^D \Theta, \chi\rangle_X \neq 0$, we choose an extension $\widehat{\chi} \in \mathrm{Irr}(I_B(\chi))$. We have that $\widehat{\chi}^{b\sigma} = \widehat{\chi}$ whenever $(b, \sigma) \in (B \times \Gamma)_\chi$.

Proof. First, write representation $\rho : H \to \mathrm{GL}(V)$ with character ϑ. We choose generator yH of Y/H, with $|Y/H| = m$ and choose $T_0 \in \mathrm{GL}(V)$, with $T_0\rho(h)T_0^{-1} = \rho(yhy^{-1})$ for $h \in H$. Thus, $T_0^m \rho(y^m)^{-1} = \kappa I_V$ for $\kappa \in \mathbb{C}^\times$. Choose $\beta \in \mathbb{C}^\times$, with $\beta^m = \kappa^{-1}$ and put $T = \beta T_0$, with $T^m = \rho(y^m)$. Moreover, we choose an extension $\widehat{\rho} : Y \to \mathrm{GL}(V)$, with $\widehat{\rho}(y) = T$ and $\widehat{\rho}|_H = \rho$. Write character χ of $\widehat{\rho}$, and put $E = \mathrm{Irr}(Y/H)$, and we put $\mathrm{Ext}_Y(\vartheta) = \{\chi\lambda : \lambda \in E\}$. In particular, by [43], $\chi\lambda = \chi\lambda' \iff \lambda = \lambda'$ for $\lambda, \lambda' \in E$. Write $\chi^a = \overline{\chi}^{\delta^t}$. Then, $\vartheta^a = \vartheta^{\delta^{2t}} = \vartheta$. Since δ^2 is inner on Y, $\chi^{a^2} = \chi$. Since δ is trivial on Y/H, $\lambda^a = \lambda^{-1}$ for $\lambda \in E$. With $\chi^a = \chi\alpha$, we consider $\alpha \in E$. Thus, $(\chi\lambda)^a = \chi\lambda \iff \lambda^2 = \alpha$. Since $|E| = m$ is odd, we write the bijection $E \to E$, with $\lambda \mapsto \lambda^2$. By $\lambda \mapsto \lambda^2$, choose the unique λ with $\lambda^2 = \alpha$, and put $\widehat{\vartheta} = \chi\lambda$. Moreover, we consider $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\vartheta))$. $(\widehat{\vartheta}^\sigma)^a = (\widehat{\vartheta}^a)^\sigma = \widehat{\vartheta}^\sigma$ and $\mathrm{Res}_H^Y \widehat{\vartheta}^\sigma = \vartheta$. Thus, $\widehat{\vartheta}^\sigma = \widehat{\vartheta}$ by the unique a-fixed choice. Thus, $\mathbb{Q}(\widehat{\vartheta}) \subseteq \mathbb{Q}(\vartheta)$. $\mathbb{Q}(\vartheta) \subseteq \mathbb{Q}(\widehat{\vartheta})$ from $\mathrm{Res}_H^Y \widehat{\vartheta} = \vartheta$.

Moreover, put $B = DA$, with extension $\widehat{\Theta}$. With $\mathrm{Res}_X^D \Theta = \chi \in \mathrm{Irr}(X)$, $I_B(\chi) = B$. Thus, choose $\widehat{\chi} = \widehat{\Theta}$, with $\widehat{\chi}^{b\sigma} = \widehat{\chi}$ for $b \in B, \sigma \in \Gamma$. Alternatively, $\mathrm{Res}_X^D \Theta = \chi_+ + \chi_-$, with $\chi_+ \neq \chi_-$. With $d \in D \setminus X$, $\chi_+^d = \chi_-$. We write the action $A/X \to \mathrm{Sym}(\{\chi_+, \chi_-\})$. Since $|A/X|$ is odd, $\chi_\pm^a = \chi_\pm$ for $a \in A$. $I_B(\chi_+) = I_B(\chi_-) = A$. With character $\widehat{\Theta}$, we consider W and X-isotypic spaces W_+, W_-. Thus, $W = W_+ \oplus W_-$, with $\chi_\pm$ on $W_\pm|_X$. On A, choose irreducible modules W_+, W_-, with characters $\widehat{\chi}_+, \widehat{\chi}_-$. $\mathrm{Res}_X^A \widehat{\chi}_\pm = \chi_\pm$. Moreover, $\mathrm{Res}_A^B \widehat{\Theta} = \widehat{\chi}_+ + \widehat{\chi}_-$. $\mathrm{Res}_X^A : \{\widehat{\chi}_+, \widehat{\chi}_-\} \to \{\chi_+, \chi_-\}$ is bijective. Since $\widehat{\Theta}^\sigma = \widehat{\Theta}$, $\{\widehat{\chi}_+, \widehat{\chi}_-\}^{b\sigma} = \{\widehat{\chi}_+, \widehat{\chi}_-\}$. $\mathrm{Res}_X^A(\widehat{\chi}_\pm^{b\sigma}) = \chi_\pm^{b\sigma}$. Thus, $\chi_\pm^{b\sigma} = \chi_\pm \Longrightarrow \widehat{\chi}_\pm^{b\sigma} = \widehat{\chi}_\pm$. With $\chi_+^b = \chi_- = \chi_+^\sigma$, we consider mixed stabilizer operations (b, σ). Then, $\widehat{\chi}_+^{b\sigma} = \widehat{\chi}_+$ and $\widehat{\chi}_-^{b\sigma} = \widehat{\chi}_-$. □

Moreover, we consider $q = r^f \equiv 3 \pmod 4$, with $r \equiv 3 \pmod 4$ and odd f. Put $D = \mathrm{SO}_{2n+1}(q)$ and $Y = D \rtimes \langle F_r\rangle$. Suppose that $\theta_D \in \mathrm{Irr}(D)$ is rational and that $\theta_D^{F_r} = \theta_D$. By Lemma 4.32, choose $\widehat{\theta}_D \in \mathrm{Irr}(Y)$, with $\mathbb{Q}(\widehat{\theta}_D) = \mathbb{Q}$, as in Zhang [208, Lemma 3.1]. Take $t = 0$ and $H = D$. Moreover, on G, choose the field representative $\Phi = gF_r$ with $g \in G$, $\Phi Q \Phi^{-1} = Q$ and $\Phi T \Phi^{-1} = T$. Thus, $\Phi^f = c = gF_r(g) \cdots F_r^{f-1}(g) \in \mathrm{N}_G(Q)$. Let $D_Q = \mathrm{N}_D(Q)$, with active representation $\rho_{\mathrm{act}} : D_Q \to \mathrm{GL}(V_{\mathrm{act}})$. Moreover, we consider active unitary K, with $d = \mathrm{ord}_\ell(q) = 2e$ and odd e. In D_Q, take quartic rotations w_4, with $\xi^{w_4} = \xi^{-1}$. $w_4 I_\xi w_4^{-1} = I_{\xi^{-1}} = I_\xi$. On V_{act}, choose a permutation operator $U_4 \in \mathrm{GL}(V_{\mathrm{act}})$, with $U_4^2 = I$. Moreover, write $\rho_{\mathrm{act}}(c) = \gamma U_4$, with central rotation $z \in I_\xi$. Then, $\rho_{\mathrm{act}}(z) = \gamma I$. Indeed, the exponent is $e + 1$ on quartic blocks W. With the inverse convention it is $e - 1$. Moreover, it is 1 on the other two colours $\mathfrak{c}_1, \mathfrak{c}_2$. Thus, $\gamma \in \mu_d$, where $\mu_m = \{z \in \mathbb{C}^\times : z^m = 1\}$. Put $D_r = \mathrm{ord}_\ell(r)$. Since $q = r^f$, $d = D_r/\gcd(D_r, f)$. By $d = D_r/\gcd(D_r, f)$, $\mu_{D_r} \to \mu_d$, with $\alpha \mapsto \alpha^f$, is onto. Choose $\alpha \in \mu_{D_r}$, with $\alpha^f = \gamma$. Since $D_r \mid \ell - 1$, $\alpha^\sigma = \alpha$ for $\sigma \in \mathcal{H}_\ell$. On V_{act}, put $T_{\mathrm{act}} = \alpha U_4$, with $T_{\mathrm{act}}^f = \gamma U_4$. When $\xi^\sigma = \xi$, put the coefficient matrix $C_\sigma = I$. Alternatively, when $\xi^\sigma = \xi^{-1}$, put $C_\sigma = U_4$. Thus, $C_\sigma T_{\mathrm{act}}^\sigma C_\sigma^{-1} = T_{\mathrm{act}}$. Moreover, $\rho_{\mathrm{act}}(c^{-1}) = \gamma^{-1} U_4$. With no quartic coordinates W, put $U_4 = I$. Moreover, we choose residual extension $\widehat{\theta}_{\mathrm{res}}$ on the remaining odd orthogonal parent D_{res}. On D_{res}, we

write residual image $\Phi_{\rm res}\in D_{\rm res}\rtimes\langle\varphi_{\rm res}\rangle$, with $|\varphi_{\rm res}|\mid f$ and $\Phi_{\rm res}^f=c_{\rm res}$. Put $T_{\rm res}=\widehat{\rho}_{\rm res}(\Phi_{\rm res})$. We write local parent representation $\rho_{D_Q}:D_Q\to{\rm GL}(V_{\rm act}\otimes V_{\rm res})$. Put $\mathcal{T}=T_{\rm act}\otimes T_{\rm res}$, with $\mathcal{T}^f=\rho_{D_Q}(c)$. By Lemma 4.32, we choose extensions $\widehat{\chi}_+,\widehat{\chi}_-$ above χ_+,χ_-. For $b\in{\rm N}_D(P)\setminus G$ inducing δ, $\chi_\pm^{\delta\sigma}=\chi_\pm\Longrightarrow\widehat{\chi}_\pm^{\delta\sigma}=\widehat{\chi}_\pm$, including simultaneous diagonal/coefficient operations (δ,σ) fixing $\chi_\pm$.

Proposition 4.33. *Moreover, we consider $n=2a+b\geq 3$, $b>0$ and $q\equiv 3\bmod 4$. We write ℓ,s from* Section 4.2. *Moreover, we put $\mathcal{G}^*=(\mathbf{G}^*)^{F^*}$ and $m_s=|\mathcal{G}^*:{\rm C}_{\mathcal{G}^*}(s)|_{r'}$, with $\ell\nmid m_s$. Assume that $P\in{\rm Syl}_\ell(G)$ lies in companion H. By* Section 4.1, *we write the domain $\mathcal{D}_{\rm deg}$. Then, on $\mathcal{P}_s=\mathcal{E}(G,s)\cap{\rm Irr}_{\ell'}(G)$, $F=I_QJI^{-1}:\mathcal{P}_s\to Y_{s,Q}(M)$ is bijective. With $\chi\in\mathcal{P}_s$, we put $E_\chi=({\rm Aut}(G)_P)_{\chi^{\mathcal{H}_\ell}}$. Thus, for $\psi=F(\chi)$, $(G\rtimes E_\chi,G,\chi)_{\mathcal{H}_\ell}\succeq_{\rm c}(M\rtimes E_\chi,M,\psi)_{\mathcal{H}_\ell}$. With $Z=\ker(\pi:\widetilde{G}\to G)$, we put $\mathcal{P}_{\rm triv}=\{\widetilde{\chi}\in{\rm Irr}_{\ell'}(\widetilde{G}):Z\leq\ker\widetilde{\chi}\}$. Moreover, we put $\widetilde{\chi}=\chi\circ\pi$, $\widetilde{M}=\pi^{-1}(M)$ and $\widetilde{\psi}=\psi\circ\pi|_{\widetilde{M}}$. By* Section 4, *on E_χ, we choose the lifted action $E_\chi\to{\rm Aut}(\widetilde{G})$, with $\pi(x^a)=\pi(x)^a$. We have that $(\widetilde{G}\rtimes E_\chi,\widetilde{G},\widetilde{\chi})_{\mathcal{H}_\ell}\succeq_{\rm c}(\widetilde{M}\rtimes E_\chi,\widetilde{M},\widetilde{\psi})_{\mathcal{H}_\ell}$.*

Proof. First, $I(\mathcal{X})=\mathcal{E}(G,s)\cap{\rm Irr}_{\ell'}(G)$ under the Lusztig construction for I. With $u\neq 1$, we consider an ℓ-element $u\in{\rm C}_{\mathcal{G}^*}(s)$. $|s|\mid 4$ and $su=us$. Since $\ell\neq 2$, $(su)_\ell=u\neq 1=s_\ell$, so $su\not\sim s$. By I, we write full-defect correspondence $\mathfrak{M}$. $\ell\nmid\vartheta(1)\Longleftrightarrow\ell\nmid I(\vartheta)(1)$. Moreover, we write the quotient twists $\nu_I=\nu_{I_Q}=\nu$ for I,I_Q. If $\sigma\in\mathcal{H}_\ell$, $F(\chi^\sigma)=F(\chi)^\sigma$. By [161, Lemma 1.3], we write diagonal action $\mathfrak{d}$ on $\mathcal{C}$, with $\mathfrak{d}(Q)=Q$. We find ${\rm Br}_{\Delta Q}(\mathcal{C}^{\mathfrak{d}})\simeq\mathcal{C}_Q^{\mathfrak{d}}$. Thus, $I(\vartheta^{\mathfrak{d}})=I(\vartheta)^{\mathfrak{d}}$ and $I_Q(\varphi^{\mathfrak{d}})=I_Q(\varphi)^{\mathfrak{d}}$. With $N_0=L_0\rtimes\langle\gamma_h\rangle$, we put $\widehat{D}_0=D_0\rtimes\langle\gamma_h\rangle$. Put $\mathcal{N}_-=D_0\setminus L_0$. Then, by the Frattini argument on P, choose $d_-\in\mathcal{N}_-\cap{\rm N}_{D_0}(Q)$. Moreover, we write quotient sign $\varepsilon_0\in{\rm Irr}(N_0/L_0)$, with $\varepsilon_0(\gamma_h)=-1$. If $\rho_1=\rho_2$, we put $\chi_0=(\xi\rho_\lambda)\boxtimes\Sigma$. $\chi_0^{\gamma_h}=\chi_0(\varepsilon_K\boxtimes\varepsilon_R)\neq\chi_0$. By [95, Theorem 6.11], $\Xi={\rm Ind}_{D_0}^{\widehat{D}_0}\chi_0\in{\rm Irr}(\widehat{D}_0)$. If ${\rm Res}_{L_0}^{N_0}\mu_\pm=\mu$ and $\mu_+\neq\mu_-$, we consider $\mu_+,\mu_-\in{\rm Irr}(N_0)$. Since $\widehat{D}_0=D_0N_0$ and $D_0\cap N_0=L_0$, [95, Problem 5.2] gives ${\rm Res}_{N_0}^{\widehat{D}_0}\Xi={\rm Ind}_{L_0}^{N_0}\mu=\mu_++\mu_-$. Thus, $\mu_\pm^{d_-}=\mu_\mp=\mu_\pm\varepsilon_0$. With $\rho_1\neq\rho_2$, $\Theta_{\rm glob}^{d_-}=\Theta_{\rm glob}=\Theta_{\rm glob}\varepsilon_0$. Over $\mathcal{O}_N$, with $\rho_1=\rho_2$, $\mu_{{\rm loc},\pm}^{d_-}=\mu_{{\rm loc},\mp}=\mu_{{\rm loc},\pm}\varepsilon_0|_U$. Alternatively, with $\rho_1\neq\rho_2$, $\Theta_{\rm loc}^{d_-}=\Theta_{\rm loc}=\Theta_{\rm loc}\varepsilon_0|_U$. Moreover, $J(\vartheta\varepsilon_0)=J(\vartheta)\varepsilon_0|_U$. We obtain $F(\chi^{\mathfrak{d}})=F(\chi)^{\mathfrak{d}}$.

Moreover, we put $D={\rm SO}_{2n+1}(q)$ and $B=D\rtimes\langle F_r\rangle$. Above $\chi\in\mathcal{P}_s$, we consider a D-parent θ_D. With $\rho_\lambda^{\gamma_h}=\rho_\lambda$, choose rational unipotent $\rho_1,\rho_2,\rho_\lambda$. Moreover, consider $g_{\rm inv}\in{\rm Sp}_{2n}(q)$, with $g_{\rm inv}\widetilde{s}g_{\rm inv}^{-1}=\widetilde{s}^{-1}$. Thus, $g_{\rm inv}(V_1)=V_1$ and $g_{\rm inv}(V_{-1})=V_{-1}$. $g_{\rm inv}(V_i)=V_{-i}$ and $g_{\rm inv}(V_{-i})=V_i$. On K, we write the graph action $k\mapsto\gamma_h(k)$. By the Jordan labels $(\rho_1,\rho_2,\rho_\lambda)$, $\mathbb{Q}(\theta_D)=\mathbb{Q}$ and $\theta_D^{F_r}=\theta_D$. Alternatively, consider the other lift $s'=-\widetilde{s}$ of s. At s', write the label pair (ρ_2,ρ_1), with $\mathbb{Q}(\theta_D)=\mathbb{Q}$ and $\theta_D^{F_r}=\theta_D$. By Lemma 4.32, we choose $\widehat{\theta}_D\in{\rm Irr}(B)$, with $\mathbb{Q}(\widehat{\theta}_D)=\mathbb{Q}$. Next, we put $D_Q={\rm N}_D(Q)$ and $B_Q={\rm N}_B(Q)$. Write $\Phi=gF_r\in B_Q$, with $g\in G$ and ΦD_Q a generator of B_Q/D_Q. Moreover, we put $T_{\rm act}=\alpha U_4$ on the active parent $\theta_{\rm act}$. On V_O, we write cyclic/symmetric matrices $M_{\rm cyc},M_{\rm sym}$, with entries in $\mathbb{Q}(\mu_{2e})$. On the unitary coefficient space V_U, we write $M_{\rm cyc},M_{\rm sym}$ over $\mathbb{Q}(\mu_e)$. If d is odd, $e=d$ and $2e\mid\ell-1$. Alternatively, with even d, $2e=d\mid\ell-1$. $\mathbb{Q}(\mu_{2e}),\mathbb{Q}(\mu_e)$ are fixed by $\mathcal{H}_\ell$. $M_{\rm cyc}^\sigma=M_{\rm cyc}$ and $M_{\rm sym}^\sigma=M_{\rm sym}$. Moreover, choose residual parent $\theta_{\rm res}$ on $D_{\rm res}={\rm SO}(V_{\rm res})$. On $D_{\rm res}$, we write field automorphism $\varphi_{\rm res}$, with $|\varphi_{\rm res}|\mid f$. By the Jordan labels of $\theta_{\rm res}$, $\mathbb{Q}(\theta_{\rm res})=\mathbb{Q}$ and $\theta_{\rm res}^{\varphi_{\rm res}}=\theta_{\rm res}$. By Lemma 4.32, choose $\widehat{\theta}_{\rm res}$, with $\mathbb{Q}(\widehat{\theta}_{\rm res})=\mathbb{Q}$. Then, put $T_{\rm res}=\widehat{\rho}_{\rm res}(\Phi_{\rm res})$. Thus, put $\mathcal{T}=T_{\rm act}\otimes T_{\rm res}$, with $\mathcal{T}^f=\rho_{D_Q}(c)$. If $x\in D_Q$, $\mathcal{T}\rho_{D_Q}(x)\mathcal{T}^{-1}=\rho_{D_Q}(\Phi x\Phi^{-1})$. We choose extension $\widehat{\theta}_{D_Q}\in{\rm Irr}(B_Q)$, with $\widehat{\theta}_{D_Q}^\sigma=\widehat{\theta}_{D_Q}$ for $\sigma\in\mathcal{H}_\ell$.

By Lemma 4.32, put $A=G\rtimes\langle F_r\rangle$ and $A_Q=A\cap B_Q$. Then, $B=DA$ and $D\cap A=G$. Moreover, $[D:G]=[D_Q:M]=2$. Since $g\in G$, $B_Q=D_QA_Q$ and $D_Q\cap A_Q=M$. Thus, $A_Q/M\simeq B_Q/D_Q$. On G and M, we write $\langle F_r\rangle\to{\rm Sym}(\{\chi_+,\chi_-\})$ and $\langle\Phi M\rangle\to{\rm Sym}(\{\psi_+,\psi_-\})$. Since f is odd, $\chi_\pm^{F_r}=\chi_\pm$ and $\psi_\pm^\Phi=\psi_\pm$. With $\chi^\delta=\chi$, we choose full parent extensions $\widehat{\theta}_D,\widehat{\theta}_{D_Q}$. Alternatively, with $\chi^\delta\neq\chi$, choose the constituent extensions on A,A_Q. Put $\psi=F(\chi)$. Then, by Lemma 4.32, we choose $\widehat{\chi},\widehat{\psi}$. With $S_\chi=(B\times\mathcal{H}_\ell)_\chi$, put $S_\psi=(B_Q\times\mathcal{H}_\ell)_\psi$. If $(a,\sigma)\in S_\chi$, $\widehat{\chi}^{a\sigma}=\widehat{\chi}$. Moreover, with $(a,\sigma)\in S_\psi$, $\widehat{\psi}^{a\sigma}=\widehat{\psi}$. Choose $\rho_\chi:I_B(\chi)\to{\rm GL}(V_\chi)$ and $\rho_\psi:I_{B_Q}(\psi)\to{\rm GL}(V_\psi)$. $\widehat{\chi}={\rm tr}\,\rho_\chi$ and $\widehat{\psi}={\rm tr}\,\rho_\psi$. Thus, $\alpha_\chi=\alpha_\psi=1$ and $\tau_\chi=\tau_\psi=1$. We write $E_\chi=({\rm Aut}(G)_P)_{\chi^{\mathcal{H}_\ell}}=({\rm Aut}(G)_P)_{\psi^{\mathcal{H}_\ell}}$. Since ${\rm Aut}(P)_Q={\rm Aut}(P)$, $a(Q)=Q$ and $a(M)=M$ for $a\in E_\chi$. Put semidirect products $\widetilde{A}=G\rtimes E_\chi$ and $\widetilde{A}_Q=M\rtimes E_\chi$ in the $\succeq_c$ relation. Moreover, we write $\kappa:\widetilde{A}\to A_{\rm cong}\leq{\rm Aut}(G)$, with $\kappa(g,a)={\rm Int}(g)a$. With centreless simple $S=G$, we put $A_{\rm cong}=\kappa(\widetilde{A})$. We find ${\rm C}_{A_{\rm cong}}(G)=1$. Thus, $K_{\rm diag}=\ker\kappa={\rm C}_{\widetilde{A}}(G)$. With $(g,a)\in K_{\rm diag}$, $g\in{\rm N}_G(P)\leq M$, since $a\in{\rm Aut}(G)_P$. Thus, $K_{\rm diag}\leq\widetilde{A}_Q$, with $\widetilde{A}=G\widetilde{A}_Q$ and $G\cap\widetilde{A}_Q=M$. Moreover, we write ordinary inertia groups $\widetilde{I}_\chi\leq\widetilde{A}$ and $\widetilde{I}_\psi\leq\widetilde{A}_Q$. Write conjugation maps $\kappa_\chi:\widetilde{I}_\chi\to I_B(\chi)$ and $\kappa_\psi:\widetilde{I}_\psi\to I_{B_Q}(\psi)$ and put $\widetilde{\rho}_\chi=\rho_\chi\circ\kappa_\chi$ and $\widetilde{\rho}_\psi=\rho_\psi\circ\kappa_\psi$. Thus, $\widetilde{\rho}_\chi(k)=I_{V_\chi}$ and $\widetilde{\rho}_\psi(k)=I_{V_\psi}$ for $k\in K_{\rm diag}$. With $|Z|=2$, put

$Z = \langle z \rangle = \ker(\pi : \widetilde{G} \to G)$. Since $\ell \neq 2$, we choose $\widetilde{P} \in \mathrm{Syl}_\ell(\pi^{-1}(P))$, with $\pi(\widetilde{P}) = P$. $\pi^{-1}(P) = Z \times \widetilde{P}$, so $\widetilde{P}$ is unique. Thus, $\pi^{-1}(\mathrm{N}_G(P)) = \mathrm{N}_{\widetilde{G}}(\widetilde{P}) = N_\ell$. With the unique lift $\widetilde{Q} \leq \widetilde{P}$ of Q, $\widetilde{M} = \pi^{-1}(M) = \mathrm{N}_{\widetilde{G}}(\widetilde{Q})$. On $\mathrm{GSpin}_{2n+1}(q)$, we write $\pi_{\mathrm{SO}} : \mathrm{GSpin}_{2n+1}(q) \twoheadrightarrow \mathrm{SO}_{2n+1}(q)$. If $a = dF_r^j \in E_\chi$, the map π_{SO} gives a lift $h \in \mathrm{GSpin}_{2n+1}(q)$ with $\pi_{\mathrm{SO}}(h) = d$. Whenever $x \in \widetilde{G}$, we put $\widehat{a}(x) = hF_r^j(x)h^{-1}$. Since $\ker \pi_{\mathrm{SO}} = Z(\mathrm{GSpin}_{2n+1}(q))$, $\widehat{ab} = \widehat{a}\widehat{b}$ and $\pi\widehat{a} = a\pi$. By Section 4, $(\widetilde{G} \rtimes E_\chi, \widetilde{G}, \widetilde{\chi})_{\mathcal{H}_\ell} \succeq_c (\widetilde{M} \rtimes E_\chi, \widetilde{M}, \widetilde{\psi})_{\mathcal{H}_\ell}$. Moreover, $\mathrm{Inf}_G^{\widetilde{G}}(\mathcal{P}_s) \subseteq \mathcal{P}_{\mathrm{triv}}$. □

4.3 Faithful characters

We fix $q = r^f \equiv 3 \pmod 4$, with r prime. We put $X = \mathrm{Spin}_{2n+1}(q)$, with central involution $z = -1$. Moreover, we put $\widetilde{X} = \mathrm{GSpin}_{2n+1}(q)$. We put square scalars $Z_{\mathrm{odd}} = (\mathbb{F}_q^\times)^2 \leq Z(\widetilde{X})$, with $|Z_{\mathrm{odd}}| = (q-1)/2$ odd. Then, we find $D = \widetilde{X}/Z_{\mathrm{odd}}$ by Lemma 4.7. We have that $X \cap Z_{\mathrm{odd}} = 1$ and $[D : X] = 2$. Moreover, we write the orthogonal map $\pi : D \to \mathrm{SO}_{2n+1}(q)$, with $\ker \pi = \langle z \rangle$ and $\pi(X) = \Omega_{2n+1}(q)$. We write natural module $W = \mathbb{F}_q^{2n+1}$, with $\mathrm{End}_{\mathbb{F}_q \pi(X)}(W) = \mathbb{F}_q I_W$. Since $\dim W = 2n+1$ is odd, we have that $tI_W \in \mathrm{SO}(W) \iff t = 1$. Thus, we find $\mathrm{C}_{\mathrm{SO}(W)}(\pi(X)) = 1$. Thus, we find $\mathrm{C}_D(X) \leq \ker \pi = \langle z \rangle$. We have that $Z(D) = \mathrm{C}_D(X) = \langle z \rangle$. Moreover, $r \equiv 3 \pmod 4$ and f is odd.

Lemma 4.34. *We fix $H \lhd A$, with A/H cyclic of odd order m, and we choose $\chi \in \mathrm{Irr}(H)$, with $\chi^a = \chi$ for $a \in A$. Moreover, we consider the extension set $\mathcal{X} = \{\theta \in \mathrm{Irr}(A) : \mathrm{Res}_H^A \theta = \chi\}$. We let Γ act on $\mathcal{X}$ by group and coefficient automorphisms (α, σ). We choose $\alpha \in \mathrm{Aut}(A)$, with $\alpha(H) = H$. Moreover, we choose $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, with $\gamma(\theta) = (\theta \circ \alpha^{-1})^\sigma$ for $\gamma = (\alpha, \sigma) \in \Gamma$. We fix $\Gamma_0 \lhd \Gamma$, with $V = \Gamma/\Gamma_0$ a finite 2-group. If we have that $\mathcal{X}^{\Gamma_0} \neq \varnothing$, we also have that $\mathcal{X}^\Gamma \neq \varnothing$.*

Proof. First, put $\mathcal{X}_0 = \mathcal{X}^{\Gamma_0}$ and $E_0 = \mathrm{Irr}(A/H)$. $|E_0| = m$ is odd, with multiplication $\theta\lambda$ for $\theta \in \mathcal{X}, \lambda \in E_0$. Then, by [43] $\mathcal{X} = \{u\lambda : \lambda \in E_0\}$ for $u \in \mathcal{X}_0$. In particular, $u\lambda = u\mu$ if and only if $\lambda = \mu$. Whenever $\gamma \in \Gamma$, $\gamma(\theta\lambda) = \gamma(\theta)\gamma(\lambda)$. Thus, $\mathcal{X}_0 = \{u\lambda : \lambda \in E\}$, where $E = E_0^{\Gamma_0}$ has odd order $|E|$. Write E additively, with $u + \lambda$ denoting $u\lambda$. Since $\Gamma_0 \lhd \Gamma$, we write the induced action $V \to \mathrm{Sym}(\mathcal{X}_0) \times \mathrm{Aut}(E)$. Write $g(u) = u + c_g$, with $c_g \in E$ for $g \in V$. Then, $g(u+\lambda) = u + c_g + g(\lambda)$. Thus, $c_{hg} = c_h + h(c_g)$ for $h, g \in V$. Since $|V|$ is a power of 2, $\gcd(|V|, |E|) = 1$. We write inverse $|V|^{-1} \in \mathrm{Aut}(E)$ and put $b = |V|^{-1} \sum_{g \in V} c_g$. By $\sum_{g \in V} c_{hg} = \sum_{g \in V} c_g$, $h(b) = b - c_h$. Thus, $h(u+b) = u + c_h + h(b) = u + b$. Put $\widehat{\chi} = u + b \in \mathcal{X}^\Gamma$. Moreover, we put $V_{\mathcal{G}} = \mathcal{G}\Gamma_0/\Gamma_0 \leq V$ for a coefficient subgroup $\mathcal{G} \leq \Gamma$. $|V_{\mathcal{G}}|$ is a power of 2. Given any $\sigma \in \mathcal{G}$, $\widehat{\chi}^\sigma = \widehat{\chi}$. □

Moreover, we consider a faithful series $\mathcal{E}_{\mathrm{faith}} = \mathcal{E}(X, s)$ from Zhang [208, Lemma 6.12]. We write Frobenius $F = F_q$ on $\mathbf{G}^*_{\mathrm{sim}} = \mathrm{GSp}_{2n}$. We choose quasi-isolated $s \in (\mathbf{G}^*_{\mathrm{ad}})^F$, with $\mathbf{G}^*_{\mathrm{ad}} = \mathrm{PSp}_{2n}$. By Zhang [208, Lemma 6.12], choose a lift $y \in \mathrm{Sp}_{2n}$ with $F(y) = -y$. We write $i^2 = -1$, with $F(i) = -i$. Thus, we find $\widetilde{y} = iy \in (\mathbf{G}^*_{\mathrm{sim}})^F$. We choose $\Theta \in \mathrm{Irr}(\widetilde{X})$, with $\mathrm{Res}_X^{\widetilde{X}} \Theta = \chi_+ + \chi_-$ for $\chi_+ \neq \chi_-$ in $\mathcal{E}_{\mathrm{faith}}$. Moreover, we choose rational unipotent character $\rho_\Theta = J_{\widetilde{y}}(\Theta)$ of $\mathrm{C}_{\mathbf{G}^*_{\mathrm{sim}}}(\widetilde{y})^F$. We write Clifford norm $\nu : \widetilde{X} \to \mathbb{F}_q^\times$ and quadratic character $\eta_q \in \mathrm{Irr}(\mathbb{F}_q^\times)$. We put $\kappa = \eta_q \circ \nu$, with $\nu_1 = 1$ and $\nu_{-1} = \kappa$. By Zhang [208, proof of Lemma 6.13], $\Theta^\sigma = \Theta\nu_c$, where $c = i^{k-1} \in \{1, -1\}$ for $\sigma(i) = i^k$. By Zhang [208, proof of Lemma 6.12], F_r fixes the label ρ_Θ and gives $\Theta^{F_r} = \Theta\nu_{i^{r-1}}$. Since $\Theta|_X$ has two constituents, Clifford [43] gives $\Theta\kappa = \Theta$. Thus, Θ is rational and F_r-fixed. We write scalar central character $\omega_\Theta(t) = \Theta(t)/\Theta(1)$ for $t \in Z(\widetilde{X})$. We have that $\omega_\Theta(t) \in \{1, -1\}$ and $\omega_\Theta(z) = -1$. Since $|Z_{\mathrm{odd}}|$ is odd, $Z_{\mathrm{odd}} \leq \ker \Theta$. We write quotient map $\pi_{\mathrm{odd}} : \widetilde{X} \to D$ and character $\Theta_D \in \mathrm{Irr}(D)$, with $\Theta_D \circ \pi_{\mathrm{odd}} = \Theta$.

Then, we choose $\widehat{\Theta} \in \mathrm{Irr}(B)$ as a rational extension of Θ_D by Lemma 4.32, with $B = D \rtimes \langle F_r \rangle$. We put $A = X \rtimes \langle F_r \rangle$, with $B = DA$ and $D \cap A = X$. We have that $|D/X| = 2$ and $A/X \simeq C_f$, with f odd. Moreover, by Zhang [208, Lemma 6.12], we have that $\chi_\epsilon^{F_r} = \chi_\epsilon$ for $\epsilon \in \{+, -\}$. Thus, we find $I_B(\chi_\epsilon) = A$. We write mixed stabilizer $S_\epsilon = (B \times \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}))_{\chi_\epsilon}$. Then, by Lemma 4.32, we choose extension $\widehat{\chi}_\epsilon \in \mathrm{Irr}(A)$ of χ_ϵ. Whenever $(b, \sigma) \in S_\epsilon$, $\widehat{\chi}_\epsilon^{b\sigma} = \widehat{\chi}_\epsilon$. Moreover, we have that $\widehat{\Theta}(z) = -\widehat{\Theta}(1)$ and $\widehat{\chi}_\epsilon(z) = -\widehat{\chi}_\epsilon(1)$. Thus, we have that $\rho_V(z) = -I_V$ for ordinary representations ρ_V of $\widehat{\Theta}, \widehat{\chi}_+, \widehat{\chi}_-$.

We now suppose that $\mathcal{W}$ consists of positive cycles $\mathcal{C}_1, \ldots, \mathcal{C}_w$. Suppose that $d = \mathrm{ord}_p(q) = e$ is odd. By Zhang [208, Lemma 6.12(ii)(b)], we have that $C^* = \mathrm{Sp}_{2a}(q^2) \times \mathrm{GL}_b(q)$. We put $n = 2a + b$, with $b = ew + s_0$ and $h = 2a + s_0 < e$. Since $2a < e$, $p \nmid |\mathrm{Sp}_{2a}(q^2)|$. Thus, $\mathcal{C}_i$ is in the linear factor $\mathrm{GL}_b(q)$ for $1 \leq i \leq w$. We write unipotent labels $\mathcal{U}_a$ of $\mathrm{Sp}_{2a}(q^2)$. We fix $\rho \in \mathcal{U}_a$ and residual character Σ on the general spin group R of rank $2a$. Moreover, we choose Σ over ρ globally and locally. We write norm $\nu_R : R \to \mathbb{F}_q^\times$. Moreover, we write scalar character $\omega(t) = \Sigma(t)/\Sigma(1)$ for

$t \in \mathbb{F}_q^\times \leq Z(R)$. We have that $\omega(-1) = \eta_q(-1) = -1$. Since $(\omega\eta_q)(-1) = 1$, we have that $\omega\eta_q = \xi^2$ for $\xi \in \mathrm{Irr}(\mathbb{F}_q^\times)$. We choose ξ with $\xi^2\eta_q = \omega$. Moreover, with $a > 0$, we choose representation $\rho_\Sigma : R \to \mathrm{GL}(V_\Sigma)$ of character Σ. We choose operator $J \in \mathrm{GL}(V_\Sigma)$, with $J^2 = I_{V_\Sigma}$. Whenever $x \in R$, $J\rho_\Sigma(x)J^{-1} = (\eta_q \circ \nu_R)(x)\rho_\Sigma(x)$.

Moreover, with $b > 0$, we choose partition $\lambda \vdash b$ and unipotent character $\rho_\lambda \in \mathrm{Irr}(\mathrm{GL}_b(q))$. We choose Levi subgroup $\widetilde{L} = \mathrm{GL}_b(q) \times R$ of $\widetilde{X}$. We put $\mu_\lambda = (\xi \circ \det)\rho_\lambda \boxtimes \Sigma \in \mathrm{Irr}(\widetilde{L})$. Thus, we find $\nu|_{\widetilde{L}}(A, x) = \det(A)\nu_R(x)$. We put kernel $H_{\mathrm{norm}} = \ker(\nu|_{\widetilde{L}})$. We have that $\mu_\lambda\gamma = \mu_\lambda$ implies $\gamma \in \{1, \nu_2\}$ for $\gamma \in \mathrm{Irr}(\widetilde{L}/H_{\mathrm{norm}})$. Moreover, $\nu_2 = (\eta_q \circ \det) \boxtimes (\eta_q \circ \nu_R)$. However, we consider linear scalars $(s_{\mathrm{lin}}, -s_{\mathrm{lin}})$ for $(\mu_\lambda, \mu_\lambda\nu_2)$. We have that $s_{\mathrm{lin}} \neq -s_{\mathrm{lin}}$. Thus, $\mu_\lambda\nu_2 \neq \mu_\lambda$. By Clifford theory (Clifford [43]), we have that $\mathrm{Res}^{\widetilde{L}}_{H_{\mathrm{norm}}}\mu_\lambda \in \mathrm{Irr}(H_{\mathrm{norm}})$. We choose parabolic $\widetilde{P} = \widetilde{L}\mathcal{U}$ and inflate μ_λ by $\mu_\lambda(lu) = \mu_\lambda(l)$ for $l \in \widetilde{L}, u \in \mathcal{U}$. Moreover, we choose dual Levi $\mathbf{L}^*$ for $\widetilde{L}$. We choose semisimple $s_\lambda \in (\mathbf{L}^*)^F$, with $\mu_\lambda \in \mathcal{E}(\widetilde{L}, s_\lambda)$. $\mathrm{C}_{\mathbf{G}^*_{\mathrm{sim}}}(s_\lambda) \leq \mathbf{L}^*$. Thus, we find $\Theta_\lambda = \mathrm{Ind}^{\widetilde{X}}_{\widetilde{P}}\mu_\lambda \in \mathrm{Irr}(\widetilde{X})$. We have that $\mathrm{Res}^{\widetilde{X}}_X\Theta_\lambda = \chi_{\lambda,+} + \chi_{\lambda,-}$, with $\chi_{\lambda,+} \neq \chi_{\lambda,-}$. Whenever $\epsilon \in \{+,-\}$, $\langle \mathrm{Res}^{\widetilde{X}}_X\Theta_\lambda, \chi_{\lambda,\epsilon}\rangle_X = 1$. Alternatively, with $b = 0$, we put $\Theta_\lambda = \Sigma$. We put $\chi_{\lambda,+} = \Sigma_+$ and $\chi_{\lambda,-} = \Sigma_-$, with $\mathrm{Res}^R_X\Sigma = \Sigma_+ + \Sigma_-$.

Moreover, we put $K_j = j(j+1)/2$ and take $b > 0$. We choose hyperbolic spaces $U_+ = \langle e_1, \ldots, e_b\rangle$ and $U_- = \langle f_1, \ldots, f_b\rangle$. By Subsection 4.1, we choose coordinate lifts $w_i = R_i$ in $\mathrm{Cl}^0(W)$ (Milne [137, §24.h]). Then, $w_i^2 = z$ and $w_iw_j = zw_jw_i$ for $i \neq j$. Moreover, we write residual coordinates $b+1, \ldots, b+2a$. We put $E_R = \prod_{j=b+1}^{b+2a} T_j(-1)$ and $\delta_b = E_R^{b \bmod 2}$. We have that $\delta_b^2 = 1$ and $\delta_b w_i = w_i\delta_b$. Thus, $\delta_b = 1$ for even b. We put $w_* = w_1 \cdots w_b\delta_b$, with $F_r(w_*) = w_*$ and $\nu(w_*) = 1$. We have that $\pi(w_*)(U_+) = U_-$ and $\pi(w_*)(U_-) = U_+$. Moreover, $w_*x = xw_*$ for $x \in R$. We have that $w_*^2 = z^{b+\binom{b}{2}} = z^{K_b}$. We write the injection $\iota : \mathrm{GL}_b(q) \to \widetilde{L}$, with $\iota(A) = (A, 1)$. Then, we find $w_*\iota(A)w_*^{-1} = (\det A)\iota(A^{-T})$ by Milne [137, §24.h] and Subsection 4.1. Since $2a$ is even, $\nu(\delta_b) = 1$.

Moreover, we choose representation $D_\lambda : \mathrm{GL}_b(q) \to \mathrm{GL}(V_\lambda)$ of character ρ_λ. We choose graph operator $S_\lambda \in \mathrm{GL}(V_\lambda)$, with $S_\lambda^2 = I_{V_\lambda}$. In particular, $S_\lambda D_\lambda(A)S_\lambda^{-1} = D_\lambda(A^{-T})$. Since $\xi^2\eta_q = \omega$, we set $A_\lambda = S_\lambda \otimes J$. If $a = 0$, we set $V_\Sigma = \mathbb{C}$, $\rho_\Sigma = \Sigma = \eta_q$ and $J = I$. Thus, we find $A_\lambda = S_\lambda$. We put $P = \widetilde{P} \cap X$, with Levi subgroup H_{norm}. Moreover, we choose representation $\rho_\mu : P \to \mathrm{GL}(V_\mu)$ of $\mathrm{Res}^{\widetilde{P}}_P\mu_\lambda$. On H_{norm}, $\rho_\mu(A, x) = \xi(\det A)D_\lambda(A) \otimes \rho_\Sigma(x)$. Whenever $(A, x) \in H_{\mathrm{norm}}$, $\det(A)\nu_R(x) = 1$. Thus, we have that $A_\lambda\rho_\mu(w_*^{-1}lw_*)A_\lambda^{-1} = \rho_\mu(l)$ for $l \in H_{\mathrm{norm}}$. We have that $A_\lambda^2 = I_{V_\mu}$. We put $V_{\mathrm{ind}} = \mathrm{Ind}^X_P V_\mu$. Since $\widetilde{X} = X\widetilde{P}$, we choose an ordinary model $\widetilde{\rho}_\lambda : \widetilde{X} \to \mathrm{GL}(V_{\mathrm{ind}})$ of Θ_λ. We have that $\mathrm{ch}(V_{\mathrm{ind}}) = \chi_{\lambda,+} + \chi_{\lambda,-}$. Thus, $\dim \mathcal{E}_\lambda = 2$, with $\mathcal{E}_\lambda = \mathrm{End}_X(V_{\mathrm{ind}})$.

We write Hecke kernel $K_T : X \to \mathrm{End}(V_\mu)$, with $\mathrm{supp}(K_T) = Pw_*P$ and $K_T(w_*) = A_\lambda$. Moreover, $K_T(p_1gp_2) = \rho_\mu(p_1)K_T(g)\rho_\mu(p_2)$ for $p_1, p_2 \in P$. We write product $(K_1 * K_2)(g) = |P|^{-1}\sum_{x \in X} K_1(x)K_2(x^{-1}g)$. We write $T \in \mathcal{E}_\lambda$ for K_T and $I \in \mathcal{E}_\lambda$ for K_I. Since $K_I(p) = \rho_\mu(p)$ for $p \in P$, $\mathrm{supp}(K_I) = P$. We have that $P \cap Pw_*P = \varnothing$. Thus, we find $\mathcal{E}_\lambda = \mathbb{C}I \oplus \mathbb{C}T$. We choose $u \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$ and $g_{\mathrm{ns}} = \iota(\mathrm{diag}(u, 1, \ldots, 1))$. We have that $\nu(g_{\mathrm{ns}}) = u$. Since $g_{\mathrm{ns}} \in \widetilde{L}$, $g_{\mathrm{ns}}Pg_{\mathrm{ns}}^{-1} = P$. We have that $g_{\mathrm{ns}}w_*g_{\mathrm{ns}}^{-1}w_*^{-1} = u^{-1}\iota(\mathrm{diag}(u^2, 1, \ldots, 1)) \in H_{\mathrm{norm}}$. Consequently, we find $g_{\mathrm{ns}}(Pw_*P)g_{\mathrm{ns}}^{-1} = Pw_*P$. Thus, we find $\widetilde{\rho}_\lambda(g_{\mathrm{ns}})\mathbb{C}T\widetilde{\rho}_\lambda(g_{\mathrm{ns}})^{-1} = \mathbb{C}T$. By Clifford theory (Clifford [43]), we have that $\chi_{\lambda,+}^{g_{\mathrm{ns}}} = \chi_{\lambda,-}$. We obtain that $\widetilde{\rho}_\lambda(g_{\mathrm{ns}})T\widetilde{\rho}_\lambda(g_{\mathrm{ns}})^{-1} = -T$. Thus, $T^2 \in \mathbb{C}I$. We have that $P \cap w_*Pw_*^{-1} = H_{\mathrm{norm}}$ and $|\mathcal{U}| = q^{4ab+K_b}$. Moreover, $\rho_\mu(w_*^2) = (-1)^{K_b}I_{V_\mu}$. Since $A_\lambda^2 = I_{V_\mu}$, we deduce that $K_{T^2}(1) = |\mathcal{U}|\rho_\mu(w_*^2)^{-1} = (-1)^{K_b}q^{4ab+K_b}I_{V_\mu}$. Thus, we find $T^2 = (-1)^{K_b}q^{4ab+K_b}I_{V_{\mathrm{ind}}}$.

Moreover, we choose coefficient operation σ with $\rho^\sigma = \rho$. We choose $\delta_\sigma \in \mathrm{Irr}(\mathbb{F}_q^\times)$, with $\Sigma^\sigma = \Sigma(\delta_\sigma \circ \nu_R)$. We choose intertwiner C_σ on V_Σ, with $C_\sigma\rho_\Sigma(x)^\sigma C_\sigma^{-1} = \delta_\sigma(\nu_R(x))\rho_\Sigma(x)$. Since $J^2 = I$, $C_\sigma J^\sigma C_\sigma^{-1} = u_\rho(\sigma)J$ for $u_\rho(\sigma) \in \{1, -1\}$. We write eigenspaces $V_\epsilon = \ker(J - \epsilon I)$, with $\epsilon \in \{1, -1\}$. Then, we find $C_\sigma(V_\epsilon^\sigma) = V_{\epsilon u_\rho(\sigma)}$. Moreover, $\omega^\sigma = \omega\delta_\sigma^2$ on $\mathbb{F}_q^\times$. We put $\tau_\sigma = \xi^\sigma/(\xi\delta_\sigma)$. Since $\xi^2\eta_q = \omega$, $\tau_\sigma^2 = 1$. We have that $\tau_\sigma \in \{1, \eta_q\}$. Hence, we find $\mu_\lambda^\sigma = \mu_\lambda(\delta_\sigma \circ \nu)(\tau_\sigma \circ \det)$ on $\widetilde{L}$. Since $\nu|_{H_{\mathrm{norm}}} = 1$, we have that $\mu_\lambda^\sigma|_{H_{\mathrm{norm}}} = (\mu_\lambda(\tau_\sigma \circ \det))|_{H_{\mathrm{norm}}}$. Whenever $(A, x) \in H_{\mathrm{norm}}$, $\eta_q(\det A) = \eta_q(\nu_R(x))$. Thus, with $\tau_\sigma = \eta_q$, we choose extra operator $I \otimes J$. We have that $(I \otimes J)A_\lambda = A_\lambda(I \otimes J)$.

Moreover, we choose coefficient intertwiner $B_{\lambda,\sigma}$, with $B_{\lambda,\sigma}D_\lambda(A)^\sigma B_{\lambda,\sigma}^{-1} = D_\lambda(A)$. We define $s_\lambda(\sigma) \in \{1, -1\}$ by $B_{\lambda,\sigma}S_\lambda^\sigma B_{\lambda,\sigma}^{-1} = s_\lambda(\sigma)S_\lambda$. Since $e = \mathrm{ord}_p(q)$ is odd, $(q/p) = 1$. Whenever $\sigma \in \mathcal{H}_p$, we have that $\sigma(\sqrt{q}) = \sqrt{q}$. By Dudas–Malle [61, Theorem 3], $s_\lambda(\sigma) = 1$ for $\sigma \in \mathcal{H}_p$ when $b \geq 3$. If $b = 1$, we put $S_\lambda = I$. Moreover, with $b = 2$, we choose $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $S_\lambda = D_\lambda(J_2)$. We have that $D_\lambda(-I) = I$ and $J_2^2 = -I$. Thus, $s_\lambda(\sigma) = 1$ also for $b \in \{1, 2\}$. We write $\tau_\sigma = \eta_q^{t_\sigma}$, with $t_\sigma \in \{0, 1\}$. On V_μ, we put $D_\sigma = (I \otimes J)^{t_\sigma}(B_{\lambda,\sigma} \otimes C_\sigma)$. Thus, $D_\sigma\rho_\mu(l)^\sigma D_\sigma^{-1} = \rho_\mu(l)$ for $l \in H_{\mathrm{norm}}$. We choose induced coefficient intertwiner $C_{\mathrm{ind},\sigma}$ on V_{ind}. Then, we find $C_{\mathrm{ind},\sigma}T^\sigma C_{\mathrm{ind},\sigma}^{-1} = u_\rho(\sigma)s_\lambda(\sigma)T$. By $T^2 = (-1)^{K_b}q^{4ab+K_b}I$, we choose $\beta_\lambda = i^{K_b}q^{(4ab+K_b)/2}$. Thus, we put $T_0 = \beta_\lambda^{-1}T$, with $T_0^2 = I$. Whenever $\sigma \in \mathcal{H}_p$, $\sigma(\beta_\lambda)/\beta_\lambda = (\sigma(i)/i)^{K_b}$. Thus, we write permutation sign $\epsilon_\lambda(\sigma) = u_\rho(\sigma)(\sigma(i)/i)^{K_b}$ on $\{\chi_{\lambda,+}, \chi_{\lambda,-}\}$. We have that

$\chi^{\sigma}_{\lambda,\epsilon} = \chi_{\lambda,\epsilon\epsilon_\lambda(\sigma)}$ with sign indices $\epsilon \in \{1,-1\}$. We write e-core κ of λ, with $|\kappa| = s_0$. Moreover, with $s_0 > 0$, we put $\widetilde{X}_0 = \mathrm{GSpin}_{2h+1}(q)$. We choose parabolic $\widetilde{P}_0 = \widetilde{L}_0\mathcal{U}_0$, with $\widetilde{L}_0 = \mathrm{GL}_{s_0}(q) \times R$. We choose unipotent $\rho_\kappa \in \mathrm{Irr}(\mathrm{GL}_{s_0}(q))$ and $\mu_\kappa = (\xi \circ \det)\rho_\kappa \boxtimes \Sigma$. Accordingly, by the construction for μ_λ, we have that $\Theta_\kappa = \mathrm{Ind}_{\widetilde{P}_0}^{\widetilde{X}_0} \mu_\kappa \in \mathrm{Irr}(\widetilde{X}_0)$. We have that $\mathrm{Res}^{\widetilde{X}_0}_{\mathrm{Spin}_{2h+1}(q)} \Theta_\kappa = \chi_{\kappa,+} + \chi_{\kappa,-}$. Moreover, with $a > 0, s_0 = 0$, we put $\chi_{\kappa,\epsilon} = \Sigma_\epsilon$. With $h > 0$, we write permutation sign $\epsilon_\kappa(\sigma) = u_\rho(\sigma)(\sigma(i)/i)^{K_{s_0}}$ on $\{\chi_{\kappa,+}, \chi_{\kappa,-}\}$. Thus, $\epsilon_\kappa(\sigma) = u_\rho(\sigma)$ when $s_0 = 0$. With $a = 0$, we put $u_\rho(\sigma) = 1$ and $A_\lambda = S_\lambda$. Moreover, with $a = s_0 = 0$, we put $\epsilon_\kappa(\sigma) = 1$. We have that $|\mathcal{U}| = q^{K_b}$ in the linear parabolic P. $T^2 = (-1)^{K_b}q^{K_b}I$ when $a = 0$.

We choose active normalizer M_{act}, with w blocks and residual rank h. We choose active torus $T_{\mathrm{act}} \triangleleft M_{\mathrm{act}}$. Moreover, we put $a_0 = (e+1)/2$ and take quadratic weight $\vartheta_Q \in \mathrm{Irr}(T_{\mathrm{act}})$. We choose block reflections $r_1, \dots, r_w$ in the Clifford model (R_i, T_i). Moreover, we put $R_w = r_1 \cdots r_w$ and inertia group $I_{\vartheta_Q} = I_{M_{\mathrm{act}}}(\vartheta_Q)$. We choose distinguished extensions $\widehat{\vartheta}_+, \widehat{\vartheta}_-$ of ϑ_Q to I_{ϑ_Q}. $\widehat{\vartheta}_\pm(r_i^2) = (-1)^{a_0+h}$. We write $\widehat{\vartheta}_+(R_w) = i^{w(w+e)/2+hw}$ and $\widehat{\vartheta}_-(R_w) = -\widehat{\vartheta}_+(R_w)$. Since $r_ir_j = zr_jr_i$, $\widehat{\vartheta}_\pm(R_w)^2 = (-1)^{\binom{w}{2}+w(a_0+h)}$. We choose adjacent block permutations s_i with $\widehat{\vartheta}_\pm(s_i) = 1$. We write $\widehat{\vartheta}_Q = \widehat{\vartheta}_+$. Moreover, we choose relative character $\eta \in \mathrm{Irr}(C_e \wr \mathfrak{S}_w)$, with $\mathbb{Q}(\eta) \subseteq \mathbb{Q}(\mu_e)$. Since $e \mid p-1$, $\eta^\sigma = \eta$ for $\sigma \in \mathcal{H}_p$. We have that $w(w+e)/2 \equiv K_{ew} \pmod 2$. Thus, we write permutation sign $\epsilon_{\mathrm{act}}(\sigma) = (\sigma(i)/i)^{K_{ew}+hw}$ on $\{\widehat{\vartheta}_+, \widehat{\vartheta}_-\}$. With $h > 0$, we write residual sign $\epsilon_\kappa(\sigma)$ from $\{\chi_{\kappa,+}, \chi_{\kappa,-}\}$. We have that $K_b - K_{s_0} = K_{ew} + ews_0$ and $h \equiv s_0 \pmod 2$. Consequently, $\epsilon_{\mathrm{act}}(\sigma)\epsilon_\kappa(\sigma) = u_\rho(\sigma)(\sigma(i)/i)^{K_b} = \epsilon_\lambda(\sigma)$ for every $\rho \in \mathcal{U}_a$.

Moreover, we put $M = \mathrm{N}_X(Q)$ and a toral cycle $\mathcal{C}_{\mathrm{tor}}$. We write eigenvalues $\lambda_t \in \{1,-1,i,-i\}$ of y along $\mathcal{C}_{\mathrm{tor}}$. Since $F(y) = -y$, $\lambda_{t+1} = -\lambda_t^q$. We put $\phi : \{1,-1,i,-i\} \to \{1,-1,i,-i\}$, with $\phi(t) = -t^q$. Thus, we find $(\phi(1), \phi(-1), \phi(i), \phi(-i)) = (-1, 1, i, -i)$. With cycle length e, $\lambda_e = \lambda_0$ for a positive cycle $\mathcal{C}_{\mathrm{tor}}$. Alternatively, for a negative cycle $\mathcal{C}_{\mathrm{tor}}$, we have that $\lambda_e = \lambda_0^{-1}$. With odd e, $\lambda_0 \in \{i, -i\}$ for a positive cycle $\mathcal{C}_{\mathrm{tor}}$. Thus, we write residual factor $\mathrm{GL}_{b-e}(q)$ after one cycle $\mathcal{C}_{\mathrm{tor}}$. With even $e = 2j$, $\lambda_0 \in \{1,-1\}$ for a negative cycle $\mathcal{C}_{\mathrm{tor}}$. We write residual factor $\mathrm{Sp}_{2(a-j)}(q^2)$ after one cycle $\mathcal{C}_{\mathrm{tor}}$. Alternatively, with odd e, we have that $\phi^e(t) \neq t^{-1}$ for $t \in \{1,-1,i,-i\}$. Therefore, $\mathcal{C}_{\mathrm{tor}}$ is positive when e is odd. With even $e = 2j$, we write $a = jw + c$, with $0 \leq c < j$. We write residual rank $h = 2c + b < e$. Moreover, we write the action of M on $\{\mathcal{C}_1, \dots, \mathcal{C}_w\}$ by permutations π and orientations $\varepsilon_i \in \{1,-1\}$. We have that $\mathcal{C}_i \mapsto \mathcal{C}_{\pi(i)}^{\varepsilon_i}$. Moreover, we choose active toral character θ_{act} for Q, with orbit $\mathcal{O}_Q = \theta_{\mathrm{act}}^M$.

We put $X^* = (\mathbf{G}^*_{\mathrm{ad}})^F$. Then, we choose $s \in X^*$ quasi-isolated by Proposition 4.39, with $p \nmid |s|$ and rational class $\mathcal{C}_s = s^{X^*}$. We put $C = \mathrm{C}_X(Q)$ and ordinary series $\mathcal{E}_C(s)$ above $\mathcal{C}_s$. Accordingly, put $\mathcal{U}_s = \bigcup_{m \in M} \mathcal{E}_C(s)^m$. Moreover, $Q \in \mathrm{Syl}_p(C)$ and $Q \leq Z(C)$. We have that $C = Q \times C_0$, with $p \nmid |C_0|$. We choose block $b_C \in \mathrm{Bl}_p(C)$. Thus, we choose unique $\chi_{b_C} \in \mathrm{Irr}(C, b_C)$ with $Q \leq \ker \chi_{b_C}$. Whenever $\beta_{b_C} \in \mathrm{Irr}(C_0)$, $\chi_{b_C} = 1_Q \boxtimes \beta_{b_C}$. We have that $p \nmid \chi_{b_C}(1)$. Moreover, we write $\bar{e}_s^X$ for the p-block union at s. We write $\bar{b}_{s,i}$ on C above $\mathcal{C}_s$. By Bonnafé et al. [19, Theorem 4.14], we have that $\mathrm{br}_Q(\bar{e}_s^X) = \sum_i \bar{b}_{s,i}$. Moreover, we put $\mathcal{B}_s = \{b_C : \chi_{b_C} \in \mathcal{U}_s\}$. We choose ordinary characters $\chi_{b_C} \in \mathcal{U}_s$ within the block union $\sum_i \bar{b}_{s,i}$. We have that $\mathcal{U}_s \subseteq \{\chi_{b_C} : b_C \in \mathcal{B}_s\}$, since $p \nmid |s|$. Thus, we find $Y_s^X(M) = \{\psi \in \mathrm{Irr}_{p'}(M) : \exists b_C \in \mathcal{B}_s, \langle \mathrm{Res}_C^M \psi, \chi_{b_C}\rangle_C \neq 0\}$. Whenever $\psi \in Y_s^X(M)$, $Q \leq \ker \psi$.

Moreover, with odd e, we write residual labels (ρ, λ_0), with $\rho \in \mathcal{U}_a$ and $\lambda_0 \vdash s_0$. We have that $s_0 < e$, so $\mathrm{core}_e(\lambda_0) = \lambda_0$. With even $e = 2j$, we choose residual symbol Σ_0 of rank c and partition $\lambda \vdash b$. Since $c < j$, $\mathrm{cocore}_j(\Sigma_0) = \Sigma_0$. We write CQ from partitions λ to their e-cores κ and relative characters η. By Lemma 4.23, we write cohook reconstruction CH from (Σ_0, η) to symbols Σ. Then, we consider all ordinary labels $\mathcal{L}_s$ above the cores κ, Σ_0. We choose branches $\psi_{C,+}, \psi_{C,-}$ above $\mathcal{O}_Q$. Then, we choose $\eta \in \mathrm{Irr}(C_e \wr \mathfrak{S}_w)$ by Clifford theory (Clifford [43]). Moreover, $p \nmid |R_{\mathrm{res}}|$ for the residual group R_{res}. We write normalizer indices ι, with $\iota \mid 2^k$ for $k \geq 0$. Thus, $\psi(1)_p = \eta(1)_p$ for the characters ψ above $\mathcal{O}_Q$. With $h = 0$, we put $R_{\mathrm{res}} = 1$ in the degree formula $\psi(1)_p = \eta(1)_p$. We obtain that $Y_s^X(M)$ consists of the characters ψ from $\mathcal{L}_s$ with $p \nmid \eta(1)$. Moreover, we choose the positive pairing J_{odd} by $(\kappa, \eta) = \mathsf{CQ}(\lambda)$. We put $J_{\mathrm{odd}}(\chi_{\rho,\lambda,\epsilon}) = \psi_{\rho,\kappa,\eta,\epsilon}$, with $\epsilon \in \{+,-\}$. By the sign comparison $\epsilon_{\mathrm{act}}\epsilon_\kappa = \epsilon_\lambda$, $J_{\mathrm{odd}}(\chi^\sigma) = J_{\mathrm{odd}}(\chi)^\sigma$. Moreover, we choose diagonal representative $\delta \in \mathrm{N}_D(Q) \setminus M$. We have that $\chi^\delta_{\rho,\lambda,\epsilon} = \chi_{\rho,\lambda,-\epsilon}$ and $\psi^\delta_{\rho,\kappa,\eta,\epsilon} = \psi_{\rho,\kappa,\eta,-\epsilon}$.

Proposition 4.35. *We choose $\chi \in \mathcal{E}(X,s) \cap \mathrm{Irr}_{p'}(X)$ and $\psi \in Y_s^X(M)$ in a faithful pair (χ, ψ). If $d = e$ is odd, we put $\psi = J_{\mathrm{odd}}(\chi)$. Alternatively, with even $e = 2j$, we put $\psi = J_{\mathrm{even}}(\chi)$ from* Proposition 4.25. *We put $B = D \rtimes \langle F_r\rangle$ and $A = X \rtimes \langle F_r\rangle$ and put $B_Q = \mathrm{N}_B(Q)$ and $A_Q = B_Q \cap A$. Moreover, we choose extensions $\widehat{\chi} \in \mathrm{Irr}(A)$ and $\widehat{\psi} \in \mathrm{Irr}(A_Q)$ of χ, ψ. We put field inertia groups $I_\chi = I_A(\chi)$ and $I_\psi = I_{A_Q}(\psi)$. Thus, $I_\chi = A$ and $I_\psi = A_Q$. We write common mixed stabilizer $S = (B_Q \times \mathcal{H}_p)_{\chi,\psi}$. Thus, $\widehat{\chi}^{b\sigma} = \widehat{\chi}$ and $\widehat{\psi}^{b\sigma} = \widehat{\psi}$ for $(b,\sigma) \in S$. We write factor sets $\alpha_\chi = \alpha_\psi = 1$ on*

I_χ, I_ψ. *Moreover, we choose common central character* $\omega \in \mathrm{Irr}(\langle z\rangle)$, *with* $\omega(z) = -1$.

Proof. First, choose $P \in \mathrm{Syl}_p(X)$ and characteristic subgroup $Q \le P$ from Proposition 4.4. We put $D_Q = B_Q \cap D$, with $X \cap B_Q = M$, and we choose $b \in B$ and choose $x \in X$ with $P^{bx} = P$. Since Q is characteristic in P, $Q^{bx} = Q$. Thus, $B = XB_Q$. We have that $D = XD_Q$ and $A = XA_Q$, and that $D_Q/M \simeq C_2$ and $A_Q/M \simeq C_f$, and $B_Q = D_Q A_Q$ and $D_Q \cap A_Q = M$. By Lemma 4.11, choose $\Phi = gF_r \in A_Q$, with $g \in X$. Thus, $\Phi^f = c = gF_r(g)\cdots F_r^{f-1}(g) \in M$. We write Φ as the field generator F_0 on M.

With odd $d = e$, choose block lifts $k_l \in D$ for $1 \le l \le w$. Moreover, we put block factors $c_l = z^{a_0(1+h)}(k_l^2)^{a_0}$ in $c = \Phi^f$. We find $\widehat{\vartheta}_Q(k_l^2) = (-1)^{a_0+h}$. Since $a_0(1+h) + (a_0+h)a_0 = a_0(a_0+1) + 2a_0h$, $\widehat{\vartheta}_Q(c_l) = 1$. We choose representation $\rho_\eta : C_e \wr \mathfrak{S}_w \to \mathrm{GL}(V_\eta)$ of η. Moreover, we choose the remaining rotation c_{W} in the centre of $C_e \wr \mathfrak{S}_w$. $|c_{\mathrm{W}}| \mid e$. Thus, put scalar $\gamma = \eta(c_{\mathrm{W}})/\eta(1) \in \mu_e$ on the source space V_η. We put $D_r = \mathrm{ord}_p(r)$, with $D_r/\gcd(D_r, f) = e$. Then, $\{\zeta^f : \zeta \in \mu_{D_r}\} = \mu_e$. Since $D_r \mid p-1$, we choose $\alpha \in \mu_{D_r}$ with $\alpha^f = \gamma$. Whenever $\sigma \in \mathcal{H}_p$, $\sigma(\alpha) = \alpha$. If the carry is c_{W}^{-1}, choose α^{-1}. Thus, we put field operator $T_\eta = \alpha I_{V_\eta}$ on V_η, with $T_\eta^f = \rho_\eta(c_{\mathrm{W}})$. We choose the coefficient matrices $C_{\eta,\sigma}$ for η on V_η. Then, $C_{\eta,\sigma}T_\eta^\sigma C_{\eta,\sigma}^{-1} = T_\eta$. By induction from V_η, we choose extension $\widehat{\varphi}_{\mathrm{act}}$ of the active character $\varphi_{\mathrm{act}} = \vartheta_{\mathrm{act}}$. Consequently, $\widehat{\varphi}_{\mathrm{act}}^\sigma = \widehat{\varphi}_{\mathrm{act}}$ for $\sigma \in (\mathcal{H}_p)_{\varphi_{\mathrm{act}}}$.

Alternatively, by Lemma 4.9 with even $e = 2j$, we choose lifts k_l. By Lemma 4.24, write active sign $\epsilon_{\mathrm{a}}(\sigma) = \eta_q(c_\sigma)^{jw}$ and extension ratio $\epsilon_{\mathrm{f}}(\sigma) = (c_\sigma/r)^{jw}$. Choose $c_\sigma \in \mathbb{F}_r^\times$, with $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$. Since f is odd, $\eta_q(c_\sigma) = (c_\sigma/r)$. Thus, $\epsilon_{\mathrm{f}}(\sigma) = \epsilon_{\mathrm{a}}(\sigma)$. On $(\mathcal{H}_p)_{\varphi_{\mathrm{act}}}$, $\epsilon_{\mathrm{a}}(\sigma) = \epsilon_{\mathrm{f}}(\sigma) = 1$. We choose extension $\widehat{\varphi}_{\mathrm{act}}$ with $\widehat{\varphi}_{\mathrm{act}}^\sigma = \widehat{\varphi}_{\mathrm{act}}$ for $\sigma \in (\mathcal{H}_p)_{\varphi_{\mathrm{act}}}$. Moreover, in both cases $d = e$ and $e = 2j$, we choose operator T_{act} for Φ. $T_{\mathrm{act}}^f = \rho_{\mathrm{act}}(c_{\mathrm{act}})$ for the active component c_{act} of c.

Moreover, with residual rank $h > 0$, we put $R_{\mathrm{res}} = \mathrm{Spin}_{2h+1}(q)$. Write active norm kernel L_{act} and central product $L = (L_{\mathrm{act}} \times R_{\mathrm{res}})/\langle (z,z)\rangle$. We write active norm class $\bar{\nu}_{\mathrm{act}} : M \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. Thus, $\ker \bar{\nu}_{\mathrm{act}} = L$ and $\mathrm{im}\, \bar{\nu}_{\mathrm{act}} \simeq C_2$. Thus, $L \lhd M$ and $[M : L] = 2$. We choose residual member $\varphi_{\mathrm{res}} \in \mathcal{E}(R_{\mathrm{res}}, s_{\mathrm{res}})$ from (ρ, κ) or (Σ_0, λ), with quasi-isolated s_{res}, and $\varphi_{\mathrm{res}}(z) = -\varphi_{\mathrm{res}}(1)$. By [208, Lemma 6.13], choose extension $\widehat{\varphi}_{\mathrm{res}}$ with $\mathbb{Q}(\widehat{\varphi}_{\mathrm{res}}) = \mathbb{Q}(\varphi_{\mathrm{res}})$, since f is odd. We choose ordinary models $\widehat{\rho}_{\mathrm{act}}, \widehat{\rho}_{\mathrm{res}}$ of $\widehat{\varphi}_{\mathrm{act}}, \widehat{\varphi}_{\mathrm{res}}$. Moreover, $\widehat{\rho}_{\mathrm{act}}(z) = -I$ and $\widehat{\rho}_{\mathrm{res}}(z) = -I$. We write the common generator $F_0 = \Phi$ on $L_{\mathrm{act}}, R_{\mathrm{res}}$. Moreover, we write component field elements $\Phi_{\mathrm{act}}, \Phi_{\mathrm{res}}$ of F_0, with $\Phi_{\mathrm{act}}^f = c_{\mathrm{act}}$ and $\Phi_{\mathrm{res}}^f = c_{\mathrm{res}}$. We put $T_{\mathrm{res}} = \widehat{\rho}_{\mathrm{res}}(\Phi_{\mathrm{res}})$, with $T_{\mathrm{res}}^f = \rho_{\mathrm{res}}(c_{\mathrm{res}})$. Thus, $(\rho_{\mathrm{act}} \otimes \rho_{\mathrm{res}})(z,z) = I$. Write quotient map $\pi_L : L_{\mathrm{act}} \times R_{\mathrm{res}} \to L$, with $\ker \pi_L = \langle (z,z)\rangle$. Moreover, we choose character $\varphi \in \mathrm{Irr}(L)$, with $\varphi \circ \pi_L = \varphi_{\mathrm{act}} \boxtimes \varphi_{\mathrm{res}}$. Put $V_L = V_{\mathrm{act}} \otimes V_{\mathrm{res}}$ and $\rho_L : L \to \mathrm{GL}(V_L)$, and $\rho_L \circ \pi_L = \rho_{\mathrm{act}} \boxtimes \rho_{\mathrm{res}}$. On V_L, we put $T_L = T_{\mathrm{act}} \otimes T_{\mathrm{res}}$. Thus, $T_L \rho_L(l) T_L^{-1} = \rho_L(F_0 l F_0^{-1})$ for $l \in L$. Moreover, $T_L^f = \rho_L(c_{\mathrm{act}}c_{\mathrm{res}}) = \rho_L(c)$. We find $c = c_{\mathrm{act}}c_{\mathrm{res}} \in L$. Thus, put $H = \langle L, F_0\rangle \le A_Q$, with $H \cap M = L$ and $A_Q = MH$. We choose extension $\widehat{\varphi} \in \mathrm{Irr}(H)$, with $\widehat{\rho}_L(F_0) = T_L$. Thus, $\widehat{\rho}_L(F_0)^f = \widehat{\rho}_L(c)$. Choose $c_{\mathrm{norm}} \in M \setminus L$. Write factor pairs $\mathcal{P}_{\mathrm{act}} = \{\varphi_{\mathrm{act}}^+, \varphi_{\mathrm{act}}^-\}$ and $\mathcal{P}_{\mathrm{res}} = \{\varphi_{\mathrm{res}}^+, \varphi_{\mathrm{res}}^-\}$. Since $\bar{\nu}_{\mathrm{act}}(c_{\mathrm{norm}}) \neq 1$, $(\varphi_{\mathrm{act}}^\epsilon, \varphi_{\mathrm{res}}^\delta)^{c_{\mathrm{norm}}} = (\varphi_{\mathrm{act}}^{-\epsilon}, \varphi_{\mathrm{res}}^{-\delta})$. Thus, $I_M(\varphi) = L$. By Clifford theory [43], $\psi = \mathrm{Ind}_L^M \varphi \in \mathrm{Irr}(M)$. With $\Pi_1 = \varphi$, put $\Pi_2 = \varphi^{c_{\mathrm{norm}}}$. Accordingly, put $\widehat{\psi}_0 = \mathrm{Ind}_H^{A_Q} \widehat{\varphi}$. Since $A_Q = MH$, $\mathrm{Res}_M^{A_Q} \widehat{\psi}_0 = \psi$. Thus, $\widehat{\psi}_0 \in \mathrm{Irr}(A_Q)$.

We put $S_{\mathrm{pure}} = (\mathcal{H}_p)_\psi$. Moreover, choose $S_0 \le S_{\mathrm{pure}}$ fixing $\varphi_{\mathrm{act}}, \varphi_{\mathrm{res}}$ separately. Whenever $\sigma \in S_0$, $\widehat{\psi}_0^\sigma = \widehat{\psi}_0$. Since $\mathrm{Res}_L^M \psi = \Pi_1 + \Pi_2$, we write action $S_{\mathrm{pure}} \to \mathrm{Sym}(\{\Pi_1, \Pi_2\})$. $S_0 \lhd S_{\mathrm{pure}}$ and $[S_{\mathrm{pure}} : S_0] \le 2$. By Lemma 4.34, we choose extension $\widehat{\psi}_1$ of ψ to A_Q, fixed by S_{pure}. If $h = 0$, we put $\psi = \varphi_{\mathrm{act}}$ and $\widehat{\psi}_1 = \widehat{\varphi}_{\mathrm{act}}$. Thus, $\widehat{\psi}_1^\sigma = \widehat{\psi}_1$ for $\sigma \in S_{\mathrm{pure}}$ and $h \ge 0$.

Moreover, we put $S_\psi = (B_Q \times \mathcal{H}_p)_\psi$ and $S_{\mathrm{mix}} = S_\psi/(A_Q \times \{1\})$. Put $\mathcal{X}_\psi = \{\theta \in \mathrm{Irr}(A_Q) : \mathrm{Res}_M^{A_Q}\theta = \psi\}$. Moreover, write action $\theta \mapsto \theta^{b\sigma}$ on $\mathcal{X}_\psi$ for $(b,\sigma) \in S_\psi$. $A_Q \times \{1\}$ acts trivially on $\mathrm{Irr}(A_Q)$. We embed S_{pure} by $\sigma \mapsto (1,\sigma)(A_Q \times \{1\})$. Since $B_Q/A_Q \simeq C_2$, $S_{\mathrm{pure}} \lhd S_{\mathrm{mix}}$ and $[S_{\mathrm{mix}} : S_{\mathrm{pure}}] \le 2$. Moreover, $A_Q/M \simeq C_f$, with f odd. $\widehat{\psi}_1 \in \mathcal{X}_\psi^{S_{\mathrm{pure}}}$. By Lemma 4.34, choose $\widehat{\psi} \in \mathrm{Irr}(A_Q)$ fixed by S_{mix}. Whenever $(b,\sigma) \in S_\psi$, $\widehat{\psi}^{b\sigma} = \widehat{\psi}$. Moreover, put rational parent $\Xi_B = \widehat{\Theta} \in \mathrm{Irr}(B)$ from Lemma 4.32. Choose constituent extension $\widehat{\chi} \in \mathrm{Irr}(A)$ of χ from Ξ_B. By Lemma 4.32, we have that $\widehat{\chi}^{b\sigma} = \widehat{\chi}$ for $(b,\sigma) \in S_\chi = (B \times \mathcal{H}_p)_\chi$.

Moreover, we choose $\delta \in D_Q \setminus M$. We have that $\chi^\delta \neq \chi$ and $\psi^\delta \neq \psi$. Thus, $I_B(\chi) = A$ and $I_{B_Q}(\psi) = A_Q$. Choose ordinary models $\rho_\chi : A \to \mathrm{GL}(V_X)$ and $\rho_\psi : A_Q \to \mathrm{GL}(V_M)$ of $\widehat{\chi}, \widehat{\psi}$. Moreover, $\alpha_\chi = \alpha_\psi = 1$. With $(b,\sigma) \in S$, $\rho_\chi^{b\sigma} \simeq \rho_\chi$ and $\rho_\psi^{b\sigma} \simeq \rho_\psi$. Thus, write mixed functions $m_X = 1_A$ and $m_M = 1_{A_Q}$. $\mathrm{C}_B(X) \le D$, since F_r^i is outer modulo $\mathrm{Inndiag}(X)$ for $0 < i < f$. By $\mathrm{C}_D(X) = \langle z\rangle$, $\mathrm{C}_B(X) = \langle z\rangle$. In particular, $\rho_\chi(z) = -I_{V_X}$ and $\rho_\psi(z) = -I_{V_M}$. We choose $\omega \in \mathrm{Irr}(\langle z\rangle)$, with $\omega(z) = -1$.

Moreover, we put $\mathcal{B} = B_{\chi^{\mathcal{H}_p}}$ and $\mathcal{B}_Q = (B_Q)_{\psi^{\mathcal{H}_p}}$. $\mathcal{B}_Q = (B_Q)_{\chi^{\mathcal{H}_p}}$ by the correspondence $\chi \mapsto \psi$, and $\mathcal{B} = X\mathcal{B}_Q$ and $X \cap \mathcal{B}_Q = M$. Thus, $(\mathcal{B}, X, \chi)_{\mathcal{H}_p} \succeq_c (\mathcal{B}_Q, M, \psi)_{\mathcal{H}_p}$. We put $E = (\mathrm{Aut}(X)_P)_{\chi^{\mathcal{H}_p}}$. Since Q is characteristic in P, $a(Q) = Q$ and $a(M) = M$ for $a \in E$. Write conjugation map $\kappa : B \to \mathrm{Aut}(X)$, with $\kappa(b)(x) = bxb^{-1}$. Then,

by [208, Lemma 6.12], $\kappa(B) = \mathrm{Aut}(X)$. $\mathcal{B} = X\mathrm{N}_{\mathcal{B}}(P)$ and $\mathcal{B}_Q = M\mathrm{N}_{\mathcal{B}}(P)$, and we put $\mathrm{Int}_X(M) = \{\mathrm{Int}(m) : m \in M\}$. Thus, $\kappa(\mathcal{B}) = \mathrm{Inn}(X)E$ and $\kappa(\mathcal{B}_Q) = \mathrm{Int}_X(M)E$. Put $\widehat{\mathcal{B}} = X \rtimes E$ and $\widehat{\mathcal{B}}_Q = M \rtimes E$. Moreover, we put $\widehat{\kappa}(x, a) = \mathrm{Int}(x)a$ for $(x, a) \in \widehat{\mathcal{B}}$. With $(x, a) \in \ker \widehat{\kappa}$, $a = \mathrm{Int}(x^{-1})$. Since $a(P) = P$, $x \in \mathrm{N}_X(P) \leq M$. Thus, $\mathrm{C}_{\widehat{\mathcal{B}}}(X) = \ker \widehat{\kappa} \leq \widehat{\mathcal{B}}_Q$. $\widehat{\kappa}(\widehat{\mathcal{B}}) = \kappa(\mathcal{B})$ and $\widehat{\kappa}(\widehat{\mathcal{B}}_Q) = \kappa(\mathcal{B}_Q)$. Then, by [141, Theorem 2.9], $(\widehat{\mathcal{B}}, X, \chi)_{\mathcal{H}_p} \succeq_c (\widehat{\mathcal{B}}_Q, M, \psi)_{\mathcal{H}_p}$. We choose projective models $\widehat{\mathcal{P}}_\chi, \widehat{\mathcal{P}}_\psi$ of χ, ψ on V_X, V_M. Moreover, the common scalar function is $\widehat{\lambda}$ on $\mathrm{C}_{\widehat{\mathcal{B}}}(X)$, with $\widehat{\lambda}|_{\langle z \rangle} = \omega$. Thus, $\widehat{\mathcal{P}}_\chi(z) = -I_{V_X}$ and $\widehat{\mathcal{P}}_\psi(z) = -I_{V_M}$. □

We set $q = r^f \equiv 3 \pmod 4$, with prime r and $n \geq 3$. We put $X = \mathrm{Spin}_{2n+1}(q)$ and $\widetilde{X} = \mathrm{GSpin}_{2n+1}(q)$. Moreover, we choose the Clifford norm $\nu : \widetilde{X} \to \mathbb{F}_q^\times$ (Milne [137, §24.h]). We have that $X = \ker \nu$ and $z = -1 \in X$. By Zhang [208, Lemma 6.12], we choose $(a, b) = (0, n)$. Consequently, we set $a = 0$ in Subsection 4.3, with no residual character ρ_{res}. We compute the value fields $K_\lambda = \mathbb{Q}(\chi_{\lambda,+}) = \mathbb{Q}(\chi_{\lambda,-})$. Moreover, we compute $\chi^\sigma_{\lambda,\pm}$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proposition 4.36. *We choose a partition $\lambda \vdash n$, with conjugate partition λ'. We take its unipotent character $\rho_\lambda \in \mathrm{Irr}(G_0)$, with $G_0 = \mathrm{GL}_n(q)$. Moreover, we put $K = n(n+1)/2$. We fix $m_\lambda \in \{0, 1\}$ with $m_\lambda \equiv \binom{n}{2} + \sum_j (\binom{\lambda_j}{2} - \binom{\lambda'_j}{2}) \pmod 2$, and we choose the constituents $\chi_{\lambda,+}, \chi_{\lambda,-}$ at $(a, b) = (0, n)$. Thus, we find $\chi_{\lambda,+}(1) = \chi_{\lambda,-}(1) = \frac{\rho_\lambda(1)}{2} \prod_{j=1}^n (q^j + 1)$. We have that $\mathbb{Q}(\chi_{\lambda,+}) = \mathbb{Q}(\chi_{\lambda,-}) = K_\lambda = \mathbb{Q}(\sqrt{(-1)^K q^{K+m_\lambda}})$. We choose the prime-field automorphism $F_r \in \mathrm{Aut}(X)$, of order f. Moreover, we choose extensions $\widehat{\chi}_{\lambda,\pm} \in \mathrm{Irr}(X \rtimes \langle F_r \rangle)$ with $\widehat{\chi}_{\lambda,\pm}|_X = \chi_{\lambda,\pm}$ and $\mathbb{Q}(\widehat{\chi}_{\lambda,\pm}) = K_\lambda$.*

Proof. We choose the maximal isotropic space W and its stabilizer $\widetilde{P} = \widetilde{L}U$ in $\widetilde{X}$. By Subsection 4.1, choose $\mathcal{E} = \bigwedge W$. Then, write $(c, A) = cL(A) \in \widetilde{L}$, with $c \in \mathbb{F}_q^\times$, $A \in G_0$ and $L(A) = \bigwedge A$. We find $\nu(c, A) = c^2 \det A$. Moreover, we choose the quadratic character $\eta_q \in \mathrm{Irr}(\mathbb{F}_q^\times)$ and put $\varepsilon = \eta_q \circ \nu$. Fix $\mu \in \mathrm{Irr}(\widetilde{P})$, with $\mu((c, A)u) = \eta_q(c)\rho_\lambda(A)$ for $u \in U$. Choose the dual group $G^* = \mathrm{GSp}_{2n}(q)$. Moreover, we choose the semisimple element $s_\mu \in G^*$ for μ, with multiplier -1. Take eigenspaces V_+, V_-, with $s_\mu|_{V_+} = 1$, $s_\mu|_{V_-} = -1$ and $\dim V_+ = \dim V_- = n$. Choose V_+, V_- as paired maximal isotropic spaces in $\mathbb{F}_q^{2n}$. Thus, $C_{G^*}(s_\mu)$ is the Siegel Levi $\mathrm{GL}_n(q) \times \mathbb{F}_q^\times$. $g|_{V_-}$ is determined by $g|_{V_+}$ and the multiplier $\nu^*(g)$, for $g \in C_{G^*}(s_\mu)$. By Harish-Chandra induction of μ, $\Theta = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{X}} \mu \in \mathrm{Irr}(\widetilde{X})$. Moreover, take its Jordan label (s_μ, ρ_λ). The image $s_\mu Z(G^*) \in G^*_{\mathrm{ad}}$ belongs to $(a, b) = (0, n)$.

Choose $w_* = R_1 \cdots R_n$ from Subsection 4.1. Moreover, $\nu(w_*) = 1$. We take paired spaces $W_+ = W, W_-$, with $w_*(W_\pm) = W_\mp$. $w_*(c, A)w_*^{-1} = (c \det A, A^{-T})$ and $w_*^2 = z^{n+\binom{n}{2}} = z^K$. Thus, the operators on $\mathcal{E}$ give that $\bigwedge^n A = \det A$ on $\bigwedge^n W$. $R_j^2 = z$ and $R_j R_k = z R_k R_j$ for $j \neq k$ [137, §24.h]. Thus, we count n squared factors R_j^2 and $\binom{n}{2}$ factors z in w_*^2. Fix $\tau : G_0 \to G_0$, $A \mapsto A^{-T}$. Then, $\rho_\lambda^\tau = \rho_\lambda$ and $(\mu|_{\widetilde{L}})^{w_*} = (\mu|_{\widetilde{L}})(\varepsilon|_{\widetilde{L}})$. By opposite parabolics $\widetilde{P}, w_*\widetilde{P}w_*^{-1}$, $\Theta\varepsilon = \Theta$. If $\gamma \in \mathrm{Irr}(\mathbb{F}_q^\times)$ and $\Theta(\gamma \circ \nu) = \Theta$, $\gamma(c^2) = 1$ for $c \in \mathbb{F}_q^\times$. Thus, we have that $\gamma \in \{1, \eta_q\}$. Thus, $T_\Theta = \{\gamma \circ \nu : \Theta(\gamma \circ \nu) = \Theta\} = \{1, \varepsilon\}$.

We put $P = \widetilde{P} \cap X$ and $L = \widetilde{L} \cap X$ and choose $\gamma \in \mathrm{Irr}(\mathbb{F}_q^\times)$ with $\mu(\gamma \circ \nu)|_{\widetilde{L}} = \mu|_{\widetilde{L}}$. Thus, $\gamma(c^2) = 1$ and $\gamma \in \{1, \eta_q\}$. $\rho_\lambda(\eta_q \circ \det) \neq \rho_\lambda$, with dual semisimple labels -1 and 1. Then, by Clifford theory for $\widetilde{L}/L$, $\mu|_L \in \mathrm{Irr}(L)$. We find $\nu(\widetilde{P}) = \mathbb{F}_q^\times = \nu(\widetilde{X})$. Thus, $|\widetilde{P}\backslash\widetilde{X}/X| = 1$ and $\Gamma = \Theta|_X = \mathrm{Ind}_P^X(\mu|_P)$. We find $|T_\Theta| = 2$. By Clifford theory for $\widetilde{X}/X$, $\Gamma = \chi_{\lambda,+} + \chi_{\lambda,-}$, with distinct $\widetilde{X}$-conjugate constituents $\chi_{\lambda,\pm}$. $\eta_q(-1) = -1$ and $\chi_{\lambda,\pm}(z) = -\chi_{\lambda,\pm}(1)$. Next, $[X : P] = \prod_{j=1}^n (q^j + 1)$ from $|X|, |L|, |U|$. Thus, $\chi_{\lambda,\pm}(1) = \rho_\lambda(1)[X : P]/2$.

By [208, Lemma 4.4], the model $\overline{D}_\lambda$ on V_λ is rational on $\mathrm{PGL}_n(q)$, and its graph operator S_λ satisfies $S_\lambda^2 = I$ over $\overline{\mathbb{Q}}$. Its inflation D_λ to G_0 affords ρ_λ. Write $\mathcal{L}_\lambda = \{B \in \mathrm{End}_{\overline{\mathbb{Q}}}(V_\lambda \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) : BD_\lambda(A) = D_\lambda(A^{-T})B \ (A \in G_0)\}$. By [95, Corollary 1.6], we have that $\dim_{\overline{\mathbb{Q}}} \mathcal{L}_\lambda = 1$. Thus, the normalized operators are $\mathcal{S}_\lambda = \{S_\lambda, -S_\lambda\}$. The contents satisfy $\sum_{(i,j)\in\lambda}(j - i) = \sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2}$, so the exponent in [208, (4.1)] has parity m_λ. Thus, [208, (4.2)] gives $S_\lambda^\sigma = (\sigma(\sqrt{q})/\sqrt{q})^{m_\lambda} S_\lambda$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The two extensions $\widehat{\rho}_\lambda^\pm$ have values $\pm \mathrm{Tr}(D_\lambda(A)S_\lambda)$ on $A\tau$. Thus, their fields of values are $\mathbb{Q}$ if $m_\lambda = 0$ and $\mathbb{Q}(\sqrt{q})$ if $m_\lambda = 1$.

Take the based graph $\tau_{\mathrm{b}} = \mathrm{Int}(J)\tau$ with antidiagonal $J \in G_0$. Since $J^2, J\tau(J) \in Z(G_0)$, $S_{\lambda,\mathrm{b}} = D_\lambda(J)S_\lambda$ has square I_{V_λ}. Moreover, $D_\lambda(J)^{\pm 1}$ are rational, so $\mathbb{Q}(S_{\lambda,\mathrm{b}}) = \mathbb{Q}(S_\lambda)$. We put $D_\mu(c, A) = \eta_q(c)D_\lambda(A)$ and $D_\mu(u) = I_{V_\lambda}$ for $u \in U$ and choose the rational space $\mathcal{V} = \{f_0 : X \to V_\lambda : f_0(pg) = D_\mu(p)f_0(g) \ (p \in P, g \in X)\}$. Thus, we fix $R_\Gamma : X \to \mathrm{GL}_{\mathbb{Q}}(\mathcal{V})$ with $(R_\Gamma(x)f_0)(g) = f_0(gx)$. We choose $\mathcal{V}_{\overline{\mathbb{Q}}} = \mathcal{V} \otimes \overline{\mathbb{Q}}$ and $T \in \mathrm{End}_{\overline{\mathbb{Q}}}(\mathcal{V}_{\overline{\mathbb{Q}}})$, with $(Tf_0)(g) = \sum_{u\in U} S_\lambda f_0(w_*^{-1}ug)$, and we have that $(Tf_0)(u_0 g) = (Tf_0)(g)$ for $u_0 \in U$ and that $lUl^{-1} = U$ for $l \in L$. $S_\lambda D_\mu(w_*^{-1}lw_*) = D_\mu(l)S_\lambda$, since $\varepsilon|_L = 1$. Thus, $(Tf_0)(lg) = D_\mu(l)(Tf_0)(g)$. Whenever $x \in X$, $TR_\Gamma(x) = R_\Gamma(x)T$. We choose the endomorphism

kernels K_T, K_I for T and $I = I_{\mathcal{V}}$. Moreover, $\operatorname{supp} K_T = Pw_*P$ and $\operatorname{supp} K_I = P$. Fix f_0 supported on P, with $f_0(1) = v \in V_\lambda$. Since $w_*^{-1}Uw_* \cap P = 1$, $(Tf_0)(w_*) = S_\lambda v$. $K_T(w_*) \neq 0$ and $T \notin \overline{\mathbb{Q}}I$. Since $\Gamma = \chi_{\lambda,+} + \chi_{\lambda,-}$, $\operatorname{End}_{\overline{\mathbb{Q}}X}(\mathcal{V}_{\overline{\mathbb{Q}}}) = \overline{\mathbb{Q}}I \oplus \overline{\mathbb{Q}}T$.

Set $t = L(\operatorname{diag}(-1, 1, \dots, 1))$. Then, $tPt^{-1} = P$ and $tUt^{-1} = U$. Moreover, fix $J_t \in \mathrm{GL}_{\mathbb{Q}}(\mathcal{V})$, with $(J_t f_0)(g) = D_\mu(t) f_0(t^{-1}gt)$. We compute the kernel $K_T(w_*)$ with $D_\mu(t)$ and $D_\mu(w_*^{-1}tw_*)$. Since $w_*(c, A)w_*^{-1} = (c \det A, A^{-T})$, $J_t T J_t^{-1} = \varepsilon(t)T = -T$. We write $T^2 = A_0 I + B_0 T$, with $A_0, B_0 \in \overline{\mathbb{Q}}$, and $J_t T^2 J_t^{-1} = A_0 I - B_0 T = T^2$. Thus, $B_0 = 0$. We fix f_0 supported on P, with $f_0(1) = v$, and we compute $(T^2 f_0)(1)$ at $w_*^{-1}u'w_*^{-1}u$, for $u, u' \in U$. Since $w_*^{-2} \in Z(X)$, $w_*^{-1}u'w_*^{-1}u \in P$ exactly when $u' = 1$. We have that $D_\mu(w_*^{-2}) = (-1)^K I_{V_\lambda}$ and $S_\lambda^2 = I_{V_\lambda}$. Thus, $(T^2 f_0)(1) = |U|(-1)^K v$. We count n short positive roots ϵ_j and $\binom{n}{2}$ roots $\epsilon_j + \epsilon_k$ in U. We find $|U| = q^{n+\binom{n}{2}} = q^K$. Consequently, $T^2 = (-q)^K I$. We choose $b \in \overline{\mathbb{Q}}$ with $b^2 = (-q)^K$, and we choose the projectors $E_\pm = (I \pm T/b)/2$ and $\mathcal{V}_\pm = E_\pm \mathcal{V}_{\overline{\mathbb{Q}}}$. Since $T \notin \overline{\mathbb{Q}}I$, choose $\chi_{\lambda,\pm}$ on $\mathcal{V}_\pm$.

Fix $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then, $T^\sigma = (\sigma(\sqrt{q})/\sqrt{q})^{m_\lambda} T$. Moreover, we put $\epsilon_\lambda(\sigma) = (\sigma(\sqrt{q})/\sqrt{q})^{m_\lambda}\sigma(b)/b$. $E_\pm^\sigma = E_{\pm\epsilon_\lambda(\sigma)}$ and $\chi_{\lambda,\pm}^\sigma = \chi_{\lambda,\pm\epsilon_\lambda(\sigma)}$. Put $d_\lambda = b(\sqrt{q})^{m_\lambda}$. Thus, $\epsilon_\lambda(\sigma) = \sigma(d_\lambda)/d_\lambda$ and $d_\lambda^2 = (-1)^K q^{K+m_\lambda}$. $\chi_{\lambda,+} \neq \chi_{\lambda,-}$. We obtain $\operatorname{Stab}(\chi_{\lambda,\pm}) = \ker \epsilon_\lambda = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(d_\lambda))$. Thus, $\mathbb{Q}(\chi_{\lambda,\pm}) = \mathbb{Q}(d_\lambda) = K_\lambda$. When $(-1)^K q^{K+m_\lambda} \in (\mathbb{Q}^\times)^2$, $K_\lambda = \mathbb{Q}$. Also, $T/(\sqrt{q})^{m_\lambda}$ has rational entries. Since $(\sqrt{q})^{m_\lambda}/b = d_\lambda/b^2$, $E_\pm \in \operatorname{End}_{K_\lambda}(\mathcal{V} \otimes K_\lambda)$.

Finally, choose the field permutation $F_r : \mathcal{F} \to \mathcal{F}$ and its rational matrix P_{flag} on Π_{flag}. F_r fixes relative positions of $\mathcal{F}$. Thus, $P_{\mathrm{flag}} h = h P_{\mathrm{flag}}$ for $h \in \mathcal{H}_q$. $P_{\mathrm{flag}} e_\lambda = e_\lambda P_{\mathrm{flag}}$, and we set $P_L = P_{\mathrm{flag}}|_{V_\lambda}$, with $P_L^f = I_{V_\lambda}$. Moreover, $P_L D_\lambda(A) P_L^{-1} = D_\lambda(F_r(A))$ for $A \in G_0$. We find $\tau F_r = F_r \tau$. Since $\dim \mathcal{L}_\lambda = 1$, $P_L S_\lambda P_L^{-1} = a_\lambda S_\lambda$ for $a_\lambda \in \overline{\mathbb{Q}}^\times$. We find $a_\lambda^2 = a_\lambda^f = 1$. Since f is odd, $a_\lambda = 1$. Choose the rational operator Φ_r on $\mathcal{V}$, with $(\Phi_r f_0)(g) = P_L f_0(F_r^{-1}(g))$, and $\Phi_r^f = I$ and $\Phi_r R_\Gamma(g) \Phi_r^{-1} = R_\Gamma(F_r(g))$. Moreover, $F_r(w_*) = w_*$ and $F_r(U) = U$. $\Phi_r T = T \Phi_r$ and $\Phi_r E_\pm = E_\pm \Phi_r$, and we choose $\widehat{R}_\pm(g F_r^k) = (R_\Gamma(g)\Phi_r^k)|_{\mathcal{V}_\pm}$, with character $\widehat{\chi}_{\lambda,\pm}$. Thus, $\widehat{\chi}_{\lambda,\pm}|_X = \chi_{\lambda,\pm}$. $\widehat{\chi}_{\lambda,\pm}(g F_r^k) = \operatorname{Tr}(E_\pm R_\Gamma(g)\Phi_r^k)$ for $g \in X$ and $0 \leq k < f$, and $E_\pm R_\Gamma(g)\Phi_r^k$ has entries in K_λ. Since $\mathbb{Q}(\chi_{\lambda,\pm}) = K_\lambda$, $\mathbb{Q}(\widehat{\chi}_{\lambda,\pm}) = K_\lambda$. □

Proposition 4.37. *We choose $X, q, r, f, n, \chi_{\lambda,\epsilon}$ from* Proposition 4.36*, with $\epsilon \in \{1, -1\}$. We choose an odd prime $p \neq r$ with odd $e = \operatorname{ord}_p(q)$ and $n = ew$. Moreover, we set $M = N_X(Q)$ from* Proposition 4.4*. We put $W_0 = C_e \wr \mathfrak{S}_w$. Moreover, we choose the local set $\mathcal{P}_M = \{\psi_{\delta_\epsilon,\eta} : \epsilon \in \{1, -1\},\ \eta \in \operatorname{Irr}_{p'}(W_0)\}$. We choose $\mathcal{U}_p = \{\chi_{\lambda,\epsilon} : p \nmid \chi_{\lambda,\epsilon}(1)\}$ and the e-quotient $\boldsymbol{\lambda}$ of λ. Thus, we choose the bijection $\Omega_{\mathrm{Sg}} : \mathcal{U}_p \to \mathcal{P}_M$, $\chi_{\lambda,\epsilon} \mapsto \psi_{\delta_\epsilon,\eta_{\boldsymbol{\lambda}}}$. We fix $i \in \mathbb{C}$ with $i^2 = -1$. Moreover, we have that $\Omega_{\mathrm{Sg}}(\chi^\sigma) = \Omega_{\mathrm{Sg}}(\chi)^\sigma$ for $\chi \in \mathcal{U}_p$ and $\sigma \in \mathcal{H}_p$. We choose the sign action $\epsilon \mapsto \epsilon(\sigma(i)/i)^K$ on $\chi_{\lambda,\epsilon}$ and $\psi_{\delta_\epsilon,\eta_{\boldsymbol{\lambda}}}$ and choose a diagonal automorphism δ of X, with $\delta \notin \operatorname{Inn}(X)$. Thus, $\chi_{\lambda,\epsilon}^\delta = \chi_{\lambda,-\epsilon}$ and $\psi_{\delta_\epsilon,\eta}^\delta = \psi_{\delta_{-\epsilon},\eta}$. We choose the field extensions $(\widehat{\chi}, \widehat{\psi})$ from* Proposition 4.35 *for $\psi = \Omega_{\mathrm{Sg}}(\chi)$.*

Proof. $\langle q \bmod p \rangle$ has odd order e. Thus, $p \nmid q^j + 1$ for $j \geq 1$. $p \nmid \chi_{\lambda,\epsilon}(1)$ exactly when $p \nmid \rho_\lambda(1)$, and we choose $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(e-1)})$, with $c = \sum_{j=0}^{e-1} |\lambda^{(j)}|$. Moreover, choose the abacus $\mathcal{A}_\lambda$, with $N \equiv 0 \pmod e$ beads. We number the runners of $\mathcal{A}_\lambda$ by $0, \dots, e-1$, with residue $b \bmod e$ for a bead b. Choose the hook multiset $\mathcal{H}(\alpha)$ for a partition α. Thus, put $\mathcal{B} = \biguplus_{j=0}^{e-1} \mathcal{H}(\lambda^{(j)})$. $\{h \in \mathcal{H}(\lambda) : e \mid h\} = e\mathcal{B}$, with multiplicities in $\mathcal{B}$, and we choose a bead b and gap $g < b$, with $b \equiv g \pmod e$. Indeed, we take hook length $h = b - g$ for λ and h/e for $\lambda^{(b \bmod e)}$. We put $d_p = v_p(q^e - 1)$. Then, $v_p(q^{ej} - 1) = d_p + v_p(j)$ for $j \geq 1$. By the hook formula for $\rho_\lambda(1)$, $v_p(\rho_\lambda(1)) = (w - c)d_p + v_p(w!) - \sum_{h \in \mathcal{B}} v_p(h)$. Moreover, we take $\varphi_j \in \operatorname{Irr}(\mathfrak{S}_{|\lambda^{(j)}|})$ for the partition $\lambda^{(j)}$. With $d_j = \varphi_j(1)$, we put $m_{\boldsymbol{\lambda}} = c!/\prod_j |\lambda^{(j)}|!$. Thus, $c!/\prod_{h \in \mathcal{B}} h = m_{\boldsymbol{\lambda}} \prod_j d_j \in \mathbb{Z}$. $v_p(\rho_\lambda(1)) \geq (w - c)d_p + v_p(w!) - v_p(c!)$. With $c < w$, $v_p(\rho_\lambda(1)) > 0$. Alternatively, with $c = w$, $\operatorname{core}_e(\lambda) = \varnothing$. We fix $\eta_{\boldsymbol{\lambda}} \in \operatorname{Irr}(W_0)$, with $\eta_{\boldsymbol{\lambda}}(1) = m_{\boldsymbol{\lambda}} \prod_j d_j$, and $v_p(\rho_\lambda(1)) = v_p(\eta_{\boldsymbol{\lambda}}(1))$. Conversely, we choose $\boldsymbol{\lambda}$ of total size w. We place the beads on runner j according to $\lambda^{(j)}$. $\lambda \vdash ew$ is unique with $\operatorname{core}_e(\lambda) = \varnothing$ and quotient $\boldsymbol{\lambda}$. Thus, we choose the bijection $\lambda \mapsto \eta_{\boldsymbol{\lambda}}$, with $p \nmid \rho_\lambda(1)$ and $\eta_{\boldsymbol{\lambda}} \in \operatorname{Irr}_{p'}(W_0)$.

We take ordered coordinate blocks $B_j = \{(j-1)e + 1, \dots, je\}$ for $1 \leq j \leq w$. We choose the cycle $\pi_j = ((j-1)e + 1, \dots, je)$. Moreover, put $v_j = L(\pi_j)$, $I_j = \prod_{k \in B_j} R_k$ and $E_j = \prod_{k \in B_j} T_k(-1)$. We order the factors R_k by increasing $k \in B_j$. With $c_j = I_j v_j$, put $v = \prod_{j=1}^w v_j$ and $F' = \operatorname{Int}(v) F_q$. Since e is odd, $\nu(v_j) = (-1)^{e-1} = 1$. $v_j I_j = I_j v_j$, with permutation sign $(-1)^{e-1} = 1$. By Lemma 4.11, we calculate X, M, Q in the F'-model. Moreover, choose $\widetilde{\mathbf{T}}_0 = \{c \prod_{k=1}^n T_k(t_k) : c, t_k \in \overline{\mathbb{F}}_q^\times\}$. Put $\mathbf{T}_0 = \ker(\nu|_{\widetilde{\mathbf{T}}_0})$ and $T = \mathbf{T}_0^{F'}$. Moreover, we choose $\iota \in \overline{\mathbb{F}}_q$ with $\iota^2 = -1$ and put $h_j = (-\iota)^e E_j$. $h_j h_k = h_k h_j$ and $h_j^2 = z$. Since $F'(h_j) = z h_j$, choose $H = \langle z, h_1 h_2, \dots, h_{w-1} h_w \rangle \leq T$. $H/\langle z \rangle \simeq E_{\mathrm{even}} = \{(x_1, \dots, x_w) \in \mathbb{F}_2^w : \sum_j x_j = 0\}$ and $(h_j h_{j+1})^2 = 1$. Thus, $H \simeq C_2^w$, including $w = 1$. $|T| = (q^e - 1)^w$ from the positive cycles $\pi_1, \dots, \pi_w$ on $X(\mathbf{T}_0)$, and $q^e \equiv 3 \pmod 4$ and $|T|_2 = 2^w$. Thus, $H = T_2$.

We choose $\widehat{H} = \langle z, h_1, \dots, h_w \rangle$. Take $\xi_{\widehat{H}} \in \operatorname{Irr}(\widehat{H})$, with $\xi_{\widehat{H}}(z) = -1$ and $\xi_{\widehat{H}}(h_j) = i$. Since $h_j^2 = z$ and

$i^2=-1$, $\xi_{\widehat{H}}(h_j)^2=\xi_{\widehat{H}}(z)=-1$. Choose the basis $\{h_j\langle z\rangle : 1\le j\le w\}$ of $\widehat{H}/\langle z\rangle\simeq\mathbb{F}_2^w$ and set $T=H\times T_{\rm odd}$ and $\theta(ht)=\xi_{\widehat{H}}(h)$ for $h\in H, t\in T_{\rm odd}$. Thus, $\theta(z)=-1$ and $\theta|_{T_{\rm odd}}=1$. We choose the toral orbit $\mathcal{O}_{\rm quad}$ at $(a,b)=(0,n)$. Moreover, on every positive cycle π_j, the dual eigenvalues are $\{\iota,-\iota\}$ by Subsection 4.3. $\bigcup_{j=1}^w B_j=\{1,\dots,n\}$ and $n=ew$. We choose the block sign group $D_{\rm sign}=\langle c_j^e T : 1\le j\le w\rangle\simeq C_2^w$. Moreover, $D_{\rm sign}$ is transitive on $\{\vartheta\in{\rm Irr}(H):\vartheta(z)=-1\}$. $\theta\in\mathcal{O}_{\rm quad}$. Moreover, $\mathcal{O}_{\rm quad}=\{\vartheta\in{\rm Irr}(T):\vartheta(z)=-1,\ \vartheta|_{T_{\rm odd}}=1\}$.

We choose the adjacent block permutation $s_j=(B_j,B_{j+1})$, with index permutation $\overline{s}_j=(j,j+1)\in\mathfrak{S}_w$. We take matching coordinate positions in $s_j(B_j)=B_{j+1}$ and $s_j(B_{j+1})=B_j$. Moreover, we put $b_j=E_jL(s_j)$ for $1\le j<w$. $\nu(E_j)=\nu(L(s_j))=-1$ and $\nu(b_j)=1$, and $b_jv=vb_j$, $c_jv=vc_j$ and $F_r(b_j)=b_j, F_r(c_j)=c_j$. Thus, we choose the images b_jT, c_jT generating $M/T=C_{2e}\wr\mathfrak{S}_w$. We find $M/T=C_{W(B_n)}(\pi_1\cdots\pi_w)$. Choose the full sign change ϵ_j on B_j. Moreover, $c_jT=\pi_j\epsilon_j$, with $|\pi_j|=e$ and $|\epsilon_j|=2$. $\pi_j\epsilon_j=\epsilon_j\pi_j$ and $\langle\pi_j,\epsilon_j\rangle=C_{2e,j}$. $b_jC_{2e,j}b_j^{-1}=C_{2e,j+1}$ and $b_jC_{2e,j+1}b_j^{-1}=C_{2e,j}$ in M/T. By Subsection 4.1, $c_j^{2e}=z^{e(e+1)/2}$ and $[c_j,c_k]=z$ for $j\ne k$. Moreover, $b_j^2=zh_jh_{j+1}$. $b_jc_jb_j^{-1}=c_{j+1}$ and $b_jc_{j+1}b_j^{-1}=zc_j$. Indeed, $I_j^2=z^{e(e+1)/2}$ and $I_jI_k=z^{e^2}I_kI_j$ for $j\ne k$. $L(s_j)R_kL(s_j)^{-1}=R_{s_j(k)}$ and $E_jR_kE_j^{-1}=zR_k$ for $k\in B_j$. Thus, $z^e=z$ in $b_jc_{j+1}b_j^{-1}$. $[E_j,E_k]=1$ and $L(s_j)E_kL(s_j)^{-1}=E_{\overline{s}_j(k)}$. $b_jb_{j+1}b_j=b_{j+1}b_jb_{j+1}$ and $b_jb_k=b_kb_j$ for $|j-k|>1$. Moreover, we choose $N_0=\langle H,c_1,\dots,c_w,b_1,\dots,b_{w-1}\rangle$. Choose one reduced permutation lift b_π for $\pi\in\mathfrak{S}_w$ and write each $x\in N_0$ as $hc_1^{u_1}\cdots c_w^{u_w}b_\pi$, with $h\in H$ and $0\le u_j<2e$. In particular, from the Weyl image xT, we determine $(u_1,\dots,u_w,\pi)$. Thus, $N_0\cap T=H$ and $M=TN_0$.

$c_jh_jc_j^{-1}=zh_j$ and $c_jh_kc_j^{-1}=h_k$ for $k\ne j$. Moreover, we set $x=hc_1^{u_1}\cdots c_w^{u_w}b_\pi\in N_0$. $\theta^x|_H=\theta|_H$ exactly when $u_1\equiv\cdots\equiv u_w\pmod 2$, and $xT_{\rm odd}x^{-1}=T_{\rm odd}$ and $\theta|_{T_{\rm odd}}=1$. Thus, $\theta^x=\theta$ exactly when $u_1\equiv\cdots\equiv u_w\pmod 2$. We put $c_0=c_1\cdots c_w$ and $M_\theta=\{m\in M:\theta^m=\theta\}$, and $M_\theta/T=\langle c_j^2T,b_jT,c_0T\rangle$. Moreover, we choose $W_0=\langle c_j^2T,b_jT\rangle\simeq C_e\wr\mathfrak{S}_w$. We have that $c_0^eT\in Z(M_\theta/T)$ and $|c_0^eT|=2$, and that $c_0^eT\notin W_0$. Thus, $M_\theta/T=\langle c_0^eT\rangle\times W_0$ and $[M:M_\theta]=2^{w-1}$.

With $a=(e+1)/2$, we put $\beta=(-1)^a$ and $\kappa=i^{w(w+e)/2}$. We choose $N_\theta=N_0\cap M_\theta=\langle H,c_j^2,b_j,c_0\rangle$. Moreover, take $\Lambda_0|_H=\theta|_H$ and $\Lambda_0(c_j^2)=\beta$, $\Lambda_0(b_j)=1$, $\Lambda_0(c_0)=\kappa$. $\beta^e=(-1)^{e(e+1)/2}$ and $\theta(zh_jh_{j+1})=1$, and $c_0^2=z^{\binom{w}{2}}\prod_j c_j^2$. Thus, $\kappa^2=(-1)^{\binom{w}{2}}\beta^w$. $c_j^2c_k^2=c_k^2c_j^2$ and $b_jc_k^2b_j^{-1}=c_{\overline{s}_j(k)}^2$, and $b_jc_0b_j^{-1}=c_0$, with the two central signs $z^2=1$. By the normal form in N_0, we fix $\Lambda_0\in{\rm Irr}(N_\theta)$ with $\Lambda_0(1)=1$. We have that $N_\theta\cap T=H$ and $\Lambda_0|_H=\theta|_H$. Thus, we fix $\Lambda:M_\theta\to\mathbb{C}^\times$, with $\Lambda(tn)=\theta(t)\Lambda_0(n)$ for $t\in T, n\in N_\theta$. Whenever $n\in N_\theta$, $\theta^n=\theta$. Whenever $x,y\in M_\theta$, $\Lambda(xy)=\Lambda(x)\Lambda(y)$. Moreover, we choose the quotient projections $\pi_2:M_\theta\to\langle c_0^eT\rangle\simeq C_2$ and $\pi_0:M_\theta\to W_0$. Choose $\delta_\epsilon\in{\rm Irr}(C_2)$ with $\delta_\epsilon(c_0^eT)=\epsilon$, and we fix $\eta\in{\rm Irr}(W_0)$ and $\psi_{\delta_\epsilon,\eta}={\rm Ind}_{M_\theta}^M(\Lambda(\delta_\epsilon\circ\pi_2)(\eta\circ\pi_0))$. Then, by Clifford theory for (M,T,θ) [43], we choose the bijection $(\epsilon,\eta)\mapsto\psi_{\delta_\epsilon,\eta}$ onto ${\rm Irr}(M\mid\theta)$. We find $\psi_{\delta_\epsilon,\eta}(1)=2^{w-1}\eta(1)$. Thus, $\mathcal{P}_M={\rm Irr}_{p'}(M\mid\theta)$.

We fix $\eta\in{\rm Irr}(W_0)$. We take its representation $D_\eta:W_0\to{\rm GL}_{\mathbb{Q}(\zeta_e)}(V_\eta)$ from linear characters of C_e. Moreover, we take rational models $D_j:\mathfrak{S}_{|\lambda^{(j)}|}\to{\rm GL}_{d_j}(\mathbb{Q})$ of φ_j. $\mathbb{Q}(\eta)\subseteq\mathbb{Q}(\zeta_e)$. $e\mid p-1$. Thus, $\sigma(\zeta_e)=\zeta_e$ and $\eta^\sigma=\eta$ for $\sigma\in\mathcal{H}_p$. $\theta^\sigma=\theta$, $\sigma(\beta)=\beta$ and $\sigma(\Lambda(b_j))=\Lambda(b_j)$, and we choose $\epsilon_\sigma=\epsilon\,\sigma(\kappa)/\kappa$. Moreover, $\psi_{\delta_\epsilon,\eta}^\sigma=\psi_{\delta_{\epsilon_\sigma},\eta}$. Since e is odd, $K-w(w+e)/2=(e^2-1)w^2/2\equiv 0\pmod 2$. Thus, $\sigma(\kappa)/\kappa=(\sigma(i)/i)^K$. Since ${\rm ord}_p(q)=e$ is odd, [156, Lemma 2.3(d)] gives $\sigma(\sqrt{q})=\sqrt{q}$ for $\sigma\in\mathcal{H}_p$. By the global projectors $E_\pm$ in Proposition 4.36, $\chi_{\lambda,\epsilon}^\sigma=\chi_{\lambda,\epsilon(\sigma(i)/i)^K}$. Thus, $\Omega_{\rm Sg}(\chi^\sigma)=\Omega_{\rm Sg}(\chi)^\sigma$.

We have that $\nu(E_1)=-1\notin(\mathbb{F}_q^\times)^2$, and that $[E_1,c_0]=z$ and $[E_1,c_j^2]=1$. Moreover, with $w\ge 2$, $[E_1,b_1]=E_1E_2=zh_1h_2$. $\theta(E_1E_2)=1$ and $[E_1,b_j]=1$ for $j>1$, and we choose the diagonal action δ on M from conjugation by E_1. Thus, $\psi_{\delta_\epsilon,\eta}^\delta=\psi_{\delta_{-\epsilon},\eta}$. We choose the global diagonal representative t from Proposition 4.36. We find $J_tE_\epsilon J_t^{-1}=E_{-\epsilon}$. Thus, $\chi_{\lambda,\epsilon}^\delta=\chi_{\lambda,-\epsilon}$. We choose the signs ϵ in $\Omega_{\rm Sg}(\chi_{\lambda,\epsilon})=\psi_{\delta_\epsilon,\eta_{\boldsymbol{\lambda}}}$.

Moreover, we choose the field lift u in the F'-model from Lemma 4.11. $uxu^{-1}=F_r(x)$ for $x\in M$ and $u^f=v^{-1}$. We find $c_j^2=z^{e(e+1)/2}v_j^2$. Since $2a=e+1$, $v_j=z^a(c_j^2)^a$. We find $\Lambda(v_j)=(-1)^a\beta^a=1$. We choose the central rotation $\overline{v}=vT\in Z(W_0)$, with $|\overline{v}|\mid e$. Thus, $D_\eta(\overline{v}^{-1})=\xi I_{V_\eta}$, with $\xi=\eta(\overline{v}^{-1})/\eta(1)\in\mu_e$. We put $D_r={\rm ord}_p(r)$, with $D_r/\gcd(D_r,f)=e$, and $\{\gamma^f:\gamma\in\mu_{D_r}\}=\mu_e$. Moreover, choose $\gamma\in\mu_{D_r}$ with $\gamma^f=\xi$. $D_r\mid p-1$ and $\sigma(\gamma)=\gamma$ for $\sigma\in\mathcal{H}_p$. $F_r(h)=h$ for $h\in H$ and $F_r(c_j)=c_j, F_r(b_j)=b_j$. Thus, $\theta^{F_r}=\theta$. Moreover, choose $V_{\delta_\epsilon,\eta}=V_\eta$ and $R_{\delta_\epsilon,\eta}:M_\theta\to{\rm GL}(V_\eta)$. Whenever $x\in M_\theta$, $R_{\delta_\epsilon,\eta}(x)=\Lambda(x)\delta_\epsilon(\pi_2(x))D_\eta(\pi_0(x))$. $R_{\delta_\epsilon,\eta}(F_r(x))=R_{\delta_\epsilon,\eta}(x)$ and $R_{\delta_\epsilon,\eta}(v^{-1})=\xi I_{V_\eta}$. Thus, we choose $\widehat{R}_{\delta_\epsilon,\eta}$ on $M_\theta\langle u\rangle$, with $\widehat{R}_{\delta_\epsilon,\eta}(u)=\gamma I_{V_\eta}$. $\widehat{R}_{\delta_\epsilon,\eta}|_{M_\theta}=R_{\delta_\epsilon,\eta}$ and $(\gamma I_{V_\eta})^f=\xi I_{V_\eta}$, and we choose $\widehat{\psi}_0={\rm Ind}_{M_\theta\langle u\rangle}^{M\langle u\rangle}{\rm Tr}\,\widehat{R}_{\delta_\epsilon,\eta}$. Moreover, $\widehat{\psi}_0|_M=\psi_{\delta_\epsilon,\eta}$ and $\widehat{\psi}_0^\sigma=\widehat{\psi}_0$ for $\sigma\in(\mathcal{H}_p)_{\psi_{\delta_\epsilon,\eta}}$. We choose the paired character $\chi=\chi_{\lambda,\epsilon}$ and $\psi=\psi_{\delta_\epsilon,\eta_{\boldsymbol{\lambda}}}$, and $\chi(z)/\chi(1)=\psi(z)/\psi(1)=-1$. Since f is odd, we take diagonal roots $\alpha_X=\alpha_M=-1$ in Lemma 4.11. $\alpha_X^f=\alpha_M^f=-1$ and $\sigma(\alpha_X)/\alpha_X=\sigma(\alpha_M)/\alpha_M=1$. Choose the pairing

$\Omega_{\mathrm{Sg}} = J_{\mathrm{odd}}|_{\mathcal{U}_p}$, with $\mathrm{core}_e(\lambda) = \varnothing$. By Proposition 4.35, set $A = X \rtimes \langle F_r \rangle$, $B = D \rtimes \langle F_r \rangle$ and $B_Q = N_B(Q)$. We set $D = \widetilde{X}/(\mathbb{F}_q^\times)^2$ and $A_Q = B_Q \cap A = M\langle u \rangle$ and choose the common mixed stabilizer $S = (B_Q \times \mathcal{H}_p)_{\chi,\psi}$. Moreover, we choose extensions $\widehat{\chi} \in \mathrm{Irr}(A)$ and $\widehat{\psi} \in \mathrm{Irr}(A_Q)$ with $\widehat{\chi}^{b\sigma} = \widehat{\chi}$ and $\widehat{\psi}^{b\sigma} = \widehat{\psi}$ for $(b,\sigma) \in S$. $\widehat{\chi}|_X = \chi$ and $\widehat{\psi}|_M = \psi$. Thus, $\alpha_\chi = \alpha_\psi = 1$ and $\omega_\chi(z) = \omega_\psi(z) = -1$. □

4.4 Regular spin characters

We take odd $q = r^f$, with prime r. We put $X = \mathrm{Spin}_{2n+1}(q)$, $\widetilde{X} = \mathrm{GSpin}_{2n+1}(q)$ and $z = -1$. Moreover, we set $F = F_q$ on $\mathbf{X}^* = \mathrm{PSp}_{2n}$. Choose the rational class $\mathcal{C}_s$ from Zhang [208, Lemma 6.12]. We choose the regular character $\Theta \in \mathrm{Irr}(\widetilde{X})$ for $\mathcal{C}_s$, with $\Theta|_X = \chi_{\mathrm{reg},+} + \chi_{\mathrm{reg},-}$. We choose a diagonal automorphism α, with $\chi^\alpha_{\mathrm{reg},+} = \chi_{\mathrm{reg},-}$. Moreover, we choose the symplectic factors $\mathcal{S}, \mathcal{S}$ in case (i). We choose $\mathcal{P}_{\mathrm{ii}} = \{\chi, \chi^\alpha\}$ in case (ii). We take semisimple $s \in (\mathbf{X}^*)^F$ and $C_s = C^\circ(s)^F$. Thus, we choose the regular label St_{C_s} and the semisimple label 1_{C_s}. We put $K_n = n(n+1)/2$. We choose $\mathcal{G} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and a primitive root ζ_r of order r. Moreover, we fix $\sigma \in \mathcal{G}$ with $\sigma(\zeta_r) = \zeta_r^b$, $b \in \mathbb{F}_r^\times$. We choose the quadratic character $\eta_q \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Moreover, we establish Proposition 4.38 for $\mathcal{C}_s$.

Proposition 4.38. *We choose the regular pair $\{\chi_{\mathrm{reg},+}, \chi_{\mathrm{reg},-}\}$ and the semisimple pair $\{\chi_{\mathrm{ss},+}, \chi_{\mathrm{ss},-}\}$ for Θ. Given $\sigma \in \mathcal{G}$, we choose $\epsilon_\sigma = \eta_q(b)^{K_n}$. Thus, $\chi^\sigma_{\mathrm{reg},\epsilon} = \chi_{\mathrm{reg},\epsilon\epsilon_\sigma}$ and $\chi^\sigma_{\mathrm{ss},\epsilon} = \chi_{\mathrm{ss},\epsilon\epsilon_\sigma}$ for $\epsilon \in \{1,-1\}$. We choose χ from either pair $\{\chi_{\mathrm{reg},\pm}\}, \{\chi_{\mathrm{ss},\pm}\}$, and we have that $\mathbb{Q}(\chi) = \mathbb{Q}$ when fK_n is even. Moreover, $\mathbb{Q}(\chi) = K_r = \mathbb{Q}(\sqrt{(-1)^{(r-1)/2}r})$ when fK_n is odd. We choose $\chi \in \{\chi_{\mathrm{reg},+}, \chi_{\mathrm{reg},-}\}$ from the source (U,ϑ), with simple-root coefficients 1, and we set $A = X \rtimes \langle F_r \rangle$ and $\Sigma_\chi = \mathrm{Stab}_{\mathcal{G}}(\chi)$. Thus, we choose an extension $\widehat{\chi} \in \mathrm{Irr}(A)$ with $\widehat{\chi}|_X = \chi$. We fix $a_\chi(\sigma) \in \mathrm{Irr}(A/X)$, with $\widehat{\chi}^\sigma = a_\chi(\sigma)\widehat{\chi}$ for $\sigma \in \Sigma_\chi$, and we have that $a_\chi(\sigma)(F_r) = (b/r)^{K_n}$ when $\omega_\chi(z) = -1$. Moreover, $a_\chi(\sigma)(F_r) = 1$ when $\omega_\chi(z) = 1$. We choose the corresponding semisimple constituent χ_{ss} and choose an extension $\widehat{\chi}_{\mathrm{ss}} \in \mathrm{Irr}(A)$ with $\widehat{\chi}_{\mathrm{ss}}|_X = \chi_{\mathrm{ss}}$. Thus, $\widehat{\chi}^\sigma_{\mathrm{ss}} = a_\chi(\sigma)\widehat{\chi}_{\mathrm{ss}}$ for $\sigma \in \Sigma_\chi$. We choose the second constituent $\chi' = \chi^\alpha$ of either pair $\{\chi_{\mathrm{reg},\pm}\}, \{\chi_{\mathrm{ss},\pm}\}$. By* Lemma 4.11*, we fix $a_{\chi'}$.*

Proof. First, choose a maximal unipotent subgroup $U \leq X$ for the roots $\alpha_j = e_j - e_{j+1}$, $1 \leq j < n$, and $\alpha_n = e_n$. We choose the simple-root coordinates $\pi_j : U \to \mathbb{F}_q$ on $U/[U,U]$. Moreover, take $\vartheta \in \mathrm{Irr}(U)$, with $\vartheta(u) = \zeta_r^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(\sum_{j=1}^n \pi_j(u))}$. We fix $H \in \{X, \widetilde{X}\}$ and $\mathcal{I}_H = \{f : H \to \overline{\mathbb{Q}} : f(ug) = \vartheta(u)f(g)\ (u \in U, g \in H)\}$. Moreover, choose the representation $R_H : H \to \mathrm{GL}(\mathcal{I}_H)$, with $(R_H(h)f)(g) = f(gh)$. We choose $\widetilde{\mathbf{X}} = \mathrm{GSpin}_{2n+1}$, with $\widetilde{\mathbf{X}}^{F_q} = \widetilde{X}$. In particular, $Z(\widetilde{\mathbf{X}}) \simeq \mathbb{G}_m$. By [17, §1.C], $\langle \mathrm{Ind}_U^{\widetilde{X}} \vartheta, \Theta \rangle_{\widetilde{X}} = 1$. Choose the unique submodule $V_\Theta \subseteq \mathcal{I}_{\widetilde{X}}$ with character Θ. By [95, Lemma 5.2] for U, $\langle \mathrm{Ind}_U^X \vartheta, \chi_{\mathrm{reg},+} + \chi_{\mathrm{reg},-} \rangle_X = 1$. We choose the unique occurring constituent χ with $\langle \mathrm{Ind}_U^X \vartheta, \chi \rangle_X = 1$.

We put $\varepsilon = \eta_q \circ \nu$. Then, by Clifford theory for $\widetilde{X}/X$ [43], $\Theta\varepsilon = \Theta$. Choose J on $\mathcal{I}_{\widetilde{X}}$, with $(Jf)(g) = \varepsilon(g)f(g)$. Thus, $J^2 = I$ and $JR_{\widetilde{X}}(g) = \varepsilon(g)R_{\widetilde{X}}(g)J$. Moreover, $J(V_\Theta) = V_\Theta$. We choose $V_\Theta|_X = V_+ \oplus V_-$, with characters $\chi_{\mathrm{reg},+}, \chi_{\mathrm{reg},-}$. Since $\varepsilon|_X = 1$, $J(V_\pm) = V_\pm$. We choose $g_0 \in \widetilde{X}$ with $\eta_q(\nu(g_0)) = -1$. Thus, $JR_{\widetilde{X}}(g_0) = -R_{\widetilde{X}}(g_0)J$. We choose the signs V_+, V_- with $J|_{V_+} = I_{V_+}$ and $J|_{V_-} = -I_{V_-}$. Accordingly, by [208, proof of Lemma 6.13], we choose $\widetilde{s} \in G^* = \mathrm{GSp}_{2n}(q)$. We choose odd $m \in \mathbb{Z}$, with $\sigma(\zeta) = \zeta^m$ for $\zeta \in \mu_{|G^*|}$. Moreover, $\widetilde{s}^m \sim_{G^*} c_m\widetilde{s}$, with $c_m \in Z(G^*)$. We find $(\mathrm{St}_{C_s})^\sigma = \mathrm{St}_{C_s}$. Thus, $\Theta^\sigma = \Theta(\delta_\sigma \circ \nu)$ for $\delta_\sigma \in \mathrm{Irr}(\mathbb{F}_q^\times)$.

Choose the torus operators $T_j(a)$ from Subsection 4.1. We put $D_n(b) = \prod_{j=1}^n T_j(b^{n+1-j})$. Moreover, we have that $D_n(b)x_{\alpha_j}(c)D_n(b)^{-1} = x_{\alpha_j}(bc)$ for $c \in \mathbb{F}_q$ and $1 \leq j \leq n$. We include the short root $\alpha_n = e_n$ in B_n. We find $\nu(D_n(b)) = b^{K_n}$. Thus, $\vartheta^\sigma(u) = \vartheta(D_n(b)uD_n(b)^{-1})$ for $u \in U$. Choose the semilinear map C_σ on $\mathcal{I}_{\widetilde{X}}$, with $(C_\sigma f)(g) = \sigma(f(D_n(b)^{-1}g))$. Thus, $C_\sigma JC_\sigma^{-1} = \eta_q(b)^{K_n}J$. We choose M_{δ_σ} on $\mathcal{I}_{\widetilde{X}}$, with $(M_{\delta_\sigma}f)(g) = \delta_\sigma(\nu(g))f(g)$. Moreover, $M_{\delta_\sigma}J = JM_{\delta_\sigma}$ and $M_{\delta_\sigma}(V_\Theta) = V_{\Theta^\sigma}$. Choose $S_\sigma = M_{\delta_\sigma}^{-1}C_\sigma|_{V_\Theta}$, and $S_\sigma JS_\sigma^{-1} = \eta_q(b)^{K_n}J$. Thus, $\chi^\sigma_{\mathrm{reg},\epsilon} = \chi_{\mathrm{reg},\epsilon\eta_q(b)^{K_n}}$. $\eta_q(b)^{K_n} = (b/r)^{fK_n}$, and we choose the Gauss sum $G_r = \sum_{a\in\mathbb{F}_r^\times}(a/r)\zeta_r^a$. Moreover, $G_r^2 = (-1)^{(r-1)/2}r$ and $\sigma(G_r) = (b/r)G_r$. $\mathbb{Q}(\chi) = \mathbb{Q}$ for even fK_n. $\chi_{\mathrm{reg},+} \neq \chi_{\mathrm{reg},-}$. Since every $b \in \mathbb{F}_r^\times$ occurs under σ, $\mathbb{Q}(\chi) = \mathbb{Q}(G_r) = K_r$ for odd fK_n.

We choose P on $\mathcal{I}_X$, with $(Pf)(g) = f(F_r^{-1}(g))$. Then, $\vartheta^{F_r} = \vartheta$ and $P^f = I$. By [208, Lemma 6.12], $\chi^{F_r} = \chi$. Thus, choose the unique submodule $V_\chi \subseteq \mathcal{I}_X$ with character χ. We find $P(V_\chi) = V_\chi$. Thus, we choose $P_\chi = P|_{V_\chi}$ and $\rho_\chi = R_X|_{V_\chi}$. We choose $\widehat{\rho}_\chi(gF_r^i) = \rho_\chi(g)P_\chi^i$ on $A = X \rtimes \langle F_r \rangle$, and $\widehat{\chi} = \mathrm{Tr}\,\widehat{\rho}_\chi$ and $\widehat{\chi}|_X = \chi$. Moreover, we fix $\sigma \in \Sigma_\chi$. $\eta_q(b^{K_n}) = 1$, and we choose $d \in \mathbb{F}_q^\times$ with $d^2 = b^{K_n}$ and put $g = d^{-1}D_n(b) \in X$. Thus, $\mathrm{Int}(g)|_U = \mathrm{Int}(D_n(b))|_U$. We choose $C_{g,\sigma}$ on $\mathcal{I}_X$, with $(C_{g,\sigma}f)(x) = \sigma(f(g^{-1}x))$, and we have that $C_{g,\sigma}(V_\chi) = V_\chi$ and that $F_r(D_n(b)) = D_n(b)$.

Since $d^2 \in \mathbb{F}_r^\times$, $h = F_r(g)g^{-1} = d^{1-r} \in \{1, z\}$. $h = (b/r)^{K_n}$ as a scalar in X. Thus, $C_{g,\sigma}P_\chi C_{g,\sigma}^{-1} = \omega_\chi(h)P_\chi$. Fix $a_\chi(\sigma) \in \mathrm{Irr}(A/X)$, with $\widehat{\chi}^\sigma = a_\chi(\sigma)\widehat{\chi}$. Thus, $a_\chi(\sigma)(F_r) = \omega_\chi(h)$. When $\omega_\chi(z) = -1$, $\omega_\chi(h) = (b/r)^{K_n}$. When $\omega_\chi(z) = 1$, $\omega_\chi(h) = 1$. $2 \mid f$ whenever $h = z$, since $d \in \mathbb{F}_q$ and $d^2 \notin (\mathbb{F}_r^\times)^2$. Thus, $\omega_\chi(h)^f = 1$.

We take Alvis–Curtis duality D_X, with $\mathsf{D}_X(\chi) = \pm\chi_{\mathrm{ss}}$. Choose $I \subseteq \Delta = \{\alpha_1, \dots, \alpha_n\}$ and the standard parabolic L_IU_I. Moreover, we put $e_I = |U_I|^{-1}\sum_{u\in U_I} u$ and $\mathcal{B}_I = \mathbb{Q}Xe_I \otimes_{\mathbb{Q}L_I} e_I\mathbb{Q}X$. Choose $(x, y) \in X \times X$ on $\mathcal{B}_I$, with $(x, y)(v \otimes w) = xv \otimes wy^{-1}$. $F_r(I) = I$, $F_r(L_I) = L_I$ and $F_r(e_I) = e_I$. Thus, we choose the simultaneous action $v \otimes w \mapsto F_r(v) \otimes F_r(w)$. Take $\pi_A : A \to \langle F_r\rangle$, $xF_r^i \mapsto F_r^i$. Thus, choose the fibre product $\mathfrak{F} = A \times_{\langle F_r\rangle} A$, with kernel $\mathfrak{K} = X \times X$. Choose a rational representation $R_I : \mathfrak{F} \to \mathrm{GL}_{\mathbb{Q}}(\mathcal{B}_I)$. Moreover, we choose the virtual character $\mathcal{D} = \sum_{I\subseteq\Delta}(-1)^{|I|}\,\mathrm{ch}(\mathcal{B}_I)$. $\mathcal{D}|_{\mathfrak{K}}$ is the bimodule character for D_X. Thus, $\langle \mathcal{D}|_{\mathfrak{K}}, \chi_{\mathrm{ss}} \boxtimes \chi^\vee\rangle_{\mathfrak{K}} = \pm 1$. Choose $\alpha = \mathrm{Int}(t)|_X$, with $t = T_1(a_0)$ and $\eta_q(a_0) = -1$, and $\alpha(L_IU_I) = L_IU_I$. Thus, $(\mathcal{D}|_{\mathfrak{K}})^{(\alpha,\alpha)} = \mathcal{D}|_{\mathfrak{K}}$ and $\mathcal{D}^\sigma = \mathcal{D}$. $\chi^\sigma_{\mathrm{ss},\epsilon} = \chi_{\mathrm{ss},\epsilon\epsilon_\sigma}$ and $\chi^\alpha_{\mathrm{ss},+} = \chi_{\mathrm{ss},-}$. Consequently, $\Sigma_{\chi_{\mathrm{ss}}} = \Sigma_\chi$ and $\mathbb{Q}(\chi_{\mathrm{ss}}) = \mathbb{Q}(\chi)$. By Proposition 2.4 and Corollary 2.5, we take $\widehat{\chi}_{\mathrm{ss}} \in \mathrm{Irr}(A)$. In particular, $\widehat{\chi}_{\mathrm{ss}}|_X = \chi_{\mathrm{ss}}$ and $\widehat{\chi}^\sigma_{\mathrm{ss}} = a_\chi(\sigma)\widehat{\chi}_{\mathrm{ss}}$ for $\sigma \in \Sigma_\chi$. Moreover, we choose the partner $\chi' = \chi^\alpha$ from the source $(U^\alpha, \vartheta^\alpha)$. By Lemma 4.11, we fix $a_{\chi'}$ for every f, including even f. □

Suppose that $q = r^f$ is odd with r prime. We put $X = \mathrm{Spin}_{2n+1}(q)$ and take an odd prime $p \mid |X|$, with $p \neq r$. Moreover, we choose $P \in \mathrm{Syl}_p(X)$ and $Q \leq P$ from Proposition 4.4. With $M = N_X(Q)$ and $C = C_X(Q)$, we put $R = \widetilde{X}/Z_o$ and $\widetilde{X} = \mathrm{GSpin}_{2n+1}(q)$. In particular, by Lemma 4.7, we choose $Z_o \leq Z(\widetilde{X})$. We write $F = F_q$ on $\mathbf{X}^* = \mathrm{PSp}_{2n}$. Moreover, we write $X^* = (\mathbf{X}^*)^F$. We consider a rational semisimple class $[s] = s^{X^*}$ and $\psi \in \mathrm{Irr}_{p'}(M)$. Thus, $\psi \in Y_s^X(M)$ exactly when $\langle \psi|_C, \vartheta\rangle_C > 0$ for some $\vartheta \in \mathcal{E}(C, t)$ with $t \in [s]$. Moreover, every constituent ϑ' of $\psi|_C$ belongs to $\mathcal{E}(C, t')$ with $t' \in [s]$. We choose a rational Levi L with $Q \leq L$ and put $K = N_L(Q)$. Moreover, we write the dual Levi $L^* \leq X^*$. We consider $s \in L^*$ and $\psi_L \in \mathrm{Irr}_{p'}(K)$. Thus, $\psi_L \in Y_s^L(K)$ exactly when $\langle \psi_L|_{C_L(Q)}, \beta\rangle_{C_L(Q)} > 0$ for some $\beta \in \mathcal{E}(C_L(Q), t)$ with $t \in s^{L^*}$. We write the full semisimple element $s = s_ps_{p'}$ for $Y_s^X(M), Y_s^L(K)$, including s_p. Moreover, we establish Proposition 4.39 for $Y_s^X(M)$.

Proposition 4.39. *We write* $\mathbf{X} = \mathrm{Spin}_{2n+1}$ *and* $\mathbf{X}^* = \mathrm{PSp}_{2n}$ *over* $\overline{\mathbb{F}}_q$ *and write* $X = \mathbf{X}^F$, $X^* = (\mathbf{X}^*)^F$ *with* $F = F_q$. *Moreover, we fix odd* $p \mid |X|$, *with* $p \neq r$. *We choose* $Q \leq P \in \mathrm{Syl}_p(X)$ *from* Proposition 4.4. *Moreover,* $\mathrm{Irr}_{p'}(M) = \bigsqcup_{[s]} Y_s^X(M)$ *over semisimple classes* $[s] = s^{X^*}$. *Moreover,* $|X^* : C_{X^*}(s)|_p = 1$ *when* $Y_s^X(M) \neq \varnothing$. *By* Proposition 4.4, *we write* $Q^* \leq P^* \in \mathrm{Syl}_p(X^*)$ *on the dual torus* $\mathbf{T}^*$. *We put* $M^*_{\mathrm{tor}} = N_{X^*}(Q^*)$. *Thus, we choose* $s \in [s]$ *with* $s \in C_{X^*}(P^*)$. *We write a centralizer series* $\mathcal{E}_C(t)$ *occurring below* $Y_s^X(M)$. *Moreover,* $\mathcal{E}_C(t)$ *belongs to the* M*-orbit of* $\mathcal{E}_C(s)$. *Moreover, we choose an* F*-stable dual Levi* $\mathbf{L}^*$, *with* $C_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$ *and* $Q^* \leq \mathbf{L}^*$. *Choose dual rational tori* $\mathbf{T}, \mathbf{T}^*$ *in* $\mathbf{L}, \mathbf{L}^*$. *We put* $L = \mathbf{L}^F$, $C_0 = C_L(Q)$ *and* $K = N_L(Q)$, *and we consider* $\beta \in \mathcal{E}(C_0, s)$ *and its positive-degree signed induction* $\xi \in \mathcal{E}(C, s)$. *Thus, we find* $I_M(\xi) = CI_K(\beta)$. *We write the isomorphism* $\varphi : I_K(\beta)/C_0 \xrightarrow{\sim} I_M(\xi)/C,\ kC_0 \mapsto kC$.

Proof. The signed cycles $\mathcal{C}_j$ of Proposition 4.4 give the Sylow d-tori $\mathbf{S}_d, \mathbf{S}_d^*$ in $\mathbf{T}, \mathbf{T}^*$. Thus, $Q = (\mathbf{S}_d^F)_p$ and $Q^* = (\mathbf{S}_d^{*F})_p$ have maximal toral p-rank w by [132, Propositions 5.6–5.7]. Put $E = \{x \in Q : x^p = 1\}$ and $E^* = \{x \in Q^* : x^p = 1\}$. In types B_n, C_n with $p \neq r, 2$, [7, Theorem 3.4] gives connected $C_{\mathbf{X}}(E)$ and $C_{\mathbf{X}^*}(E^*)$. By [132, proof of Proposition 5.13], $\mathbf{S}_d \leq Z(C_{\mathbf{X}}(E))^\circ$ and $\mathbf{S}_d^* \leq Z(C_{\mathbf{X}^*}(E^*))^\circ$. Thus, $E \leq Q \leq \mathbf{S}_d$ gives $\mathbf{C} = C_{\mathbf{X}}(Q) = C_{\mathbf{X}}(\mathbf{S}_d)$, and $E^* \leq Q^* \leq \mathbf{S}_d^*$ gives $\mathbf{C}^* = C_{\mathbf{X}^*}(Q^*) = C_{\mathbf{X}^*}(\mathbf{S}_d^*)$. Moreover, Q, Q^* are central Sylow subgroups of C, C^*. Indeed, a larger p-subgroup of C containing Q would contain a larger abelian p-subgroup $\langle Q, x\rangle$ for $x \notin Q$, contrary to Lemma 4.3. For C^*, adjoining $x \notin Q^*$ gives the same contradiction. By Proposition 4.4, $M = N_X(\mathbf{S}_d)$. On the dual side, $\mathbf{S}_d^*$ is the unique Sylow Φ_d-torus through E^* by [132, Proposition 5.13]. Since $E^* = Q^*[p]$ and $Q^* = (\mathbf{S}_d^{*F})_p$, $M^* = N_{X^*}(\mathbf{S}_d^*)$.

For the signed-cycle representative $w_d \in W$, put $V_d = \ker(w_d - \zeta_d)$ with ζ_d a primitive d-th root. By [63, Proposition 2.19(2)], M/C and M^*/C^* are both isomorphic to $N_W(V_d)/C_W(V_d)$, giving $\iota : M/C \xrightarrow{\sim} M^*/C^*$. Put $L^* = (\mathbf{L}^*)^F$, $K^* = N_{L^*}(Q^*)$ and $C_0^* = L^* \cap C^*$. In $\mathbf{L}, \mathbf{L}^*$, the geometric quotient calculation of [63, proof of Proposition 2.19(2)] gives $N_{W_L}(V_d)/C_{W_L}(V_d)$, with $W_L = W(\mathbf{L}) = W(\mathbf{L}^*)$ in the common W. On this quotient, F acts through w_d. The action is trivial on $N_W(V_d)/C_W(V_d)$, hence also on $N_{W_L}(V_d)/C_{W_L}(V_d)$. Moreover, the torus centralizers $C_{\mathbf{L}}(\mathbf{S}_d), C_{\mathbf{L}^*}(\mathbf{S}_d^*)$ are connected. By [114], K/C_0 and K^*/C_0^* are the rational quotients. Thus, $\iota(KC/C) = K^*C^*/C^*$. Given $g \in M$, put $g^*C^* = \iota(gC)$, and let a denote conjugation by g on C. By [160, Lemma 2.23 and the following paragraph], pullback by a acts on rational C^*-classes through the inverse dual automorphism $(a^*)^{-1}$. Thus, the M-orbits on the series $\mathcal{E}(C, t)$ correspond to M^*-orbits on rational C^*-classes.

By [95, Theorem 6.2], the constituents of $\psi|_C$, for $\psi \in \mathrm{Irr}(M)$, form one M-orbit. Thus, all labels lie in $[s] = s^{X^*}$ for some s. Hence, $\mathrm{Irr}_{p'}(M) = \bigsqcup_{[s]} Y_s^X(M)$. Given $\psi \in Y_s^X(M)$, choose ξ below $\psi|_C$ and $t \in [s] \cap C^*$ with $\xi \in \mathcal{E}(C,t)$. We write $e_\psi \in \mathbb{Z}_{>0}$, with $\psi(1) = [M : I_M(\xi)]e_\psi \xi(1)$ [43]. Thus, $p \nmid [M : I_M(\xi)]$. Put $J_M = \{g \in M : \mathcal{E}(C,t)^g = \mathcal{E}(C,t)\}$. Since $I_M(\xi) \leq J_M$, $|J_M|_p = |M|_p$. Write $I^* = C^*(M^* \cap C_{X^*}(t))$. Moreover, $I^* = \{g^* \in M^* : t^{g^*} \in t^{C^*}\}$. Since $Q^* \leq C_{C^*}(t)$ is Sylow in C^*, $[I^* : M^* \cap C_{X^*}(t)] = [C^* : C_{C^*}(t)]$ is prime to p. Moreover, $\iota(J_M/C) = I^*/C^*$ and $[M : J_M]$ is prime to p. By [132, Proposition 5.21], M^* contains a Sylow p-subgroup of X^*. Thus, $|M^* \cap C_{X^*}(t)|_p = |I^*|_p = |M^*|_p = |X^*|_p$. Thus, choose $s \in [s] \cap C_{X^*}(P^*)$, where $P^* \in \mathrm{Syl}_p(X^*)$ contains Q^*. By [132, Proposition 5.11], $M^* = \mathrm{N}_{X^*}(\mathbf{S}_d^*)$ controls X^*-fusion in C^*. Thus, $s^{X^*} \cap C^* = s^{M^*}$, and the local series below $Y_s^X(M)$ lie in the orbit of $\mathcal{E}(C,s)$.

Since $C_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$, $C_{\mathbf{C}^*}(s) \leq \mathbf{C}^* \cap \mathbf{L}^*$. By [18, Theorem B′ and Remark 11.3], over a splitting field E of characteristic 0, signed induction $\mathcal{R}_s : \mathcal{E}(C_0,s) \to \mathcal{E}(C,s)$ is bijective. Both series have label $s = s_p s_{p'}$. Let $K_s = \mathrm{Stab}_K(\mathcal{E}(C_0,s))$ and $M_s = \mathrm{Stab}_M(\mathcal{E}(C,s))$. If $g^* \in M^*$ stabilizes s^{C^*}, some $g_0^* \in C^* g^*$ centralizes s. Since $C_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$, we have $g_0^* \in K^*$. Thus, $C_{K^*}(s)$ maps onto the class stabilizer in M^*/C^*. By the dual action and $\iota(KC/C) = K^*C^*/C^*$, this gives $M_s = CK_s$. By [160, Lemma 2.23], the map $\mathcal{R}_s$ is K_s-equivariant. Thus, $\xi^{ck} = \xi$ for $c \in C$ and $k \in K_s$ exactly when $\beta^k = \beta$. Hence, $I_M(\xi) = CI_K(\beta)$. Since $K \cap C = C_0$, the map $kC_0 \mapsto kC$ gives the stated isomorphism. □

Proposition 4.40. *In characteristic* 0*, we choose a splitting field* E*. Moreover, we write* L, K, C_0, C, M, s *from* Proposition 4.39*, with* $P \leq L$*. We consider* $\beta \in \mathcal{E}(C_0,s)$ *and* $\xi = \mathcal{R}_s(\beta)$*. Thus, we write* $U_\beta = I_K(\beta)/C_0 \simeq I_M(\xi)/C$*. We choose associated models* $\mathcal{P}_\beta, \mathcal{P}_\xi$ *with factor* $\alpha : U_\beta \times U_\beta \to E^\times$*, and we choose* $\eta \in \mathrm{Irr}_{\alpha^{-1}}(U_\beta)$*. Moreover, we write the ordinary bijection* $I_Q : Y_s^L(K) \to Y_s^X(M)$ *with the label* η *on both inertia quotients* U_β*. We choose the scalar function* $\kappa_{\xi,\beta} : U_\beta \to E^\times$ *from* (ξ,β)*, before the choice of* η*. We consider a character* $\varphi \in Y_s^L(K)$ *above* β*, with label* η*. Thus, we find* $I_Q(\varphi)(1)/\varphi(1) = [M : K]|C_0|_r/|C|_r$*.*

Proof. Let $\mathbf{LU}$ be a parabolic with radical $\mathbf{U}$, and put $\mathbf{U}_0 = C_{\mathbf{U}}(Q)$, $\mathbf{C}_0 = \mathbf{C} \cap \mathbf{L}$. By [160, Example 2.3], $\mathbf{C}_0\mathbf{U}_0$ is parabolic in $\mathbf{C}$. Since $K \leq L$ normalizes Q and $\mathbf{U}$, it normalizes $\mathbf{U}_0 = C_{\mathbf{U}}(Q)$. Thus, $I_K(\beta)$ acts on $Y = Y_{\mathbf{U}_0}^{\mathbf{C}}$ by conjugation as in [160, proof of Theorem 5.2]. The C, C_0 actions then give $\Delta_0 = (C \times C_0)\Delta I_K(\beta)$ acting by $(ck,k)y = c(kyk^{-1})$. Thus, Δ_0 is the fibre product over U_β with quotient maps $q_K(k) = kC_0$, $q_M(ck) = kC_0$.

By [18, Theorem B′], $e_{\beta^\vee} H_c^\bullet(Y,E)$ is concentrated in degree $j = \dim \mathbf{U}_0 - \dim(\mathbf{U}_0 \cap F(\mathbf{U}_0))$. Thus, the Euler character with sign $(-1)^j$ is the character of $e_{\beta^\vee} H_c^j(Y,E)$. By the signed bijection $\mathcal{R}_s$, its restriction to $C \times C_0$ is $\xi \boxtimes \beta^\vee$. Thus, Proposition 2.4 gives $E_{\xi,\beta}^{\mathrm{ord}}$ with multiplicity 1. By Corollary 2.5, we choose $\mathcal{P}_\beta : I_K(\beta) \to \mathrm{GL}(V_\beta)$ and $\mathcal{P}_\xi : I_M(\xi) \to \mathrm{GL}(V_\xi)$. Thus, both models have factor α on U_β. Given an initial associated model $\mathcal{P}_\xi^0$ with factor α_ξ^0, [141, Remark 1.3] gives a scalar function $\kappa_{\xi,\beta} : U_\beta \to E^\times$ with $\alpha(u,v) = \alpha_\xi^0(u,v)\kappa_{\xi,\beta}(u)\kappa_{\xi,\beta}(v)/\kappa_{\xi,\beta}(uv)$. Moreover, $\kappa_{\xi,\beta}$ is chosen from (ξ,β) before any η. By Lemma 4.8, the common label η gives a bijection above β^K, ξ^M, with degree ratio $\frac{[M:C]}{[K:C_0]}\frac{\xi(1)}{\beta(1)}$. By [18, Remark 10.12], $\xi(1)/\beta(1) = [C : C_0]_{r'}$. Thus, the ratio is $[M : K]|C_0|_r/|C|_r$. Since Q is characteristic in $P \leq L$, we have $P \leq K$. Therefore, $p \nmid [M : K]$ and the degree ratio has p-valuation zero.

Since $\mathbf{S}_d^*$ is Sylow in $\mathbf{L}^*$ and $C_0^* = C_{\mathbf{L}^*}(\mathbf{S}_d^*)^F$, [132, Proposition 5.11] gives $s^{L^*} \cap C_0^* = s^{K^*}$. Thus, each K-orbit below $Y_s^L(K)$ meets $\mathcal{E}(C_0,s)$. By fusion on C^*, each M-orbit below $Y_s^X(M)$ meets $\mathcal{E}(C,s)$. By the stabilizer argument in Proposition 4.39, $M_s = CK_s$ and $\mathcal{R}_s$ is K_s-equivariant. Since C acts trivially on $\mathrm{Irr}(C)$, the M_s-orbits on $\mathcal{E}(C,s)$ are the K_s-orbits. Thus, $\mathcal{R}_s$ pairs the K_s-orbits. By [95, Theorem 6.11], common quotient labels give $I_Q : Y_s^L(K) \xrightarrow{\sim} Y_s^X(M)$. □

Fix non-quasi-isolated $[s] = s^{X^*}$ from Proposition 4.39. Choose $P^* \leq L^*$ and C, C_0, K, M from Proposition 4.39. Moreover, let $\lambda \in \mathcal{E}(L,s)$ and $\beta \in \mathcal{E}(C_0,s)$. The induction pairs are (χ,λ) and (ξ,β), with $\chi = I_G(\lambda)$, $\xi = \mathcal{R}_s(\beta)$. Thus, we construct the comparison for (ξ,β) before the choice of $\psi_0 \in \mathrm{Irr}(K \mid \beta)$.

Proposition 4.41. *Suppose that either* $(\mathbf{H},\mathbf{J};\theta,\gamma) = (\mathbf{X},\mathbf{L};\chi,\lambda)$ *or* $(\mathbf{H},\mathbf{J};\theta,\gamma) = (\mathbf{C},\mathbf{C}_0;\xi,\beta)$*. With* $H = \mathbf{H}^F$*, we put* $J = \mathbf{J}^F$*. Moreover, we choose the source overgroup* $\widehat{J}$ *for* γ*. Define* $\mathrm{ad} : \widehat{J} \to \mathrm{Inndiag}(X)\langle F_r \rangle$*, with* $\mathrm{ad}(b)(x) = bxb^{-1}$*. Moreover, we choose the ordinary inertia* $B = I_{\widehat{J}}(\gamma)$*. For* $(\mathbf{C},\mathbf{C}_0)$*, take* $K \leq \widehat{J}$ *and* $B \leq N_{\widehat{J}}(Q)$*. Thus, we set* $A = HB$*, with* $H \cap B = J$*. We choose the quotient maps* $\pi_A : A \to B/J$*,* $hb \mapsto bJ$ *and* $\pi_B : B \to B/J$*,* $b \mapsto bJ$ *and put* $\Delta = A \times_{B/J} B$*. Moreover, we choose an ordinary representation* $E_{\theta,\gamma} : \Delta \to \mathrm{GL}(V_{\theta,\gamma})$*, with* $\mathrm{Tr}(E_{\theta,\gamma}|_{H \times J}) = \theta \boxtimes \gamma^\vee$*. Its character is* $\Sigma_{\theta,\gamma} = \mathrm{Tr}\, E_{\theta,\gamma}$*. We have that* $\Sigma_{\theta,\gamma}^{b\sigma} = \Sigma_{\theta,\gamma}$ *for every* (b,σ) *in the mixed stabilizer of the marked source and*

series $[s]$. *Thus,* $E_{\theta,\gamma}(z,z) = I$ *for every diagonal element* (z,z) *acting trivially on* $\mathbf{H}$. *Whenever* $q \in Q$, $E_{\xi,\beta}(q,q) = I$. *We set* $\Delta_0 = (C \times C_0)\Delta I_K(\beta)$. *Moreover, we have that* $E_{\xi,\beta}|_{\Delta_0} \simeq E^{\mathrm{ord}}_{\xi,\beta}$, *with* $E^{\mathrm{ord}}_{\xi,\beta}$ *from* Proposition 4.40.

Proof. We fix $\mathbf{X} \leq \widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{L}} = \mathbf{L}Z(\widetilde{\mathbf{X}})$. We choose the degree map $\deg : B \to \langle F_r \rangle$ and put $B^\circ = \ker \deg$. By [208, proof of Proposition 4.16, Steps 2–3], $N_X(\mathbf{L}, \mathcal{E}(L,s)) = L$ and the actions of $b \in B^\circ$ have algebraic lifts $K_b \in \mathrm{Int}(\widetilde{\mathbf{L}}^F)$. Moreover, we choose a unipotent radical $\mathbf{U}$ with Levi $\mathbf{L}$ in $\mathbf{X}$. Whenever $b \in B^\circ$, $b\mathbf{U}b^{-1} = \mathbf{U}$. Since $bQb^{-1} = Q$ for $b \in B^\circ$, we also have that $b\mathbf{V}b^{-1} = \mathbf{V}$, with $\mathbf{V} = C_{\mathbf{U}}(Q)$. Whenever $k \in I_K(\beta)$, $k\mathbf{V}k^{-1} = \mathbf{V}$. By Proposition 4.40, fix $\mathbf{C}$, with action $(ck,k)y = c(kyk^{-1})$. Moreover, $C_{\mathbf{C}^*}(s) \leq \mathbf{C}_0^*$. Choose $\mathbf{W} = \mathbf{U}$ for $(\mathbf{X},\mathbf{L})$ and $\mathbf{W} = \mathbf{V}$ for $(\mathbf{C},\mathbf{C}_0)$, and $C_{\mathbf{H}^*}(s) \leq \mathbf{J}^*$ for both pairs $(\mathbf{H},\mathbf{J})$.

Moreover, we put $m = |B/B^\circ|$. We choose $m > 1$ and $a \in B$ with field degree f/m. Moreover, put $v = a^m \in B^\circ$ and take $\varphi = \mathrm{Int}(u)F_r^{f/m}$ that induces a. We choose K_b as the inner-diagonal map of $b \in B^\circ$, and $\varphi^m = K_v F$ and $\varphi K_b = K_{aba^{-1}}\varphi$. Thus, put $u_m = uF_r^{f/m}(u)\cdots F_r^{(m-1)f/m}(u)$. $\mathrm{Int}(u_m) = K_v$ and $\mathrm{Int}(z) = 1$ for central scalars z. We choose K_b, φ on $\mathbf{X}$ before their action on $\mathbf{C}$. Moreover, with $m = 1$, put $a = v = 1$ and $\varphi = F$. By [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], on $\mathbf{H}^m$ we have $\tau(x_0,\dots,x_{m-1}) = (x_1,\dots,x_{m-1},K_v(x_0))$ and $\mathcal{F} = \tau^{-1}\,\mathrm{diag}(\varphi)$, with $\mathcal{F}^m = \mathrm{diag}(F)$. The action D of B satisfies $D(b)\mathcal{F} = \mathcal{F}D(b)$ for $b \in B$ and has $D(a) = \tau$ when $m > 1$, and $D(t)_j = K_{a^j t a^{-j}}$ for $t \in B^\circ$. Thus, $\iota_H : x \mapsto \widehat{x} = (x, \varphi x, \dots, \varphi^{m-1}x)$ gives $H \simeq (\mathbf{H}^m)^{\mathcal{F}}$. On the induction variety $\mathcal{Y}_0$ for $\mathbf{W}^m$ and $\mathcal{F}$, the action of $(hb,b) \in \Delta$ is $y \mapsto \widehat{h}D(b)(y)$. Moreover, on $H \times J$, $(h,j)y = \widehat{h}y\widehat{j}^{-1}$. If $K_z = 1$ and $z \in B^\circ$, then $D(z) = 1$. Every $q \in Q$ also satisfies $D(q) = 1$ in the local case, since a normalizes Q and $\mathbf{H} = C_{\mathbf{X}}(Q)$. Moreover, $C_{\widetilde{X} \rtimes \langle F_r \rangle}(X) \leq \widetilde{X}$, since $F_r^i \notin \mathrm{Inndiag}(X)$ for $0 < i < f$.

Let $\mathcal{Y}_1$ be the induction variety for $\mathbf{W}_1 = \prod_{j=0}^{m-1}\varphi^j\mathbf{W}$ and $\mathcal{F}$. By [208, proof of Lemma 4.19(b)], the first projection pr_0 takes $\mathcal{Y}_1$ isomorphically to $Y^{\mathbf{H}}_{\mathbf{W}}$, with $\mathrm{pr}_0(\widehat{x}) = x$. We choose $\ell_0 \nmid r|H||J||B|$ and take $E = \overline{\mathbb{Q}}_{\ell_0}$ and choose the ordinary projector $e_s \in Z(E[H])$ for $[s]$. In particular, by [208, proof of Lemma 4.19(b)], choose $\widehat{e}_s = \iota_H(e_s)$ on $(\mathbf{H}^m)^{\mathcal{F}}$. We put $\mathcal{H}_0 = \sum_i (-1)^i \,\mathrm{ch}\, H_c^i(\mathcal{Y}_0, E)$. By [19, Corollary 6.5 and Remark 6.7], $m_{\theta,\gamma} = \langle \mathcal{H}_0|_{H\times J}, \theta \boxtimes \gamma^\vee\rangle_{H\times J} \in \{1,-1\}$. We put $\epsilon_{\theta,\gamma} = m_{\theta,\gamma}$ and take the right projector $e_{\gamma^\vee} \in Z(E[J])$. Thus, put $\Xi_{\theta,\gamma} = \epsilon_{\theta,\gamma}\sum_i (-1)^i\,\mathrm{ch}(e_{\gamma^\vee}H_c^i(\mathcal{Y}_0,E))$. In the local case, $I_K(\beta) \leq B^\circ$ has coordinate actions $K_{a^j k a^{-j}}$ for $k \in I_K(\beta)$. These fix $\mathbf{V}, \varphi^j\mathbf{V}$ setwise, and $\mathrm{pr}_0\, D(k) = K_k\,\mathrm{pr}_0$. By [160, proof of Theorem 5.2], the comparison is equivariant under $\Delta_0 = (C \times C_0)\Delta I_K(\beta)$. After normalization at $\xi \boxtimes \beta^\vee$, it gives $\Xi_{\xi,\beta}|_{\Delta_0} = \Xi^{\mathrm{ord}}_{\xi,\beta}$.

Choose (b,σ) in the mixed stabilizer with $\gamma^{b\sigma} = \gamma$ and $\theta^{b\sigma} = \theta$. Then, we choose b after correction to the marked source $\mathbf{J}$ and the ambient series $[s]$. Thus, $bBb^{-1} = B$ and $bB^\circ b^{-1} = B^\circ$. $(\gamma^b)^\sigma = (\gamma^\sigma)^b$, and b acts trivially on B/B°. Moreover, we choose the algebraic lift Z of b with nonnegative field degree $\deg Z \geq 0$. Moreover, put $h_j = a^j b a^{-j} b^{-1} \in B^\circ$ and $Z_j = K_{h_j}Z$. Z_j is a morphism on $\mathbf{H}$, and $Z_j = \varphi^j Z \varphi^{-j}$ on geometric points $\mathbf{H}(\overline{\mathbb{F}}_q)$. Thus, $Z_j\varphi = \varphi Z_{j-1}$ for $1 \leq j < m$. $ZK_v^{-1}\varphi = K_v^{-1}\varphi Z_{m-1}$, and $\varphi^m = K_v F$ and $ZF = FZ$. Moreover, put $Z_\bullet = (Z_0,\dots,Z_{m-1})$. Whenever $t \in B^\circ$, $Z_\bullet D(t) = D(btb^{-1})Z_\bullet$. We find $Z_\bullet\tau = D(bab^{-1}a^{-1})\tau Z_\bullet$. Thus, $Z_\bullet(\mathbf{W}^m) = (Z\mathbf{W})^m$. We have that $t\mathbf{W}t^{-1} = \mathbf{W}$ and $t(Z\mathbf{W})t^{-1} = Z\mathbf{W}$ for $t \in B^\circ$.

Put $\mathbf{P}_0 = \mathbf{J}^m\mathbf{W}^m$ and $\mathbf{P}_Z = \mathbf{J}^m(Z\mathbf{W})^m$. Thus, $D(B)$ stabilizes $\mathbf{P}_0, \mathbf{P}_Z$. The returned label s_σ is conjugate to s^k for $\gcd(k,|s|) = 1$, and B fixes e_{s_σ}. Since $C_{\mathbf{H}^*}(s^k) = C_{\mathbf{H}^*}(s) \leq \mathbf{J}^*$, [160, Lemma 3.1] compares the e_{s_σ}-parts of the bimodules for $\mathbf{W}^m$ and $(Z\mathbf{W})^m$ with their diagonal $D(B)$-actions. $Z_\bullet$ accounts for b as in Lemma 7.168. The trace on Δ is integral before projection by [53, Proposition 3.3]. Thus, σ takes the transported projector to $e_{\gamma^\vee}$, since $(\gamma^b)^\sigma = \gamma$. Thus, normalization at $\theta \boxtimes \gamma^\vee$ gives $\Xi^{b\sigma}_{\theta,\gamma} = \Xi_{\theta,\gamma}$. Accordingly, by Proposition 2.4, we fix $E_{\theta,\gamma}$. We find $\Sigma^{b\sigma}_{\theta,\gamma} = \Sigma_{\theta,\gamma}$. Moreover, $E_{\theta,\gamma}(z,z) = I$ when $K_z = 1$, and $E_{\xi,\beta}(q,q) = I$ for $q \in Q$. By the restriction to Δ_0, Proposition 2.4 gives $E_{\xi,\beta}|_{\Delta_0} \simeq E^{\mathrm{ord}}_{\xi,\beta}$. □

Proposition 4.42. *We choose the marked source* $\widehat{L}$ *of* Proposition 4.41. *Next, we choose* $\psi_0 \in \mathrm{Irr}(K \mid \beta)$ *and* $\psi = I_Q(\psi_0)$ *from* Proposition 4.40. *Moreover, we choose* $\widehat{K}, \widehat{M}$ *as inertia groups of* ψ_0, ψ. *We choose the common quotient* $T = \widehat{K}/K \simeq \widehat{M}/M$. *Fix any associated model* $\Pi_0 : \widehat{K} \to \mathrm{GL}(V_0)$ *with factor* $\alpha : T \times T \to \overline{\mathbb{Q}}^\times$. *Moreover, we choose* $\Pi : \widehat{M} \to \mathrm{GL}(V_\Pi)$, *with* $\mathrm{Tr}(\Pi|_M) = \psi$ *and factor* α. *Thus,* $\mu_a^{\Pi_0}(t) = \mu_a^{\Pi}(t)$ *for every* a *in the mixed stabilizer and* $t \in T$. *We fix* $z \in C_{\widehat{K}}(K) \cap C_{\widehat{M}}(M)$, *and we have that* $\Pi_0(z) = \lambda_z I$ *and* $\Pi(z) = \lambda_z I$, *with* $\lambda_z \in \overline{\mathbb{Q}}^\times$. *Moreover, we choose* $\zeta \in \mathrm{Irr}(Q)$, *with* $\beta|_Q = \beta(1)\zeta$ *and* $\xi|_Q = \xi(1)\zeta$.

Proof. Take $\mathcal{P}_\beta, \mathcal{P}_\xi$ from $E_{\xi,\beta}$ by Corollary 2.5. Proposition 4.41 gives $\mu_a^{\mathcal{P}_\beta} = \mu_a^{\mathcal{P}_\xi}$ on U_β for a jointly stabilizing (β,ξ). Thus, by Lemma 4.8, the common label η gives matching stabilizers of ψ_0, ψ in the marked source. Put $B = (\widehat{K})_\beta$.

By Clifford conjugacy for $\psi_0|_{C_0}$ and $\psi|_C$, the marked stabilizers are $\widehat{K} = KB$, $\widehat{M} = MB$. Moreover, $B \cap K = I_K(\beta)$, and $(CB) \cap M = CI_K(\beta) = I_M(\xi)$. Thus, $T \simeq B/I_K(\beta)$ in both cases. Write $E = E_{\xi,\beta}|_{(CB)\times_{B/C_0}B}$ on V_E.

Given Π_0, set $V_{0,\beta} = e_\beta V_0$ and $U_\xi = (V_E \otimes V_{0,\beta})^{C_0}$, with action $t \mapsto E(1,t) \otimes \Pi_0(t)$ for $t \in C_0$. Since B fixes β, apply Corollary 2.5 on $V_{0,\beta}$ by its tensor formula $\rho_\xi(x) = (E(x,b) \otimes \Pi_0(b))|_{U_\xi}$ for $x \in CB$ and $b \in B$ with $xC = bC$. Thus, $\rho_\xi(x)$ is independent of b and has factor $\alpha(xM, yM)$. By $E|_{\Delta_0} \simeq E^{\mathrm{ord}}_{\xi,\beta}$, the restriction of ρ_ξ to $I_M(\xi)$ has label η, since $V_\beta^\vee$ contracts with V_β and leaves the multiplicity module of Π_0.

Take $\Pi = \mathrm{Ind}_{CB}^{\widehat{M}} \rho_\xi$ in the twisted group algebra for α. Since $\widehat{M} = M(CB)$, left cosets are represented in M, and restriction to M is ordinary induction from $I_M(\xi)$. Thus, $\mathrm{Tr}(\Pi|_M) = \psi$ by [95, Theorem 6.11]. For a normalized cochain u on T, the algebra map $e_g^u \mapsto u(gM)e_g$ gives $\mathrm{Ind}_{CB}^{\widehat{M}}(u\rho_\xi) \simeq u\Pi$, with the new factor $\alpha^u(s,t) = u(s)u(t)u(st)^{-1}\alpha(s,t)$. Consequently, a joint stabilizer element a with $E^a \simeq E$ gives $\rho_\xi^a \simeq \mu_a \rho_\xi$ by the tensor construction, where $\mu_a = \mu_a^{\Pi_0}$ on T. Thus, $\Pi^a \simeq \mu_a \Pi$, and uniqueness in [141, Lemma 1.4] gives $\mu_a^\Pi = \mu_a^{\Pi_0}$. Given a stabilizing the orbit, choose $k \in K$ so that ak jointly stabilizes (β, ξ). By [141, Lemma 1.7(b)], the functions on T for a and ak coincide. Moreover, we fix $z \in C_{\widehat{K}}(K) \cap C_{\widehat{M}}(M)$. $z \in B$ and $E(z,z) = I$, and we choose λ_z, with $\Pi_0(z) = \lambda_z I$. Thus, $\rho_\xi(z) = \lambda_z I$. $zt = tz$ for coset representatives $t \in M$, and $\Pi(z) = \lambda_z I$. Moreover, $Q \le Z(C) \cap Z(C_0)$ and $E(q,q) = I$ for $q \in Q$. We choose the linear character $\zeta \in \mathrm{Irr}(Q)$ of $V_\beta|_Q$, and $\xi|_Q = \xi(1)\zeta$ and $\beta|_Q = \beta(1)\zeta$. □

Moreover, write $s = s_{p'}t$ with $t = s_p$, and put $\mathbf{H}^* = C_{\mathbf{X}^*}(s)$ and $\mathbf{L}^* = C_{\mathbf{X}^*}(Z(\mathbf{H}^*)^\circ)$. For the covering $\pi : \mathrm{Sp}_{2n} \to \mathbf{X}^*$, choose a lift $\widetilde{s}$ and put $\widetilde{\mathbf{H}} = C_{\mathrm{Sp}_{2n}}(\widetilde{s})$. By Breuer et al. [25, proof of Proposition 2.9], $\widetilde{\mathbf{H}} \simeq \prod_i \mathrm{Sp}(V_i) \times \prod_j \mathrm{GL}(W_j)$. By Bonnafé [20, (2.2) and Corollary 2.9], $\pi(\widetilde{\mathbf{H}}) = (\mathbf{H}^*)^\circ$ and $[\mathbf{H}^* : (\mathbf{H}^*)^\circ] \le 2$. If $z \in \widetilde{\mathbf{H}}$ lifts an element of $Z((\mathbf{H}^*)^\circ)$, then $[z, \widetilde{\mathbf{H}}] \subseteq \ker \pi$. Since $\widetilde{\mathbf{H}}$ is connected, $[z, \widetilde{\mathbf{H}}] = 1$. Consequently, $\pi(Z(\widetilde{\mathbf{H}})) = Z((\mathbf{H}^*)^\circ)$, so its component group is a 2-group. Thus, $t \in Z(\mathbf{H}^*)$ has odd order and lies in $\mathbf{S} = Z((\mathbf{H}^*)^\circ)^\circ$. The component group acts on $\mathbf{S}$ through an involution ι, taking $\iota = 1$ in the connected case. Since $\iota(t) = t$, the norm map $x \mapsto x\iota(x)$ takes t^k to t when $2k \equiv 1 \pmod{|t|}$. Its image lies in $Z(\mathbf{H}^*)^\circ$, so $t \in Z(\mathbf{L}^*)$. Consequently, $C_{\mathbf{L}^*}(s_{p'}) = C_{\mathbf{L}^*}(s) = \mathbf{H}^*$. The dual character of $t \in Z(\mathbf{L}^*)$ is $\vartheta_t : L \to \overline{\mathbb{Q}}^\times$. Choose $a = (b, \sigma)$ in the joint stabilizer of the full Levi class $[s]_L$. Then, we find $t^a = t$ by the p-part of s. Thus, we find $\vartheta_t^a = \vartheta_t$. Let B_0 be its inertia group and $B_1 = B_0 \cap N(Q)$, with $B_0 = LB_1$ and $B_1 \cap L = K$. By Navarro et al. [141, Corollary 1.2], take an associated projective representation T of B_0 for ϑ_t, and put $T_1 = T|_{B_1}$. Thus, T, T_1 have the same factor on $T_t = B_0/L \simeq B_1/K$. For $a = (b, \sigma)$ in the joint stabilizer, Navarro et al. [141, Lemma 1.4] gives $T^a = u_a T$ on B_0. Consequently, $T_1^a = u_a T_1$ under the quotient T_t. Moreover, $T_1(z) = T(z)$ on $C_{B_1}(K) \cap C_{B_0}(L)$, and $T_1(q) = \vartheta_t(q)$ on Q. Thus, we choose the full series $\mathcal{E}(L, s) = \vartheta_t \mathcal{E}(L, s_{p'})$ with projector e_s.

4.5 Norm operators

Lemma 4.43. *Let V, W be modules with splitting field F_* of characteristic 0. On their joint stabilizer D, suppose that the systems P_a, Q_a have coefficient action σ_a and common factor α, with $P_1 = I_V$ and $Q_1 = I_W$. Let $a, x \in D$ satisfy $ax = xa$, $x^2 = 1$ and $\sigma_x = 1$. If $P_x^2 = I_V$ and $Q_x^2 = I_W$, then $P_a P_x P_a^{-1} = \epsilon_a P_x$ and $Q_a Q_x Q_a^{-1} = \epsilon_a Q_x$, with $\epsilon_a = \alpha(a,x)\alpha(x,a)^{-1} \in \{1, -1\}$. With $b^2 = \alpha(x,x)^{-1}$, we put $P'_x = bP_x$ and $Q'_x = bQ_x$ over $F_*(b)$. Then, the common sign is $\sigma_a(b)b^{-1}\alpha(a,x)\alpha(x,a)^{-1}$ for any extension of σ_a to $F_*(b)$. Moreover, suppose that a permutes the paired factors of $\mathcal{T}_V = \bigotimes_{i\in I} V_i$ and $\mathcal{T}_W = \bigotimes_{i\in I} W_i$ by π. If $P_{x,i}, Q_{x,i}$ have common conjugation sign $\epsilon_{a,i}$, both tensor signs equal $\prod_{i\in I} \epsilon_{a,i}$.*

Proof. Since $ax = xa$, the factor set formula in [141, Lemma 1.7(a)] gives $\kappa = \alpha(a,x)/\alpha(x,a)$ in both systems. Moreover, $P_x^2 = \alpha(x,x)I_V$ and $Q_x^2 = \alpha(x,x)I_W$. Thus, $\sigma_a(\alpha(x,x)) = \kappa^2\alpha(x,x)$. For $b^2 = \alpha(x,x)^{-1}$, this gives $\sigma_a(b) = \pm\kappa^{-1}b$, so bP_x, bQ_x have conjugation scalar $\sigma_a(b)b^{-1}\kappa \in \{1, -1\}$. If $\alpha(x,x) = 1$, take $b = 1$ to obtain $\kappa \in \{1, -1\}$. Thus, after π permutes the factors, both signs are $\prod_{i\in I} \epsilon_{a,i}$. □

We choose the norm $\nu : \widetilde{\mathbf{X}} = \mathrm{GSpin}_{2n+1} \to \mathbf{G}_m$ of Lemma 4.5 (Milne [137, §24.i, p. 532]). We have that $\ker \nu = \mathbf{X}$ and $\nu(c) = c^2$ for scalars c. By definition, the twist $\lambda \circ \nu$ has value $\lambda(\nu(a))$ at a. Moreover, we set $F = F_0^m$, with $q = r^{km}$ and $F_0(c) = c^{r^k}$ on $\mathbf{G}_m$. We set $\varphi = \mathrm{Int}(l)F_0$ and $\varphi^m = \mathrm{Int}(v)F$ from Zhang [208, Lemma 4.19]. We take either $(\mathbf{H}, \mathbf{J}) = (\widetilde{\mathbf{X}}, \widetilde{\mathbf{L}})$ or $(\mathbf{H}, \mathbf{J}) = (C_{\widetilde{\mathbf{X}}}(Q), C_{\widetilde{\mathbf{L}}}(Q))$. Thus, we choose $H = \mathbf{H}^F, J = \mathbf{J}^F$ and the induction variety Y for the cyclic Frobenius $\mathcal{F}$. Since $\nu(\mathbf{U}) = 1$, applying ν to the condition for Y gives $\mathcal{F}((\nu(y_j))_j) = (\nu(y_j))_j$ on $\mathbf{G}_m^m$. Thus, $\nu(y_j) = \nu(y_0)^{r^{kj}}$ and $\nu(y_0)^q = \nu(y_0)$. With $C = \mathbb{F}_q^\times$, put $Y_c = \{y \in Y : \nu(y_0) = c\}$. Thus, Y_c is open and closed in Y. Take $E = \overline{\mathbb{Q}}_{\ell_0}$ with $\ell_0 \ne r$, and write $V_{Y,c} = \bigoplus_i H_c^i(Y_c, E)$, $V_Y = \bigoplus_c V_{Y,c}$. For $\lambda \in \widehat{C} = \mathrm{Irr}(C)$, let

$T_\lambda|_{V_{Y,c}} = \lambda(c)I$. Thus, $T_\lambda T_\mu = T_{\lambda\mu}$. Let R denote the action of Δ of Proposition 4.41 on V_Y. By its product-variety action, $xY_c = Y_{u_x(c)}$, with $u_x(c) = d_x c^{r^{j_x}}$ and $d_x = u_x(1)$. In particular, $d_{(g,h)} = \nu(g)/\nu(h)$ on $H \times J$ and $u_a(c) = c^{r^k}$ for the marked field generator a. Thus, $a^m = v$ in the source gives $u_a^m = u_v$, also for even m.

Moreover, we choose the source stabilizer B_{tw} of the orbit $\{\gamma(\lambda \circ \nu) : \lambda \in \widehat{C}\}$. We impose $B_{\mathrm{tw}} \leq N(Q)$ in the local case $\mathbf{H} = C_{\widetilde{\mathbf{X}}}(Q)$. Moreover, we take its field kernel B_{tw}°. We choose a regular dual lift $\widetilde{s}$ and $\pi^* : \widetilde{\mathbf{X}}^* \to \mathbf{X}^*$. Since $z_\lambda \in \ker \pi^*$, the central twist satisfies $\pi^*(z_\lambda \widetilde{s}) = \pi^*(\widetilde{s}) = s$. By Proposition 4.41 and $C_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$, we have that $b\mathbf{U}b^{-1} = \mathbf{U}$ for $b \in B_{\mathrm{tw}}^\circ$. We construct the cyclic product Y for B_{tw} before the right projector e_{orb}. Thus, we choose the fibre product Δ. We put $A = \{a_{z,\lambda} : c \mapsto z\lambda(c) : z \in \mu_{q-1}, \lambda \in \widehat{C}\}$. Given $x \in \Delta, a \in A$, we choose $x \cdot a = a \circ u_x^{-1}$. Moreover, $x \cdot a \in A$ and $a|_{V_{Y,c}} = a(c)I$. We choose the semidirect product $A \rtimes \Delta$ on V_Y. Write $\mathrm{Tr}_{\mathrm{alt}}$ for the alternating trace. Thus, ax has contributions only from fibres Y_c fixed by x: $\mathrm{Tr}_{\mathrm{alt}}(ax; Y) = \sum_{u_x(c)=c} a(c)\,\mathrm{Tr}_{\mathrm{alt}}(x; Y_c)$. On each such Y_c, the automorphism x has finite order. Thus, by Deligne–Lusztig [53, Proposition 3.3], $\mathrm{Tr}_{\mathrm{alt}}(x; Y_c) \in \mathbb{Z}$; only $a(c)$ is affected by σ. Choose an ordinary stabilizer element b of the marked source $\mathbf{J}$. We have that $b(Y_c) = Y_{u_b(c)}$. Moreover, $\mathrm{Tr}_{\mathrm{alt}}(bxb^{-1}; Y_{u_b(c)}) = \mathrm{Tr}_{\mathrm{alt}}(x; Y_c)$.

We choose the distinct right series $\mathcal{E}' \in \widehat{C} \cdot \mathcal{E}(J, \widetilde{s})$. We put $e_{\mathrm{orb}} = \sum_{\mathcal{E}'} e_{\mathcal{E}'}$. Thus, $T_\lambda e_{\mathrm{orb}} T_\lambda^{-1} = e_{\mathrm{orb}}$ for $\lambda \in \widehat{C}$. Whenever $c \in C$, we obtain that $e_{\mathrm{orb}} V_{Y,c} \subseteq V_{Y,c}$. Since z_λ is central, $C_{\mathbf{H}^*}(z_\lambda \widetilde{s}) = C_{\mathbf{H}^*}(\widetilde{s})$. By Bonnafé et al. [19, Corollary 6.5], we choose the incidence correspondence Z_{inc} for radicals $\mathbf{W}, \mathbf{W}'$. Then, we choose representatives g_i, g_{i+1} with $g_{i+1} = u_i g_i v_i$ for products of unipotent elements u_i, v_i. As $\nu(u_i) = \nu(v_i) = 1$, the norm tuples $\boldsymbol{\nu}(g_i) = (\nu(g_{i,j}))_j$ are constant along Z_{inc}. Thus, both projections have the same $c = \nu(g_{0,0})$. Thus, $\mathcal{C}_{\mathbf{W},\mathbf{W}'} T_\lambda = T_\lambda \mathcal{C}_{\mathbf{W},\mathbf{W}'}$. By Ruhstorfer [160, proof of Lemma 3.1], it also intertwines $R(d)$ for $d \in N_\Delta(\mathbf{W}) \cap N_\Delta(\mathbf{W}')$. In particular, d may lie in $C_{\mathbf{U}}(Q) \subseteq \mathbf{U}$.

Put $N = \mu_{q-1} \times (H \times J) \lhd A \rtimes \Delta$ and $\vartheta = \omega \boxtimes \theta \boxtimes \gamma^\vee$, where $\omega(z) = z$. The signed Lusztig induction pairs θ, γ, so the alternating character has multiplicity ± 1 at ϑ. Let Ξ_ϑ be its signed projection to the ϑ-part on $I_{\theta,\gamma} = (A \rtimes \Delta)_\vartheta$. By the determinant formula, Ξ_ϑ gives $\widehat{E}_{\theta,\gamma}$ extending ϑ. For $a = (b, \sigma)$ jointly stabilizing (θ, γ), put $a \cdot f = \sigma \circ f \circ u_b^{-1}$ on A. Thus, $\vartheta^a = \vartheta$. The trace and radical comparison give $\Xi_\vartheta^a = \Xi_\vartheta$ and $\widehat{E}_{\theta,\gamma}^a \simeq \widehat{E}_{\theta,\gamma}$. On the ordinary inertia product $\Delta_{\theta,\gamma}$, restrict $\widehat{E}_{\theta,\gamma}$ and put $T = \Delta_{\theta,\gamma}/(H \times J)$. By Corollary 2.5, $\alpha_\theta = \alpha_\gamma$ and $\mu_a^\theta = \mu_a^\gamma$ on T. If a returns $\widehat{C} \cdot (\theta, \gamma)$, take λ_a with $(\theta, \gamma)^{aT_{\lambda_a}} = (\theta, \gamma)$. Thus, the same comparison at aT_{λ_a} gives $\mu^\theta_{aT_{\lambda_a}} = \mu^\gamma_{aT_{\lambda_a}}$.

Let $\varepsilon \in \widehat{C}$ be quadratic and suppose that $\varepsilon \circ \nu$ satisfies $\theta(\varepsilon \circ \nu) = \theta$ and $\gamma(\varepsilon \circ \nu) = \gamma$. Thus, $T_\varepsilon \in I_{\theta,\gamma}$ and $T_\varepsilon^2 = 1$. By Isaacs [95, Corollary 1.6], each intertwiner for $\varepsilon \circ \nu$ is unique up to scalar. Normalize Q_ε on V_γ by $Q_\varepsilon^2 = I$. Then, $\widehat{E}_{\theta,\gamma}(T_\varepsilon) = P_\varepsilon \otimes Q_\varepsilon^\vee$, where $Q_\varepsilon^\vee(f) = f \circ Q_\varepsilon^{-1}$ and $P_\varepsilon^2 = I$ by $T_\varepsilon^2 = 1$. Since $u_b(c) = d_b c^{r^{j_b}}$ and r is odd, $\varepsilon \circ u_b^{-1} = \varepsilon(d_b)\varepsilon$. The coefficient action fixes ε, so in A the stabilizing action takes T_ε to $\varepsilon(d_b)T_\varepsilon$. By the intertwiner for $\widehat{E}_{\theta,\gamma}^a \simeq \widehat{E}_{\theta,\gamma}$, the two normalized factor signs satisfy $\epsilon_a^\theta \epsilon_a^\gamma = \varepsilon(d_b)$. In particular, a common diagonal stabilizer element with $d_b = 1$ gives $\epsilon_a^\theta = \epsilon_a^\gamma$.

Proposition 4.44. *Moreover, we suppose that $P \leq L$, with factors $(D_i)_i$ of* Proposition 4.6. *We write f_i of* Section 3 *on $D_i, N_i = N_{D_i}(Q_i)$. Moreover, we write their rational labels s_i and ordinary local labels η_i. We write the residual spin factor S and its marked subgroup Q_S. We suppose that Φ_S satisfies $\succeq_c$ for $S, N_S(Q_S)$. Thus, we write a regular correspondence $\widetilde{\Phi}_S$ with full labels $(\widetilde{s}_S, \eta_S)$ on $S, N_S(Q_S)$. We suppose that $J_S^2 = (J_S')^2 = I$ for the operators J_S, J_S'. We suppose that $\epsilon_{a,S}^{\mathrm{glo}} = \epsilon_{a,S}^{\mathrm{loc}}$ on the marked preimages $\widehat{S}, N_{\widehat{S}}(Q_S)$. By* Lemma 4.43, *we have that $\epsilon_{a,i}^{\mathrm{glo}} = \epsilon_{a,i}^{\mathrm{loc}}$ on matrix factors D_i. We write central twists $\lambda_i, \lambda_i|_{\widehat{N}_i}$ on $\widehat{D}_i, \widehat{N}_i = N_{\widehat{D}_i}(Q_i)$. Thus, $\Phi_L : \mathcal{E}(L, s)_{p'} \xrightarrow{\sim} Y_s^L(K)$ on the chosen factor series $(s_i)_i$. Moreover, we write marked actions $\mathcal{A}_L$ on the pairs (D_i, N_i). Moreover, $\Phi_L(\chi^a) = \Phi_L(\chi)^a$ for $a \in \mathcal{A}_L \mathcal{H}_p$. $\alpha_\chi = \alpha_{\Phi_L(\chi)}$ and $\mu_a^\chi = \mu_a^{\Phi_L(\chi)}$ on the inertia quotient T_χ. Thus, we have that $\lambda_z^\chi = \lambda_z^{\Phi_L(\chi)}$ on the common centralizer Z_χ. We allow arbitrary $f \geq 1$.*

Proof. We write the geometric preimages $\widehat{D}_i$ and signs $\delta_i : \widehat{D}_i \to \{1, -1\}$. Moreover, by Proposition 4.6, write $\widehat{D}$ with common involution z. On $\widehat{D}$, put $\delta = \prod_i \delta_i$. $L = \ker \delta$ and $K = \ker(\delta|_{N_{\widehat{D}}(Q)})$. Since $L \lhd \widehat{D}$, fix $g \in \widehat{D}$ and $l \in L$, with $lgPg^{-1}l^{-1} = P$. Moreover, $lg \in N_{\widehat{D}}(Q)$, since Q is characteristic in P. Thus, $\delta(\widehat{D}) = \delta(N_{\widehat{D}}(Q))$. Thus, $[\widehat{D} : L] = [N_{\widehat{D}}(Q) : K] \leq 2$.

We write $\pi : \widehat{D} \to \prod_i D_i$, with $\ker \pi = \langle z \rangle$. Moreover, $\pi(P) = \prod_i P_i$, with $P_i \in \mathrm{Syl}_p(D_i)$. Since $p \neq 2$, $\pi|_P : P \xrightarrow{\sim} \prod_i P_i$. Consider an abelian subgroup $B \leq \prod_i P_i$. Moreover, $|B| \leq \prod_i |\,\mathrm{pr}_i(B)|$. Moreover, we put $a_i = \max\{|A| : A \leq P_i,\ [A, A] = 1\}$. $|B| \leq \prod_i a_i$. We choose subgroups $Q_i \leq P_i$, with $[Q_i, Q_i] = 1$ and $|Q_i| = a_i$. Thus, $|\prod_i Q_i| = \prod_i a_i$. Moreover, $\pi(Q) = \prod_i Q_i$. Consider another abelian subgroup $Q_i' \leq P_i$ of order a_i. Since

Q is unique of order $\prod_i a_i$, $Q_i' = Q_i$. We write $D_i = \mathrm{GL}_m(Q_0)$ or $\mathrm{GU}_m(Q_0)$, with $Q_0 = q^a$, and we put $\epsilon = 1$ for $D_i = \mathrm{GL}_m(Q_0)$ and $\epsilon = -1$ for $D_i = \mathrm{GU}_m(Q_0)$. Moreover, put $b = \mathrm{ord}_p(\epsilon Q_0)$ and $v = \lfloor m/b \rfloor$. Write the cyclic torus T_b of order $Q_0^b - \epsilon^b$ and its Sylow subgroup $A \in \mathrm{Syl}_p(T_b)$ and choose $V \in \mathrm{Syl}_p(\mathfrak{S}_v)$. By [207, §§2 and 4], $P_i \simeq A^v \rtimes V$. We write the v torus blocks $W_1, \dots, W_v$, each of dimension b. Moreover, V acts by permutations of $W_1, \dots, W_v$. By Lemma 4.3, $Q_i = A^v$. Assume $v > 0$ and let G_b be the block group of sign ϵ. By [198, Lemma 1.11], $C_{G_b}(A) = T_b$ and $N_{G_b}(A)/T_b \simeq C_b$. On $Q_i = \prod_k A_k$, each nontrivial weight of W_j has kernel $\prod_{k \neq j} A_k$. Thus, $N_{D_i}(Q_i)$ permutes the blocks W_j and acts on $\mathrm{Fix}(Q_i)$ of dimension $m - bv$. The chosen block maps $W_j \simeq W_1$ lift $\mathfrak{S}_v$. Consequently, $N_{D_i}(Q_i) = D_{i,0} \times (N_{G_b}(A) \wr \mathfrak{S}_v)$, with full residual group $D_{i,0}$ on $\mathrm{Fix}(Q_i)$. Moreover, $N_{D_i}(Q_i)$ is the parent in Section 3. If $v = 0$, then $N_{D_i}(Q_i) = D_i$ and f_i is the identity.

Write the a geometric linear factors $\mathbf{G}_j$ on paired isotropic spaces $U_j \oplus U_j^\vee$ and put $t_j = \det(g_j)$ for $g_j \in \mathbf{G}_j$, with $0 \leq j < a$. Moreover, we write the cover coordinate w, with $w^2 = \prod_j t_j$. We write the cocharacter lattice $\{(n_i)_i \in \mathbb{Z}^n : \sum_i n_i \in 2\mathbb{Z}\}$ in the spin cover $\mathbf{Spin}_{2n+1}$ and write its generators $e_i - e_j, e_i + e_j, 2e_i$. Thus, we write the cover equation $w^2 = \prod_j t_j$ on the selected torus $\prod_j T_j$. Write the derived factors $\mathrm{SL}(U_j)$. Moreover, $\mathrm{SL}(U_j)$ lifts isomorphically, since $\mathrm{SL}(U_j)$ is simply connected. Moreover, we write (g_j, w) of $\prod_j \mathbf{G}_j$, with $z = (1, -1)$. Choose the isotropic spaces U_j so that $F(t_0, \dots, t_{a-1}) = (t_{a-1}^{\epsilon q}, t_0^q, \dots, t_{a-2}^q)$. Consider $h \in \widehat{D}_i$, with $t = \det(\pi(h))$. Moreover, $t_j = t^{q^j}$ and $t^{Q_0 - \epsilon} = 1$. If $\epsilon = 1$, then $F(w) = w^q$. Thus, $\delta_i(h) = w^{q-1} = t^{(Q_0-1)/2}$. If $\epsilon = -1$, the spinor norm on D_i is the square class of $N_{\mathbb{F}_{Q_0^2}/\mathbb{F}_q}(\beta)$, with $t = \beta/\beta^{Q_0}$, by [146, Theorem and Remark 3.3]. Consequently, $\delta_i(h) = \beta^{(Q_0^2-1)/2} = t^{(Q_0+1)/2}$. In both cases, $\delta_i(h) = t^{(Q_0-\epsilon)/2}$.

Moreover, we choose a faithful character $\eta_i : w(\widehat{D}_i) \to \mathbb{C}^\times$ of the finite cyclic group $w(\widehat{D}_i)$. We put $\lambda_i(h) = \eta_i(w(h))$. Thus, $\lambda_i(z) = -1$. Thus, $\Theta_i \mapsto \lambda_i \Theta_i$ is a bijection between sectors $\Theta_i(z) = \pm\Theta_i(1)$. Moreover, write $\lambda_i|_{\widehat{N}_i}$ on $\widehat{N}_i = N_{\widehat{D}_i}(Q_i)$. We consider $\theta_i \in \mathrm{Irr}(D_i)$ and $j \in \{0, 1\}$. Thus, we put $f_i(\lambda_i^j(\theta_i \circ \pi)) = (\lambda_i^j|_{\widehat{N}_i})(f_i(\theta_i) \circ \pi)$. Moreover, we write a marked map $a : i \mapsto j$ between $\widehat{D}_i, \widehat{D}_j$. $w = 1$ on the lifted derived product $\prod \mathrm{SL}(U_j)$. We put $\gamma_{a,i} = \lambda_i^a/\lambda_j$. Since $\gamma_{a,i}(z) = 1$, write a determinant character $\gamma_{a,i} = \overline{\gamma}_{a,i} \circ \det \circ \pi$. Moreover, $\gamma_{a,i} = 1$ on the lifted derived factors $\mathrm{SL}(U_j)$. Write a coefficient operation σ on λ_i. Thus, $\lambda_i^\sigma/\lambda_i = \overline{\gamma}_{\sigma,i} \circ \det \circ \pi$. Write the paired restrictions $\gamma_{a,i}|_{\widehat{N}_j}$ and $\gamma_{\sigma,i}|_{\widehat{N}_i}$. If D_i is unitary, by Lemma 3.48, we obtain $T_{a,i}, T_{a,i}'$ before determinant restriction $\mathrm{GU}_m(Q_0) \to \mathrm{SU}_m(Q_0)$. For linear D_i, we choose $T_{a,i}, T_{a,i}'$ from Propositions 3.7, 3.14 and 3.20. Moreover, we write T_i as the marked inertia quotient of $(\Theta_i, f_i(\Theta_i))$. We write $(\mathcal{P}_i, \mathcal{P}_i')$ from the proof of Proposition 3.22, with $u_{F,i} : T_i \to \overline{\mathbb{Q}}^\times$. Thus, write models on D_i, N_i before the special subgroups $\mathrm{SL}_m(Q_0), \mathrm{SU}_m(Q_0)$.

Write the residual spin factor S and the geometric preimage $\widehat{S}$ of $\mathrm{SO}(V_0)^F$. Moreover, $[\widehat{S} : S] = 2$. By Sylow conjugacy in S, $N_{\widehat{S}}(Q_S)/N_S(Q_S) \simeq \widehat{S}/S$. Consequently, the comparison on $\widehat{S}, N_{\widehat{S}}(Q_S)$ is from [141, Theorem 2.9]. Let (θ, ψ) be a matched pair with factor α on quotient T_S. Moreover, we choose $\eta \in \mathrm{Irr}_{\alpha^{-1}}(T_S)$ for both θ, ψ. Consider a diagonal element γ, with $\chi_1^\gamma = \chi_2$ and $\chi_2^\gamma = \chi_1$. Write the two coset summands V_1, V_2 of V. Thus, put $J_{\mathrm{res}} = \mathrm{diag}(I_{V_1}, -I_{V_2})$. Moreover, $T_a J_{\mathrm{res}} T_a^{-1} = \epsilon_a J_{\mathrm{res}}$. Thus, a fixes (χ_1, χ_2) when $\epsilon_a = 1$, and has $\chi_1^a = \chi_2$ when $\epsilon_a = -1$. Moreover, we consider $\chi^\gamma = \chi$. Choose extensions $\widehat{\chi}_1, \widehat{\chi}_2$ with $\widehat{\chi}_2 = \widehat{\chi}_1 \varepsilon_{\mathrm{res}}$, and we write $\varepsilon_{\mathrm{res}} \in \mathrm{Irr}(\widehat{S}/S) \setminus \{1\}$. Thus, we write the comparison on $\widehat{S}, N_{\widehat{S}}(Q_S)$.

We write the regular Levi $\mathbf{H}$ of L and consider $\widetilde{\Theta} \in \mathrm{Irr}(\mathbf{H}^F/Z_o)$ with factor representatives (Θ_i, τ_i). Then, by Lemma 4.7, put $C^+ = \ker \kappa$. If $J_L = \widehat{D} \times_{\{1,-1\}} C$ and $J_K = N_{\widehat{D}}(Q) \times_{\{1,-1\}} C$, the scalar inclusion is $c \mapsto [1, c]$ for $c \in C^+$. Moreover, we write the central character $\omega_C \in \mathrm{Irr}(C^+)$ of $\widetilde{\Theta}$. Moreover, we choose representatives (Θ_i, τ_i), with $\tau_i|_{C^+} = \omega_C$. The two extensions $\tau, \tau\kappa$ of ω_C are characters of C, and we choose $\tau \in \mathrm{Irr}(C)$. Thus, we write $(\Theta_i, \tau_i) \sim (\Theta_i \delta_i, \tau_i \kappa)$ where $\tau_i \neq \tau$. Moreover, $f_i(\Theta_i \delta_i) = f_i(\Theta_i)\delta_i$. Write the local replacements $(f_i(\Theta_i), \tau_i) \sim (f_i(\Theta_i)\delta_i, \tau_i\kappa)$. Moreover, $\prod_i \tau(c_i) = \tau(\prod_i c_i)$. We represent $l \in L$ by $[l, 1]$, with $\prod_i c_i = 1$. Choose the other extension $\tau\kappa$ of ω_C. Thus, we put the product twist $\delta = \prod_i \delta_i$, with $\delta|_L = \delta|_K = 1$.

Write the algebraic regular factors $\mathbf{H}_i$ and regular Levi $\mathbf{H}$ and write multiplication $\mu : \prod_i \mathbf{H}_i \to \mathbf{H}$. Since $\ker \mu = \{(c_i)_i : \prod_i c_i = 1\}$ in the scalar tori $\prod_i \mathbf{G}_m$, $\ker \mu \simeq \mathbf{G}_m^{t-1}$. By Lang, $\mu^F : (\prod_i \mathbf{H}_i)^F \to \mathbf{H}^F$ is onto $\mathbf{H}^F$. Write the regular dual element $\widetilde{s}$ and tuple $\mu^*(\widetilde{s}) = (s_i)_i$. Moreover, $Z(\mathbf{H}_i)$ is connected for every i. Thus, $C_{\mathbf{H}_i^*}(s_i)$ is connected. $\#([s_i]_{\mathbf{H}_i^*}^F/\mathbf{H}_i^{*F}) = 1$ for the geometric class $[s_i]_{\mathbf{H}_i^*}$. We write $\lambda_i(h)\tau(c)$ on the faithful regular preimage (h, c). Since $\tau(-1) = \lambda_i(z)$, $\lambda_i(z)\tau(-1) = 1$ on $\langle (z, -1) \rangle$. We put $\lambda_i = 1$ in the sector $\Theta_i(z) = \Theta_i(1)$. Moreover, $\lambda_i(h)\tau(c) = 1$ on the derived factors $w = c = 1$. By Lang on $\mathbf{H}_i'$, write the character on $(\mathbf{H}_i/\mathbf{H}_i')^F$. Accordingly, write inflation $\mathrm{Inf}_{\mathbf{H}_i^F/Z_o}^{\mathbf{H}_i^F}$. Choose $\tau_o \in \mathrm{Irr}(Z_o)$. Since $\nu(c) = c^2$ on Z_o, choose $\xi \in \mathrm{Irr}(\mathbb{F}_q^\times)$, with $(\xi \circ \nu)|_{Z_o} = \tau_o$. Thus, we

write the central element $z_{\tau,i} \in Z(\mathbf{H}_i^*)^F$. Write the label $z_{\tau,i}s_i$ for both regular characters $\widetilde{\Theta}_i, \widetilde{f}_i(\widetilde{\Theta}_i)$. We consider an active colour $\mathcal{O}$ of length $e \mid b$. Moreover, we write the signed norm $N_{b/e} : T_b \to T_e$. The multiplicity of $\zeta \in \mathcal{O}$ is b/e. Moreover, we write the partition λ, core κ and quotient size w_λ. Moreover, $|\kappa| + (b/e)w_\lambda = |\lambda|$. We write the torus centralizer $C_{D_i}(Q_i)$. Thus, we write the multiplicities of $\zeta \in \mathcal{O}$, including the p-parts of the full dual element s_i. We write rotation labels k_i and permutation labels η_i. Moreover, the multiplicity $\mathrm{mult}_\zeta(s_i)$ is independent of the labels k_i, η_i. Moreover, write the residual correspondence $\widetilde{\Phi}_S$ with its full rational element $\widetilde{s}_S$. By the common character τ, we write full labels $(s_i)_i$ on the paired regular tuples $(\Theta_i)_i, (f_i(\Theta_i))_i$. We inflate from the quotient by Z_o and take the regular restriction to L. Moreover, each constituent χ lies in $\mathcal{E}(L, s)$, where s is the image of $\widetilde{s}$.

Let $\Xi \in \mathrm{Irr}(\widehat{D})$ and denote its paired parent by Ξ'. When $\delta \neq 1$ and $\Xi\delta = \Xi$, Section 4.1 gives the branch spaces $V_\pm = \ker(\mathcal{J} \mp I)$, where $\mathcal{J} = \bigotimes_i J_i$. For a in the mixed stabilizer, the sign on $\mathcal{J}$ is $\prod_i \epsilon_{a,i}$. The two products agree, so a induces matching permutations on $V_\pm, V'_\pm$, with joint inertia $A_\pm$. If $\delta = 1$ or $\Xi\delta \neq \Xi$, the quotient formulas give $\Xi|_L \in \mathrm{Irr}(L)$. Thus, $\chi(1)$ is $\Xi(1)$ or $\Xi(1)/2$. Since p is odd, $p \nmid \chi(1)$ exactly when $p \nmid \Xi(1)$. Thus, the sets $\mathcal{E}(L, s)_{p'}$ and $Y_s^L(K)$ come from the restrictions $\Xi|_L, \Xi'|_K$. Write functions $\alpha_i : T_\chi^2 \to \overline{\mathbb{Q}}^\times$ and $\mu_{a,i} : T_\chi \to \overline{\mathbb{Q}}^\times$ on the joint quotient T_χ. Thus, $\alpha = \prod_i \alpha_i$ and $\mu_a = \prod_i \mu_{a,i}$ on tensor products $\bigotimes_i V_i$. In the split case, write $T_\pm \leq T_\chi$ for $V_\pm, V'_\pm$. Moreover, we write $\alpha_\pm = \alpha|_{T_\pm^2}$ and $\mu_{a,\pm} = \mu_a|_{T_\pm}$. $\lambda_{z,i}^{\mathrm{glo}} = \lambda_{z,i}^{\mathrm{loc}}$ and that $\prod_i \lambda_{z,i}^{\mathrm{glo}}\tau(c) = \prod_i \lambda_{z,i}^{\mathrm{loc}}\tau(c)$. $\langle (z, -1) \rangle$ acts trivially on both models ρ, ρ'. Moreover, consider $H = L$ or $H = K$. We consider a class $[h, c]$ centralizing the embedded group $[H, 1]$. Since $[(h, c), (t, 1)] \in (\widehat{D} \times 1) \cap \langle (z, -1) \rangle = 1$, $ht = th$ for $t \in H$. Moreover, $\lambda_z^{\mathrm{glo}} = \lambda_z^{\mathrm{loc}}$ on the quotients $J_L/\langle (z, -1) \rangle, J_K/\langle (z, -1) \rangle$. By Lemma 4.7, on J_L, J_K the field relation is $\varphi^f(h, c) = (h, c)(z, -1)^u$, with $(-1)^u = \delta(h)$. □

We set $D = \mathrm{SO}_{2n+1}(q)$ and the quadratic character $\eta_q : \mathbb{F}_q^\times \to \{1, -1\}$. Given $g \in \mathrm{GSpin}_{2n+1}(q)$, we choose $\varepsilon(\pi(g)) = \eta_q(\nu(g))$. Then, by Subsection 4.5, we have that $\nu(c) = c^2$ on the scalar kernel $\ker \pi$. We put $S = \ker \varepsilon$. We set $Q_D = \pi(Q)$, with $\pi|_Q : Q \xrightarrow{\sim} Q_D$. Moreover, we choose $\pi_M : M = N_X(Q) \to N_S(Q_D)$, with $\ker \pi_M = \langle z \rangle$. Inflation gives $\mathrm{Irr}(S) \xrightarrow{\sim} \{\chi \in \mathrm{Irr}(X) : \chi(z) = \chi(1)\}$. Moreover, we fix $y \in \mathrm{Sp}_{2n}(q)$ with eigenspaces V_+, V_- of dimensions $2a, 2b$. We have that $a + b = n$. We choose the ordered unipotent labels (ρ_0, ρ_1) on $\mathrm{Sp}(V_+), \mathrm{Sp}(V_-)$. Thus, $\Theta_{a,\rho_0;b,\rho_1}\varepsilon = \Theta_{b,\rho_1;a,\rho_0}$, with dual elements $y, -y$. We choose an irreducible restriction $\Theta|_S$ unless $(a, \rho_0) = (b, \rho_1)$. Moreover, we choose the order $(a, \rho_0; b, \rho_1)$ on the parent group D.

Proposition 4.45. *We put $d = \mathrm{ord}_p(q)$, $e = \mathrm{ord}_p(q^2)$ and $n = ew + h$, with $0 \leq h < e$. We choose $a = eu + c_0, b = ev + c_1$, with $0 \leq c_i < e$. Since $p \nmid \Theta(1)$, $u + v = w$, $c_0 + c_1 = h$ and $p \nmid \binom{w}{u}$. We fix u trivial toral characters and v quadratic characters on the blocks of T_A. We choose the residual cores κ_0, κ_1 and quotient labels η_0, η_1. Thus, we choose the labels $\eta_0, \eta_1\nu_1$, where $\nu_i = \varepsilon_A|_{W_{j_i}}$ and $(j_0, j_1) = (u, v)$. The parent restrictions $\Psi|_{N_S(Q_D)}$ have ordered-label action $(a, \rho_0; b, \rho_1) \mapsto (b, \rho_1; a, \rho_0)$. We choose a field extension $\widehat{\Psi}_A$ of the active parent Ψ_A. Moreover, $\widehat{\Psi}_A^\sigma = \widehat{\Psi}_A$ for $\sigma \in \mathrm{Stab}_{\mathcal{H}_p}(\Psi_A)$.*

Proof. We put $A = v_p(q^{2e} - 1)$ and $h_p(m) = A\lfloor m/e \rfloor + v_p(\lfloor m/e \rfloor!)$, and $v_p(|\mathrm{Sp}_{2m}(q)|) = h_p(m)$. Thus, compare $h_p(n), h_p(a), h_p(b)$. When $c_0 + c_1 \geq e$, $h_p(n) - h_p(a) - h_p(b) \geq A$. When $c_0 + c_1 < e$, $h_p(n) - h_p(a) - h_p(b) = v_p\binom{w}{u}$. Since $p \nmid \Theta(1)$, $u + v = w$ and $c_0 + c_1 = h$. We choose a symbol $\Lambda_i = (A_i, B_i)$ for ρ_i. Take its e-hook core κ_i when d is odd. Moreover, take its e-cohook core κ_i when d is even. Take residue rows $(A_{i,t}, B_{i,t})$ for $t \in \mathbb{Z}/e\mathbb{Z}$. We choose the row order $(A_{i,t}, B_{i,t}) \mapsto (B_{i,t}, A_{i,t})$ after a vertical cohook step $x \mapsto x - e$. Thus, we choose the quotient $\boldsymbol{\lambda}_i = (\lambda_{i,t})_{0 \leq t < 2e}$. Take its character η_i of $W_{j_i} = C_{2e} \wr \mathfrak{S}_{j_i}$, with $(j_0, j_1) = (u, v)$. A runner hook t contributes valuation $A + v_p(t)$ to the degree denominator of ρ_i. Meanwhile, the numerator has valuation $Aj_i + v_p(j_i!)$. By Lemma 4.23, $p \nmid \rho_i(1)$ exactly when $\mathrm{rk}(\kappa_i) = c_i$ and $p \nmid \eta_i(1)$. We keep the rows $A_{i,t}, B_{i,t}$ separate in the hook case d odd. Thus, we reconstruct Λ_i from $\kappa_i, \boldsymbol{\lambda}_i$. Choose the original labels ρ_0, ρ_1 from $(\kappa_0, \eta_0), (\kappa_1, \eta_1)$.

Take the blocks $(W_l)_{l=1}^w$ in the inclusion of Section 4.1 for Q_D. Thus, $D_A = T_A \rtimes W_w$ after the quotient by $\langle z \rangle$. Moreover, choose the lifts $R_t, T_t(x), L(\pi_l)$ for the coordinates t and cycle π_l. $\nu(T_t(x)) = x$, and we put $E_l = \prod_{t \in l} T_t(-1)$ and take the first coordinate $t_l \in l$. Thus, we choose $\widehat{k}_l = E_l^h(\prod_{t \in l} R_t)L(\pi_l)$ when $d = e$ is odd. We choose $\widehat{k}_l = T_{t_l}((-1)^{h+e-1})R_{t_l}L(\pi_l)$ when $d = 2e$, and we choose the injection j into the centralizer, with $\nu(j(x)) = (-1)^{hs(x)}\nu(x)$. Moreover, we choose the parity $s(x) \in \mathbb{Z}/2\mathbb{Z}$ of coordinate sign changes in x. Choose the orthogonal image $k_l = \pi(\widehat{k}_l)$. Let Π_l permute adjacent blocks W_l, W_{l+1}. Thus, $k_l^{2e} = 1$ and $\Pi_l k_j \Pi_l^{-1} = k_{(l,l+1)(j)}$. $[k_l, k_j] = 1$ for $l \neq j$, since $\pi(z) = 1$. With $\varepsilon_A = \varepsilon|_{D_A}$, we set $\epsilon_q = \eta_q(-1)$. Moreover, $\varepsilon_A|_{T_A}$ is the product of quadratic characters ϑ_l on the blocks W_l. $(\varepsilon_A(k_l), \varepsilon_A(\Pi_l)) = (1, \epsilon_q)$ when $d = e$ is odd. When $d = 2e$, $(\varepsilon_A(k_l), \varepsilon_A(\Pi_l)) = ((-1)^{w-1}, -\epsilon_q^e)$. Since

$\nu(\widehat{k}_l) = 1$ in the split case $d = e$, $\varepsilon_A(k_l) = 1$. $\det(\Pi_l) = (-1)^e = -1$ when $d = e$ is odd, and $\varepsilon_A(\Pi_l) = \epsilon_q$. Moreover, choose the block product $v_A = a_1 \cdots a_w$ in the nonsplit Frobenius $F = \mathrm{Int}(v_A)F_q$. Whenever $j \neq l$, $[a_j, \widehat{k}_l] = z$. We obtain $F(\widehat{k}_l)\widehat{k}_l^{-1} = z^{w-1}$. Thus, $\widehat{\Pi}_l v_A \widehat{\Pi}_l^{-1} = z v_A$ and $F(\widehat{\Pi}_l)\widehat{\Pi}_l^{-1} = z$. $\nu(\widehat{k}_l) = 1$ and $\nu(\widehat{\Pi}_l) = (-1)^e$. Moreover, we choose a raw lift $h \in \mathbf{Spin}$, with $F(h) = zh$. We choose $c \in \overline{\mathbb{F}}_q^{\times}$, with $c^{q-1} = -1$. Thus, $F(ch) = ch$ and $\eta_q(c^2) = -1$. We choose the nonsplit values $(-1)^{w-1}, -\epsilon_q^e$ for $\varepsilon_A(k_l), \varepsilon_A(\Pi_l)$.

We fix $\theta_A \in \mathrm{Irr}(T_A)$, trivial on the first u blocks, and we choose $\theta_A|_{T_l} = \vartheta_l$ on the last v blocks. Moreover, we choose $\widetilde{\theta}_A(tw) = \theta_A(t)$ for $t \in T_A, w \in W_u \times W_v$. With $\nu_0 = \varepsilon_A|_{W_u}$, set $\nu_1 = \varepsilon_A|_{W_v}$. Then, we put $\Psi_A = \mathrm{Ind}_{T_A \rtimes (W_u \times W_v)}^{D_A}(\widetilde{\theta}_A \otimes (\eta_0 \boxtimes \eta_1\nu_1))$. By Clifford theory [43], $\Psi_A(1) = \binom{w}{u}\eta_0(1)\eta_1(1)$. $\theta_A\varepsilon_A|_{T_A}$ has quadratic values on the first u blocks. Whenever $t \in T_l$, $u < l \leq w$, $(\theta_A\varepsilon_A)(t) = 1$. Thus, choose the labels $\eta_0\nu_0, \eta_1$ before the block permutation $(u, v) \mapsto (v, u)$. Set $D_R = \mathrm{SO}_{2h+1}(q)$ and its parent Ξ_R with ordered cores (κ_0, κ_1). Moreover, set $\varepsilon_R = \varepsilon|_{D_R}$. On $D_A \times D_R$, we set $\Psi = \Psi_A \boxtimes \Xi_R$. Moreover, we restrict Ψ to $N_S(Q_D) = \ker(\varepsilon_A\varepsilon_R)$. We choose the residual twist $(\kappa_0, \kappa_1) \mapsto (\kappa_1, \kappa_0)$. Then, $\Psi\varepsilon = \Psi$ if and only if $u = v$, $\eta_0 = \eta_1$, $\kappa_0 = \kappa_1$. By reconstruction $(\kappa_i, \eta_i) \mapsto \rho_i$, $(a, \rho_0) = (b, \rho_1)$. Thus, we choose one irreducible restriction $\Psi|_{N_S(Q_D)}$ for each unequal unordered pair $\{(a, \rho_0), (b, \rho_1)\}$. Choose two distinct constituents ψ_+, ψ_-, each with multiplicity 1, for $(a, \rho_0) = (b, \rho_1)$. By Proposition 4.39, we choose $Y_s^X(M)$ from $(\kappa_i, \eta_i, \pm)$, with inflation along π_M.

Choose the source matrices of W_j over $\mathbb{Q}(\mu_{2e})$. The model combines tensor induction from C_{2e} and rational Specht modules S^λ. Since $2e \mid p-1$, $\sigma|_{\mathbb{Q}(\mu_{2e})} = 1$ for $\sigma \in \mathcal{H}_p$. $\theta_A(t), \nu_i(w) \in \{1, -1\}$. We choose the field carry $z_{\mathrm{car}} = \prod_l k_l^{e+1}$ when $d = e$ is odd. Moreover, we choose $z_{\mathrm{car}} = \prod_l k_l$ when $d = 2e$. $z_{\mathrm{car}} \in Z(W_w) \cap (W_u \times W_v)$, and $\theta_A^{z_{\mathrm{car}}} = \theta_A$. Thus, $R_{\Psi_A}(z_{\mathrm{car}}) = \zeta_{\mathrm{car}} I$, with $\zeta_{\mathrm{car}} \in \mu_d$. Take coset representatives t_i given by block permutations $t_i \in W_w$. $t_i^{-1} z_{\mathrm{car}} t_i = z_{\mathrm{car}}$ on every source summand V_i. Moreover, consider the field action F_r on the toral signs $\theta_A(t)$ and complement matrices $R_{W_j}(w)$. $\theta_A(t^r) = \theta_A(t)$ and $R_{W_j}(F_r(w)) = R_{W_j}(w)$, and we take $P_F^f = R_{\Psi_A}(z_{\mathrm{car}})$, with $q = r^f$. Thus, set $D_r = \mathrm{ord}_p(r)$ and $D_r / \gcd(D_r, f) = d$. $\{\alpha^f : \alpha \in \mu_{D_r}\} = \mu_d$ and $D_r \mid p-1$, and we choose $\alpha_{\mathrm{car}} \in \mu_{D_r}$, with $\alpha_{\mathrm{car}}^f = \zeta_{\mathrm{car}}$. Consequently, $\sigma(\alpha_{\mathrm{car}})/\alpha_{\mathrm{car}} = 1$ and $\widehat{\Psi}_A^\sigma = \widehat{\Psi}_A$. □

We choose $(a, \rho_0) = (b, \rho_1)$, with $u = v$. We choose an involution $g \in W_{2u}$ with $g\{1, \ldots, u\} = \{u+1, \ldots, 2u\}$. Moreover, we choose the operator J on the induced space V_{Ψ_A}. We choose the coset permutation $tH \mapsto tgH$ for $H = W_u \times W_u$. We choose the tensor flip $v_0 \otimes v_1 \mapsto v_1 \otimes v_0$ and the coset multiplier $\varepsilon_A(t)$. Thus, we find $J^2 = \varepsilon_A(g)I$. We have that $\varepsilon_A(g) = \epsilon_q^u$ when $d = e$ is odd. When $d = 2e$, $\varepsilon_A(g) = (-\epsilon_q^e)^u$. Moreover, we fix $a_A \in \mu_4$, with $a_A^2 = \varepsilon_A(g)$. We put $J_A = a_A^{-1}J$, with $J_A^2 = I$. Whenever $\sigma \in \mathcal{H}_p$, $J^\sigma = J$. Moreover, we fix c_σ, with $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$. Thus, we find $\sigma(a_A)/a_A = \eta_q(c_\sigma)^{eu}$. If $d = e$ is odd, $(q/p) = 1$. If $d = 2e$ and e is odd, $(q/p) = (-1)^{(p-1)/2}$. By quadratic reciprocity for $(p/r), (r/p)$, we choose the ratio $\sigma(a_A)/a_A$ for the p-power action on a_A. Its powers cover every $\sigma \in \mathcal{H}_p$. If e is even, $4 \mid p-1$. Thus, $\sigma(a_A) = a_A$ and $\eta_q(c_\sigma)^{eu} = 1$. The residual ranks satisfy $c_0 = c_1$ and $a - c_0 = eu$. We choose the residual exchange ratio $\eta_q(c_\sigma)^{c_0}$. Then, by Lemma 4.26, we have that $\eta_q(c_\sigma)^{eu}\eta_q(c_\sigma)^{c_0} = \eta_q(c_\sigma)^a$. If $c_0 = 0$, the residual operator is $J_R = I$. We have that $\sigma(J_R) = J_R$ and residual ratio 1.

Lemma 4.46. *Moreover, we fix $X, D \lhd B$, with $X \leq D$ and $[D : X] = 2$. We choose $A \lhd B$, with $B = DA$ and $D \cap A = X$. Choose $\Theta \in \mathrm{Irr}(D)$ and an extension $\widehat{\Theta} \in \mathrm{Irr}(B)$. We suppose that $\Theta|_X = \sum_i \chi_i$ with distinct A-invariant characters $\chi_i \in \mathrm{Irr}(X)$. We choose the unique constituent $\widehat{\chi}_i \in \mathrm{Irr}(A \mid \chi_i)$ of $\widehat{\Theta}|_A$, with $\widehat{\chi}_i|_X = \chi_i$. Thus, we choose a mixed operation t fixing X, D, A, B. We suppose that $\widehat{\Theta}^t = \widehat{\Theta}\lambda_t$, with $\lambda_t \in \mathrm{Irr}(B/D)$. We put $\chi_i^t = \chi_{i^t}$. Moreover, $\widehat{\chi}_i^t = \widehat{\chi}_{i^t}\lambda_t|_A$.*

Proof. Choose a representation $R : B \to \mathrm{GL}(V)$ of $\widehat{\Theta}$. Whenever $e_{\chi_i} \in Z(\mathbb{C}[X])$, put $V_i = e_{\chi_i}V$. Since $\Theta|_X = \sum_i \chi_i$, $\dim V_i = \chi_i(1)$. Whenever $a \in A$, $R(a)V_i = V_i$. Choose $\widehat{\chi}_i = \mathrm{Tr}(R|_{A,V_i})$. Thus, $\widehat{\chi}_i \in \mathrm{Irr}(A)$ and $\widehat{\chi}_i|_X = \chi_i$. We find $\widehat{\Theta}|_A = \sum_i \widehat{\chi}_i$. Take distinct restrictions $\widehat{\chi}_i|_X = \chi_i$. Consequently, $\{\xi \in \mathrm{Irr}(A) : \langle \widehat{\Theta}|_A, \xi\rangle_A > 0,\ \xi|_X = \chi_i\} = \{\widehat{\chi}_i\}$. We choose $\widehat{\Theta}^t|_A = (\widehat{\Theta}|_A)(\lambda_t|_A)$, and we have that $\lambda_t|_X = 1$ and $\chi_i^t = \chi_{i^t}$. Thus, $\widehat{\chi}_i^t = \widehat{\chi}_{i^t}\lambda_t|_A$. □

We set $D = \mathrm{SO}_{2n+1}(q)$, $X = S$ and $B = D \rtimes \langle F_r \rangle$. We set $A = S \rtimes \langle F_r \rangle$. Then, by Lemma 4.46, we choose extensions $\widehat{\chi}_i$. We put $B_Q = N_B(Q_D)$, $A_Q = A \cap B_Q$ and $D_Q = N_D(Q_D)$. We put $S_Q = N_S(Q_D)$. Consequently, by Sylow conjugacy, $B_Q/D_Q \simeq B/D$ and $A_Q/S_Q \simeq A/S$. Suppose f is even. Then, $\eta_q(c_\sigma) = 1$ for $c_\sigma \in \mathbb{F}_r^\times$. We have that $\eta_q(c_\sigma)^a = 1$, so $\chi_i^\sigma = \chi_i$. Moreover, $P_F J_A = J_A P_F$ on the active space V_{Ψ_A}. We choose the global field action $\chi_i^{F_r} = \chi_i$ and the corresponding local branches $\psi_i^{F_r} = \psi_i$. We choose the single irreducible restriction $\Theta|_S$ when $(a, \rho_0) \neq (b, \rho_1)$. Thus, $\widehat{\chi}_i^t = \widehat{\chi}_{i^t}(\lambda_t|_A)$ for the full parent ratio λ_t, including $\lambda_t \neq 1$. The active parent ratio is 1 by Proposition 4.45. The residual ratio is $\sigma(\alpha_{k,l})/\alpha_{k,l}$ from Lemma 4.26. If $2 \nmid d$, both ordinary supports are (k, l). By Ruhstorfer et

al. [156, Lemma 8.2], we choose one root $\alpha_{k,l}$ for the two supports (k,l). If d is even, $v_2(\mathrm{ord}_p(r)) = v_2(f) + v_2(d)$. Thus, we have that $2f_2 \mid p-1$. Whenever $\sigma \in \mathcal{H}_p$, $\sigma(\alpha_{k,l}) = \alpha_{k,l}$. Consider residual ordinary supports (k',l') that may differ from (k,l) after e-cohooks. Moreover, we have that $\sigma(\alpha_{k',l'}) = \alpha_{k',l'}$ on the residual source (k',l'). We choose $D, S, \varepsilon, D_A = T_A \rtimes W_w$ from Proposition 4.45. We choose the centralizer $C^{\circ}_{\mathrm{Sp}_{2n}}(y)^F \simeq \mathrm{Sp}_{2a}(q)^2 \times \mathrm{GL}_b(q)$. Moreover, we take its unipotent labels $(\rho_0, \rho_1, \lambda)$. We order the eigenspaces V_+, V_- of y for $1, -1$. We take $\iota \in \mathbb{F}_q^{\times}$, with $\iota^2 = -1$, and order $V_\iota, V_{-\iota}$. Thus, we choose the graph map $\gamma_{\mathrm{gr}}(g) = (g^{-1})^{\mathsf{T}}$ when ι is replaced by $-\iota$. We have that $\chi_\lambda^{\gamma_{\mathrm{gr}}} = \chi_\lambda$ for the unipotent character χ_λ of $\mathrm{GL}_b(q)$.

Proposition 4.47. *We suppose that $q = r^f \equiv 1 \pmod 4$ and $d = \mathrm{ord}_p(q) = e$ is odd. Moreover, we put $n = 2a+b = ew+h$, with $0 \leq h < e$. Moreover, we write $a = eu+c$ and $b = ev+s_0$, with $0 \leq c, s_0 < e$. We suppose that $p \nmid \chi(1)$ in the series of y. We have that $2c+s_0 = h$ and $w = 2u+v$. Thus, we fix $\eta_0, \eta_1 \in \mathrm{Irr}(W_u)$ and $\zeta \in \mathrm{Irr}(C_e \wr \mathfrak{S}_v)$. We have that $\Psi_A(1) = 2^v w! \eta_0(1)\eta_1(1)\zeta(1)/(u!^2 v!)$, and we choose the twist $\Psi_A \varepsilon_A$ with ordered labels (η_1, η_0, ζ). Moreover, we set $\eta_0 = \eta_1$ and the normalized operator $J_A^2 = I$. We have that $U_\sigma J_A^\sigma U_\sigma^{-1} = J_A$. We choose the full field extension $\widehat{\Psi}_A$. Thus, $\widehat{\Psi}_A^\sigma = \widehat{\Psi}_A$ for $\sigma \in \mathrm{Stab}_{\mathcal{H}_p}(\Psi_A)$. Moreover, we choose the residual operators J_R, Φ_R. We have that $J = J_A \otimes J_R$ and $\Phi = \Phi_A \otimes \Phi_R$ on $V_{\Psi_A} \otimes V_{\Psi_R}$.*

Proof. Put $A = v_p(q^e - 1) = v_p(q^{2e} - 1)$. Then, by Proposition 4.45 $v_p(|\mathrm{Sp}_{2n}(q)|) = Aw + v_p(w!)$. We find $v_p(|\mathrm{GL}_b(q)|) = Av + v_p(v!)$. Indeed, $v_p(q^{et} - 1) = A + v_p(t)$ for $t \geq 1$. Moreover, $v_p(|\mathrm{Sp}_{2a}(q)|) = Au + v_p(u!)$. We choose the semisimple index I_y with $v_p(I_y) = A(w - 2u - v) + v_p(w!/(u!^2 v!))$, and $w!/(u!^2 v!) \in \mathbb{Z}$ when $w \geq 2u + v$. Since $v_p(I_y) = 0$, $w = 2u + v$ and $2c + s_0 = h$. $p \nmid w!/(u!^2 v!)$. Take ordered e-hook cores $\kappa_0, \kappa_1, \kappa_2$ for ρ_0, ρ_1, λ. Thus, $\mathrm{rk}(\kappa_0) = \mathrm{rk}(\kappa_1) = c$ and $|\kappa_2| = s_0$. We choose the quotient labels η_0, η_1, ζ and choose the inverse bijections $(\kappa_0, \eta_0) \mapsto \rho_0$, $(\kappa_1, \eta_1) \mapsto \rho_1$ and $(\kappa_2, \zeta) \mapsto \lambda$. Moreover, we include weight 0 in each bijection $(\kappa_i, \eta_i) \mapsto \rho_i$.

Choose the cyclic block tori T_l, with $|T_l| = q^e - 1$, and we choose a quadratic character ϑ and quartic character τ on T_l, with $\tau^2 = \vartheta$. Thus, fix $\theta_A \in \mathrm{Irr}(T_A)$ with labels $1, \vartheta, \tau$ on u, u, v blocks. Choose the rotation k_l of Proposition 4.45. $\tau^q = \tau$ and $\tau^{k_l} = \tau^{-1}$. Moreover, k_l^2 fixes τ and $|\langle k_l^2 \rangle| = e$. With $I = I_{W_w}(\theta_A) = W_u \times W_u \times (C_e \wr \mathfrak{S}_v)$, set $C_e = \langle k_l^2 \rangle$ on each quartic block l. Thus, we choose $\widetilde{\theta}_A(tw) = \theta_A(t)$ for $t \in T_A, w \in I$. Put $\nu_1 = \varepsilon_A|_{W_u}$ on the second factor W_u. Then, put $\Psi_A = \mathrm{Ind}_{T_A \rtimes I}^{D_A}(\widetilde{\theta}_A \otimes (\eta_0 \boxtimes \eta_1\nu_1 \boxtimes \zeta))$. By Clifford theory [43], $\Psi_A \in \mathrm{Irr}(D_A)$. Moreover, we set $R = R_{\Psi_A} : D_A \to \mathrm{GL}(V_{\Psi_A})$, with $\mathrm{Tr}\, R = \Psi_A$. $[W_w : I] = 2^v w!/(u!^2 v!)$, and $\Psi_A(1) = 2^v w! \eta_0(1)\eta_1(1)\zeta(1)/(u!^2 v!)$. Thus, we consider all characters in $\mathrm{Irr}(D_A \mid \theta_A^{W_w})$ from the labels (η_0, η_1, ζ).

We choose the block involution g with $g\{1, \ldots, u\} = \{u+1, \ldots, 2u\}$, and we put $h_4 = \prod_{l \text{ quartic}} k_l^e$. Moreover, $g^2 = h_4^2 = 1$ and $gh_4 = h_4 g$. Whenever $x \in I$, $h_4 x = x h_4$. $\tau^{h_4} = \tau^{-1}$ on every quartic block l. Thus, $(\theta_A \varepsilon_A)^{gh_4} = \theta_A$. Accordingly, the label map is $(\eta_0, \eta_1, \zeta) \mapsto (\eta_1, \eta_0, \zeta)$. By Proposition 4.45 and $q \equiv 1 \pmod 4$, $\varepsilon_A(k_l) = \varepsilon_A(\Pi_l) = 1$. Moreover, $\varepsilon_A(h_4) = \varepsilon_A(gh_4) = 1$. We choose the residual parent Ψ_R with ordered cores $(\kappa_0, \kappa_1, \kappa_2)$. Moreover, choose the twist $(\kappa_0, \kappa_1, \kappa_2) \mapsto (\kappa_1, \kappa_0, \kappa_2)$. $(\Psi_A \boxtimes \Psi_R)\varepsilon = \Psi_A \boxtimes \Psi_R$ exactly when $\rho_0 = \rho_1$.

Moreover, we set $\eta_0 = \eta_1$ and the source space V_{Ψ_A}. We choose the coset map $t(T_A \rtimes I) \mapsto tgh_4(T_A \rtimes I)$. Thus, we choose the tensor flip $v_0 \otimes v_1 \mapsto v_1 \otimes v_0$ and coset multiplier $\varepsilon_A(t)$. We choose all representatives $t \in W_w$ and choose their operator J_A, with $J_A R(x) J_A^{-1} = \varepsilon_A(x) R(x)$. Moreover, $J_A^2 = \varepsilon_A(gh_4) I = I$. We choose U_4 as the coset permutation $tI \mapsto th_4 I$. We fix $U_\sigma : V_{\Psi_A}^\sigma \to V_{\Psi_A}$. Thus, $U_\sigma = I$ when $\tau^\sigma = \tau$, and $U_\sigma = U_4$ when $\tau^\sigma = \tau^{-1}$. Choose the quotient matrices over $\mathbb{Q}(\mu_{2e})$, and $2e \mid p-1$, so $\sigma|_{\mathbb{Q}(\mu_{2e})} = 1$ for $\sigma \in \mathcal{H}_p$. Moreover, $J_A^\sigma = J_A$ and $U_4 J_A = J_A U_4$. Choose the rational tensor flip $v_0 \otimes v_1 \mapsto v_1 \otimes v_0$. We obtain $U_\sigma J_A^\sigma U_\sigma^{-1} = J_A$. Thus, we choose $J_A \otimes J_R$ on the full parent $V_{\Psi_A} \otimes V_{\Psi_R}$. The sign is $\epsilon_{\sigma,R}$ of the residual operator J_R.

Moreover, we put $v_A = \prod_l k_l^{e+1}$. $v_A \in Z(W_w) \cap I$ and $v_A^e = 1$. Thus, $R_{\Psi_A}(v_A) = \gamma I$, with $\gamma \in \mu_e$. $t^{-1} v_A t = v_A$ for all representatives $t \in W_w$. We set $U_F = I$ when $r \equiv 1 \pmod 4$ and $U_F = U_4$ when $r \equiv 3 \pmod 4$. Since $q = r^f \equiv 1 \pmod 4$, $2 \mid f$ in the case $r \equiv 3 \pmod 4$. $U_F^f = I$, and we choose the source relation $a^f = v_A^{-1}$. Moreover, put $D_r = \mathrm{ord}_p(r)$ and $D_r / \gcd(D_r, f) = e$. $\{\alpha^f : \alpha \in \mu_{D_r}\} = \mu_e$, and we choose $\alpha \in \mu_{D_r}$, with $\alpha^f = \gamma^{-1}$. Thus, set $\Phi_A = \alpha U_F$, with $\Phi_A^f = R_{\Psi_A}(v_A^{-1})$. $D_r \mid p-1$, so $\sigma(\alpha) = \alpha$, and $U_\sigma U_F = U_F U_\sigma$. Consequently, $U_\sigma \Phi_A^\sigma U_\sigma^{-1} = \Phi_A$. We include $v = 0$, with $h_4 = 1$ and $U_4 = I$. □

Moreover, we take even $d = 2e$ in the split series $q \equiv 1 \pmod 4$. We choose a quartic eigenvalue ι, with $\iota^2 = -1$. Since $\iota^{q^e} = \iota \neq \iota^{-1}$, ι is not a weight on W_l. We choose the full partition λ in the residual rank h. We take active labels $(\kappa_0, \eta_0), (\kappa_1, \eta_1)$ from Proposition 4.45 on W_l. If $2 \mid f$, the ratio is $\sigma(\alpha_{k,l})/\alpha_{k,l}$ from residual source (k,l) by Lemma 4.26. If d is odd, the ordinary supports (k,l) are fixed by the hooks $\rho_i \mapsto \kappa_i$. If d is even, we have that $2f_2 \mid p-1$. Thus, we have that $\sigma(\alpha_{k,l}) = \alpha_{k,l}$ and $\sigma(\alpha_{k',l'}) = \alpha_{k',l'}$ on $(k,l), (k',l')$. We allow arbitrary partitions λ

and arbitrary two-cores $\mathrm{core}_2(\lambda)$.

Proposition 4.48. *We suppose that $q \equiv 1 \pmod 4$ and take nonidentity quasi-isolated $s \in X^*$. Moreover, we write the bijection $\Omega_s : \mathcal{E}(X,s)_{p'} \xrightarrow{\sim} Y_s^X(M)$. Moreover, we put $R = \widetilde{X}/Z_o$ and $M_R = \widetilde{M}/Z_o$. We write its regular lift $\widetilde{\Omega}_s : \mathrm{Irr}_{p'}(R \mid \mathcal{E}(X,s)) \xrightarrow{\sim} \mathrm{Irr}_{p'}(M_R \mid Y_s^X(M))$ and write full rational labels $[\widetilde{s}]$ on $(\Theta, \widetilde{\Omega}_s(\Theta))$, with $\pi^*(\widetilde{s}) = s$. Thus, $\widetilde{\Omega}_s(\Theta\lambda) = \widetilde{\Omega}_s(\Theta)(\lambda|_{M_R})$ for norm characters $\lambda \in \mathrm{Irr}(R/X)$. Moreover, we have that $\alpha_\Theta = \alpha_{\widetilde{\Omega}_s(\Theta)}$ and $\mu_t^\Theta = \mu_t^{\widetilde{\Omega}_s(\Theta)}$ on the marked inertia quotient T_Θ.*

Proof. Put $d = \mathrm{ord}_p(q)$, $e = \mathrm{ord}_p(q^2)$ and $n = ew + h$, with $0 \le h < e$. Moreover, by [208, Lemma 6.12], choose a lift y in the dual group Sp_{2n}. We choose $\iota \in \mathbb{F}_q^\times$, with $\iota^2 = -1$. $\mathrm{Spec}(y) \subseteq \{1, -1, \iota, -\iota\}$. Thus, we put the faithful sector $F(y) = -y$. Write the transition $\lambda \mapsto -\lambda^q$ on toral blocks W_l and consider a positive block W_l, with $d = e$ odd. Since $(-1)^e\lambda^{q^e} = -\lambda$, we would have that $\lambda = -\lambda$. Moreover, $\lambda \ne 0$ and $r \ne 2$. Consider a negative block W_l, with endpoint λ^{-1}. Thus, $\lambda \in \{\iota, -\iota\}$ when e is odd, and $\lambda \in \{1, -1\}$ when e is even. Write $2a$ for each multiplicity of $1, -1$ and b for each multiplicity of $\iota, -\iota$. Moreover, $n = 2a + b$. Moreover, we consider odd e, with $b = ew + b_0$ and $h = 2a + b_0$. We write the unitary quotient on b and residual labels (a, b_0) and consider even $e = 2j$, with $a = jw + a_0$ and $h = 2a_0 + b$. Since $e = 2j$, we write the cohook quotient on a and residual labels (a_0, b). We put the sector $\chi(z) = \chi(1)$, with $F(y) = y$, and we write its transition $\lambda \mapsto \lambda^q$. Moreover, we consider $\lambda \in \{1, -1\}$ on either block sign $\pm$. $\lambda \in \{\iota, -\iota\}$ occurs only on positive blocks W_l. Consider the involution case $b = 0$ and mixed linear case $b > 0$. Thus, write the full partition λ in the residual factor when $d = 2e$. Choose all core labels κ with quotient and branch labels $(\eta, \pm)$ over the toral orbit $\theta_A^{W_w}$. By Clifford theory [43], write all $\psi \in Y_s^X(M)$ over $(\kappa, \eta, \pm)$. Moreover, $p \nmid |s|$ and that $\xi|_Q = \xi(1)1_Q$ for constituents $\xi \in \mathcal{E}(C_X(Q), s)$. By Section 4.1, we have that $\mathrm{Im}\,\Omega_s = Y_s^X(M)$ with ambient rational class $[s]$.

Choose the parent Θ from Proposition 4.45, with support (k, l). Then, by Lemma 4.26, write $\gamma_{k,l}(\sigma) = \sigma(\alpha_{k,l})/\alpha_{k,l}$, with $\alpha_{k,l}^f = \xi_{k,l}^{-1}$. Write the source scalar $\xi_{k,l}$ for (k, l). If d is odd, both supports equal (k, l). Thus, we choose $\alpha_{k,l}^{\mathrm{glo}} = \alpha_{k,l}^{\mathrm{res}}$. Consider even d and two-power roots $\alpha_{k,l}, \alpha_{k',l'}$. Moreover, $\sigma(\alpha_{k,l}) = \alpha_{k,l}$ and $\sigma(\alpha_{k',l'}) = \alpha_{k',l'}$ for $\sigma \in \mathcal{H}_p$. By Propositions 4.45 and 4.47, $\gamma_A(\sigma) = 1$. We choose the doubled space $I_+ \oplus I_-$ for the mixed linear parent when $2 \mid f$. Moreover, $\gamma_{\mathrm{glo}}(\sigma) = \gamma_{\mathrm{res}}(\sigma)$ on the residual involution source (k, l). Thus, we write symplectic hooks $\rho_i \mapsto \kappa_i$ with fixed supports (k, l) when d is odd. $\lambda_{\mathrm{res}} = \lambda$ when d is even. We allow every $\mathrm{core}_2(\lambda)$ in $\gamma_{\mathrm{glo}} = \gamma_{\mathrm{res}}$.

Moreover, we choose equal labels $\rho_0 = \rho_1$ of rank a. With $b = |\lambda|$, put $K_t = |\,\mathrm{core}_2(\lambda)|$. Moreover, we put $b = K_t = 0$ in the involution series $[s]$. By Lemma 4.26, write the sign $s_{\mathrm{glob}}(\sigma) = \eta_q(c_\sigma)^{a+(b+K_t)/2}$. With $\eta_q : \mathbb{F}_q^\times \to \{1, -1\}$ and $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$, write the residual sign $s_{\mathrm{res}}(\sigma)$. At odd d, put $g = \sum_{x \in \mathbb{F}_q^\times} \eta_q(x)\zeta_r^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x)}$. By [44, Corollary 2.3], $g^2 = \eta_q(-1)q = q$, since $q \equiv 1 \bmod 4$. Hence, $\mathcal{H}_p$ fixes g by [156, Lemma 2.3(d)]. The substitution $x \mapsto c_\sigma x$ gives $\sigma(g)/g = \eta_q(c_\sigma) = 1$. Thus, $s_{\mathrm{glob}}(\sigma) = s_{\mathrm{res}}(\sigma) = 1$. We allow $\mathrm{core}_2(\lambda) \ne \mathrm{core}_2(\mu)$ for the residual partition μ. In particular, on active involution blocks W_l, the sign is 1 by Proposition 4.45. Write R_{h_4}, J_A from Proposition 4.47 on $V_A = V_{\Psi_A}$. We choose $i \in \mathbb{C}$, with $i^2 = -1$. Thus, $R_{h_4}J_A = J_A R_{h_4}$, including $\sigma(i) = -i$.

We consider $d = 2e$, with $\iota^{q^e} = \iota \ne \iota^{-1}$. Moreover, $\lambda_{\mathrm{res}} = \lambda$. Thus, write u active blocks of each colour $1, -1$. With $a_0 = a - eu$, $s_{\mathrm{glob}}(\sigma)/s_{\mathrm{res}}(\sigma) = \eta_q(c_\sigma)^{eu}$. Moreover, write s_A from the calculation after Proposition 4.45, with $s_A(\sigma) = \eta_q(c_\sigma)^{eu}$. Thus, $s_{\mathrm{glob}}(\sigma) = s_A(\sigma)s_{\mathrm{res}}(\sigma)$. We write $s_{\mathrm{res}}(\sigma) = 1$ when the residual rank is 0. Moreover, we include the involution series $b = 0$.

Consider unequal labels $\rho_0 \ne \rho_1$. Moreover, $\Theta|_{\ker\varepsilon} \in \mathrm{Irr}(\ker\varepsilon)$. Thus, we consider equal labels $\rho_0 = \rho_1$ and paired constituents $(\chi_+, \chi_-), (\psi_+, \psi_-)$. $s_{\mathrm{glob}} = s_A s_{\mathrm{res}}$ is the permutation sign on $\{\chi_+, \chi_-\}, \{\psi_+, \psi_-\}$. We consider even f, with q a square. Since $\eta_q(c_\sigma) = 1$, we have that $\chi_\pm^\sigma = \chi_\pm$ and $\psi_\pm^\sigma = \psi_\pm$. By Lemma 4.46, if χ_i is field-stable, $\widehat{\chi}_i^t = \widehat{\chi}_{i^t}(\lambda_t|_A)$ with the full parent ratio λ_t. Consider odd f and ordered parents $\Theta, \Theta_{\mathrm{res}}$. Moreover, $\mathbb{Q}(\Theta) = \mathbb{Q}(\Theta_{\mathrm{res}}) = \mathbb{Q}$ and $\Theta^{F_r} = \Theta$, $\Theta_{\mathrm{res}}^{F_r} = \Theta_{\mathrm{res}}$. By Lemma 4.26, choose I_+, I_- with inverse quartic twists τ, τ^{-1}. $\mathrm{ch}(I_+) = \mathrm{ch}(I_-)$ and that Θ_{res} extends rationally over the field quotient C_f. By Lemma 4.32 with $\delta = 1$, choose rational extensions $\widehat{\Theta}, \widehat{\Theta}_{\mathrm{res}}$. We write the active correction $\alpha_A \in \mu_{\mathrm{ord}_p(r)}$. Moreover, $\mathrm{ord}_p(r) \mid p - 1$ and $\sigma(\alpha_A) = \alpha_A$ for $\sigma \in \mathcal{H}_p$. Thus, $\mu_t^{\chi_i} = \mu_t^{\psi_i}$ on the paired branches χ_i, ψ_i.

Consider the faithful pair (χ, ψ) from Propositions 4.22 and 4.25, with indices (d, e). Moreover, put $K_m = m(m+1)/2$ and $\epsilon_\sigma = (c_\sigma/r)$. Consider odd e, with $d = 2e$. We write cuspidal rank $k(k+1)$ of the symplectic label ρ. Thus, write exchange sign $(\sigma(i)/i)^{K_n+K_k}$ and field ratio $\epsilon_\sigma^{K_n+K_k}$. Write the unitary core κ of λ. Moreover, $\rho_{\mathrm{glob}} = \rho_{\mathrm{res}} = \rho$ of rank $k(k+1)$. Moreover, write every λ with quotient degree $p \nmid \eta_\lambda(1)$. We write the active exponent

$K_n - K_h$ between global rank n and residual rank h. With $d = 2e$, we consider even $e = 2j$. Thus, write $\lambda \vdash b$, its two-core size K_t and $u_\lambda = a + (b + K_t)/2$. Write exchange sign $\epsilon_\sigma^{fu_\lambda}$ and field ratio $\epsilon_\sigma^{u_\lambda}$. We write the cohook core κ_ρ of rank a_0. Moreover, $a - a_0 = jw$, the active exponent jw. Write only the active pair (χ_A, ψ_A) when $h = 0$, and we consider even f in both faithful cases (d, e). Thus, we write branch signs 1. Moreover, q is a square modulo p of order $2e$ when e is odd. Thus, $p \equiv 1 \pmod 4$ and $\sigma(i) = i$. Since f is even, $\epsilon_\sigma^{fu_\lambda} = 1$. We allow the full field ratio $\gamma_F(\sigma) \neq 1$. If f is odd, choose extensions $\widehat{\chi}$ by [208, Lemma 6.13], with matched local extensions $\widehat{\psi}$.

We write the geometric preimage $\widehat{D}$ of $\mathrm{SO}_{2n+1}(q)$ in **Spin**. Moreover, $(F_r|_{\widehat{D}})^{2f} = 1$. Thus, with odd f, we put $\phi_0 = F_r^{f+1}$. $\phi_0^f = 1$ on $\widehat{D}$ and $\phi_0|_X = F_r$. We put $\widehat{A} = \widehat{D} \rtimes \langle \phi_0 \rangle$ and $\widehat{A}_Q = N_{\widehat{A}}(Q)$. Then, by Sylow conjugacy in X, $N_{\widehat{D}}(Q)/M \simeq \widehat{D}/X$. We consider a field-stable branch χ and $\mathcal{H}_\chi = \{\sigma \in \mathcal{H}_p : \chi^\sigma = \chi\}$. Accordingly, write the nonempty torsor $\mathcal{T}_\chi = \{\widehat{\chi} \in \mathrm{Irr}(X \rtimes \langle \phi_0 \rangle) : \widehat{\chi}|_X = \chi,\ \widehat{\chi}^\sigma = \widehat{\chi}\ (\sigma \in \mathcal{H}_\chi)\}$. Since $2 \nmid f$, write $L_\chi \leq \mathrm{Irr}(C_f)$, with $2 \nmid |L_\chi|$. Write the action $(\widehat{\chi}, \lambda) \mapsto \widehat{\chi}\lambda$ of L_χ on $\mathcal{T}_\chi$ and write the quotient Γ_χ of the mixed stabilizer of χ modulo inner operations of $X \rtimes \langle \phi_0 \rangle$ and pure stabilizer $\mathcal{H}_\chi$. Moreover, $|\Gamma_\chi| \leq 2$, including the diagonal coset $\widehat{D} \setminus X$. By Lemma 4.34, choose $\widehat{\chi} \in \mathcal{T}_\chi^{\Gamma_\chi}$. We choose the paired local extension $\widehat{\psi}$ over ψ. Moreover, $\widehat{\chi}(z)/\widehat{\chi}(1) = \chi(z)/\chi(1)$ and $\widehat{\psi}(z)/\widehat{\psi}(1) = \psi(z)/\psi(1)$. Moreover, consider even f and faithful branches ϑ_+, ϑ_-. $\vartheta_\pm^\sigma = \vartheta_\pm$ for $\sigma \in \mathcal{H}_p$. We choose b_δ, with $\vartheta_\pm^{b_\delta} = \vartheta_\mp$. Thus, $\vartheta_\pm^{b_\delta \sigma} = \vartheta_\mp \neq \vartheta_\pm$. Choose the inner-field extensions $\widehat{\vartheta}_\pm$. Moreover, $\gamma_{F,\mathrm{glob}} = \gamma_{F,\mathrm{loc}}$ on their joint stabilizers $T_{\vartheta_\pm}$.

Thus, we write the spin correspondence Ω_s, with image $Y_s^X(M)$. Moreover, the pair $(\chi, \Omega_s(\chi))$ satisfies $\succeq_c$. By Proposition 4.50, write $\widetilde{\Omega}_s$ on R, M_R. We write full rational labels $[\widetilde{s}]$ through inflation $\mathrm{Inf}_R^{\widetilde{X}}$. Moreover, $\alpha_\Theta = \alpha_{\widetilde{\Omega}_s(\Theta)}$ and $\mu_t^\Theta = \mu_t^{\widetilde{\Omega}_s(\Theta)}$ for each t with $\Theta^t = \Theta\lambda$ for a norm character λ. Thus, we write all twists $\lambda \in \mathrm{Irr}(R/X)$ and full rational classes $[\widetilde{s}]$. □

Corollary 4.49. *We suppose that $q \equiv 3 \pmod 4$. We consider nonidentity quasi-isolated classes $\mathcal{S}$. Moreover, we write one subgroup $M = N_X(Q)$. Moreover, $\Omega = \bigsqcup_{[s]\in\mathcal{S}} \Omega_s$ is a bijection $\bigsqcup_{[s]\in\mathcal{S}} \mathcal{E}(X, s)_{p'} \xrightarrow{\sim} \bigsqcup_{[s]\in\mathcal{S}} Y_s^X(M)$. Thus, we have that $(\chi, \Omega(\chi))$ satisfies $\succeq_c$ for $\chi \in \mathrm{Dom}(\Omega)$.*

Proof. With n, p and $P \in \mathrm{Syl}_p(X)$, write $n = ew + h$ and $Q = \prod_{l=1}^w A_l$. Then, by Proposition 4.4, Q is characteristic in P. Write the block signs $+$ when $d = e$ is odd, and $-$ when $d = 2e$. Write labels $(\theta_{s,l}, \rho_s)$ on the fixed blocks $(W_l)_l$, for s. Thus, we write $M = N_X(Q)$ independently of $[s]$. Write $\pi : X \to G = \Omega_{2n+1}(q)$, with $\ker \pi = \langle z \rangle$, and we put $\overline{P} = \pi(P)$ and $\overline{Q} = \pi(Q)$. Since $p \neq 2$, $\pi^{-1}(\overline{Q}) = Q\langle z \rangle$ with unique Sylow subgroup Q. Thus, $M/\langle z \rangle \simeq \overline{M} = N_G(\overline{Q})$. We write inflation along $M \to \overline{M}$ in the sector $\chi(z) = \chi(1)$.

Moreover, we consider the mixed sector $\chi(z) = \chi(1)$ with multiplicities $(2a, 2a, b, b)$. We consider $\chi \in \mathcal{E}(X, s)$, with $p \nmid \chi(1)$. By Jordan decomposition, $p \nmid j_s$ for the rational semisimple index j_s. Moreover, write the isogeny kernel K_s and component group A_s. $|K_s|, |A_s| \in \{1, 2\}$. Thus, $|X|_p = |\mathrm{Sp}_{2a}(q)|_p^2 |\mathrm{GU}_b(q)|_p$. Write $J = \mathrm{SO}_{4a+1}(q) \times \mathrm{GU}_b(q)$ in the orthogonal quotient $\mathrm{SO}_{2n+1}(q)$. We write the companion $H \leq G$ of Section 4.2, with $J \cap G \leq H$. Since $[J : J \cap G] \leq 2$, $|J \cap G|_p = |J|_p$. Moreover, $|\mathrm{SO}_{4a+1}(q)| = |\mathrm{Sp}_{4a}(q)|$. We write the inclusion $\mathrm{Sp}_{2a}(q)^2 \leq \mathrm{Sp}_{4a}(q)$ on the symplectic sum $V_1 \perp V_2$. Thus, $|G|_p \geq |H|_p \geq |J \cap G|_p \geq |G|_p$. Thus, $|H|_p = |G|_p$, including $a = 0$. We choose a Sylow subgroup $P_H \in \mathrm{Syl}_p(H)$. By conjugacy in G, choose $P_H = \overline{P}$ after conjugation of H. Moreover, $\overline{Q} \leq H$ by the unique abelian subgroup $\overline{Q}$ of largest order in $\overline{P}$. We write $U = N_H(\overline{Q})$. By Proposition 4.33, we write $I : \mathrm{Irr}(H, e_s^H) \to \mathrm{Irr}(G, e_s)$ and $I_Q : \mathrm{Irr}(U, e_U) \to \mathrm{Irr}(\overline{M}, e_{\overline{M}})$. In particular, their Sylow hypothesis $\overline{P} \leq H$ holds on every nonempty degree cut $p \nmid \chi(1)$. If $\rho_1 = \rho_2$, by Lemma 4.29, $\epsilon_{\mathrm{glob}} = \epsilon_{\mathrm{loc}}$ in H. Choose the residual character ρ and its orthogonal sign ϵ_ρ. Thus, we write the tensor operator $J_A \otimes J_\rho$ with sign $\epsilon_A \epsilon_\rho$. Write $L \lhd H$, with $[H : L] = 2$. Moreover, we write the block twist $\nu_\sigma \in \mathrm{Irr}(H/L)$ for I. We write $\nu_{\sigma,Q} = \nu_\sigma|_U$ for I_Q. We write $\chi_G \in \mathrm{Irr}(G)$, with $\chi = \chi_G \circ \pi$. Moreover, we put $\theta = I^{-1}(\chi_G)$. Write the map J_H from Section 4.2, on H, U. Thus, $J_H(\theta\nu_\sigma) = J_H(\theta)\nu_{\sigma,Q}$. Moreover, $\nu_{\sigma,Q}\nu_{\sigma,Q}^{-1} = 1$ in $I_Q J_H I^{-1}$, before the field extensions $\widehat{\chi}, \widehat{\psi}$.

We consider the involution sector $\chi(z) = \chi(1)$. By Proposition 4.45, write cores (κ_0, κ_1). Moreover, $c_0 + c_1 = h$ and $u + v = w$. Thus, we write exactly w toral blocks W_l. We write the ordered orthogonal parent Θ with rational involution s. Moreover, $s^k = s$ when $\gcd(k, |s|) = 1$. Moreover, $\mathbb{Q}(\rho_i) = \mathbb{Q}$ and $\rho_i^{F_r} = \rho_i$ for $i \in \{0, 1\}$. Thus, $\mathbb{Q}(\Theta) = \mathbb{Q}$ and $\Theta^{F_r} = \Theta$. $2 \nmid f$, since $q = r^f \equiv 3 \pmod 4$. By Lemma 4.32, we choose $\widehat{\Theta}$, with $\mathbb{Q}(\widehat{\Theta}) = \mathbb{Q}$. We write the active root $\alpha_A \in \mu_{\mathrm{ord}_p(r)}$. Moreover, $\sigma(\alpha_A) = \alpha_A$, since $\mathrm{ord}_p(r) \mid p - 1$. Moreover, we choose $\widehat{\Theta}_{\mathrm{res}}$, with $\widehat{\Theta}_{\mathrm{res}}^\sigma = \widehat{\Theta}_{\mathrm{res}}$ and $\mathbb{Q}(\widehat{\Theta}_{\mathrm{res}}) = \mathbb{Q}$. Write the branch restrictions $\chi_\pm, \psi_\pm$ or the irreducible restrictions χ, ψ. By Lemma 4.26, $\epsilon_\sigma^{\mathrm{glob}} = \epsilon_\sigma^{\mathrm{loc}}$. Moreover, $\chi(z)/\chi(1) = \psi(z)/\psi(1) = 1$ in both sectors s.

Moreover, we consider the faithful sector $\chi(z) = -\chi(1)$. Choose $\iota \in \mathbb{F}_{q^2}$, with $\iota^2 = -1$ and $\iota^q = -\iota$. Thus, we write the transition $x \mapsto -x^q$ for $x \in \{1, -1, \iota, -\iota\}$. Consider a negative block W_l of odd length e. We would have that $-x^{q^e} = x^{-1}$. Since $x \in \{1, -1\}$, $-x^{q^e} = -x \neq x^{-1} = x$. Consider $x \in \{\iota, -\iota\}$. Moreover, $q^e \equiv 3 \pmod 4$ and $-x^{q^e} = x \neq x^{-1} = -x$. Thus, $\{x \in \{1, -1, \iota, -\iota\} : -x^{q^e} = x^{-1}\} = \varnothing$. $w \geq 1$, since $p \mid |X|$. Consider positive blocks $d = e$ or negative blocks of even length $2 \mid e$. By Proposition 4.35, $\alpha_\chi = \alpha_\psi$, $\mu_t^\chi = \mu_t^\psi$. Moreover, $\chi(z)/\chi(1) = \psi(z)/\psi(1) = -1$. By [208, Lemma 6.12], we exhaust the nonidentity classes $[s]$.

Moreover, $p \nmid |s|$ in every class $[s]$ above. We write torus duality in $C_X(Q)$. Thus, $s_p = 1$ corresponds to 1_Q. Choose a splitting ring $\mathcal{O}$ with residue field k of characteristic p. By [19, Theorem 4.14], write $[s]$ as the ambient class in $Y_{s,Q}(M)$. Thus, $Y_{s,Q}(M) = Y_s^X(M)$. We choose idempotents $e_s, e_t \in Z(\mathcal{O}X)$ for $\mathcal{E}_p(X, s), \mathcal{E}_p(X, t)$. Moreover, we consider classes $[s] \neq [t]$, with $p \nmid |s||t|$. $e_s e_t = 0$ and $\overline{e}_s \overline{e}_t = 0$ over kX. Thus, write the Brauer map $\mathrm{Br}_Q : (kX)^Q \to kC_X(Q)$. Moreover, $\mathrm{Br}_Q(\overline{e}_s)\,\mathrm{Br}_Q(\overline{e}_t) = 0$. Thus, $Y_s^X(M) \cap Y_t^X(M) = \varnothing$. Moreover, write the marked actions $\mathcal{A}_Q = N_{\mathrm{Aut}(X)}(Q)$. $\Omega_{s^{a\sigma}}(\chi^{a\sigma}) = \Omega_s(\chi)^{a\sigma}$ for $(a, \sigma) \in \mathcal{A}_Q \times \mathcal{H}_p$. We write the disjoint union $\Omega = \bigsqcup_{[s]\in\mathcal{S}} \Omega_s$. Thus, $(\chi, \Omega(\chi))$ satisfies $\succeq_c$. □

Proposition 4.50. *We set $q = r^f$ odd and an odd prime $p \nmid q$ and put $\widetilde{X} = \mathrm{GSpin}_{2n+1}(q)$ and $\widetilde{M} = N_{\widetilde{X}}(Q)$. Moreover, we choose the marked map $f : \mathcal{X} \xrightarrow{\sim} \mathcal{Y}$ on the nonidentity quasi-isolated classes $\mathcal{S}$. We choose $\mathcal{X} = \bigcup_{[s]\in\mathcal{S}} \mathcal{E}(X, s)_{p'}$ and $\mathcal{Y} = \bigcup_{[s]\in\mathcal{S}} Y_s^X(M)$. We suppose that $(\chi, f(\chi))$ satisfies $\succeq_c$ on the marked groups X, M. Thus, we take $\widetilde{f} : \mathrm{Irr}_{p'}(\widetilde{X} \mid \mathcal{X}) \xrightarrow{\sim} \mathrm{Irr}_{p'}(\widetilde{M} \mid \mathcal{Y})$. We have that $\omega_\Theta = \omega_{\widetilde{f}(\Theta)}$ on the scalar centre Z. We choose the full rational class $[\widetilde{s}]$ for both $\Theta, \widetilde{f}(\Theta)$. Moreover, $\widetilde{f}(\Theta(\lambda \circ \nu)) = \widetilde{f}(\Theta)(\lambda \circ \nu)|_{\widetilde{M}}$ for $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$. We have that $\alpha_\Theta = \alpha_{\widetilde{f}(\Theta)}$ and $\mu_a^\Theta = \mu_a^{\widetilde{f}(\Theta)}$ on their marked inertia quotient T_Θ. We choose a quadratic self-twist $\Theta(\eta \circ \nu) = \Theta$. Thus, we choose normalized operators $J_X^2 = J_M^2 = I$ with equal signs $\epsilon_a^X = \epsilon_a^M$. With $R = \widetilde{X}/Z_o$, we set $M_R = \widetilde{M}/Z_o$. Moreover, we fix f_R, functions α, μ_a and signs ϵ_a on R, M_R, with labels from inflation $\mathrm{Inf}_R^{\widetilde{X}}$.*

Proof. With $C = \mathbb{F}_q^\times$, put $m = q-1$. By Lemma 4.5 and [137, §24.i, p. 532], we fix $\nu : \widetilde{X} \to C$ with $\nu(\widetilde{X}) = \nu(\widetilde{M}) = C$. In particular, $\ker \nu = X$ and $\ker(\nu|_{\widetilde{M}}) = M$. Moreover, we choose the scalar centre $Z \simeq C$, with $\nu(c) = c^2$. $Z \cap X = \langle z \rangle$, with $z = -1$. Thus, we put $A = \mu_m \subset \mathbb{C}^\times$ and $N = A \times \widetilde{X}$. We embed X by $x \mapsto (1, x)$. Fix $s_\lambda \in \mathrm{Aut}(N)$, with $s_\lambda(a, g) = (a\lambda(\nu(g)), g)$ for $\lambda \in \mathrm{Irr}(C)$. Moreover, we choose $e_k(a, g) = (a^k, g)$ for $k \in (\mathbb{Z}/m\mathbb{Z})^\times$. We choose $F_r(a, g) = (a, F_r(g))$ and put $B = \langle s_\lambda, e_k, F_r \rangle$ and $U = N \rtimes B$. Thus, we put $V = N_U(Q)$ and $N' = N \cap V = A \times \widetilde{M}$. $P \in \mathrm{Syl}_p(X) \cap \mathrm{Syl}_p(M)$. By Lemma 4.3, $|B_0| \leq |Q|$ for $B_0 \leq P$, $[B_0, B_0] = 1$. Moreover, $\{B_0 \leq P : [B_0, B_0] = 1,\ |B_0| = |Q|\} = \{Q\}$. By Sylow conjugacy in X, $U = XV$. By Sylow conjugacy in M, $V = MN_U(P)$. Moreover, $N_U(P) \leq N_U(Q) = V$. Moreover, choose $\deg_F : B \to \mathbb{Z}/f\mathbb{Z}$ from the action on $\widetilde{X}$. We fix $p_U : U \to \widetilde{X} \rtimes \langle F_r \rangle$, with $p_U((a, g), b) = (g, F_r^{\deg_F(b)})$, and $p_U|_X = \mathrm{id}_X$ and $\ker p_U \leq C_U(X)$. Thus, $p_U(V) = N_{\widetilde{X}\rtimes\langle F_r\rangle}(Q)$. We choose the images $p_U(U_{\chi^{\mathcal{H}_p}}), p_U(V_{f(\chi)^{\mathcal{H}_p}})$. By [141, Theorem 2.9], $(U_{\chi^{\mathcal{H}_p}}, X, \chi)_{\mathcal{H}_p} \succeq_c (V_{f(\chi)^{\mathcal{H}_p}}, M, f(\chi))_{\mathcal{H}_p}$. Consequently, $\mathcal{P}_\chi(c) = \lambda_c I$ and $\mathcal{P}_{f(\chi)}(c) = \lambda_c I$ when $p_U(c)$ centralizes X.

Moreover, take $\iota_A \in \mathrm{Irr}(A)$, with $\iota_A(a) = a$. By Lemma 2.15, fix $F : \mathrm{Irr}(N \mid \mathcal{X}) \xrightarrow{\sim} \mathrm{Irr}(N' \mid \mathcal{Y})$. We choose the sector $\iota_A \boxtimes \Theta$ and its partner $\iota_A \boxtimes \Theta'$ on N, N'. Moreover, we choose $\chi \in \mathcal{X}$, with $\langle \Theta|_X, \chi \rangle_X > 0$. $\Theta(1)/\chi(1) = \Theta'(1)/f(\chi)(1) \in \{1, 2\}$. Thus, $p \nmid \Theta(1)$ exactly when $p \nmid \Theta'(1)$. We choose the centre $A \times Z$ of N. Whenever $c \in Z$, $\omega_\Theta(c) = \Theta(c)/\Theta(1) = \Theta'(c)/\Theta'(1) = \omega_{\Theta'}(c)$. Moreover, $(\iota_A \boxtimes \Theta)^{s_\lambda} = \iota_A \boxtimes \Theta(\lambda \circ \nu)$. We choose $\sigma(a) = a^j$ on A, with $\sigma \in \mathcal{H}_p$, and we choose j^{-1} modulo m. Thus, $(\iota_A \boxtimes \Theta)^{e_{j^{-1}}\sigma} = \iota_A \boxtimes \Theta^\sigma$. We include field and diagonal operations gF_r^k in $p_U(V)$. We take every operation $gF_r^k\sigma$ satisfying $\Theta^{gF_r^k\sigma} = \Theta(\lambda \circ \nu)$ through $s_\lambda, e_{j^{-1}}, gF_r^k$. Moreover, $\alpha_\Theta = \alpha_{\Theta'}$ and $\mu_a^\Theta = \mu_a^{\Theta'}$ on the ordinary inertia quotient T_Θ.

We choose the quadratic character $\eta \in \mathrm{Irr}(C)$ and $H = XZ = \nu^{-1}(C^2)$ and choose $t \in \widetilde{M}$, with $\nu(t) \notin C^2$. First, we suppose that $\chi^t \neq \chi$ and take $\omega \in \mathrm{Irr}(Z)$ with $\omega(z) = \chi(z)/\chi(1)$. We choose $(\chi \cdot \omega)(xc) = \chi(x)\omega(c)$ for $x \in X, c \in Z$, and $\Theta = \mathrm{Ind}_H^{\widetilde{X}}(\chi \cdot \omega)$. Thus, choose $\Theta' = \mathrm{Ind}_{MZ}^{\widetilde{M}}(f(\chi) \cdot \omega)$ with representatives $t_0 = 1, t_1 = t$. Choose their two coset spaces V_0, V_1 and V_0', V_1'. $s_\eta|_{1\times H} = 1$ and $(1, t)^{-1} s_\eta(1, t) = (-1, 1)s_\eta$. Then, by [141, Lemma 1.6], choose the scalars $\mathcal{P}_\Theta(s_\eta)|_{V_0} = I$ and $\mathcal{P}_{\Theta'}(s_\eta)|_{V_0'} = I$. $s_\eta \in C_U(X)$. By Lemma 2.15, we choose τ on $N_\chi/X \simeq N'_{f(\chi)}/M$ for the first summands V_0, V_0'. We choose the operators $J_X = \mathrm{diag}(I_{V_0}, -I_{V_1})$ and $J_M = \mathrm{diag}(I_{V_0'}, -I_{V_1'})$. Thus, $J_X^2 = J_M^2 = I$. $\sigma(J_X) = J_X$ and $\sigma(J_M) = J_M$. We choose the permutation sign $\epsilon_a \in \{1, -1\}$ of χ, χ^t under a mixed operation a stabilizing $\{\chi, \chi^t\}$. Moreover, $T_a J_X^a T_a^{-1} = \epsilon_a J_X$ and $T_a' J_M^a (T_a')^{-1} = \epsilon_a J_M$.

We choose $\lambda^{[k]}(c) = \lambda(c^{r^k})$ for $\lambda \in \mathrm{Irr}(C)$ and choose a field lift $u = gF_r^k \in V$. Thus, $u^{-1}s_\lambda u = (\lambda(\nu(g)), 1)s_{\lambda^{[k]}}$ in U, including $\Theta^{s_\lambda} \neq \Theta$. We suppose that $\lambda^{[k]}|_{C^2} = \lambda|_{C^2}$. Fix $b \in \{0,1\}$, with $\lambda^{[k]}/\lambda = \eta^b$. Moreover, we put $D_\lambda = \mathrm{diag}(\lambda(\nu(t_0))I, \lambda(\nu(t_1))I)$. We choose $gF_r^k(t_i) = t_j h$, with $h \in H$, and $\nu(t_j)\nu(h) = \nu(g)\nu(t_i)^{r^k}$. Thus, $D_{\lambda^{[k]}} = D_\lambda J_X^b$ and the local formula $D'_{\lambda^{[k]}} = D'_\lambda J_M^b$. We choose the scalar $\lambda(\nu(g))$ for the common representative $u = gF_r^k \in V$, and $u, t_0, t_1 \in V$. Moreover, we choose $u\sigma$ with $\Theta^{u\sigma} = \Theta(\lambda \circ \nu)$ and its indicated matrix on $V_0 \oplus V_1$. We choose the product $u_1u_2 = g_1F_r^{k_1}(g_2)F_r^{k_1+k_2}$, and $\nu(g_1F_r^{k_1}(g_2)) = \nu(g_1)\nu(g_2)^{r^{k_1}}$. Thus, we choose the factor $\lambda(\nu(g_1)\nu(g_2)^{r^{k_1}})$ in the comparison for u_1u_2.

We take $\Theta \in \mathcal{E}(\widetilde{X}, \widetilde{s})$. Moreover, put $\widetilde{C} = C_{\widetilde{X}}(Q)$. We take $\xi \in \mathrm{Irr}(\widetilde{C})$, with $\langle \Theta'|_{\widetilde{C}}, \xi\rangle_{\widetilde{C}} > 0$. We choose the dual map $\pi^* : \widetilde{\mathbf{X}}^* \to \mathbf{X}^*$. Since $f(\chi) \in Y_s^X(M)$, $\pi^*(\widetilde{s}_1) \in [s]$ for the full label $\widetilde{s}_1$ of ξ. We choose the multiplier $\mu : \widetilde{\mathbf{X}}^* \to \mathbf{G}_m$. Moreover, we fix $m_\omega \in \mathbb{F}_q^\times$ dual to $\omega = \omega_\Theta$. $\mu(\widetilde{s}) = \mu(\widetilde{s}_1) = m_\omega$, and we choose an adjoint conjugator $\overline{g}$ for $\pi^*(\widetilde{s}_1), \pi^*(\widetilde{s})$. Since $\ker \pi^*$ is the scalar kernel, we lift $\overline{g}$ to $g \in (\widetilde{\mathbf{X}}^*)^{F^*}$ by Lang's theorem. $g\widetilde{s}_1 g^{-1} = c\widetilde{s}$ and $c^2 = 1$, since both multipliers are m_ω. $c \in \{1, -1\}$. We choose the quadratic norm twist $\eta \circ \nu$, with dual action $\widetilde{s} \mapsto -\widetilde{s}$. Since $\Theta(\eta \circ \nu) = \Theta$, $[\widetilde{s}] = [-\widetilde{s}]$ as rational classes $[\widetilde{s}]$.

Alternatively, we set $\chi^t = \chi$. By [208, Lemma 6.12], $\chi(z) = \chi(1)$. Take parents Θ_0, Θ_1, with $\Theta_i|_Z = \Theta_i(1)1_Z$, from $D = \widetilde{X}/Z \simeq \mathrm{SO}_{2n+1}(q)$. Thus, $\Theta_1 = \Theta_0(\eta \circ \nu)$. Moreover, $\mu(\widetilde{s}_0) = \mu(\widetilde{s}_1) = 1$. Moreover, take local choices Θ'_0, Θ'_1 with $\Theta'_1 = \Theta'_0(\eta \circ \nu)$. We suppose that $[\widetilde{s}_0] \neq [\widetilde{s}_1]$. We choose the ordering (Θ'_0, Θ'_1) with the matched full classes $[\widetilde{s}_0], [\widetilde{s}_1]$. Thus, take $(\Theta'_0, \Theta'_1) \mapsto (\Theta'_1, \Theta'_0)$ under $\eta \circ \nu$. Alternatively, we choose $[\widetilde{s}_0] = [\widetilde{s}_1]$ and choose the local order over $\chi, f(\chi)$. $\mathbb{Q}(\Theta_i) = \mathbb{Q}$ and $\Theta_i^{F_r} = \Theta_i$. Moreover, put $V_\pm = \ker(\widetilde{s}_i \mp I)$. By [208, Lemma 6.12 and proof of Proposition 6.15], choose $\widetilde{s}_i^k|_{V_+} = I$ and $\widetilde{s}_i^k|_{V_-} = -I$ for odd k, with labels (ρ_0, ρ_1). We set $\iota^2 = -1$ and inversion $\iota \mapsto -\iota$ on the quartic eigenspaces $V_\iota, V_{-\iota}$. Then, $\lambda^{\gamma_{\mathrm{gr}}} = \lambda$ on the linear or unitary factor D_i. Thus, $(\Theta'_i)^{F_r} = \Theta'_i$ and $(\Theta'_i)^\sigma = \Theta'_i$ under the matched actions F_r, σ. $\{\lambda \in \mathrm{Irr}(C) : (\lambda \circ \nu)|_Z = 1\} = \{1, \eta\}$, and we choose the two orders $(\Theta_0, \Theta_1), (\Theta'_0, \Theta'_1)$ with their common action of $\eta \circ \nu$. Moreover, $\omega(c) = \lambda(c^2)$ for every scalar character ω with $\omega(z) = 1$. Choose the corresponding norm translates $\Theta_i(\lambda \circ \nu), \Theta'_i(\lambda \circ \nu)$.

Moreover, choose the ordering change $\Theta'_i \mapsto \Theta'_i(\eta \circ \nu)$ on an entire marked orbit $\mathcal{O}$. On U, we choose $\widehat{\eta}((a, g), b) = \eta(\nu(g))$. Since $\eta(\nu(F_r(g))) = \eta(\nu(g))$, $\widehat{\eta} \in \mathrm{Irr}(U)$. $\widehat{\eta}|_A = \widehat{\eta}|_B = 1$ and $\widehat{\eta}^\sigma = \widehat{\eta}$. We fix $c \in C_U(N)$. Thus, $p_U(c) = (d, 1)$, with $d \in Z$. $\widehat{\eta}(c) = \eta(d^2) = 1$ for the scalar component $d \in Z$, and we choose the local model $\mathcal{P}'\widehat{\eta}|_V$. Moreover, $\alpha_{\mathcal{P}'\widehat{\eta}} = \alpha_{\mathcal{P}'}$ and $\mu_a^{\mathcal{P}'\widehat{\eta}} = \mu_a^{\mathcal{P}'}$. Whenever $c \in C_U(N)$, $\mathcal{P}'(c)\widehat{\eta}(c) = \mathcal{P}'(c)$. $\widehat{\eta}(gF_r^k) = \eta(\nu(g))$ for every carried field element gF_r^k. Thus, we include its product relation u_1u_2 with the full norm $\nu(g_1)\nu(g_2)^{r^{k_1}}$.

We choose the odd scalar subgroup $Z_o \leq Z$, and $Z_o \leq \ker\Theta$ exactly when $Z_o \leq \ker\Theta'$. Since $Z_o \cap X = 1$, $QZ_o \cap X = Q$. Take $\pi_o : \widetilde{X} \to R$ and $\overline{Q} = QZ_o/Z_o$. We take $\pi_o(g) \in N_R(\overline{Q})$, so $gQg^{-1} \leq QZ_o$. Since $gQg^{-1} \leq X$, $gQg^{-1} = Q$. $\pi_o^{-1}(N_R(\overline{Q})) = \widetilde{M}$, and we set $M_R = \widetilde{M}/Z_o$ and the quotient correspondence f_R. Thus, $\mathrm{Inf}_{M_R}^{\widetilde{M}} f_R(\theta) = \widetilde{f}(\mathrm{Inf}_R^{\widetilde{X}}\theta)$. Choose the full rational class $[\widetilde{s}]$ from $\mathrm{Inf}_R^{\widetilde{X}}\theta$. We include all $\lambda \in \mathrm{Irr}(C)$ with $(\lambda \circ \nu)|_{Z_o} = 1$. Moreover, choose the functions α, μ_a and quadratic signs ϵ_a on R, M_R. □

Lemma 4.51. *We consider $m \in \{1, 2\}$ and odd q, p, with $p \nmid q$, and we put $X_m = \mathrm{Spin}_{2m+1}(q)$ and $\widetilde{X}_m = \mathrm{GSpin}_{2m+1}(q)$. Moreover, we write the marked subgroup $Q_m \leq X_m$ and $\widetilde{M}_m = N_{\widetilde{X}_m}(Q_m)$. We write $\widetilde{\Phi}_m$ of* Proposition 4.44 *on $\widetilde{X}_m, \widetilde{M}_m$. We write their full rational labels $[\widetilde{s}]$ and norm twists $\lambda \circ \nu_m$. Thus, we write the correspondence on $R_m = \widetilde{X}_m/Z_{o,m}$, with $Z_{o,m} \leq Z(\widetilde{X}_m)$ odd.*

Proof. We write the pinned isomorphisms $\iota_1 : \mathrm{Spin}_3 \to \mathrm{SL}_2$ and $\iota_2 : \mathrm{Spin}_5 \to \mathrm{Sp}_4$, over $\mathbb{F}_r$. Moreover, $\iota_m(z) = -I$. By Lemma 4.7, write $\varphi_m([g, c]) = c\iota_m(g)$ on $(\mathrm{Spin}_{2m+1} \times \mathbf{G}_m)/\langle(z, -1)\rangle$. With $\mathbf{H}_1 = \mathrm{GL}_2$ and $\mathbf{H}_2 = \mathrm{GSp}_4$, write $\delta_1(A) = \det A$ and $A^{\mathsf{T}}\mathsf{J}A = \delta_2(A)\mathsf{J}$, with symplectic matrix J. Thus, $\ker((g, c) \mapsto c\iota_m(g)) = \langle(z, -1)\rangle$. We consider $A \in \mathbf{H}_m$ over $\overline{\mathbb{F}}_r$ and $c^2 = \delta_m(A)$. Moreover, $c^{-1}A \in \iota_m(\mathrm{Spin}_{2m+1})$. Moreover, we write $\varphi_m^{-1}(A) = [\iota_m^{-1}(c^{-1}A), c]$. $[\iota_m^{-1}((-c)^{-1}A), -c] = [\iota_m^{-1}(c^{-1}A), c]$ and that $\varphi_m F_r = F_r\varphi_m$. Thus, we write $\widetilde{X}_1 \simeq \mathrm{GL}_2(q)$ and $\widetilde{X}_2 \simeq \mathrm{GSp}_4(q)$. Moreover, $\nu_m([g, c]) = c^2 = \delta_m(\varphi_m([g, c]))$. We include rational points $[g, c]$, with $(g, c)^F = (gz^\epsilon, (-1)^\epsilon c)$ and $\epsilon \in \{0, 1\}$. Moreover, $\varphi_m(N_{\widetilde{X}_m}(\mathbf{T})) = N_{\mathbf{H}_m^F}(\varphi_m(\mathbf{T}))$ for rational tori $\mathbf{T}$. We write the dual isomorphism φ_m^* on the full rational classes $[\widetilde{s}]$. Moreover, $\varphi_m^* F_r = F_r\varphi_m^*$.

Moreover, we suppose that $p \mid |X_1|$ and put $d = \mathrm{ord}_p(q)$. $d \in \{1, 2\}$. We write $P_1 = Q_1$ in the corresponding torus $\mathbf{T}$. Moreover, we choose $Q_1 = \langle t\rangle$, with $\mathrm{Spec}(t) = \{\zeta, \zeta^{-1}\}$ and $\zeta \in \mathbb{F}_{q^2}^\times$. Since $p \neq 2$, $\zeta \neq \zeta^{-1}$. Thus, $C_{\mathbf{H}_1}(Q_1) = \mathbf{T}$ and $\widetilde{M}_1 = N_{\mathbf{H}_1^F}(\mathbf{T})$. We choose split weights $\alpha, \beta \in \mathrm{Irr}(\mathbb{F}_q^\times)$ and nonsplit weights $\vartheta \in \mathrm{Irr}(\mathbb{F}_{q^2}^\times)$. Thus, consider $d = 1$

and $\alpha \neq \beta$, with degree $q+1$. Consider $\alpha = \beta$, with degrees $1, q$, and we consider $\vartheta^q \neq \vartheta$, with degree $q-1$. Since $p \mid q-1$, $\Theta(1) = q-1 \Longrightarrow \Theta \notin \mathrm{Irr}_{p'}(\widetilde{X}_1)$. We consider $d = 2$, with degrees $1, q$ at scalar labels $\alpha = \beta$, and we consider nonsplit labels $\vartheta^q \neq \vartheta$, with degree $q-1$. Since $p \mid q+1$, $\Theta(1) = q+1 \Longrightarrow \Theta \notin \mathrm{Irr}_{p'}(\widetilde{X}_1)$. We write the Weyl group $W_T = N_{\mathbf{H}_1^F}(\mathbf{T})/\mathbf{T}^F \simeq \mathfrak{S}_2$. Moreover, $(\alpha, \beta)^w = (\beta, \alpha)$ or $\vartheta^w = \vartheta^q$ for its generator w. Moreover, we choose $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. Thus, write $\mathrm{Ind}_{\mathbf{T}^F}^{\widetilde{M}_1}\theta$ when $|\theta^{W_T}| = 2$. If $|\theta^{W_T}| = 1$, we choose two extensions $\widehat{\theta}_+, \widehat{\theta}_-$. We include each scalar class $[aI]$, with $a \in \mathbb{F}_q^\times$, above the adjoint identity class $[1]$.

By Section 3.2, we choose matrix models $\mathcal{P}_\Theta, \mathcal{P}_{\Theta'}$ for $\mathrm{GL}_2(q), \widetilde{M}_1$. If $d = 1$, we find two linear cuspidal factors α, β. We write their permutation $(\alpha, \beta) \mapsto (\beta, \alpha)$. By Proposition 3.14, write $1_{\mathfrak{S}_2}, \mathrm{sgn}_{\mathfrak{S}_2}$ at equal weights $\alpha = \beta$. If $d = 2$, we find one torus coordinate T_2 and residual dimension 0. Write $\phi = \{\vartheta, \vartheta^q\}$, with $|\phi| = 2$ and partition $\lambda_\phi = (1)$. Moreover, we write scalar partitions $(2), (1,1)$, with empty two-core $\varnothing$. Write their ordered two-quotients $((1), \varnothing), (\varnothing, (1))$ for $\widehat{\theta}_+, \widehat{\theta}_-$. Accordingly, by Lemma 3.13 and Proposition 3.20 choose $\mathcal{P}_\Theta, \mathcal{P}_{\Theta'}$. We write their cuspidal factors ρ_i in matrix rank 2. We write the cochain c, with $\alpha_{\mathrm{glo}}/\alpha_{\mathrm{loc}} = \partial c$. Moreover, we put $s_\lambda(\Theta) = \Theta(\lambda \circ \det)$. $c(s_\lambda) = 1$ when $s_\lambda(\Theta) = \Theta$. Moreover, $c(av) = c(a)$, with torus operator $v = u^f$. We write the comparisons on $\mathrm{GL}_2(q), \widetilde{M}_1$ before the restrictions $\Theta|_{X_1}, \Theta'|_{N_{X_1}(Q_1)}$.

By Proposition 4.4, we write the subgroup $Q_2 \leq P_2$. Similarly, take $Q_{\mathrm{sp}} \leq \mathrm{Sp}_4(q)$ from Proposition 5.3. By Lemma 4.3, choose $\iota_2(Q_2) = Q_{\mathrm{sp}}$ after conjugation of $\iota_2(P_2)$. Moreover, Q_{sp} is the unique abelian subgroup $B \leq \iota_2(P_2)$ with $|B| = \max\{|A| : A \leq \iota_2(P_2), [A, A] = 1\}$. By Propositions 5.45 and 5.43, we write $\widetilde{\Phi}_2$ on $\mathrm{GSp}_4(q), \widetilde{M}_2$. Moreover, consider $\Theta \in \mathcal{E}(\widetilde{X}_2, \widetilde{s})_{p'}$ and $\Theta' = \widetilde{\Phi}_2(\Theta)$. Choose a constituent ξ of $\Theta'|_{C_{\widetilde{X}_2}(Q_2)}$ and write ξ from inverse Jordan decomposition at $(\widetilde{s}, \rho)$. Moreover, we write its relative label η on the inertia quotient $I_\xi/C_{\widetilde{X}_2}(Q_2)$. Write the full label $[\widetilde{s}]$ for Θ, Θ' and write $z_\lambda \in Z(\widetilde{\mathbf{X}}_2^*)^{F^*}$ dual to $\lambda \circ \nu_2$. Thus, write $\widetilde{s} \mapsto z_\lambda \widetilde{s}$ for $\Theta(\lambda \circ \nu_2), \Theta'(\lambda \circ \nu_2)|_{\widetilde{M}_2}$. We choose P_λ, Q_λ from the proof of Theorem 5.2 for $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$. We include P_η, Q_η when $\Theta(\eta \circ \nu_2) = \Theta$ and $\eta^2 = 1 \neq \eta$. Moreover, choose $J_X^2 = J_M^2 = I$ in either rank m. We write their eigenspaces V_+, V_- and W_+, W_-. $T_a J_X^a T_a^{-1} = \epsilon_a J_X$ and $T_a' J_M^a (T_a')^{-1} = \epsilon_a J_M$. Thus, write $\epsilon_a = 1$ on fixed partners $(V_\pm, W_\pm)$, and $\epsilon_a = -1$ on $(V_\pm, W_\pm) \mapsto (V_\mp, W_\mp)$.

Write the scalar centre Z_m of $\widetilde{X}_m$. Moreover, $\omega_\Theta = \omega_{\Theta'}$ on Z_m. Thus, $Z_{o,m} \leq \ker\Theta$ exactly when $Z_{o,m} \leq \ker\Theta'$. With $\pi_m : \widetilde{X}_m \to R_m$ and $\overline{Q}_m = Q_m Z_{o,m}/Z_{o,m}$, $Z_{o,m} \cap X_m = 1$. Since $gQ_m g^{-1} \leq X_m$, $gQ_m g^{-1} \leq Q_m Z_{o,m}$ implies $gQ_m g^{-1} = Q_m$. Thus, $\pi_m^{-1}(N_{R_m}(\overline{Q}_m)) = \widetilde{M}_m$. Write the quotient models $\mathcal{P}_\Theta, \mathcal{P}_{\Theta'}$ on their vector spaces V, W. Moreover, write field operators $\overline{T}_a, \overline{T}_a'$. $\overline{T}_a = T_a$ on V and $\overline{T}_a' = T_a'$ on W. Alternatively, we consider $p \nmid |X_m|$ in the remaining case $m \in \{1, 2\}$. Moreover, $Q_m = 1$ and $N_{\widetilde{X}_m}(Q_m) = \widetilde{X}_m$. Thus, we write $\widetilde{\Phi}_m = \mathrm{id}$, including central p-scalars in Z_m. □

Lemma 4.52. *With $\widetilde{X} = \mathrm{GSpin}_{2n+1}(q)$ and $\widetilde{M} = N_{\widetilde{X}}(Q)$, we consider unipotent characters $\rho_0 \in \mathrm{Irr}_{p'}(\widetilde{X})$ with $\rho_0|_Z = \rho_0(1)1_Z$. Moreover, we write their norm translates $\rho_\lambda = \rho_0(\lambda \circ \nu)$, with $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$. We write their local partners $\tau_\lambda = \tau_0((\lambda \circ \nu)|_{\widetilde{M}})$. Moreover, $Y_1^X(M) = \{\tau_0|_M : \rho_0 \in \mathcal{E}(\widetilde{X}, 1)_{p'}\}$. Thus, we have that $\mu_b^{\rho_\lambda} = \mu_b^{\tau_\lambda}$ for each b with $\rho_\lambda^b = \rho_\lambda(\kappa \circ \nu)$, where $\kappa \in \mathrm{Irr}(\mathbb{F}_q^\times)$.*

Proof. By Proposition 4.4, we write $M = N_X(\mathbf{T})$ for the marked Sylow torus $\mathbf{T}$. By [156, proof of Theorem D], the unipotent pair (χ, ψ) has parents ρ_0, τ_0 from $\widetilde{X}/Z, \widetilde{M}/Z$. Thus, $\rho_0|_X = \chi$ and $\tau_0|_M = \psi$. The local source λ_0 on $C_X(\mathbf{T})$ extends to $\widetilde{\lambda}_0$ on $C_{\widetilde{X}}(\mathbf{T})$ with $Z \leq \ker\widetilde{\lambda}_0$. By [156, Theorem B and §4.4], the labels are $(1, \lambda_0, \eta)$. Hence, $Y_1^X(M) = \bigcup_{\lambda_0}\{\psi_{\lambda_0,\eta}\}_\eta$.

Put $C = \mathbb{F}_q^\times$. The norm ν gives $\widetilde{X}/X \simeq C \simeq \widetilde{M}/M$. By [95, Corollary 6.17], $\lambda \mapsto \rho_\lambda$ and $\lambda \mapsto \tau_\lambda$ are bijections from $\mathrm{Irr}(C)$ onto the parent sets over χ and ψ, respectively. Let $\eta_2 \in \mathrm{Irr}(C)$ be quadratic. Thus, $\rho_{\eta_2} \neq \rho_0$ and $\tau_{\eta_2} \neq \tau_0$, with $\eta_2(\nu(c)) = \eta_2(c^2) = 1$ for $c \in Z$. Moreover, write $z_\lambda \in Z(\widetilde{\mathbf{X}}^*)^{F^*}$ dual to $\lambda \circ \nu$. Write the full classes $[z_\lambda]$ for $\rho_\lambda, \tau_\lambda$ and write the centralizer label $1 \mapsto z_\lambda$ under $(\lambda \circ \nu)|_{C_{\widetilde{X}}(\mathbf{T})}$.

Moreover, we choose a field generator a of the marked inertia group I. Choose its corrected lift $la \in N_{\widetilde{X} \rtimes \langle a \rangle}(Q)$, with $l \in \widetilde{X}$. Since $\nu(l) \in C$, the scalar is $\lambda(\nu(l))$ in both field models $\mathcal{P}_{\rho_\lambda}, \mathcal{P}_{\tau_\lambda}$. Consider a mixed operation b, with $\lambda^b \in \mathrm{Irr}(C)$. Moreover, $\lambda^b/\lambda : C \to \mathbb{C}^\times$ is the quotient character for both $\rho_\lambda, \tau_\lambda$. By [156, §8.1 and proof of Theorem D], choose matched extensions $\widehat{\rho}_0, \widehat{\tau}_0$ on the field inertia groups I_X, I_M. Thus, $\mu_b^{\rho_0} = \mu_b^{\tau_0}$. We write the restoring twist λ/λ^b on the common quotient C. Thus, $(\lambda^b/\lambda)(\lambda/\lambda^b) = 1$ and $\mu_b^{\rho_\lambda} = \mu_b^{\tau_\lambda}$ on the marked inertia quotient T. We write $R = \widetilde{X}/Z_o$, with $Z_o \leq Z$ odd. Moreover, $Z_o \leq \ker\rho_\lambda$ exactly when $Z_o \leq \ker\tau_\lambda$. Moreover, write the quotient pair $\rho_\lambda, \tau_\lambda$ and their functions μ_b on $R, N_R(QZ_o/Z_o)$. □

Proof of Theorem 4.2. We induct on $n = \mathrm{rk}(\mathbf{X})$ for the spin map Ω_X and regular map $\widetilde{\Omega}_X$. Consider a residual factor X_0, with $p \nmid |X_0|$. Thus, $Q_0 = 1$, $N_{X_0}(Q_0) = X_0$ and $\Omega_{X_0} = \mathrm{id}$. If $m \in \{1, 2\}$, by Lemma 4.51, we obtain $\widetilde{\Omega}_{X_m}$. By Lemma 4.52, we write the series $\mathcal{E}(X, 1)_{p'}$ and regular norm translates $\rho_0(\lambda \circ \nu)$. We consider nonidentity quasi-isolated classes $[s]$. If $q \equiv 1 \pmod 4$, we choose Ω_X from Proposition 4.48. If $q \equiv 3 \pmod 4$, we choose Ω_X from Corollary 4.49. The regular lifts $\widetilde{\Omega}_X$ come from Proposition 4.50. Thus, $\alpha_\Theta = \alpha_{\widetilde{\Omega}_X(\Theta)}$ and $\mu_a^\Theta = \mu_a^{\widetilde{\Omega}_X(\Theta)}$, including norm twists $\lambda \circ \nu$.

We consider non-quasi-isolated s and $\mathbf{L}^* = C_{\mathbf{X}^*}(Z(C_{\mathbf{X}^*}(s))^\circ)$. Then, by [208, Proposition 4.16, Step 1], $C_{\mathbf{X}^*}(s) \leq \mathbf{L}^* < \mathbf{X}^*$. Write the dual rational Levi $L = \mathbf{L}^F$. Moreover, by Section 4.4, $s_p \in Z(\mathbf{L}^*)$. We write the projective linear twist ζ_s on the two marked inertia groups I_L, I_K. Then, we apply signed superregular induction $I_G : \mathcal{E}(L, s) \to \mathcal{E}(X, s)$. Thus, $I_G(\theta)(1) = |X|_{r'}|L|_{r'}^{-1}\theta(1)$. We consider $\chi = I_G(\theta)$, with $p \nmid \chi(1)$. Thus, $|X : L|_p = 1$. By conjugacy of Sylow subgroups P, write $P \leq L$. Write the residual spin space V_R and a linear or unitary block $W \neq 0$. Moreover, $W \perp \mathsf{V}_R$, since $\mathbf{L}^* < \mathbf{X}^*$. Thus, $n_R = \mathrm{rk}(\mathrm{Spin}(\mathsf{V}_R)) < n$. We choose the lower-rank maps $\Phi_S, \widetilde{\Phi}_S$ on the residual factor S and choose their involutions J_S, J_S', with $J_S^2 = (J_S')^2 = I$. Moreover, $\epsilon_{a,S}^{\mathrm{glo}} = \epsilon_{a,S}^{\mathrm{loc}}$. We write determinant comparisons f_i from Proposition 4.44, on the matrix factors D_i. Moreover, we put $D_i^0 = \ker \det_i$ and $N_i^0 = N_{D_i}(Q_i) \cap D_i^0$. We write $\Theta_i|_{D_i^0}, f_i(\Theta_i)|_{N_i^0}$ after the parent comparisons on $D_i, N_{D_i}(Q_i)$. Moreover, we consider the faithful sector $\Theta_i(z) = -\Theta_i(1)$. We write its character $\lambda_i \in \mathrm{Irr}(\widehat{D}_i)$, with $\lambda_i(1) = 1$. Moreover, write the models $\mathcal{P}_i\lambda_i$ and $\mathcal{Q}_i\lambda_i|_{\widehat{N}_i}$, with $\widehat{N}_i = N_{\widehat{D}_i}(Q_i)$. Write the permutations $\widehat{D}_i^a = \widehat{D}_{\pi_a(i)}$ and the field or graph actions a_i. Moreover, we put $\delta_{a,i} = \lambda_i^a/\lambda_{\pi_a(i)}$. $\delta_{a,i} \in \mathrm{Irr}(\widehat{D}_{\pi_a(i)}/[\widehat{D}_{\pi_a(i)}, \widehat{D}_{\pi_a(i)}])$. By Lemma 4.43, $\epsilon_{a,i}^{\mathrm{glo}} = \epsilon_{a,i}^{\mathrm{loc}}$ for the quadratic operators J_i, J_i'. Write the linear twists $\delta_{a,i}$ in the relative comparisons $f_{\pi_a(i)}$. Thus, we write $\Phi_L : \mathcal{E}(L, s)_{p'} \to Y_s^L(K)$ of Proposition 4.44, with $K = N_L(Q)$.

We write paired constituents $\beta \in \mathrm{Irr}(C_L(Q))$ and $\xi \in \mathrm{Irr}(C_X(Q))$. By the inertia relations, write $I_\beta/C_L(Q) \simeq I_\xi/C_X(Q)$. Accordingly, by Proposition 4.40, we write I_Q. Write its projective models $\mathcal{P}_\beta, \mathcal{P}_\xi$ of Proposition 4.42. Thus, we put $\Omega_s = I_Q\Phi_L I_G^{-1}$. With $\Omega_X|_{\mathcal{E}(X,s)_{p'}} = \Omega_s$, we choose a full lift $\widetilde{s}$ of s and inertia $I_{\widetilde{s}}$ of the marked pair Θ, Θ'. Accordingly, by Section 4.5, we write $\widehat{E}_G, \widehat{E}_Q$ on $I_{\widetilde{s}}$, including operations a with the norm twist $\Theta^a = \Theta(\lambda \circ \nu)$. Write the local space $(E \otimes U_\beta)^{C_0}$. Moreover, the two C_0-twists are κ_0, κ_0^{-1}, with $\kappa_0\kappa_0^{-1} = 1$. Moreover, we consider $G \in \{\widetilde{X}, \widetilde{M}\}$ and $J \leq G$. We choose representatives $t_i \in G$ for G/J. Moreover, we choose $\rho : J \to \mathrm{GL}(V_0)$ and $V_i = t_i \otimes V_0$. Write $gt_i = t_j h$, with $h \in J$. Moreover, $\nu(g)\nu(t_i) = \nu(t_j)\nu(h)$. Thus, we write $D_\lambda|_{V_i} = \lambda(\nu(t_i))I_{V_i}$ for $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$. With $R_{\rho,\lambda} = \mathrm{Ind}_J^G((\lambda \circ \nu)|_J \cdot \rho)$, we put $R_\rho = \mathrm{Ind}_J^G \rho$. Thus, $D_\lambda R_{\rho,\lambda}(g) D_\lambda^{-1} = \lambda(\nu(g))R_\rho(g)$. Moreover, $D_\lambda D_\kappa = D_{\lambda\kappa}$ for $\lambda, \kappa \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Write the field action $t_i \mapsto a(t_i)$ and coefficient action $\lambda(\nu(t_i)) \mapsto \sigma(\lambda(\nu(t_i)))$. By Lemma 4.7, we include $a^m = v$ in both induction extensions $\widehat{E}_G, \widehat{E}_Q$. We write the induction varieties Y_G, Y_Q of $\widehat{E}_G, \widehat{E}_Q$. Moreover, write their gradings $V_{Y_j} = \bigoplus_{c \in \mathbb{F}_q^\times} V_{Y_j,c}$, with $j \in \{G, Q\}$. $T_{\lambda,j}|_{V_{Y_j,c}} = \lambda(c)I$ and $u_a^m = u_v$ on the norm coordinates c. Thus, write the regular comparison $\widetilde{\Omega}_X$ at rank n, with full labels $[\widetilde{s}]$.

Consider a mixed stabilizer operation a and the Levi $\mathbf{L}^*(s)$. Moreover, $\mathbf{L}^*(s^a) = \mathbf{L}^*(s)^a$ and $\mathbf{L}^*(s^k) = \mathbf{L}^*(s)$ for $\gcd(k, |s|) = 1$. By rational fusion of Q, choose $m \in M$ with $(ma)(L) = L$. We choose two choices m_1, m_2, with $a_i = m_i a$. Moreover, $m_2 m_1^{-1} \in N_M(L)$ and that $\Phi_L(\theta^{a_i}) = \Phi_L(\theta)^{a_i}$. Thus, $\Omega_X(\chi^a) = \Omega_X(\chi)^a$ for both choices m_1, m_2. In the regular case, we include every a with $\Theta^a = \Theta(\lambda \circ \nu)$ for $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$.

Moreover, we write ordinary inertia $A_\chi = XB_0$ and source inertia B_0, with $X \cap B_0 = L$. With $B_1 = B_0 \cap N_{A_\chi}(Q)$, consider $b \in B_0$, with $bPb^{-1}, P \in \mathrm{Syl}_p(L)$. By Sylow conjugacy in L, we choose $l \in L$ with $lbPb^{-1}l^{-1} = P$. Then, $lb \in N_{B_0}(P) \leq N_{B_0}(Q)$. $B_0 = LB_1$ and $B_1 \cap L = K$. Moreover, $X \cap N_{A_\chi}(Q) = M$. We write paired characters $\chi, \Omega_X(\chi)$ and their stabilizers on $N_{A_\chi}(Q)$, and $\Omega_X(\chi)^a = \Omega_X(\chi)$ exactly when $\chi^a = \chi$, by the bijection Ω_X. Thus, we write local inertia MB_1. $(MB_1)/M \simeq B_1/K \simeq B_0/L \simeq A_\chi/X$, and we put $T = A_\chi/X$ and choose representatives $r_t \in B_1$ for $t \in T$. Moreover, $k(t, u) = r_t r_u r_{tu}^{-1} \in K$. We write the Levi models $\mathcal{P}_L, \mathcal{P}_K$ with factor $\alpha : T \times T \to \mathbb{C}^\times$. The mixed functions are $\mu_a^L = \mu_a^K$ on T, with representatives $(r_t)_{t \in T}$. By Corollary 2.5 and Proposition 4.41, we write $\mathcal{P}_X$ from $\mathcal{P}_L$. By Proposition 4.42, we write $\mathcal{P}_M$ from $\mathcal{P}_K$. Thus, $\alpha_X = \alpha_M = \alpha$ and $\mu_a^X = \mu_a^M = \mu_a^L$. Moreover, choose a cochain $c : T \to \mathbb{C}^\times$. With $\mathcal{P}_L^c = c\mathcal{P}_L$ and $\mathcal{P}_K^c = c\mathcal{P}_K$, $\mathcal{P}_X^c = c\mathcal{P}_X$ and $\mathcal{P}_M^c = c\mathcal{P}_M$. Thus, we write the common centralizer $Z_\chi = C_{A_\chi}(X)$. $\mathcal{P}_X(z) = \lambda_z I$ and $\mathcal{P}_M(z) = \lambda_z I$ for $z \in Z_\chi$. Accordingly, choose λ_z from the Levi pair $\mathcal{P}_L, \mathcal{P}_K$, with trivial diagonal action I on $\widehat{E}_G, \widehat{E}_Q$. By [141, Theorem 2.9], $\chi, \Omega_X(\chi)$ satisfy the $\succeq_c$ relation on the inductive overgroups A_X, A_M.

If $[s] \in \mathcal{S}_X = \{[s] : \mathcal{E}(X, s)_{p'} \neq \varnothing\}$, write $Y_s^X(M)$ for the packet. By Proposition 4.42 on $C_X(Q) \lhd M$, we choose pairs ξ, ψ. Consequently, $\mathrm{Irr}_{p'}(M) = \bigsqcup_{[s] \in \mathcal{S}_X} Y_s^X(M)$. By Proposition 4.39, we write rational classes $[s]$

and centralizer-series orbits ξ^M. Moreover, $|C_{\mathbf{X}^*}(s)^{F^*}|_p = |X|_p$ for every nonempty set $Y_s^X(M)$. Thus, we write $\Omega_X = \bigsqcup_{[s]\in\mathcal{S}_X} \Omega_s$, with $\Omega_s = I_Q \Phi_L I_G^{-1}$ on non-quasi-isolated $[s]$. Write the preceding Ω_s on quasi-isolated classes $[s]$. By Proposition 4.4, $N_X(P) \le M < X$. Thus, $M^a = M$ for $a \in \mathrm{Aut}(X)_P$, as in the $\succeq_c$ relation. □

4.6 Proof of Theorem 4.1

Proof. If $q = 2^f$, the inseparable isogeny of [179, §17.2.2] gives $S = \Omega_{2n+1}(q) \cong S_C = \mathrm{Sp}_{2n}(q)$ on $\mathbb{F}_q$-points. The induced isomorphism $\widehat{\theta} : \widehat{S} \to \widehat{S}_C$ of universal covers transports iMN(S_C, p) from Theorem 5.1 to iMN(S, p), including $(n, q) = (3, 2)$. If $S = \Omega_7(3)$, then $|\,\mathrm{Mult}(S)| = 6$ [45]. By [157, Lemma 4.1(2)], iMN(S, p) holds for every $p \mid |S|$. Otherwise, for odd q, the universal cover is $X = \mathrm{Spin}_{2n+1}(q)$ by [208, §6.6, proof of Theorem 6.2]. At $p \mid q$, [162, Theorem 1.3] gives iMN(S, p). At $p = 2$, apply [158, Theorem A and §6]. If $p \nmid 2q$, Theorem 4.2 gives Ω_X with $\succeq_c$. For a residual spin factor X_0 of rank $m < n$, the induction takes $X_0 = \widehat{S}_0/Z_0$ when $S_0 = X_0/Z(X_0)$ is simple, with $Z_0 \le Z(\widehat{S}_0)$. □

5 Symplectic groups

Theorem 5.1. *Let $S = \mathrm{PSp}_{2n}(q)$ be a nonabelian simple group, where $n \ge 2$ and q is a prime power. Then* iMN(S, p) *holds for every prime $p \mid |S|$.*

Take $X = \mathrm{Sp}_{2n}(q)$ with odd q, p and $p \nmid q$. If $p \mid q - 1$, Malle [132, Theorem 5.14] gives $\mathrm{N}_X(P) \le M = \mathrm{N}_X(T)$ for a split maximal torus T. On this M, Schaeffer Fry [166, Theorem 6.2] gives the equivariant map Ω_X, with each pair above a common $\omega \in \mathrm{Irr}(\mathrm{Z}(X))$. By Theorem 5.2, the relation $\succeq_c$ holds. When q is even, Section 5.5 treats rational series $\mathcal{E}(X, s)$ for $n \ge 3$. Finally, Theorem 5.1 includes $p \mid q$, $p = 2$ and the exceptional cover $2.\,\mathrm{Sp}_6(2)$.

Theorem 5.2. *Let q be odd and $n \ge 2$, with nonabelian simple $S = \mathrm{PSp}_{2n}(q)$ and universal cover $X = \mathrm{Sp}_{2n}(q)$. Let p be odd with $p \mid |S|$ and $p \nmid q$. Next, take $P \in \mathrm{Syl}_p(X)$ and an abelian normal subgroup D containing every abelian normal subgroup of P. Put $M = N_X(D)$. There is a bijection $\Omega_X : \mathrm{Irr}_{p'}(X) \xrightarrow{\sim} \mathrm{Irr}_{p'}(M)$ with $\Omega_X(\chi^{a\sigma}) = \Omega_X(\chi)^{a\sigma}$ for $(a, \sigma) \in \mathrm{Aut}(X)_P \times \mathcal{H}_p$. Moreover, $(\chi, \Omega_X(\chi))$ satisfies $\succeq_c$ in* Definition 2.1. *Thus,* iMN(S, p) *holds.*

Section 5.4 pairs parents $\widetilde{\chi}, \widetilde{\psi}$ on $\mathrm{GSp}_{2n}(q), N_{\mathrm{GSp}_{2n}(q)}(D)$. Thus, Theorem 5.2 takes the restrictions $\widetilde{\chi}|_X, \widetilde{\psi}|_M$. The field lift has $a^f = v$ on $M = N_X(D)$ for either torus sign $\tau \in \{1, -1\}$. At odd $p \nmid q$, put $M_0 = \mathrm{N}_X(\mathbf{S}_d)$, where $\mathbf{S}_d$ is a Sylow Φ_d-torus for $d = \mathrm{ord}_p(q)$. By the normalizer argument in Section 5.1, $M_0 = M$. If $X \ne \mathrm{Sp}_4(2^f)$, Ruhstorfer et al. [156, Theorem D] gives $\succeq_c$ for the pairs of Malle [132, Theorem 7.8] on $\mathcal{U} = \mathcal{E}(X, 1)_{p'}$. If $S = \mathrm{Sp}_4(2^f)$ with $f \ge 2$ and $p \ne 2$, Johansson [100, Lemma 5.1] gives the equivariant parametrization Ω_X. For odd f, Johansson [100, Theorem 5.4] gives iMN(S, p) for all p. At even f, Lemma 5.49 gives lifts on T_d for $d \in \{1, 2, 4\}$, and Theorem 5.51 gives $\succeq_c$ for odd p. For $S = \mathrm{Sp}_6(2)$, Ruhstorfer–Schaeffer Fry [157, Lemma 4.1(2)] gives iMN(S, p) at every p.

5.1 Normalizers

We set $V = \mathbb{F}_q^{2n}$ and $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Then, we put $\widetilde{X} = \mathrm{GSp}(V)$ and $C = \mathbb{F}_q^\times$. Thus, we fix $\mu : \widetilde{X} \to C$, with $g^{\mathsf{T}} J_n g = \mu(g) J_n$. We have that $\ker \mu = X$. We set $h_a = \mathrm{diag}(aI_n, I_n)$, with $\mu(h_a) = a$ for $a \in C$. Thus, we obtain $\mu(\widetilde{X}) = C$. Next, let $\mathrm{Z}(\widetilde{X}) \le \widetilde{H} \le \widetilde{X}$ with $\mu(\widetilde{H}) = C$. We put $H = \widetilde{H} \cap X$. We take $(G, K) \in \{(\widetilde{X}, X), (\widetilde{H}, H)\}$, with $G/K \simeq C$. Moreover, for $\rho \in \mathrm{Irr}(G)$, put $T_\rho = \{\delta \in \mathrm{Irr}(C) : \rho(\delta \circ \mu) = \rho\}$. Since $\mu(cI_{2n}) = c^2$, the cyclic-quotient formulas give $t_\rho = |T_\rho| \mid (2, q - 1)$. The t_ρ constituents of $\rho|_K$ each have degree $\rho(1)/t_\rho$. If q is odd, $|C : C^2| = 2$. Then, choose $h_m \in \widetilde{H}$ with $\mu(h_m) = m \in C \setminus C^2$.

We write $q = \ell^f$, with prime ℓ. We fix an odd prime $p \ne \ell$, with $p \mid |X|$. Moreover, we put $X = \mathrm{Sp}(V)$, with $\dim_{\mathbb{F}_q} V = 2n$. We put $d = \mathrm{ord}_p(q)$, $e = d/(d, 2)$ and $n = ew + r$, with $0 \le r < e$. We write $\tau = 1$ for odd d and $\tau = -1$ for even d. Thus, we put $a = v_p(q^e - \tau)$, with $a, w \ge 1$. We write the additive valuation $v_p : \mathbb{Q}^\times \to \mathbb{Z}$. We construct $D \le P$ from the abstract group P. Moreover, we prove that $\alpha(D) = D$ for $\alpha \in \mathrm{Aut}(P)$, without a field model for α on V. We put $K = \mathbb{F}_{q^e}$, $Q = q^e$ and $L = \mathbb{F}_{Q^2}$.

First, we consider $\tau = 1$ and $V_1 = K \oplus K$. We put $t_b(x, y) = (bx, b^{-1}y)$, with $b \in K^\times$. Thus, we write $B((x, y), (x', y')) = \mathrm{Tr}_{K/\mathbb{F}_q}(xy' - yx')$. Alternatively, we consider $\tau = -1$ and $V_1 = L$. We choose $c \in L^\times$, with $c^Q = -c$. When $\ell = 2$, we choose $c = 1$. Moreover, we fix $c \ne 0$ in $\ker \mathrm{Tr}_{L/K}$ when $\ell \ne 2$. We put $B(x, y) = \mathrm{Tr}_{L/\mathbb{F}_q}(cxy^Q)$ and $t_b(x) = bx$, with $b^{Q+1} = 1$. Moreover, $B(x, x) = \mathrm{Tr}_{K/\mathbb{F}_q}(x^{Q+1} \mathrm{Tr}_{L/K}(c)) = 0$. By the perfect

trace pairing $(x,y)\mapsto \mathrm{Tr}_{E/\mathbb{F}_q}(xy)$, $\mathrm{rad}(B)=0$, for $E\in\{K,L\}$. $B(t_bx,t_by)=B(x,y)$ in both models V_1. We write the torus $T_1=\{t_b\}$, with $T_1\simeq C_{q^e-\tau}$. Thus, we choose $D_1\le T_1$, with $|D_1|=p^a$. We fix the orthogonal sum $V=V_0\perp V_1\perp\cdots\perp V_w$, with $\dim V_0=2r$ and $V_i\simeq V_1$. We choose fixed isometries $\iota_i:V_1\xrightarrow{\sim}V_i$ and $D_i=\iota_iD_1\iota_i^{-1}$. Moreover, we put $D=\prod_{i=1}^w D_i$ and $P=D\rtimes U$, with $U\in\mathrm{Syl}_p(\mathfrak{S}_w)$. We write $\pi(V_i)=V_{\pi(i)}$ for $\pi\in\mathfrak{S}_w$, with $\pi|_{V_0}=I$ and $\pi|_{V_i}=\iota_{\pi(i)}\iota_i^{-1}$.

Proposition 5.3. *Thus, $P\in\mathrm{Syl}_p(X)$. Its subgroup D contains every abelian normal subgroup of P and is characteristic in P. Thus, $M=N_X(D)$ contains $N_X(P)$ and is $\mathrm{Aut}(X)_P$-stable. If X is quasisimple with $|Z(X)|=(2,q-1)$, then $M<X$.*

Proof. By [207], $|X|_p=p^{aw}(w!)_p=|P|$. Lemma 4.3 makes D the largest abelian normal subgroup of P. Thus, D is characteristic in P, giving $N_X(P)\le M$ and $M^\alpha=M$ for $\alpha\in\mathrm{Aut}(X)_P$. Under the hypothesis on X, $|D|=p^{aw}>1$ gives $D\not\le Z(X)$. The abelian image $DZ(X)/Z(X)$ is nontrivial and proper in simple $X/Z(X)$. Hence, $M<X$. □

We write t_b on $V_1=K\oplus K$. We put $R(x,y)=(x^q,y^q)$ and $J(x,y)=(y,-x)$. Thus, we find $B(Rx,Ry)=B(x,y)=B(Jx,Jy)$. Moreover, $RJ=JR$, $R^e=I$ and $J^2=t_{-1}$. $Rt_bR^{-1}=t_{b^q}$ and $Jt_bJ^{-1}=t_{b^{-1}}$. Moreover, we write $h_m(x,y)=(mx,y)$, with $m\in\mathbb{F}_q^\times$. Thus, we find $B(h_mx,h_my)=mB(x,y)$. Moreover, $h_mRh_m^{-1}=R$ and $h_mJh_m^{-1}=t_mJ$. Thus, we write $u(x,y)=(x^\ell,y^\ell)$ on K^2. $B(ux,uy)=B(x,y)^\ell$ and $uR=Ru, uJ=Ju$. Thus, we find $u^f=R$.

Alternatively, we consider $\tau=-1$, with $V_1=L$. Thus, we find $(c^{\ell-1})^Q=c^{\ell-1}$. By the norm map $N_{L/K}:L^\times\to K^\times$, we choose $b_0\in L^\times$, with $b_0^{Q+1}=c^{\ell-1}$. If $\ell=2$, we set $b_0=1$. We put $u(x)=b_0x^\ell$ and $v=u^f$. By the form B, $B(ux,uy)=B(x,y)^\ell$. Thus, $v\in\mathrm{Sp}(V_1)$. Moreover, $ut_bu^{-1}=t_{b^\ell}$ and $vt_bv^{-1}=t_{b^q}$. By the formula for u, we write the power u^{2ef}. Moreover, $v^{2e}=u^{2ef}=t_{N_{L/\mathbb{F}_\ell}(b_0)}=t_{N_{K/\mathbb{F}_\ell}(c^{\ell-1})}=t_{c^{Q-1}}=t_{-1}$. When $\ell=2$, we write $t_{-1}=I$. Moreover, we write $\widetilde{T}_1=\{b\in L^\times:b^{Q+1}\in\mathbb{F}_q^\times\}$. Thus, $\widetilde{T}_1\simeq C_{(Q+1)(q-1)}$. We write $\mu:\widetilde{T}_1\to\mathbb{F}_q^\times$, with $\mu(b)=b^{Q+1}$. Thus, we find $\mu(\widetilde{T}_1)=\mathbb{F}_q^\times$.

Theorem 5.4. *With $K_1=N_{\mathrm{Sp}(V_1)}(D_1)$, we consider $\tau=1$, with $K_1=\langle T_1,R,J\rangle$ and $K_1/T_1\simeq C_e\times C_2$. Moreover, we consider $\tau=-1$, with $K_1=\langle T_1,v\rangle$ and $K_1/T_1\simeq C_{2e}$. Moreover, $C_X(D)=\mathrm{Sp}(V_0)\times\prod_{i=1}^w T_i$. Thus, we find $M=\mathrm{Sp}(V_0)\times(K_1^w\rtimes\mathfrak{S}_w)$. Thus, we put $\widetilde{M}=N_{\mathrm{GSp}(V)}(D)$. Moreover, $Z(\mathrm{GSp}(V))\le\widetilde{M}$. $\mu(\widetilde{M})=\mathbb{F}_q^\times$ and $\ker(\mu|_{\widetilde{M}})=M$.*

Proof. First, we write $d_a=\mathrm{ord}_{p^a}(q)$. Thus, $d\mid d_a$, by reduction modulo p. Since $q^e\equiv\tau\pmod{p^a}$, $d_a\mid d$. Thus, $d_a=d$. Choose $D_1=\langle t_b\rangle$, with $|b|=p^a$, and put $t=t_b$. Thus, $\mathbb{F}_q[b]=K$ in the split model and $\mathbb{F}_q[b]=L$ in the nonsplit model. Write K_+,K_- for the summands in $V_1=K\oplus K$. Thus, K_+,K_- are irreducible $\mathbb{F}_qD_1$-modules. Thus, $K_+\not\simeq K_-$, since $-1\notin\langle q\rangle$ modulo p. We put $g_{a_+,a_-}(x,y)=(a_+x,a_-y)$, with $a_+,a_-\in K^\times$. Thus, $C_{\mathrm{GL}_{\mathbb{F}_q}(K^2)}(D_1)=\{g_{a_+,a_-}:a_+,a_-\in K^\times\}$. Thus, $B(g_{a_+,a_-}u,g_{a_+,a_-}v)=B(u,v)$ for $u,v\in K^2$ exactly when $a_+a_-=1$. Put $m_\beta(x)=\beta x$ on $V_1=L$, with $\beta\in L^\times$. Moreover, $C_{\mathrm{GL}_{\mathbb{F}_q}(L)}(D_1)=\{m_\beta:\beta\in L^\times\}$. Thus, $B(\beta x,\beta y)=B(x,y)$ exactly when $\beta^{Q+1}=1$. Thus, $C_{\mathrm{Sp}(V_1)}(D_1)=T_1$.

Since $C_{\mathrm{Sp}(V_1)}(D_1)=T_1$ and D_1 is the Sylow p-subgroup of T_1, their normalizers coincide. For odd q, [34, §5.A and Notation 5.4] gives $K_1/T_1\simeq C_e\times C_2$ for $\tau=1$ and $K_1/T_1\simeq C_{2e}$ for $\tau=-1$. Write $E_1=\{\pm q^j:0\le j<e\}$ and $E_{-1}=\{q^j:0\le j<2e\}$ as subsets of $(\mathbb{Z}/p^a\mathbb{Z})^\times$. If $2\mid q$, the spectrum of t is $\{b^j:j\in E_\tau\}$. Hence, a conjugate t^k of t satisfies $kE_\tau=E_\tau$. Since $1\in E_\tau$, this implies $k\in E_\tau$. In both characteristics, the powers $q^i(-1)^\epsilon\in E_1$ have representatives R^iJ^ϵ, with $0\le i<e$. Moreover, $q^i\in E_{-1}$ has representative v^i, with $0\le i<2e$. Thus, the same quotient holds for $2\mid q$. We include the relations $R^e=I, J^2=t_{-1}, v^{2e}=t_{-1}$.

The fixed space is $V_0=V^D$, and each active block V_i collects the summands with kernel $H_i=\prod_{j\ne i}D_j$. Thus, $C_X(D)=\mathrm{Sp}(V_0)\times\prod_i T_i$. By Lemma 4.3, D is largest abelian in P. Thus, D has maximal toral p-order, and [132, Proposition 5.3(b)] gives $D\in\mathrm{Syl}_p(\mathbf{S}_d^F)$ for a Sylow Φ_d-torus $\mathbf{S}_d$. By [132, Propositions 5.6–5.7 and 5.13], $D[p]$ has maximal toral rank and determines $\mathbf{S}_d$ uniquely. Thus, the characteristic subgroup $D[p]\le D$ gives $\mathrm{N}_X(D)\le\mathrm{N}_X(\mathbf{S}_d)$. Conversely, D is characteristic in $\mathbf{S}_d^F$, so $M=\mathrm{N}_X(\mathbf{S}_d)$. Thus, for odd q, [34, §5.A and proof of Theorem 5.1] gives $M=\mathrm{Sp}(V_0)\times(K_1^w\rtimes\mathfrak{S}_w)$. If $2\mid q$, $N_X(D)$ permutes the kernels H_i and the blocks V_i. Thus, ι_i give lifts of $\mathfrak{S}_w$ and the same product for M.

Fix $m\in\mathbb{F}_q^\times$ and $h_{m,i}(x,y)=(mx,y)$ on positive V_i and write $h_{m,i}(x)=b_ix$ on negative V_i, with $b_i^{Q+1}=m$. Since the norm $L^\times\to K^\times$ is onto, each b_i exists. Write $h_{m,0}=\mathrm{diag}(mI_r,I_r)$ on V_0, with $h_{m,0}=I$ if $r=0$. We put

$h_m = \mathrm{diag}(h_{m,0}, h_{m,1}, \dots, h_{m,w})$. Thus, $\mu(h_m) = m$ and $h_m \in C_{\mathrm{GSp}(V)}(D)$. Moreover, $cI_{2n} \in C_{\mathrm{GSp}(V)}(D)$ for $c \in \mathbb{F}_q^\times$. Thus, $\mu(\widetilde{M}) = \mathbb{F}_q^\times$. Since $\ker\mu = X$, $\ker(\mu|_{\widetilde{M}}) = N_X(D) = M$. □

Corollary 5.5. *We fix $(G,H) \in \{(\mathrm{GSp}(V), X), (\widetilde{M}, M)\}$ and $\rho \in \mathrm{Irr}(G)$. Thus, $\rho|_H = \sum_{i=1}^t \chi_i$ with distinct $\chi_i \in \mathrm{Irr}(H)$. Thus, $t \mid (2, q-1)$ and $\chi_i(1) = \rho(1)/t$. We consider even q, with $t = 1$. Thus, $p \nmid \chi_i(1)$ exactly when $p \nmid \rho(1)$, for odd p.*

Proof. By Theorem 5.4, μ gives $G/H \simeq \mathbb{F}_q^\times$, and $\mu(cI_{2n}) = c^2$ gives $(\mathbb{F}_q^\times)^2 \le \mu(\mathrm{Z}(G))$. Hence, by the cyclic-quotient formulas, $t \mid (2, q-1)$ with constituent degrees $\rho(1)/t$ and multiplicity 1. Thus, $t = 1$ for even q, and $p \nmid t$ for odd p. □

We put $m_0 = \mathrm{ord}_p(\ell)$ and $N_0 = \mathrm{lcm}(q^e - 1, (q^e+1)(q-1), 2m_0, 4)$. We write $\mu_h = \{z \in \mathbb{C}^\times : z^h = 1\}$ for $h \ge 1$. Thus, we write the cyclotomic field $\mathbb{K} = \mathbb{Q}(\mu_{N_0})$ for the matrices A and semilinear operators S_a. We put $\chi_A(g) = \mathrm{tr}(A(g))$ for each representation $A : G \to \mathrm{GL}_{\mathbb{K}}(V_A)$ of finite G. We fix a cyclic torus T and $\theta \in \mathrm{Irr}(T)$. Moreover, we put $\theta^{k\sigma}(t) = \sigma(\theta(t^k))$ for $k \in \mathbb{Z}$ and $\sigma \in \mathcal{H}_p$. Choose $\overline{q}q \equiv \overline{\ell}\ell \equiv 1 \pmod{|T|}$. $\theta^{q^{-j}}(t) = \theta(t^{\overline{q}^j})$ and $\theta^{\ell^{-h}}(t) = \theta(t^{\overline{\ell}^h})$, for $j, h \ge 0$.

First, we consider odd $d = e$ and $\theta \in \mathrm{Irr}(T_1)$. We put $c_0 = |\theta^{\langle q\rangle}|$, with $c_0 \mid d$. We suppose that $\theta^{-1} = \theta^{q^j}$. Since $c_0 \mid 2j$ and $2 \nmid c_0$, $c_0 \mid j$. Thus, we obtain $\theta^{-1} = \theta$. We consider $\theta^2 \ne 1$ and $I_\theta = \langle T_1, R^{c_0}\rangle$. Thus, we find $I_\theta = (K_1)_\theta$. We choose extensions $\widehat{\theta}_\lambda \in \mathrm{Irr}(I_\theta)$ with $\widehat{\theta}_\lambda(R^{c_0}) = \lambda \in \mu_{d/c_0}$. We choose $\kappa \in \mu_d$, with $\kappa^{c_0} = \lambda$. Moreover, we put $s = \theta(-1)$ and take the basis $\{e_{\pm,j} : j \in \mathbb{Z}/c_0\mathbb{Z}\}$. We put $P_0 e_{\pm,j} = e_{\pm,j+1}$ and $W_s = \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix} \otimes I_{c_0}$.

We write $A(t_b)e_{\pm,j} = \theta(b^{\pm q^{-j}})e_{\pm,j}$, $A(R) = \kappa P_0$ and $A(J) = W_s$. Thus, $P_0 W_s = W_s P_0$ and $W_s^2 = sI = A(t_{-1})$. Moreover, $A(R)A(t_b)A(R)^{-1} = A(t_{b^q})$ and $W_s A(t_b) W_s^{-1} = A(t_{b^{-1}})$. The line $\mathbb{K}e_{+,0}$ affords $\widehat{\theta}_\lambda$ on I_θ. Thus, A affords $\chi_A = \mathrm{Ind}_{I_\theta}^{K_1} \widehat{\theta}_\lambda$. By Isaacs [95, Theorem 6.11 and Corollary 6.17], these exhaust $\mathrm{Irr}(K_1 \mid \theta)$. We choose $\kappa' \in \mu_d$, with $(\kappa')^{c_0} = \kappa^{c_0}$. We put $z = \kappa'/\kappa$ and $D_z e_{\pm,j} = z^j e_{\pm,j}$. Thus, $D_z A_\kappa(x) D_z^{-1} = A_{\kappa'}(x)$ for $x \in K_1$. We consider $\theta^2 = 1$, with $c_0 = 1$. We choose linear extensions A, with $A(R) = \kappa \in \mu_d$ and $A(J) = \xi$, where $\xi^2 = s$. Since $K_1/T_1 \simeq C_d \times C_2$, Isaacs [95, Corollary 6.17] gives all $2d$ characters $\chi_{\kappa,\xi}$ over θ, with $\chi_{\kappa,\xi}(R) = \kappa$ and $\chi_{\kappa,\xi}(J) = \xi$.

Moreover, we put $\widetilde{K}_1 = N_{\mathrm{GSp}(V_1)}(D_1)$. Thus, $\widetilde{K}_1 = K_1 \rtimes \{h_m : m \in \mathbb{F}_q^\times\}$ on $V_1 = K \oplus K$. By the matrices A, we write $\theta_0 = \theta|_{\mathbb{F}_q^\times}$. We put $A(h_m)e_{+,j} = \theta_0(m)e_{+,j}$ and $A(h_m)e_{-,j} = e_{-,j}$. Moreover, $A(h_m)A(R) = A(R)A(h_m)$. $A(h_m)W_s A(h_m)^{-1} = A(t_m)W_s$. We write $A(h_m)$ for $m \in \mathbb{F}_q^\times$. Moreover, $A(h_m)A(h_n) = A(h_{mn})$ and $A|_{K_1}$ is the preceding representation A. Thus, we choose A on $\widetilde{K}_1$. Thus, $zI = t_{z^{-1}} h_{z^2}$ and $A(zI) = \theta_0(z)I$ for $z \in \mathbb{F}_q^\times$. We consider nontrivial quadratic θ, with $\theta^2 = 1$. Since $(q^d-1)/(q-1)$ is odd, $\theta_0^2 = 1 \ne \theta_0$. We choose A with $c_0 = 1$ and $A(h_m)$ on $\mathbb{K}^2$. Moreover, $\mathrm{Spec}(W_s) = \{\xi, -\xi\}$, with $\xi^2 = s$. Choose $m \notin (\mathbb{F}_q^\times)^2$, with $\theta_0(m) = -1$. $A(h_m)\ker(W_s - \xi I) = \ker(W_s + \xi I)$. Thus, $\chi_A \in \mathrm{Irr}(\widetilde{K}_1)$ and $\chi_A|_{K_1} = \chi_{\kappa,\xi} + \chi_{\kappa,-\xi}$. We consider $\theta = 1$, with $A(h_m) = 1$. Thus, we choose each linear extension $\chi_{\kappa,\xi}$ separately on $\widetilde{K}_1$. Since $\widetilde{K}_1/K_1 \simeq \mathbb{F}_q^\times$, the cyclic-quotient formulas give $\mathrm{Irr}(\widetilde{K}_1 \mid \chi) = \{\chi_A(\delta \circ \mu) : \delta \in \mathrm{Irr}(\mathbb{F}_q^\times)\}$ for each constituent χ of $\chi_A|_{K_1}$.

Alternatively, we consider $d = 2e$ and $(H_1, B_1) \in \{(K_1, T_1), (\widetilde{K}_1, \widetilde{T}_1)\}$. Moreover, $\widetilde{K}_1 = \langle \widetilde{T}_1, v\rangle$ and $H_1/B_1 \simeq C_d$. Moreover, we fix $\theta \in \mathrm{Irr}(B_1)$ and $c_0 = |\theta^{\langle q\rangle}|$, with $c_0 \mid d$. We put $I_\theta = \langle B_1, v^{c_0}\rangle = (H_1)_\theta$. By $v^d = t_{-1}$, we choose $\widehat{\theta}_\nu(v^{c_0}) = \nu$, with $\nu^{d/c_0} = s = \theta(-1)$. Moreover, $\theta^{q^{c_0}} = \theta$. Thus, we consider every $\widehat{\theta}_\nu \in \mathrm{Irr}(I_\theta)$ with the indicated value ν. We choose $\xi \in \mu_{2d}$, with $\xi^{c_0} = \nu$ and $\xi^d = s$. We choose the basis $\{e_j : j \in \mathbb{Z}/c_0\mathbb{Z}\}$ and $P_0 e_j = e_{j+1}$. Moreover, we put $A(t_b)e_j = \theta(b^{q^{-j}})e_j$ and $A(v) = \xi P_0$. $A(v)^{c_0} = \nu I$ and $A(v)^d = sI = A(t_{-1})$. The line $\mathbb{K}e_0$ affords $\widehat{\theta}_\nu$ on I_θ. Thus, $\chi_A = \mathrm{Ind}_{I_\theta}^{H_1} \widehat{\theta}_\nu$. By Isaacs [95, Theorem 6.11 and Corollary 6.17], $\mathrm{Irr}(H_1)$ is indexed by θ up to H_1-conjugacy and $\nu^{d/c_0} = \theta(-1)$.

Moreover, we fix $\widetilde{\theta} \in \mathrm{Irr}(\widetilde{T}_1)$ and $\theta = \widetilde{\theta}|_{T_1}$. We put $\widetilde{c} = |\widetilde{\theta}^{\langle q\rangle}|$ and $c_0 = |\theta^{\langle q\rangle}|$. Thus, we write $\widetilde{\theta}^{q^{c_0}}/\widetilde{\theta} = \lambda \circ \mu$, with $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Thus, $\lambda(z^2) = 1$ for $z \in \mathbb{F}_q^\times$, since $z^q = z$. Thus, we obtain $\lambda^2 = 1$. Since q fixes $\lambda \circ \mu$, $\widetilde{c} \in \{c_0, 2c_0\}$. We consider $\widetilde{c} = c_0$. Thus, $\chi_A|_{K_1} \in \mathrm{Irr}(K_1)$ by the distinct weights $\theta^{q^{-j}}$. Alternatively, we consider $\widetilde{c} = 2c_0$ and $J_0 = P_0^{c_0}$. Thus, $J_0^2 = I$, $J_0 A(v) = A(v) J_0$ and $J_0 A(t_b) = A(t_b) J_0$ for $b \in T_1$. Thus, we write $V_\pm = \langle e_j \pm e_{j+c_0} : 0 \le j < c_0\rangle_{\mathbb{K}}$. Given $x \in K_1$, we write $A_\pm(x) = A(x)|_{V_\pm}$. Thus, $\chi_A|_{K_1} = \chi_{A_+} + \chi_{A_-}$, with $\chi_{A_\pm} \in \mathrm{Irr}(K_1)$. We write the weights $\theta^{q^{-j}}$ with multiplicity 1 in $V_\pm$. Moreover, $A(v)^{c_0}|_{V_\pm} = \pm\xi^{c_0} I$. Moreover, we consider odd q and nontrivial quadratic θ, with $c_0 = 1$. We put $N_T = (Q+1)(q-1)$ and choose a generator $\widetilde{t}$ of $\widetilde{T}_1$. We write $\widetilde{\theta}(\widetilde{t}) = \zeta_{N_T}^k$, with $\mathrm{ord}(\zeta_{N_T}) = N_T$

and $k \in \mathbb{Z}/N_T\mathbb{Z}$. Since $|\mu(\widetilde{t})| = q-1$, $T_1 = \langle \widetilde{t}^{q-1} \rangle$. Thus, $k \equiv (Q+1)/2 \pmod{Q+1}$. Thus, $(q-1)k \equiv N_T/2 \pmod{N_T}$. We write the quadratic character $\lambda_2 \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Consequently, $\widetilde{\theta}^q/\widetilde{\theta} = \lambda_2 \circ \mu \neq 1$ and $\widetilde{c} = 2$.

We fix $h_a \in \mathbb{Z}$, $\sigma_a \in \mathcal{H}_p$ and $\delta_a \in \mathrm{Irr}(\mathbb{F}_q^\times)$. We put $a = (h_a, \sigma_a, \delta_a)$ for the powers u^{h_a} of the fixed map u. Thus, we write $\mathcal{R}_a(A)(x) = \delta_a(\mu(x))\sigma_a(A(u^{-h_a}xu^{h_a}))$. We put $h_{ab} = h_a + h_b$ and $\sigma_{ab} = \sigma_a\sigma_b$. Moreover, $\delta_{ab}(c) = \delta_a(c)\sigma_a(\delta_b(c^{\ell^{-h_a}}))$ for $c \in \mathbb{F}_q^\times$. $\mathcal{R}_{ab} = \mathcal{R}_a \circ \mathcal{R}_b$. We write $\delta_a = 1$ on the isometry group K_1. With $N' = N_0/p^{v_p(N_0)}$, we choose $k_a \in \mathbb{Z}$ for $\sigma_a|_{\mu_{N'}}$. Thus, $\sigma_a(z) = z^{p^{k_a}}$ for $z \in \mu_{N'}$. We choose k_a modulo $\mathrm{ord}_{N'}(p)$. We put $\mathcal{G}_A = \{a : \mathcal{R}_a(A) \simeq A\}$ and $a_f = (f, 1, 1)$. Moreover, we choose S_a to be σ_a-semilinear. We write the matrix relation $\delta_a(\mu(x))S_aA(u^{-h_a}xu^{h_a})S_a^{-1} = A(x)$. Moreover, $u^{h+f} = u^hR$ or $u^{h+f} = u^hv$.

Proposition 5.6. *We consider A and $A(h_m)$, with $\theta \neq 1$. When $\theta^2 = 1$, we choose $\dim_{\mathbb{K}} V_A = 2$. Thus, $a \in \mathcal{G}_A$ exactly when $\theta^{\ell^{-h_a}\sigma_a} = \theta^{(-1)^{\epsilon_a}q^{j_a}}$ and $\delta_a = \theta_0^{\epsilon_a}$. Moreover, $(\epsilon_a, j_a) \in \{0,1\} \times \mathbb{Z}/c_0\mathbb{Z}$ is unique. We choose operators S_a, with $S_aS_b = s^{\epsilon_a\epsilon_b}S_{ab}$ and $S_{a_f} = A(R)$. Moreover, we put $s = (-1)^{b_0}$, with $b_0 \in \{0,1\}$. Moreover, $\widehat{S}_a\widehat{S}_b = (-1)^{b_0((p-1)/2)k_a\epsilon_b}\widehat{S}_{ab}$ and $\widehat{S}_{a_f} = A(R)$.*

Proof. By the condition on θ, we write the orbit $\theta^{\ell^{-h_a}\sigma_a}$. We consider $\theta^2 \neq 1$, with disjoint orbits $\theta^{\langle q \rangle}, (\theta^{-1})^{\langle q \rangle}$. Thus, ϵ_a, j_a are unique. By the matrices $A(h_m)$, $\delta_a = \theta_0^{\epsilon_a}$. Consider nontrivial quadratic θ, with $j_a = 0$. Thus, $\theta_0^2 = 1 \neq \theta_0$. Thus, we write $\delta_a = 1$ for $\epsilon_a = 0$, and $\delta_a = \theta_0$ for $\epsilon_a = 1$. By the condition on θ, $j_{ab} = j_a + j_b \pmod{c_0}$ and $\epsilon_{ab} = \epsilon_a + \epsilon_b \pmod 2$. We write the powers ℓ^{-h_a}, q^{j_a} and coefficient operations σ_a on θ. Moreover, $(\theta^{q^j})^{\ell^{-h}\sigma} = (\theta^{\ell^{-h}\sigma})^{q^j}$ and that $\theta_0^{\ell^{-h_a}\sigma_a} = \theta_0^{(-1)^{\epsilon_a}}$. Thus, $\delta_{ab} = \theta_0^{\epsilon_{ab}}$.

By $q = \ell^f$, $d = m_0/(m_0, f)$. Thus, $\{z^f : z \in \mu_{m_0}\} = \mu_d$. Thus, we choose $\eta \in \mu_{m_0}$, with $\eta^f = \kappa$. Thus, $\sigma(\eta) = \eta$ and $\sigma(\kappa) = \kappa$, since $m_0 \mid p-1$. We write entrywise σ_a on the basis $e_{\pm,j}$. Moreover, put $S_a = \eta^{h_a}P_0^{-j_a}W_s^{\epsilon_a}\sigma_a$. Consider $x = t_b$, with $A(t_b)e_{\pm,j} = \theta(b^{\pm q^{-j}})e_{\pm,j}$. Thus, $P_0^{-j_a}\mathbb{K}e_{\pm,j} = \mathbb{K}e_{\pm,j-j_a}$ and $W_s\mathbb{K}e_{\pm,j} = \mathbb{K}e_{\mp,j}$. Thus, $\delta_a(\mu(x))S_aA(u^{-h_a}xu^{h_a})S_a^{-1} = A(x)$. We consider $x = R, J$, with $uR = Ru, uJ = Ju$. Thus, $P_0W_s = W_sP_0$ and $\sigma_a(P_0) = P_0, \sigma_a(W_s) = W_s$. Thus, $S_aA(x)S_a^{-1} = A(x)$ for $x \in \{R, J\}$. We consider $x = h_m$. Thus, $\theta_0^{\ell^{-h_a}\sigma_a}(m) = \theta_0(m)^{(-1)^{\epsilon_a}}$. Moreover, we consider $\epsilon_a = 1$, with diagonal entries $\theta_0(m)^{-1}, 1$. Thus, $\theta_0(m)W_s\,\mathrm{diag}(\theta_0(m)^{-1}, 1)W_s^{-1} = \mathrm{diag}(\theta_0(m), 1)$. We consider $\epsilon_a = 0$, with entries $\theta_0(m), 1$. Thus, $\delta_a(m)S_aA(u^{-h_a}h_mu^{h_a})S_a^{-1} = A(h_m)$.

Thus, $S_aS_b = s^{\epsilon_a\epsilon_b}S_{ab}$, since $W_s^2 = sI$. We consider $a = a_f$, with $j_a = -1 \pmod{c_0}$ and $\epsilon_a = 0$. Thus, $S_{a_f} = \eta^f P_0 = \kappa P_0 = A(R)$, including $c_0 = 1$. We choose $\mathrm{i} \in \mu_4$, with $\mathrm{i}^2 = -1$, and we put $\widehat{S}_a = \mathrm{i}^{b_0\epsilon_a}S_a$. Since $\mathrm{i}^{2b_0} = s$, $\mathrm{i}^{b_0(\epsilon_a+\epsilon_b-\epsilon_{ab})}s^{\epsilon_a\epsilon_b} = 1$. Thus, $\sigma_a(\mathrm{i})/\mathrm{i} = (-1)^{((p-1)/2)k_a}$. Thus, $\widehat{S}_a\widehat{S}_b = (-1)^{b_0((p-1)/2)k_a\epsilon_b}\widehat{S}_{ab}$. Thus, $\widehat{S}_{a_f} = A(R)$. □

Proposition 5.7. *We fix $H_1 \in \{K_1, \widetilde{K}_1\}$ in* the representation *A. Thus, $a \in \mathcal{G}_A$ exactly when $\theta^{\ell^{-h_a}\sigma_a}(\delta_a \circ \mu) = \theta^{q^{j_a}}$ and $\sigma_a(\nu) = \nu$. Moreover, $j_a \in \mathbb{Z}/c_0\mathbb{Z}$ is unique. We put $s = (-1)^{b_0}$, with $b_0 \in \{0,1\}$. Thus, we have that $S_aS_b = \left(\frac{\ell}{p}\right)^{b_0k_a(h_b+fj_b)}S_{ab}$. Thus, $S_{a_f} = A(v)$ and $S_{a_f}^d = A(t_{-1})$.*

Proof. Write the weight orbit $\theta^{\langle q \rangle}$. Thus, j_a is unique modulo c_0. Since $uv = vu$ and $\mu(v) = 1$, $\sigma_a(\nu) = \nu$. On I_θ, choose $\widehat{\theta}_\nu$. Thus, $\mathrm{Ind}_{I_\theta}^{H_1}\widehat{\theta}_\nu = \chi_A$. Thus, $a \in \mathcal{G}_A$ exactly when $(\theta^{\ell^{-h_a}\sigma_a}(\delta_a \circ \mu), \sigma_a(\nu)) = (\theta^{q^{j_a}}, \nu)$. Thus, $j_{ab} = j_a + j_b \pmod{c_0}$, since $\mu(t_b^q) = \mu(t_b)$. We consider $s = 1$ and $\eta \in \mu_{m_0}$, with $\eta^f = \xi$. Thus, $\xi^d = 1$ in μ_d. Alternatively, we consider $s = -1$, with even d. Thus, $v_2(f) < v_2(m_0)$ and $(2m_0, f) = (m_0, f)$. Thus, $\{z^f : z \in \mu_{2m_0}\} = \mu_{2d}$. Thus, choose $\eta \in \mu_{2m_0}$, with $\eta^f = \xi$. Suppose that $\eta^{m_0} = 1$. Thus, $\xi^d = \eta^{fd} = 1$, contrary to $\xi^d = -1$. Thus, $\eta^{m_0} = -1$. We put $\beta_a = \sigma_a(\eta)/\eta$ in both cases s. Moreover, write the Legendre symbol $\left(\frac{\ell}{p}\right)$. We find $\beta_a = \left(\frac{\ell}{p}\right)^{b_0k_a}$. Since $m_0 \mid p-1$, $\eta^p/\eta = (-1)^{(p-1)/m_0}$ when $s = -1$. We write the subgroup $(\mathbb{F}_p^\times)^2$ of $\mathbb{F}_p^\times$, and $\ell \in (\mathbb{F}_p^\times)^2$ exactly when $m_0 \mid (p-1)/2$. Thus, $(-1)^{(p-1)/m_0} = \left(\frac{\ell}{p}\right)$.

We put $\Delta_a = \xi/\sigma_a(\xi)$ and $E_ae_j = \Delta_a^j e_j$, and $\Delta_a^{c_0} = \nu/\sigma_a(\nu) = 1$. Thus, write E_a on indices $j \in \mathbb{Z}/c_0\mathbb{Z}$. We put $S_a = \eta^{h_a}P_0^{-j_a}E_a\sigma_a$. Then, by the condition on θ, $\delta_a(\mu(t_b))S_aA(u^{-h_a}t_bu^{h_a})S_a^{-1} = A(t_b)$. $E_aP_0E_a^{-1} = \Delta_aP_0$ and $\sigma_a(\xi)\Delta_a = \xi$. Thus, $S_aA(v)S_a^{-1} = A(v)$. $E_aP_0^{-j_b} = \Delta_a^{-j_b}P_0^{-j_b}E_a$ and $\eta^{h_a}\sigma_a(\eta^{h_b})/\eta^{h_a+h_b} = \beta_a^{h_b}$. Moreover, $\Delta_a\sigma_a(\Delta_b) = \Delta_{ab}$ and $E_a\sigma_a(E_b) = E_{ab}$. $S_aS_b = \beta_a^{h_b}\Delta_a^{-j_b}S_{ab}$, and $\Delta_a = \beta_a^{-f}$. Thus, $S_aS_b = \left(\frac{\ell}{p}\right)^{b_0k_a(h_b+fj_b)}S_{ab}$. $\beta_a^{fc_0} = \sigma_a(\nu)/\nu = 1$, and $\beta_a^{f(j_b+c_0)} = \beta_a^{fj_b}$. Moreover, we consider $a = a_f$, with $j_a = -1$ and $\Delta_a = 1$. Consequently, $S_{a_f} = \eta^f P_0 = \xi P_0 = A(v)$. By $v^d = t_{-1}$, $S_{a_f}^d = A(t_{-1})$. □

Corollary 5.8. *Suppose that $\ell = 2$, and fix signed coordinates V_i. We choose field powers u^h, coefficient operations $\sigma \in \mathcal{H}_p$ and multiplier twists $\delta \circ \mu$. Thus, we choose operators S_a, with $S_aS_b = S_{ab}$ and $S_{a_f} = A(u^f)$.*

Proof. $2 \nmid |T_1|$ and $t_{-1} = 1$. Hence, $s = 1$ and $\theta^2 = 1 \Longrightarrow \theta = 1$. By Propositions 5.6 and 5.7, $S_aS_b = S_{ab}$. Then, we consider the positive weight $\theta = 1$. Choose a linear representation A, with $A(J) \in \{1, -1\}$ and $A(R) = \kappa \in \mu_d$. Moreover, choose $\eta \in \mu_{m_0}$, with $\eta^f = \kappa$. Put $S_a = \eta^{h_a}\sigma_a$, and $\sigma_a(\eta) = \eta$ and $\sigma_a(\kappa) = \kappa$. Thus, $S_aS_b = S_{ab}$ and $S_{a_f} = A(R)$. We fix an additional character $\lambda \in \mathrm{Irr}(\mathbb{F}_q^\times)$ and $A_\lambda = A(\lambda \circ \mu)$. Whenever $c \in \mathbb{F}_q^\times$, we put $\lambda^a(c) = \sigma_a(\lambda(c^{\ell^{-h_a}}))$. Moreover, we write $\delta'_a = \delta_a\lambda^a/\lambda$ and $a' = (h_a, \sigma_a, \delta'_a)$. $\mathcal{R}_a(A_\lambda) = (\lambda \circ \mu)\mathcal{R}_{a'}(A)$, and we write $S^\lambda_a = S_{a'}$ on the stabilizer $\mathcal{G}_{A_\lambda}$. Thus, $S^\lambda_a S^\lambda_b = S^\lambda_{ab}$ and $S^\lambda_{a_f} = A_\lambda(u^f)$. We fix ordinary field powers u^h. By Lemma 5.49, the graph lifts are γ_d. □

We fix odd $q = \ell^f$. We choose the quadratic character $\varepsilon_q : \mathbb{F}_q \to \{0, 1, -1\}$, with $\varepsilon_q(0) = 0$. Moreover, we put $q^* = \varepsilon_q(-1)q$. We choose $\zeta \in \mu_\ell$, with $\mathrm{ord}(\zeta) = \ell$. We write $\omega(x) = \zeta^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_\ell}(x)}$. Then, the Gauss sum is $G_q = \sum_{x\in\mathbb{F}_q} \varepsilon_q(x)\omega(x)$. Thus, we put $M_G = \mathrm{lcm}(\ell, N', 4, 2d)$ and choose $k_\sigma \geq 0$ with $\sigma(z) = z^{p^{k_\sigma}}$ on μ_{M_G} for $\sigma \in \mathcal{H}_p$.

Lemma 5.9. *We fix the quadratic torus character θ, with $\theta^2 = 1 \neq \theta$, and its extension value ξ. We write $\xi = \widehat{\theta}(J)$ for odd d, or $\xi = \widehat{\theta}(v)$ for $d = 2e$. Thus, $\sigma(\xi)/\xi = \sigma(G_q)/G_q = (q^*/p)^{k_\sigma}$ for $\sigma \in \mathcal{H}_p$.*

Proof. By [44, Corollary 2.3], $G_q^2 = q^*$. By [165, Lemma 4.3], $\sigma(G_q)/G_q = (q^*/p)^{k_\sigma}$. At $d = 1$, this is the sign $\gamma_{\lambda,\sigma}\delta_{\lambda,\sigma}$ for a nontrivial quadratic $\lambda \in \mathrm{Irr}(T)$ by [165, Lemma 14.2]. If d is odd, $\xi^2 = \varepsilon_q(-1)$ and q is a square modulo p. Thus, $\xi^{p-1} = \varepsilon_q(-1)^{(p-1)/2} = (q^*/p)$. If $d = 2e$, put $m = (p-1)/d$ and $A \equiv (q-1)/2 \bmod 2$. Then, $\xi^d = (-1)^{(q^e+1)/2} = (-1)^{1+Ae}$. Moreover, $(q/p) = (-1)^m$ and $(-1/p) = (-1)^{em}$. Consequently, $\xi^{p-1} = (-1)^{m(1+Ae)} = (q^*/p)$. In both cases, $\xi \in \mu_{M_G}$, so $\sigma(\xi)/\xi = (\xi^{p-1})^{\sum_{j=0}^{k_\sigma-1} p^j}$. Thus, the ratio is $(q^*/p)^{k_\sigma}$. □

Lemma 5.10. *Moreover, we fix finite H and $w \in \mathbb{Z}_{\geq 0}$. We choose a splitting field $\mathbb{F}$, with $\mathrm{char}\,\mathbb{F} = 0$. Moreover, we write $r = |\mathrm{Irr}(H)|$, $\mathrm{Irr}(H) = \{\chi_i : 1 \leq i \leq r\}$ and $\rho_i : H \to \mathrm{GL}_{\mathbb{F}}(V_i)$, with $\mathrm{tr}\,\rho_i = \chi_i$. With $G_w = H^w \rtimes \mathfrak{S}_w$, we choose $m_i \in \mathbb{Z}_{\geq 0}$, with $\sum_i m_i = w$, and partitions $\lambda_i \vdash m_i$. Let S^{λ_i} be the Specht module over $\mathbb{Q}$. By* James–Kerber [99, Theorem 4.3.34]*, the distinct labels $(\mathbf{m}, \boldsymbol{\lambda})$ parametrize $\mathrm{Irr}(G_w)$, with $\psi_{\mathbf{m},\boldsymbol{\lambda}}(1) = \frac{w!}{\prod_i m_i!}\prod_i\big((\dim_{\mathbb{F}} V_i)^{m_i}\dim_{\mathbb{Q}} S^{\lambda_i}\big)$. Moreover, we fix homomorphisms $\pi : \Gamma \to \mathfrak{S}_w$ and $\sigma : \Gamma \to \mathrm{Aut}(\mathbb{F})$. We put $aj = \pi(a)(j)$ for $a \in \Gamma$ and fix $\mathbb{F}$-spaces W_j at positions $j \in \{1, \ldots, w\}$. Moreover, we choose σ_a-semilinear maps $T_{a,j} : W_j \xrightarrow{\sim} W_{aj}$, with $\sigma_a\sigma_b = \sigma_{ab}$. We suppose that $T_{a,bj}T_{b,j} = \alpha_j(a,b)T_{ab,j}$, with $\alpha_j(a,b) \in \mathbb{F}^\times$. Thus, we put T_a on $\bigotimes_j W_j$, with $T_aT_b = (\prod_j \alpha_j(a,b))T_{ab}$. We include maps $T_{a,j}$ between different factors W_j, W_{aj}. Moreover, $\Pi_\pi\Pi_\tau = \Pi_{\pi\tau}$.*

Proof. By [99, Theorem 4.3.34], take $\mathcal{V}_{\mathbf{m},\boldsymbol{\lambda}} = \mathrm{Ind}_{I_{\mathbf{m}}}^{G_w}\mathcal{E}_{\mathbf{m},\boldsymbol{\lambda}}$, where $I_{\mathbf{m}} = H^w \rtimes \prod_i \mathfrak{S}_{m_i}$ and $\mathcal{E}_{\mathbf{m},\boldsymbol{\lambda}} = \widehat{U}_{\mathbf{m}} \otimes_{\mathbb{Q}} \mathcal{S}_{\boldsymbol{\lambda}}$. In this model, $\widehat{U}_{\mathbf{m}}$ extends $\bigotimes_i V_i^{\otimes m_i}$ by factor permutations Π_π, and $\mathcal{S}_{\boldsymbol{\lambda}} = \bigotimes_{i,\mathbb{Q}} S^{\lambda_i}$. At $w = 0$, the model is $\mathbb{F}$. On $\bigotimes_j W_j$, put $T_a(\bigotimes_j w_j) = \bigotimes_j T_{a,a^{-1}j}w_{a^{-1}j}$. The factor at position abj under T_aT_b is $T_{a,bj}T_{b,j}w_j = \alpha_j(a,b)T_{ab,j}w_j$, so $T_aT_b = (\prod_j \alpha_j(a,b))T_{ab}$. On $\mathcal{S}_{\boldsymbol{\lambda}}$, permutations satisfy $\Pi^{\mathcal{S}}_a\Pi^{\mathcal{S}}_b = \Pi^{\mathcal{S}}_{ab}$. Thus, $B_a = T_a \otimes_{\mathbb{Q}} \Pi^{\mathcal{S}}_a$ has the same multiplier. Write $I = I_{\mathbf{m}}$ and $\mathcal{E} = \mathcal{E}_{\mathbf{m},\boldsymbol{\lambda}}$. If $a \in \mathrm{Aut}(G_w)$, put $I_a = a(I)$ and let $B_a : \mathcal{E} \to \mathcal{E}_a$ satisfy $B_a\rho(i) = \rho_a(a(i))B_a$. Then, the induced map is $\widehat{B}_a(g \otimes v) = a(g) \otimes B_av$. In coset coordinates $a(g) = g'i_a$, it is $g' \otimes \rho_a(i_a)B_av$. By the formulas for $\widehat{B}_a, \widehat{B}_b$ on pure tensors, $\widehat{B}_a\widehat{B}_b = (\prod_j \alpha_j(a,b))\widehat{B}_{ab}$. If $a^m = \mathrm{Int}(g_0)$, then $\widehat{B}_{a^m}(g \otimes v) = g_0gg_0^{-1} \otimes B_{a^m}v$. □

We fix odd q and $C = \mathbb{F}_q^\times$. Choose finite groups $\widetilde{H}_i$, indexed by $1 \leq i \leq t$ and maps $\mu_i : \widetilde{H}_i \twoheadrightarrow C$, with $H_i = \ker\mu_i$. Moreover, let $z_i : C \hookrightarrow Z(\widetilde{H}_i)$ be scalar embeddings, with $\mu_i(z_i(c)) = c^2$ for $c \in C$. We put $H_0 = \prod_i H_i$ and $H^\dagger = \{(g_i) \in \prod_i \widetilde{H}_i : \mu_i(g_i) = \mu_j(g_j) \text{ for all } i, j\}$. We choose irreducible $A_i : \widetilde{H}_i \to \mathrm{GL}(V_i)$ and $\chi_i = \mathrm{tr}\,A_i$. By the quotient formulas for H_i, $\chi_i|_{H_i} = \sum_{j=1}^{t_i}\chi_{i,j}$, with distinct $\chi_{i,j} \in \mathrm{Irr}(H_i)$ and $t_i \in \{1, 2\}$, from the scalars $z_i(c)$. We fix a factor i with two constituents χ_i^+, χ_i^-. Moreover, $\chi_i(\varepsilon_q \circ \mu_i) = \chi_i$. Moreover, we choose an intertwiner Y_i, with $Y_iA_i(g) = \varepsilon_q(\mu_i(g))A_i(g)Y_i$. By Schur's lemma for A_i, $Y_i^2 = c_iI$, with $c_i \neq 0$. We choose $b_i^2c_i = 1$ and put $X_i = b_iY_i$. Thus, $X_i^2 = I$ and $X_iA_i(g) = \varepsilon_q(\mu_i(g))A_i(g)X_i$. We put $X_i = \xi^{-1}W_s$ in the positive quadratic model A_i. We put $X_i = P_0^{c_0}$ in the negative model $\widetilde{c} = 2c_0$. Moreover, we put $\mathcal{V} = \bigotimes_i V_i$ and $\widehat{X}_i = I \otimes \cdots \otimes X_i \otimes \cdots \otimes I$. On $\prod_i \widetilde{H}_i$, we put $A = \bigotimes_i A_i$. $\widehat{X}_i^2 = I$ and $\widehat{X}_i\widehat{X}_j = \widehat{X}_j\widehat{X}_i$.

Proposition 5.11. *First, we consider $k > 0$, with $\chi_i|_{H_i} = \chi_i^+ + \chi_i^-$ for $1 \leq i \leq k$. Moreover, $\chi_i|_{H_i} \in \mathrm{Irr}(H_i)$ for $i > k$. We fix $\eta = (\eta_2, \ldots, \eta_k) \in \{1, -1\}^{k-1}$. Thus, we put $\mathcal{V}_\eta = \{v \in \mathcal{V} : \widehat{X}_1\widehat{X}_jv = \eta_jv, 2 \leq j \leq k\}$.*

Moreover, $\mathcal{V}|_{H^\dagger} = \bigoplus_\eta \mathcal{V}_\eta$ has 2^{k-1} distinct irreducible constituents. Alternatively, we consider $k = 0$, with $\mathcal{V}_\varnothing = \mathcal{V}$ and $\mathcal{V}|_{H^\dagger}$ irreducible. Moreover, we choose σ_a-semilinear isomorphisms $S_a : \mathcal{V} \xrightarrow{\sim} \mathcal{V}$, with $S_a(\mathcal{V}_\eta) = \mathcal{V}_\eta$. Moreover, $(S_a|_{\mathcal{V}_\eta})(S_b|_{\mathcal{V}_\eta}) = \alpha(a,b)S_{ab}|_{\mathcal{V}_\eta}$ whenever $S_aS_b = \alpha(a,b)S_{ab}$.

Proof. If $k > 0$, let $V_i^\pm = \ker(X_i \mp I)$ for $1 \le i \le k$. An element $g_i \in \widetilde{H}_i$ with nonsquare multiplier exchanges V_i^+, V_i^-, so both are nonzero. Thus, $V_i^\pm$ afford $\chi_i^\pm$. Given $\epsilon \in \{1,-1\}^k$, put $V_\epsilon = (\bigotimes_{i=1}^k V_i^{\epsilon_i}) \otimes \bigotimes_{i>k} V_i$. Hence, $\mathcal{V}|_{H_0} = \bigoplus_\epsilon V_\epsilon$ is multiplicity free. Given $g \in H^\dagger$ with common multiplier m, conjugation by $A(g)$ gives $\varepsilon_q(m)\widehat{X}_i$ from $\widehat{X}_i$. Thus, $A(g)V_\epsilon = V_{\varepsilon_q(m)\epsilon}$, with both signs of $\varepsilon_q(m)$ attained by surjectivity. With $\eta_j = \epsilon_1\epsilon_j$, the eigenspace $\mathcal{V}_\eta = V_\epsilon \oplus V_{-\epsilon}$ is $H^\dagger$-stable. By [95, Theorem 6.2], each irreducible summand of $\mathcal{V}_\eta$ restricts to a multiple of this $H^\dagger$-orbit. Multiplicity 1 on H_0 gives one summand for each η, with distinct restrictions to H_0. If $k = 0$, $\mathcal{V}$ is irreducible on H_0, hence also on $H^\dagger$. If S_a, S_b stabilize $\mathcal{V}_\eta$, then $S_aS_b = \alpha(a,b)S_{ab}$ holds on $\mathcal{V}_\eta$. Moreover, if $S_a^m = \lambda A(g_0)$ with $g_0 \in H^\dagger$, then $(S_a|_{\mathcal{V}_\eta})^m = \lambda A(g_0)|_{\mathcal{V}_\eta}$. □

5.2 Semisimple elements

Suppose that q is odd and let $H = \mathrm{SO}_{2n+1}(q)$ be dual to X. If $\mathcal{E}(X,s)_{p'} \ne \varnothing$, Proposition 5.12 places a conjugate of s in $C_H(D)$. For quasi-isolated $s \ne 1$, Proposition 5.13 gives the valuation of $[H : C_H(s)]$. We put $W_+ = K \oplus K$, with $B_+((x,y),(x',y')) = \mathrm{Tr}_{K/\mathbb{F}_q}(xy' + yx')$. Alternatively, we put $W_- = L$, with $B_-(x,y) = \mathrm{Tr}_{L/\mathbb{F}_q}(xy^Q)$. Moreover, $B_\tau(x,y) = B_\tau(y,x)$ for $\tau \in \{1,-1\}$. By the trace pairing $(x,y) \mapsto \mathrm{Tr}_{E/\mathbb{F}_q}(xy)$, $\mathrm{rad}(B_\tau) = 0$, with $E \in \{K,L\}$. Then, we put $W = W_0 \perp W_1 \perp \cdots \perp W_w$, with $\dim W_0 = 2r+1$ and $(W_i,\beta_i) \simeq (W_\tau, B_\tau)$. We choose β_0 on W_0, with $\det[\beta_0] \equiv \det[\beta]\prod_i \det[\beta_i]^{-1} \pmod{(\mathbb{F}_q^\times)^2}$, for the ambient form β on W. Thus, we write $H = \mathrm{SO}(W,\beta)$. We put $t_b(x,y) = (bx, b^{-1}y)$ on W_+, with $b \in K^\times$. Moreover, $B_+(t_bu, t_bv) = B_+(u,v)$ and $\det(t_b) = N_{K/\mathbb{F}_q}(b)N_{K/\mathbb{F}_q}(b^{-1}) = 1$. Alternatively, we put $t_b(x) = bx$ on W_-, with $b^{Q+1} = 1$. We have that $B_-(t_bx, t_by) = B_-(x,y)$ and $\det(t_b) = N_{L/\mathbb{F}_q}(b) = N_{K/\mathbb{F}_q}(b^{Q+1}) = 1$. Thus, we write $T_\tau = \{t_b\}$ on W_τ, with $|T_\tau| = q^e - \tau$. We choose $D_\tau \in \mathrm{Syl}_p(T_\tau)$, with $|D_\tau| = p^a$. Moreover, we choose fixed isometries $j_i : W_\tau \xrightarrow{\sim} W_i$. We put $T_i = j_iT_\tau j_i^{-1}$ and $D_i = j_iD_\tau j_i^{-1}$. Moreover, we put $\pi|_{W_i} = j_{\pi(i)}j_i^{-1}$ and $\pi|_{W_0} = I$, for $\pi \in \mathfrak{S}_w$. $\det(\pi) = (\mathrm{sgn}\,\pi)^{2e} = 1$. Thus, we put $P = D \rtimes U \le H$, with $D = \prod_i D_i$ and $U \in \mathrm{Syl}_p(\mathfrak{S}_w)$. Moreover, $|H| = q^{n^2}\prod_{i=1}^n (q^{2i}-1)$. By Proposition 5.3, we have that $v_p(|H|) = aw + v_p(w!) = v_p(|P|)$ and $P \in \mathrm{Syl}_p(H)$. We consider $g \in C_{\mathrm{O}(W,\beta)}(D)$. Then, by Theorem 5.4, $g(W_i) = W_i$ and $g(W_0) = W_0$. Moreover, $C_{\mathrm{O}(W_i,\beta_i)}(D_i) = T_i$, from $m_{b_+,b_-}(x,y) = (b_+x, b_-y), b_+b_- = 1$ on W_+ and $m_b(x) = bx, b^{Q+1} = 1$ on W_-. Moreover, $\det(g|_{W_i}) = 1$ for $1 \le i \le w$. Thus, $g \in H$ exactly when $\det(g|_{W_0}) = 1$. Thus, we obtain that $C_H(D) = \mathrm{SO}(W_0,\beta_0) \times \prod_i T_i$, with $\mathrm{SO}(W_0,\beta_0) \simeq \mathrm{SO}_{2r+1}(q)$. Moreover, we include $r = 0$, with $\mathrm{SO}_1(q) = 1$.

Proposition 5.12. *We fix $\chi \in \mathrm{Irr}_{p'}(X)$ and semisimple $s \in H$, with $\chi \in \mathcal{E}(X,s)$. Moreover, $h^{-1}sh = (s_0; t_1, \ldots, t_w) \in \mathrm{C_H(D)}$ for some $h \in H$. Moreover, $t_i = t_{u(i)}$ for $u \in U$. Moreover, we have that $p \nmid |\,\mathrm{SO}_{2r+1}(q)|$.*

Proof. Put $\widetilde{X} = \mathrm{GSp}_{2n}(q)$ and $\rho \in \mathrm{Irr}(\widetilde{X})$, with $\langle \rho|_X, \chi\rangle_X > 0$. By Corollary 5.5, $p \nmid \rho(1)$. We write $\widetilde{\mathbf{H}} = \mathrm{GSpin}_{2n+1}$ and $\mathbf{H} = \mathrm{SO}_{2n+1}$, with Frobenius F, and we put $\widetilde{H} = \widetilde{\mathbf{H}}^F$ and $H = \mathbf{H}^F$. Moreover, we write the central map $\pi : \widetilde{\mathbf{H}} \twoheadrightarrow \mathbf{H}$, with $Z = \ker\pi \simeq \mathbb{G}_m$. Moreover, we choose the Jordan pair $(\widetilde{s}, \varphi)$ for ρ, with semisimple $\widetilde{s} \in \widetilde{H}$. $\varphi \in \mathcal{E}(C_{\widetilde{H}}(\widetilde{s}), 1)$ and $\rho(1) = [\widetilde{H} : C_{\widetilde{H}}(\widetilde{s})]_{\ell'}\varphi(1)$ by [208, §6.2, proof of Lemma 6.5]. By [175, Proposition 3.17(b)], s is H-conjugate to $\pi(\widetilde{s})$. Moreover, [114] gives $H^1(F,Z) = 1$. Thus, $\pi(\widetilde{H}) = H$. Since $p \ne \ell$, $p \nmid [\widetilde{H} : C_{\widetilde{H}}(\widetilde{s})]$ and $p \nmid \varphi(1)$. We choose $\widetilde{P} \in \mathrm{Syl}_p(C_{\widetilde{H}}(\widetilde{s}))$. Moreover, $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{H})$. Thus, $\pi(\widetilde{P}) \in \mathrm{Syl}_p(H)$. Choose $h \in H$, with $h^{-1}\pi(\widetilde{P})h = P$, and we put $s' = h^{-1}\pi(\widetilde{s})h$. Thus, $s' \in C_H(P) \le C_H(D)$. By $C_H(D)$, we write $s' = (s_0; t_1, \ldots, t_w)$. Moreover, $u^{-1}s'u = s'$ for $u \in U$. Thus, $t_i = t_{u(i)}$ for $1 \le i \le w$. Since $r < e$, every $q^{2j} - 1$ with $1 \le j \le r$ is prime to p. Thus, $v_p(|\,\mathrm{SO}_{2r+1}(q)|) = 0$. □

Proposition 5.13. *We consider quasi-isolated $s \in H \setminus \{1\}$ and put $W_- = \ker(s+I)$, $W_+ = \ker(s-I)$ and $\dim W_- = 2A, \dim W_+ = 2B+1$. Moreover, we write the type $\epsilon \in \{1,-1\}$ of W_-, with $A + B = n$. We put $A_0 = \lfloor A/e \rfloor$ and $B_0 = \lfloor B/e \rfloor$ and put $m = A_0 - 1$ when $e \mid A$ and $\epsilon \ne \tau^{A_0}$. Alternatively, we put $m = A_0$ when $e \nmid A$ or $\epsilon = \tau^{A_0}$. Moreover, $\mathcal{E}(X,s)_{p'} \ne \varnothing$ exactly when $m + B_0 = w$ and $p \nmid \binom{w}{m}$. Moreover, we consider $\chi \in \mathcal{E}(X,s)$ and an ambient Jordan pair $(\widetilde{s},\varphi)$ above χ. We suppose that $m + B_0 = w$ and $p \nmid \binom{w}{m}$. Moreover, $p \nmid \chi(1)$ exactly when $p \nmid \varphi(1)$.*

Proof. By Lemma 6.5(i)–(iv) of [208] for s, $s^2=1$. We put $H_-=\mathrm{SO}^{\epsilon}_{2A}(q)$ and $H_+=\mathrm{SO}_{2B+1}(q)$. By [34, proof of Proposition 3.5], $C_0=C_{\mathbf{H}}(s)^{\circ F}=H_-\times H_+$ and $[C_H(s):C_0]=2$. The other component has $\det(g_-)=\det(g_+)=-1$. Moreover, $v_p(|H_-|)=v_p((q^A-\epsilon)\prod_{i=1}^{A-1}(q^{2i}-1))$. We consider $e\nmid A$. $p\nmid q^A-\epsilon$ and $v_p(|H_-|)=aA_0+v_p(A_0!)$. Alternatively, we fix $A=eA_0$, with $A_0\geq 1$. Moreover, $v_p(\prod_{i=1}^{A-1}(q^{2i}-1))=a(A_0-1)+v_p((A_0-1)!)$ and $q^A\equiv\tau^{A_0}\pmod p$. Thus, $v_p(q^A-\epsilon)=a+v_p(A_0)$ when $\epsilon=\tau^{A_0}$, and $v_p(q^A-\epsilon)=0$ otherwise. Thus, $v_p(|H_-|)=am+v_p(m!)$ in every case A,e,ϵ. We include $A=1$, with $|H_-|=q-\epsilon$. Moreover, $v_p(|H_+|)=aB_0+v_p(B_0!)$. Put $\delta=v_p([H:C_0])$. Moreover, $\delta=a(w-m-B_0)+v_p(w!/(m!B_0!))$. Since $A+B=n$, $m+B_0\leq w$. Moreover, $w!/(m!B_0!)\in\mathbb{Z}$ and $a\geq 1$. Thus, $\delta=0$ exactly under $m+B_0=w$ and $p\nmid\binom{w}{m}$.

Fix semisimple $\widetilde{s}\in\widetilde{H}$, with $\pi(\widetilde{s})=s$, and we put $\widetilde{\mathbf{C}}=C_{\widetilde{\mathbf{H}}}(\widetilde{s})$ and $\mathbf{C}=C_{\mathbf{H}}(s)^{\circ}$. By [175, Proposition 3.17(b)], $\pi(\widetilde{\mathbf{C}}^F)=C_0$. Since $\ker(\pi|_{\widetilde{\mathbf{C}}})=Z$, this gives $[\widetilde{H}:C_{\widetilde{H}}(\widetilde{s})]=[H:C_0]$. Moreover, $[H:C_0]=2[H:C_H(s)]$. We consider $\rho\in\mathcal{E}(\widetilde{X},\widetilde{s})$ and $\chi\in\mathrm{Irr}(X)$, with $\langle\rho|_X,\chi\rangle_X>0$, and we write the Jordan pair $(\widetilde{s},\varphi)$ for ρ. Then, by Corollary 5.5, $\rho(1)=t\chi(1)$, with $t\mid 2$. Moreover, $\rho(1)=[H:C_0]_{\ell'}\varphi(1)$ and $p\neq 2,\ell$. Thus, $v_p(\chi(1))=\delta+v_p(\varphi(1))$. $p\nmid\chi(1)$ exactly when $\delta=0$ and $p\nmid\varphi(1)$. Put $\varphi=1_{C_{\widetilde{H}}(\widetilde{s})}$. Thus, $\mathcal{E}(X,s)_{p'}\neq\varnothing$ exactly when $\delta=0$. □

We fix odd q, cuspidal $\rho\in\mathrm{Irr}(\widetilde{X})$ and $\widetilde{X}=\mathrm{GSp}(V)$. We choose semisimple $\widetilde{s}$, with $\rho\in\mathcal{E}(\widetilde{X},\widetilde{s})$ and quasi-isolated $s=\pi(\widetilde{s})\in H$. Moreover, we write the class $\mathcal{O}_\rho$ and partition λ_ρ from Taylor [187, Proposition 14.4 and Theorem 14.6]. We choose representatives $\mathcal{R}_\rho$ for $\widetilde{X}\backslash\mathcal{O}_\rho^F$ and generalized Gelfand–Graev characters Γ_u, with $u\in\mathcal{R}_\rho$. $\sum_{u\in\mathcal{R}_\rho}\langle\Gamma_u,\rho\rangle_{\widetilde{X}}=1$ and $j\in 2\mathbb{Z}_{>0}$ for every part j of λ_ρ. Thus, we put the unique $\Gamma_\rho=\Gamma_u$, with $\langle\Gamma_\rho,\rho\rangle_{\widetilde{X}}=1$. We write the weights $1-j,3-j,\dots,j-1$ on each Jordan string L_j. Moreover, $V=\bigoplus_{k\text{ odd}}V[k]$. Since $\ell\neq 2$, we write $\mathfrak{g}=\mathfrak{sp}(V)\simeq\mathrm{Sym}^2(V)$. $\mathfrak{g}[k]=0$ for odd k. We put $U=U_{\geq 1}=U_{\geq 2}$ and $q_2:U\to\mathfrak{g}[2]$, with $\ker q_2=U_{\geq 3}$. Then, by Taylor [188, §5, especially Lemma 5.15], we put $\eta=\omega\circ\lambda_2\circ q_2\in\mathrm{Irr}(U)$, with $\eta(1)=1$. We choose opposite Jordan elements $N_\pm\in\mathfrak{g}[\pm 2]$ and $\lambda_2(x)=\mathrm{tr}(N_-x)$. Moreover, $q_2(uu')=q_2(u)+q_2(u')$ and $\eta(uu')=\eta(u)\eta(u')$. Thus, $\Gamma_\rho=\mathrm{Ind}_U^{\widetilde{X}}\eta$, without a subspace choice in $\mathfrak{g}[1]=0$. We choose a cyclotomic splitting field E, with $\mu_{2f\ell(q-1)}\subseteq E$, and $\mathcal{V}=\{F:\widetilde{X}\to E:F(ux)=\eta(u)F(x)\text{ for }u\in U\}$. Given $g,x\in\widetilde{X}$, we put $[A(g)F](x)=F(xg)$. Since $\langle\Gamma_\rho,\rho\rangle=1$, we choose the irreducible summand $V_\rho\leq\mathcal{V}$. Moreover, $\mathrm{tr}(A(g)|_{V_\rho})=\rho(g)$. We put $W_\eta=\{v\in V_\rho:A(u)v=\eta(u)v\text{ for }u\in U\}$. By Isaacs [95, Lemma 5.2] for $\mathrm{Ind}_U^{\widetilde{X}}\eta$, $\dim_E W_\eta=1$.

We fix $t\in\mathbb{F}_\ell^\times$ and $D(t)|_{V[k]}=t^{(k+1)/2}I$. Moreover, $(k+1)/2\in\mathbb{Z}$ for odd k. Since $B(V[k],V[k'])=0$ unless $k+k'=0$, $B(D(t)x,D(t)y)=tB(x,y)$. Thus, $\mu(D(t))=t$ and $D(t)D(t')=D(tt')$. Moreover, $t^{(k+3)/2}/t^{(k+1)/2}=t$ on arrows $V[k]\to V[k+2]$. Thus, we find $q_2(D(t)uD(t)^{-1})=tq_2(u)$. Thus, we obtain $\eta(D(t)uD(t)^{-1})=\eta(u)^t$. We interpret $\eta(u)^t=\eta(u)^{n_t}$, with $1\leq n_t<\ell$ and $\overline{n_t}=t\in\mathbb{F}_\ell$. Moreover, we write $V=\perp_{j\text{ even}}(L_j\otimes W_j)$ and $B|_{L_j\otimes W_j}=J_j\otimes\beta_j$. We choose alternating J_j over $\mathbb{F}_\ell$ and symmetric β_j on W_j. Moreover, we choose $b\in\mathbb{F}_q^\times$, with $\varepsilon_q(b)=-1$. By the top and bottom vectors $e_r,N_+^{j-1}e_s$, we write the Gram matrix $M_j=(B(e_r,N_+^{j-1}e_s))_{r,s}$ of β_j. $M_j\mapsto g^{\mathsf{T}}M_jg$ under a basis matrix $g\in\mathrm{GL}(W_j)$. Thus, $g^{\mathsf{T}}M_jg=\mathrm{diag}(1,\dots,1,b_j)$ for some $g\in\mathrm{GL}(W_j)$, with $b_j\in\{1,b\}$. Moreover, we choose the maps $N_\pm|_{L_j}$ and grading $L_j=\bigoplus_k L_j[k]$ over $\mathbb{F}_\ell$. We consider odd f and choose $b\in\mathbb{F}_\ell^\times$, with $\varepsilon_\ell(b)=-1$. $\varepsilon_q(b)=\varepsilon_\ell(b)^f=-1$. Thus, we write coordinate Frobenius $v(x)=x^\ell$ and $v^f=1$. Alternatively, we consider even f and $b\in\mathbb{F}_q^\times$. We put $v(x)=x^\ell$ on coordinates of coefficient 1, and $v(x)=b^{(\ell-1)/2}x^\ell$ on coordinates of coefficient b. Since $b(b^{(\ell-1)/2})^2=b^\ell$, $B(vx,vy)=B(x,y)^\ell$. Moreover, $v(V[k])=V[k]$ and $vN_-=N_-v$. Thus, we obtain $\lambda_2(q_2(vuv^{-1}))=\lambda_2(q_2(u))^\ell$. By $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_\ell}(z^\ell)=\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_\ell}(z)$, $vUv^{-1}=U$ and $\eta(vuv^{-1})=\eta(u)$. Moreover, $vD(t)=D(t)v$ for $t\in\mathbb{F}_\ell^\times$. We put $c=v^f$. Since $\prod_{i=0}^{f-1}(b^{(\ell-1)/2})^{\ell^i}=b^{(q-1)/2}=-1$, $c=-I$ on coordinates of coefficient b, for even f. Moreover, $c=I$ on coordinates of coefficient 1, and $c=I$ throughout when f is odd. Thus, $c\in X$, $c^2=I$ and $v^f(g)=cgc^{-1}$ on $\widetilde{X}$. Moreover, $v(c)=c$ and $cD(t)=D(t)c$. We write $C=\mathbb{F}_q^\times$, $F_\ell:m\mapsto m^\ell$ and $\delta^{\ell^{-h}}\sigma(m)=\sigma(\delta(F_\ell^{-h}(m)))$. We fix triples $a=(h_a,\sigma_a,\delta_a)\in\mathbb{Z}\times\mathcal{H}_p\times\mathrm{Irr}(C)$. Moreover, we put $ab=(h_a+h_b,\sigma_a\sigma_b,\delta_a\delta_b^{\ell^{-h_a}\sigma_a})$.

Proposition 5.14. *We choose $\zeta\in\mu_\ell$, with $\mathrm{ord}(\zeta)=\ell$, and $\sigma(\zeta)=\zeta^{t_\sigma}$, with $t_\sigma\in\mathbb{F}_\ell^\times$. We fix $a=(h_a,\sigma_a,\delta_a)$, with $h_a\in\mathbb{Z}$ and $\delta_a\in\mathrm{Irr}(\mathbb{F}_q^\times)$. Moreover, we write the stabilizer $\Gamma_\rho=\{a:(\delta_a\circ\mu)\rho^{v^{-h_a}\sigma_a}=\rho\}$. We write $\kappa_\rho\in\{1,-1\}$, with $A(c)|_{W_\eta}=\kappa_\rho I$, and $\gamma^f=\kappa_\rho^{-1}$, and we choose σ_a-semilinear operators T_a on V_ρ. Moreover, $T_aT_b=\alpha(a,b)T_{ab}$, with $\alpha(a,b)=\sigma_a(\delta_b)(t_{\sigma_a})^{-1}(\sigma_a(\gamma)/\gamma)^{h_b}$. Moreover, $T_{(f,1,1)}=A(c)|_{V_\rho}$. Moreover, we consider odd f or $v_2(f)<v_2(p-1)$. We choose γ with $\sigma(\gamma)=\gamma$ for $\sigma\in\mathcal{H}_p$. Moreover, $v_2(f)<v_2(p-1)$ for even d.*

Proof. First, we put $[E_\delta F](x)=\delta(\mu(x))F(x)$ and $[F_vF](x)=F(v^{-1}(x))$ on $\mathcal{V}$. Moreover, put $[S_\sigma F](x)=$

$\sigma(F(D(t_\sigma)^{-1}x))$ and $[L_cF](x) = F(c^{-1}x)$. $\mu(u) = 1$ for $u \in U$. By $\eta(D(t)uD(t)^{-1}) = \eta(u)^t$, $\sigma(\eta(D(t_\sigma)^{-1}uD(t_\sigma))) = \eta(u)$. Thus, $E_\delta, F_v, S_\sigma, L_c : \mathcal{V} \to \mathcal{V}$. Moreover, $S_\sigma S_\tau = S_{\sigma\tau}$, since $D(t_\sigma)D(t_\tau) = D(t_{\sigma\tau})$. Moreover, $S_\sigma F_v = F_v S_\sigma$. Moreover, $S_\sigma E_\delta = \sigma(\delta)(t_\sigma)^{-1}E_{\sigma(\delta)}S_\sigma$ and $F_vE_\delta = E_{\delta^{\ell^{-1}}}F_v$ and $F_v^f = L_cA(c)$. Since $\mu(D(t_\sigma)^{-1}x) = t_\sigma^{-1}\mu(x)$, $\sigma(\delta(t_\sigma^{-1}\mu(x))) = \sigma(\delta)(t_\sigma)^{-1}\sigma(\delta)(\mu(x))$. Moreover, $[F_v^fF](x) = F(c^{-1}xc) = [L_cA(c)F](x)$. $L_cA(g) = A(g)L_c$ and $L_c^2 = I$. We choose $0 \neq m \in V_\rho^*$, with $mA(u) = \eta(u)m$ for $u \in U$, and we put $[\Phi_m(v)](g) = m(A(g)v)$. By the definition of Φ_m, $L_c\Phi_m = \Phi_{mA(c)^{-1}} = \kappa_\rho^{-1}\Phi_m$. Thus, $L_c|_{V_\rho} = \kappa_\rho I$ and $\kappa_\rho = |U|^{-1}\sum_{u\in U}\eta(u)^{-1}\rho(cu)$. We write the projector $e_\eta = |U|^{-1}\sum_{u\in U}\eta(u)^{-1}A(u)$ on V_ρ. Since e_η projects onto W_η, $\mathrm{tr}(A(c)e_\eta) = \kappa_\rho = \kappa_\rho^{-1}$. Moreover, $\mu(c) = 1$, $v(c) = c$ and $D(t)c = cD(t)$. Thus, $L_cE_\delta = E_\delta L_c$, $L_cF_v = F_vL_c$ and $L_cS_\sigma = S_\sigma L_c$. Write the orbit $\mathcal{O} = \{(\delta \circ \mu)\rho^{v^{-h}\sigma}\}$ and $\mathcal{M} = \bigoplus_{\rho'\in\mathcal{O}} V_{\rho'} \leq \mathcal{V}$. Moreover, $L_c|_{V_{\rho'}} = \kappa_\rho I$ for $\rho' \in \mathcal{O}$. On $\mathcal{M}$, put $T_a = E_{\delta_a}(\gamma F_v)^{h_a}S_{\sigma_a}$. $A(g)E_\delta = \delta(\mu(g))E_\delta A(g)$ and $A(g)F_v = F_vA(v^{-1}(g))$. Thus, $T_a(V_\rho) = V_{(\delta_a\circ\mu)\rho^{v-h_a\sigma_a}}$. Moreover, $S_{\sigma_a}\gamma^{h_b} = \sigma_a(\gamma)^{h_b}S_{\sigma_a}$. By $S_\sigma E_\delta, F_vE_\delta$, $T_aT_b = \sigma_a(\delta_b)(t_{\sigma_a})^{-1}(\sigma_a(\gamma)/\gamma)^{h_b}T_{ab}$. $T_{(f,1,1)} = \gamma^f L_cA(c) = A(c)$ on $\mathcal{M}$. We consider odd f, with $c = I$. Thus, we choose $\gamma = 1$. We consider $\kappa_\rho = -1$ and $\mu_{p-1} = \langle z\rangle$. Moreover, $(z^r)^f = -1$ exactly when $fr \equiv (p-1)/2 \pmod{p-1}$. Thus, we choose $\gamma \in \mu_{p-1}$ exactly when $(f, p-1) \mid (p-1)/2$. $(f, p-1) \mid (p-1)/2$ exactly when $v_2(f) < v_2(p-1)$. With $\gamma = 1$, consider $\kappa_\rho = 1$. Since $p \equiv 1 \pmod{p-1}$, $\sigma(\gamma) = \gamma$ for $\gamma \in \mu_{p-1}$ and $\sigma \in \mathcal{H}_p$. We consider even d and $m_0 = \mathrm{ord}_p(\ell)$. Moreover, $d = m_0/(m_0, f)$. Thus, $v_2(f) < v_2(m_0) \leq v_2(p-1)$. □

Corollary 5.15. *We put the quadratic twist $x = (0, 1, \varepsilon_q)$. Moreover, $\alpha(a, x)/\alpha(x, a) = (q^*/p)^{k_a}$ for $a, x \in \Gamma_\rho$. Moreover, we consider odd d and $E_\varepsilon = E_{\varepsilon_q}$. We choose $\xi^2 = \varepsilon_q(-1)$ and $J_\rho = \xi E_\varepsilon$ on the orbit $\mathcal{M}$. Moreover, $J_\rho^2 = \varepsilon_q(-1)I$ and $T_aJ_\rho = J_\rho T_a$.*

Proof. By $\alpha(a, b)$, $\alpha(a, x) = \varepsilon_q(t_{\sigma_a})^{-1}$ and $\alpha(x, a) = 1$. Moreover, $\varepsilon_q(t_{\sigma_a})^{-1} = \varepsilon_q(t_{\sigma_a})$. By Lemma 5.9, $\varepsilon_q(t_{\sigma_a}) = (q^*/p)^{k_a}$. We write a Levi coordinate (a, g_0), with $a \in \mathbb{F}_q^\times$ and the second line $\mu(g_0)/a$. Moreover, $\varepsilon_q(\mu(g_0)/a)\rho(g_0) = \varepsilon_q(a)((\varepsilon_q\circ\mu)\rho)(g_0)$. Thus, we put $E_\varepsilon = E_{\varepsilon_q}$ on $V_\rho \oplus V_{(\varepsilon_q\circ\mu)\rho}$ when $(\varepsilon_q\circ\mu)\rho \neq \rho$. We choose E_ε on V_ρ when $(\varepsilon_q\circ\mu)\rho = \rho$. Moreover, $E_\varepsilon^2 = I$ and $(\xi E_\varepsilon)^2 = \varepsilon_q(-1)I$, with $\xi \in \{1, i\}$. By $S_\sigma E_\delta, F_vE_\delta$, $T_aET_a^{-1} = \varepsilon_q(t_{\sigma_a})^{-1}E_\varepsilon$. We consider odd d. Moreover, $\sigma_a(\xi)/\xi = \varepsilon_q(t_{\sigma_a})$ by Lemma 5.9. Thus, $T_a(\xi E_\varepsilon)T_a^{-1} = \xi E_\varepsilon$, with $(\xi E_\varepsilon)^2 = \varepsilon_q(-1)I$ as in $J^2 = t_{-1}$. □

Lemma 5.16. *We fix odd d and a classical symbol $\Lambda = (A, B)$ and fix a d-hook $(i, i-d)$ in either row A, B and the new symbol Λ'. Thus, $\Delta(\Lambda') = \Delta(\Lambda)$ and $C(\Lambda') = C(\Lambda)$, for the ordinary core $C(\Lambda)$. Moreover, $\mathrm{rk}\, C(\Lambda) = (\Delta^2 - 1)/4$ in types B, C, and $\mathrm{rk}\, C(\Lambda) = \Delta^2/4$ in type D.*

Proof. By [48, §10.1], $\Delta(\Lambda') = \Delta(\Lambda)$. The packed rows of $C(\Lambda)$ are $\varnothing$ and $\{0, \ldots, |\Delta| - 1\}$ after a common shift of $C(\Lambda)$, in the order given by Δ. Thus, $C(\Lambda') = C(\Lambda)$. By [48, §10.1], the rank is $\lfloor \Delta^2/4 \rfloor$. For H_+ and H_- in Proposition 5.13, the ranks are $t(t+1)$ and u^2. The degenerate labels $(\Lambda, +), (\Lambda, -)$ for $\Lambda = (A, A)$ both give rank 0. By [156, Lemma 8.2], the odd-d unipotent series in classical types B, C, D, outside ${}^3D_4$, has a common 1-cuspidal source. □

Lemma 5.17. *Moreover, we fix $\sigma : \Gamma \to \mathrm{Aut}(E)$ and isotypic spaces $\mathcal{M} = V \otimes_E L, \mathcal{M}' = V' \otimes_E L'$. We choose σ_a-semilinear operators T_a, T_a' from the common cuspidal source $\rho_0 \in \mathrm{Irr}(K_0)$. Moreover, we suppose that $T_aT_b = \alpha_0(a, b)T_{ab}$ and $T_a'T_b' = \alpha_0(a, b)T_{ab}'$, for $a, b \in \Gamma$. We choose σ_a-semilinear $B_a : L \to L$ and $B_a' : L' \to L'$. Moreover, $T_a(1 \otimes z)T_a^{-1} = 1 \otimes B_azB_a^{-1}$ for $z \in \mathrm{End}_E(L)$. Moreover, $T_a'(1 \otimes z')(T_a')^{-1} = 1 \otimes B_a'z'(B_a')^{-1}$ for $z' \in \mathrm{End}_E(L')$. We write the character operators A_a, A_a', with $T_a = A_a \otimes B_a$ and $T_a' = A_a' \otimes B_a'$. We write the factors $\alpha_V, \alpha_{V'}, \alpha_L, \alpha_{L'} : \Gamma^2 \to E^\times$ on V, V', L, L'. Thus, we find $\alpha_V(a, b)/\alpha_{V'}(a, b) = \alpha_{L'}(a, b)/\alpha_L(a, b)$.*

Proof. We choose bases $(v_i), (l_j)$ of V, L and put $C_a^V(\sum_i c_iv_i) = \sum_i \sigma_a(c_i)v_i$ and $C_a^L(\sum_j d_jl_j) = \sum_j \sigma_a(d_j)l_j$. Moreover, we write $C_a = C_a^V \otimes C_a^L$ on $V \otimes_E L$. With $R_a = T_aC_a^{-1}$, we put $Q_a = B_a(C_a^L)^{-1}$. R_a, Q_a are E-linear. By the action on $1 \otimes \mathrm{End}_E(L)$, $R_a(1 \otimes z)R_a^{-1} = 1 \otimes Q_azQ_a^{-1}$. Thus, $R_a(1 \otimes Q_a^{-1}) \in \mathrm{End}_{\mathrm{End}(L)}(V \otimes L)$. Moreover, $\mathrm{End}_{\mathrm{End}(L)}(V \otimes L) = \mathrm{End}(V) \otimes 1$, by the matrix units $e_{ij} \in \mathrm{End}(L)$. Thus, we write $R_a = P_a \otimes Q_a$ and $A_a = P_aC_a^V$. $T_a = A_a \otimes B_a$, with A_a, B_a both σ_a-semilinear. Moreover, $A_a(cv) \otimes B_a(l) = A_a(v) \otimes B_a(cl)$ for $c \in E$. Thus, write the tensor map T_a over E. We write $A_aA_b = \alpha_V(a, b)A_{ab}$ and $B_aB_b = \alpha_L(a, b)B_{ab}$. Moreover, $\alpha_V(a, b)\alpha_L(a, b) = \alpha_0(a, b)$ on pure tensors $v \otimes l$ and that $\alpha_{V'}(a, b)\alpha_{L'}(a, b) = \alpha_0(a, b)$ on $v' \otimes l'$. Put $c_\gamma(a, b) = (\sigma_a(\gamma)/\gamma)^{h_b}$ from $\alpha(a, b)$. We write $\alpha_0(a, b) = \beta_0(a, b)c_\gamma(a, b)$. Thus, $\alpha_V/\alpha_{V'} = (\beta_0c_\gamma/\alpha_L)/(\beta_0c_\gamma/\alpha_{L'}) = \alpha_{L'}/\alpha_L$. □

We consider rank 0 and the Levi factor $C = \mathbb{F}_q^\times$. We put $U = 1$ and $\mathcal{V}_0 = \{F : C \to E\}$. Moreover, we put $[A_0(m)F](x) = F(xm)$ for $m, x \in C$. $\mathcal{V}_0 = \bigoplus_{\nu\in\mathrm{Irr}(C)} EF_\nu$, with $F_\nu(x) = \nu(x)$. We put $D(t) = t \in C$ and $v(x) = x^\ell$

on C. Then, by E_δ, F_v, S_σ, we put $[E_\delta F](x) = \delta(x)F(x)$ and $[F_v F](x) = F(v^{-1}(x))$. We put $[S_\sigma F](x) = \sigma(F(t_\sigma^{-1}x))$. Moreover, $E_{\varepsilon_q}F_\nu = F_{\varepsilon_q\nu}$. Thus, we include $E_{\varepsilon_q}F_1 = F_{\varepsilon_q}$ and $E_{\varepsilon_q}F_{\varepsilon_q} = F_1$. By Corollary 5.15, we put $J_0 = \xi E_{\varepsilon_q}$, with $J_0^2 = \varepsilon_q(-1)I$.

Suppose that q is odd. Let $\widetilde{B} = \widetilde{T}U_B$ be a split Borel of $\widetilde{X}$. We write $\widetilde{T} \simeq (\mathbb{F}_q^\times)^{n+1}$, with coordinates $(a_1, \ldots, a_n; m)$. Moreover, we write the two entries $a_i, m/a_i$ on plane i, with $\mu(t) = m$. We fix $\lambda \in \mathrm{Irr}(\widetilde{T})$ and $\lambda(tu) = \lambda(t)$ for $t \in \widetilde{T}, u \in U_B$. We put $I(\lambda) = \{F : \widetilde{X} \to E : F(bx) = \lambda(b)F(x) \text{ for } b \in \widetilde{B}\}$. Thus, we put $[A_\lambda(g)F](x) = F(xg)$ on $I(\lambda)$. We put $N = N_{\widetilde{X}}(\widetilde{T})$ and $W = N/\widetilde{T} \simeq C_2^n \rtimes \mathfrak{S}_n$. We write $\Phi = \{\pm e_i \pm e_j : i < j\} \cup \{\pm 2e_i\}$ and positive roots Φ^+. Moreover, we choose a simple root α, its coroot $\alpha^\vee : \mathbb{F}_q^\times \to \widetilde{T}$ and reflection $s_\alpha \in W$. We choose Chevalley maps $\varphi_\alpha : \mathrm{SL}_2(\mathbb{F}_q) \to \widetilde{X}$ from a fixed pinning over $\mathbb{F}_\ell$. We put $n_\alpha = \varphi_\alpha(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ and $u_\alpha(t) = \varphi_\alpha(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right))$. Thus, $n_\alpha^2 = \alpha^\vee(-1)$ and $\mu(n_\alpha) = \mu(u_\alpha(t)) = 1$. We put $[I_\alpha^\lambda F](x) = \sum_{t\in\mathbb{F}_q} F(n_\alpha u_\alpha(t)x)$. We put $U^\alpha = \langle U_\beta : \beta \in \Phi^+ \setminus \{\alpha\}\rangle$. Since $s_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$, $U_B = U^\alpha U_\alpha$ and $n_\alpha U^\alpha n_\alpha^{-1} = U^\alpha$. Moreover, $F(n_\alpha u_\alpha(t)zx) = F(n_\alpha u_\alpha(t)x)$ for $z \in U^\alpha$. $u_\alpha(t)u_\alpha(s) = u_\alpha(t+s)$ and $(I_\alpha^\lambda F)(zx) = (I_\alpha^\lambda F)(x)$ for $z \in U_B$. Moreover, $(I_\alpha^\lambda F)(yx) = (s_\alpha\lambda)(y)(I_\alpha^\lambda F)(x)$ for $y \in \widetilde{T}$. Thus, $I_\alpha^\lambda : I(\lambda) \to I(s_\alpha\lambda)$. Moreover, $I_\alpha^\lambda A_\lambda(g) = A_{s_\alpha\lambda}(g)I_\alpha^\lambda$.

Lemma 5.18. *Moreover, we put $\theta = \lambda \circ \alpha^\vee$. We consider $\theta \neq 1$, with $I_\alpha^{s_\alpha\lambda}I_\alpha^\lambda = q\lambda(n_\alpha^2)I$. Alternatively, we consider $\theta = 1$, with $s_\alpha\lambda = \lambda$ and $I_\alpha^{s_\alpha\lambda}I_\alpha^\lambda = qI + (q-1)I_\alpha^\lambda$. We put $J_\alpha^\lambda = q^{-1/2}I_\alpha^\lambda$. Moreover, $J_\alpha^{s_\alpha\lambda}J_\alpha^\lambda = \lambda(n_\alpha^2)I$ for $\theta \neq 1$. Moreover, we consider odd d. Moreover, $\sigma(\sqrt{q}) = \sqrt{q}$ for $\sigma \in \mathcal{H}_p$.*

Proof. On the rank-one subgroup $\varphi_\alpha(\mathrm{SL}_2(q))$, I_α^λ is q times the double-coset average at n_α. Thus, the Yokonuma relation in [191, Theorem 6.1] gives $I_\alpha^{s_\alpha\lambda}I_\alpha^\lambda = q\lambda(n_\alpha^2)I + S_\theta I_\alpha^\lambda$, where $S_\theta = \sum_{t\in\mathbb{F}_q^\times}\theta(t^{-1})$. Thus, $S_\theta = 0$ for $\theta \neq 1$, and $S_\theta = q-1$ for $\theta = 1$. $(s_\alpha\lambda)(y) = \lambda(y)\theta(\alpha(y))^{-1}$ for $y \in \widetilde{T}$. Thus, write $s_\alpha\lambda = \lambda$ and $\lambda(n_\alpha^2) = \theta(-1) = 1$ when $\theta = 1$. Consider $\theta \neq 1$. Thus, $J_\alpha^{s_\alpha\lambda}J_\alpha^\lambda = \lambda(n_\alpha^2)I$. Moreover, we fix odd d and $\xi^2 = \varepsilon_q(-1)$. By Lemma 5.9, $(G_q/\xi)^2 = q$ and $\sigma(G_q/\xi) = G_q/\xi$. Choose $\sqrt{q} \in \{G_q/\xi, -G_q/\xi\}$. Thus, $\sigma(\sqrt{q}) = \sqrt{q}$ for $\sigma \in \mathcal{H}_p$. □

Proposition 5.19. *We write $\lambda(a_1, \ldots, a_n; m) = \nu(m)\prod_i \theta_i(a_i)$, with $\nu, \theta_i \in \mathrm{Irr}(\mathbb{F}_q^\times)$, and we choose λ in its orbit $W\lambda$ with $\theta_i \in \{1, \varepsilon_q\} \cup \Theta$. Moreover, we put $\Theta = \{\theta_i : \theta_i^2 \neq 1\}$, with $\Theta \cap \Theta^{-1} = \varnothing$. With $r_0 = |\{i : \theta_i = 1\}|$, we put $r_\varepsilon = |\{i : \theta_i = \varepsilon_q\}|$ and $r_\theta = |\{i : \theta_i = \theta\}|$. Moreover, $W_\lambda = W(B_{r_0}) \times W(D_{r_\varepsilon}) \times \prod_{\theta\in\Theta}\mathfrak{S}_{r_\theta}$. We put $\mathcal{R} = \{\alpha \in \Phi : \lambda \circ \alpha^\vee = 1\}$. Thus, we find $W_\lambda = \langle s_\alpha : \alpha \in \mathcal{R}\rangle = W(\mathcal{R})$. We include $W(B_0) = W(D_0) = W(D_1) = \mathfrak{S}_0 = 1$.*

Proof. First, we write $w_i \in W$, with $w_i(a_i) = m/a_i$ and $w_i(m) = m$. Moreover, $(w_i\lambda)(a_1, \ldots, a_n; m) = (\nu\theta_i)(m)\theta_i(a_i)^{-1}\prod_{j\neq i}\theta_j(a_j)$. By [34, Proposition 5.5(b)–(c)] at $d = 1$, the trivial labels give $W(B_{r_0})$, and $\theta \in \Theta$ gives $\mathfrak{S}_{r_\theta}$. At ε_q, the sign kernel on $C_2^{r_\varepsilon} \rtimes \mathfrak{S}_{r_\varepsilon}$ is $W(D_{r_\varepsilon})$. This gives the product formula for W_λ. Moreover, $\lambda\circ(e_i - e_j)^\vee = \theta_i\theta_j^{-1}$, $\lambda\circ(e_i + e_j)^\vee = \theta_i\theta_j$ and $\lambda\circ(2e_i)^\vee = \theta_i$. Thus, we consider $\theta_i = \theta_j = 1$, with roots $\pm e_i \pm e_j, \pm 2e_i$ of C_{r_0}. We consider $\theta_i = \theta_j = \varepsilon_q$, with roots $\pm e_i \pm e_j$ of D_{r_ε}. Moreover, we consider $\theta_i = \theta_j = \theta \in \Theta$, with roots $e_i - e_j$ of $A_{r_\theta - 1}$. $W(C_{r_0}) = W(B_{r_0})$ and $W(A_{r_\theta - 1}) = \mathfrak{S}_{r_\theta}$. Thus, $W(\mathcal{R}) = W_\lambda$. □

We put $\mathcal{R}^+ = \mathcal{R} \cap \Phi^+$. We write the ordinary field map $\phi(g) = (g_{ij}^\ell)_{ij}$, with $\phi^f = 1$. Moreover, we write $(a\lambda)(t) = \delta_a(\mu(t))\sigma_a(\lambda(\phi^{-h_a}(t)))$ for $a = (h_a, \sigma_a, \delta_a)$. $\delta_a(\mu(\alpha^\vee(x))) = 1$ for $x \in \mathbb{F}_q^\times$. $(a\lambda)\circ\alpha^\vee = 1$ exactly when $\lambda\circ\alpha^\vee = 1$. Thus, we find $\mathcal{R}_{a\lambda} = \mathcal{R}$. We write $\Gamma_\lambda = \{a : a\lambda \in W\lambda\}$ and $w_a \in W$, with $w_a(a\lambda) = \lambda$. Moreover, $w_a\mathcal{R} = \mathcal{R}$ and $\{w \in W : w(a\lambda) = \lambda\} = W_\lambda w_a$. By the Weyl chamber action of $W(\mathcal{R})$, we have that $|\{v \in W(\mathcal{R}) : v(w_a\mathcal{R}^+) = \mathcal{R}^+\}| = 1$. By $W_\lambda = W(\mathcal{R})$, we choose the unique $c(a) \in W_\lambda w_a$ with $c(a)\mathcal{R}^+ = \mathcal{R}^+$. Moreover, $c(a)(a\lambda) = \lambda$. $w(a\lambda) = a(w\lambda)$ for $w \in W$. Thus, we obtain $c(a)c(b)((ab)\lambda) = c(a)a(c(b)(b\lambda)) = \lambda$. Moreover, $c(a)c(b)\mathcal{R}^+ = \mathcal{R}^+$. Thus, we obtain $c(ab) = c(a)c(b)$.

Lemma 5.20. *We choose $c = s_{\alpha_1}\cdots s_{\alpha_r}$, with $r = \mathrm{length}(c)$ and $c\mathcal{R}^+ = \mathcal{R}^+$, and we put $\lambda_i = s_{\alpha_{i+1}}\cdots s_{\alpha_r}\lambda$ and $R_c^\lambda = J_{\alpha_1}^{\lambda_1}\cdots J_{\alpha_r}^{\lambda_r}$. Moreover, we put $u_c = n_{\alpha_r}\cdots n_{\alpha_1}$ and $[K_c^\lambda F](x) = F(u_c x)$ on $I_N(\lambda) = \{F : N \to E : F(tx) = \lambda(t)F(x) \text{ for } t \in \widetilde{T}\}$. Moreover, $I_N(\lambda) \simeq \mathrm{Ind}_{\widetilde{T}}^N\lambda$. Thus, we write $R_c^\lambda : I(\lambda) \to I(c\lambda)$ and $K_c^\lambda : I_N(\lambda) \to I_N(c\lambda)$. We include $r = 0$, with $u_1 = 1, R_1^\lambda = K_1^\lambda = I$. Moreover, we consider $c_1, c_2 \in W$, with $c_i\mathcal{R}^+ = \mathcal{R}^+$. With $t = u_{c_1c_2}^{-1}u_{c_2}u_{c_1} \in \widetilde{T}$, we put $\beta = (c_1c_2\lambda)(t)$. Thus, $R_{c_1}^{c_2\lambda}R_{c_2}^\lambda = \beta R_{c_1c_2}^\lambda$ and $K_{c_1}^{c_2\lambda}K_{c_2}^\lambda = \beta K_{c_1c_2}^\lambda$.*

Proof. By [208, §4.1, Hypothesis 4.5], the lifts n_α satisfy braid relations of length $m_{\alpha\beta}$. By [191, Theorem 6.1], the operators I_α^λ satisfy the same relations on $I(\lambda)$. Moreover, R_c^λ has the factor $q^{-r/2}$ for every reduced word of length r. Thus, each reduced word gives $u_c, R_c^\lambda, K_c^\lambda$. Choose the ambient chamber $\mathcal{C} \subset \mathbb{R}^n$ and $\mathcal{D} = \{x : \alpha(x) > 0 \text{ for } \alpha \in \mathcal{R}^+\}$. Moreover, $\mathcal{C}, c^{-1}\mathcal{C} \subseteq \mathcal{D}$. Since $\mathcal{D}$ is convex, we consider reduced galleries $\mathcal{G}$ between $\mathcal{C}, c^{-1}\mathcal{C}$ in $\mathcal{D}$. $\mathcal{G}$ crosses

$H_\beta = \{x : \beta(x) = 0\}$ only for $\beta \in \Phi \setminus \mathcal{R}$. We put $\beta_i = s_{\alpha_r} \cdots s_{\alpha_{i+1}}(\alpha_i)$. Thus, $\beta_i \notin \mathcal{R}$ and $\lambda_i \circ \alpha_i^\vee \neq 1$. We write the concatenation $\mathcal{G}_2 * (c_2^{-1}\mathcal{G}_1)$ for c_1, c_2, with every chamber in $\mathcal{D}$. We consider braid moves in rank 2 and adjacent pairs $s_\alpha s_\alpha$. Since W has the Coxeter word property, c_1c_2 has a reduced expression from $\mathcal{G}_2 * (c_2^{-1}\mathcal{G}_1)$. We have that a braid move $\mathcal{G}' \leftrightarrow \mathcal{G}''$ crosses the hyperplanes H_β separating its endpoint chambers $\mathcal{C}', \mathcal{C}'' \subset \mathcal{D}$. We write $\lambda' = w\lambda$ at a pair $s_\alpha s_\alpha$. Thus, $w^{-1}\alpha \notin \mathcal{R}$ and $\lambda' \circ \alpha^\vee \neq 1$. By Lemma 5.18, $J_\alpha^{s_\alpha \lambda'} J_\alpha^{\lambda'} = \lambda'(n_\alpha^2) I$ at the character λ'. Moreover, $K_{s_\alpha}^{s_\alpha \lambda'} K_{s_\alpha}^{\lambda'} = \lambda'(n_\alpha^2) I$. Moreover, put $\mathcal{P}$ as the set of pairs $s_{\alpha_P} s_{\alpha_P}$. We write $\nu_P = w_P \lambda$, with $w_P \in W$, at $P \in \mathcal{P}$. Thus, $\beta_R = \beta_N = \beta = \prod_{P \in \mathcal{P}} \nu_P(n_{\alpha_P}^2)$. Moreover, $u_{c_2} u_{c_1} = u_{c_1 c_2} t$ and that $F(u_{c_2} u_{c_1} x) = (c_1 c_2 \lambda)(t) F(u_{c_1 c_2} x)$ for $F \in I_N(\lambda)$. Thus, $\beta = (c_1 c_2 \lambda)(t)$ for any reduced expression of $c_1 c_2$. □

We put $\mathcal{I} = \bigoplus_{\lambda' \in \mathrm{Irr}(\widetilde{T})} I(\lambda')$ and $\mathcal{I}_N = \bigoplus_{\lambda' \in \mathrm{Irr}(\widetilde{T})} I_N(\lambda')$. On $\mathcal{I}, \mathcal{I}_N$, we put $[E_a F](x) = \delta_a(\mu(x)) \sigma_a(F(\phi^{-h_a}(x)))$. Thus, substitution in $E_a E_b F$ gives that $E_a E_b = E_{ab}$, with $ab = (h_a + h_b, \sigma_a \sigma_b, \delta_a \delta_b^{\ell^{-h_a}\sigma_a})$. Moreover, $\phi(n_\alpha) = n_\alpha$ and $\phi(u_\alpha(t)) = u_\alpha(t^\ell)$ and $\mu(n_\alpha) = \mu(u_\alpha(t)) = 1$. By Lemma 5.18, we consider odd d and $\sigma(\sqrt{q}) = \sqrt{q}$ for $\sigma \in \mathcal{H}_p$. Thus, $E_a R_{c(b)}^{b\lambda} = R_{c(b)}^{ab\lambda} E_a$ and $E_a K_{c(b)}^{b\lambda} = K_{c(b)}^{ab\lambda} E_a$. We put $T_a = R_{c(a)}^{a\lambda} E_a$ and $T_{N,a} = K_{c(a)}^{a\lambda} E_a$, with $a \in \Gamma_\lambda$. Then, by Lemma 5.20, $T_a T_b = \alpha_\lambda(a,b) T_{ab}$ and $T_{N,a} T_{N,b} = \alpha_\lambda(a,b) T_{N,ab}$. We put $u_a = u_{c(a)}$ and $t(a,b) = u_{ab}^{-1} u_b u_a \in \widetilde{T}$. Moreover, $\alpha_\lambda(a,b) = \lambda(t(a,b))$. Thus, we include the lifts u_a, u_b, u_{ab} in $\alpha_\lambda(a,b)$. We put $a_f = (f, 1, 1)$, with $a_f \lambda = \lambda$. Moreover, $c(a_f) = 1$, $u_{a_f} = 1$ and $E_{a_f} = I$. Thus, we obtain $T_{a_f} = T_{N,a_f} = I$.

We put $\mathcal{H}_\lambda = \mathrm{End}_{\widetilde{X}}(I(\lambda))$ and $W_\lambda = W(\mathcal{R})$. Moreover, $Z(\mathrm{GSp}_{2n}) \simeq \mathbb{G}_m$ and $Z(\mathrm{GSp}_{2n}) = Z(\mathrm{GSp}_{2n})^\circ$. By Zhang [208, §4.1, (4.3)], we write the generators $H_s \in \mathcal{H}_\lambda$ on the Bruhat cells $\widetilde{B} n_s \widetilde{B}$. $H_s^2 = (q-1)H_s + qI$ for simple reflections s of $W(\mathcal{R})$. We consider an ambient simple root $\alpha \in \mathcal{R}$. Then, by Lemma 5.18, we deduce that $H_{s_\alpha} = I_\alpha^\lambda$ and $H_{s_\alpha}^2 = (q-1)H_{s_\alpha} + qI$. We fix a simple root β of $\mathcal{R}$ and a chamber $\mathcal{C}_\beta$ adjacent to H_β. We choose a minimal gallery $\mathcal{G}$ to $\mathcal{C}_\beta$ in the chamber $\mathcal{D}$ of $\mathcal{R}$. Thus, we choose an invertible link $J_\mathcal{G}$ with $H_{s_\beta} = J_\mathcal{G} H_{s_\alpha} J_\mathcal{G}^{-1}$, for an ambient simple α. Thus, we obtain $H_{s_\beta}^2 = (q-1)H_{s_\beta} + qI$. Moreover, we consider a second generator cH_{s_β} on $\widetilde{B} n_{s_\beta} \widetilde{B}$. $c^2 = 1$ and $c^2(q-1) = c(q-1)$ under the quadratic relation $(cH_{s_\beta})^2 = (q-1)cH_{s_\beta} + qI$. Since $q - 1 \neq 0$, $c = 1$. We consider the quadratic block D_r, with $r \geq 2$, $\alpha = e_{r-1} - e_r$, $\beta = e_{r-1} + e_r$ and $b(e_r) = -e_r$. We choose $J_b : I(b\lambda) \xrightarrow{\sim} I(\lambda)$. Thus, $H_{s_\beta}^\lambda = J_b H_{s_\alpha}^{b\lambda} J_b^{-1}$ for either value $J_b^2 = \varepsilon_q(-1) I$. Thus, we have that $\mathcal{H}_\lambda \simeq \mathcal{H}(B_{r_0}, q) \otimes \mathcal{H}(D_{r_\varepsilon}, q) \otimes \bigotimes_{\theta \in \Theta} \mathcal{H}(A_{r_\theta - 1}, q)$ on W_λ. We consider $c\mathcal{R}^+ = \mathcal{R}^+$ and $\lambda' = c^{-1}\lambda$. Moreover, $R_c^{\lambda'} H_s^{\lambda'} (R_c^{\lambda'})^{-1} = H_{csc^{-1}}^\lambda$. By Lemma 5.20, we choose the gallery $\mathcal{G}$ with its one subsystem wall H_β. Thus, $\mathrm{supp}(H_{csc^{-1}}) \subseteq \widetilde{B} n_{csc^{-1}} \widetilde{B}$ and $H_{csc^{-1}}^2 = (q-1)H_{csc^{-1}} + qI$. Moreover, we consider a block $\theta_i = \theta$, with $\theta^2 \neq 1$ and $r = r_\theta$. We write its generators $T_1, \ldots, T_{r-1}$ in $\mathcal{H}(A_{r-1}, q)$. Moreover, we fix $\Lambda \vdash r$, $N_r = \binom{r}{2}$ and $c(\Lambda) = \sum_{(i,j) \in \Lambda} (j - i)$. We put $w_A(i) = r + 1 - i$ in $\mathfrak{S}_r$ and $U_\Lambda = q^{-(N_r + c(\Lambda))/2} T_{w_A}$. Moreover, we consider c returning to the block $\{i : \theta_i = \theta\}$, with $c = I$ or $c(e_i) = -e_{r+1-i}$ for $1 \leq i \leq r$.

Lemma 5.21. *We choose a simple $\mathcal{H}_\mathbb{Q}(A_{r-1}, q)$-module $M_\Lambda^0(q)$. Moreover, we put $M_\Lambda(q) = \mathbb{Q}(\sqrt{q}) \otimes_\mathbb{Q} M_\Lambda^0(q)$. Moreover, $U_\Lambda \in \mathrm{GL}_{\mathbb{Q}(\sqrt{q})}(M_\Lambda(q))$ and $U_\Lambda^2 = I$. Moreover, $U_\Lambda T_i U_\Lambda^{-1} = T_{r-i}$. We write the coordinate map $i \mapsto r + 1 - i$, with all signs -1, on the block $\theta_i = \theta$. Moreover, $L_{c\lambda} L_\lambda = K_{c\lambda} K_\lambda = \theta(-1)^r I$ for paired global and local links L_λ, K_λ.*

Proof. First, we put $L_1 = I$ and $L_k = q^{-(k-1)} T_{k-1} \cdots T_2 T_1^2 T_2 \cdots T_{k-1}$ for $k \geq 2$. By [90, §3], choose a tableau basis $v_\mathfrak{t}$ for $M_\Lambda^0(q)$. Moreover, $L_k v_\mathfrak{t} = q^{j-i} v_\mathfrak{t}$ when $\mathfrak{t}(i,j) = k$. Put $P_k = T_{k-1} \cdots T_1^2 \cdots T_{k-1}$ and take the longest element $w_k \in \mathfrak{S}_k$. $T_{w_k}^2 = T_{w_{k-1}}^2 P_k$ by the braid relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. Thus, $T_{w_A}^2 = \prod_{k=2}^r P_k = q^{N_r} \prod_{k=1}^r L_k$. Moreover, $\prod_{k=1}^r L_k v_\mathfrak{t} = q^{c(\Lambda)} v_\mathfrak{t}$. Thus, $T_{w_A}^2 = q^{N_r + c(\Lambda)} I$ on $M_\Lambda(q)$. Moreover, $T_{w_A} T_i = T_{r-i} T_{w_A}$. Thus, $U_\Lambda^2 = I$ and $U_\Lambda T_i U_\Lambda^{-1} = T_{r-i}$. Choose the rational Specht module $M_\Lambda(1)$, with $U_\Lambda(1) = w_A$. Thus, $U_\Lambda(1)^2 = I$ and $U_\Lambda(1) s_i U_\Lambda(1)^{-1} = s_{r-i}$. Choose plane lifts J_i, with $J_i^2 = -I$ on plane i, and we put $j_r = (\prod_{i=1}^r J_i) P_{w_A}$, with P_{w_A} the plane permutation w_A. Since $P_{w_A} (\prod_i J_i) P_{w_A}^{-1} = \prod_i J_i$, $j_r^2 = -I_{2r}$. Moreover, $\lambda(j_r^2) = \theta(-1)^r$. We write the separating roots $2e_i, e_i + e_j$, with coroot characters $\theta, \theta^2 \neq 1$. Then, by Lemma 5.20, we write u_c and $j_r = u_c t$, with $t \in \widetilde{T}$. We put $L_\lambda = (c\lambda)(t) R_c^\lambda$ and $[K_\lambda F](x) = F(j_r x)$. Thus, $L_{c\lambda} L_\lambda = K_{c\lambda} K_\lambda = \theta(-1)^r I$. We consider the multiplicity operators $U_\Lambda, U_\Lambda(1)$. Since $U_\Lambda^2 = U_\Lambda(1)^2 = I$, the character links $C_\lambda, C_{c\lambda}$ satisfy $C_{c\lambda} C_\lambda = \theta(-1)^r I$. □

We consider $r \geq 2$, $A = \mathcal{H}_\mathbb{Q}(D_r, q)$ and $B = \mathcal{H}_\mathbb{Q}(B_r, 1, q)$, with $T_0^2 = I$. Moreover, $(T_i - q)(T_i + 1) = 0$ for $1 \leq i < r$. Moreover, we put $S_0 = T_0 T_1 T_0$, $S_1 = T_1$ and $S_i = T_i$ for $2 \leq i < r$. $S_0 S_1 = S_1 S_0$ by $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$. $S_0 S_2 S_0 = S_2 S_0 S_2$ for $r \geq 3$, from $T_0 T_2 = T_2 T_0$ and $T_1 T_2 T_1 = T_2 T_1 T_2$. Thus, we write $A = \langle S_0, S_1, \ldots, S_{r-1} \rangle$ and $\tau(a) = T_0 a T_0$. Moreover, $\tau(S_0) = S_1$, $\tau(S_1) = S_0$ and $\tau(S_i) = S_i$ for $i \geq 2$. Thus, $B = A \rtimes C_2$, with $C_2 = \langle T_0 : T_0^2 = 1 \rangle$. Moreover, we include $r = 1$, with $A = \mathbb{Q}$ and $B = \mathbb{Q}[C_2]$. Moreover, we include $r = 2$, with $S_0 S_1 = S_1 S_0$ and $A \simeq \mathcal{H}_\mathbb{Q}(A_1, q) \otimes \mathcal{H}_\mathbb{Q}(A_1, q)$.

Lemma 5.22. *We fix* $\mathrm{char}\, K = 0$, $A \simeq \prod_i M_{n_i}(K)$ *and* $\tau \in \mathrm{Aut}_K(A)$, *with* $\tau^2 = 1$. *We suppose that* $B_1 = A \rtimes C_2$ *is split over* K, *with* $|C_2| = 2$. *Moreover, we put* $B_s = A \oplus AJ$, *with* $Ja = \tau(a)J, J^2 = s, s \in \{1, -1\}$. *We choose* $\xi^2 = s$ *and* $K_s = K(\xi)$. *First, we consider a simple* A*-module* M, *with* $M^\tau \not\simeq M$. *We write the matrix* $J = \left(\begin{smallmatrix} 0 & sI \\ I & 0 \end{smallmatrix}\right)$ *on* $M \oplus M^\tau$. *Thus,* $M \oplus M^\tau$ *is simple over* B_s. *Alternatively, we consider* $M^\tau \simeq M$. *We write the two extensions* $J = \pm\xi U$ *over* $K(\xi)$, *with* $U \in \mathrm{GL}_K(M)$ *and* $U^2 = I$.

Proof. The algebra B_s has C_2-grading with homogeneous unit J. By [50, Proposition 11.16], if $M^\tau \not\simeq M$, then $B_s \otimes_A M$ is the only simple module over M, with restriction $M \oplus M^\tau$. On $1 \otimes M$ and $J \otimes M$, the matrix of J follows from $J^2 = s$. If $M^\tau \simeq M$, the corresponding factor of A is $M_m(K)$, with $\tau(a) = U_0 a U_0^{-1}$ and $U_0^2 = tI$. Given the generator u in B_1, put $z = U_0^{-1}u$. Thus, the centre of $M_m(K) \rtimes C_2$ is $K[z]/(z^2 - t^{-1})$. Since B_1 is split, $t = b^2$ for $b \in K^\times$, and $U = b^{-1}U_0$ has $U^2 = I$. On the same factor in B_s, the central element $U^{-1}J$ has square s. Thus, over K_s, its two scalar values are $\pm\xi$, giving $J = \pm\xi U$. □

Suppose that $q > 1$ and $r \geq 1$, and put $B = \mathcal{H}_{\mathbb{Q}}(B_r, 1, q)$. We fix a bipartition (Λ^+, Λ^-) of r and its standard bitableaux $\mathfrak{t}$. Then, the rational module M_{Λ^+,Λ^-} from Ariki–Koike [8] has basis $v_{\mathfrak{t}}$. We put $R_{\mathfrak{t}}(k) = q^{j-i}$ or $-q^{j-i}$ when k lies in box (i, j) of Λ^+ or Λ^-. Moreover, $T_0 v_{\mathfrak{t}} = v_{\mathfrak{t}}$ or $-v_{\mathfrak{t}}$ according to the component of 1. Moreover, we put $a = R_{\mathfrak{t}}(i)$, $b = R_{\mathfrak{t}}(i+1)$ and $\mathfrak{s} = s_i\mathfrak{t}$. We put $T_i v_{\mathfrak{t}} = \frac{(q-1)b}{b-a} v_{\mathfrak{t}} + \frac{qb-a}{b-a} v_{\mathfrak{s}}$ when $\mathfrak{s}$ is standard. $b \neq a$ for consecutive entries $i, i+1$ in distinct addable boxes $(u, v), (u', v')$ of one component Λ^+ or Λ^-. Moreover, $b/a < 0$ when $i, i+1$ lie in different components Λ^+, Λ^-. Thus, $b - a \neq 0$ for every standard $\mathfrak{s}$. We put $T_i v_{\mathfrak{t}} = q v_{\mathfrak{t}}$ for one row $i, i+1$, and $T_i v_{\mathfrak{t}} = -v_{\mathfrak{t}}$ for one column $i, i+1$. Thus, $T_i \in M_{\dim M_{\Lambda^+,\Lambda^-}}(\mathbb{Q})$ for $q > 1$. Thus, $B \simeq \prod_{(\Lambda^+,\Lambda^-)\vdash r} \mathrm{End}_{\mathbb{Q}}(M_{\Lambda^+,\Lambda^-})$. We put $e(v_{\mathfrak{t}}) = v_{\mathfrak{t}^\circ}$, with $\mathfrak{t}^\circ = (\mathfrak{t}^-, \mathfrak{t}^+)$. Moreover, $R_{\mathfrak{t}^\circ}(k) = -R_{\mathfrak{t}}(k)$. Thus, $eT_i = T_i e$ for $i > 0$ and $eT_0 = -T_0 e$. We put $\epsilon(T_0) = -T_0$ and $\epsilon(T_i) = T_i$ for $i > 0$. Thus, $M^\epsilon_{\Lambda^+,\Lambda^-} \simeq M_{\Lambda^-,\Lambda^+}$. By Lemma 5.22, we consider $\Lambda^+ \neq \Lambda^-$ and one simple A-module V under both $M_{\Lambda^+,\Lambda^-}, M_{\Lambda^-,\Lambda^+}$. Moreover, $T_0^2 = I$ on either extension of V. Alternatively, we consider $\Lambda^+ = \Lambda^-$, with $e \in \mathrm{End}_{\mathbb{Q}}(M_{\Lambda^+,\Lambda^-})$ and $e^2 = I$. We put $M^+ = \ker(e - I)$ and $M^- = \ker(e + I)$. Since $eT_0 = -T_0 e$, $T_0(M^+) = M^-$ and $T_0(M^-) = M^+$. By Lemma 5.22, M^+, M^- are distinct simple A-modules and $M = M^+ \oplus M^-$. If $q = 1$, the group is $C_2 \wr \mathfrak{S}_r$. By Lemma 5.10, we write $M_{\Lambda^+,\Lambda^-}(1)$ over $\mathbb{Q}$ with the component maps T_0, e above. If $r_0 > 0$, we consider $B_0 = \mathcal{H}_{\mathbb{Q}}(B_{r_0}, q)$, with residues $q^{j-i+1}, -q^{j-i}$. We put $T_0 v_{\mathfrak{t}} = q v_{\mathfrak{t}}$ or $-v_{\mathfrak{t}}$ according to the component of 1. Moreover, $b - a \neq 0$ for the standard pairs $\mathfrak{t}, s_i\mathfrak{t}$ by the content and sign conditions $q > 1$. Thus, we write its matrices T_i over $\mathbb{Q}$. Moreover, we include $r_0 = 0$, with $B_0 = \mathbb{Q}$.

Proposition 5.23. *We consider* $d = 1$ *and a split constituent* $\rho_{\lambda,\eta} \in \mathrm{Irr}(\widetilde{X})$. *We write* $\tau_{\lambda,\eta} \in \mathrm{Irr}(N)$, *with ordinary label* $\eta \in \mathrm{Irr}(W_\lambda)$ *on* W_λ. *Moreover, we choose* $\widehat{\lambda} \in \mathrm{Irr}(N_\lambda)$, *with* $\widehat{\lambda}(n_\alpha) = 1$ *for simple roots* α *of* $\mathcal{R}$. *We write the common stabilizer* $\Gamma_{\lambda,\eta}$ *under multiplier, field and coefficient actions* $a = (h_a, \sigma_a, \delta_a)$. *Moreover,* $P_a P_b = \alpha_\lambda(a, b) P_{ab}$ *and* $Q_a Q_b = \alpha_\lambda(a, b) Q_{ab}$ *for* σ_a*-semilinear operators* P_a, Q_a *on the two character spaces* $V_{\lambda,\eta}, V'_{\lambda,\eta}$. *Thus, we find* $P_{(f,1,1)} = Q_{(f,1,1)} = I$. *Moreover,* $\rho_{\lambda,\eta}(z)/\rho_{\lambda,\eta}(1) = \lambda(z) = \tau_{\lambda,\eta}(z)/\tau_{\lambda,\eta}(1)$ *for* $z \in Z(\widetilde{X})$.

Proof. First, we put $N_\lambda = \{n \in N : \lambda^n = \lambda\}$ and $I_\theta = \{i : \theta_i = \theta\}$. With $I_0 = I_1$, put $I_\varepsilon = I_{\varepsilon_q}$ and $s = \varepsilon_q(-1)$. We put $J_i(e_i) = f_i, J_i(f_i) = -e_i$, with $J_i(e_j) = e_j, J_i(f_j) = f_j$ for $j \neq i$, and $P_\pi(e_i) = e_{\pi(i)}, P_\pi(f_i) = f_{\pi(i)}$. Moreover, we put $J_S = \prod_{i\in S} J_i$ and $J_i^2 = t_i(-1)$, with $t_i(-1)e_j = (-1)^{\delta_{ij}} e_j, t_i(-1)f_j = (-1)^{\delta_{ij}} f_j$. We fix $S \subseteq I_0 \cup I_\varepsilon$, with $|S \cap I_\varepsilon| \in 2\mathbb{Z}$, and $\pi(I_\theta) = I_\theta$ for $\theta \in \{1, \varepsilon_q\} \cup \Theta$. Then, by Proposition 5.19, $N_\lambda = \bigsqcup_{S,\pi} \widetilde{T} J_S P_\pi$. We write $l_\theta(\pi) \in \{0, 1\}$, with $(-1)^{l_\theta(\pi)} = \mathrm{sgn}(\pi|_{I_\theta})$. We put $\widehat{\lambda}(tJ_S P_\pi) = \lambda(t) s^{|S\cap I_\varepsilon|/2} \prod_{\theta\in\{1,\varepsilon_q\}\cup\Theta} \theta(-1)^{l_\theta(\pi)}$. Moreover, $P_\pi J_S P_\pi^{-1} = J_{\pi(S)}$ and $J_S J_{S'} = (\prod_{i\in S\cap S'} t_i(-1)) J_{S\triangle S'}$, with $\lambda((J_S P_\pi) t (J_S P_\pi)^{-1}) = \lambda(t)$. Since $|S \cap I_\varepsilon| + |S' \cap I_\varepsilon| = |(S \triangle S') \cap I_\varepsilon| + 2|S \cap S' \cap I_\varepsilon|$, $\widehat{\lambda}(xy) = \widehat{\lambda}(x)\widehat{\lambda}(y)$. With $n_{e_i - e_j} = P_{(ij)} t_j(-1)$, we write $n_{e_i+e_j} = P_{(ij)} J_i J_j$. Moreover, $\widehat{\lambda}(n_{e_i-e_j}) = \theta(-1)^2 = 1$ when $i, j \in I_\theta$. Moreover, $\widehat{\lambda}(n_{e_i+e_j}) = \theta(-1)^2 = 1$ for $\theta \in \{1, \varepsilon_q\}$, and $\widehat{\lambda}(J_i) = 1$ for $i \in I_0$. Thus, $\widehat{\lambda}(n_\alpha) = 1$ for every simple α of $\mathcal{R}$. $\widehat{\lambda} \in \mathrm{Irr}(N_\lambda)$, $\widehat{\lambda}(1) = 1$ and $\widehat{\lambda}|_{\widetilde{T}} = \lambda$. Thus, $\mathrm{Ind}_{\widetilde{T}}^{N_\lambda} \lambda \simeq \widehat{\lambda} \otimes E[W_\lambda]$. Write its distinct $\widetilde{T}$-weights λ^n, with $n \in N/N_\lambda$, after induction $\mathrm{Ind}_{N_\lambda}^N$. Thus, $\mathrm{End}_N(I_N(\lambda)) \simeq E[W_\lambda]^{\mathrm{op}} \simeq E[W_\lambda]$. We fix $E_0 = \mathbb{Q}(\sqrt{q})$ and models $M^0_\eta(q), M^0_\eta(1)$ over $E_0, \mathbb{Q}$ and put $M_\eta(q) = E \otimes_{E_0} M^0_\eta(q)$ and $M_\eta(1) = E \otimes_{\mathbb{Q}} M^0_\eta(1)$. Moreover, we write $\eta = \eta_0 \otimes \eta_\varepsilon \otimes \bigotimes_{\theta\in\Theta} \eta_{\Lambda_\theta}$. We put $M_\eta(q) = M_{\eta_0}(q) \otimes M_{\eta_\varepsilon}(q) \otimes \bigotimes_{\theta\in\Theta} M_{\Lambda_\theta}(q)$. Moreover, we write the isotypic spaces $V_{\lambda,\eta} \otimes M_\eta(q)$ and $V'_{\lambda,\eta} \otimes M_\eta(1)$. $\mathrm{tr}(A(g)|_{V_{\lambda,\eta}}) = \rho_{\lambda,\eta}(g)$ and $\mathrm{tr}(A(n)|_{V'_{\lambda,\eta}}) = \tau_{\lambda,\eta}(n)$. By Lemma 5.21, write $U_\Lambda(q), U_\Lambda(1)$ on each block $\theta^2 \neq 1$. Moreover, the tensor permutations are Π_π, with $\Pi_\pi \Pi_\nu = \Pi_{\pi\nu}$. We put $U_{\Lambda,i} = I \otimes \cdots \otimes U_\Lambda \otimes \cdots \otimes I$. Thus, $\Pi_\pi U_{\Lambda,i} \Pi_\pi^{-1} = U_{\Lambda,\pi(i)}$ on the factors $M_{\Lambda,i}(q) \simeq M_\Lambda(q)$. Moreover, $U_{\Lambda,i} U_{\Lambda,j} = U_{\Lambda,j} U_{\Lambda,i}$ for $i \neq j$, and $U_{\Lambda,i}^2 = I$. We consider $r_\varepsilon > 0$ and T_0 on the quadratic modules M_{Λ^+,Λ^-} and both labels M^+, M^-

when $\Lambda^+ = \Lambda^-$. $T_0^2 = I$, including $T_0 : M^+ \xrightarrow{\sim} M^-$. Alternatively, consider $r_\varepsilon = 0$, with multiplicity space E and operator I. Moreover, $c(a)|_{\{i:\theta_i=1\}} = 1$, with multiplicity operator I on $M_{\eta_0}(q)$. Thus, we write $B_a(q), B_a(1)$ on $\mathcal{M}(q) = \bigoplus_\eta M_\eta(q), \mathcal{M}(1) = \bigoplus_\eta M_\eta(1)$, with $B_a(q)B_b(q) = B_{ab}(q)$ and $B_a(1)B_b(1) = B_{ab}(1)$. $B_a(q)$ has coefficients in $\mathbb{Q}(\sqrt{q})$ and $B_a(1)$ has coefficients in $\mathbb{Q}$. By Lemma 5.18, $\sigma(\sqrt{q}) = \sqrt{q}$ for $\sigma \in \mathcal{H}_p$. Then, put the coefficient maps $C_\sigma(\sum_i c_i v_i) = \sum_i \sigma(c_i)v_i$. Put $D_a(q) = B_a(q)C_{\sigma_a}$ and $D_a(1) = B_a(1)C_{\sigma_a}$. Thus, $\alpha_{M(q)}(a,b) = \alpha_{M(1)}(a,b) = 1$. Moreover, $a\eta = \eta'$ on $M_\eta(q)$ exactly when $a\eta = \eta'$ on $M_\eta(1)$. Thus, we write $\Gamma_{\lambda,\eta} = \{a \in \Gamma_\lambda : a\eta = \eta\}$ for both families $M_\eta(q), M_\eta(1)$. By Lemma 5.20, $\alpha_{I(\lambda)} = \alpha_{I_N(\lambda)} = \alpha_\lambda$. We write the tensor maps $P_a \otimes D_a(q)$ and $Q_a \otimes D_a(1)$. Accordingly, by Lemma 5.17, $\alpha_P = \alpha_Q = \alpha_\lambda$. Consider $a_f = (f,1,1)$, with $E_{a_f} = I$ and $c(a_f) = 1$. Moreover, $D_{a_f}(q) = D_{a_f}(1) = I$. Thus, $P_{(f,1,1)} = Q_{(f,1,1)} = I$. Fix $z \in Z(\widetilde{X})$, with $F(xz) = F(zx) = \lambda(z)F(x)$ on $I(\lambda), I_N(\lambda)$. Thus, $\rho_{\lambda,\eta}(z)/\rho_{\lambda,\eta}(1) = \lambda(z) = \tau_{\lambda,\eta}(z)/\tau_{\lambda,\eta}(1)$. □

We write the canonical trace t_λ on $\mathcal{H}_\lambda$. We put $[j]_q = (q^j - 1)/(q-1)$ and $P_W(q) = \prod_{i=1}^n [2i]_q$. Thus, $P_W(q) = [\widetilde{X} : \widetilde{B}]$ for the split Borel subgroup $\widetilde{B}$. We fix $\eta \in \mathrm{Irr}(W_\lambda)$ and the simple module $M_\eta(q)$. We write its character $\chi_{M_\eta(q)}$ and Schur element $s_\eta(q)$. Thus, the canonical trace is $t_\lambda = \sum_{\eta \in \mathrm{Irr}(W_\lambda)} s_\eta(q)^{-1} \chi_{M_\eta(q)}$.

Proposition 5.24. *We fix $p > 2$ with $p \mid (q-1)$ and the pairs $(\rho_{\lambda,\eta}, \tau_{\lambda,\eta})$ of* Proposition 5.23. *Then, $v_p(\rho_{\lambda,\eta}(1)) = v_p\big([W : W_\lambda]\eta(1)\big) = v_p(\tau_{\lambda,\eta}(1))$.*

Proof. By Lemmas 5.18 and 5.20, the support of H_w is $\widetilde{B}w\widetilde{B}$. The fork generator $H_{s'}$ has support $\widetilde{B}s'\widetilde{B}$. Thus, H_w has zero diagonal unless $w = 1$. The normalized trace on $I(\lambda)$ gives $t_\lambda(H_w) = \delta_{w,1}$. Hence, the trace expansion gives $\rho_{\lambda,\eta}(1) = P_W(q)/s_\eta(q)$. By [41, Theorem 3.2], for type A and for type B with roots $(q,-1)$ or $(1,-1)$, the Schur factors are $[h_x]_q$ within components and $1 + q^{j_x}$ across components, up to $\pm q^a$. We have $h_x > 0$, whereas j_x may be negative. Since $q \equiv 1 \pmod p$ and $p > 2$, each $1 + q^{j_x}$ is a p-adic unit, as at $q = 1$. By [39, pp. 42–43], $v_p([h_x]_q) = v_p(h_x)$. Thus, $v_p(s_\eta(q)) = v_p(s_\eta(1))$ in types A, B.

Consider $\mathcal{H}_D$ in $\mathcal{H}_B$ with $T_0^2 = 1$. The canonical traces satisfy $t_B|_{\mathcal{H}_D} = t_D$, as both take H_w to $\delta_{w,1}$ for $w \in W_D$. By [41, §6.3], each constituent on $\mathcal{H}_D$ has $m \in \{1,2\}$ parents on $\mathcal{H}_B$, each with multiplicity 1. Their Schur element is $s_B(q)$, since $T_0 \mapsto -T_0$ fixes t_B. By [126, Lemma 5.10], $s_D(q) = s_B(q)/m$. Thus, $v_p(s_D(q)) = v_p(s_D(1))$ because the same m occurs at $q = 1$ and $p > 2$. Moreover, we put $\mathcal{H}_D = E$ at ranks $r_\varepsilon = 0, 1$, with Schur element 1. We put the product trace $t_\lambda = \bigotimes_i t_i$ and Schur element $s_\eta(q) = \prod_i s_{\eta_i}(q)$. $s_\eta(1) = |W_\lambda|/\eta(1)$ by the regular character of W_λ. Thus, $v_p(s_\eta(q)) = v_p(|W_\lambda|/\eta(1))$. Moreover, $v_p(P_W(q)) = \sum_{i=1}^n v_p(2i) = v_p(2^n n!) = v_p(|W|)$. Thus, $v_p(\rho_{\lambda,\eta}(1)) = v_p([W : W_\lambda]\eta(1))$ in the formula for v_p. Moreover, $\tau_{\lambda,\eta}(1) = [W : W_\lambda]\eta(1)$ by induction $\mathrm{Ind}_{N_\lambda}^{\widetilde{M}}$. □

We write $\rho = \rho_{\lambda,\eta}$ and $\tau = \tau_{\lambda,\eta}$ on the spaces V_ρ, V_τ over E. We fix $\widetilde{M}$ from Theorem 5.4 with $d = 1$ and $M = \widetilde{M} \cap X$. Thus, we put $G_0 = \widetilde{X} \times \widetilde{M}$ and $\nu(g,h) = \mu(g)\mu(h)^{-1}$. We write $\nu : G_0 \longrightarrow \mathbb{F}_q^\times$ and $F_0 = \ker \nu = \widetilde{X} \times_{\mathbb{F}_q^\times} \widetilde{M}$. We put $\mathcal{W} = V_\rho \otimes_E V_\tau^\vee$ and $R(g,h) = A_\rho(g) \otimes A_\tau(h)^\vee$. Then, by Proposition 5.23, we write $Y = \{y \in \mathrm{Irr}(\mathbb{F}_q^\times) : \rho(y \circ \mu) = \rho\} = \{y \in \mathrm{Irr}(\mathbb{F}_q^\times) : \tau(y \circ \mu) = \tau\}$. Moreover, by Corollary 5.5, $|Y| \leq 2$. We choose the paired maps P_a, Q_a with coefficient action σ_a. Given $l \in V_\tau^\vee$, we put $Q_a^\vee(l) = \sigma_a \circ l \circ Q_a^{-1}$. Thus, $Q_a^\vee(cl) = \sigma_a(c)Q_a^\vee(l)$ for $c \in E$. On $\mathcal{W}$, we put $T_a = P_a \otimes Q_a^\vee$. Moreover, $P_aP_b = \alpha(a,b)P_{ab}$ and $Q_aQ_b = \alpha(a,b)Q_{ab}$. Thus, $T_aT_b = T_{ab}$, since the dual contributes $\alpha(a,b)^{-1}$. We write T_y for $y \in Y$, with $\sigma_y = 1$. Moreover, $T_yR(g,h)T_y^{-1} = y(\nu(g,h))R(g,h)$. Thus, we obtain that $T_yR(g,h) = R(g,h)T_y$ for $(g,h) \in F_0$.

Proposition 5.25. *We have that $\mathcal{W}^Y$ is irreducible on F_0. Moreover, $\mathcal{W}^Y|_{X \times M}$ contains each pair $V_\chi \otimes V_{\Omega(\chi)}^\vee$ with multiplicity* 1. *Thus, $\Omega : \mathrm{Irr}(\rho|_X) \longrightarrow \mathrm{Irr}(\tau|_M)$ is bijective. Moreover, $\Omega(\chi^g) = \Omega(\chi)^g$ for $g \in \widetilde{M}$. Moreover, $\Omega(\chi^a) = \Omega(\chi)^a$ for every mixed operation a stabilizing the multiplier orbits $\rho\,\mathrm{Irr}(\mathbb{F}_q^\times), \tau\,\mathrm{Irr}(\mathbb{F}_q^\times)$.*

Proof. By the quotient formulas, a that fixes $\chi \in \mathrm{Irr}(\rho|_X)$ fixes the multiplier orbit of ρ. The same holds for τ and $\psi \in \mathrm{Irr}(\tau|_M)$. Proposition 5.23 gives $\rho^a = \rho(\delta_a \circ \mu)$ and $\tau^a = \tau(\delta_a \circ \mu)$ with common δ_a. Moreover, P_a, Q_a have common factor α. Thus, Lemma 3.4 applies with $(T, B, H) = (F_0, Y, \mathcal{W}^Y)$. In particular, $\mathcal{W}^Y$ is irreducible, and its constituents on $X \times M$ give the indicated bijection Ω.

If $Y = \{1, y\}$, let J_ρ, J_τ be the operators at y. By Proposition 5.23, both squares are κI. Choose b with $b^2\kappa = 1$. Thus, Ω matches equal signs of bJ_ρ, bJ_τ. The diagonal action of $(g,g) \in F_0$ gives equivariance for $g \in \widetilde{M}$. Moreover, δ_a is unique modulo Y, which acts trivially on $\mathcal{W}^Y$. By Lemma 3.4, T_a gives the action of a. Thus, $R(g,g)T_a$ gives the action for ag. □

Moreover, we consider $d = 1$ and the full split torus $\widetilde{T}$. $C_{\widetilde{X}^*}(\widetilde{T}^*) = \widetilde{T}^*$ and the unipotent character set $\{1_{\widetilde{T}^*}\}$. Thus, we write ordinary principal series $\rho_{\lambda,\eta}$ and local induction from N_λ. By Cabanes–Späth [34, Theorem 6.2], the bijection $\widetilde{\Omega}$ is $(\widetilde{X} \rtimes \langle\phi\rangle)_{\widetilde{T}}$-equivariant. It matches central characters on $Z(\widetilde{X})$ and satisfies $\widetilde{\Omega}(\rho(\delta \circ \mu)) = \widetilde{\Omega}(\rho)(\delta \circ \mu)$. By Cabanes–Späth [34, Theorem 6.1], write $\widetilde{\Lambda}_{\mathrm{CS}}$ for its extension map on $\widetilde{T}$. It is $\widetilde{M} \rtimes \langle\phi\rangle$-equivariant and compatible with multiplication by $\delta \circ \mu$ on $\widetilde{T}$. Next, take $\widehat{\lambda}$ from Proposition 5.23. We put $\pi_\lambda : N_\lambda \longrightarrow N_\lambda/\widetilde{T} \simeq W_\lambda$. $\tau_{\lambda,\eta} = \mathrm{Ind}_{N_\lambda}^{\widetilde{M}}(\widehat{\lambda}(\eta \circ \pi_\lambda))$ for our extension $\widehat{\lambda}$. By Corollary 5.5, we have that $\mathrm{Irr}(\widetilde{X} \mid \mathrm{Irr}_{p'}(X)) = \mathrm{Irr}_{p'}(\widetilde{X})$ and $\mathrm{Irr}(\widetilde{M} \mid \mathrm{Irr}_{p'}(M)) = \mathrm{Irr}_{p'}(\widetilde{M})$. By Proposition 5.24, $p \nmid \rho_{\lambda,\eta}(1)$ exactly when $p \nmid \tau_{\lambda,\eta}(1)$. Moreover, we consider $d > 1$ and a Sylow d-torus $\mathbf{S}_d$. We put the centralizer $L_d = C_{\mathrm{CSp}_{2n}(\overline{\mathbb{F}}_q)}(\mathbf{S}_d)^F$ with its possible residual factor $L_{d,0} \neq 1$. By Proposition 5.14, choose cuspidal sources ζ of $L_{d,0}$, under the quasi-isolated hypotheses for ζ. By Proposition 5.25, we write the pair (χ, ψ) of (ρ, τ). We prove that (χ, ψ) satisfies $\succeq_c$ under the field condition $|\phi| = f$.

Proposition 5.26. *Moreover, we fix odd q and $X = \mathrm{Sp}_{2n}(q)$, the universal cover of $X/\mathrm{Z}(X)$. Moreover, $X/\mathrm{Z}(X)$ is simple. We fix $p > 2$ with $p \mid (q-1)$. Moreover, we choose $P \in \mathrm{Syl}_p(X)$ and the split subgroup $M = \mathrm{N}_X(D)$ from* Proposition 5.3. *We write the surjection $\iota : A = \widetilde{X} \rtimes \langle\phi\rangle \longrightarrow \mathrm{Aut}(X)$, with $\phi(x) = x^{(\ell)}$ and $|\phi| = f$. Moreover, $\ker\iota = Z = \mathrm{Z}(\widetilde{X})$. By* Proposition 5.23, *we write the pair (ρ, τ) with the models P_a, Q_a. Then, by* Proposition 5.25, *we write (χ, ψ) and $\psi = \Omega(\chi)$. With $\Delta = \mathrm{Aut}(X)_P$, we put $\Delta_0 = \Delta_{\chi^{\mathcal{H}_p}}$. Thus, we have that $(X \rtimes \Delta_0, X, \chi)_{\mathcal{H}_p} \succeq_c (M \rtimes \Delta_0, M, \psi)_{\mathcal{H}_p}$.*

Proof. Put $B = \widetilde{M} \rtimes \langle\phi\rangle$. Moreover, $\mu(\widetilde{M}) = \mathbb{F}_q^\times$ and $A = XB$. Thus, $X \cap B = M$ and $Z \leq B$. We fix the automorphism ϕ, with $\phi(M) = M$ and period $|\phi| = f$, and we put $D_{\mathrm{amb}} = A \times_{A/X} B$ and $\Phi = \langle(\phi, \phi)\rangle$. Thus, $(a, b) \in D_{\mathrm{amb}}$ exactly when $aX = bX$. Write $a = x\phi^j, b = m\phi^k$ with $x \in \widetilde{X}, m \in \widetilde{M}$. Moreover, $aX = bX$ exactly when $j = k \pmod f$ and $\mu(x) = \mu(m)$. Thus, $D_{\mathrm{amb}} = F_0 \rtimes \Phi$. Put $\mathcal{O}_{\rho,\tau} = \{(\rho(\delta \circ \mu), \tau(\delta \circ \mu)) : \delta \in \mathrm{Irr}(\mathbb{F}_q^\times)\}$ and put $C_{\rho,\tau} = \Phi_{\mathcal{O}_{\rho,\tau}}$ and $D_{\rho,\tau} = F_0 C_{\rho,\tau}$. Then, the maps T_a from Proposition 5.25 act on $\mathcal{W} = V_\rho \otimes V_\tau^\vee$. We fix $c = (\phi^{h_c}, \phi^{h_c}) \in C_{\rho,\tau}$ and $a_c = (h_c, 1, \delta_c)$. Accordingly, choose $\delta_c \in \mathrm{Irr}(\mathbb{F}_q^\times)$, with $\delta_c(\mu(g))\rho(\phi^{-h_c}(g)) = \rho(g)$ and $\delta_c(\mu(m))\tau(\phi^{-h_c}(m)) = \tau(m)$. Moreover, fix $g \in \widetilde{X}$ and $m \in \widetilde{M}$ in the identities for a_c. Put $U_c = T_{a_c}|_{\mathcal{W}^Y}$ and $\mathcal{R}(u) = R(u)|_{\mathcal{W}^Y}$ for $u \in F_0$. Since $T_y|_{\mathcal{W}^Y} = I$, $T_{(h_c,1,\delta_c y)}|_{\mathcal{W}^Y} = U_c$ for $y \in Y$. We fix $c, d \in C_{\rho,\tau}$ and the discrepancy $y(c, d) \in Y$ between the choices. We have that $U_c U_d = T_{y(c,d)} U_{cd} = U_{cd}$ on $\mathcal{W}^Y$ and that $U_c \mathcal{R}(u) U_c^{-1} = \mathcal{R}(cuc^{-1})$ for $u \in F_0$. Moreover, $\delta_c(\mu(x))\delta_c(\mu(m))^{-1} = 1$ when $\mu(x) = \mu(m)$. By Proposition 5.23, $P_{(f,1,1)} = Q_{(f,1,1)} = I$. Thus, $\mathcal{R}(uc) = \mathcal{R}(u)U_c$ is an ordinary representation of $D_{\rho,\tau}$ on $\mathcal{W}^Y$.

With $K = X \times M$, put $W_{\chi,\psi} = V_\chi \otimes V_\psi^\vee$. Then, by Proposition 5.25, $W_{\chi,\psi}$ occurs with multiplicity 1 in $\mathcal{W}^Y|_K$. With $L_{\chi,\psi} = \{(a, b) \in D_{\mathrm{amb}} : \chi^a = \chi,\ \psi^b = \psi\}$, fix $(a, b) \in L_{\chi,\psi}$ with common field component $c \in \Phi$. Since $\rho^x = \rho$ for $x \in \widetilde{X}$, $\rho^c = \rho(\delta \circ \mu)$ for some $\delta \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Moreover, $\tau^c = \tau(\delta \circ \mu)$ by Proposition 5.23. $c \in C_{\rho,\tau}$ and $L_{\chi,\psi} \leq D_{\rho,\tau}$. Thus, we put $\mathcal{R}_{\chi,\psi}(u) = \mathcal{R}(u)|_{W_{\chi,\psi}}$ for $u \in L_{\chi,\psi}$. Moreover, $\mathcal{R}_{\chi,\psi}|_K$ has character $\chi \boxtimes \psi^\vee$. We write the mixed operators T_a of Proposition 5.25 and the diagonal maps $R(b, b)$ for $b \in \widetilde{M}$. Thus, $\mathcal{R}^a_{\chi,\psi} \simeq \mathcal{R}_{\chi,\psi}$ for every mixed operation a stabilizing (χ, ψ). Moreover, $\delta_a(\mu(x))\delta_a(\mu(m))^{-1} = 1$ for $(x, m) \in F_0$. We have that every discrepancy $y \in Y$ acts by $T_y|_{\mathcal{W}^Y} = I$. Thus, $T_a T_b = T_{ab}$ on $\mathcal{W}^Y$ for the joint operators T_a, T_b. We fix $z \in Z$ and the pair $(z, z) \in F_0$. Moreover, $\mathcal{R}(z, z) = I$, since $\rho(z)/\rho(1) = \tau(z)/\tau(1)$.

We fix $a = x\phi^j \in \iota^{-1}(\Delta)$, with $x \in \widetilde{X}$. By Proposition 5.3, $aPa^{-1} = P$ and $aDa^{-1} = D$. Moreover, $\phi^j(D) = D$ and $x \in \mathrm{N}_{\widetilde{X}}(D) = \widetilde{M}$. Thus, $\iota^{-1}(\Delta) \leq B$. Choose a section $t : \Delta \longrightarrow B$, with $\iota(t(i)) = i$ and $t(1) = 1$. Whenever $i, j \in \Delta$, put $z(i, j) = t(i)t(j)t(ij)^{-1}$. Thus, $z(i, j) \in \ker\iota = Z$. By Proposition 5.25, $\Delta_\chi = \Delta_\psi$ and $(\Delta \times \mathcal{H}_p)_\chi = (\Delta \times \mathcal{H}_p)_\psi$, with $\Delta_0 = \Delta_{\psi^{\mathcal{H}_p}}$. We fix $(g, m, i) \in K \rtimes \Delta_\chi$ and put $s(g, m, i) = (g\,t(i), m\,t(i)) \in L_{\chi,\psi}$. Then, we put $\mathcal{T}(g, m, i) = \mathcal{R}_{\chi,\psi}(s(g, m, i))$. Moreover, $s(g, m, i)s(g', m', j)$ differs from $s(g\,i(g'), m\,i(m'), ij)$ by the pair $(z(i, j), z(i, j))$. Since $\mathcal{R}(z, z) = I$, $\mathcal{T}$ is an ordinary representation of $K \rtimes \Delta_\chi$. $\phi(Z) = Z$, so each conjugate defect $(\phi^j(z), \phi^j(z))$ also acts by I, and we choose another section t', with $t'(i)t(i)^{-1} \in Z$. Thus, $\mathcal{T}_{t'} = \mathcal{T}_t$ from $\mathcal{R}(z, z) = I$. We fix $i_0 \in \Delta$ and $b = t(i_0)$, and $b\,t(i)b^{-1}t(i_0 i i_0^{-1})^{-1} \in Z$. Thus, $\mathcal{T}^a \simeq \mathcal{T}$ for the mixed operations a stabilizing (χ, ψ). We include the local inner maps $(m, m) \in F_0$ through $\mathcal{R}(m, m)$. The conjugation on $K \rtimes \Delta_\chi$ is the conjugation by $(m, 1)$ on both factors $X \rtimes \Delta_\chi, M \rtimes \Delta_\chi$. Thus, $\mathcal{T}^a \simeq \mathcal{T}$ also for such operations $a = (m, i, \sigma)$, with $m \in M, i \in \Delta_0, \sigma \in \mathcal{H}_p$.

By [141, Corollary 1.2], choose a projective representation $\mathcal{P}$ of $X \rtimes \Delta_\chi$ associated with χ. We write the matrices $\mathcal{P}(g, i)$ over $\mathbb{Q}^{\mathrm{ab}}$ and the factor set α with values in roots of unity μ_∞. Moreover, we write $\mathcal{P}(g, 1) = A_\chi(g)$ for $g \in X$. We put $\mathcal{T}(i) = \mathcal{T}(1, 1, i)$ and $\mathcal{P}(i) = \mathcal{P}(1, i)$. By Schur's lemma for V_χ, $\mathcal{T}(i) = \mathcal{P}(i) \otimes Q(i)$ with $Q(i) \in \mathrm{GL}(V_\psi^\vee)$. $Q(i) \in \mathrm{GL}_{\psi(1)}(\mathbb{Q}^{\mathrm{ab}})$, since $\mathcal{T}(i), \mathcal{P}(i)$ are matrices over $\mathbb{Q}^{\mathrm{ab}}$. Whenever $(m, i) \in M \rtimes \Delta_\psi$, we put $\mathcal{P}'(m, i) =$

$A_\psi(m)Q(i)^\vee$. Thus, $\mathcal{P}'$ is associated with ψ and $\mathcal{T}(g,m,i)=\mathcal{P}(g,i)\otimes\mathcal{P}'(m,i)^\vee$. $Q(i)Q(j)=\alpha(i,j)^{-1}Q(ij)$ because $\mathcal{T}(i)\mathcal{T}(j)=\mathcal{T}(ij)$. Thus, $\alpha'(i,j)=\alpha(i,j)$ and $\alpha'=\alpha|_{(M\rtimes\Delta_\psi)^2}$ for the factor set α' of $\mathcal{P}'$. Fix $a=(m,i,\sigma)$ with $\chi^a=\chi$ and $\psi^a=\psi$. Then, by [141, Lemma 1.4], write the functions β_a,β'_a, constant on the cosets X,M. $\mathcal{P}^a\simeq\beta_a\mathcal{P}$ and $(\mathcal{P}')^a\simeq\beta'_a\mathcal{P}'$. Whenever $j\in\Delta_\psi$, we put $\omega_a(j)=\beta_a(j)/\beta'_a(j)$. Since $\alpha'|_{\Delta_\psi\times\Delta_\psi}=\alpha|_{\Delta_\psi\times\Delta_\psi}$, ω_a is a linear character of Δ_ψ. $\mathcal{T}^a\simeq\omega_a\mathcal{T}$ by the tensor product $\mathcal{P}\otimes(\mathcal{P}')^\vee$. Moreover, $\mathcal{T}^a\simeq\mathcal{T}$ by the mixed operators on $W_{\chi,\psi}$. We choose an equivalence C_a between $\mathcal{T}$ and $\omega_a\mathcal{T}$. $C_a\mathcal{T}(k)=\mathcal{T}(k)C_a$ for $k\in K$, since $\omega_a|_K=1$. By the irreducible character $\chi\boxtimes\psi^\vee$, $C_a=c_aI$ with $c_a\neq0$. Whenever $j\in\Delta_\psi$, $\omega_a(j)\mathcal{T}(j)=\mathcal{T}(j)$. Thus, $\omega_a(j)=1$ and $\beta_a(j)=\beta'_a(j)$. We include $i\in\Delta_0$ with $\chi^i\neq\chi$ and $\chi^{i\sigma}=\chi$. $\beta_a|_{M\rtimes\Delta_\psi}=\beta'_a$ for every $a=(m,i,\sigma)$ stabilizing (χ,ψ).

Moreover, we fix $(g,i)\in C_{X\rtimes\Delta_0}(X)$. We find $z=g\,t(i)\in\ker\iota=Z$. Since $Z\leq B$ and $t(i)\in B$, $g\in X\cap B=M$. Consider the diagonal case $m=g$ of the pair $s(g,m,i)$. $s(g,g,i)=(z,z)$ and $\mathcal{T}(g,g,i)=I$. Thus, $\mathcal{P}(g,i)=\lambda_{g,i}I$ and $\mathcal{P}'(g,i)=\lambda_{g,i}I$. $C_{X\rtimes\Delta_0}(X)\leq M\rtimes\Delta_0$ and $X(M\rtimes\Delta_0)=X\rtimes\Delta_0$ and $X\cap(M\rtimes\Delta_0)=M$. By [141, Definitions 1.5 and 3.1], we obtain that $\succeq_c$ is valid for $(X\rtimes\Delta_0,X,\chi)\mathcal{H}_p$ and $(M\rtimes\Delta_0,M,\psi)\mathcal{H}_p$. □

5.3 Negative blocks

By Section 5.1, we fix odd $q=\ell^f$ and odd $p\neq\ell$. Suppose that $d=\mathrm{ord}_p(q)=2e$ and put $Q=q^e$. We write the Legendre symbol $\varepsilon=(\ell/p)$ and $\bar{p}=p\cdot1_{\mathbb{F}_\ell}$. Thus, we write the multiplication maps $b\in\mathbb{F}_{Q^2}^\times$ on the block $\mathbb{F}_{Q^2}$. We put $\widetilde{T}=\{b\in\mathbb{F}_{Q^2}^\times:b^{Q+1}\in\mathbb{F}_q^\times\}$ and $\mu:\widetilde{T}\longrightarrow\mathbb{F}_q^\times$, with $\mu(b)=b^{Q+1}$. We write the subgroup $T=\ker\mu$ and the field lift u. Moreover, we put $v=u^f$, with $ubu^{-1}=b^\ell$ for $b\in\widetilde{T}$. We have that $v^d=-I$ and $\mu(v)=1$.

Lemma 5.27. *We have that some $z\in\mathbb{F}_{\ell^2}^\times\cap\widetilde{T}$ satisfies $z^{\ell-1}=\varepsilon$ and $\mu(z)=\bar{p}$. Moreover, we have that* $\mathrm{ord}(z)\mid2(\ell-1)$ *and $p\nmid\mathrm{ord}(z)$.*

Proof. Put $m=fe$, so that $Q=\ell^m\equiv-1\pmod p$ and $\varepsilon^m=(-1)^{(p-1)/2}$. By [92, Ch. 5, §3], quadratic reciprocity gives $(p/\ell)=\varepsilon\,(-1)^{((p-1)/2)((\ell-1)/2)}=\varepsilon^{1+m(\ell-1)/2}$. Set $a_0=\bar{p}\varepsilon^m\in\mathbb{F}_\ell^\times$. Then, Euler's criterion gives $a_0^{(\ell-1)/2}=\varepsilon$. Choose $z\in\mathbb{F}_{\ell^2}$ with $z^2=a_0$. Thus, $z^{\ell-1}=\varepsilon$ and $z^Q=\varepsilon^m z$, so $z^{Q+1}=\bar{p}$. Hence, $z\in\widetilde{T}$ with $\mu(z)=\bar{p}$. Moreover, $z^{2(\ell-1)}=1$. If $p\mid\ell-1$, then $d=\mathrm{ord}_p(q)=1$, contrary to $d=2e$. Thus, $\mathrm{ord}(z)$ is prime to p. □

We fix a colour (θ,ν) with $\theta\in\mathrm{Irr}(\widetilde{T})$. From each orbit $\theta^{\langle q\rangle}$, we choose one representative θ. Then, we put $c=|\theta^{\langle q\rangle}|$, with $c\mid d$. Moreover, we write $I_\theta=\langle\widetilde{T},v^c\rangle$ and $\widehat{\theta}_\nu\in\mathrm{Irr}(I_\theta)$, with $\widehat{\theta}_\nu(1)=1$ and $\widehat{\theta}_\nu|_{\widetilde{T}}=\theta$. We have that $\widehat{\theta}_\nu(v^c)=\nu$ and $\nu^{d/c}=\theta(-1)=(-1)^{b_0}$, with $b_0\in\{0,1\}$. Accordingly, by Section 5.1, we write $a=(h_a,\sigma_a,\delta_a)$, with $\delta_a\in\mathrm{Irr}(\mathbb{F}_q^\times)$. We put $N'=N_0/p^{v_p(N_0)}$ and $N_1=\mathrm{lcm}(N',\ell)$. We choose k_a with $\sigma_a(\xi)=\xi^{p^{k_a}}$ for $\xi\in\mu_{N_1}$. Thus, we put $L_a=p^{k_a}$ and $t_a=\bar{p}^{k_a}\in\mathbb{F}_\ell^\times$. We have that $\sigma_a(\zeta_\ell)=\zeta_\ell^{t_a}$ for a primitive root ζ_ℓ. Moreover, we choose $s_{a,r}\in(\mathbb{Z}/p^r\mathbb{Z})^\times$ for $r\geq1$, with $\sigma_a(\xi)=\xi^{s_{a,r}}$ on μ_{p^r}. We write $\theta(t)=\xi_0\xi_1$ for $t\in\widetilde{T}$, with $\xi_0\in\mu_{N'}$ and $\xi_1\in\mu_{p^r}$. Thus, we find $\sigma_a(\theta(t))=\xi_0^{L_a}\xi_1^{s_{a,r}}$. We put $\mathcal{G}_{\theta,\nu}=\{a:\theta^{\ell^{-h_a}\sigma_a}(\delta_a\circ\mu)\in\theta^{\langle q\rangle},\ \sigma_a(\nu)=\nu\}$. Thus, we put $A(a,b)=\sigma_a(\delta_b)(t_a)^{-1}$ for $a,b\in\mathbb{Z}\times\mathcal{H}_p\times\mathrm{Irr}(\mathbb{F}_q^\times)$.

Proposition 5.28. *We choose z from* Lemma 5.27 *and put $w=\theta(z)$. Given $a\in\mathcal{G}_{\theta,\nu}$, we put $C_\theta(a)=w^{-k_aL_a}$. Then, we choose S_a by* Proposition 5.7 *and put $S'_a=C_\theta(a)S_a$. Thus, we find $S'_aS'_b=\mathrm{A(a,b)}S'_{ab}$ for $a,b\in\mathcal{G}_{\theta,\nu}$. Moreover, $S'_{(f,1,1)}=S_{(f,1,1)}=A(v)$ and $(S'_{(f,1,1)})^d=A(-I)$. We have that $S'_{(0,1,\delta)}=S_{(0,1,\delta)}$ for $(0,1,\delta)\in\mathcal{G}_{\theta,\nu}$.*

Proof. Then, by Lemma 5.27, $w^{\ell-1}=\varepsilon^{b_0}$ and $w^{2(\ell-1)}=1$. We put $M=2(\ell-1)\in\mathbb{Z}_{>0}$. Since $p\nmid M$, choose $B>0$ with $M\mid B$ and $\mathrm{ord}_M(p)\mid B$. Moreover, we take $\mathrm{ord}_{N_1}(p)\mid B$ and the field $\mathbb{K}(\mu_\ell,\mu_{p^B-1})$. We have that $\mu_M,\mu_{N_1}\subseteq\mu_{p^B-1}$. $\mathrm{ord}_{p^B-1}(p)=B$, since $0<p^j-1<p^B-1$ for $0<j<B$. Thus, we choose $k_a\pmod B$ from $\sigma_a|_{\mu_{p^B-1}}$. We have that $k_a\pmod M$ and $L_a\pmod M$ depend only on $k_a\pmod B$. Thus, $C_\theta(a)=w^{-k_aL_a}$ is defined on the full stabilizer $\mathcal{G}_{\theta,\nu}$.

By the conditions on θ,ν, we write $j_b\pmod c$ for $b\in\mathcal{G}_{\theta,\nu}$. Then, $\delta_b(\bar{p})=w^{q^{j_b}-L_b\ell^{-h_b}}$ by evaluation at z. We read ℓ^{-h_b} modulo $\mathrm{ord}(w)$, with $\mathrm{ord}(w)\mid M$. Since $t_a=\bar{p}^{k_a}$, $A(a,b)=w^{k_aL_a(L_b\ell^{-h_b}-q^{j_b})}$. Moreover, $k_{ab}=k_a+k_b\pmod B$ and $L_{ab}=L_aL_b\pmod M$. Thus, $C_\theta(a)\sigma_a(C_\theta(b))/C_\theta(ab)=w^{k_aL_a(L_b-1)}$. $(\ell-1)^2\equiv0\pmod M$ and $(\ell,M)=1$, and $\ell^{-h}-1\equiv-h(\ell-1)\pmod M$ and $q^j-1\equiv fj(\ell-1)\pmod M$. Since L_a,L_b are odd, $A(a,b)\big/\big(C_\theta(a)\sigma_a(C_\theta(b))/C_\theta(ab)\big)=\varepsilon^{b_0k_a(h_b+fj_b)}$. By the factor of S_aS_b, $S'_aS'_b=A(a,b)S'_{ab}$. Consider $\sigma_a=1$, with $k_a=0\pmod B$. Thus, $C_\theta(a)=1$ for the field maps $a=(h_a,1,1)$ and self-twists $a=(0,1,\delta_a)$. We write $a_f=(f,1,1)$

and $a \in \mathcal{G}_{\theta,\nu}$. We find $k_{aa_f} = k_{a_f a} = k_a \pmod B$. Thus, $C_\theta(aa_f) = C_\theta(a_f a) = C_\theta(a)$. $S'_{a_f} = S_{a_f} = A(v)$ and $(S'_{a_f})^d = A(v)^d = A(-I)$. □

Moreover, we fix $X = \mathrm{Sp}_{2n}(q)$ and $\pi \in \mathrm{Irr}(X)$. We write generalized Gelfand–Graev characters $\Gamma_{\mathcal{O}}$ for rational nilpotent orbits $\mathcal{O}$. Thus, we put $m_\pi(\mathcal{O}) = \langle \Gamma_{\mathcal{O}}, \pi \rangle_X$ and $\mathcal{S}(\pi) = \{\mathcal{O} : m_\pi(\mathcal{O}) > 0\}$. By Wang [206, §3.2], we write the rational order $\leq_{\mathrm{rat}}$ on the orbits $\mathcal{O}$. Accordingly, write $\mathcal{S}_{\max}(\pi)$ as the maximal elements $\mathcal{O}$ of $\mathcal{S}(\pi)$ under $\leq_{\mathrm{rat}}$. By Wang [206, Theorem 7.7(ii)], we write the geometric orbit $\mathcal{O}_\pi^{\mathrm{st}}$ of $\mathcal{S}_{\max}(\pi)$. We write the partition λ_π of $\mathcal{O}_\pi^{\mathrm{st}}$. Moreover, we write the eigenvalue orbit $\mathcal{C}_a = \{a^{q^j} : j \geq 0\}$ for $a \in \overline{\mathbb{F}}_q^\times \setminus \{1, -1\}$. We put $[a] = \mathcal{C}_a \cup \mathcal{C}_{a^{-1}}$ with one count per class $[a]$. When $\mathcal{C}_a \neq \mathcal{C}_{a^{-1}}$, we write $c[a] = |\mathcal{C}_a|$. Alternatively, we write $c[a] = |\mathcal{C}_a|/2$ when $\mathcal{C}_a = \mathcal{C}_{a^{-1}}$. By Wang [206, §2.3], we write the partition $\lambda[a]$ for the unipotent character at $[a]$.

Proposition 5.29. *Suppose that $\ell > 3(2n-1)$ and let Λ_+, Λ_- be the cuspidal symbols at $+1, -1$ for π. We have that $\mathrm{rank}(\Lambda_+) = t(t+1)$ and $\mathrm{rank}(\Lambda_-) = s^2$. Moreover, we put $R_i = \sum_{[a] \neq [\pm 1]} c[a]\lambda[a]_i^{\mathsf{t}}$ for $i \geq 1$. We put $A_i = \max\{0, t+1-\lceil i/2 \rceil\}$ and $B_i = \max\{0, s - \lfloor i/2 \rfloor\}$ and put $L_i = R_i + A_i + B_i$. Thus, we find $\lambda_\pi = (2L_i)_{L_i > 0}$. Whenever $\mathcal{O} \in \mathcal{S}_{\max}(\pi)$, $m_\pi(\mathcal{O}) = 1$.*

Proof. Then, by [206, §§1.5, 2.1 and 6.1, Corollaries 6.8–6.9 and Theorem 7.7], we choose the sequence $\pi = \pi_0 \to \cdots \to \pi_k = 1$. Write $\kappa_j = (\{0, \ldots, j-1\}, \varnothing)$ for $j \geq 1$. Then, $\mathrm{rank}(\kappa_j) = \lfloor j^2/4 \rfloor$. Moreover, we write $\Lambda_+ = \kappa_{2t+1}$ and $\Lambda_- = \kappa_{2s}$ for $s > 0$, up to $(A, B)^{\mathsf{t}} = (B, A)$. $D(\kappa_j) \in \{\kappa_j, \kappa_j^{\mathsf{t}}\}$ for the Alvis–Curtis dual D. Then, by [206, §6.1], write κ_{j-1} or $\kappa_{j-1}^{\mathsf{t}}$ after κ_j, for $j > 1$. We obtain $r_j = \lfloor j^2/4 \rfloor - \lfloor (j-1)^2/4 \rfloor = \lfloor j/2 \rfloor$. We write $(\{0\}, \varnothing)$ at symplectic rank 0 and, for further descent i, $(\{0\}, \{0\})$ at orthogonal rank 0. Thus, we write the classical contributions D_i^+, D_i^- with $D_i^{(-1)^{i-1}} = A_i$ and $D_i^{(-1)^i} = B_i$. $A_i = r_{2t+2-i}$ and $B_i = r_{2s+1-i}$ until the respective rank 0. Then, by [206, Corollary 6.9], we choose $\lambda[a]_j^{(i)} = \max\{\lambda[a]_j - i, 0\}$ after i columns of $\lambda[a]$. We find $\sum_j (\lambda[a]_j^{(i-1)} - \lambda[a]_j^{(i)}) = \lambda[a]_i^{\mathsf{t}}$. Thus, $R_i = \sum_{[a] \neq [\pm 1]} c[a]\lambda[a]_i^{\mathsf{t}}$ and $L_i = R_i + A_i + B_i$.

By [206, Theorem 7.7(iii),(iv)], write the index sets $\mathsf{N}_\pi, \mathsf{M}_\pi$. If $i \in \mathsf{N}_\pi \cup \mathsf{M}_\pi$, $D_i^\epsilon = D_{i+1}^{-\epsilon} - 1$ for some $\epsilon \in \{1, -1\}$. $A_i = A_{i+1} - 1$ or $B_i = B_{i+1} - 1$. Since $A_i \geq A_{i+1}$ and $B_i \geq B_{i+1}$, $\mathsf{N}_\pi = \mathsf{M}_\pi = \varnothing$. $\lambda_\pi = (2L_i)_{L_i > 0}$ and $m_\pi(\mathcal{O}) = 2^{|\mathsf{M}_\pi|} = 1$ for $\mathcal{O} \in \mathcal{S}_{\max}(\pi)$. Moreover, $m_\pi(\mathcal{O}) \leq 1$ for cuspidal π on $\mathcal{O} \subset \mathcal{O}_\pi^{\mathrm{st}}$ by [150, Theorems 1.1(iii), 1.3(iii), 6.16 and 8.5]. □

Proposition 5.30. *We fix $\ell > 3(2n-1)$ and $n \geq 1$. Moreover, we choose cuspidal $\rho \in \mathrm{Irr}(\widetilde{X})$, with $\widetilde{X} = \mathrm{GSp}_{2n}(q)$. We write generalized Gelfand–Graev characters $\Gamma_Y = \mathrm{Ind}_U^{\widetilde{X}} \eta$, and we have that $\langle \Gamma_Y, \rho \rangle = 1$ for some $\eta \in \mathrm{Irr}(U)$, with $\eta(1) = 1$. Moreover, we choose the nilpotent representative $Y \in \mathfrak{sp}_{2n}(q)$ with even Jordan blocks $j \in 2\mathbb{Z}_{>0}$. We choose the semilinear isometry υ_Y, with $c_* = \upsilon_Y^f$ and $\varphi_Y(x) = \upsilon_Y x \upsilon_Y^{-1}$. Thus, we write $\Gamma_\rho = \{a = (h_a, \sigma_a, \delta_a) : (\delta_a \circ \mu)\rho^{\varphi_Y^{-h_a}\sigma_a} = \rho\}$. We have that the source maps T_a satisfy $T_a T_b = \mathrm{A(a,b)} T_{ab}$ for $a, b \in \Gamma_\rho$. Moreover, $T_{(f,1,1)} = A(c_*)$ on the space V_ρ.*

Proof. By Corollary 5.5, $\rho|_X = \pi$ or $\rho|_X = \pi + \pi^g$, with $\pi^g \neq \pi$. Then, choose $g \in \widetilde{X}$ with $\mu(g) \notin (\mathbb{F}_q^\times)^2$. Consider a proper parabolic $Q < X$ and the corresponding parabolic $\widetilde{Q} < \widetilde{X}$. Since their unipotent radical U_Q is common, $V_\pi^{U_Q} \subseteq V_\rho^{U_Q} = 0$. π is cuspidal on X. By [150, proof of Theorem 6.16], we write the cuspidal symbols Λ_+, Λ_- of π. Then, the partition is $\lambda_\pi = (2L_i)_{L_i > 0}$ by Proposition 5.29. Consider $\rho|_X = \pi$ and choose $\mathcal{O} \in \mathcal{S}_{\max}(\pi)$, and $m_\pi(\mathcal{O}) = 1$ in the case $\rho|_X = \pi$.

Alternatively, consider $\rho|_X = \pi + \pi^g$. We suppose that $s = 0$, with absent eigenspace $V_{-1} = 0$ in the dual group SO_{2n+1}. Moreover, choose the semisimple element s_π and its centralizer $C_{\mathrm{SO}_{2n+1}}(s_\pi)$. The orthogonal factor $\mathrm{SO}(V_1)$ is connected, and the other factors $\mathrm{GL}(V_a)$ are connected. Thus, $C_{\mathrm{SO}_{2n+1}}(s_\pi)$ is connected. By [169, Lemma 4.8], $\rho|_X$ is irreducible, contrary to $\rho|_X = \pi + \pi^g$. Thus, $s > 0$. We have that $A_1 = A_2 = t$ and $B_1 = s, B_2 = s - 1$. $L_1 - L_2 = R_1 - R_2 + 1 > 0$, since $R_1 \geq R_2$. Thus, we write the largest block $j = 2L_1$ with multiplicity 1. We write the unique largest entries $2t, 2s-1$ of Λ_+, Λ_-, including $t = 0$. In particular, we write the sign $a_1(\mathcal{O}) \in \{1, -1\}$ of the block $2L_1$. By [206, Theorem 7.7(i)], $a_1(\mathcal{O}) = a_1(\mathcal{O}')$ for $\mathcal{O}, \mathcal{O}' \in \mathcal{S}_{\max}(\pi)$.

Next, we fix $V = \mathbb{F}_q^{2n}$ with alternating form $\langle\ ,\ \rangle$ and $Y \in \mathcal{O}$. We put $W_j = \ker Y^j / (\ker Y^{j-1} + Y \ker Y^{j+1})$ for even j. Thus, we write the symmetric form $\beta_j^Y([x], [y]) = \langle x, Y^{j-1} y \rangle$ on W_j. We find $\beta_j^{gYg^{-1}}(g[x], g[y]) = \mu(g)\beta_j^Y([x], [y])$. We write $j = 2L_1$, with $\dim W_j = 1$. Thus, $\det \beta_j^{gYg^{-1}} / \det \beta_j^Y = \mu(g) \notin (\mathbb{F}_q^\times)^2$ in the bases $[x], g[x]$. We obtain $\mathcal{S}_{\max}(\pi) \cap \mathcal{S}_{\max}(\pi^g) = \varnothing$. Moreover, choose $\mathcal{O} \in \mathcal{S}_{\max}(\pi)$, with $\lambda_{\pi^g} = \lambda_\pi$. $\dim \mathcal{O}' = \dim \mathcal{O}$ for every rational form

$\mathcal{O}' \subset \mathcal{O}^{\mathrm{st}}_\pi$. Then, by [206, Theorem 1.1(i)], $m_{\pi^\vartheta}(\mathcal{O}) = 0$ and $m_\pi(\mathcal{O}) = 1$. Write the even partition $\lambda_Y = \lambda_\pi$ for the chosen $Y \in \mathcal{O}$. Moreover, the string weights $1-j, 3-j, \dots, j-1$ are odd for $j \in \lambda_Y$. Thus, $\mathfrak{sp}(V)[1] \simeq \mathrm{Sym}^2(V)[1] = 0$. We choose $\eta \in \mathrm{Irr}(U)$, with $\eta(1) = 1$ and $\Gamma_{\mathcal{O}} = \mathrm{Ind}_U^X \eta$. By [95, Lemma 5.2] for $U \leq X \leq \widetilde{X}$, $\langle \mathrm{Ind}_U^{\widetilde{X}} \eta, \rho\rangle = \langle \Gamma_{\mathcal{O}}, \rho|_X\rangle = 1$.

By Proposition 5.14, we write the space $\mathcal{V} = \{F : \widetilde{X} \to E : F(ux) = \eta(u)F(x)\}$ and the summand V_ρ. Whenever $g, x \in \widetilde{X}$, $[A(g)F](x) = F(xg)$. Moreover, we write the grading $V = \bigoplus_k V[k]$ and $D(t)|_{V[k]} = t^{(k+1)/2} I$ for $t \in \mathbb{F}_\ell^\times$. $\mu(D(t)) = t$ and $\eta(D(t)uD(t)^{-1}) = \eta(u)^t$ for $u \in U$. Thus, we write the maps E_δ, S_σ on $\mathcal{V}$. By the relation for $S_\sigma E_\delta$, $S_{\sigma_a} E_{\delta_b} = \sigma_a(\delta_b)(t_a)^{-1} E_{\sigma_a(\delta_b)} S_{\sigma_a}$. We consider odd f and a representative $Y \in \mathfrak{sp}_{2n}(\mathbb{F}_\ell)$. Thus, we choose $\upsilon_Y^f = I$, with $c_* = I$. Alternatively, consider even f. We write $m_j = \dim W_j$ and the basis $(e_{j,1}, \dots, e_{j,m_j})$ of W_j and write $\beta_j = \mathrm{Gram}_{(e_{j,k})}(\beta_j^Y) = \mathrm{diag}(1, \dots, 1, d_j)$, with $d_j \in \mathbb{F}_q^\times$, and put $b_j = d_j^{(1-\ell)/2}$. Moreover, $b_j^2 d_j^\ell = d_j$ and $b_j^{1+\ell+\cdots+\ell^{f-1}} = d_j^{(1-q)/2} \in \{1, -1\}$. Thus, choose υ_Y with coefficient action $x \mapsto x^\ell$ and scalar b_j on the final coordinate $e_{j,m_j} \in W_j$. $\upsilon_Y Y \upsilon_Y^{-1} = Y$ and $\upsilon_Y^f = c_*$, with $c_*|_{L_{j-1} \otimes e_{j,m_j}} = d_j^{(1-q)/2} I$ on the string L_{j-1} of dimension j. We have that $c_* = I$ on $L_{j-1} \otimes e_{j,k}$ for $1 \leq k < m_j$. Moreover, $d_j^{(1-q)/2} = -1$ exactly when $d_j \notin (\mathbb{F}_q^\times)^2$. Put $\varphi_Y(x) = \upsilon_Y x \upsilon_Y^{-1}$ and $[\mathcal{F}_Y F](x) = F(\varphi_Y^{-1}(x))$, and we have that $\mathcal{F}_Y^f = L_{c_*} A(c_*)$ and $L_{c_*}|_{V_\rho} = \kappa I$, with $\kappa \in \{1, -1\}$. Moreover, we put $r = \mathrm{ord}_p(\ell)$, with $d = r/(r, f)$ even. Thus, $v_2(f) < v_2(r) \leq v_2(p-1)$. We choose $\gamma \in \mu_{p-1}$ with $\gamma^f = \kappa^{-1}$. Whenever $\sigma \in \mathcal{H}_p$, $\sigma(\gamma) = \gamma$. Thus, write $T_a = E_{\delta_a}(\gamma \mathcal{F}_Y)^{h_a} S_{\sigma_a}$ on V_ρ for $a \in \Gamma_\rho$. $T_a T_b = A(a,b) T_{ab}$ and $T_{(f,1,1)} = \gamma^f L_{c_*} A(c_*) = A(c_*)$. □

Moreover, we put $k = \mathbb{F}_q$, with $q = \ell^f$ odd. We prove that $\langle \mathrm{Ind}_U^{\mathrm{GSp}_{2n}(q)} \eta_Y, \rho\rangle = 1$ for cuspidal $\rho \in \mathrm{Irr}(\mathrm{GSp}_{2n}(q))$. Moreover, we set $k_m = \mathbb{F}_{q^m}$ with odd prime degree $[k_m : k] = m$. We have that $\mathrm{char}\, k_m = \ell$. Moreover, we choose the string $L_{2b-1} = \langle e_0, \dots, e_{2b-1}\rangle_k$ with weights $2b-1, 2b-3, \dots, 1-2b$. We have that $\langle e_i, e_{2b-1-i}\rangle = (-1)^i$ and $\langle e_i, e_j\rangle = 0$ for $i + j \neq 2b-1$. Moreover, we choose the symmetric space (W_{2b}, β_{2b}) and $V = \bigoplus_b L_{2b-1} \otimes W_{2b}$. We put $Y(e_i \otimes w) = e_{i+1} \otimes w$ and $Y(e_{2b-1} \otimes w) = 0$. We have that $\langle Yx, y\rangle = -\langle x, Yy\rangle$. Thus, we fix $\lambda : \mathbf{G}_m \longrightarrow \mathrm{Sp}(V)$ with $\lambda(t)(e_i \otimes w) = t^{2b-1-2i} e_i \otimes w$. We have that $\lambda(t) Y \lambda(t)^{-1} = t^{-2} Y$, including strings $2b > \ell$. We choose the positive subgroup $U = \mathbf{U}_\lambda(k)$ and $\mathrm{End}_{\geq r}(V) = \{T : T(V[s]) \subseteq \bigoplus_{t \geq s+r} V[t]\}$. Since all natural weights $2b-1-2i$ are odd, $u - I \in \mathrm{End}_{\geq 2}(V)$ for $u \in U$. We fix $\psi : k \to \mathbb{C}^\times$ with $\psi \neq 1$. We put $\mathcal{C}(u) = 2(u-I)(u+I)^{-1}$ and $\eta_Y(u) = \psi(\frac{1}{2}\mathrm{tr}(Y\mathcal{C}(u)))$. Since $\ell \neq 2$, $u + I$ is invertible. We have that $\mathcal{C}(u) \equiv u - I \pmod{\mathrm{End}_{\geq 4}(V)}$ and $\mathcal{C}(uv) \equiv \mathcal{C}(u) + \mathcal{C}(v) \pmod{\mathrm{End}_{\geq 4}(V)}$. Whenever $T \in \mathrm{End}_{\geq 4}(V)$, $\mathrm{tr}(YT) = 0$. Thus, $\eta_Y(uv) = \eta_Y(u)\eta_Y(v)$ and $\eta_Y(1) = 1$. We choose the model $\mathrm{Ind}_U^{\mathrm{Sp}(V)} \eta_Y$ and $\mathrm{Wh}_Y(\pi) = \mathrm{Hom}_U(\mathbb{C}_{\eta_Y}, V_\pi)$. Given $c \in k^\times$, we put $\psi_c(x) = \psi(cx)$. Thus, $\eta_{cY}^{(\psi)} = \eta_Y^{(\psi_c)}$ for any trace normalization c. Given $a \geq 1, c \in k^\times$, we choose $\mathcal{D}^{\mathrm{FJ}}_{2a,\psi_c} = \mathcal{D}^{\mathrm{FJ}}_{2a,c}$.

Lemma 5.31. *Moreover, we fix $a \geq 1$ and $\epsilon \in k^\times$ for the rational form $\epsilon(k^\times)^2$. We set $Y = (2a, \epsilon) \oplus Y_0$, with even parts $2b < 2a$ for Y_0. Thus, $\dim \mathrm{Wh}_{Y_0}(\mathcal{D}^{\mathrm{FJ}}_{2a,\epsilon}(\pi)) = \dim \mathrm{Wh}_Y(\pi)$ for every ordinary module π of $\mathrm{Sp}(V)$.*

Proof. Since $2b < 2a$, [206, Lemma 5.4] applies for large ℓ. At other odd ℓ, set $V = A \oplus V_0 \oplus A^*$, with $\dim A = a$, and we choose the Gram matrix $J = \begin{pmatrix} 0 & 0 & I_a \\ 0 & J_0 & 0 \\ -I_a & 0 & 0 \end{pmatrix}$, with J_0 on V_0. Moreover, we take weights $u_i = 2a + 1 - 2i$ on A, for $1 \leq i \leq a$. We fix $V_v \leq V_0$, with $|v| \leq 2a - 3$. We put $F_0 = Y_0$, of degree -2. Thus, choose $\eta_A(n) = \psi(\sum_{i=1}^{a-1} n_{i,i+1})$ on the upper group $U_A \leq \mathrm{GL}(A)$. Choose the last coefficient ϵ on the root coordinate Z_{aa}. We fix $B : V_0 \to A$ and $D : A \to V_0$, with companions $-J_0^{-1} B^{\mathrm{t}}$ and $-D^{\mathrm{t}} J_0$. Moreover, set $Z = Z^{\mathrm{t}}$ and the upper block $Z - \frac{1}{2} B J_0^{-1} B^{\mathrm{t}}$. We consider all matrices B, D, Z over k.

We fix C_0 generated by U_A, U_0, all coordinates Z and the entries $B_{i,v}$ with $v < u_i$. Thus, choose $\eta_0|_{U_A} = \eta_A$ and $\eta_0|_{U_0} = \eta_{Y_0}$. We have that $\eta_0(Z) = \psi(\epsilon Z_{aa})$ and $\eta_0(B_{i,v}) = 1$ for $v < u_i$. We put $X_j = \bigoplus_{i=1}^{a-1} V^*_{u_i+j-2}$ and $Y_j = \bigoplus_{i=2}^{a} V_{u_i+j}$ for even $j \geq 2$. Thus, we choose the rows B in X_j and the columns D in Y_j. $u_i + j - 2 > 0$ for $i < a$ and $u_i + j > 0$ for $i \geq 2$. $B J_0^{-1} (B')^{\mathrm{t}} = 0$ and $D^{\mathrm{t}} J_0 D' = 0$ within X_j, Y_j. Thus, X_j, Y_j are additive groups. $u_i - 2 = u_{i+1}$ and $\dim X_j = \dim Y_j$, and we put $H_X = \langle C_0, X_j : j \geq 2\rangle$ and $H_Y = \langle C_0, Y_j : j \geq 2\rangle$. Moreover, choose $\eta_X|_{C_0} = \eta_Y|_{C_0} = \eta_0$ and $\eta_X|_{X_j} = \eta_Y|_{Y_j} = 1$. By the positive coordinates $D_{v,i}$ with $v > u_i$, $H_Y = U$ for Y. Then, we choose H_X for $\mathcal{D}^{\mathrm{FJ}}_{2a,\epsilon}$ followed by Wh_{Y_0}. The groups X_j include all rows B_i with $i < a$. Moreover, we choose the negative entries $B_{a,v}$, with $v < 0$, as a polarization of the oscillator ω_ψ. Put $L = V_{0,>0} = \bigoplus_{v>0} V_v$, and $L = L^\perp$ in V_0 and $U_0 L = L$. Thus, we fix $\mathcal{S}(L)$, the Schrödinger model on L, and the functional $\mathrm{ev}_0(f) = f(0)$. $\det(u|_L) = 1$ for $u \in U_0$, and $\psi(0) = 1$. Whenever $u \in U_0$, $\mathrm{ev}_0(\omega_\psi(u) f) = \mathrm{ev}_0(f)$. Thus, we choose the residual character η_{Y_0} on U_0 in

the model for H_X.

We choose the nonzero pairs (X_j, Y_j) in decreasing order j and put $C_j = \langle C_0, X_s\ (s<j), Y_s\ (s>j)\rangle$. Thus, choose $\eta_j|_{C_0} = \eta_0$, with $\eta_j|_{X_s} = 1$ for $s<j$ and $\eta_j|_{Y_s} = 1$ for $s>j$. Take temporary weights $u_i + j - 1$ on A, the weights v on V_v and $-u_i - j + 1$ on A^*. $\deg(X_j) = \deg(Y_j) = 1$ and $\deg(C_j) \geq 2$. Moreover, $(u-I)V[r] \subseteq \bigoplus_{s>r} V[s]$ for $u \in U_A U_0$ in the original grading $(u_i, v, -u_i)$. Choose the extra entries X_s, with $s<j$, or positive entries B, under the action on X_j. Take entries Y_s, with $s>j$, under the action on Y_j. Thus, upper entries B, B' contribute only to Z. We have that $i<a$ for every row X_j, so the contribution at Z_{aa} is 0. The lower images $D(A)$ are isotropic in V_0. Moreover, choose $[x,y] = xyx^{-1}y^{-1}$. Take diagonal degrees ≥ 4 for $[X_j, Y_s]$ with $s>j$ and $[X_s, Y_j]$ with $s<j$. $[Y_j, B]$ has diagonal degree at least 4 for each positive entry B. Thus, $\eta_j(c) = 1$ on diagonal coordinates c of degree ≥ 4. $[Y_j, Z]$ contributes entries B with source weights $-v$ for $v>0$, and the entries B lie in C_0. Thus, $xC_jx^{-1} = yC_jy^{-1} = C_j$ and $\eta_j(xcx^{-1}) = \eta_j(ycy^{-1}) = \eta_j(c)$ for $x \in X_j, y \in Y_j, c \in C_j$. $X_j \cap C_j = Y_j \cap C_j = 1$ by temporary degrees 1 and ≥ 2. Moreover, $[X_j, Y_j] \leq C_j$.

We fix $b_i \in V^*_{u_i+j-2}$ and $y_i \in V_{u_i+j}$, with $y_1 = 0$. Consequently, in the upper block BD, $c_A(B,D) = \sum_i (BD)_{i,i+1} = \sum_{i=1}^{a-1} b_i(y_{i+1})$. We choose the residual block $J_0^{-1}(DB)^{\mathrm{t}}J_0 - DB$, and $J_0F_0 = -F_0^{\mathrm{t}}J_0$. Thus, $\frac{1}{2}\operatorname{tr}(F_0(J_0^{-1}(DB)^{\mathrm{t}}J_0 - DB)) = -\operatorname{tr}(F_0DB)$. Higher commutators $[x,y]$ contribute off-diagonal entries with character 1, or diagonal degree ≥ 4, and $\eta_j([x(b), y(y)]) = \psi(\sum_{i=1}^{a-1} b_i(y_{i+1} - F_0y_i))$. Moreover, we take arbitrary $z_i \in V_{u_i+j-2}$ and put $y_1 = 0$. Choose $y_{i+1} = z_i + F_0y_i$ for $1 \leq i < a$, and $F_0y_i \in V_{u_i+j-2}$ and $y_{i+1} \in V_{u_{i+1}+j}$. Thus, $\Phi_j : y \mapsto (y_{i+1} - F_0y_i)_{i=1}^{a-1}$ is a bijection $Y_j \xrightarrow{\sim} \bigoplus_{i=1}^{a-1} V_{u_i+j-2}$. $\eta_j([x(b), y(y)])$ is a perfect pairing $X_j \times Y_j \to \mathbb{C}^\times$. We include the full expression $-\operatorname{tr}(F_0DB)$, also for small ℓ.

Moreover, we put $G_j = C_jX_jY_j$ and take the characters η_j^X, η_j^Y on C_jX_j, C_jY_j. $\eta_j^X|_{C_j} = \eta_j^Y|_{C_j} = \eta_j$, with $\eta_j^X|_{X_j} = \eta_j^Y|_{Y_j} = 1$. Since X_j, Y_j occupy distinct degree-one blocks, $G_j/C_j \simeq X_j \oplus Y_j$. The perfect pairing gives $\mathrm{Z}(G_j/\ker\eta_j) = C_j/\ker\eta_j \simeq \mu_\ell$. Thus, $G_j/\ker\eta_j$ is a Heisenberg group over $\mathbb{F}_\ell$, with Lagrangians X_j, Y_j modulo its centre. By [86, Theorem 3.2.1 and Remark 3.2.2], $\operatorname{Ind}_{C_jX_j}^{G_j} \eta_j^X \simeq \operatorname{Ind}_{C_jY_j}^{G_j} \eta_j^Y$. We take induction in stages through j from C_0 to H_X, H_Y. Thus, $\operatorname{Ind}_{H_X}^{\mathrm{Sp}(V)} \eta_X \simeq \operatorname{Ind}_{H_Y}^{\mathrm{Sp}(V)} \eta_Y$. We obtain $\dim \mathrm{Wh}_{Y_0}(\mathcal{D}^{\mathrm{FJ}}_{2a,\epsilon}(\pi)) = \dim \mathrm{Wh}_Y(\pi)$. Alternatively, set $X_j = Y_j = 0$ for every j, with $H_X = H_Y = C_0$. □

Lemma 5.32. *We set $A = \langle a\rangle$, with odd prime order $|A| = m$, acting on finite H with $(m, |H|) = 1$. We fix $U \leq H$ and $\eta \in \mathrm{Irr}(U)$, with $U^a = U$, $\eta^a = \eta$ and $\eta(1) = 1$. Moreover, we choose $\chi \in \mathrm{Irr}(H)$, with $\chi^a = \chi$ and $\langle \operatorname{Ind}_U^H \eta, \chi\rangle = 1$. We have that $\langle \operatorname{Ind}_{U^A}^{H^A}(\eta|_{U^A}), \chi^*\rangle = 1$ for the Glauberman correspondent $\chi^* \in \mathrm{Irr}(H^A)$. Moreover, we set $H = \mathbf{G}(k_m)$ and $A = \langle F_q\rangle$, with $\mathbf{G} \in \{\mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}\}$. We have that the inverse correspondent χ is cuspidal whenever χ^* is cuspidal and $m > |\mathbf{G}(k)| + |W(C_n)|$.*

Proof. First, we put $\widehat{\eta}(ua^i) = \eta(u)$ on $U \rtimes A$. We choose $\widehat{V} = \operatorname{Ind}_{U\rtimes A}^{H\rtimes A} \widehat{\eta}$. Moreover, we fix $c \in H^A$, with $[a,c] = 1$ and $(|a|, |c|) = 1$. We find $\langle ac\rangle = \langle a\rangle \times \langle c\rangle$. Consequently, an ac-fixed coset hU is fixed by a and c. Then, by coprime conjugacy in $U \rtimes A$, we choose $u \in U$ with $hu \in H^A$ for each A-fixed coset hU. a acts by 1 on the line at huU. Thus, $\widehat{V}(ac) = \operatorname{Ind}_{U^A}^{H^A}(\eta|_{U^A})(c)$. By [77, Theorem 3], choose the canonical extension $\widehat{\chi}$. Then, $\widehat{\chi}(ac) = \epsilon_\chi\chi^*(c)$, with $\epsilon_\chi \in \{1, -1\}$. We choose the multiplicity space M_χ in $\widehat{V}$ relative to $\widehat{\chi}$. Thus, choose an ordinary action $A \to \mathrm{GL}(M_\chi)$. $\tau_1 \neq \tau_2$ implies $\tau_1^* \neq \tau_2^*$ for $\tau_1, \tau_2 \in \mathrm{Irr}(H)^A$, and nonfixed orbits τ^A contribute trace 0 at ac. Thus, $\langle \operatorname{Ind}_{U^A}^{H^A}(\eta|_{U^A}), \chi^*\rangle = \epsilon_\chi \operatorname{tr}(a|_{M_\chi})$. $\dim M_\chi = 1$ and $|\operatorname{tr}(a|_{M_\chi})| = 1$, and $\epsilon_\chi \operatorname{tr}(a|_{M_\chi}) \in \mathbb{Z}_{\geq 0}$ so $\langle \operatorname{Ind}_{U^A}^{H^A}(\eta|_{U^A}), \chi^*\rangle = 1$.

By [46, Proposition 4.2.2], we consider cuspidal $\chi^* \in \mathrm{Irr}(\mathbf{G}(k))$. We choose $m > |\mathbf{G}(k)| + |W(C_n)|$ for the trace calculation on aH^A. Then, $q^m \equiv q \pmod m$. Thus, $|\mathbf{G}(k_m)| \equiv |\mathbf{G}(k)| \pmod m$ and $(m, |H|) = 1$. Suppose that χ belongs to a proper series $R_{\mathbf{L}(k_m)}^{\mathbf{G}(k_m)}(\lambda)$, with cuspidal λ. We take standard $\mathbf{L}$ and the orbit $\lambda^{N_H(\mathbf{L})/\mathbf{L}(k_m)}$. Since the orbit is A-stable and its size is $\leq |W(C_n)| < m$, every member λ' satisfies $(\lambda')^a = \lambda'$. We choose the canonical extension $\widehat{\lambda}$ and its composition with $\mathbf{P}(k_m) \to \mathbf{L}(k_m)$. We induce to $H \rtimes A$ and take the fibre trace $\widehat{\lambda}(ac)$. Thus, we choose the signed character $\epsilon_\lambda R_{\mathbf{L}(k)}^{\mathbf{G}(k)}(\lambda^*)$ on aH^A. We put $b = \langle R_{\mathbf{L}(k_m)}^{\mathbf{G}(k_m)}(\lambda), \chi\rangle > 0$, and $b^2 \leq \dim \mathrm{End}_H(R_{\mathbf{L}(k_m)}^{\mathbf{G}(k_m)}(\lambda)) \leq |W(C_n)|$. Thus, $0 < b < m$. Choose ζ_m of order m and the eigenvalue polynomial $P(X) = \sum_{r=0}^{m-1} n_rX^r$ for the action $a|_{M_\chi}$. In particular, $n_r \in \mathbb{Z}_{\geq 0}$ and $P(1) = b < m$. By the minimal polynomial $1 + X + \cdots + X^{m-1}$, $P(\zeta_m) \neq 0$. $\langle R_{\mathbf{L}(k)}^{\mathbf{G}(k)}(\lambda^*), \chi^*\rangle \neq 0$, contrary to cuspidal χ^*. □

Proposition 5.33. *Moreover, we fix $n \geq 1$, odd q and cuspidal $\rho \in \mathrm{Irr}(\mathrm{GSp}_{2n}(q))$. We have that $\langle \mathrm{Ind}_U^{\mathrm{GSp}_{2n}(q)} \eta_Y, \rho \rangle = 1$ for an even partition λ_Y and linear η_Y. Moreover, we take even $d = \mathrm{ord}_p(q)$. We choose the notation Γ_ρ, c_* of* Proposition 5.30. *We have that $T_a T_b = A(a,b) T_{ab}$ and $T_{(f,1,1)} = A(c_*)$ on V_ρ.*

Proof. First, fix $q \geq q_0(n,\ell)$ for every branching formula of rank $\leq n$. Put $\mathbf{Z} = Z(\mathbf{G})$ and $Z = Z(\mathbf{G}^F)$. By [205, Introduction] and [204, Introduction], we choose the group $X_T = (\mathbf{T}^F/Z)^\wedge$ at each regular character choice θ. Then, put $r_T = \dim \mathbf{T} - \dim \mathbf{Z} > 0$ and $\mathcal{B}_T = \bigcup_{1 \neq w \in W_T} \{\theta \in X_T : w\theta = \theta\}$. We take proper subtori and translates S for $\mathcal{B}_T$, with bounded numbers $|\pi_0(S)|$. Thus, $|\mathcal{B}_T| \leq C_T q^{r_T - 1}$ and $|X_T| \geq c_T q^{r_T}$, for constants $C_T, c_T > 0$. Fix $q_0(n,\ell)$ with $c_T q^{r_T} - C_T q^{r_T - 1} > 2|W_T|$ at every rank $\leq n$, and $|X_T \setminus \mathcal{B}_T| > 2|W_T|$ for fixed odd characteristic ℓ.

Moreover, we choose a cuspidal constituent π of $\rho|_{\mathrm{Sp}_{2n}(q)}$. By Proposition 5.29, we choose the notation R_i, A_i, B_i and put $L_i = R_i + A_i + B_i$. Choose a maximizing sign ψ for $\mathcal{D}^{\mathrm{FJ}}_{2L_1,\psi}(\pi)$ at L_1. Then, by [150, Theorem 6.16], we have that $\mathcal{D}^{\mathrm{FJ}}_{2L_1,\psi}(\pi) = \pi_1$ with cuspidal π_1 and multiplicity 1. Take initial classical ranks $t(t+1), s^2$ and $L_1 = R_1 + t + s$. Moreover, choose the remaining ranks $s(s-1), t^2$ on the classical sides $(+1,-1)$. If $s = 0$, the rank is 0. By [205, Theorem 1.1 and Corollary 1.3], we take cuspidal π_1 at the extreme ranks $s(s-1), t^2$. We repeat with cuspidal π_i and indices L_i. Accordingly, the classical losses are $t, t, t-1, t-1, \ldots$ and $s, s-1, s-1, \ldots$. By [150, §6.1 and proof of Theorem 6.16], choose the partitions $\lambda \mapsto (\lambda_2, \lambda_3, \ldots)$ at each step i. Then, take ordinary branching multiplicities $\langle\ ,\ \rangle$ in the finite groups $\mathrm{Sp}_{2n_i}(q)$.

We take staircase partitions $\lambda[a]$ on the unitary factors $C_{[a]}$, and singletons (1) on the linear factors $C_{[a]}$. Thus, $R_i > 0$ implies $R_i > R_{i+1}$. $A_i \geq A_{i+1}$ and $B_i \geq B_{i+1}$. $L_i > L_{i+1}$ whenever $R_i > 0$. Moreover, we put $r = \min\{j \geq 0 : R_{j+1} = 0\}$. By Section 5.2, we fix Γ_r, Taylor's model on π_r, with $\langle \Gamma_r, \pi_r \rangle = 1$. Choose the even parts $2(A_{r+j} + B_{r+j})$ for $j \geq 1$. Moreover, choose the orbit $\mathcal{O}$ for the chosen constituent π_r. Choose a diagonal element g and the model $\mathcal{O}^g$ for π_r^g. Thus, we choose the rational sign ϵ_i at each preceding strict part $2L_i$. By Lemma 5.31, $\langle \mathrm{Ind}_U^{\mathrm{Sp}_{2n}(q)} \eta_Y, \pi \rangle = 1$.

Choose $\rho|_{\mathrm{Sp}_{2n}(q)} = \pi$. Then, by induction in stages for $U \leq \mathrm{Sp}_{2n}(q)$, $\langle \mathrm{Ind}_U^{\mathrm{GSp}_{2n}(q)} \eta_Y, \rho \rangle = 1$. Alternatively, we fix $r \geq 1$ and $\rho|_{\mathrm{Sp}_{2n}(q)} = \pi + \pi^g$. By the argument for $s = 0$ in Proposition 5.30, $s > 0$. Consequently, the partitions are $(\lambda_2, \lambda_3, \ldots)$, with ranks $s(s-1), t^2$ at $2L_1$. By [205, Corollary 1.3(i)], we choose the sign pair (ϵ_+, ϵ_-) for the fixed character ψ. Thus, the orthogonal signs are $\epsilon_-, -\epsilon_-$ for π, π^g. Thus, $\mathcal{D}^{\mathrm{FJ}}_{2L_1,\psi}(\pi^g) = 0$. We find $\dim \mathrm{Hom}_{H_X}(\eta_X, \pi^g) = \dim \mathrm{Hom}_{H_Y}(\eta_Y, \pi^g) = 0$. Alternatively, we set $r = 0$ and Taylor's model $\mathrm{Ind}_U^{\mathrm{GSp}_{2n}(q)} \eta_Y$ on ρ. Thus, $\langle \mathrm{Ind}_U^{\mathrm{GSp}_{2n}(q)} \eta_Y, \rho \rangle = 1$ for $q \geq q_0(n,\ell)$.

We take arbitrary k and an odd prime $m > |\mathrm{GSp}_{2n}(k)| + |W(C_n)|$. We fix $q^m \geq q_0(n,\ell)$. Then, choose ρ_m corresponding to ρ by Lemma 5.32 and cuspidal over k_m. Choose the model $\mathrm{Ind}_{U_m}^{\mathrm{GSp}_{2n}(k_m)} \eta_{Y_m}^{(\widetilde{\psi})}$ of multiplicity 1 on ρ_m. Moreover, put $\psi_m = \psi \circ (m^{-1} \mathrm{Tr}_{k_m/k})$. Fix $c \in k_m^\times$, with $\widetilde{\psi}(x) = \psi_m(cx)$ and $\eta_{Y_m}^{(\widetilde{\psi})} = \eta_{cY_m}^{(\psi_m)}$. The symmetric forms β_{2b} are classified by $\dim W_{2b}$ and $\det \beta_{2b} (k_m^\times)^2$. Since m is odd, we choose the isomorphism $k^\times/(k^\times)^2 \xrightarrow{\sim} k_m^\times/(k_m^\times)^2$. We choose the strings L_{2b-1} and forms β_{2b} over k and fix $g \in \mathrm{Sp}_{2n}(k_m)$ with $g(cY_m)g^{-1} = Y \in \mathfrak{sp}_{2n}(k)$. Moreover, $\langle \mathrm{Ind}_{U(k_m)}^{\mathrm{GSp}_{2n}(k_m)} \eta_{Y,m}, \rho_m \rangle = 1$. Thus, $U(k_m), \eta_{Y,m}$ are stable under $A = \langle F_q \rangle$. $U(k_m)^A = U(k)$ and $\eta_{Y,m}|_{U(k)} = \eta_Y$, since $m^{-1} \mathrm{Tr}_{k_m/k}(x) = x$ for $x \in k$. By Lemma 5.32, $\langle \mathrm{Ind}_{U(k)}^{\mathrm{GSp}_{2n}(k)} \eta_Y, \rho \rangle = 1$.

We choose the induced function space $\mathcal{V}$ over k, and its summand V_ρ. Then, by the formula for $D(t)$, $D(t)|_{V[w]} = t^{(w+1)/2} I$ for $t \in \mathbb{F}_\ell^\times$. Moreover, $\mu(D(t)) = t$ and $\eta_Y(D(t) u D(t)^{-1}) = \eta_Y(u)^t$ for $u \in U$. Thus, choose the coefficient maps S_{σ_a} and multiplier maps E_{δ_b} on $\mathcal{V}$. By the relation for $S_{\sigma_a} E_{\delta_b}$, $\sigma_a(\delta_b)(t_a)^{-1} = A(a,b)$. Take odd f, with $\gamma = 1$ and $c_* = I$. Alternatively, we take even f and the period sign $\kappa \in \{1,-1\}$. By the calculation for κ, γ in Proposition 5.30, we take $\gamma \in \mu_{p-1}$, with $\gamma^f = \kappa^{-1}$. $v_2(\mathrm{ord}_p(\ell)) > v_2(f)$ because d is even. Whenever $\sigma \in \mathcal{H}_p$, $\sigma(\gamma) = \gamma$. Moreover, we put $T_a = E_{\delta_a}(\gamma \mathcal{F}_Y)^{h_a} S_{\sigma_a}$ on V_ρ for $a = (h_a, \sigma_a, \delta_a) \in \Gamma_\rho$. Thus, $T_a T_b = A(a,b) T_{ab}$ and $T_{(f,1,1)} = \gamma^f L_{c_*} A(c_*) = A(c_*)$. □

5.4 Unitary Levi descent

Moreover, we fix negative colour modules V_i with representatives (θ_i, ν_i). We prove that $T_a T_b = A(a,b) T_{ab}$ for maps $T_a : V_i \to V_j$ between colours i, j. We write $b = (h_b, \sigma_b, \delta_b)$ and a rotation j_b for an arrow $b : i \to j$. Thus, $\theta_i^{\ell^{-h_b}\sigma_b}(\delta_b \circ \mu) = \theta_j^{q^{j_b}}$ and $\sigma_b(\nu_i) = \nu_j$. We put $c_i = |\theta_i^{\langle q \rangle}|$ and $c = c_i = c_j$ along $b : i \to j$. We have that $\delta_b \circ \mu$ is fixed by q. Thus, $j_b \in \mathbb{Z}/c\mathbb{Z}$ is unique and $j_{ab} = j_a + j_b \pmod c$. We fix a connected component I_0 and $b_0 \in \{0,1\}$. Since $\mu(-I) = 1$, we have that $\theta_i(-1) = (-1)^{b_0}$ for $i \in I_0$.

Proposition 5.34. *We choose the σ_a-semilinear maps $T_a : V_i \to V_j$ between negative colours i, j, and we have that $T_aT_b = \mathrm{A(a,b)}T_{ab}$ for $b : i \to j, a : j \to k$. Moreover, we consider $\sigma_a = 1$ for the field and multiplier maps a. By* Proposition 5.7*, $T_a = R_a$. We write the tensor space $\mathcal{V} = \bigotimes_i V_i$ and the ordinary permutation maps P_π. Thus, we find $\mathcal{T}_a\mathcal{T}_b = \sigma_a(\prod_i \delta_{b,i})(t_a)^{-1}\mathcal{T}_{ab}$. By* Proposition 5.11*, we have that $\mathcal{T}_a\mathcal{T}_b|_{\mathcal{V}_\epsilon} = \sigma_a(\prod_i \delta_{b,i})(t_a)^{-1}\mathcal{T}_{ab}|_{\mathcal{V}_\epsilon}$ on each stable common-multiplier constituent $\mathcal{V}_\epsilon$.*

Proof. First, we put $w_i = \theta_i(z)$ for $i \in I_0$. Since $\theta_i^{q^c} = \theta_i$, $1 = w_i^{q^c-1} = \varepsilon^{b_0fc}$ and $w_i^{\ell-1} = \varepsilon^{b_0}$. By Proposition 5.7, choose $\xi^c = \nu_i$ and $\eta^f = \xi$. Whenever $\sigma_a \in \mathcal{H}_p$, $\sigma_a(\eta)/\eta = \varepsilon^{b_0k_a}$. Thus, $\sigma_a(\nu_i)/\nu_i = \varepsilon^{b_0k_afc} = 1$. By the formula for ν_i, $\nu_i = \nu_j$ for $i, j \in I_0$. Fix ξ, η throughout I_0 and bases $\{e_{i,t} : t \in \mathbb{Z}/c\mathbb{Z}\}$ of V_i. Moreover, we put $Pe_{i,t} = e_{i,t+1}$ and $D_ae_{i,t} = (\xi/\sigma_a(\xi))^t e_{i,t}$. We write $R_a = \eta^{h_a}P^{-j_a}D_a\sigma_a : V_i \to V_j$, with $\sigma_a(\sum_t x_te_{i,t}) = \sum_t \sigma_a(x_t)e_{j,t}$. Whenever $t \in \mathbb{Z}/c\mathbb{Z}$, we put $\lambda_{i,t} = \theta_i^{q^{-t}}$. Then, by the formula for θ_i, $\lambda_{i,t}^{\ell^{-h_a}\sigma_a}(\delta_a \circ \mu) = \lambda_{j,t-j_a}$. $D_aPD_a^{-1} = (\xi/\sigma_a(\xi))P$ and $(\xi/\sigma_a(\xi))^c = 1$. Thus, we write the rotation matrix R_a on the lines $\mathbb{K}e_{j,t-j_a}$. We put $r(a,b) = (\sigma_a(\eta)/\eta)^{h_b}(\xi/\sigma_a(\xi))^{-j_b}$, and $R_aR_b = r(a,b)R_{ab}$ and $r(a,b) = \varepsilon^{b_0k_a(h_b+fj_b)}$. Since $P^c = I$ and $\varepsilon^{b_0fc} = 1$, $R_a(j_a+c) = R_a(j_a)$ and $r(a,b;j_b+c) = r(a,b;j_b)$.

Moreover, put $C_i(a) = w_i^{-k_aL_a}$ and $T_a = C_i(a)R_a$ for $a : i \to j$. Fix $b : i \to j$ and $a : j \to k$ and put $B(a,b) = C_j(a)\sigma_a(C_i(b))/C_i(ab)$. Thus, $B(a,b) = w_i^{k_aL_aL_b}w_j^{-k_aL_a}$. By the formula at z, $\delta_b(\bar{p}) = w_j^{q^{j_b}}w_i^{-L_b\ell^{-h_b}}$. Then, $A(a,b)/B(a,b) = w_i^{k_aL_aL_b(\ell^{-h_b}-1)}w_j^{-k_aL_a(q^{j_b}-1)}$. By Proposition 5.28, $A(a,b)/B(a,b) = \varepsilon^{b_0k_a(h_b+fj_b)} = r(a,b)$. Consequently, $T_aT_b = A(a,b)T_{ab}$ for every $b : i \to j, a : j \to k$. We consider $\sigma_a = 1$, with $k_a = 0$ and $C_i(a) = 1$. Thus, $T_a = R_a$ for field maps $(h_a, 1, 1)$ and multiplier maps $(0, 1, \delta_a)$.

By Lemma 5.10, we write $\mathcal{T}_a$ with the ordinary permutations P_{π_a}. We write $v = \bigotimes_i v_i \in \mathcal{V}$. Thus, $\mathcal{T}_a\mathcal{T}_b(v) = \prod_i A_i(a,b)\mathcal{T}_{ab}(v)$. We obtain $\prod_i A_i(a,b) = \sigma_a(\prod_i \delta_{b,i})(t_a)^{-1}$. Moreover, we write quadratic involution products J_α from Proposition 5.11. We put $\mathcal{V}_\epsilon = \bigcap_\alpha \ker(J_\alpha - \epsilon_\alpha I)$, with $\epsilon_\alpha \in \{1, -1\}$. We suppose that $\mathcal{T}_a\mathcal{V}_\epsilon = \mathcal{V}_\epsilon$ and $\mathcal{T}_b\mathcal{V}_\epsilon = \mathcal{V}_\epsilon$. Thus, $\mathcal{T}_a\mathcal{T}_b|_{\mathcal{V}_\epsilon} = \sigma_a(\prod_i \delta_{b,i})(t_a)^{-1}\mathcal{T}_{ab}|_{\mathcal{V}_\epsilon}$. □

Moreover, we fix the Sylow d-torus $\mathbf{S}$. Choose an active colour (θ, ν) on a norm-one block of $\mathbf{S}$. Let $x \neq x^{-1}$ be its eigenvalue with $x^{q^e} = x^{-1}$. Thus, inversion on $[x] = x^{\langle q \rangle}$ has order 2. We write $|[x]| = 2c$ and $e \equiv c \pmod{2c}$. We put $D = e/c$, with $2 \nmid D$. Moreover, we fix $[x]$ with multiplicity m and the hermitian space W_x. Over $\mathbb{F}_{q^{2c}}/\mathbb{F}_{q^c}$, we have that $\mathrm{Isom}(W_x) = \mathrm{GU}_m(q^c)$. Since $\mathrm{ord}_p(q^c) = 2D$, we obtain that $\mathrm{ord}_p(-q^c) = D$. We consider every pair $[y] \neq [y^{-1}]$ in the residual space V_{res}.

Lemma 5.35. *Moreover, we fix the unipotent partition λ of $\mathrm{GU}_m(q^c)$. We put $\mathrm{core}_2(\lambda) = \Delta_t = (t, t-1, \ldots, 1)$ and $r = (|\lambda| - |\Delta_t|)/2$ and choose the cuspidal character $\tau = \tau_{[x]} \in \mathrm{Irr}(\mathrm{GL}_{2c}(q))$, with $\tau^\vee = \tau$. Thus, we write $\tau^{\otimes r}$ and the label Δ_t on the symplectic cuspidal core ρ_0. We fix $r > 0$ and the Hecke algebra $\mathcal{H}_{t,r}$ of type B_r. In particular, $q^{c(2t+1)}$ occurs at s_0 and q^{2c} at s_i, for $1 \leq i < r$.*

Proof. By Jordan decomposition in $\mathrm{GSp}_{2n}(q)$, we write the ordinary series $R_L^{\mathrm{GSp}_{2n}(q)}(\tau^{\otimes r} \otimes \rho_0)$. We write $\lambda \rightsquigarrow \mu$ when $\lambda \setminus \mu$ is a rim domino of size 2. Thus, write one Levi step $\mathrm{GL}_{2c}(q)$ for $\lambda \rightsquigarrow \mu$. Write the staircase Δ_t at the cuspidal unitary rank $|\Delta_t|$. Then, by [181, Lemma 2.1 and its proof], we write $\mathrm{quot}_2(\lambda) = (\alpha, \beta)$, with $|\alpha| + |\beta| = r$. Fix the orbit $[x]$ of length $2c$ for each cuspidal block τ. Since $[x] = [x^{-1}]$, $\tau^\vee = \tau$. We write the permutations $\mathfrak{S}_r$ and the dualities C_2^r on $\tau^{\otimes r}$. Thus, write $W(B_r) = C_2^r \rtimes \mathfrak{S}_r$. By [121, §§7.2 and 7.6], we write $n_Q = 2c$ and $E_0 = n_Q(2t+1)/2$. Thus, $E_0 = c(2t+1)$ at the sign change s_0. Fix adjacent equal blocks $\mathrm{GL}_{2c}(q)$ with exponent $E_i = 2c$. Thus, $T_{s_i}^2 = q^{E_i} + (q^{E_i} - 1)T_{s_i}$ for $0 \leq i < r$. □

We consider $\lambda \rightsquigarrow \lambda'$, with $\lambda \setminus \lambda'$ a D-hook. Thus, $|\lambda| - |\lambda'| = D$. Since D is odd, we allow $\mathrm{core}_2(\lambda) \neq \mathrm{core}_2(\lambda')$. We compute (t, r) and (t', r') separately for the two ordinary series λ, λ'.

Lemma 5.36. *Moreover, we fix $m = 2c > 1$ and $H = \mathrm{GL}_m(q)$. We choose an arbitrary cuspidal $\tau \in \mathrm{Irr}(H)$ with $\tau^\vee = \tau$. We choose simultaneous maps F_h, S_σ, C for field, coefficient and duality actions h, σ, ι. Thus, $F_hF_k = F_{h+k}$, $S_\sigma S_\upsilon = S_{\sigma\upsilon}$ and $C^2 = I$. We have that $F_hS_\sigma = S_\sigma F_h$, $F_hC = CF_h$ and $CS_\sigma = S_\sigma C$. Moreover, we choose a symplectic involution n on a similitude Levi $\widetilde{L}$. We write $C \otimes I$ on $V_\tau \otimes V_{\pi_0}$, with identity I on the residual factor V_{π_0}.*

Proof. By [84], choose the regular character ϑ of $\mathbb{F}_{q^m}^\times$ for τ. Since $\tau^\vee = \tau$, $\vartheta^{q^c} = \vartheta^{-1}$ on $\vartheta^{\langle q \rangle}$. $\mathrm{ord}(\vartheta) \mid q^c + 1$ and $q^c + 1 \mid (q^{2c} - 1)/(q - 1)$, and we choose a generator z of $\mathbb{F}_{q^{2c}}^\times$ and $\mathbb{F}_q^\times = \langle z^{(q^{2c}-1)/(q-1)} \rangle$. Thus, $\omega_\tau = \vartheta|_{\mathbb{F}_q^\times} = 1$. Fix the upper unitriangular subgroup $U_H \leq H$ and $\psi(y) = \zeta_\ell^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_\ell}(\sum_{i=1}^{m-1} y_{i,i+1})}$. By [40, §8.4], $\langle \mathrm{Ind}_{U_H}^H \psi, \tau \rangle = 1$ for each

cuspidal τ. Moreover, write the summand $\mathcal{V}_H$ of $\mathrm{Ind}_{U_H}^H \psi$ with trivial scalar action $\omega = 1$. We put $J_{i,m+1-i} = (-1)^{i-1}$ and $J_{ij} = 0$ for $i + j \neq m + 1$. Put $\iota(x) = Jx^{-\mathsf{t}}J^{-1}$ and $x^{-\mathsf{t}} = (x^{-1})^{\mathsf{t}}$. Since $J^{\mathsf{t}} = -J$, $\iota^2 = 1$. $\iota(U_H) = U_H$ and $\iota(y)_{i,i+1} = y_{m-i,m+1-i}$, and $\psi \circ \iota = \psi$. Moreover, we put $D_H(t) = \mathrm{diag}(t^{m-1}, t^{m-2}, \ldots, t, 1)$. $\iota(D_H(t)) = t^{1-m}D_H(t)$ and $(D_H(t)yD_H(t)^{-1})_{i,i+1} = ty_{i,i+1}$, and we write $t_\sigma \in \mathbb{F}_\ell^\times$, with $\sigma(\zeta_\ell) = \zeta_\ell^{t_\sigma}$. Thus, put $[CF](x) = F(\iota(x))$ and $[S_\sigma F](x) = \sigma(F(D_H(t_\sigma)^{-1}x))$. Whenever $h \in \mathbb{Z}/f\mathbb{Z}$, we write $[F_h F](x) = F(x^{\ell^{-h}})$. Whenever $y \in U_H, x \in H$, $S_\sigma F(yx) = \psi(y)S_\sigma F(x)$. Since $D_H(t_\sigma t_\upsilon) = D_H(t_\sigma)D_H(t_\upsilon)$, $S_\sigma S_\upsilon = S_{\sigma\upsilon}$. $J, D_H(t)$ are defined over $\mathbb{F}_\ell$. $F_h C = CF_h$ and $F_h S_\sigma = S_\sigma F_h$. Moreover, $\iota(D_H(t)) = t^{1-m}D_H(t)$ with scalar $t^{1-m}I$. Whenever $F \in \mathcal{V}_H, z \in Z(H)$, $F(zx) = F(x)$. Thus, $CS_\sigma = S_\sigma C$ and $C^2 = I$ on $\mathcal{V}_H$. We write the multiplicity-one spaces $V_\tau \subseteq \mathcal{V}_H$ for colour maps $V_\tau \to V_{\tau'}$, including maps with $\tau' = \tau$.

Moreover, we fix the residual factor $\widetilde{X}_0$ and a representation π_0 of $\widetilde{X}_0$. We write the inclusion $(A, g) \mapsto \mathrm{diag}(A, g, \mu(g)A^{-\mathsf{t}})$ of $\widetilde{L} = H \times \widetilde{X}_0$ and put $n = \left(\begin{smallmatrix} 0 & J \\ -J^{-\mathsf{t}} & 0 \end{smallmatrix}\right)$ on the outer pair $A \oplus A^*$. Thus, we write $n \oplus I$ on $A \oplus V_0 \oplus A^*$, with identity I on V_0. $n^2 = I$ and $n \in X$. On $\widetilde{L}$, $n(A, g)n^{-1} = (\mu(g)\iota(A), g)$. Since $\omega_\tau = 1$, $C \otimes I$ is the operator on $V_\tau \otimes V_{\pi_0}$. We fix disjoint matrices n_i, n_j, with $n_i n_j = n_j n_i$, and $n_i^{\ell^h} = n_i$ and $P_\pi n_i P_\pi^{-1} = n_{\pi(i)}$. □

Proposition 5.37. *Moreover, we write ordinary multiplicity spaces M for the unitary colours λ. We write M' for the partitions λ', with $\lambda \rightsquigarrow \cdots \rightsquigarrow \lambda'$. Then, $T_a^M T_b^M = T_{ab}^M$ and $T_a^{M'} T_b^{M'} = T_{ab}^{M'}$.*

Proof. By Lemma 5.36, write an action $C_2^r \rtimes \mathfrak{S}_r$ on $V_\tau^{\otimes r}$. Moreover, put $G = \mathrm{GSp}_{2n}(q)$. We write $\mathcal{H}_{t,r} = \mathrm{End}_G(R_L^G(\tau^{\otimes r} \otimes \rho_0))$ and its double-coset line $\mathcal{H}_s$, with $\dim \mathcal{H}_s = 1$. Moreover, choose the maps F_h, S_σ, P_π from $\mathcal{H}_s$ to the target line $\mathcal{H}_{s'}$. By Lemma 5.35, we fix $E > 0$ and $T_s^2 = q^E + (q^E - 1)T_s$. Consider $aT_s \neq 0$ with $(aT_s)^2 = q^E + (q^E - 1)aT_s$. Thus, $a^2 q^E = q^E$ and $a^2(q^E - 1) = a(q^E - 1)$. Since $q^E \neq 1$, $a = 1$. By [208, Lemma 4.7], $T_s \mapsto T_{s'}$.

Fix a block of dimension $m = 2c$ and inner symplectic rank r_0. Moreover, write the crossed roots $\mathcal{R}_s$, with $|\mathcal{R}_s| = N$. We count $2mr_0$ roots with an inner coordinate, $\binom{m}{2}$ sums and m long roots. We obtain $N = 2mr_0 + m(m+1)/2 \equiv c \pmod 2$. Thus, $E = c(2t + 1) \equiv N \pmod 2$. We fix an adjacent transposition s_i of two blocks of dimension m, and $N = m^2$ and $E = m = 2c$. Moreover, write the root sum R_s with the lift n_s and intertwiner $C_s^2 = I$. The coefficient $[I]R_s^2 = q^N$ by the inverse formula for R_s, and we write $R_s = a_s T_s$ and $a_s^2 q^E = q^N$. Thus, $a_s \in \{q^{(N-E)/2}, -q^{(N-E)/2}\} \subseteq \mathbb{Q}^\times$. The linear coefficient $q^E - 1 \neq 0$ fixes the sign $a_s/q^{(N-E)/2}$ under $\mathcal{H}_s \to \mathcal{H}_{s'}$.

Then, $\mathcal{H}_{t,r,\mathbb{Q}}$ is split over $\mathbb{Q}$ by [208, Lemma 4.7(iii)]. Given a simple module M, the matrices are $r_M(T_s) \in \mathrm{Mat}_{\dim M}(\mathbb{Q})$. Moreover, we write coefficient action $T_a^M(v) = \sigma_a(v)$ on the rational model M. $T_a^M T_b^M = T_{ab}^M$ for every a, b in the mixed stabilizer. We write the ordinary flip $P_\pi(\bigotimes_i v_i) = \bigotimes_i v_{\pi^{-1}(i)}$ on factors M_i with equal labels $M_i \simeq M_j$. Thus, $P_\pi P_\omega = P_{\pi\omega}$ and $S_\sigma P_\pi = P_\pi S_\sigma$. We fix the two pairs $(t, r), (t', r')$ separately for λ, λ'. $T_a^M T_b^M = T_{ab}^M$ and $T_a^{M'} T_b^{M'} = T_{ab}^{M'}$. □

Lemma 5.38. *Moreover, we fix the ordinary Levis L_1, L_2 with ordered general linear blocks $(A_i; B_j; C_k)$. We consider the orbit pairs $[x_i] \neq [x_i^{-1}]$ on A_i, before the self-dual blocks B_j. We choose corresponding links n_1, n_2 and crossed-root counts N_1, N_2. Thus, $N_1 \equiv N_2 \pmod 2$. We have that $\sigma(q^{N_1/2})/q^{N_1/2} = \sigma(q^{N_2/2})/q^{N_2/2}$ for every coefficient automorphism σ.*

Proof. Moreover, we consider residual orbit pairs $[x_i], [x_i^{-1}]$ for A_i in L_1, L_2. We write multiplicities $m_i^{(s)}$, partitions $\lambda_i^{(s)}$ and central characters $\omega_i^{(s)}$ in L_s, for $s = 1, 2$, and we put $(m_i, \lambda_i, \omega_i) = (m_i^{(1)}, \lambda_i^{(1)}, \omega_i^{(1)}) = (m_i^{(2)}, \lambda_i^{(2)}, \omega_i^{(2)})$. Moreover, we put $\mathcal{A} = \{A_i\}, \mathcal{B} = \{B_j\}, \mathcal{C} = \{C_k\}$. $\dim B_j > 1$, with self-dual colours $\tau_j^\vee = \tau_j$, and $\dim C_k = 1$, with $\theta_k^2 = 1$. Thus, $\mathcal{A}^a = \mathcal{A}$, $\mathcal{B}^a = \mathcal{B}$ and $\mathcal{C}^a = \mathcal{C}$ under field and coefficient operations a. We fix adjacent blocks A_i, A_j of dimensions a, b, and $N_1 = N_2 = ab$ for their transposition (i, j). Moreover, consider an orientation change $A_i \leftrightarrow A_i^*$ with $\dim A_i = a$. We write inner ranks r_1, r_2 and $N_s = 2ar_s + a(a+1)/2$, for $s = 1, 2$, and $N_1 - N_2 = 2a(r_1 - r_2) \in 2\mathbb{Z}$. Alternatively, we consider a crossing $A_i \leftrightarrow B_j$ with $\dim B_j = 2c$. Whenever $s = 1, 2$, $N_s = 2ac \in 2\mathbb{Z}$. $N_1 \equiv N_2 \pmod 2$ for corresponding composite links n_1, n_2. Thus, $q^{N_1/2}/q^{N_2/2} = q^{(N_1-N_2)/2} \in \mathbb{Q}_{>0}$. We fix a coefficient automorphism σ with $\sigma(q^{(N_1-N_2)/2}) = q^{(N_1-N_2)/2}$, and $\sigma(q^{N_1/2})/q^{N_1/2} = \sigma(q^{N_2/2})/q^{N_2/2}$. □

Lemma 5.39. *Moreover, we put $T = \{x \in \mathbb{F}_{q^{2e}}^\times : x^{q^e+1} = 1\}$. We fix an active colour $\theta \in \mathrm{Irr}(T)$ with $\theta^2 \neq 1$ and $|\theta^{\langle q \rangle}| = 2c$. Moreover, we choose an arbitrary extension $\widetilde{\theta} \in \mathrm{Irr}(\widetilde{T} \mid \theta)$. We have that $|\widetilde{\theta}^{\langle q \rangle}| = 2c$. Thus, $I_{\widetilde{\theta}} = \langle \widetilde{T}, v^{2c} \rangle$*

with $I_{\widetilde{\theta}}/\widetilde{T} \simeq C_D$, for $D = e/c$. We choose $\Lambda(\widetilde{\theta}) \in \mathrm{Irr}(I_{\widetilde{\theta}} \mid \widetilde{\theta})$ with $\Lambda(\widetilde{\theta})(v^{2c}) = \widetilde{\theta}(-1)$, and we have that $\Lambda(\widetilde{\theta}^a) = \Lambda(\widetilde{\theta})^a$ for field maps $a = u^h$, coefficient maps $a = \sigma \in \mathcal{H}_p$, multiplier twists $a = \delta \circ \mu$ and rotations $a = v^j$.

Proof. Moreover, we put $\widetilde{\theta}^{q^{2c}}/\widetilde{\theta} = \delta \circ \mu$, with $\delta \in \mathrm{Irr}(\mathbb{F}_q^\times)$. $(\delta \circ \mu)(zI) = 1$ and $\mu(zI) = z^2$, for $z \in \mathbb{F}_q^\times$. Thus, $\delta^2 = 1$. $(\delta \circ \mu)^q = \delta \circ \mu$ and $|\widetilde{\theta}^{\langle q \rangle}| \in \{2c, 4c\}$, and we write $d = 2cD$, with $2 \nmid D$. Since $|\widetilde{\theta}^{\langle q \rangle}| \mid d$, $|\widetilde{\theta}^{\langle q \rangle}| = 2c$. By the model A, write $\widetilde{K} = \langle \widetilde{T}, v \rangle$ and $K = \langle T, v \rangle$. We fix $\chi \in \mathrm{Irr}(\widetilde{K} \mid \widetilde{\theta})$ and its space V_χ. Moreover, we choose the lines $\mathbb{K}e_t$ with distinct weights $\theta^{q^{-t}}$, for $t \in \mathbb{Z}/2c\mathbb{Z}$. $v(\mathbb{K}e_t) = \mathbb{K}e_{t+1}$ and $\chi|_K \in \mathrm{Irr}(K)$. Suppose that $\chi(\delta \circ \mu) = \chi$ for $\delta \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Since $\widetilde{K}/K \simeq \mathbb{F}_q^\times$, $\delta = 1$ by the quotient formulas for $\widetilde{K}/K$.

Moreover, $(v^{2c})^D = -I$ and $s = \widetilde{\theta}(-1) \in \{1, -1\}$. Since $2 \nmid D$, $s^D = s$. We write every inertia value $\zeta = s\kappa$, with $\kappa \in \mu_D$ and $\zeta^D = s$. Thus, $\zeta \in \mu_{2D} \subseteq \mu_d$ and $\sigma(\zeta) = \zeta$, for $\sigma \in \mathcal{H}_p$. $d \mid p - 1$ and $uv = vu$ for the field lift u, and we consider multiplier twists $\delta \circ \mu$ and rotations v^j. Since $\mu(v) = 1$, $(\delta \circ \mu)(v^{2c}) = 1$. We have that $u^{-h}(-I)u^h = -I$, $v^{-j}(-I)v^j = -I$ and $\delta(\mu(-I)) = 1$, and $\Lambda(\widetilde{\theta}^a)(v^{2c}) = \Lambda(\widetilde{\theta})^a(v^{2c}) = s$. Thus, $\Lambda(\widetilde{\theta}^a) = \Lambda(\widetilde{\theta})^a$ on $I_{\widetilde{\theta}^a} = I_{\widetilde{\theta}}$. □

Moreover, we put $\widetilde{\mathbf{G}} = \mathrm{GSp}_{2n}$ over $\mathbb{F}_q$, with $q = \ell^f$ and $2 \nmid q$. We put $\widetilde{G} = \widetilde{\mathbf{G}}(\mathbb{F}_q)$. We fix a split rational Borel $\widetilde{\mathbf{B}}$ and $\widetilde{B} = \widetilde{\mathbf{B}}(\mathbb{F}_q)$. Moreover, we choose a standard Levi $\mathbf{L}$ and $L = \mathbf{L}(\mathbb{F}_q)$. We fix a cuspidal pair (L, V) and $P = LU$. Moreover, we write $r_V : L \to \mathrm{GL}(V)$ with $r_V(lu) = r_V(l)$, for $l \in L, u \in U$. We put $I_P(V) = \{v : \widetilde{G} \to V : v(xg) = r_V(x)v(g),\ x \in P\}$. Moreover, we put $V(x) = r_V(x)$ for $x \in P$. We write the reflection group $W_V = \mathrm{N}_{\widetilde{G}}(L, V)/L$ and a positive system $\mathcal{R}_V^+$. Since $Z(\widetilde{\mathbf{G}}) \simeq \mathbf{G}_m$, W_V is the full relative group $\mathrm{N}_{\widetilde{G}}(L, V)/L$. By Zhang [208, §4.1, (4.3)], we have that $E_s > 0$ in $T_s^2 = q^{E_s} + (q^{E_s} - 1)T_s$.

Lemma 5.40. *Then, we fix $w \in W(C_n)$ with $w\Delta_L = \Delta_{L'}$ and $w\mathcal{R}_V^+ = \mathcal{R}_{{}^nV}^+$. We write the positive systems $\Delta_L, \Delta_{L'}$ from the ambient chamber $\widetilde{B}$. Moreover, we choose a lift $n \in \widetilde{G}$ of w and $P' = L'U'$. We put $J_n : I_P(V) \to I_{P'}({}^nV)$, with $[J_n v](g) = q^{l(w)/2}|U'|^{-1} \sum_{y \in U'} v(n^{-1}yg)$. We write the absolute Weyl length $l(w)$ of $w\Delta_L = \Delta_{L'}$. Thus, $J_{n_s} \cdots J_{n_1} = J_{n_s \cdots n_1}$ on the naturally conjugated sources ${}^{n_j \cdots n_1}V$. We write $n' = hn$, with $h \in L'$, and we have that $J_{hn} = r_{{}^nV}(h^{-1})J_n$. Moreover, we fix prime-field isometries n and coefficient automorphisms σ. We have that $\mathcal{S}_\sigma J_n = \sigma(q^{l(w)/2})q^{-l(w)/2} J_n \mathcal{S}_\sigma$. We consider field maps ϕ^h on U' and multiplier twists $\delta \circ \mu$, with $\delta(\mu(n)) = 1$. Moreover, $\mathcal{F}_h J_n = J_{\phi^h(n)} \mathcal{F}_h$ and $M_\delta J_n = J_n M_\delta$.*

Proof. First, we put $[R(x)v](g) = v(gx)$ for $x, g \in \widetilde{G}$. $J_n R(x) = R(x) J_n$ and $r_{{}^nV}(l') = r_V(n^{-1}l'n)$, for $l' \in L'$. Since $l'U'(l')^{-1} = U'$, $[J_n v](l'u'g) = r_{{}^nV}(l')[J_n v](g)$ for $l' \in L', u' \in U'$. We put $K_n = q^{-l(w)/2} J_n$ and $\mathcal{I}(w) = \{\alpha \in \Phi^+ : w^{-1}\alpha \in \Phi^-\}$ and write the ordered root product $U(w) = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha$, with $|U(w)| = q^{l(w)}$. Moreover, we put $U_c = U' \cap nUn^{-1}$. $U' = U_c U(w)$ and $|U'| = |U_c| q^{l(w)}$. Thus, $[K_n v](g) = q^{-l(w)} \sum_{y \in U(w)} v(n^{-1}yg)$. We choose $w = w_s \cdots w_1$ with $l(w) = \sum_j l(w_j)$. By [40, Propositions 10.2.2 and 10.7.2], we choose elementary parabolic changes w_j, with $b_j = l(w_j)$. Suppose that $l(w_2 w_1) = l(w_2) + l(w_1)$. $(y_1, y_2) \mapsto n_2 y_1 n_2^{-1} y_2$ is a bijection $U(w_1) \times U(w_2) \to U(w_2 w_1)$. Thus, $K_{n_2} K_{n_1} = K_{n_2 n_1}$ and $J_{n_2} J_{n_1} = J_{n_2 n_1}$. We find $q^{l(w_2)/2} q^{l(w_1)/2} = q^{l(w_2 w_1)/2}$. Thus, $J_{\mathbf{s}} = J_{\mathbf{t}}$ for braid paths $\mathbf{s}, \mathbf{t}$ with $n_{\mathbf{s}} = n_{\mathbf{t}}$.

Moreover, we choose a shortest path $\mathcal{C}_0, \ldots, \mathcal{C}_s$ for w. Put $w_{<j} = w_{j-1} \cdots w_1$ and take a current relative root β_j. We suppose that its wall $\ker \beta_j$ occurs at step j. Thus, $\alpha = w_{<j}^{-1} \beta_j \in \mathcal{R}_V^+$. $w\alpha < 0$ by $l(w) = l(w_s \cdots w_{j+1}) + l(w_j w_{<j})$, and $w\alpha > 0$ by $w\mathcal{R}_V^+ = \mathcal{R}_{{}^nV}^+$. Thus, $\ker \beta_j$ is absent from the shortest path $\mathcal{C}_0, \ldots, \mathcal{C}_s$. We write the split-centre coordinates $x_1, \ldots, x_t$ of $\mathbf{L}$, modulo $Z(\widetilde{\mathbf{G}})$. We consider the walls $x_i = x_j, x_i = -x_j, x_i = 0$ in the restricted arrangement $\mathcal{H}$. Moreover, we choose a chamber $\mathcal{C}$ with $0 < |x_{\pi(1)}| < \cdots < |x_{\pi(t)}|$ and $\mathrm{sgn}(x_i) = \epsilon_i$. We consider one crossing $A_i \leftrightarrow A_j$ or $A_i \leftrightarrow A_i^*$ at each wall H_j, and $H_i \neq H_j$ for $i \neq j$ along a shortest path $\mathcal{C}_0, \ldots, \mathcal{C}_s$. Thus, write b_j for the positive root count at H_j, with $l(w) = \sum_j b_j$. We have that $b_j = (\dim A_i)(\dim A_k)$ for $A_i \leftrightarrow A_k$, including $\dim A_i \neq \dim A_k$ and that $b_j = 2(\dim A_i) r_0 + (\dim A_i)(\dim A_i + 1)/2$ for $A_i \leftrightarrow A_i^*$ across inner rank r_0.

Moreover, we put $\mathcal{H}_{\mathrm{rel}} = \{\ker \alpha : \alpha \in \mathcal{R}_V^+\}$. By [40, Propositions 10.5.1, 10.7.3], we choose an elementary lift n at $H_j \notin \mathcal{H}_{\mathrm{rel}}$. $K_{n^{-1}} K_n = q^{-b_j} I$ on the corresponding cuspidal pair (L, V). Then, by [208, proof of Proposition 4.16, Step 1], $J_{n^{-1}} J_n = q^{b_j} K_{n^{-1}} K_n = I$. Fix two paths $\mathcal{P}_1, \mathcal{P}_2$ with positive relative systems $\mathcal{R}_{V_j}^+$. Then, we consider the concatenation $\mathcal{P}_2 * \mathcal{P}_1$ in one relative chamber $\mathcal{C}_{\mathrm{rel}}$. Thus, choose rank 2 braid steps $\mathbf{s} \leftrightarrow \mathbf{t}$ and adjacent inverse crossings (n, n^{-1}). The wall sets $\mathcal{H}(\mathbf{s}) = \mathcal{H}(\mathbf{t})$ exclude $\mathcal{H}_{\mathrm{rel}}$. $J_{\mathbf{s}} = J_{\mathbf{t}}$ and $J_{n^{-1}} J_n = I$. Thus, $J_{n_s} \cdots J_{n_1} = J_{n_s \cdots n_1}$ for compatible lifts $n_1, \ldots, n_s$.

We write $n' = hn$, with $h \in L'$, and $v(n^{-1}h^{-1}yg) = r_{{}^nV}(h^{-1})v(n^{-1}yg)$. Thus, we obtain $J_{hn} = r_{{}^nV}(h^{-1})J_n$. We consider a repeated reflection lift n with $n^2 = h \in L$, and $J_n J_n = J_h = r_V(h^{-1})$ on the naturally conjugated source

${}^nV$. Moreover, we fix linear source matrices $S_1,\dots,S_s$ after the links $J_{n_1},\dots,J_{n_s}$. We find $(S_sJ_{n_s})\cdots(S_1J_{n_1}) = S_s\cdots S_1J_{n_s\cdots n_1}$. We suppose that $S_s\cdots S_1 = cI$ and obtain that $(S_sJ_{n_s})\cdots(S_1J_{n_1}) = cJ_{n_s\cdots n_1}$.

Moreover, put $[\mathcal{S}_\sigma v](g) = \sigma(v(g))$ and $[\mathcal{F}_h v](g) = v(\phi^{-h}(g))$. Write the bijection $\phi^h : U' \to \phi^h(U')$ and $|\phi^h(U')| = |U'|$, and $\mathcal{F}_hJ_n = J_{\phi^h(n)}\mathcal{F}_h$ on the field-conjugate parabolics $\phi^h(P), \phi^h(P')$. Since $n \in \mathrm{Sp}_{2n}(\mathbb{F}_\ell)$, $\phi^h(n) = n$ and $\mu(n) = 1$. Whenever $\delta \in \mathrm{Irr}(\mathbb{F}_q^\times)$, we put $[M_\delta v](g) = \delta(\mu(g))v(g)$. $\mu(y) = 1$ for $y \in U'$, and $\delta(\mu(n)) = 1$. Thus, $M_\delta J_n = J_nM_\delta$. $\sigma(|U'|^{-1}) = |U'|^{-1}$ and $\mathcal{S}_\sigma K_n = K_n\mathcal{S}_\sigma$, and $\mathcal{S}_\sigma J_n = \sigma(q^{l(w)/2})q^{-l(w)/2}J_n\mathcal{S}_\sigma$. □

Moreover, we fix a mixed operation a on the cuspidal pairs (L,V). We write the returning coset $\mathcal{C}(a) = W_{V'}w_a$ and the positive systems $\mathcal{R}^+_{V^a}, \mathcal{R}^+_{V'}$. Moreover, we put $\mathcal{C}_{V'} = \{w\mathcal{R}^+_{V'} : w \in W_{V'}\}$. We have that $W_{V'}$ acts simply transitively on $\mathcal{C}_{V'}$. Thus, we choose the unique $c(a) \in \mathcal{C}(a)$ with $c(a)\mathcal{R}^+_{V^a} = \mathcal{R}^+_{V'}$. We obtain that $c(a)c(b) = c(ab)$ for composable operations a, b. Thus, we choose one block word $(n_1,\dots,n_s)$ at the cuspidal labels V_j, V'_j of both calculations. We consider $\mathcal{H}_s, \mathcal{H}_{s'}$ for simple reflections s, s', with $s' = c(a)sc(a)^{-1}$. We have that $J_{c(a)}T_sJ^{-1}_{c(a)} \in \mathcal{H}_{s'}$ by the adjacent-wall path $(c(a), s, c(a)^{-1})$. Thus, $\dim \mathcal{H}_s = \dim \mathcal{H}_{s'} = 1$ and $J_{c(a)}T_sJ^{-1}_{c(a)} = a_sT_{s'}$. We have that $T_s^2 = q^E + (q^E-1)T_s$, with $E > 0$ and $q^E \in \mathbb{Q}$. By Zhang [208, Lemma 4.7], we obtain that $a_s = 1$ under group and coefficient actions a.

Lemma 5.41. *Moreover, we fix a cuspidal character π of the similitude group G_0. With $\epsilon = \varepsilon_q$, we put $s = \epsilon(-1)$ and $\xi^2 = s$. Moreover, we write $\pi_\epsilon = \pi \otimes (\epsilon\circ\mu)$ for the quadratic coordinate $\theta = \epsilon$. We fix the outer reflection j. Thus, we find $(\epsilon\otimes\pi)^j = \epsilon\otimes\pi_\epsilon$. We suppose that $\pi_\epsilon = \pi$ and choose E, with $E^2 = I$. Whenever $g \in G_0$, $Er_\pi(g)E^{-1} = \epsilon(\mu(g))r_\pi(g)$. Moreover, we consider odd d and mixed maps $S_a : V_\pi \to V_{\pi^a}$. We choose the signs E_π with $S_a(\xi E_\pi)S_a^{-1} = \xi E_{\pi^a}$. By* Proposition 5.33*, we consider even d. Moreover, we include both cases $d \equiv 0, 1 \pmod 2$. We have that $A^\chi_aA^\chi_b = \mathrm{A}(\mathrm{a},\mathrm{b})A^\chi_{ab}$ on each ordinary Harish–Chandra character χ, including $\pi_\epsilon \neq \pi$.*

Proof. First, choose a basis (e,f) on the outer symplectic plane H. Put $j(e) = -f$, $j(f) = e$ and $j|_{H^\perp} = I$. Thus, $j^2|_H = -I$ and $j(x,g)j^{-1} = (\mu(g)x^{-1}, g)$ on $\mathrm{GL}_1(q)\times G_0$. $\epsilon(-1) = s$ is the scalar at j^2. Moreover, we write ϕ for the field map and S_a for the source map on V. Define $(\widehat{S}_av)(x) = \delta_a(\mu(x))S_a(v(\phi^{-h_a}(x)))$ before the parabolic change $P \to P'$. Moreover, put $r_{V^a}(l) = \delta_a(\mu(l))S_ar_V(\phi^{-h_a}(l))S_a^{-1}$ for $l \in L$. $(\widehat{S}_av)(lx) = r_{V^a}(l)(\widehat{S}_av)(x)$, and $\mu(u) = 1$ for $u \in U$, and $\widehat{S}_av \in I_{P^a}(V^a)$.

Moreover, suppose that $\pi_\epsilon = \pi$ and fix the root-sum operator $\mathcal{I}$. Write its kernel f_j with $f_j(j) = E$. We fix the crossed radical U_j with $|U_j| = q^N$, and $E^2 = I$ and $[1]\mathcal{I}^2 = |U_j|\epsilon(-1) = sq^N$. Thus, we choose the normalized generator T with $T^2 = q^{e_H} + (q^{e_H}-1)T$, for $e_H > 0$. $\mathcal{I} = aT$ and $a^2 = sq^{N-e_H}$, and we choose the sign E with $\mathcal{I} = \xi q^{(N-e_H)/2}T$. Accordingly, by Lemma 5.40, $T \mapsto T'$ on the target double-coset line $\mathcal{H}_{j'}$. We write $S_aE_\pi S_a^{-1} = c_aE_{\pi^a}$ and $\widehat{S}_a\mathcal{I}_\pi\widehat{S}_a^{-1} = c_a\mathcal{I}_{\pi^a}$, and $c_a = \sigma_a(\xi q^{(N-e_H)/2})/(\xi q^{(N-e_H)/2})$. Moreover, consider odd $d = \mathrm{ord}_p(q)$. $(q/p) = 1$ and $\sigma(\sqrt{q}) = \sqrt{q}$, for $\sigma \in \mathcal{H}_p$. We include square q, with $\sqrt{q} \in \mathbb{Q}$. Thus, $c_a = \sigma_a(\xi)/\xi \in \{1,-1\}$. $\sigma_a(\xi)c_a = \xi$ and $S_a(\xi E_\pi)S_a^{-1} = \xi E_{\pi^a}$. We allow rescaling $S'_a = u_aS_a$, with $u_a \in \mathbb{K}^\times$. Thus, $S'_aE_\pi(S'_a)^{-1} = S_aE_\pi S_a^{-1}$.

If d is even, we choose the model $\mathcal{V}$ common to all $\pi\otimes(\delta\circ\mu)$. By Lemma 5.17, we include rank 0 with $\mathcal{V}_0 = \{F : \mathbb{F}_q^\times \to \mathbb{K}\}$. We put $[E_\epsilon F](x) = \epsilon(\mu(x))F(x)$ on $\mathcal{V}$, and $[E_\epsilon F](z) = \epsilon(z)F(z)$ on $\mathcal{V}_0$. Then, by the formula for E_ϵ, $S_aE_\epsilon S_a^{-1} = \epsilon(t_a)^{-1}E_\epsilon$. Thus, we write $\delta_{ab}(z) = \delta_a(z)\sigma_a(\delta_b(z^{\ell^{-h_a}}))$ for $z \in \mathbb{F}_q^\times$. $S_aS_b = A(a,b)S_{ab}$ and $\widehat{S}_a\widehat{S}_b = A(a,b)\widehat{S}_{ab}$.

We suppose that $\pi_\epsilon = \pi$ and write the relative groups $W(B_{r_+}), W(B_{r_-})$. By [208, Lemma 4.7(iii)], choose rational models M of the corresponding Hecke algebras $\mathcal{H}_+, \mathcal{H}_-$. The normalized generators T_s satisfy $T_s \mapsto T_{s'}$ under the mixed maps a. Moreover, we put $B_a = C_{\sigma_a}P_{\omega_a}$ on the rational factors M. We write $[C_\sigma v]_i = \sigma(v_i)$ and $P_\omega(\bigotimes_i v_i) = \bigotimes_i v_{\omega^{-1}(i)}$. Thus, $B_aB_b = B_{ab}$. Put $\widehat{S}_a = A^\chi_a \otimes B_a$ on the isotypic space $V_\chi \otimes M$. By Lemma 5.17, $A^\chi_aA^\chi_b = A(a,b)A^\chi_{ab}$.

Alternatively, we suppose that $\pi_\epsilon \neq \pi$ and fix the number r_- of quadratic coordinates. On $C_2^{r_-} \rtimes \mathfrak{S}_{r_-}$, we put $\eta((u_1,\dots,u_{r_-}),\omega) = \prod_i u_i$. $W(D_{r_-}) = \ker\eta$ because $\eta = -1$ has source action $\pi \mapsto \pi_\epsilon$. Moreover, we consider $r_- > 0$ and the sources $V_0 = V, V_1 = {}^jV$. Whenever $i = 0,1$, put $\mathcal{I}_i = I_P(V_i)$. Moreover, write $J_{10} : \mathcal{I}_0 \to \mathcal{I}_1$ and $J_{01} : \mathcal{I}_1 \to \mathcal{I}_0$ for the last reflection j. We include E_ϵ and the divisor $q^{N/2}$ in J_{10}, J_{01}. Thus, $N = 2m+1$ for inner symplectic rank m. By Lemma 5.40, $J_{01}J_{10} = J_{10}J_{01} = sI$ from $j^2|_H = -I$ and $E_\epsilon^2 = I$. Write the coefficient ratio c_a of J_{10}, J_{01}. Then, $c_a = \epsilon(t_a)^{-1}(\sqrt{q}/\sigma_a(\sqrt{q}))^N$. By Lemma 5.9, $G_q^2 = sq$ and $\sigma_a(G_q)/G_q = \epsilon(t_a)$. Thus, $\sigma_a(\xi)/\xi \in \{1,-1\}$ and $\sigma_a(\sqrt{q})/\sqrt{q} \in \{1,-1\}$. Since N is odd, $c_a = \sigma_a(\xi)/\xi$. We put $J'_{10} = \xi^{-1}J_{10}$ and $J'_{01} = \xi^{-1}J_{01}$, and $J'_{01}J'_{10} = J'_{10}J'_{01} = I$. Moreover, we fix the stabilizer $\Gamma_{\{\pi,\pi_\epsilon\}}$ of the unordered pair $\{\pi,\pi_\epsilon\}$. We write $j(a) \in \{0,1\}$

for its permutation $i \mapsto i + j(a) \pmod 2$. Thus, $j(ab) = j(a) + j(b) \pmod 2$. Thus, $\widehat{S}_a J'_{ij} \widehat{S}_a^{-1} = J'_{a(i),a(j)}$.

With $J'_{00} = J'_{11} = I$, we put $R_a = J'_{0,j(a)} \widehat{S}_a$. Then, $R_a R_b = A(a,b) R_{ab}$ on the returned induced space $\mathcal{I}_0$. By Section 5.2, write φ for the fork action on $\mathcal{H}(D_{r_-})$. We put $\mathcal{H}_i = \mathrm{End}_{\widetilde{G}}(\mathcal{I}_i)$ and $\mathcal{H}_i \simeq \mathcal{H}(D_{r_-})$. Consequently, $\varphi : \mathcal{H}_1 \to \mathcal{H}_0$ is $h \mapsto J'_{01} h J'_{10}$. Moreover, choose rational tableau maps B_a, with $B_a B_b = B_{ab}$, including the two degenerate labels M^+, M^-. Fix $\Gamma_\chi = \{a : \chi^a = \chi\}$ before the factorization $R_a = A_a^\chi \otimes B_a$. Whenever $a, b \in \Gamma_\chi$, $A_a^\chi A_b^\chi = A(a,b) A_{ab}^\chi$. We include $r_- = 0$, with no J_{10}, J_{01}, φ. Moreover, we fix a field period $a = (h, 1, 1)$ with $\phi^h = \mathrm{Int}(z_a)$, $z_a \in L$ and $\mu(z_a) = 1$. $j(a) = 0$ and $S_a = r_V(z_a)$, and $(\widehat{S}_a v)(x) = r_V(z_a) v(z_a^{-1} x z_a) = v(x z_a)$. □

Lemma 5.42. *Moreover, we fix an ordered symbol $\Lambda = (B_0, B_1)$. We consider negative period $d = 2e$ and its ordered core $\Lambda_0 = (B_0^0, B_1^0)$ and write $\lambda_{j,t}$, for $0 \le j < e, t \in \{0,1\}$, for the $2e$ quotient partitions of Λ. Thus, $(B_0, B_1) \mapsto (B_1, B_0)$ has action $(\Lambda_0, \lambda_{j,t}) \mapsto (\Lambda_0^{\mathrm{row}}, \lambda_{j,1-t})$. We have that $\lambda_{j,t} \mapsto \lambda_{j,1-t}$ is the quadratic twist $\chi_s \mapsto \chi_{s+e}$ on C_{2e}. Alternatively, we consider positive odd d and the d-hook core Λ_0. We write the core exchange $\Lambda_0 \mapsto \Lambda_0^{\mathrm{row}}$ and the orientation sign $\chi_{j,t} \mapsto \chi_{j,1-t}$. Moreover, we include fixed labels $\Lambda = \Lambda^{\mathrm{row}}$ and the index 2 sign kernel $\ker \varepsilon$. We write the constituents V^+, V^-. Moreover, we fix g with odd diagram action $\varepsilon(g) = -1$. We have that $gV^+ = V^-$ and $gV^- = V^+$.*

Proof. First, we fix finite beta-sets B_0, B_1 and negative period $d = 2e$. We put $C_{j,t} = \{k \ge 0 : j + ek \in B_{t+k \bmod 2}\}$, with $0 \le j < e, t \in \{0,1\}$, and we consider a cohook $(j + ek, r) \mapsto (j + e(k-1), 1 - r)$, with $k \ge 1$ and $r = t + k \pmod 2$. Thus, $C_{j,t} \mapsto (C_{j,t} \setminus \{k\}) \cup \{k-1\}$ when $k - 1 \notin C_{j,t}$. We write $C_{j,t} = \{k_1 < \cdots < k_m\}$ and $\lambda_{j,t} = (k_m - m + 1, \ldots, k_2 - 1, k_1)$ and put $C_{j,t}^0 = \{0, \ldots, m-1\}$ for the packed runner $C_{j,t}$. Moreover, $k_i = \lambda_{j,t,m+1-i} + i - 1$ for $1 \le i \le m$. Write the inverse construction $(C_{j,t}^0, \lambda_{j,t}) \mapsto C_{j,t}$, and $\mathrm{rk}(\Lambda) - \mathrm{rk}(\Lambda_0) = e \sum_{j,t} |\lambda_{j,t}|$. Thus, we write the block weight $w = \sum_{j,t} |\lambda_{j,t}|$. Put $\delta(\Lambda) = |B_0| - |B_1|$ and $\delta(\Lambda') = \delta(\Lambda) \pm 2$ for one cohook $\Lambda \to \Lambda'$, and $\delta(\Lambda') = \delta(\Lambda) + 2 \pmod 4$ when $\delta(\Lambda) \in 2\mathbb{Z}$. Thus, write the orthogonal types $\varepsilon(\Lambda), \varepsilon(\Lambda')$, with $\varepsilon(\Lambda') = -\varepsilon(\Lambda)$ and $\varepsilon(\Lambda_0) = (-1)^w \varepsilon(\Lambda)$.

Write the runner index $s = j + et \pmod{2e}$, and $(B_0, B_1) \mapsto (B_1, B_0)$ has runner action $s \mapsto s + e$. Moreover, we choose z generating C_{2e} and $\chi_s(z) = \zeta_{2e}^s$. Whenever $\varepsilon_b(z) = -1$, $\chi_{s+e} = \chi_s \varepsilon_b$. We consider the common shift $B'_r = \{0\} \cup (B_r + 1)$, for $r \in \{0,1\}$. Thus, $C'_{j+1,t} = C_{j,t}$ for $j < e - 1$. At $j = e - 1$, $C'_{0,1-t} = \{0\} \cup \{k + 1 : k \in C_{e-1,t}\}$. $\lambda'_{j+1,t} = \lambda_{j,t}$ and $\lambda'_{0,1-t} = \lambda_{e-1,t}$. Thus, $s' = s + 1 \pmod{2e}$. We choose a common origin $\lfloor (|B_0| + |B_1|)/2 \rfloor \equiv 0 \pmod{2e}$. Moreover, we write the shift counts n_a, n_b for two choices $B_r^{(a)}, B_r^{(b)}$. $n_a - n_b \in 2e\mathbb{Z}$. Moreover, we consider positive odd d and $C_{j,t} = \{k \ge 0 : j + dk \in B_t\}$, for $0 \le j < d$. We write the hook step $(j + dk, t) \mapsto (j + d(k-1), t)$, with $k \ge 1$ and $j + d(k-1) \notin B_t$, and $\mathrm{rk}(\Lambda) - \mathrm{rk}(\Lambda_0) = dw$, with $w = \sum_{j,t} |\lambda_{j,t}|$. $(B_0, B_1) \mapsto (B_1, B_0)$ has quotient action $\lambda_{j,0} \leftrightarrow \lambda_{j,1}$ at each j. Thus, $B_0 = B_1$ if and only if $B_0^0 = B_1^0$ and $\lambda_{j,0} = \lambda_{j,1}$ for every j. We verify both directions through $k_i = \lambda_{j,t,m+1-i} + i - 1$.

We put $W_{\mathrm{loc}} = C_{2e} \wr \mathfrak{S}_w$ in the negative case $d = 2e$. Alternatively, we put $W_{\mathrm{loc}} = (C_d \times C_2) \wr \mathfrak{S}_w$ for positive odd d. We write the induced tensor model M_{loc} with Specht factors $S^{\lambda_{j,t}}$, and $\chi_s \mapsto \chi_s \varepsilon_b$ permutes the factors $S^{\lambda_{j,t}}$. Moreover, we write $\chi_{j,t}(z,u) = \zeta_d^j u^t$ in the positive case $(z,u) \in C_d \times C_2$. Whenever $\varepsilon_b(z,u) = u$, $\chi_{j,t} \varepsilon_b = \chi_{j,1-t}$. Suppose that the quotient label $(\lambda_{j,t})$ is fixed by $t \mapsto 1 - t$. Thus, choose the tensor flip E_b on M_{loc}, with $E_b^2 = I$. We write $K_b = \mathbb{Q}(\mu_{2e})$ for $d = 2e$ and $K_b = \mathbb{Q}(\mu_{2d})$ for odd d. Moreover, E_b is defined over K_b. $d \mid p - 1$ and $\mathbb{Q}(\mu_{2d}) = \mathbb{Q}(\mu_d)$ for odd d. Consequently, $\sigma|_{K_b} = 1$, for $\sigma \in \mathcal{H}_p$. By Section 5.2, choose E_r on a fixed residual parent V_r, with $E_r^2 = I$. Then, put $E = E_r \otimes E_b$ on $V = V_r \otimes M_{\mathrm{loc}}$. $E r_V(g) E^{-1} = \varepsilon_r(g) \varepsilon_b(g) r_V(g)$ for the total reflection sign $\varepsilon = \varepsilon_r \varepsilon_b$. Thus, we put $V^+ = \ker(E - I)$ and $V^- = \ker(E + I)$. When $\varepsilon(g) = -1$, $r_V(g) E r_V(g)^{-1} = -E$. $gV^+ = V^-$ and $gV^- = V^+$. Moreover, write $J \subseteq \{r, b\}$ for the present factors $V_r, V_b = M_{\mathrm{loc}}$. Whenever $i \notin J$, we put $V_i = \mathbb{K}, E_i = I$. We include $J = \varnothing$, with $\varepsilon = 1, E = I, V^- = 0$.

Moreover, fix the parent algebra $\mathcal{A}$, with $\mathcal{A} \simeq \bigoplus_\chi \mathrm{Mat}_{n_\chi}(\mathbb{K})$ and sign automorphism α. Consider two simple blocks $\mathcal{A}_1, \mathcal{A}_2$ with $\alpha(\mathcal{A}_1) = \mathcal{A}_2$. Thus, $(\mathcal{A}_1 \oplus \mathcal{A}_2)^\alpha \simeq \{(X, X) : X \in \mathrm{Mat}_m(\mathbb{K})\}$. $\mathbb{K}^m$ is simple over $(\mathcal{A}_1 \oplus \mathcal{A}_2)^\alpha$. Alternatively, fix $\mathcal{A}_0 = \mathrm{End}(V)$ with $\alpha|_{\mathcal{A}_0} = \mathrm{Ad}(E)$ and $\varepsilon \ne 1$. We find $\mathcal{A}_0^\alpha = \mathrm{End}(V^+) \oplus \mathrm{End}(V^-)$. Thus, V^+, V^- are the two constituents, with $V^+ \ne 0$ and $V^- \ne 0$ by the invertible odd generator g. We allow projective tensor factors U with $T_a^U T_b^U = c_U(a,b) T_{ab}^U$. We find $(T_a \otimes T_a^U)(T_b \otimes T_b^U) = c(a,b) c_U(a,b)(T_{ab} \otimes T_{ab}^U)$. Moreover, write $T_a^{(i)} E_i (T_a^{(i)})^{-1} = \eta_i(a) E_i$ on the two factors $i = 1, 2$. We put $\mathcal{T}_a = T_a^{(1)} \otimes T_a^{(2)}$, and $\mathcal{T}_a (E_1 \otimes E_2) \mathcal{T}_a^{-1} = \eta_1(a) \eta_2(a) (E_1 \otimes E_2)$. Moreover, we fix the stabilizer Γ_χ of the constituent χ before restriction to V^+ or V^-. Thus, $T_a T_b|_{V^\pm} = c(a,b) T_{ab}|_{V^\pm}$ for $a, b \in \Gamma_\chi$. We include a nonfixed parent $\mathcal{A}_1$ with $\alpha(\mathcal{A}_1) = \mathcal{A}_2$. Moreover, write the parent characters χ_1, χ_2 for $\mathcal{A}_1, \mathcal{A}_2$. $\chi_1|_{\ker \varepsilon} = \chi_2|_{\ker \varepsilon} = \chi$ through $\mathcal{A}_1 \simeq \mathcal{A}_2$. Consequently, we include every mixed operation $a \in \Gamma_\chi$, also with a coefficient map σ_a followed by an odd diagram action g. □

We write $\mathbf{G} = \mathrm{Sp}_{2n}$ and $X = \mathbf{G}^F$, with $F = F_q$. Under the regular inclusion $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$, we write the dual map $\pi^* : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$. Moreover, we choose dual Sylow d-tori $\mathbf{S}, \mathbf{S}^*$. We put $\mathbf{C} = C_{\widetilde{\mathbf{G}}}(\mathbf{S})$, $C = \mathbf{C}^F$ and $N = \mathrm{N}_{\widetilde{G}}(\mathbf{S})$. In $\widetilde{\mathbf{G}}^*$, we write $\mathbf{C}^*$ for the group dual to $\mathbf{C}$. Then, by Ruhstorfer et al. [156, §4.D], we fix the tuple $(\mathbf{S}^*, s, \lambda_0, \eta_0)$. With semisimple $s \in \mathbf{C}^{*F}$, we put $\mathbf{H}_s = C_{\widetilde{\mathbf{G}}^*}(s)$. Next, we write $\mathbf{L}_s = C_{\mathbf{C}^*}(s)$ and $\mathcal{W}_{s,\lambda_0} = \mathrm{N}_{\mathbf{H}_s^F}(\mathbf{L}_s, \lambda_0)/\mathbf{L}_s^F$. Moreover, $\lambda_0 \in \mathrm{UCh}(\mathbf{L}_s^F) \cap \mathrm{Irr}_{p'}(\mathbf{L}_s^F)$ is d-cuspidal and $\eta_0 \in \mathrm{Irr}_{p'}(\mathcal{W}_{s,\lambda_0})$. $|\mathbf{G}^{*F} : C_{\mathbf{G}^*}(\pi^*(s))^F|_p = 1$. We fix $x \in C_{\mathbf{G}^*}(\pi^*(s))^F$ with $|x| = p^r$, for $r \geq 0$. Thus, $(\mathbf{S}^*, \lambda_0, \eta_0)^x \sim_{C^\circ_{\mathbf{G}^*}(\pi^*(s))^F} (\mathbf{S}^*, \lambda_0, \eta_0)$ from Ruhstorfer et al. [156, §4.D]. In $\mathbf{H}_s^F$, we write ψ_{η_0} for the unipotent d-Harish–Chandra character. We put $\rho = J^{-1}_{\widetilde{G},s}(\psi_{\eta_0})$ and $\xi_0 = J^{-1}_{C,s}(\lambda_0)$. Moreover, we write $I_{\xi_0} = N_{\xi_0}$ and $\mathcal{W}_{s,\lambda_0} \simeq I_{\xi_0}/C$. We choose $\widehat{\xi_0} \in \mathrm{Irr}(I_{\xi_0})$ with $\widehat{\xi_0}|_C = \xi_0$. Then, we define $\tau = \mathrm{Ind}_{I_{\xi_0}}^N(\widehat{\xi_0} \otimes \mathrm{Inf}_{I_{\xi_0}/C}^{I_{\xi_0}} \eta_0)$. By Ruhstorfer et al. [156, Corollary 4.7 and proof of Theorem B], we write $\widetilde{\Omega}(\rho) = \tau$ for the ordinary correspondence $\rho \leftrightarrow \tau$.

Proposition 5.43. *If q is odd and $d = 2e$, we fix the characters ρ, τ. By* Proposition 5.33*, we choose T_a on the cores π_0. Moreover, we write $\Gamma_{\rho,\tau} = \{a = (h_a, \sigma_a, \delta_a) : \rho^a = \rho,\ \tau^a = \tau\}$. We choose $\widehat{\xi_0}$ and character maps R_a^ρ, R_a^τ with factors α_ρ, α_τ, and we have that $\alpha_\rho(a,b) = \alpha_\tau(a,b) = A(a,b)\beta(a,b)$ for $a, b \in \Gamma_{\rho,\tau}$.*

Proof. First, fix the outer blocks A_i with nonreciprocal pairs $[x_i] \neq [x_i^{-1}]$. On A_i, we write the cuspidal characters ϑ_i and central characters ω_i. In the models ρ, τ, $(m_i, \lambda_i, \omega_i)_\rho = (m_i, \lambda_i, \omega_i)_\tau$. Moreover, by Lemma 5.36, the inclusion is $(A, g) \mapsto \mathrm{diag}(A, g, \mu(g)A^{-\mathsf{t}})$. Under $A_i \leftrightarrow A_i^*$, $\vartheta_i \otimes \pi_0 \mapsto \vartheta_i^\vee \otimes (\pi_0 \otimes (\omega_i \circ \mu))$. Accordingly, in ρ, τ, we include $\omega_i \circ \mu$ on the core π_0. Moreover, we choose ordered bases $\mathcal{B}_i$ on the corresponding blocks A_i. On $\bigoplus_i (A_i \oplus A_i^*)$, write block permutations P_ω and inverse-transpose matrices j_i. With the bases $\mathcal{B}_i$, $(P_\omega, j_i)_\rho = (P_\omega, j_i)_\tau$. Then, by Lemma 5.40, we choose J_n with the chosen positive relative systems $\mathcal{R}_V^+$. Along $(n_1, \ldots, n_s)$, $J_{n_s} \cdots J_{n_1} = J_{n_s \cdots n_1}$ and $J_{hn} = r_{n_V}(h^{-1})J_n$. By Lemma 5.38, $q^{N_\rho/2}/q^{N_\tau/2} \in \mathbb{Q}^\times$. Under $\sigma \in \mathcal{H}_p$, $\sigma(q^{N_\rho/2})/q^{N_\rho/2} = \sigma(q^{N_\tau/2})/q^{N_\tau/2}$. Moreover, we write U_a for the outer source maps and B_a for their multiplicity maps on M_A. After the links J_n, we define $\mathcal{U}_a = U_a \otimes B_a$ and $\mathcal{U}_a\mathcal{U}_b = \beta_A(a,b)\mathcal{U}_{ab}$. We obtain $\beta_A^\rho(a,b) = \beta_A^\tau(a,b) = \beta_A(a,b)$.

Moreover, fix the quadratic character $\epsilon \in \mathrm{Irr}(\mathbb{F}_q^\times)$ and a quadratic norm-one colour θ. Under v_i, $\widetilde{\theta}^{v_i}/\widetilde{\theta} = \epsilon \circ \mu$. Then, by the source relation for E, $T_a E T_a^{-1} = \epsilon(t_a)^{-1}E$ when $\pi_0 \otimes (\epsilon \circ \mu) = \pi_0$ and $E^2 = I$. At $-I$, $\theta(-1) = (-1)^{(Q+1)/2} = (-1)^{b_0}$. Then, by Proposition 5.7, we choose ξ with $\xi^d = \theta(-1)$. We find $\sigma_a(\xi)/\xi = \varepsilon^{b_0 f k_a}$. By Lemma 5.27, $\varepsilon^{b_0 f k_a} = (p/\ell)^{f k_a} = \epsilon(t_a)$. On $\mathbb{F}_\ell^\times$, $\epsilon|_{\mathbb{F}_\ell^\times} = \varepsilon_\ell^f$ and $t_a = p^{k_a} \cdot 1_{\mathbb{F}_\ell}$. Thus, $T_a(\xi E)T_a^{-1} = \xi E$ and $(\xi E)^d = \theta(-1)I$. Alternatively, we suppose that $\pi_0 \otimes (\epsilon \circ \mu) \neq \pi_0$. On quadratic coordinates i, we fix $v_{\mathbf{m}} = \prod_i v_i^{m_i}$. Since $\epsilon^2 = 1$, $\pi_0 \otimes (\epsilon \circ \mu)^{\sum_i m_i} = \pi_0$ if and only if $\sum_i m_i \in 2\mathbb{Z}$. Thus, $\sigma_a(\prod_i \xi_i^{m_i})/\prod_i \xi_i^{m_i} = 1$ for each such rotation word $v_{\mathbf{m}}$. In the index 2 parent, we keep $\pi_0, \pi_0 \otimes (\epsilon \circ \mu)$. By Lemma 5.39, we include $\theta_i^2 \neq 1$ with $\Lambda(\widetilde{\theta}_i)(v_i^{2c_i}) = \widetilde{\theta}_i(-1)$.

Moreover, we order the colours (θ_i, ν_i) in adjacent positions i. Above θ_i, choose $\widetilde{\theta}_i \in \mathrm{Irr}(\widetilde{T}_i \mid \theta_i)$ with $\widetilde{\theta}_i = \widetilde{\theta}_j$ on equal coordinate models i, j. Moreover, we fix $\lambda_i^\mu \in \mathrm{Irr}(\mathbb{F}_q^\times)$. With $\widetilde{\theta}'_i = \widetilde{\theta}_i(\lambda_i^\mu \circ \mu)$, we write $\pi'_0 = \pi_0 \otimes ((\prod_i \lambda_i^\mu)^{-1} \circ \mu)$. Thus, $\pi'_0 \prod_i \widetilde{\theta}'_i = \pi_0 \prod_i \widetilde{\theta}_i$ on the common-multiplier centralizer C. Moreover, we include residual rank 0 with a character $\lambda_\mu \in \mathrm{Irr}(\mathbb{F}_q^\times)$ in place of π_0. With v_i, P_ω, we fix a word w. Moreover, we write $\widetilde{\theta}_i^w = \widetilde{\theta}_{w(i)}(\lambda_{w,i} \circ \mu)$ and $\lambda_w = \prod_i \lambda_{w,i}$ when $\theta_i^w = \theta_{w(i)}$. $w \in I_{\xi_0}$ exactly when $\theta_i^w = \theta_{w(i)}$ and the net multiplier twist λ_w satisfies $\pi_0 \otimes (\lambda_w \circ \mu) = \pi_0$. Then, by $\widetilde{\theta}^{v_i}/\widetilde{\theta} = \epsilon \circ \mu$, $\lambda_{v_i} = \epsilon$ on quadratic coordinates $\theta_i^2 = 1 \neq \theta_i$. By Lemma 5.39, $\lambda_{v_i^{2c_i}} = 1$ on nonquadratic coordinates $\theta_i^2 \neq 1$. If $\theta_i = 1$, we choose $\widetilde{\theta}_i = 1$ and rotation value 1. Thus, we write the generators $v_i^{r_i}$, $v_{\mathbf{m}}$ and P_ω for I_{ξ_0}/C. Write coordinate operators R_i and permutation matrices P_ω. $R_iR_j = R_jR_i$ because disjoint coordinates i, j share only E. Moreover, $P_\omega R_i P_\omega^{-1} = R_{\omega(i)}$. We write $R(v_{\mathbf{m}})$ for an allowed rotation word $v_{\mathbf{m}}$, and $R(v_{\mathbf{m}})R(v_{\mathbf{n}}) = R(v_{\mathbf{m}+\mathbf{n}})$ and $R(v_{\mathbf{m}+2\mathbf{e}_i}) = \widetilde{\theta}_i(-I)R(v_{\mathbf{m}})$. Thus, $\widehat{\xi_0}|_C = \xi_0$. We choose $\widehat{R}_{\xi_0} : I_{\xi_0} \to \mathrm{GL}(V_{\xi_0})$ with character $\widehat{\xi_0}$, and $\Lambda(\xi_0^n) = \Lambda(\xi_0)^n$ for $n \in N$ and ordered representatives ξ_0. Then, by Proposition 5.34, $C_i(m_\lambda) = 1$ for pure multiplier maps $m_\lambda = (0, 1, \lambda)$. Thus, $\Lambda(\xi_0 \otimes (\lambda \circ \mu)) = \Lambda(\xi_0) \otimes (\lambda \circ \mu)$.

Next, from $\mathcal{U}_b$, write the net core twist ω_b on π_0. Put $\Delta_b = \delta_b\omega_b$ and $A_\Delta(a,b) = \sigma_a(\Delta_b)(t_a)^{-1}$. By Proposition 5.37, B_a^{U} satisfies $B_a^{\mathrm{U}}B_b^{\mathrm{U}} = B_{ab}^{\mathrm{U}}$ for both staircase sizes $(t, r), (t', r')$. By Lemma 5.41, T_a has factor $A_\Delta(a,b)$ in both ordinary series ρ, τ. Moreover, we decompose the local multiplier $\Delta_b = \delta_{b,0} \prod_i \delta_{b,i}$. By Proposition 5.34, $\sigma_a(\delta_{b,0})(t_a)^{-1} \prod_i \sigma_a(\delta_{b,i})(t_a)^{-1} = A_\Delta(a,b)$. We put $\beta(a,b) = \beta_A(a,b)A_\Delta(a,b)/A(a,b)$. Thus, $\alpha_\rho(a,b) = \alpha_\tau(a,b) = A(a,b)\beta(a,b)$. On a stable common multiplier space V_χ, $(R_aR_b)|_{V_\chi} = A(a,b)\beta(a,b)R_{ab}|_{V_\chi}$. Moreover, we consider a self-twist $m_\lambda = (0, 1, \lambda)$. $\mathcal{U}_{m_\lambda} = I$, $\omega_{m_\lambda} = 1$ and $t_{m_\lambda} = 1$. Thus, $\beta(a, m_\lambda) = \beta(m_\lambda, a) = 1$.

By Lemma 5.10, choose coset representatives r_i for the induced character τ. Then, on $r_iI_{\xi_0}$, we write $a(r_i) =$

$r_{\kappa_a(i)}h_a(i)$ with $h_a(i)\in I_a$. Choose $\widehat{R}_a : I_a \to \mathrm{GL}(V_{\xi_0^a})$ with character $\Lambda(\xi_0^a)$. Thus, $h_{ab}(i) = h_a(\kappa_b(i))a(h_b(i))$. Thus, we include the matrix $\widehat{R}_{ab}(h_{ab}(i))$ in the coset operator product $R_a^\tau R_b^\tau$. Over $\mathbb{Q}$, choose Specht modules S^{λ_i} with permutation maps $P_\omega P_\nu = P_{\omega\nu}$. We obtain $R_a^\tau R_b^\tau = A(a,b)\beta(a,b)R_{ab}^\tau$.

Moreover, $\Lambda(\xi_0^a) = \Lambda(\xi_0)^a$ on $I_{\xi_0^a} = I_{\xi_0}^a$. By Lemma 5.42, we include $\Lambda \mapsto \Lambda^{\mathrm{row}}$ on cores and quotients $(\kappa_i, \lambda_{i,j})$. We keep both eigenspaces V^+, V^- when the ordered parent label Λ is fixed. Then, we write $D_i = e/c_i$ on each unitary factor $\mathrm{GU}_{m_i}(q^{c_i})$. From the D_i-core κ_i and quotient $(\lambda_{i,j})$, λ_i is unique. By [156, Corollary 4.7 and proof of Theorem B], we write $\rho \leftrightarrow \tau$ from the connected-centre Jordan maps $J_{\widetilde{G},s}, J_{C,s}$. Under fields, $\sigma_a \in \mathcal{H}_p$ and multiplier maps m_λ, $\widetilde{\Omega}(\rho^a) = \widetilde{\Omega}(\rho)^a$. Moreover, at $z \in Z(\widetilde{G})$, $\rho(z)/\rho(1) = \tau(z)/\tau(1)$. Moreover, we include every $a = (h_a, \sigma_a, \delta_a)$ in $\Gamma_{\rho,\tau}$ through the cochains $C_i(a)$ on all composable colour arrows a, b. At $a_f = (f,1,1)$, $C_i(aa_f) = C_i(a_f a) = C_i(a)$. With $E^2 = I$, $C_i(a_f) = 1$. Thus, $T_{a_f,i} = A_i(v_i)$ with $u_i^f = v_i$ and $v_i^{2e} = -I$ on every active coordinate i. □

We fix $X = \mathrm{Sp}_{2n}(q)$ and $\widetilde{G} = \mathrm{GSp}_{2n}(q)$. From Section 5.1, $D \le P$, $M = \mathrm{N}_X(D)$ and $\widetilde{M} = \mathrm{N}_{\widetilde{G}}(D)$. Moreover, write $\mathbf{G} = \mathrm{Sp}_{2n}$ with $F = F_q$. On V_i, write $\mathbf{T}_i$ for the active torus. Their Φ_d-subtori $\mathbf{S}_i$ give the Sylow Φ_d-torus $\mathbf{S} = \prod_i \mathbf{S}_i$ of Theorem 5.4, with $D = (\mathbf{S}^F)_p$ and $D[p]$ of maximal toral rank w. Thus, Malle [132, Proposition 5.13] gives uniqueness of $\mathbf{S}$ through $D[p]$. The group $\widetilde{G}$ normalizes $\mathbf{G}$ and its elements are F-fixed. Thus, $\widetilde{M}$ normalizes $\mathbf{S}$. Conversely, $\mathrm{N}_{\widetilde{G}}(\mathbf{S})$ normalizes $D = (\mathbf{S}^F)_p$, so $\widetilde{M} = \mathrm{N}_{\widetilde{G}}(\mathbf{S})$ and $M = \mathrm{N}_X(\mathbf{S})$. Moreover, $Z(\widetilde{G}) \le \widetilde{M}$.

Lemma 5.44. *With odd d, we write $n = dw + r$, with $0 \le r < d$. On a nonquadratic colour $[z] \ne [z^{-1}]$, we fix its degree c and put $D_c = d/c$. Moreover, we choose the cuspidal character $\tau \in \mathrm{Irr}(\mathrm{GL}_c(q))$ and its central character $\omega \in \mathrm{Irr}(\mathbb{F}_q^\times)$. We write the ordinary multiplicities m, m_0 and the active count $b = (m - m_0)/D_c$, and we have that $\omega^m = \omega^{m_0}(\omega^{D_c})^b$. Thus, $\omega(-1)^m = \omega(-1)^{m_0}\theta_d(-1)^b$, with torus character θ_d on $\mathbb{F}_{q^d}^\times$. At $a_f = (f,1,1)$, we choose local maps $S_{a_f,i} = A_i(R)$.*

Proof. First, we fix an eigenvalue $z \in \mathbb{F}_{q^d}^\times$ and write $c = |z^{\langle q\rangle}|$. $c \mid d$ and $2 \nmid c$. If $z^{-1} = z^{q^j}$, $c \mid 2j$ and $c \mid j$. Thus, $z = z^{-1}$ and $z \in \{1, -1\}$. On $[z] \ne [z^{-1}]$, we choose the regular character θ_c of $\mathbb{F}_{q^c}^\times$ for τ. Put $\theta_d = \theta_c \circ N_{\mathbb{F}_{q^d}/\mathbb{F}_{q^c}}$. Since $N_{\mathbb{F}_{q^d}/\mathbb{F}_{q^c}}(t) = t^{D_c}$ for $t \in \mathbb{F}_q^\times$, $\theta_d|_{\mathbb{F}_q^\times} = \omega^{D_c}$. Write the partition λ on the connected Jordan factor $\mathrm{GL}_m(q^c)$ and put $\kappa = \mathrm{core}_{D_c}(\lambda)$ and write its quotient $(\nu_0, \dots, \nu_{D_c-1})$. Thus, $m = |\lambda|$, $m_0 = |\kappa|$ and $b = \sum_j |\nu_j|$. $|\lambda| = |\kappa| + D_c \sum_j |\nu_j|$. Moreover, $b = (m - m_0)/D_c$ for the degree d torus blocks $\mathbf{T}_i$.

With odd c, we put $T(t) = \mathrm{diag}(t^{(c+1)/2-i})_{i=1}^c$. On $W_c = \mathbb{F}_q^c$, we write $w_0(e_i) = e_{c+1-i}$ and $H = T(-1)w_0$. Thus, $H^{\mathsf{t}} = H$ and $H^2 = I$. With $A \in \mathrm{GL}_c(q)$, put $\iota(A) = HA^{-\mathsf{t}}H^{-1}$. $\iota(T(t)) = T(t)$ and $\det T(t) = 1$. Moreover, we fix $U_c = \{u \in \mathrm{GL}_c(q) : u_{ii} = 1,\ u_{ij} = 0 \text{ for } i > j\}$ and $\psi(u) = \zeta_\ell^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_\ell}(\sum_{i=1}^{c-1} u_{i,i+1})}$. Over $E = \mathbb{Q}(\mu_{\ell|\,\mathrm{GL}_c(q)|})$, we put $\mathcal{V}_c = \{v : \mathrm{GL}_c(q) \to E : v(ux) = \psi(u)v(x),\ u \in U_c\}$. Thus, $\mathcal{V}_c \simeq \mathrm{Ind}_{U_c}^{\mathrm{GL}_c(q)} \psi$. Under $\sigma \in \mathcal{H}_p$, we write $t_\sigma \in \mathbb{F}_\ell^\times$ with $\sigma(\zeta_\ell) = \zeta_\ell^{t_\sigma}$. On $\mathcal{V}_c$, we put $[F_h v](x) = v(x^{\ell^{-h}})$ and $[S_\sigma v](x) = \sigma(v(T(t_\sigma)^{-1}x))$. Moreover, put $[Cv](x) = v(\iota(x))$. $(T(t)uT(t)^{-1})_{i,i+1} = tu_{i,i+1}$ and $\iota(u)_{i,i+1} = u_{c-i,c+1-i}$. Thus, $\psi \circ \iota = \psi$ and $\sigma(\psi(T(t_\sigma)^{-1}uT(t_\sigma))) = \psi(u)$. $T(t), H \in \mathrm{GL}_c(\mathbb{F}_\ell)$. By $\iota(T(t)) = T(t)$, $CS_\sigma = S_\sigma C$. Moreover, $F_h F_k = F_{h+k}$, $S_\sigma S_\upsilon = S_{\sigma\upsilon}$ and $C^2 = I$. $F_h S_\sigma = S_\sigma F_h$ and $F_h C = C F_h$. $\langle \mathrm{Ind}_{U_c}^{\mathrm{GL}_c(q)} \psi, \tau\rangle = 1$ for every cuspidal τ. Thus, we choose $V_\tau \subseteq \mathcal{V}_c$ and $r_\tau : \mathrm{GL}_c(q) \to \mathrm{GL}(V_\tau)$. We put $[r_\tau(A)v](x) = v(xA)$ and have that $\mathrm{tr}\, r_\tau = \tau$, and we write $\tau_h(A) = \tau(A^{\ell^{-h}})$ and $\tau^\sigma(A) = \sigma(\tau(A))$. Moreover, $F_h : V_\tau \to V_{\tau_h}$, $S_\sigma : V_\tau \to V_{\tau^\sigma}$ and $C : V_\tau \to V_{\tau^\vee}$.

In the Levi $\widetilde{L}$, we write $(A, g) \mapsto \mathrm{diag}(A, g, \mu(g)A^{-\mathsf{t}})$. On $W_c \oplus W_c^*$, put $j_c = \left(\begin{smallmatrix} 0 & H \\ -H & 0 \end{smallmatrix}\right)$. Moreover, choose $j_c \oplus I$ on $W_c \oplus V_{\mathrm{mid}} \oplus W_c^*$, with residual space V_{mid} for g. $j_c \in \mathrm{Sp}_{2c}(\mathbb{F}_\ell)$ and $j_c^2 = -I_{2c}$, and $j_c(A,g)j_c^{-1} = (\mu(g)\iota(A), g)$. Moreover, we fix $\pi : \mathrm{GSp}(V_{\mathrm{mid}}) \to \mathrm{GL}(V_\pi)$. Consequently, $(r_\tau \boxtimes \pi)^{j_c} \simeq r_\tau^\vee \boxtimes (\pi \otimes (\omega \circ \mu))$. On repeated coordinates i, we write the lifts j_i. With $i \ne j$, $j_i j_j = j_j j_i$ and $P_\alpha j_i P_\alpha^{-1} = j_{\alpha(i)}$. Thus, $\omega^m = \omega^{m_0}(\omega^{D_c})^b$. Since $2 \nmid D_c$, $\theta_d(-1) = \omega(-1)^{D_c} = \omega(-1)$. Consequently, $\omega(-1)^m = \omega(-1)^{m_0}\theta_d(-1)^b$.

By $W_s^2 = sI$, choose W_s with $s = \theta_d(-1)$ on the local block V_i. Write $W_s = \left(\begin{smallmatrix} 0 & s \\ 1 & 0 \end{smallmatrix}\right) \otimes I_c$ and $P_\alpha P_\beta = P_{\alpha\beta}$ on equal coordinates i. Thus, $W_s^2 = \theta_d(-1)I$ in both directions $[z] \leftrightarrow [z^{-1}]$. With rotation value $\kappa \in \mu_d$, we put $t = \mathrm{ord}_p(\ell)$. By $q = \ell^f$, $d = t/(t,f)$ and $t \mid p-1$. Thus, $\{\eta^f : \eta \in \mu_t\} = \mu_d$. Choose $\eta \in \mu_t$ with $\eta^f = \kappa$. On repeated colours i, j, choose $\eta_i = \eta_j$. Since $t \mid p-1$, $\sigma(\eta) = \eta$ for $\sigma \in \mathcal{H}_p$. By Proposition 5.6, $S_{a_f,i} = A_i(R)$ with $u_i^f = R$ at $a_f = (f,1,1)$. □

Proposition 5.45. *With odd q, p, we suppose that $p \nmid q$ and $d = \mathrm{ord}_p(q) \in 2\mathbb{Z}+1$. We choose $\widetilde{\Omega} : \mathrm{Irr}_{p'}(\widetilde{G}) \xrightarrow{\sim} \mathrm{Irr}_{p'}(\widetilde{M})$ with $\widetilde{\Omega}(\rho) = \tau$. Moreover, we write $\Gamma_\rho = \{a = (h_a, \sigma_a, \delta_a) : \rho^a = \rho\}$ and $\Gamma_\tau = \{a : \tau^a = \tau\}$. We have that $\Gamma_\rho = \Gamma_\tau$*

for field, $\sigma_a \in \mathcal{H}_p$ and multiplier actions $\delta_a \circ \mu$. Moreover, we choose $r_\rho : \widetilde{G} \to \mathrm{GL}(V_\rho)$ and $r_\tau : \widetilde{M} \to \mathrm{GL}(V_\tau)$ with characters ρ, τ. We choose character maps R_a^ρ, R_a^τ with $\alpha_\rho(a,b) = \alpha_\tau(a,b)$ for $a, b \in \Gamma_\rho$. Thus, $\rho(z)/\rho(1) = \tau(z)/\tau(1)$ for $z \in Z(\widetilde{G})$. At the field period f, we have that $T^f_{u,\rho}|_{V_\rho} = r_\rho(u^f)$ and $T^f_{u,\tau}|_{V_\tau} = r_\tau(u^f)$, with $u^f \in M$.

Proof. First, fix $(\mathbf{S}^*, s, \lambda_0, \eta_0)$ from Proposition 5.43. We include $s = s_p s_{p'}$ with its full p-part s_p. On $C = C_{\widetilde{G}}(\mathbf{S})$, we write $\xi_0 = J^{-1}_{C,s}(\lambda_0)$ and $I_{\xi_0} = (\widetilde{M})_{\xi_0}$. Moreover, we write $\rho = J^{-1}_{\widetilde{G},s}(\psi_{\eta_0})$ and $\tau = \mathrm{Ind}_{I_{\xi_0}}^{\widetilde{M}}(\widehat{\xi_0} \otimes \mathrm{Inf}^{I_{\xi_0}}_{I_{\xi_0}/C}\, \eta_0)$. By Lemma 5.44, compare $[z] \neq [z^{-1}]$. By Proposition 5.12, include other nonclassical L_i in the residual Levi L_{res}. We write the two classical symbols Λ_+, Λ_-. By Lemma 5.16, $\mathrm{core}_1(\Lambda_\pm) = \mathrm{core}_1(\mathrm{core}_d(\Lambda_\pm))$. Thus, we choose cuspidal $\pi \in \mathrm{Irr}(\widetilde{L}_0)$ and $r_\pi : \widetilde{L}_0 \to \mathrm{GL}(V_\pi)$. On the other common factors L_i, we choose source models V_i and multiplicity models M_i. Moreover, fix $c_i(a,b)$ on V_i, M_i for both ordinary characters ρ, τ.

Near V_π, we order the common blocks V_i and fix one orientation $[z]$ in each pair $\{[z], [z^{-1}]\}$. By Lemma 5.40, we choose normalized maps J_n between the ordered Levi models L, L'. Then, $J_{n_t} \cdots J_{n_1} = J_{n_t \cdots n_1}$ and $J_{hn} = r_{nV}(h^{-1}) J_n$. Moreover, we write J_i for the coordinate maps with $J_i^2 = s_i I$. $J_i J_j = J_j J_i$ for $i \neq j$ and $P_\alpha J_i P_\alpha^{-1} = J_{\alpha(i)}$, and $P_\alpha P_\beta = P_{\alpha\beta}$. Thus, write each word as $J_1^{e_1} \cdots J_t^{e_t} P_\alpha$, with $e_i \in \{0,1\}$, after its scalar squares s_i. On a common block V_i, choose the basis $\mathcal{B}_i$ in both models ρ, τ. Consequently, $(P_\alpha, J_i, A_i(u^f))_\rho = (P_\alpha, J_i, A_i(u^f))_\tau$. By Lemma 5.36, $\omega_i = 1$ for cuspidal $\tau_i \in \mathrm{Irr}(\mathrm{GL}_{2c_i}(q))$ with $\tau_i^\vee = \tau_i$ and $2c_i > 1$. On other common colours i, we include the full twist $\omega_i^{m_i} \circ \mu$ on π when $[z_i] \mapsto [z_i^{-1}]$. Then, by Lemma 5.44, we have that $\omega_i^{m_i} = \omega_i^{m_{i,0}} (\omega_i^{D_i})^{b_i}$ and $\omega_i(-1)^{m_i} = \omega_i(-1)^{m_{i,0}} \theta_{i,d}(-1)^{b_i}$. We include every $a = (h_a, \sigma_a, \delta_a)$ in Γ_ρ in the comparison of ρ, τ. Since d is odd, $d \mid (p-1)/2$ and $(q/p) = 1$. Thus, $\sigma(\sqrt{q}) = \sqrt{q}$ and $\sigma(q^{l(w)/2}) = q^{l(w)/2}$ for $\sigma \in \mathcal{H}_p$.

On a changing colour i, we write $\Lambda_i \vdash m_i$ and its type A multiplicity module $L_{\Lambda_i}(q^{c_i})$. We choose ordinary tensor permutations P_α with $P_\alpha P_\beta = P_{\alpha\beta}$. Then, by Lemma 5.21, we choose $U_{\Lambda_i}(q^{c_i})$ with $U_{\Lambda_i}(q^{c_i})^2 = I$. The coefficients of $U_{\Lambda_i}(q^{c_i})$ lie in $\mathbb{Q}(\sqrt{q})$. Thus, we choose $U_{\Lambda_{i,0}}(q^{c_i})$ on the residual partition $\Lambda_{i,0}$. Write the quotient partitions $(\nu_{i,0}, \ldots, \nu_{i,D_i-1})$ and their cyclic and Specht modules $M_{\boldsymbol{\nu}_i}$. We put $E_0 = \mathbb{Q}(\sqrt{q}, \mu_{2d})$. Since $2d \mid p-1$, $\sigma|_{E_0} = 1$ for $\sigma \in \mathcal{H}_p$. On $L_{\Lambda_i}, L_{\Lambda_{i,0}}, M_{\boldsymbol{\nu}_i}$, choose maps B_a with $B_a B_b = B_{ab}$. We include $\Lambda_i \neq \Lambda_{i,0}$ with no extra scalar in $B_a B_b = B_{ab}$. Moreover, choose identical projective maps $B_{a,i}$ on every common multiplicity module M_i.

On quadratic coordinates i, write $\epsilon \in \mathrm{Irr}(\mathbb{F}_q^\times)$ and $s = \epsilon(-1)$. We suppose that $\pi \otimes (\epsilon \circ \mu) = \pi$. By Lemma 5.41, we choose $E : V_\pi \to V_\pi$ with $E^2 = I$ and $E r_\pi(g) E^{-1} = \epsilon(\mu(g)) r_\pi(g)$. We choose $\xi^2 = s$ and put $J = \xi E$. Then, $T_a E T_a^{-1} = \epsilon(t_a)^{-1} E$ and $\sigma_a(\xi)/\xi = \epsilon(t_a)$. Thus, $T_a J T_a^{-1} = J$ and $J^2 = sI$. If $\theta_i = 1$, we choose $J_i = I$. We write the split type B algebras $\mathcal{H}_i$ with rational positive values $Q_v > 0$. Moreover, by Lemma 5.41, we include J^{-1} in the double-coset coefficient a_v. We choose normalized generators T_v with $(\mathsf{T}_v - Q_v I)(\mathsf{T}_v + I) = 0$. Then, write ${}^a Z$ for the algebra action on $\mathcal{H}_i$ from the source map T_a. Thus, ${}^a \mathsf{T}_v = \mathsf{T}_{a(v)}$ and $Q_{a(v)} = Q_v$. On quadratic torus blocks i, $\theta_{i,d}|_{\mathbb{F}_q^\times} = \epsilon$. From Lemma 5.44, we choose $J_i = \xi E$ and the rotation roots η_i. Moreover, $P_\alpha J_i P_\alpha^{-1} = J_{\alpha(i)}$ on equal colours $\theta_i = \theta_{\alpha(i)}$.

Alternatively, we suppose that $\pi \otimes (\epsilon \circ \mu) \neq \pi$. Whenever $\nu \in \mathrm{Irr}(\mathbb{F}_q^\times)$, we put $\pi_\nu = (\nu \circ \mu)\pi$. We choose $V_{\pi_\nu} = V_\pi$ and $r_{\pi_\nu}(g) = \nu(\mu(g)) r_\pi(g)$. By Corollary 5.5, $\pi_\nu = \pi$ only for $\nu = 1$. For a field and coefficient operation $a_0 = (h_a, \sigma_a)$ stabilizing the multiplier orbit of π, we choose the unique twist γ_a with $(\gamma_a \circ \mu)\pi^{a_0} = \pi$. $\gamma_{ab} = \gamma_a \gamma_b^{a_0}$ and $\gamma_b^{a_0}(c) = \sigma_a(\gamma_b(c^{\ell - h_a}))$. Thus, choose T_a on every translate V_{π_ν}. On a pure multiplier change m_λ, we choose $I : V_{\pi_\nu} \to V_{\pi_{\lambda\nu}}$. Thus, the base factors $\alpha_{\pi_\nu}(a,b) = \alpha_\pi(a,b)$ are common to ρ, τ. By Lemma 5.40, we choose J_{10}, J_{01} with $J_{01} J_{10} = J_{10} J_{01} = sI$. The source map $V_\pi \to V_{\pi_\epsilon}$ is I in the twist models r_{π_ν}. Moreover, we choose the local matrix $W_s = \left(\begin{smallmatrix} 0 & s \\ 1 & 0 \end{smallmatrix}\right) \otimes I$. We have that $W_s^2 = sI$ and $W_s A(R) = A(R) W_s$. We include the multiplier corrections $\omega_i^{m_i} \circ \mu$ from every other oriented colour i. Thus, we write $j(a) \in \{0,1\}$ for the action on $\{\pi, \pi_\epsilon\}$. Since $\epsilon^a = \epsilon$, $j(ab) \equiv j(a) + j(b) \pmod 2$. Thus, $J^{j(a)} J^{j(b)} = s^{j(a)j(b)} J^{j(ab)}$ for the original reflection J. At quadratic rank $n_- = 0$, we include no reflection J.

We write the residual count r_-, active count w_- and total rank $n_- = r_- + d w_-$. Moreover, put $\mathcal{H}_{\mathrm{par}} = \mathcal{H}_{E_0}(B_{r_-}; 1, q) \otimes E_0[(C_d \times C_2) \wr \mathfrak{S}_{w_-}]$. We write the two sign maps $\varepsilon_{\mathrm{res}}, \varepsilon_{\mathrm{act}}$ and $\varepsilon_{\mathrm{par}} = \varepsilon_{\mathrm{res}} \otimes \varepsilon_{\mathrm{act}}$. $J_i J_j = J_j J_i$ on disjoint residual and active coordinates i, j. Thus, $J_i : V_\pi \to V_{\pi_\epsilon}$ for an odd generator J_i on either factor $\mathcal{H}_{\mathrm{par}}$. We write $\mathcal{H}_{\mathrm{even}} = \mathcal{H}_{\mathrm{par}}^{\varepsilon_{\mathrm{par}}}$ for the even words $J_{i_1} \cdots J_{i_{2k}}$. Over $E_0(\xi)$, we normalize each $J_i^2 = sI$ by $J_i' = \xi^{-1} J_i$. Then, $(J_i')^2 = I$. Since $\xi^{-2k} = s^{-k}$, $\sigma(\xi^{-2k}) = \xi^{-2k}$ for $\sigma \in \mathcal{H}_p$. We keep J_i in the source factor $s^{j(a)j(b)}$. Moreover, we write the global parent $\mathcal{H}_{E_0}(B_{n_-}; 1, q)$ and its even subalgebra $\mathcal{H}_{\mathrm{glob,even}} = \mathcal{H}_{E_0}(B_{n_-}; 1, q)^\varepsilon$. If $n_- \geq 2$, $\mathcal{H}_{\mathrm{glob,even}} = \mathcal{H}_{E_0}(D_{n_-}; q)$. By Lemma 5.42, we match parent labels $\Theta, \Theta^\varepsilon$, cores κ and quotients (ν_j). We choose one ordered pair $(\Theta, \Theta^\varepsilon)$ on each orbit $\{\Theta, \Theta^\varepsilon\}$. Moreover, both eigenspaces V^+, V^- are present when $\Theta^\varepsilon = \Theta$. Thus, we

include $r_- = 0$ or $w_- = 0$ with $\mathcal{H}(B_0) = E_0$ and $\mathfrak{S}_0 = 1$. We choose rational type B modules M_i on every other classical colour i.

On C, we now choose the extension $\widehat{\xi}_0 \in \mathrm{Irr}(I_{\xi_0})$. Moreover, we fix rotation values $\widehat{\theta}_i(R_i^{c_i}) = 1$ for the base nonquadratic characters θ_i. We put the other cyclic values κ_i in the relative label η_0. Then, by Theorem 5.4, $R_i^d = I$, $J_i^2 = t_{i,-1}$ and $P_\alpha J_i P_\alpha^{-1} = J_{\alpha(i)}$. Choose E on V_π or even quadratic words $J_{i_1} \cdots J_{i_{2k}}$. The corresponding operators $\widehat{R}(n)$ satisfy $\widehat{R}(n_1 n_2) = \widehat{R}(n_1)\widehat{R}(n_2)$. Thus, $\widehat{\xi}_0|_C = \xi_0$. On the opposite orientation $[z^{-1}]$, we choose the conjugate extension $\widehat{\xi}_0^{j_i}$ with the multiplier correction $\omega_i^{m_i} \circ \mu$. Consequently, $\Lambda(\xi_0^a) = \Lambda(\xi_0)^a$ for the extension map $\Lambda(\xi_0) = \widehat{\xi}_0$. By [156, Theorem B and §4.4], $\widetilde{\Omega}(\rho^a) = \widetilde{\Omega}(\rho)^a$. Moreover, $\rho(z)/\rho(1) = \tau(z)/\tau(1)$ for $z \in Z(\widetilde{G})$. By Lemma 5.44, we also obtain that $\omega_i^{m_i} = \omega_i^{m_{i,0}} \theta_{i,d}|_{\mathbb{F}_q^\times}^{b_i}$ on each active colour i. From core κ and quotient (ν_j), Λ_i is unique. Thus, we have that $\rho^a = \rho$ if and only if $\tau^a = \tau$, including both signs V^+, V^-. Thus, $\Gamma_\rho = \Gamma_\tau = \Gamma$ for all combined operations $a = (h_a, \sigma_a, \delta_a)$ stabilizing ρ, τ.

On the ordinary isotypic spaces $V_\rho \otimes M_\rho, V_\tau \otimes M_\tau$, choose maps $\mathcal{T}_a^\rho, \mathcal{T}_a^\tau$. $\mathcal{T}_a^\rho \mathcal{T}_b^\rho = \gamma(a,b)\mathcal{T}_{ab}^\rho$ and $\mathcal{T}_a^\tau \mathcal{T}_b^\tau = \gamma(a,b)\mathcal{T}_{ab}^\tau$. Moreover, we choose B_a^ρ, B_a^τ with factor $\beta(a,b)$. By Lemma 5.17, we write $\mathcal{T}_a^\rho = R_a^\rho \otimes B_a^\rho$ and $\mathcal{T}_a^\tau = R_a^\tau \otimes B_a^\tau$. Thus, $\alpha_\rho(a,b) = \gamma(a,b)/\beta(a,b) = \alpha_\tau(a,b)$. We include the core maps T_a and twists $\omega_i^{m_i} \circ \mu$ in $\mathcal{T}_a^\rho \mathcal{T}_b^\rho, \mathcal{T}_a^\tau \mathcal{T}_b^\tau$. Moreover, we keep the $\delta_a \circ \mu$ actions on V_i, M_i in $\gamma(a,b)$. We keep both cases $\pi_\epsilon = \pi$ and $\pi_\epsilon \neq \pi$ in $\gamma(a,b)$.

With $\mathcal{O}_\rho = \{\rho^{u^j} : j \in \mathbb{Z}\}$, we put $\mathcal{V}_\rho = \bigoplus_{\rho' \in \mathcal{O}_\rho} V_{\rho'}$. On $\mathcal{V}_\rho$, we put $r_{\mathcal{O}_\rho} = \bigoplus_{\rho' \in \mathcal{O}_\rho} r_{\rho'}$. Moreover, we write $\mathcal{O}_\tau = \widetilde{\Omega}(\mathcal{O}_\rho)$ and $\mathcal{V}_\tau = \bigoplus_{\tau' \in \mathcal{O}_\tau} V_{\tau'}$. At $a_f = (f, 1, 1)$, $S_{a_f,i} = A_i(R_i)$ on every active positive block i. On the common source V_π, we keep its chosen field map T_{v_0}. Then, by Proposition 5.33, $T_{a_f,0} = r_\pi(c_*)$ for the inner period $c_* \in \widetilde{L}_0$. Thus, $T_{u,\tau}^f|_{V_\tau} = r_\tau(u^f)$ on the local model V_τ.

On a Levi model $I_P(V)$, put $[\mathcal{F}_v f_L](g) = T_v f_L(v^{-1}gv)$. We choose the adapted field v with $v^f \in L$ and source period $T_v^f = r_V(v^f)$. Since $f_L(lg) = r_V(l) f_L(g)$, $[\mathcal{F}_v^f f_L](g) = r_V(v^f) f_L(v^{-f} g v^f) = f_L(g v^f)$. On the common tensor $M_{\mathrm{com}} = \bigotimes_i M_i$, choose the field maps B_v. Then, we choose involutions U_i by Lemmas 5.21 and 5.42 with $U_i^2 = I$. We have that $P_\alpha U_i P_\alpha^{-1} = U_{\alpha(i)}$ and $U_i U_j = U_j U_i$ for distinct factors i, j. Thus, $B(d_1)B(d_2) = B(d_1 d_2)$ for the diagram and permutation group $\mathcal{D}$. On $\mathcal{D}$, write generators p_α, u_i with $p_\alpha p_\beta = p_{\alpha\beta}$, $u_i^2 = 1$, $u_i u_j = u_j u_i$ and $p_\alpha u_i p_\alpha^{-1} = u_{\alpha(i)}$. Moreover, we put $B(p_\alpha) = P_\alpha$ and $B(u_i) = U_i$. We write the field action $\vartheta(v) \in \mathcal{D}$. Since v^f is inner, $\vartheta(v)^f = 1$ and $B_v^f = B(\vartheta(v)^f) = I$. Choose the field maps B_v^ρ, B_v^τ from the common models M_i. Thus, $T_{v,\rho}^f = r_{\mathcal{O}_\rho}(v^f)$. On $\mathcal{V}_\rho$, we write the marked field lift $u = xv$, with $x \in X$. We choose $T_{u,\rho} = r_{\mathcal{O}_\rho}(x) T_{v,\rho}$ and put $N_v(x) = x\, v(x) \cdots v^{f-1}(x)$. Thus, $u^f = N_v(x) v^f$. Thus, $T_{u,\rho}^f = r_{\mathcal{O}_\rho}(N_v(x)) T_{v,\rho}^f = r_{\mathcal{O}_\rho}(u^f)$. Since $\mu(x) = 1$, $\delta(\mu(x)) = 1$ for every multiplier character δ. We include $v^f \neq 1$ through the matrix $r_{\mathcal{O}_\rho}(v^f)$. Thus, $T_{u,\rho}^f|_{V_\rho} = r_\rho(u^f)$ and $T_{u,\tau}^f|_{V_\tau} = r_\tau(u^f)$ before either multiplier kernel X, M. □

Proof of Theorem 5.2. On $X = \mathrm{Sp}_{2n}(q)$, we fix $\widetilde{G} = \mathrm{GSp}_{2n}(q)$ and $P \in \mathrm{Syl}_p(X)$. Moreover, put $M = N_X(D)$, $\widetilde{M} = N_{\widetilde{G}}(D)$ and $\Gamma = \mathrm{Aut}(X)_P$. By Theorem 5.4, $\widetilde{G}/X \simeq C \simeq \widetilde{M}/M$, with $C = \mathbb{F}_q^\times$ and multiplier μ. By Proposition 5.3, D is characteristic in P, $N_X(P) \leq M < X$ and $M^a = M$ for $a \in \Gamma$. If $d = \mathrm{ord}_p(q)$ is odd, we choose the map $\widetilde{\Omega}$ from Proposition 5.45. Alternatively, choose $\widetilde{\Omega}$ from Proposition 5.43 when $2 \mid d$. With $\lambda_i = \lambda_{i,0}$ on every unitary factor i, we keep the classical source maps T_a and common residual models V_i, M_i. By [156, §4.D], we write the common tuple $(\mathbf{S}^*, s, \lambda_0, \eta_0)$. In particular, keep $s = s_p s_{p'}$ in the centralizer label (s, λ_0) and the relative label η_0. $\widetilde{\Omega}(\rho) = \tau$ for the corresponding parents ρ, τ. Then, by Corollary 5.5, $\rho|_X$ and $\tau|_M$ are multiplicity free with at most 2 constituents. Since $p \neq 2$, $p \nmid \chi(1)$ if and only if $p \nmid \rho(1)$ for $\chi \in \mathrm{Irr}(\rho|_X)$. Moreover, $p \nmid \psi(1)$ if and only if $p \nmid \tau(1)$ for $\psi \in \mathrm{Irr}(\tau|_M)$.

On paired parents ρ, τ, choose spaces V_ρ, V_τ over $K = \mathbb{Q}^{\mathrm{ab}}$. Over K, we choose $r_\rho : \widetilde{G} \to \mathrm{GL}_K(V_\rho)$ and $r_\tau : \widetilde{M} \to \mathrm{GL}_K(V_\tau)$. Moreover, we put $Y = \{\delta \in \mathrm{Irr}(C) : \rho(\delta \circ \mu) = \rho\}$. $Y = \{\delta : \tau(\delta \circ \mu) = \tau\}$. With cyclic C, $Y = 1$ or $Y = \{1, \eta\}$, with quadratic $\eta \in \mathrm{Irr}(C)$. Since $\mu(zI) = z^2$, $\delta(z^2) = 1$ for $\delta \in Y$ and $z \in \mathbb{F}_q^\times$. Over K, we put $F_0 = \widetilde{G} \times_C \widetilde{M}$ and $\mathcal{V} = V_\rho \otimes_K V_\tau^\vee$. On $\mathcal{V}$, we write $R(g,m) = r_\rho(g) \otimes r_\tau(m)^\vee$ for $(g,m) \in F_0$. Moreover, we choose the paired maps P_a, Q_a with $P_a P_b = \alpha(a,b) P_{ab}$ and $Q_a Q_b = \alpha(a,b) Q_{ab}$. On $V_\tau^\vee$, put $Q_a^\vee(l) = \sigma_a \circ l \circ Q_a^{-1}$ and $T_a = P_a \otimes Q_a^\vee$. Thus, $T_a T_b = T_{ab}$ and $T_y R(g,m) = R(g,m) T_y$ for $y \in Y$. On $\mathcal{V}$, we put $W = \mathcal{V}^Y$. By the proof for W in Proposition 5.25, W is irreducible on F_0. If $Y = 1$, $W|_{X \times M} = V_{\rho|_X} \otimes V_{\tau|_M}^\vee$. Alternatively, we consider $Y = \{1, \eta\}$. If $Y = \{1, \eta\}$, we choose $J_\rho = P_\eta, J_\tau = Q_\eta$ with $J_\rho^2 = J_\tau^2 = \kappa I$. By Corollary 5.5 and [43], $\mathrm{End}_X(V_\rho) \simeq K \oplus K$. Choose $c \in K$ with $c^2 \kappa = 1$ and put $E_\rho = c J_\rho, E_\tau = c J_\tau$. Thus, put $V_{\rho,\pm} = \ker(E_\rho \mp I)$ and $V_{\tau,\pm} = \ker(E_\tau \mp I)$. We write $\chi_\pm, \psi_\pm$ for the characters on $V_{\rho,\pm}, V_{\tau,\pm}$. On $\mathcal{V}$, $E_\rho \otimes E_\tau^\vee = J_\rho \otimes J_\tau^\vee$. Moreover, $V_{\rho,+}, V_{\rho,-}, V_{\tau,+}, V_{\tau,-} \neq 0$, by a nonsquare multiplier $m \in C$. Choose $g \in \widetilde{G}, h \in \widetilde{M}$ with $\mu(g) = \mu(h) = m$. Moreover, put $U_\pm = V_{\rho,\pm} \otimes V_{\tau,\pm}^\vee$.

We find $R(g,h)U_\pm = U_\mp$. Thus, $W|_{X\times M} = (V_{\rho,+}\otimes V_{\tau,+}^\vee)\oplus(V_{\rho,-}\otimes V_{\tau,-}^\vee)$. On $\mathrm{Irr}(\rho|_X)$, we put $\Omega(\chi_\pm)=\psi_\pm$ and $U_{\chi,\psi}=V_\chi\otimes V_\psi^\vee$. Since E_ρ, E_τ have common scalar c, Ω is independent of the choice $c,-c$.

For a mixed stabilizer element a, choose the common compensating character $\delta_a\in\mathrm{Irr}(C)$. Moreover, $\rho^a=\rho(\delta_a\circ\mu)$ and $\tau^a=\tau(\delta_a\circ\mu)$. With another choice δ_a', $y=\delta_a'\delta_a^{-1}\in Y$. $T_aT_yT_a^{-1}=T_y$ and $T_a(W)=W$. Since $T_y|_W=I$, $T_a|_W$ is independent of the choice $\delta_a y$. By the quotient formulas for C, $\mathrm{Irr}(\widetilde{G}\mid\chi)=\rho\,\mathrm{Irr}(C)$. Moreover, the opposite twists $\delta(\mu(g)),\delta(\mu(m))^{-1}$ have product 1 on F_0. On paired parents ρ,τ, we put $\mathcal{O}_{\rho,\tau}=(\rho,\tau)\,\mathrm{Irr}(C)$. Thus, we choose one $\mathcal{O}_{\rho,\tau}$ per $\Gamma\times\mathcal{H}_p$-orbit. On $\mathcal{O}_{\rho,\tau}^{a\sigma}$, define $\Omega(\chi^{a\sigma})=\Omega(\chi)^{a\sigma}$. Then, by the stabilizer action on W, $\Omega(\chi)^{a\sigma}=\Omega(\chi)^{b\nu}$ whenever $\chi^{a\sigma}=\chi^{b\nu}$. By $\widetilde{\Omega}$, $\Omega:\mathrm{Irr}_{p'}(X)\to\mathrm{Irr}_{p'}(M)$ is bijective. Moreover, $\Omega(\chi^{a\sigma})=\Omega(\chi)^{a\sigma}$ for $(a,\sigma)\in\Gamma\times\mathcal{H}_p$, by the action on W. We put $E=\Gamma_{\chi^{\mathcal{H}_p}}=\Gamma_{\psi^{\mathcal{H}_p}}$ and $I=\Gamma_\chi=\Gamma_\psi$, with $\psi=\Omega(\chi)$.

By Section 5.1, write $u=b\phi$, with split ϕ, $|\phi|=f$ and $b\in X$. Put $b_*=b\phi(b)\cdots\phi^{f-1}(b)=u^f\in M$. On X, $u\phi^{-1}=b$ is an $\mathbb{F}_q$-linear isometry. Thus, on V_ρ, $S_{xv}=r_\rho(x)S_v$ for a field lift v and $x\in X$. The power calculation in Proposition 5.45 gives $S_{xv}^f=r_\rho(N_v(x))S_v^f$, with $(xv)^f=N_v(x)v^f$. The inner period contributes $r_\rho(v^f)$, and multiplier twists are trivial on $X=\ker\mu$. Put $a_f=(f,1,1)$. If $2\nmid d$, $S_{a_f,i}=A_i(R_i)$ by Proposition 5.45. If $2\mid d$, $T_{a_f,i}=A_i(v_i)$ by Proposition 5.34, with $v_i^{2e}=-I$. Thus, $S_u^f|_{V_\rho}=r_\rho(b_*)$ and $(S_u')^f|_{V_\tau}=r_\tau(b_*)$, on $\mathcal{V}_\rho,\mathcal{V}_\tau$. Thus, $(P_{a_f}\otimes Q_{a_f}^\vee)|_W=R(b_*,b_*)$.

Moreover, we put $\mathcal{A}=\widetilde{G}\rtimes\langle\phi\rangle$ and $\mathcal{B}=\widetilde{M}\langle u\rangle\le\mathcal{A}$. We find $\mathcal{A}/X\simeq\mathcal{B}/M$. Moreover, we put $\mathcal{F}=\mathcal{A}\times_{\mathcal{A}/X}\mathcal{B}$ and $\widehat{\mathcal{F}}=F_0\rtimes\mathbb{Z}$. On $\widehat{\mathcal{F}}$, we define $\pi_{\mathcal{F}}((g,m),k)=(gu^k,mu^k)$ for $(g,m)\in F_0$, $k\in\mathbb{Z}$. We find $\ker\pi_{\mathcal{F}}=\langle((b_*^{-1},b_*^{-1}),f)\rangle$. Thus, $\mathcal{F}\simeq\widehat{\mathcal{F}}/\ker\pi_{\mathcal{F}}$. We write $I_{\mathrm{fld}}\le\mathbb{Z}$ for the stabilizer of $(\rho,\tau)\,\mathrm{Irr}(C)$. Since $u^f=b_*\in M$, $f\mathbb{Z}\le I_{\mathrm{fld}}$. Moreover, we choose $\delta_h\in\mathrm{Irr}(C)$ with $\delta_h(\mu(g))\rho(u^{-h}gu^h)=\rho(g)$ and $\delta_h(\mu(m))\tau(u^{-h}mu^h)=\tau(m)$. With $h\in I_{\mathrm{fld}}$, put $a_h=(h,1,\delta_h)$ and $U_h=T_{a_h}|_W$. Since $a_ha_ka_{h+k}^{-1}\in Y$, $U_hU_k=U_{h+k}$. On W, $U_hR(g,m)U_h^{-1}=R(u^hgu^{-h},u^hmu^{-h})$. Thus, put $\widehat{\mathcal{R}}((g,m),h)=R(g,m)U_h$ on $F_0\rtimes I_{\mathrm{fld}}$. We find $\widehat{\mathcal{R}}((b_*^{-1},b_*^{-1}),f)=I$. On $\mathcal{F}$, we put $\mathcal{F}_{\rho,\tau}=\pi_{\mathcal{F}}(F_0\rtimes I_{\mathrm{fld}})$. Thus, $\widehat{\mathcal{R}}=\mathcal{R}\circ\pi_{\mathcal{F}}$ with $\mathcal{R}:\mathcal{F}_{\rho,\tau}\to\mathrm{GL}_K(W)$. Moreover, $\mathcal{R}(z,z)=I$ for $z\in Z(\widetilde{G})$, by $\rho(z)/\rho(1)=\tau(z)/\tau(1)$.

On the natural cover X, write $\iota:\mathcal{A}\twoheadrightarrow\mathrm{Aut}(X)$ with $\ker\iota=Z(\widetilde{G})$. Since D is characteristic in P, $\iota^{-1}(\Gamma)\le\mathcal{B}$. We choose a section $t:E\to\mathcal{B}$ from the proof of Proposition 5.26 with $\iota(t(i))=i$ and $t(1)=1$. We put $z(i,j)=t(i)t(j)t(ij)^{-1}\in Z(\widetilde{G})$. Then, by the quotient formulas for C, $(\rho^{t(i)},\tau^{t(i)})\in(\rho,\tau)\,\mathrm{Irr}(C)$. $(xt(i),yt(i))\in\mathcal{F}_{\rho,\tau}$ for $(x,y)\in X\times M$, $i\in I$. Thus, define $\mathcal{T}(x,y,i)=\mathcal{R}(xt(i),yt(i))|_{U_{\chi,\psi}}$. Since $\mathcal{R}(z(i,j),z(i,j))=I$, $\mathcal{T}$ is ordinary on $(X\times M)\rtimes I$. Moreover, $\mathcal{T}^{a\sigma}\simeq\mathcal{T}$ for every (a,σ) in the mixed stabilizer with $a\in M\rtimes E$. We include the local inner maps $m\in M$ through $\mathcal{R}(m,m)$. Accordingly, by [141, Corollary 1.2], we choose $\mathcal{P}$ over K associated with χ on $X\rtimes I$, with $\mathcal{P}(x,1)=r_\rho(x)|_{V_\chi}$ and $\alpha(i,j)\in\mu_\infty$. On I, write $\mathcal{T}_i=\mathcal{T}(1,1,i)$ and $\mathcal{P}_i=\mathcal{P}(1,i)$. Then, by Schur's lemma for V_χ, choose $Q_i\in\mathrm{GL}_K(V_\psi^\vee)$ with $\mathcal{T}_i=\mathcal{P}_i\otimes Q_i$. Whenever $m\in M$, we put $\mathcal{P}'(m,i)=(r_\tau(m)|_{V_\psi})Q_i^\vee$. Thus, $\mathcal{T}(x,y,i)=\mathcal{P}(x,i)\otimes\mathcal{P}'(y,i)^\vee$ and $\alpha(i,j)=\alpha'(i,j)$. With $c=(x^{-1},\mathrm{Int}(x))$ centralizing X, $x\in N_X(P)\le M$. We write $t(\mathrm{Int}(x))=xz$, with $z\in Z(\widetilde{G})$. Thus, $\mathcal{R}(z,z)|_{U_{\chi,\psi}}=I$. Moreover, $\mathcal{P}(c)=\lambda_cI=\mathcal{P}'(c)$ and $C_{X\rtimes E}(X)\le M\rtimes I$.

For (a,σ) in the mixed stabilizer, write the functions β_a,β_a'. On $(X\times M)\rtimes I$, we put $r_a(x,y,i)=\beta_a(i)/\beta_a'(i)$. Since $\alpha=\alpha\prime$, $r_a(g_1g_2)=r_a(g_1)r_a(g_2)$. Moreover, choose S_a with $S_a\mathcal{T}^{a\sigma}(g)S_a^{-1}=\mathcal{T}(g)$. We choose the factor intertwiners A_a,B_a and put $D_a=(A_a\otimes B_a^\vee)S_a^{-1}$, and $D_a\mathcal{T}(g)D_a^{-1}=r_a(g)\mathcal{T}(g)$. Since $r_a|_{X\times M}=1$, $D_a\in\mathrm{End}_{X\times M}(U_{\chi,\psi})=K$. Thus, $r_a=1$ and $\beta_a=\beta_a'$. $(X\rtimes E,X,\chi)_{\mathcal{H}_p}\succeq_c(M\rtimes E,M,\psi)_{\mathcal{H}_p}$. By Definition 2.1, $\mathrm{iMN}(S,p)$. □

5.5 Even characteristic

Suppose that $q=2^f$ and $n\ge3$, and let $S=\mathrm{Sp}_{2n}(q)$. Moreover, we fix an odd prime $\ell\mid|S|$ and put $d=\mathrm{ord}_\ell(q)$. We write the split adjoint group $\mathbf{G}$ of type B_n, with $F=F_q$. By Zhang [208, proof of Proposition 6.23], the isogeny $\pi:\mathbf{G}\to\mathrm{Sp}_{2n}$ gives $\pi:\mathbf{G}^F\xrightarrow{\sim}S$, equivariant for F_2. We write $\mathbf{G}^*=\mathrm{Sp}_{2n}$. Moreover, we choose $P\in\mathrm{Syl}_\ell(S)$. By Malle [132, Theorem 5.14], Sp_{2n} has a Sylow d-torus $\mathbf{T}_{\mathrm{sc}}$ with $\mathrm{N}_S(P)\le\mathrm{N}_S(\mathbf{T}_{\mathrm{sc}})$. By Johansson [102, Proposition 2.72], $\mathrm{Aut}(S)_P$ normalizes $\mathbf{T}_{\mathrm{sc}}$. Then, we put $\mathbf{T}=(\pi^{-1}(\mathbf{T}_{\mathrm{sc}}))^\circ$. Thus, we find $\pi(\mathbf{T}^F)=\mathbf{T}_{\mathrm{sc}}^F$. Moreover, we put $C=\mathrm{C}_{\mathbf{G}}(\mathbf{T})^F$ and $N=\mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$. We have that $\pi(N)=N_{\mathrm{Sp}_{2n}}(\mathbf{T}_{\mathrm{sc}})^F$. Since $\pi F_2=F_2\pi$, $\mathbf{T}^a=\mathbf{T}$ for $a\in\mathrm{Aut}(S)_P$. Consequently, we find $\mathrm{N}_S(P)\le N<S$. We choose the local group N for $\mathcal{E}(S,s)$, with $s\in(\mathbf{G}^*)^F$. Moreover, $\mathrm{Out}(S)=\langle F_2\rangle$ for $n\ge3$. Since q is even, $z\mapsto z^2$ is bijective on $\mathbb{F}_q^\times$. We have that $\mu(zI)=z^2$ for the multiplier μ. Thus, we find $\mathrm{GSp}_{2n}(q)=S\times\mathbb{F}_q^\times$. On $S\times\mathbb{F}_q^\times$, we write the characters $\chi\boxtimes1$, with $\chi\in\mathrm{Irr}(S)$. Thus, $(\chi\boxtimes1)^{a\sigma}=\chi^{a\sigma}\boxtimes1$ for $a\in\mathrm{Aut}(S)$ and $\sigma\in\mathcal{H}_\ell$.

Lemma 5.46. *We choose rational semisimple $s\in(\mathbf{G}^*)^F$, with $s\ne1$. Moreover, we assume that $\chi\in\mathcal{E}(S,s)$ and $\ell\nmid\chi(1)$. We write $\mathbf{L}$ for the rational Levi dual to $\mathrm{C}_{\mathbf{G}^*}(s)$. With $\mathbf{T}\le\mathbf{L}$, we put $L=\mathbf{L}^F$, $\mathbf{C}=\mathrm{C}_{\mathbf{G}}(\mathbf{T})$ and $\mathbf{M}=\mathrm{C}_{\mathbf{L}}(\mathbf{T})$.*

Moreover, we put $M = \mathbf{M}^F$ and $K = \mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$. We write the linear character $\theta_s \in \mathrm{Irr}(L)$ for central $s \in \mathbf{L}^*$. On M, we choose the unipotent source λ in the label $(\mathbf{T}^*, s, \lambda, \eta)$. Moreover, we put $K_\lambda = I_K(\lambda)$ and $\beta_0 = \theta_s|_M\, \lambda$. Thus, we write the signed induction $\xi = \varepsilon R^{\mathbf{C}}_{\mathbf{M}}(\beta_0)$, with $\varepsilon \in \{1, -1\}$ and $\xi(1) > 0$. We have that $\xi \in \mathrm{Irr}(C)$, $I_N(\xi) = CK_\lambda$, $C \cap K_\lambda = M$ and $I_N(\xi)/C \simeq K_\lambda/M$.

Proof. On $V^* = \overline{\mathbb{F}}_q^{2n}$, write $V_x = \ker(s - xI)$. By [208, Proposition 6.23, (6.6)], $\mathbf{H}^* = \mathrm{C}_{\mathbf{G}^*}(s)$ is the connected proper Levi $\mathrm{Sp}(V_1) \times \prod_{\{x, x^{-1}\},\, x \neq 1} \mathrm{GL}(V_x)$. On $\{x, x^{-1}\}$, we consider the permutation by F. We write rational factors $\mathrm{GL}^{\varepsilon}_m(q^a)$ and $\mathrm{Sp}(V_1)^F$ of $(\mathbf{H}^*)^F$, with $\varepsilon \in \{1, -1\}$. Moreover, we put $m = \dim V_x$ and $a = |\mathrm{Orb}_F(\{x, x^{-1}\})|$ for each linear or unitary factor $\mathrm{GL}^{\varepsilon}_m(q^a)$. By [208, Proposition 6.23, Step 2], signed induction gives $\chi(1) = |S : L|_{2'}\vartheta(1)$ for the corresponding $\vartheta \in \mathcal{E}(L, s)$. Since $\ell \neq 2$ and $\ell \nmid \chi(1)$, $\ell \nmid [S : L]$. By [156, Theorem B and §4.4], choose the representative s in $s^{(\mathbf{G}^*)^F}$, with $\mathbf{T}^* \leq \mathbf{H}^*$. Write the full dual label $(\mathbf{T}^*, s, \lambda, \eta)$. Thus, $\mathrm{C}_{\mathbf{C}^*}(s) = \mathbf{C}^* \cap \mathbf{L}^* = \mathbf{M}^*$. $\varepsilon R^{\mathbf{C}}_{\mathbf{M}} : \mathcal{E}(M, s) \to \mathcal{E}(C, s)$ is bijective. Moreover, $\mathbf{H}^* = \mathrm{C}_{\mathbf{G}^*}(s_\ell) \cap \mathrm{C}_{\mathbf{G}^*}(s_{\ell'})$, with $s = s_\ell s_{\ell'}$.

Put $C^* = (\mathbf{C}^*)^F$. On the dual normalizer $N_{\mathbf{G}^*}(\mathbf{T}^*)^F$, we choose u stabilizing the rational class $[s]_{C^*}$. Since $[usu^{-1}]_{C^*} = [s]_{C^*}$, choose $c \in C^*$ with $cus(cu)^{-1} = s$. Thus, $cu \in N_{\mathbf{L}^*}(\mathbf{T}^*)^F$. By [114, corollary to Theorem 1] for connected $\mathbf{M}^*$, $(N_{\mathbf{L}^*}(\mathbf{T}^*)/\mathbf{M}^*)^F = N_{\mathbf{L}^*}(\mathbf{T}^*)^F/(\mathbf{M}^*)^F$. $I_N(\mathcal{E}(C, s)) = CK$ on the primal labels $(\mathbf{T}^*, s, \lambda, \eta)$. Moreover, $(\varepsilon R^{\mathbf{C}}_{\mathbf{M}}\beta)^k = \varepsilon R^{\mathbf{C}}_{\mathbf{M}}(\beta^k)$ for $k \in K$. $\theta_s^k = \theta_s$ on L for $k \in K$. Since $\varepsilon R^{\mathbf{C}}_{\mathbf{M}}$ is injective on $\mathcal{E}(M, s)$, $I_N(\xi) = CK_\lambda$. Whenever $m \in M$, $\lambda^m = \lambda$. Thus, $C \cap K_\lambda = M$ and $I_N(\xi)/C \simeq K_\lambda/M$. □

On L, we index the rational factors L_i by $i \in \mathcal{I}$, with $\mathcal{I} = \mathcal{I}_A \sqcup \mathcal{I}_C$. Write $L_i = \mathrm{GL}^{\varepsilon_i}_{m_i}(Q_i)$ for $i \in \mathcal{I}_A$, with $\varepsilon_i \in \{1, -1\}$ and $Q_i = 2^{a_i}$. Moreover, write $L_i = \mathrm{Sp}_{2m_i}(Q_i)$ for $i \in \mathcal{I}_C$. Given $i \in \mathcal{I}_A$, put $e_i = \mathrm{ord}_\ell(\varepsilon_i Q_i)$. With $i \in \mathcal{I}_C$, put $e_i = \mathrm{ord}_\ell(Q_i)/\gcd(2, \mathrm{ord}_\ell(Q_i))$. Moreover, write $m_i = e_i w_i + c_i$, with $0 \leq c_i < e_i$. On the natural module V_i, write $V_i = \bigoplus_{j=1}^{w_i} V_{i,j} \oplus V_{i,0}$. For the active summands, $\dim V_{i,j} = e_i$ and $\dim V_{i,0} = c_i$ for $i \in \mathcal{I}_A$. Moreover, for $i \in \mathcal{I}_C$, $\dim V_{i,j} = 2e_i$ and $\dim V_{i,0} = 2c_i$. Let T_i denote the image of $\mathbf{T}^F$ on V_i. Then, $V_{i,0} = V_i^{T_i}$ by the independent active weights of T_i. Thus, $kV_{i,0} = V_{i,0}$ for $k \in K$. Put $L_{i,\mathrm{res}} = \mathrm{GL}^{\varepsilon_i}_{c_i}(Q_i)$ for $i \in \mathcal{I}_A$ and $L_{i,\mathrm{res}} = \mathrm{Sp}_{2c_i}(Q_i)$ for $i \in \mathcal{I}_C$. On K, write $\mathrm{pr} : K \to \prod_i L_{i,\mathrm{res}}$, with $\mathrm{pr}(k) = (k|_{V_{i,0}})_i$. Thus, $\mathrm{pr}(kk') = \mathrm{pr}(k)\mathrm{pr}(k')$. Moreover, at $w_i = 0$, $V_{i,0} = V_i$ and $\mathrm{pr}_i(k) = k|_{V_i}$. On each extra central torus T_0, take the unipotent character 1_{T_0}. Let $\lambda_i \in \mathrm{Irr}(L_{i,\mathrm{res}})$ be the residual components of λ. Since λ is trivial on the active tori T_i, $\lambda = (\boxtimes_i \lambda_i) \circ \mathrm{pr}|_M$. Put $\Lambda_\lambda = (\boxtimes_i \lambda_i) \circ \mathrm{pr}|_{K_\lambda}$. Moreover, $\mathrm{pr}(M) = \prod_i L_{i,\mathrm{res}}$. Thus, $\Lambda_\lambda|_M = \lambda \in \mathrm{Irr}(M)$. On spaces W_i, choose $A_i : L_{i,\mathrm{res}} \to \mathrm{GL}(W_i)$ with character λ_i. Put $A_\lambda = (\boxtimes_i A_i) \circ \mathrm{pr}|_{K_\lambda}$. Moreover, for an internal active rotation r, $\mathrm{pr}(r) = 1$ and $A_\lambda(r) = I$. If a normalizes $L, \mathbf{T}$, let its permutation be $i \mapsto i^a$ on $\mathcal{I}$. Then, $\mathrm{pr}(k^a) = \mathrm{pr}(k)^a$, with inner or field maps on $L_{i,\mathrm{res}}$. On $i \in \mathcal{I}_A$, we also include $g \mapsto (g^{-1})^{\mathrm{tr}}$. Thus, $\Lambda_\lambda^{a\sigma} = \Lambda_{\lambda^{a\sigma}}$ for $\sigma \in \mathcal{H}_\ell$. If $L_{i,\mathrm{res}} = \mathrm{Sp}_4(Q_i)$ and $L^a_{i,\mathrm{res}} = L_{i,\mathrm{res}}$, $a|_{L_{i,\mathrm{res}}} \in \mathrm{Inn}(L_{i,\mathrm{res}})\langle F_2\rangle$. By Section 5.5, we write $\gamma_{\mathrm{exc}} : \mathrm{Sp}_4(q) \to \mathrm{Sp}_4(q)$, with $\gamma^2_{\mathrm{exc}} = F_2$.

On a residual $H = \mathrm{Sp}_{2m}(Q)$, we fix $Q = 2^a$ and $m \geq 2$. Then, we write $\phi : x \mapsto x^2$, with $|\phi| = a$. Moreover, we fix any unipotent $\psi \in \mathrm{Irr}(H)$. By Zhang [208, Proposition 6.23, Step 3], for ψ, we choose the ordinary cuspidal support $(\mathbf{L}_0, \lambda_0)$. Moreover, we write the Frobenius eigenvalue $\omega_\psi = \omega_{\lambda_0} \in \{1, -1\}$. We have that $Z(\mathbf{L}_0)$ is connected and $\mathbb{Q}(\lambda_0) = \mathbb{Q}(\psi) = \mathbb{Q}$. With toral $\mathbf{L}_0$, $\lambda_0 = 1$. Alternatively, we write type $B_{j(j+1)}$ on $\mathbf{L}_0/Z(\mathbf{L}_0)$, including $\mathbf{L}_0 = \mathbf{H}$ in type B_2. By Zhang [208, Hypothesis 4.12 and Proposition 6.23, Step 3], $k \in 2\mathbb{Z}$ and $2^{k/2} \in \mathbb{Q}$. We choose $\iota_\ell : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, with odd ℓ. If $\alpha^a = \omega_\psi$, in Zhang [208, Hypothesis 4.12], we choose $(\mathbb{K}_{\mathrm{F}}, \gamma_{\mathrm{F}}, \varepsilon_{\mathrm{F}}) = (\mathbb{Q}, 1, \alpha)$. By Zhang [208, Corollary 4.13(i),(iii)], we choose $Y \in \mathrm{Irr}(H \rtimes \langle\phi\rangle)$ with $Y|_H = \psi$ and $Y^\sigma = Y\gamma_\sigma$. Then, we find $\gamma_\sigma(\phi) = \sigma(\alpha)/\alpha$. If $a = 1$, we put $Y = \psi$ and $\alpha = \omega_\psi$. Moreover, on a finite cyclotomic $E \supseteq \mathbb{Q}(Y, \alpha)$, we put $E_\ell = \mathbb{Q}_\ell\iota_\ell(E)$. Since $\mathcal{H}_\ell|_E \simeq \mathrm{Gal}(E_\ell/\mathbb{Q}_\ell)$, $Y^\sigma = Y\gamma_\sigma$ for every $\sigma \in \mathcal{H}_\ell$. We put $h = \mathrm{ord}_\ell(2)$, $t = v_2(a)$ and $d_H = h/\gcd(h, a) = \mathrm{ord}_\ell(Q)$. With even d_H, $2^{t+1} \mid h \mid \ell - 1$. Thus, with $\omega_\psi = -1$, we choose primitive $\alpha \in \mu_{2^{t+1}} \subseteq \mathbb{Q}_\ell$, with $\alpha^a = -1$. With $\omega_\psi = 1$, we choose $\alpha = 1$. If d_H is even, $\alpha^\sigma = \alpha$ for $\sigma \in \mathcal{H}_\ell$ in both cases $\omega_\psi = \pm 1$. If d_H is odd, $\omega_\psi = \omega_\vartheta$ by Ruhstorfer et al. [156, Lemma 8.2 and Corollary 8.4] for the unipotent d_H-cuspidal source ϑ. On the common field period a, we choose $\alpha^a = \omega_\psi = \omega_\vartheta$. Thus, we write the ratio $\sigma(\alpha)/\alpha$ for both ordinary field extensions of ψ, ϑ. On residual $\mathrm{Sp}_2(Q)$, we write $\mathbb{Q}[\mathbb{P}^1(\mathbb{F}_Q)] = \mathbb{Q} \oplus V_{\mathrm{St}}$. Since ϕ permutes $\mathbb{P}^1(\mathbb{F}_Q)$, we choose rational extensions of $1, \mathrm{St}$ on $\mathbb{Q}, V_{\mathrm{St}}$. If T is a residual torus, we extend 1_T trivially.

Lemma 5.47. *With s, λ as in* Lemma 5.46*, we choose F_0 as an adapted field generator with $F_0^h = F$. We require that F_0 stabilizes $\mathbf{L}$ and the rational central series of s. Moreover, we lift F_0 to $a = lF_0$ with $l \in L$. We choose a to stabilize $\mathbf{T}$ and λ, and put $v = a^h \in K_\lambda$. On the cyclic field image of λ with maximal ordinary period, we extend ξ to $CK_\lambda\langle a\rangle$. Moreover, the mixed ratios for ξ match the mixed ratios for β_0 under a chosen extension to $K_\lambda\langle a\rangle$. Before choosing η, we choose the restriction to CK_λ equivariantly on the orbit of ξ.*

Proof. By [208, Proposition 6.23, Step 2], we choose F_0 on the rational series of s. Then, on the source inertia for λ, choose the corresponding power of F_0 and conjugate $\mathbf{T}, \lambda$ within L. Thus, the lifted operation $\varphi = \mathrm{Int}(l)F_0$ satisfies $\varphi^h = \mathrm{Int}(v)F$, $v = lF_0(l)\cdots F_0^{h-1}(l)$. We choose the extension Y_s of θ_s to $L\langle F_0\rangle$ with $Y_s(F_0) = 1$. Then, $Y_s(a) = \theta_s(l)$ and $Y_s(a)^h = \theta_s(v)$. Suppose that (b,σ) returns the central label s. The geometric part of (b,σ) acts by $F_0 \mapsto \mathrm{Int}(c)F_0$. The centralizer of s and the rational normalizer correction give $c \in L$. Moreover, $F_0^h = F$ and $bF = Fb$ put the F_0-norm of c in $\mathrm{Z}(\mathbf{G}) = 1$. Thus, $cF_0(c)\cdots F_0^{h-1}(c) = 1$. By [114, corollary to Theorem 1] on $\mathbf{L}$, write $c = m^{-1}F_0(m)$. The norm identity gives $F(m) = m$, so $m \in L$. Since θ_s is F_0-fixed, $\theta_s(c) = 1$. Thus, Y_s is invariant under (b,σ), including on $K_\lambda\langle a\rangle$.

The source extension Λ_λ is a-stable. The quotient $K_\lambda\langle a\rangle/K_\lambda$ is cyclic. By [95, Corollary 11.22], Λ_λ has an extension Y_λ. Its operator T at a satisfies $T^h = A_\lambda(v)$. More precisely, on an a-cycle, we compose the permutation Π_k of k residual factors with the k intertwiners. For the composite operator T_k, the components of T_k^k are cyclic products of the k intertwiners. Moreover, each component of T_k^h is the (h/k)-th power of a component of T_k^k. Thus, component periods for Λ_λ give the product period $\Lambda_\lambda(v)$. By the definition of Λ_λ, the active components of v have identity source matrix. We tensor Y_λ with the restriction of Y_s and obtain Y_0 above β_0. Moreover, the model for Y_0 has the given matrix at v, including θ_s on each active torus.

For Y_0, choose a parabolic $\mathbf{P} = \mathbf{LU}$ in $\mathbf{G}$. On $\mathbf{C}$, the radical is $\mathbf{V} = \mathrm{C}_{\mathbf{U}}(\mathbf{T})$, with Levi $\mathbf{M}$. Since $K_\lambda \le L$ normalizes $\mathbf{T}$, $k\mathbf{V}k^{-1} = \mathbf{V}$ for $k \in K_\lambda$. On $\mathbf{C}$, we choose the cyclic product for a from [208, Lemma 4.19], with $B = K_\lambda\langle a\rangle$ and $B_1 = K_\lambda$. Moreover, the ordinary fibre product $\mathcal{D}$ has base $C \times M$ and quotient B/M. We write Ξ_0 for the Euler character of $\mathcal{D}$, and $\langle \Xi_0|_{C\times M}, \xi \boxtimes \beta_0^\vee\rangle_{C\times M} \in \{1,-1\}$. The multiplicity in Ξ_0 follows from $\mathrm{C}_{\mathbf{C}^*}(s) = \mathbf{M}^*$ and the projector for $s = s_\ell s_{\ell'}$.

Given a corrected mixed stabilizer element (b,σ), the ordinary part b normalizes B and acts trivially on B/K_λ. Moreover, the commutator of b, a fixes β_0, has field exponent 0 and lies in K_λ. Before restriction to $\mathbf{C}$, take inner-field lifts of a, b on $\mathbf{G}$. At position j, the algebraic transporter is $t_j = \mathrm{Int}(a^j b a^{-j} b^{-1})b$. Thus, $t_j\varphi^j = \varphi^j b$ on geometric points. Moreover, for j at the last position, we set $\varphi^h = \mathrm{Int}(v)F$. Thus, the morphisms t_j compare product radicals with the finite B-action, as in Section 6.1. We write $\mathbf{V}_{\mathrm{prod}}$ for the original product radical and $\mathbf{V}'_{\mathrm{prod}}$ for its image under the morphisms t_j. The ordinary finite diagonal actions stabilize $\mathbf{V}_{\mathrm{prod}}$ and $\mathbf{V}'_{\mathrm{prod}}$. Moreover, we write e_s for the ordinary-series projector for s, and Ξ'_0 for the Euler character from $\mathbf{V}'_{\mathrm{prod}}$. By Lemma 7.168, the radicals $\mathbf{V}_{\mathrm{prod}}, \mathbf{V}'_{\mathrm{prod}}$ give $e_s(\Xi_0) = e_s(\Xi'_0)$. The unprojected character Ξ_0 has values in $\mathbb{Z}$. Thus, the coefficient action σ on e_s returns β_0, and the signed isotypic character $e_s(\Xi_0)$ is invariant under (b,σ).

We choose $\epsilon_0 \in \{1,-1\}$ such that $\langle (\epsilon_0 e_s(\Xi_0))|_{C\times M}, \xi \boxtimes \beta_0^\vee\rangle_{C\times M} = 1$. By Proposition 2.4, on $\mathcal{D}$, we choose the invariant tensor model V. Moreover, choose a model U_0 for Y_0. The M-fixed space $(V \otimes U_0)^M$ affords an ordinary extension X_ξ. By Corollary 2.5, the mixed ratios of X_ξ match the ratios for Y_0, an ordinary model. Moreover, on K_λ, the stabilizing source action fixes Λ_λ and the central twist θ_s. Thus, the mixed ratio for Y_0 is trivial on K_λ. Given an element jointly stabilizing ξ, correction by C matches s and the source pair $(\mathbf{M}, \beta_0)$ by Lemma 5.46. Moreover, conjugation by C fixes the extension character on CK_λ. Thus, $\Lambda(\xi) = X_\xi|_{CK_\lambda}$ is invariant under every stabilizer of ξ. Consequently, choose $\Lambda(\xi)$ equivariantly on each orbit of ξ. The choice of $\Lambda(\xi)$ does not depend on η, the relative Weyl label. □

Proposition 5.48. *Let $S = \mathrm{Sp}_{2n}(2^f)$ with $n \ge 3$. If $s \ne 1$, the Sylow d-torus $\mathbf{T}$ gives the mixed comparison on S, N for the ℓ'-degree characters in $\mathcal{E}(S,s)$. With labels (s,λ,η), the local character is $\psi_N = \mathrm{Ind}_{CK_\lambda}^N\left(\Lambda(\xi)\,\mathrm{Inf}_{CK_\lambda/C}^{CK_\lambda}\eta\right)$, with inflation along $CK_\lambda/C \cong K_\lambda/M$ by* Lemma 5.46.

Proof. Write $\chi = \varepsilon R_{\mathbf{L}}^{\mathbf{G}}(\theta_s\psi)$, with $\varepsilon \in \{1,-1\}$ chosen so that $\chi(1) > 0$. The character ψ is unipotent on L, with source λ and relative label η. Moreover, on C, we choose the character ξ from Lemma 5.46. The possible η lie in $\mathrm{Irr}(K_\lambda/M)$. Thus, ψ_N is irreducible by Clifford correspondence. For ξ, let $\Lambda'(\xi)$ be another ordinary extension to CK_λ. Then, $\Lambda'(\xi) = \Lambda(\xi)\,\mathrm{Inf}_{K_\lambda/M}^{CK_\lambda}\nu$ for a linear character ν of K_λ/M. Thus, $\eta\nu^{-1}$ gives ψ_N, and $(\eta\nu^{-1})(1) = \eta(1)$. Moreover, the choice of $\Lambda(\xi)$ in Lemma 5.47 covers every orbit of ξ equivariantly. Together with the action on (s,λ,η), the choice of $\Lambda(\xi)$ is equivariant by [156, Theorem B and §4.4].

By [156, §8.1, Proposition 8.5 and Lemma 8.6], on L the paired extensions have matching $\mathcal{H}_\ell$-actions. Moreover, η extends over $\mathbb{Q}_\ell$, and the partner is induced from (Λ_λ, η). Given an internal active rotation r, $A_\lambda(r) = I$ by the definition of Λ_λ. Given a linear character ν of K_λ/M, we put $\Lambda'_\lambda = \Lambda_\lambda\,\mathrm{Inf}_{K_\lambda/M}^{K_\lambda}\nu$ and $\eta' = \eta\nu^{-1}$. Then, $\Lambda'_\lambda\,\mathrm{Inf}_{K_\lambda/M}^{K_\lambda}\eta' = \Lambda_\lambda\,\mathrm{Inf}_{K_\lambda/M}^{K_\lambda}\eta$. Thus, the comparison on K_λ includes every $\eta \in \mathrm{Irr}(K_\lambda/M)$ above λ.

By [208, Corollary 4.13(i),(iii)], on a symplectic factor H, choose the ordinary field extension Y. Suppose that $H = \mathrm{Sp}_4(Q)$, with $Q = 2^{f_H} \geq 4$. By Lemma 5.49, we choose a_H as an adapted exceptional lift on H. The class of a_H^2 in $\mathrm{Out}(H)$ is the field class of F_2. Thus, by Proposition 5.50, restriction to a_H^2 gives the unipotent pair (χ_H, ψ_H) and the local period of a_H^2. Moreover, both H-models evaluate $a_H^{2f_H}$ in the finite normalizer. If $Q = 2$, the ordinary field automorphism on H is the identity. If $\ell = 3$, the $d_H = 2$ cut has five unipotent labels for H. Moreover, the local Weyl group W_H is dihedral of order 8. Similarly, for $\ell = 5$, the $d_H = 4$ cut has four unipotent labels, and $W_H \cong C_4$. Indeed, the six unipotent characters of H in Proposition 5.50 have degrees $(1, 16, 9, 1, 5, 5)$. The repeated degrees $1, 5$ belong to distinct unipotent characters of H. Moreover, $\mathcal{H}_\ell$ fixes the W_H-labels in both cases, with $i \in \mathbb{Q}_5$ for $W_H \cong C_4$. Given a model R_H of H, an inner lift $h \in H$ acts by $R_H(h)$. If $\ell \nmid |H|$, the natural space of H is residual and the comparison for H is the identity. Thus, when $\ell \nmid |H|$, the identity comparison covers H as a factor of L, for a Levi L of larger rank.

On an orbit $\mathcal{O}$ of isomorphic factors, fix L_0 and isomorphisms $t_i : L_0 \to L_i$. Given an operation b on $\mathcal{O}$, write the component $b_i = t_{bi}^{-1} b t_i$. Thus, for b, c, $(bc)_i = b_{ci} c_i$. Moreover, on L_i, the component scalar is $e_i(b_{ci}, c_i)$. Thus, the tensor operators for b, c have factor $\prod_i e_i(b_{ci}, c_i)$. For semilinear b, c, the composition on L_i includes the coefficient action of b. The linear tensor flips Π_π satisfy $\Pi_\pi \Pi_\tau = \Pi_{\pi\tau}$ for $\pi, \tau \in \mathfrak{S}_\mathcal{O}$. Moreover, the paired models have $e_i^{\mathrm{glo}}(b_{ci}, c_i) = e_i^{\mathrm{loc}}(b_{ci}, c_i)$ for every i. By Lemma 5.47, the power along $\mathcal{C}$ has the local matrix for $v = a^h$. If $B = K_\lambda \langle a \rangle$, let $\nu \in \mathrm{Irr}(B/K_\lambda)$ be linear. Then, each extension of Λ_λ to B is of the form $Y_\lambda \nu$. We write Y_{glo} for the paired global extension of ψ. Moreover, both twists by ν are inflated from B/K_λ. Given κ in the mixed stabilizer of Y, write $\mu_{Y,\kappa}$ for its correcting function. The ν-twists of $Y_{\mathrm{glo}}, Y_\lambda$ give $\mu_{Y_{\mathrm{glo}}\nu,\kappa}/\mu_{Y_{\mathrm{glo}},\kappa} = \nu^\kappa \nu^{-1} = \mu_{Y_\lambda \nu,\kappa}/\mu_{Y_\lambda,\kappa}$.

By Lemma 5.47, on L, K, we twist both unipotent extensions by Y_s. On the induction fibre product for $\mathbf{G}, \mathbf{L}$, we choose its product radical $\mathbf{U}_{\mathrm{prod}}$ as in [208, Proposition 6.23, Step 2 and Lemma 4.19]. Since $\mathbf{L}^*$ is the full dual centralizer of s, the stabilizer in S of the marked Levi series is L. Thus, L stabilizes the parabolic radical $\mathbf{U}$ for $\mathbf{L}$. Moreover, on the full series of s, by Lemma 5.47, we obtain $\widehat{\chi}$ by the projector argument. For $\widehat{\chi}$, the mixed ratios match those of $Y_s Y_{\mathrm{glo}}$ above $\theta_s \psi$.

At the ordinary stabilizer of η, we let J_η be the common quotient for X_ξ, Y_0. Since the field image of J_η is cyclic, we obtain $\widehat{\eta}$ by the intertwiner argument for Y_λ. We multiply X_ξ, Y_0 by $\widehat{\eta}$ on J_η. Consequently, the mixed factor of $\widehat{\eta}$ occurs in both products. Induction to N gives $\widehat{\psi}_N$ above ψ_N. Given b in the mixed stabilizer of ψ_N, we conjugate in N to fix the Clifford constituent on CK_λ. Next, we conjugate in C to fix the marked Levi source β_0. Both inner conjugations act on $\widehat{\chi}, \widehat{\psi}_N$. Thus, for the corrected element b', the ratios match those for b. Thus, for χ, the ratios equal those for $\theta_s \psi$ for the twist on K and for ψ_N on N. Since $\mathrm{Z}(S) = 1$, $\mathrm{Aut}(S)$ acts faithfully on S. We compose the models for $\widehat{\chi}, \widehat{\psi}_N$ with the inertia action maps from $S \rtimes \mathrm{Aut}(S)_P$ and $N \rtimes \mathrm{Aut}(S)_P$, respectively. Moreover, both models have matrix I on centralizing pairs. Thus, when $X = S$, we obtain the $\succeq_c$ relation. □

We set $S = \mathrm{Sp}_4(q)$, with $q = 2^f$ and $f \geq 2$. We choose an odd prime $\ell \mid |S|$ and $d = \mathrm{ord}_\ell(q) \in \{1, 2, 4\}$. Moreover, we choose the adjoint group $\mathbf{G}$ of type B_2, with $S = \mathbf{G}^F$. We choose the exceptional isogeny τ, with $\phi = \tau^2$ and $F = \phi^f$. By Zhang [208, pp. 131–132], $\mathrm{Aut}(S) = S \rtimes \langle \tau \rangle$ and $\mathrm{ord}(\tau|_S) = 2f$. Moreover, we choose $\mathbf{T}_0$ as the pinned maximal torus, with $\phi(\mathbf{T}_0) = \tau(\mathbf{T}_0) = \mathbf{T}_0$. We fix $P_S \in \mathrm{Syl}_\ell(S)$. At even f, Theorem 5.51 gives the comparison on $N_S(P_S)$.

Lemma 5.49. *We choose $W = \langle s, t : s^2 = t^2 = (st)^4 = 1 \rangle$, $w = st$ and $z = w^2$, and we fix $v \in \{1, z, w\}$ and $F_v = \mathrm{Int}(v)F$. Moreover, we put $S_v = \mathbf{G}^{F_v}$, $T_v = \mathbf{T}_0^{F_v}$ and $W_v = C_W(v)$. We take $(d, v) \in \{(1, 1), (2, z), (4, w)\}$. Thus, $P = (T_v)_\ell \in \mathrm{Syl}_\ell(S_v)$ and $M = N_{S_v}(P) = T_v \rtimes W_v$. We choose the faithful injection $\jmath_v : S_v \hookrightarrow \mathrm{Aut}(S_v)$, $x \mapsto \mathrm{Int}(x)$ and write x for $\jmath_v(x)$ in $\mathrm{Aut}(S_v)$. Thus, we fix $a \in \mathrm{Aut}(S_v)$ with the exceptional outer coset aS_v. We have that $a(T_v) = T_v$ and $a(W_v) = W_v$. At $d = 1, 2$, the map a exchanges s, t. Moreover, $(|T_v|, a^{2f}) = ((q-1)^2, 1)$ at $d = 1$ and $((q+1)^2, z)$ at $d = 2$. At $d = 4$, we have $a(w) = w$ and $(|T_v|, a^{2f}) = (q^2 + 1, w^{f-1})$.*

Proof. We choose the Tits representatives $s, t \in N_\mathbf{G}(\mathbf{T}_0)$. $\alpha^\vee(-1) = 1$ for each coroot $\alpha^\vee$ in characteristic 2. Thus, $s^2 = t^2 = (st)^4 = 1$. In $N_\mathbf{G}(\mathbf{T}_0)$, choose the subgroup $\langle s, t \rangle \simeq W$. Moreover, $\phi(s) = s$, $\phi(t) = t$, $\tau(s) = t$ and $\tau(t) = s$. With $v \in \{1, z\}$, set $a = \tau|_{S_v}$. Since $\tau(v) = v$, $\tau F_v = F_v \tau$. We find $a^{2f} = F|_{S_v} = \mathrm{Int}(v^{-1})|_{S_v}$. Next, we set $v = w$ and $a = (\mathrm{Int}(s)\tau)|_{S_w}$. Since $s\tau(w)s = w$, $(\mathrm{Int}(s)\tau)F_w = F_w(\mathrm{Int}(s)\tau)$. $a(w) = w$ and $a^2 = \mathrm{Int}(w)\phi|_{S_w}$, and $F|_{S_w} = \mathrm{Int}(w^{-1})|_{S_w}$. Thus, $a^{2f} = \mathrm{Int}(w^{f-1})|_{S_w}$. We put $s_v = 1$ for $v = 1, z$ and $s_v = s$ for $v = w$. By [114, corollary to Theorem 1], choose $x_v \in \mathbf{G}$ with $F(x_v) = x_v v$. Then, choose $\iota_v : S_v \xrightarrow{\sim} S$, $g \mapsto x_v g x_v^{-1}$. Put $u_v = x_v s_v \tau(x_v)^{-1}$. Since $v s_v = s_v \tau(v)$, $F(u_v) = u_v$. We find $\iota_v a \iota_v^{-1} = \mathrm{Int}(u_v)\tau$. Thus, $\langle aS_v \rangle = \mathrm{Aut}(S_v)/S_v \simeq C_{2f}$.

By [100, §5 and Table 5.1], $\mathbf{T}_0$ is a Sylow Φ_d-torus for F_v, with the stated $|T_v|$. The relative Weyl groups have 2-power order. Thus, [63, Theorem 3.2] gives $P = (T_v)_\ell \in \mathrm{Syl}_\ell(S_v)$. By [63, Theorem 3.3], $\mathbf{T}_0$ is the unique Sylow Φ_d-torus normalized by P. Hence, $N_{S_v}(P) = N_{\mathbf{G}}(\mathbf{T}_0)^{F_v}$, since P is characteristic in T_v. Since $F_v|_W = \mathrm{Int}(v)$, the rational Tits representatives give $N_{\mathbf{G}}(\mathbf{T}_0)^{F_v} = T_v \rtimes W_v$. □

Proposition 5.50. *We fix S_v, T_v, M, a from* Lemma 5.49. *We put $\mathcal{U}_\ell = \mathcal{E}(S_v, 1) \cap \mathrm{Irr}_{\ell'}(S_v)$ and $\mathcal{Y}_\ell = \{\psi \in \mathrm{Irr}(M) : T_v \le \ker \psi\}$. Moreover, we choose the unipotent comparison $f_{\mathrm{unip}} : \mathcal{U}_\ell \to \mathcal{Y}_\ell$. We take ordinary extensions $\widehat{\chi}, \widehat{\psi}$ for $\psi = f_{\mathrm{unip}}(\chi)$. Thus, $\mu_{\chi,g,\sigma} = \mu_{\psi,g,\sigma}$ on the common quotient $\mathcal{C}_\chi$ for $(g,\sigma) \in (\mathrm{Aut}(S_v)_P \times \mathcal{H}_\ell)_\chi$.*

Proof. On S_v, we choose $\mathcal{U} = \{1, \mathrm{St}, v_0, c, u_+, u_-\}$. We find $(1(1), \mathrm{St}(1), v_0(1), c(1), u_+(1), u_-(1)) = (1, q^4, q(q+1)^2/2, q(q-1)^2/2, q(q^2+1)/2, q(q^2+1)/2)$. In particular, by [208, pp. 131–134, Propositions 6.24–6.25], $\mathcal{U} \subseteq \mathrm{CF}(S_v, \mathbb{Q})$. $u_+^a = u_-$, $u_-^a = u_+$ and $\chi^a = \chi$ for $\chi \in \{1, \mathrm{St}, v_0, c\}$. Moreover, choose the rational extensions $\widehat{1}, \widehat{\mathrm{St}}$ on $S \rtimes \langle\tau\rangle$. Take rational extensions $\widehat{u}_+, \widehat{u}_-$ on $S \rtimes \langle\phi\rangle$. By Lemma 5.49, take $\iota_v : S_v \to S$. Then, $\iota_v a \iota_v^{-1} = \mathrm{Int}(u_v)\tau$. Fix $R_{v_0} : S \to \mathrm{GL}_{v_0(1)}(\mathbb{Q})$ and $R_{c,\ell} : S \to \mathrm{GL}_{c(1)}(\mathbb{Q}_\ell)$. By the cited models, fix $C \in \mathrm{GL}_{v_0(1)}(\mathbb{Q})$ and $A \in \mathrm{GL}_{c(1)}(\mathbb{Q}_\ell)$. Consequently, $CR_{v_0}(g)C^{-1} = R_{v_0}(\tau(g))$ and $AR_{c,\ell}(g)A^{-1} = R_{c,\ell}(\tau(g))$. We choose $I_{v_0} = I_{\mathbb{C}^{v_0(1)}}$ and $I_c = I_{\mathbb{C}^{c(1)}}$. Moreover, we put $B_{v_0} = b_v C$ and $B_c = b_c A$. Set $b_v = \sqrt{2q}$, with $C^{2f} = (2q)^{-f} I_{v_0}$, and we set $b_c = \alpha\sqrt{2q}$, with $\alpha^{2f} = (-1)^{f+1}$ and $A^{2f} = (-q)^{-(f+1)} I_c$. Thus, $B_{v_0}^{2f} = I_{v_0}$ and $B_c^{2f} = I_c$. Choose the extension $\widehat{R}_{c,\ell}$ with $\widehat{R}_{c,\ell}|_S = R_{c,\ell}$ and $\widehat{R}_{c,\ell}(\tau) = B_c$. Thus, choose $\widehat{c}_\ell = \mathrm{Tr}\,\widehat{R}_{c,\ell}$ at the chosen ℓ-adic place. We choose $\widehat{c}$ as the complex character of $\widehat{R}_{c,\ell}$ under the fixed injection $\iota_\ell : \mathbb{Q}^{\mathrm{ab}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ and fix $\sigma \in \mathcal{H}_\ell$. Then, by the decomposition action $\mathcal{H}_\ell$, choose $\widetilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ with $\widetilde{\sigma}\iota_\ell = \iota_\ell\sigma$. Thus, $\widetilde{\sigma}(A) = A$ and $\widetilde{\sigma}(R_{c,\ell}(g)) = R_{c,\ell}(g)$ for $g \in S$. Moreover, we choose the quotient characters $\varepsilon_{\chi,\sigma}$ with $\widehat{\chi}^\sigma = \varepsilon_{\chi,\sigma}\widehat{\chi}$ for $\chi = v_0, c$. $\varepsilon_{v_0,\sigma}(\tau) = \sigma(b_v)/b_v$ and $\varepsilon_{c,\sigma}(\tau) = \sigma(b_c)/b_c$. We find $\varepsilon_{\chi,\sigma}|_S = 1$. Thus, $\varepsilon_{\chi,\sigma}(u_v\tau) = \varepsilon_{\chi,\sigma}(\tau)$. We set $K = \mathbb{Q}(\zeta_{16f})$ for the scalars b_v, b_c, i. Then, take $\iota_\ell|_K : K \hookrightarrow \overline{\mathbb{Q}}_\ell$. By the finite decomposition group $D_\ell(K/\mathbb{Q})$, $K^{\mathcal{H}_\ell} = K \cap \mathbb{Q}_\ell$.

First, we set $d = 1$ and $\mathcal{U}_\ell = \{1, \mathrm{St}, v_0, u_+, u_-\}$. We set $d = 2$ and $\mathcal{U}_\ell = \{1, \mathrm{St}, c, u_+, u_-\}$. Thus, at $d = 1, 2$, $W_v = W$. We choose the rational model $\rho : W \to \mathrm{GL}_2(\mathbb{Q})$, with $\rho(s) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\rho(t) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and we put $J = \rho(s) + \rho(t)$. Thus, $J^2 = 2I_2$, $J\rho(s)J^{-1} = \rho(t)$ and $J\rho(t)J^{-1} = \rho(s)$. We find $\rho(z) = -I_2$. Choose $D_1(a) = J/\sqrt{2}$ at $d = 1$. Since $J^2 = 2I_2$, $D_1(a)^{2f} = I_2$. Choose $D_2(a) = i\alpha J/\sqrt{2}$ at $d = 2$. Thus, $D_2(a)^{2f} = (i\alpha)^{2f} I_2 = -I_2 = \rho(z)$. Thus, we consider both models D_1, D_2 on the finite group $H = \langle W_v, a\rangle$. Take scalar ratios $r_1 = b_v/(1/\sqrt{2}) = 2\sqrt{q}$ and $r_2 = b_c/(i\alpha/\sqrt{2}) = -2i\sqrt{q}$. $q \equiv 1 \pmod{\ell}$ at $d = 1$ and $-q \equiv 1 \pmod{\ell}$ at $d = 2$. Then, by Hensel, $r_1 \in K \cap \mathbb{Q}_\ell$ at $d = 1$, and $r_2 \in K \cap \mathbb{Q}_\ell$ at $d = 2$. At $d = 1$, $\sigma(r_1) = r_1$. $\sigma(r_2) = r_2$ at $d = 2$. Thus, $\sigma(b_v)/b_v = \sigma(1/\sqrt{2})/(1/\sqrt{2})$ at $d = 1$. At $d = 2$, $\sigma(b_c)/b_c = \sigma(i\alpha/\sqrt{2})/(i\alpha/\sqrt{2})$. Moreover, we put $\psi_\eta = \mathrm{Inf}_{W_v}^M \eta$. We take linear labels $\eta_{\epsilon,\epsilon'}$ with $\eta_{\epsilon,\epsilon'}(s) = \epsilon$, $\eta_{\epsilon,\epsilon'}(t) = \epsilon'$ and $\epsilon, \epsilon' \in \{1, -1\}$. Moreover, choose the pairings $f_{\mathrm{unip}}(1) = \psi_{\eta_{1,1}}$ and $f_{\mathrm{unip}}(\mathrm{St}) = \psi_{\eta_{-1,-1}}$. We choose $f_{\mathrm{unip}}(u_+) = \psi_{\eta_{1,-1}}$ and $f_{\mathrm{unip}}(u_-) = \psi_{\eta_{-1,1}}$. Moreover, we pair $f_{\mathrm{unip}}(v_0) = \psi_{\mathrm{Tr}\,\rho}$ at $d = 1$ and $f_{\mathrm{unip}}(c) = \psi_{\mathrm{Tr}\,\rho}$ at $d = 2$. $\eta_{\epsilon,\epsilon'}(z) = 1$, and we choose $\widehat{\eta}_{\epsilon,\epsilon'}|_W = \eta_{\epsilon,\epsilon'}$. Thus, put $\widehat{\eta}_{1,1}(a) = \widehat{\eta}_{-1,-1}(a) = 1$. We put $\widehat{\eta}_{1,-1}(a^2) = \widehat{\eta}_{-1,1}(a^2) = 1$, and $\eta_{\epsilon,\epsilon'}^a = \eta_{\epsilon',\epsilon}$. Since u_+, u_- are rational, $u_\epsilon^{a^{2j+1}\sigma} = u_{-\epsilon}$ for $\epsilon \in \{+, -\}$.

Next, we set $d = 4$ and $\mathcal{U}_\ell = \{1, \mathrm{St}, v_0, c\}$. By Lemma 5.49, $W_v = \langle w\rangle \simeq C_4$ and $a(w) = w$. Given $j \in \{0, 1, 2, 3\}$, choose $\rho_j(w) = i^j$. Thus, we choose extensions $D_j(a) = b_j$ with $b_j^{2f} = i^{j(f-1)}$. $\ell \equiv 1 \pmod 4$. First, we take odd f. We set $b_j = 1$ when $i^{j(f-1)} = 1$ and $b_j = i$ when $i^{j(f-1)} = -1$. By odd f, $\alpha = 1$ and $b_v = b_c = 2^{(f+1)/2} \in \mathbb{Q}$. $b_j \in \mathbb{Q}_\ell$ because $i \in \mathbb{Q}_\ell$. Next, we suppose that f is even and put $t_2 = v_2(f)$, $h = \mathrm{ord}_\ell(2)$. Since $h/\gcd(h, f) = 4$, $v_2(h) = t_2 + 2$. $\ell \equiv 1 \pmod 8$, and $(2/\ell) = 1$. Thus, $h \mid (\ell-1)/2$ and $v_2(\ell-1) \ge t_2 + 3$. $\sqrt{2} \in \mathbb{Q}_\ell$ and $\mu_{2^{t_2+3}} \subseteq \mathbb{Q}_\ell$, and we choose $\alpha \in \mu_{2^{t_2+2}}$ of order 2^{t_2+2}. Since f is even, $\alpha^{2f} = -1$. $\{z^{2f} : z \in \mu_{2^{t_2+3}}\} = \mu_4$, and we choose $b_j \in \mu_{2^{t_2+3}}$ with $b_j^{2f} = i^{j(f-1)}$. Thus, $\sigma(b_j) = b_j$, $\sigma(b_v) = b_v$ and $\sigma(b_c) = b_c$. Take any fixed pairing $f_{\mathrm{unip}} : \mathcal{U}_\ell \to \{\psi_{\rho_0}, \psi_{\rho_1}, \psi_{\rho_2}, \psi_{\rho_3}\}$. $\chi^a = \chi$ and $\rho_j^a = \rho_j$ for $\chi \in \mathcal{U}_\ell$. Thus, $\widehat{\chi}^\sigma = \widehat{\chi}$ and $\widehat{\psi}_{\rho_j}^\sigma = \widehat{\psi}_{\rho_j}$ for $\psi_{\rho_j} = f_{\mathrm{unip}}(\chi)$.

Choose $H_\eta = \mathrm{Stab}_H(\eta)$ and choose the corresponding model D_η from $D_1, D_2, D_j, \widehat{\eta}_{\epsilon,\epsilon'}$. Thus, $\mathrm{Tr}(D_\eta|_{W_v}) = \eta$. Thus, we choose $\widehat{\psi}_\eta(th) = \mathrm{Tr}\, D_\eta(h)$ for $t \in T_v, h \in H_\eta$. We choose $\mathcal{A}_v = \mathrm{Aut}(S_v)$ and $\Gamma_v = (\mathcal{A}_v)_P$, and $(\Gamma_v)_{\psi_\eta} = T_v \rtimes H_\eta$. By the label actions f_{unip}, $(\Gamma_v)_\chi = (\Gamma_v)_\psi$. By Sylow conjugacy, $\mathcal{A}_v = S_v\Gamma_v$. Take $\jmath : \Gamma_v/M \xrightarrow{\sim} \mathcal{A}_v/S_v$, $xM \mapsto xS_v$. Moreover, we put $\mathcal{C}_\chi = (\mathcal{A}_v)_\chi/S_v$ and $\bar{a} = aS_v$. $\mathcal{C}_\chi = \jmath((\Gamma_v)_\psi/M)$, and we fix $\varepsilon_{\psi,\sigma}^{\mathrm{loc}}$ with $\widehat{\psi}^\sigma = \varepsilon_{\psi,\sigma}^{\mathrm{loc}}\widehat{\psi}$. Thus, write $\varepsilon_{\psi,\sigma} = \varepsilon_{\psi,\sigma}^{\mathrm{loc}} \circ \jmath^{-1}$ on $\mathcal{C}_\chi$. We have that $\mathcal{C}_\chi \le \langle\bar{a}\rangle \simeq C_{2f}$ and that $\mathcal{C}_{u_+} = \mathcal{C}_{u_-} = \langle\bar{a}^2\rangle$, and we choose the ordinary extensions $\widehat{\chi}, \widehat{\psi}$ from the matrices $B_{v_0}, B_c, D_1, D_2, D_j$ or the rational models $\widehat{1}, \widehat{\mathrm{St}}, \widehat{u}_+, \widehat{u}_-$. Then, $\varepsilon_{\chi,\sigma}(\bar{a}^r) = \varepsilon_{\chi,\sigma}(\bar{a})^r$ for $\chi \ne u_+, u_-$. By the computed scalar ratios r_1, r_2, b_j, $\varepsilon_{\chi,\sigma} = \varepsilon_{\psi,\sigma}$ on $\mathcal{C}_\chi$. Moreover, $\chi^\sigma = \chi$ and $\psi^\sigma = \psi$. Choose $(g,\sigma) \in (\Gamma_v \times \mathcal{H}_\ell)_\chi$. Thus, $g \in (\Gamma_v)_\chi = (\Gamma_v)_\psi$. $\widehat{\chi}^g = \widehat{\chi}$ and $\widehat{\psi}^g = \widehat{\psi}$. Thus, $\mu_{\chi,g,\sigma} = \varepsilon_{\chi,\sigma} = \varepsilon_{\psi,\sigma} = \mu_{\psi,g,\sigma}$ on $\mathcal{C}_\chi$. □

Theorem 5.51. *Let $S = \mathrm{Sp}_4(2^f)$ with even $f \geq 2$. Choose an odd prime $\ell \mid |S|$ and $P \in \mathrm{Syl}_\ell(S)$. Moreover, we put $M = N_S(P)$ and $\Gamma = \mathrm{Aut}(S)_P$. We take a $\Gamma \times \mathcal{H}_\ell$-equivariant bijection $\Omega : \mathrm{Irr}_{\ell'}(S) \to \mathrm{Irr}_{\ell'}(M)$. We have that $(S \rtimes D, S, \chi)_{\mathcal{H}_\ell} \succeq_{\mathrm{c}} (M \rtimes D, M, \Omega(\chi))_{\mathcal{H}_\ell}$, with $D = \Gamma_{\chi^{\mathcal{H}_\ell}}$.*

Proof. By Lemma 5.49, the Lang model $X = S_v$ has $M_v = N_X(P_v) = N_X(T_v)$, with $P_v = (T_v)_\ell$. By Sylow conjugacy, choose $\iota_v : X \xrightarrow{\sim} S$ with $\iota_v(P_v) = P$. Then, $\iota_v(M_v) = M$. Set $\tau_0 = \iota_v^{-1}\tau\iota_v$, with $|\tau_0| = 2f$. Moreover, put $\mathcal{A} = \mathrm{Aut}(X) = X \rtimes \langle\tau_0\rangle$ and $\Gamma_v = N_{\mathcal{A}}(P_v)$. By [97, Lemma 1.13], $\mathcal{A} = X\Gamma_v$. With $W_v = C_W(v)$, put $H = \langle W_v, a\rangle$. By Lemma 5.49, a normalizes T_v, W_v, and $a^{2f} \in W_v$. Since aX has order $2f$, $H \cap X = W_v$. Thus, $T_v \cap H = 1$ and T_vH contains $N_X(P_v) = T_vW_v$ and maps onto $\mathcal{A}/X$. Hence, $\Gamma_v = T_v \rtimes H$.

By [100, Lemma 5.1], the parametrization on $\mathrm{Irr}_{\ell'}(X)$ and $\mathrm{Irr}_{\ell'}(M_v)$ is equivariant. Write $X^* = (\mathbf{X}^*)^{F^*}$ and choose $s_0 \in T_v^*$ dual to $\theta \in \mathrm{Irr}(T_v)$, with $\mathbf{C} = C_{\mathbf{X}^*}(s_0)$. Set $W_\theta = (W_v)_\theta$ and $I_\theta = T_v \rtimes W_\theta$. By [100, §§3.2 and 5], (θ, η) gives the local character $\psi_{\theta,\eta} = \mathrm{Ind}_{I_\theta}^{M_v}(\Theta_\theta \,\mathrm{Inf}_{W_\theta}^{I_\theta}\,\eta)$, where $\Theta_\theta(tx) = \theta(t)$ for $t \in T_v, x \in W_\theta$. Thus, a regular s_0 gives $W_\theta = 1$. If $s_0 \neq 1$ is nonregular, [100, Table 5.1] gives $d \in \{1, 2\}$, semisimple rank 1 for $\mathbf{C}$, and $W_\theta \simeq C_2$. Then, $1, \mathrm{St}$ match $1_{W_\theta}, \mathrm{sgn}_{W_\theta}$, respectively. At $s_0 = 1$, the equivariant map is f_{unip} of Proposition 5.50. Thus, $\Omega_v : \mathrm{Irr}_{\ell'}(X) \to \mathrm{Irr}_{\ell'}(M_v)$ combines these maps.

We fix $\chi \in \mathcal{E}(X, s_0)$ with $s_0 \neq 1$. By [100, §3.2 and Table 5.1], its Jordan label lies in $\{1_{\mathbf{C}^{F^*}}, \mathrm{St}_{\mathbf{C}^{F^*}}\}$. When $\mathbf{C} = \mathbf{T}_v^*$, $1_{\mathbf{C}^{F^*}} = \mathrm{St}_{\mathbf{C}^{F^*}}$. Thus, we choose the semisimple character χ_{s_0} or a regular character χ. We choose $n_\chi = \chi_{s_0}(1) = [X^* : C_{X^*}(s_0)]_{2'}$. Hence, n_χ is odd. By [114, corollary to Theorem 1] on $\mathbf{T}_v^*$, choose a rational representative w_- of $-1 \in W$. Then, $s_0^{w_-} = s_0^{-1}$. $\overline{\chi_{s_0}} = \chi_{s_0}$ by the unique Jordan label $1_{\mathbf{C}^{F^*}}$. Since $X = [X, X]$, $\det\chi_{s_0} = 1_X$. We put $H_s = \mathcal{A}_{\chi_{s_0}}$. By [143, Theorem 2.3], a unique real extension $\widehat{\chi}_{s_0}$ exists on H_s with $\det\widehat{\chi}_{s_0} = 1_{H_s}$. Choose $(g, \sigma) \in \mathcal{A} \times \mathcal{H}_\ell$, with $\chi_{s_0}^{g\sigma} = \chi_{s_0}$. Since $(\chi^g)^\sigma = (\chi^\sigma)^g$ on $\mathrm{Irr}(X)$, $H_s^g = H_s$. $\overline{\widehat{\chi}_{s_0}^{g\sigma}} = \widehat{\chi}_{s_0}^{g\sigma}$ and $\det\widehat{\chi}_{s_0}^{g\sigma} = 1_{H_s}$. Since $\widehat{\chi}_{s_0}$ is unique, $\widehat{\chi}_{s_0}^{g\sigma} = \widehat{\chi}_{s_0}$.

Next, we choose a regular character χ. Choose a maximal unipotent subgroup $U \leq X$ from the split pinning for τ_0. Moreover, we take its simple coordinates $x(u), y(u) \in \mathbb{F}_q$ modulo the nonsimple root groups U_{ns}. Put $\lambda_U(u) = (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(x(u)+y(u))}$, and $(x(\tau_0(u)), y(\tau_0(u))) = (y(u)^2, x(u))$. Since $\mathrm{Tr}(z^2) = \mathrm{Tr}(z)$, $\lambda_U^{\tau_0} = \lambda_U$. Choose $\widehat{\lambda}_U(u\tau_0^j) = \lambda_U(u)$ on $B_U = U \rtimes \langle\tau_0\rangle$ and choose $\mathcal{V}_{\mathrm{GG}} = \mathrm{Ind}_{B_U}^{\mathcal{A}}\widehat{\lambda}_U$ with rational model R_{GG}. Then, by [102, Propositions 3.36 and 3.40], $\langle \mathrm{Tr}\, R_{\mathrm{GG}}|_X, \chi\rangle = 1$. With $e_\chi = \frac{\chi(1)}{|X|}\sum_{x\in X}\chi(x^{-1})R_{\mathrm{GG}}(x)$, we put $V_\chi = e_\chi\mathcal{V}_{\mathrm{GG}}$. In particular, $\dim V_\chi = \chi(1)$. Thus, we choose $\widehat{\chi}(h) = \mathrm{Tr}(e_\chi R_{\mathrm{GG}}(h))$ for $h \in \mathcal{A}_\chi$. We find $\widehat{\chi}|_X = \chi$. We take $(g, \sigma) \in (\mathcal{A} \times \mathcal{H}_\ell)_\chi$. Since R_{GG} is rational, $\sigma(e_\chi) = e_{\chi^\sigma}$. We find $R_{\mathrm{GG}}(g^{-1})\sigma(e_\chi)R_{\mathrm{GG}}(g) = e_\chi$. Thus, $\widehat{\chi}^{g\sigma} = \widehat{\chi}$ by the trace formula for $\widehat{\chi}$. We choose the associated ordinary models $\mathcal{P}_\chi$ for the two cases χ_{s_0}, χ. Thus, $\alpha_{\mathcal{P}_\chi} = 1$ and $\mu_{\mathcal{P}_\chi,g,\sigma} = 1$ for every (g, σ) in the mixed stabilizer.

Fix $\theta \neq 1$ and put $H_\theta = \mathrm{Stab}_H(\theta)$, $K_\theta = T_v \rtimes H_\theta$. Since H_θ fixes θ, the rule $\widehat{\theta}(th) = \theta(t)$ extends θ to K_θ. If $\eta = 1_{W_\theta}$, take $\widehat{\eta} = 1_H$. If $\eta = \mathrm{sgn}_{W_\theta} \neq 1$, then $d \in \{1, 2\}$ and $W_v = W$. In this case, the reflection determinant δ of W has $\delta(s) = \delta(t) = -1$ and $\delta(z) = 1$. Since a exchanges s, t and $a^{2f} \in \{1, z\}$, δ extends to $\widehat{\eta}$ on H with $\widehat{\eta}(a) = 1$. Thus, both choices satisfy $\widehat{\eta}|_{W_\theta} = \eta$ and $\widehat{\eta}(H) \subseteq \{1, -1\}$. By [95, Theorem 6.11], for $\psi = \psi_{\theta,\eta}$ we have $\Gamma_{v,\psi} = M_vH_\theta = M_vK_\theta$, with $K_\theta \cap M_v = I_\theta$. Put $\widehat{\psi} = \mathrm{Ind}_{K_\theta}^{\Gamma_{v,\psi}}(\widehat{\theta}\,\mathrm{Inf}_{H_\theta}^{K_\theta}(\widehat{\eta}|_{H_\theta}))$. By [95, Problem 5.2], the single double coset gives $\widehat{\psi}|_{M_v} = \psi$. Choose (g, σ) in the mixed stabilizer of ψ in $\Gamma_v \times \mathcal{H}_\ell$ and write $g = th$, with $t \in T_v, h \in H$. Since $t \in M_v$, $\widehat{\psi}^t = \widehat{\psi}$. We choose $w_0 \in W_v$ with $\theta^{hw_0\sigma} = \theta$, and we put $r = hw_0$. Since $\theta^{r\sigma} = \theta$, $H_\theta^r = H_\theta$. On K_θ, $\widehat{\theta}^{r\sigma} = \widehat{\theta}$. $\widehat{\eta}^{r\sigma} = \widehat{\eta}$ because $r \in H$ and $\widehat{\eta}(H) \subseteq \mathbb{Q}$. Thus, $\widehat{\psi}^{r\sigma} = \widehat{\psi}$. $w_0 \in M_v \leq \Gamma_{v,\psi}$. Thus, $\widehat{\psi}^{g\sigma} = \widehat{\psi}$. We choose the ordinary local model $\mathcal{P}_\psi$ with character $\widehat{\psi}$. Thus, $\alpha_{\mathcal{P}_\psi} = 1$ and $\mu_{\mathcal{P}_\psi,g,\sigma} = 1$.

From Proposition 5.50, we set $\theta = 1$. $\alpha_\chi|_{\Gamma_{v,\psi}\times\Gamma_{v,\psi}} = \alpha_\psi$ and mixed functions $\mu_{\chi,g,\sigma} = \mu_{\psi,g,\sigma}$. Moreover, we choose the nonidentity series $s_0 \neq 1$ with $\alpha_\chi = \alpha_\psi = 1$ and $\mu_{\chi,g,\sigma} = \mu_{\psi,g,\sigma} = 1$. With $\psi = \Omega_v(\chi)$, we set $D_v = (\Gamma_v)_{\chi^{\mathcal{H}_\ell}}$. Then, $(\Gamma_v)_{\chi^{\mathcal{H}_\ell}} = (\Gamma_v)_{\psi^{\mathcal{H}_\ell}}$ by the equivariant bijection Ω_v. Choose the homomorphism $\rho : X \rtimes D_v \to \mathcal{A}$, $(x, d) \mapsto \mathrm{Int}(x)d$. Take its local map $\rho_M = \rho|_{M_v\rtimes D_v}$. Since $Z(X) = 1$, $C_{\mathcal{A}}(X) = 1$. We have that $C_{X\rtimes D_v}(X) = \ker\rho$ and that $\mathrm{Inn}(M_v) \leq D_v$. With $(x, d) \in \ker\rho$, $\mathrm{Int}(x)^{-1} = d \in \Gamma_v$. $x \in M_v$. Thus, $\ker\rho = \{(m, \mathrm{Int}(m)^{-1}) : m \in M_v\}$ and $\ker\rho = \ker\rho_M$. We choose the model spaces V_χ, V_ψ of $\mathcal{P}_\chi, \mathcal{P}_\psi$ and choose the models $\mathcal{Q}_\chi = \mathcal{P}_\chi \circ \rho$ and $\mathcal{Q}_\psi = \mathcal{P}_\psi \circ \rho_M$ on $I_\chi = \rho^{-1}(\mathcal{A}_\chi)$, $I_\psi = \rho_M^{-1}(\Gamma_{v,\psi})$. Thus, $\mathcal{Q}_\chi(k) = I_{V_\chi}$ and $\mathcal{Q}_\psi(k) = I_{V_\psi}$ for $k \in \ker\rho$. $\alpha_{\mathcal{Q}_\chi}|_{I_\psi\times I_\psi} = \alpha_{\mathcal{Q}_\psi}$ and $\mu_{\mathcal{Q}_\chi,g,\sigma} = \mu_{\mathcal{Q}_\psi,g,\sigma}$. Consequently, $(X \rtimes D_v, X, \chi)_{\mathcal{H}_\ell} \succeq_c (M_v \rtimes D_v, M_v, \psi)_{\mathcal{H}_\ell}$. By [208, Theorem 6.26], $Z(S) = 1$ and $\mathrm{Mult}(S) = 1$. Thus, choose the universal cover $X \xrightarrow{\iota_v} S$ and $\iota_v(M_v) = M$. Thus, the comparison Ω is on $M = N_S(P)$. □

5.6 Proof of Theorem 5.1

Proof. We write X for the universal cover of S. By [208, §6.6, proof of Theorem 6.1], we have that $X = \mathrm{Sp}_{2n}(q)$ except for $S = \mathrm{Sp}_6(2)$, when $X = 2.S$. If $S = \mathrm{Sp}_6(2)$, iMN(S,p) follows from [157, Lemma 4.1(2)] for every $p \mid |S|$. At $p \mid q$, apply [162, Theorem 1.3] and [101, Theorem 1.1], with the latter covering $C_2(2^f)$. If $p = 2$ and q is odd, apply [158, Theorem A and §5]. Thus, consider $X = \mathrm{Sp}_{2n}(q)$ for odd $p \nmid q$. Suppose that q is odd, and choose $P \in \mathrm{Syl}_p(X)$. By Proposition 5.3, take D. Thus, $\mathrm{N}_X(P) \leq \mathrm{N}_X(D) < X$, and $\mathrm{N}_X(D)$ is $\mathrm{Aut}(X)_P$-stable. By Theorem 5.2, Ω_X gives iMN(S,p) with $M = \mathrm{N}_X(D)$.

Suppose that $q = 2^f$ and $n \geq 3$. With $d = \mathrm{ord}_p(q)$, let $N = \mathrm{N}_X(\mathbf{T})$ from Section 5.5. Thus, $\widetilde{X} = X \times Z$ and $\widetilde{N} = N \times Z$, with $Z = \mathbb{F}_q^\times$. By [156, §4.4, proof of Theorem B], the tuples $(\mathbf{T}^*, s, \lambda, \eta)$ index the characters above $\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(N)$. On $\widetilde{C} = C \times Z$, extend $\xi \boxtimes 1_Z$ by $\Lambda(\xi) \boxtimes 1_Z$ if $s \neq 1$. For each $\delta \in \mathrm{Irr}(Z)$, extend $\xi \boxtimes \delta$ by $\Lambda(\xi) \boxtimes \delta$. Thus, restriction gives the correspondence on X, N. If $s \neq 1$, Lemma 5.46 gives the label group K_λ/M. Hence, the local map in [156, §4.4] gives ψ_N from Proposition 5.48. If $s = 1$, [156, Theorem D] gives the unipotent comparison on N. Thus, the bijection $\mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(N)$ satisfies $\succeq_c$ on both parts. If $n = 2$ and $q = 2^f$, simplicity gives $f \geq 2$. If f is odd, [100, Theorem 5.4] gives iMN(S,p). At even f, Theorem 5.51 gives the mixed comparison on $\mathrm{N}_X(P)$ for $d \in \{1,2,4\}$. The local group is the Sylow d-torus normalizer by Lemma 5.49. Consequently, iMN(S,p) holds on each universal cover X in the stated family. □

6 Even orthogonal groups

Theorem 6.1. *Let S be a nonabelian simple group equal to ${}^3D_4(q)$ or $\mathrm{P}\Omega^\epsilon_{2n}(q)$, where $n \geq 4$ and $\epsilon \in \{1,-1\}$ in the latter case, and q is a prime power. Then* iMN(S,p) *holds for every prime $p \mid |S|$. If $S = {}^3D_4(q)$ and $P \in \mathrm{Syl}_p(S)$, there is an $\mathcal{H}_p$-equivariant bijection $\mathrm{Irr}_{p'}(S) \to \mathrm{Irr}_{p'}(\mathrm{N}_S(P))$.*

Theorem 6.2. *Let $q = 2^f$, let $n \geq 4$, and let $S = \mathrm{P}\Omega^\epsilon_{2n}(q)$ be simple for $\epsilon \in \{1,-1\}$. Then* iMN(S,p) *holds for every prime $p \mid |S|$, with the comparison on the universal cover of S.*

For the universal cover X, Cabanes–Späth [35, Theorem B] gives the ordinary central comparison with $\mathrm{N}_X(P)$ for $P \in \mathrm{Syl}_p(X)$. On the simply connected group $X = \mathbf{X}^F$, let $p \nmid 2q$ and $d = \mathrm{ord}_p(q)$. By Malle [132, Theorem 5.14], choose a Sylow Φ_d-torus $\mathbf{S}_d$ with $\mathrm{N}_X(P) \leq \mathrm{N}_X(\mathbf{S}_d)$ for $P \in \mathrm{Syl}_p(X)$. At $d = 1$, Schaeffer Fry [166, Theorem 6.2] gives an $\mathrm{Aut}(X)_P \times \mathcal{H}_p$-equivariant bijection on $X, \mathrm{N}_X(\mathbf{S}_d)$. If $q = 2^f$, Section 6.4 constructs $\mathbf{S}_d$ and both centralizers. Moreover, for $S = \mathrm{P}\Omega^+_8(q)$, Section 6.4 establishes iMN(S,p) with the graph action. If q is odd, Theorems 6.147 and 6.163 treat $\mathrm{P}\Omega^+_8(q)$ and $\mathrm{P}\Omega^-_8(q)$. If $n = 5$, Theorem 6.205 treats both $\mathrm{P}\Omega^\epsilon_{10}(q)$. Moreover, if $p \mid q \pm 1$, Corollary 6.202 treats the proper Levi L, and Proposition 6.204 treats the last $Q(a,c)$ packet.

Write $X = \mathrm{Spin}^\epsilon_{2n}(q)$ and $\langle z_0 \rangle$ for the kernel of $X \to \mathrm{SO}^\epsilon_{2n}(q)$. Given $n \geq 5$, Theorem 6.37 treats characters with $\chi(z_0) = -\chi(1)$. Moreover, the natural lift for $\chi(z_0) = \chi(1)$ is rational. Given a proper Levi L, Corollary 6.276 compares the factors of L. If its zero-weight space V_0 has $\dim V_0 \leq 6$, Theorem 6.199 treats L with its shared centre. If s is quasi-isolated with sign spaces V_{+1}, V_{-1}, Theorem 6.275 treats the split case for X. For unequal symbols on V_{+1}, V_{-1}, we apply Corollary 6.95.

If $p \mid q \pm 1$, Theorem 6.208 treats odd n without eigenvalues $1, -1$. Moreover, Theorems 6.215 and 6.224 treat $I(1,b)$ and $Q(a,c)$. Corollary 6.225 treats $\mathrm{P}\Omega^-_{12}(q)$ at these primes p. If the sign spaces in $I(a,b)$ are rational, Theorem 6.254 covers $I(a,a)$ with coincident symbols. Theorems 6.255 and 6.259 compare $\mathrm{N}_X(Q)$ and $\mathrm{N}_X(P)$ at $z_0 = -1, 1$, respectively. If $\chi(z_0) = -\chi(1)$, Theorem 6.255 pairs characters trivial on U_Q, U_P. Theorem 6.273 permits arbitrary characters of U_Q, U_P if $(q^e - t_b)_2 = 2$. Moreover, for exchanged s, Proposition 6.256 treats the cuspidal fibre. If s has fourth-root eigenvalues, Proposition 6.261 treats $\mathcal{E}(X,s)$ for both values at z_0. For arbitrary s, Theorem 6.272 gives the ordinary comparison for S.

On simply connected $X = \mathbf{X}^F$ of type $D_n, {}^2D_n$ or ${}^3D_4$, we put $M_0 = \mathrm{N}_X(\mathbf{S}_d)$. Then, Ruhstorfer et al. [156, Theorem D] compares $\mathcal{U} = \mathcal{E}(X,1)_{p'}$ on M_0. If $s \neq 1$, Proposition 6.194 treats the balanced blocks of $\mathcal{E}(X,s)$, and for square q, Proposition 6.105 treats rational colours in $\mathcal{E}(X,s)$. Moreover, Section 6.1 transfers $\mathcal{E}(X,s)$ through rational stabilizers. Propositions 6.8 and 6.9 give $\mathcal{C}_{D,I}$ and the mixed comparison for $f_{L,I}$. If both sign spaces $V_{\pm 1}$ occur, Lemma 6.10 and Proposition 6.13 describe lifts $\tilde{s}$.

If the spectrum is $\{i, -i\}$ and n is odd, Proposition 6.16 gives f_s. If $\epsilon = -1$, Proposition 6.25 extends $\alpha \otimes \gamma^\vee$ to $\Delta_{K,J}$ with $\mathcal{H}_p$-invariant character $\widehat{\Xi}$. If $\chi(z_0) = -\chi(1)$, Proposition 6.29 recovers s and its symbol from M_0. If

the residual rank is 4, Proposition 6.35 gives the quartic row of $\mathcal{E}(X,s)$. If one sign symbol in $Q(a,c)$ is degenerate, Proposition 6.94 treats $\mathcal{D}_{\mathrm{uneq}}$.

If $S=\mathrm{P}\Omega_8^+(q)$, Proposition 6.114 treats the cocycle 1. Moreover, Proposition 6.123 assigns the orbit to s in the other series. Given S, Theorem 6.130 treats split principal series. For S at cyclic periods, apply Corollary 6.124. If $d=4$, Theorem 6.135 compares mixed actions. If $n=5$ and $d=4$, Theorem 6.179 covers the full p'-part for either sign. If $n=p^a$ and $p\mid q-\epsilon$, Theorem 6.182 takes the transitive p^a-cycle. If $n=5$ and $p\mid q\pm 1$, the cases of $\mathcal{E}(X,s)$ are in Propositions 6.183, 6.185, 6.187 and 6.189.

At even q, Zhang [208, Corollary 8.7] gives the ordinary relation $\succeq$ on $\widehat{S}$. At odd q, Zhang [208, Proposition 7.2] gives $\succeq$ on $\mathcal{E}(X,1)$. Moreover, Zhang [208, Corollary 7.14] treats non-quasi-isolated s on the inverse image of $B\le \mathrm{Out}(X)_\chi$ for nilpotent B. Suppose $C_{X^*}(s)\le L^*$ for a proper Levi L^*. At even n or outside the quartic case, Zhang [208, Proposition 7.3] gives $\succeq$ for $(A_\gamma)_\chi$. At odd n, Zhang [208, Propositions 7.6 and 7.13] treat the quartic series $\mathcal{E}(X,s)$. If $n=4$, Zhang [208, Theorem 7.42] gives a comparison on a parabolic P_0 for type $I(2,2)$. If $n\ge 5$, Zhang [208, Lemma 7.15] gives the types $I(a,b)$, $Q(a,c)$ and $U(n)$.

Given $I(a,b)$, Zhang [208, Propositions 7.18 and 7.20] describes steps on the symbols $\Lambda_\pm$, with endpoints in Zhang [208, Propositions 7.25 and 7.26]. Given $Q(a,c)$, Zhang [208, Propositions 7.21, 7.26, 7.34 and 7.35] treats both cases for λ. If $\epsilon=1$ and n is even, $U(n)$ occurs. Given $U(n)$, Zhang [208, Propositions 7.21 and 7.36] give the sources for λ on both rational forms. On $\mathcal{E}(X,s)$, the comparison with $M_0=\mathrm{N}_X(\mathbf{S}_d)$ also requires weights of $\mathbf{S}_d$ and the mixed actions for $\succeq_c$.

6.1 Spin restriction and Levi induction

Write $W^\pm=\{\pm1\}^n\rtimes\mathfrak{S}_n$ and $\varepsilon_W(\varepsilon,\pi)=\prod_i\varepsilon_i$. Its kernel $W^+=\ker\varepsilon_W$ is the Weyl group of type D_n. By Juteau et al. [104, §4.1.2], the bipartition character $\chi_{\alpha,\beta}$ restricts irreducibly to W^+ when $\alpha\ne\beta$. If $\alpha=\beta$, then $n=2|\alpha|$ and the restriction is $\chi_+ +\chi_-$, with $\chi_+\ne\chi_-$ of degree $\chi_{\alpha,\alpha}(1)/2$. Moreover, $W^\pm\setminus W^+$ exchanges χ_+ and χ_-. The twist satisfies $\chi_{\alpha,\beta}\varepsilon_W=\chi_{\beta,\alpha}$. For the spin cover X, $\succeq_c$ compares extension actions.

Suppose that $k=\mathbb{F}_q$ with odd q. Choose a nondegenerate quadratic space W of even dimension. Let H be a hyperbolic plane. Next, put $G=O(W\perp H)$. If $\gamma\in\mathrm{Irr}(k^\times)$ satisfies $\gamma^2=1$ and $\rho\in\mathrm{Irr}(O(W))$, choose a singular line $L\le H$ and its stabilizer $P_L\le G$. Put $U_\gamma\rho=\mathrm{Ind}_{P_L}^G(\gamma\otimes\rho)$, where $\gamma\otimes\rho$ is inflated along the actions on L and $L^\perp/L$. Write D_γ for the adjoint of U_γ in the Witt tower. If W has no singular line, set $D_\gamma\rho=0$.

Lemma 6.3. *Given $\rho\in\mathrm{Irr}(O(W))$, $\langle U_\gamma\rho,U_\gamma\rho\rangle\ \le\ 2+\langle D_\gamma\rho,D_\gamma\rho\rangle$.*

Proof. Let η be the inflation of $\gamma\otimes\rho$ to P_L. Then, $\mathcal{I}_\rho$ is its induced module with fibre V_{L_i} at L_i. By [95, Lemma 5.2 and Problem 5.6], $\langle U_\gamma\rho,U_\gamma\rho\rangle$ sums the contributions from $P_L\backslash G/P_L$. These correspond to G-orbits of ordered singular lines (L_1,L_2), by Witt's theorem. At $L_1=L_2$, the contribution is 1. If L_1,L_2 are nonorthogonal, the two quotients $L_i^\perp/L_i$ are isometric to $(L_1+L_2)^\perp$. The line scalars are inverse, so $\gamma^2=1$ gives equal fibre actions on $G_{12}=\mathrm{Stab}_G(L_1,L_2)$. Thus, the contribution is 1, since G_{12} contains $\mathrm{O}((L_1+L_2)^\perp)$.

If $L_e\ne L_x$ and $L_e\perp L_x$, complete them to hyperbolic planes H_e,H_x with residual space W'. The line radical R_x in $\mathrm{O}(H_x\perp W')$ fixes H_e. Since R_x acts trivially on V_{L_x}, a fibre map $V_{L_e}\to V_{L_x}$ factors through $(V_{L_e})_{R_x}$. The exchanged radical R_e puts its image in $V_{L_x}^{R_e}$. By averaging in characteristic 0, $V_R\cong V^R$ for each radical R. The scalars on L_e,L_x select the γ-parts, both affording $D_\gamma\rho$ on $\mathrm{O}(W')$. Thus, the contribution of (L_e,L_x) is at most $\langle D_\gamma\rho,D_\gamma\rho\rangle$. When W has Witt index 0, the contribution is 0. □

Proposition 6.4. *Write $\mathcal{S}(\rho)$ and $\mathcal{P}(\rho)$ for the supports of $U_\gamma\rho$ and $D_\gamma\rho$. With ordered partitions (α,β), suppose that at each rank $\mathcal{S}(\rho)$ lists the one-box additions to (α,β) and $\mathcal{P}(\rho)$ lists its one-box deletions. Then, $U_\gamma\rho$ and $D_\gamma\rho$ have multiplicities* 1. *In the involution series of $O(W)$, let Λ_+,Λ_- be the ordered Jordan symbols at $1,-1$. By* Pan [149, §3.3, (6.14), Theorem 8.12]*, U_γ fixes $\Lambda_{-\epsilon}$ and adds one box to (α,β) for the symbol Λ_ϵ selected by γ, at fixed defect. Thus, $U_\gamma\rho$ and $D_\gamma\rho$ have multiplicities* 1 *in this series.*

Proof. With Y the Young lattice, Y^2 is 2-differential by [183, Definition 1.1 and Corollary 1.4]. Thus, $|\mathcal{S}(\rho)|=|\mathcal{P}(\rho)|+2$, including at $\alpha=\beta$. By adjunction, $\langle D_\gamma\rho,D_\gamma\rho\rangle=|\mathcal{P}(\rho)|$ if the lower-rank multiplicities are 1, with $\mathcal{P}(\rho)=\varnothing$ at the bottom. Hence, Lemma 6.3 gives $\sum_{\rho'\in\mathcal{S}(\rho)}m_{\rho'}^2\le|\mathcal{S}(\rho)|$, where $m_{\rho'}$ is the multiplicity of ρ'. Since each $m_{\rho'}\ge1$, all are 1. □

Let A be a finite group fixing H and its ordered lines L, L' by field operations. Suppose that $\widehat{\rho}$ extends ρ to $O(W) \rtimes A$ and extend γ by 1_A. Inflate the product $\widehat{\eta} = \gamma \otimes \widehat{\rho}$ to $P_A = P_L \rtimes A$. Put $G_A = G \rtimes A$ and $\widehat{\mathcal{I}}_\rho = \mathrm{Ind}_{P_A}^{G_A} \widehat{\eta}$. Since $G_A = GP_A$, Isaacs [95, Problem 5.6] gives $\widehat{\mathcal{I}}_\rho|_G \simeq \mathcal{I}_\rho$. If A fixes a successor ρ', its isotypic space $V_{\rho'}$ affords an extension $\widehat{\rho}'$ by Proposition 6.4.

Suppose that b is a mixed operation fixing ρ, ρ', P_L, with $\widehat{\eta}^b = \widehat{\eta}\lambda$, with λ linear on A. Via $P_A/P_L \simeq G_A/G$, induction gives $\widehat{\mathcal{I}}_\rho^b \simeq \widehat{\mathcal{I}}_\rho \otimes \lambda$. Moreover, $e_{\rho'}$ is fixed by b, so $(\widehat{\rho}')^b = \widehat{\rho}'\lambda$. Thus, each line induction U_γ gives the twist λ, including $\lambda \neq 1$. If a field lift a satisfies $a^f = v \in G$, take the finite group $\langle G, a \rangle$ with that relation. Given a cover $\pi_A : \widehat{G}_A \to \langle G, a \rangle$, the kernel $\ker \pi_A$ must act trivially.

Suppose that $q = \ell^f$ is odd. Take a quadratic space V of dimension $2n$ and positive semisimple rank. Write $B = O(V)$, $K = \mathrm{SO}(V)$ and $J = \Omega(V)$. Thus, $K = \ker \Delta$ for the determinant $\Delta : B \to \{1, -1\}$, and $J = \ker \nu$ for the quadratic spinor norm $\nu : K \to \{1, -1\}$. Let $\pi : X = \mathrm{Spin}(V) \to J$ be the cover. If $\ker \pi \leq \ker \chi$, the character χ is inflated from J. Moreover, restrict the flag character to K.

Let $t^2 = 1$ in K, with nonzero sign spaces V_-, V_+. In the modified parametrization of Pan [149, Theorem 8.12], write (L_-, L_+) for the symbols of ρ. By Pan [149, Corollaries 8.3 and 8.13], the labels of ρ^m and $\rho\Delta$ are (L_-^{t}, L_+) and $(L_-^{\mathrm{t}}, L_+^{\mathrm{t}})$, with m of nonsquare multiplier. For $t^2 = 1$, the same actions occur in Waldspurger [202, §4.4, Proposition (b)(iii), (v)] on (L_-, L_+). Thus, L^{t} has the rows of L exchanged, and $L = L^{\mathrm{t}}$ means that L is degenerate. If $g \in B$, then $\rho^{gm} = \rho^m$.

Lemma 6.5. *Suppose at least one of L_-, L_+ is nondegenerate. Then, $\xi = \rho|_K$ is irreducible and $\xi^g = \xi$ for every $g \in B \setminus K$. Under the stated action of m, $\xi^m = \xi$ exactly when $L_- = L_-^{\mathrm{t}}$ or $L_+ = L_+^{\mathrm{t}}$.*

Proof. Since $\rho\Delta \neq \rho$, the cyclic-quotient formulas give $\xi \in \mathrm{Irr}(K)$ and $\mathrm{Irr}(B \mid \xi) = \{\rho, \rho\Delta\}$. In particular, ξ is B-invariant. Thus, $\xi^m = \xi$ exactly when $\rho^m \in \{\rho, \rho\Delta\}$. The two identities $\rho^m = \rho$ and $\rho^m = \rho\Delta$ hold exactly when $L_- = L_-^{\mathrm{t}}$ and $L_+ = L_+^{\mathrm{t}}$, respectively, by the action of m. □

Suppose exactly one of L_-, L_+ is degenerate. Put $\chi_B = \rho|_J$. If $\lambda \in \mathrm{Irr}(B/J)$ has $\lambda|_K = \nu$, Pan [149, Corollaries 8.13–8.14] exchanges L_-, L_+ under the twist λ, up to row transposition. Thus, $\rho\lambda, \rho\lambda\Delta, \rho\Delta$ differ from ρ because L_-, L_+ differ in degeneracy. By Isaacs [95, Lemma 5.2 and Problem 5.3], $\langle \chi_B, \chi_B \rangle_J = 1$. By Lemma 6.5, $\xi = \rho|_K$ is B-invariant and $\xi^m = \xi$. Hence, $\chi_B = \xi|_J$ is invariant under B and m. These elements and scalars generate $\mathrm{GO}(V)$. By Schaeffer Fry–Taylor [165, Theorem A(i),(ii)], ξ is rational-valued and $\xi^{F_\ell} = \xi$. Hence, $\mathbb{Q}(\chi_B) = \mathbb{Q}$ and $\chi_B^{F_\ell} = \chi_B$. By the cyclic-quotient formulas, $\mathrm{Irr}(K \mid \chi_B) = \{\xi, \xi\nu\}$. By the flag argument after Lemma 6.93, χ_B occurs with multiplicity 1. Thus, its flag summand affords $\widehat{\chi}_B$ in Proposition 6.94.

Lemma 6.6. *Let $J \leq K \lhd H$ be finite groups. Suppose that $J \lhd H$ and $[K : J] = 2$, with quotient character $\nu \neq 1$. Let Y be an ordinary H-module with character Z on K. If $\xi \in \mathrm{Irr}(K)$ satisfies $\langle Z, \xi \rangle = 1$, $\langle Z, \xi\nu \rangle = 0$, then $\chi = \xi|_J$ is irreducible and occurs with multiplicity 1 in $Y|_J$. The space $e_\chi Y$ affords an extension $\widehat{\chi}$ to H_χ. If a mixed operation a fixes χ and satisfies $Y^a \simeq Y \otimes \lambda$ for a linear character λ trivial on J, then $\widehat{\chi}^a = \widehat{\chi}\lambda|_{H_\chi}$.*

Proof. Since the two multiplicities differ, $\xi\nu \neq \xi$. By the cyclic-quotient formulas, $\chi \in \mathrm{Irr}(J)$ and $\mathrm{Ind}_J^K \chi = \xi + \xi\nu$. By [95, Lemma 5.2], $\langle Z|_J, \chi \rangle = 1 + 0 = 1$. The idempotent e_χ projects onto one irreducible J-space, and H_χ fixes e_χ. Hence, $e_\chi Y$ affords $\widehat{\chi}$. If a fixes χ, $e_\chi^a = e_\chi$. Since $\lambda|_J = 1$, the isomorphism $Y^a \simeq Y \otimes \lambda$ restricts to $e_\chi Y$, yielding $\widehat{\chi}^a = \widehat{\chi}\lambda|_{H_\chi}$. □

Let $V = H^b \perp W$, with H^b split of dimension $2b$, and put $B_- = \mathrm{O}(H^b)$ and $B_0 = \mathrm{O}(W)$. For the rational dual element $t = (-I_{2b}, t_0)$, suppose that -1 is absent from the spectrum of t_0. Choose a nondegenerate symbol at $+1$ and the remaining labels of t_0, with character $h \in \mathrm{Irr}(B_0)$. On B_-, take the unipotent d_λ with equal rows (λ, λ), so $b = 2|\lambda| > 0$. In Pan's parametrization, we let $\rho \in \mathrm{Irr}(B)$ be the ordered parent for d_λ and h. Consequently, the opposite order at $+1$ corresponds to $h\Delta_0$, with $\Delta_0 = \det_W$.

Let $P \leq B$ stabilize a maximal singular b-space $L_b \leq H^b$. To define F_τ, let η_q be the quadratic character of $\mathbb{F}_q^\times$. Moreover, for $\tau \vdash b$, we let U_τ be the unipotent character of $\mathrm{GL}_b(q)$ indexed by τ. Thus, for a class function u on B_0, we put $F_\tau(u) = \mathrm{Ind}_P^B(((\eta_q \circ \det)U_\tau) \boxtimes u)$ with inflation from $\mathrm{GL}_b(q) \times B_0$. Since U_τ occurs in split Borel induction for $\mathrm{GL}_b(q)$, the constituents of $F_\tau(u)$ occur in $U_{\eta_q}^b u$ for irreducible u. Under U_{η_q}, only the symbol at -1 can change (Pan [149, (6.11), (6.14), Theorem 8.12]). Thus, among the t_0-series characters u, only h and $h\Delta_0$ can yield ρ.

Choose a rational dual torus $T \ni t_0$ with $c = \langle R_{T,t_0}^{B_0}, h\rangle_{B_0} \neq 0$. The series of h ensures such a T. Moreover, $R_{T,t_0}^{B_0}\Delta_0 = R_{T,t_0}^{B_0}$, since $R_{T,t_0}^{B_0}$ is induced from $\mathrm{SO}(W)$ (Pan [149, (3.4)]). Thus, $h\Delta_0$ has coefficient c. Since $\Delta|_P$ is inflated from Δ_0, we deduce that $F_\tau(h\Delta_0) = F_\tau(h)\Delta$. Therefore, expanding $R_{T,t_0}^{B_0}$ yields $\langle F_\tau(R_{T,t_0}^{B_0}), \rho\rangle_B = c\,\langle F_\tau(h), \rho + \rho\Delta\rangle_B$. Given $w \in \mathfrak{S}_b$, we let T_w have type w in $\mathrm{GL}_b(q)$. By Pan [149, §3.1], the torus expansion of U_τ has coefficients $\chi^\tau(w)/b!$, with Specht character χ^τ. Since $\eta_q \circ \det$ corresponds to the dual Levi scalar $-I_b$, we deduce that the coefficients of $R_{T_w\times T,t}^{B}$ in the expansion of $F_\tau(R_{T,t_0}^{B_0})$ are $\chi^\tau(w)/b!$. The induction stages are $T_w \times T$, the connected Levi, and K, with ordinary induction $K \leq B$ at the end (Pan [149, (3.4)]). The centralizer of t has factors B_- and the centralizer of t_0. Thus, the pairing Pan [149, (1.3)] gives $\langle R_{T_w\times T,t}^{B}, \rho\rangle_B = c\,\langle R_{T_w,1}^{B_-}, d_\lambda\rangle_{B_-}$. For the residual character $h\Delta_0$, the pairing with $R_{T,t_0}^{B_0}$ is also c. Moreover, adding H^b raises both rational ranks by b, so their signs cancel.

In B_-, the torus expansion is induction of U_τ from the stabilizer P_b of L_b. Since $P_b \leq \mathrm{SO}(H^b)$, we consider $d_\lambda|_{\mathrm{SO}(H^b)}$, which has two constituents of label (λ,λ). Let $\omega_{\lambda,\lambda}$ be the $W(B_b)$-character of label (λ,λ). The two Weyl characters in $\omega_{\lambda,\lambda}|_{W(D_b)}$ correspond to the two constituents of $d_\lambda|_{\mathrm{SO}(H^b)}$. Thus, by Geck–Malle [72, Theorem 3.2.7], the coefficient is $\langle \omega_{\lambda,\lambda}|_{\mathfrak{S}_b}, \chi^\tau\rangle$. The source subgroup in $W(B_b)$ contains the signs C_2^b. Hence, Isaacs [95, Problem 5.2] gives $\omega_{\lambda,\lambda}|_{\mathfrak{S}_b} = \mathrm{Ind}_{\mathfrak{S}_{b/2}\times\mathfrak{S}_{b/2}}^{\mathfrak{S}_b}(\chi^\lambda \boxtimes \chi^\lambda)$, with Frobenius characteristic s_λ^2 for the Schur function s_λ. Moreover, for $b = 2$, the two factors of $W(D_2)$ have rank 1.

Cancelling c gives $\langle F_\tau(h), \rho + \rho\Delta\rangle_B = [s_\tau]s_\lambda^2$. With the row orders of (6.14), $F_\tau(h)$ has the positive order of h. Moreover, under $h \mapsto h\Delta_0$, the positive rows exchange. Given $\alpha \in \mathrm{Irr}(B/J)$ with $\alpha|_K = \nu$, $\rho\alpha$ lies over $-t$. If t and $-t$ are rationally conjugate, $\rho\alpha$ has the degenerate symbol at $+1$. Since the symbol of $F_\tau(h)$ at $+1$ is nondegenerate, $\langle F_\tau(h), \rho\alpha\rangle_B = 0$ also when H^b and the $+1$ space are isometric (Pan [149, Corollaries 8.13–8.14]). The symbol at $+1$ also gives $\rho\Delta \neq \rho$. Thus, by the quotient twist formula, $\chi_B = \rho|_J \in \mathrm{Irr}(J)$. By Isaacs [95, Lemma 5.2], $\langle F_\tau(h)|_J, \chi_B\rangle_J = \sum_{\alpha\in\mathrm{Irr}(B/J)}\langle F_\tau(h), \rho\alpha\rangle_B = [s_\tau]s_\lambda^2$. If $\tau = 2\lambda = (2\lambda_i)_i$, the weight-$2\lambda$ space in the tensor square is $L_\lambda \otimes L_\lambda$, where L_λ is the highest weight line for λ. Hence, we find $[s_{2\lambda}]s_\lambda^2 = 1$. If a flag module Y for $\widehat{B}$ has character $F_{2\lambda}(h)$ on B, its isotypic summand Y_{χ_B} affords an extension to $\widehat{B}_{\chi_B}$. Thus, under the hypotheses of Lemma 6.6, $Y^a \simeq Y \otimes \kappa$ on $\widehat{B}$ gives $(Y_{\chi_B})^a \simeq Y_{\chi_B} \otimes \kappa|_{\widehat{B}_{\chi_B}}$.

With $q = p^f$ odd, we let η_q be the quadratic character of $\mathbb{F}_q^\times$. Take $J = \Omega(V)$ and $K = \mathrm{SO}(V)$ from Section 6.1. Thus, the action on J is by semilinear similarities of V, including in rank 4. Suppose that the rational involution t is noncentral, with split space H^b at -1, where $b = 2|\lambda|$. Let H_+ be the nondegenerate unipotent character of $K_0 = \mathrm{SO}(W)$, where $W \neq 0$ is the positive space. Choose $h \in \mathrm{Irr}(\mathrm{O}(W))$ above H_+. Thus, we compare Pan's parent $\rho \in \mathrm{Irr}(\mathrm{O}(V))$ with the lift on $H = \mathrm{GSO}(V)$ before restriction to J. Put $H = \mathrm{GSO}(V)$. On $\mathrm{GL}_b(q) \times \mathrm{GSO}(W)$, the source has factors $\eta_q \circ \det$ and the unipotent lift $\widetilde{H}_+$ of H_+. Since a nontrivial multiplier twist of $\widetilde{H}_+$ lies in a different series, $\widetilde{H}_+|_{K_0} = H_+$. Let $\widetilde{t}$ be its dual element. At $\widetilde{t}$, the characters $\widetilde{\rho}_\pm \in \mathrm{Irr}(H)$ have connected labels $(D_\lambda^\pm, H_+)$, where $D_\lambda^\pm$ are the constituents of $d_\lambda|_{\mathrm{SO}(H^b)}$. Moreover, $H^* \to K^*$ has connected central kernel. In $C_{K^*}(t)$, both components stabilize H^b and W, even when the ranks of H^b and W coincide. Thus, the other component of $C_{K^*}(t)$ has determinant -1 on both H^b and W. In H^*, we write $\widetilde{t} = z\,\mathrm{vol}(H^b)$, where z is central. Thus, an element g above the nonidentity component gives $g\widetilde{t}g^{-1} = -\widetilde{t}$, fixes H_+, and exchanges D_λ^+ with D_λ^-. Consequently, the twists in the $\widetilde{t}$-series under H/K are $\widetilde{\rho}_\pm$, with common irreducible restriction $\xi_H = \widetilde{\rho}_\pm|_K$.

Let F_τ^H be the induction to H with linear factor $(\eta_q \circ \det)U_\tau$ and positive factor $\widetilde{H}_+$. The orbits of L_b under K, H, and $\mathrm{O}(V)$ coincide. Indeed, a reflection on W fixes L_b and has determinant -1. Both sources restrict to $((\eta_q \circ \det)U_\tau) \boxtimes H_+$ on the Levi in K. Hence, $F_\tau^H|_K = F_\tau(h)|_K$, and reciprocity adds the coefficients of $\widetilde{\rho}_+$ and $\widetilde{\rho}_-$ in F_τ^H to give the coefficient of ξ_H. By Geck–Malle [72, §4.7], Jordan decomposition in H intertwines rational parabolic induction. Moreover, $\langle H_+, H_+\rangle_{K_0} = 1$. Thus, the two coefficients on H are the Siegel coefficients of $D_\lambda^\pm$, whose sum is $[s_\tau]s_\lambda^2$ by the preceding flag calculation. Consequently, $\langle F_\tau(h)|_K, \xi_H\rangle_K = [s_\tau]s_\lambda^2$ for each τ.

At $\tau = 2\lambda$, the coefficient of ξ_H is 1. Hence, we can choose $\rho' \in \mathrm{Irr}(\mathrm{O}(V))$ in $F_{2\lambda}(h)$ above ξ_H. At each step U_{η_q}, the symbol of h at $+1$ persists, with initial negative rank 0, so the label d' of ρ' is in the principal series of $\mathrm{O}(H^b)$. Since the symbol at $+1$ is nondegenerate, we deduce that $\rho'|_K = \xi_H$ and the parents are ρ' and $\rho'\Delta$. Let (μ,ω) be the Weyl bipartition of d' up to row order. By Pan [149, §§2.1, 3.3], the $W(D_b)$-character comes from (μ,ω) from Section 6. Thus, pairing with $\rho'|_K$ gives $[s_\tau](s_\mu s_\omega)$ for the coefficient in $F_\tau(h)|_K$. Consequently, we find $s_\mu s_\omega = s_\lambda^2$.

The largest partition in $s_\mu s_\omega$ under dominance is $\mu + \omega$. For the smallest partition of $s_\mu s_\omega$, transpose $\mu' + \omega'$. Consequently, we find $\mu_i + \omega_i = 2\lambda_i$ for all i, and the multiset of positive parts $\{\mu_i, \omega_i\}_{>0}$ is twice $\{\lambda_i\}_{>0}$. Thus, with zero padding, we find $\sum_i(\mu_i - \omega_i)^2 = 2\sum_i(\mu_i^2 + \omega_i^2) - \sum_i(2\lambda_i)^2 = 0$. Hence, $d' = d_\lambda$. A reflection exchanges D_λ^+ and

D_λ^-, so d_λ is the sole $\mathrm{O}(H^b)$-parent above $D_\lambda^\pm$. At $+1$, the two orders give $\rho, \rho\Delta$. Thus, $\xi_H = \rho|_K$ and $\xi_H|_J = \chi_B$.

Let Y have linear label $(\eta_q \circ \det)U_{2\lambda}$ on the semilinear group. Let k_σ be the field twist on the W-source, inflated to Y. Since b is even, $(\eta_q \circ \det)(zI_b) = \eta_q(z)^b = 1$ for $z \in \mathbb{F}_q^\times$. Moreover, $U_{2\lambda}$ has a rational field extension. Hence, $Y^\sigma = Yk_\sigma$ on the semilinear flag overgroup. Thus, χ_B has multiplicity 1 by the folded multiplicity formula. Thus, $\widehat{\chi}_B$ has $\widehat{\chi}_B^\sigma = \widehat{\chi}_B k_\sigma$ on the stabilizer of χ_B. The residual involution has twist k_σ if precisely one of L_+, L_- has equal rows.

Let V_+, V_- be isometric sign spaces of rank a with equal unordered symbols. Assume $V = V_+ \perp V_-$. Thus, we have that $V_+ \perp V_-$ is split. Let ℓ be an odd prime distinct from p, put $d = \mathrm{ord}_\ell(q)$ and $e = d/\gcd(d, 2)$, and set $\tau = 1$ if d is odd and $\tau = -1$ otherwise. The common rank r is the core rank for $\tau = 1$ and the cocore rank for $\tau = -1$. Thus, the common weight is $u = (a - r)/e$.

We consider cores of rank r and quotients of weight u in the normalizer with factor $\mathfrak{S}_{2u}$, before choosing a Sylow ℓ-subgroup. If $u > 0$, we require $2r < e$ or $2r = e$ with $\tau = -1$, together with $\pi \in \mathrm{Irr}_{\ell'}(C_{2e} \wr \mathfrak{S}_u)$ and $\ell \nmid \binom{2u}{u}$. Indeed, the doubled residue of rank $2r$ must have order prime to ℓ, and the two sets of u blocks give the index $\binom{2u}{u}$. If $\overline{Q} = 1$, we choose $\chi \mapsto \chi$.

The torus weight is 1 on u blocks and quadratic on the other u. The orientation is stable under ν. Since ν swaps the weights, the partner of π is π times ν on the one-colour complement. On $\mathfrak{S}_u$, the permutation sign is given by ν. Keep the ordered cores of t on the residue U. Thus, the signed inverse has π and its twist by ν, with the ordered cores of rank r.

Write $B = \mathrm{O}(V)$ and $B_Q = \mathrm{N}_B(\overline{Q})$, where $\overline{Q}$ is the image of Q on V. Thus, the residue is $U = V^{\overline{Q}}$. On $U^\perp$, the coordinate kernels distinguish the $2u$ blocks. By Lemma 6.61, in B_Q, each cyclic factor comes from $\overline{Q}$-weight orbits paired by inversion. Hence, B_Q is the direct product of $\mathrm{O}(U)$ and the torus wreath normalizer on $U^\perp$. Since the odd-order lift of $\overline{Q}$ is unique, $B_Q \cap J$ is the image of $\mathrm{N}_X(Q)$ on V. By Frattini with characteristic Q, we have that $B_Q/(B_Q \cap J) \simeq B/J$.

Suppose that the symbols of $V_\pm$, or those of the nonzero residue U, are principal-series symbols on split $V_\pm$ or on the split signs of U. Take L over $\mathbb{F}_p$ and the character $\xi \in \mathrm{Irr}(\mathrm{GL}(L))$ of Lemma 6.249. Let F_0 be the split field map. Indeed, ξ occurs in Borel induction with weights $1, \eta_q$, with permutation representatives over $\mathbb{F}_p$ for the supporting double cosets. Thus, F_0 fixes their normalized kernels. Given $V_\xi \otimes L_\xi$, the rational isotypic space, L_ξ is the multiplicity space. Thus, F_0 acts trivially on $\mathrm{End}(L_\xi)$, with operator $A \otimes 1$. By Proposition 6.242, the character of $V_\xi \otimes L_\xi$ on the extended group divided by $\dim L_\xi$ gives a rational extension of ξ.

Let P_L be the semilinear stabilizer of L. Thus, we inflate the extended ξ trivially on the radical of P_L and induce. Since $F_0(L) = L$, restriction is one flag induction on $\mathrm{O}(V)$ or $\mathrm{O}(U)$, containing ρ_L with multiplicity 1 by Lemma 6.249. By Proposition 6.274, ρ_L is rational and F_0-stable. Thus, the space for ρ_L affords a rational extension of ρ_L. With equal rows on V or U, induce from the semilinear stabilizer of L with Levi acting through $\mathrm{GL}(L)$ with trivial independent multiplier. Since L has stabilizer with multiplier group $\mathbb{F}_q^\times$, restriction to $\mathrm{O}(V)$ or $\mathrm{O}(U)$ gives one induction. Moreover, ρ_L is fixed by row transposition. Since a is even, the scalar zI acts on ξ by $\eta_q(z)^a = 1$.

The constituents $\xi_\pm$ of $\xi|_{\mathrm{SL}(L)}$ from Lemma 6.249 are F_0-fixed. Indeed, restriction to $\mathrm{SL}(L)$ gives one Borel induction, with w for each supporting coset, where $w \in \mathrm{SL}(L)(\mathbb{F}_p)$. Thus, the normalized kernels are F_0-fixed. For the determinant-square Levi, $\xi'_\pm$ restrict irreducibly to the distinct $\xi_\pm$, so are F_0-fixed. Each occurs with multiplicity 1 in ρ_L on the smaller flag, so above $\xi_\pm$, the spin-image constituent is F_0-fixed. Since F_0 is trivial on B/J and on $\mathrm{O}(U)/\Omega(U)$, all conjugates are F_0-fixed. For equal rows, every m occurs for a proper similarity g fixing L pointwise. Thus, g fixes the branch above ξ. Hence, g fixes both special orthogonal branches. Consequently, g fixes the determinant involution of ρ_L.

Write $W = V$ or $W = U \neq 0$, with $W = W_+ \perp W_-$ and $W_+ \simeq W_-$ of common rank j. Thus, W is split of rank $2j$, including for nonsplit $W_\pm$. Choose the split pinning for F_0 on W. If the symbol on W is nondegenerate, its parent on $\mathrm{O}(W)$ restricts irreducibly to $\mathrm{SO}(W)$. Given $g \in \mathrm{SO}(W)$ with $\nu(g) = -1$, conjugation by g exchanges the two $\Omega(W)$-constituents. Since g acts diagonally on $X_W = \mathrm{Spin}(W)$, both inflations to X_W lie in one inner-diagonal orbit $\mathcal{O}$. By Proposition 6.274, the parent on $\mathrm{SO}(W)$ is F_0-fixed, so $F_0(\mathcal{O}) = \mathcal{O}$.

If f is even and $j \geq 2$, then $q \geq 9$ and X_W is universal in split type D_{2j}, including D_4. By Späth [175, Theorem A], $\mathcal{O}$ contains χ fixed by F_0. Moreover, on X_W, the diagonal outer group is $H^1(F_q, Z(\mathbf{X}_W))$ for the algebraic spin group $\mathbf{X}_W$. Both F_0 and F_q fix the centre $\mu_2 \times \mu_2$ pointwise. Thus, every diagonal translate is F_0-fixed and $\mathcal{O}$ is pointwise fixed. If $j = 1$, we have that $\Omega_4^+(q) = (\mathrm{SL}_2(q) \times \mathrm{SL}_2(q))/\langle(-I, -I)\rangle$. In Lemma 6.57, take Weil models at $c \in \mathbb{F}_q^\times$. Since

$c^p/c = c^{p-1}$ is square, the models for c, c^p are equivalent by rescaling with $\sqrt{c^p/c}$ on each parity space. Thus, F_0 fixes the tensor constituents. The flag L gives the F_0-fixed constituents for equal rows. If f is odd, the field permutation centralizes $\mathrm{O}(W)/\Omega(W)$, acting transitively. Thus, its order divides 2. Its order divides f, so the permutation is trivial. If $U = 0$, the residual character is 1.

In the norm orientation, ρ, ρ_U are rational-valued and F_0-stable. If $2 \nmid f$, the real extensions over C_f are unique and rational by Navarro–Tiep [143, Lemma 2.1 and Corollary 2.2]. If $2 \mid f$, compare the extensions before projection to $\Omega(V), \Omega(U)$. If d is odd, extend ρ_U to $\widehat{\rho}_U$ across the split field. Write $\widehat{\rho}_U^{\sigma} = \lambda_\sigma \widehat{\rho}_U$, with $\lambda_\sigma \in \mathrm{Irr}(C_f)$ inflated from the field quotient. Deleting e-hooks in row α gives its core row β. Thus, after padding, $\beta \subseteq \alpha$. From β to α, choose a saturated chain with U_1 at $+1$ and U_{η_q} at -1. Thus, each successor has multiplicity 1 by Proposition 6.4. Give the line weight F_0-value 1. Transitivity on singular lines in rank i gives one flag induction. The rational F_0-stable successor has multiplicity 1, so its twist is λ_σ. Consequently, $\widehat{\rho}^{\sigma} = \lambda_\sigma \widehat{\rho}$, also for equal rows. Both take ρ_U on both sides.

If d is even, we construct $\mathcal{H}_\ell$-fixed extensions of ρ and ρ_U separately. Suppose that $L_\pm$ are nondegenerate. After deleting ordinary 1-hooks in $L_\pm$, the endpoint is cuspidal or rank 1 by Proposition 6.58. Thus, add boxes by U_1 or U_{η_q} on one ordered symbol with multiplicity 1. At a principal endpoint, we choose L for equal rows. In rank 1, Lemma 6.57 gives the norm extension. Moreover, $\det(g)$ on isometries g extends by value 1 on F_0. Under a similarity of multiplier m, the field lift has multiplier m^{1-p}. Hence, multiplication by $m^{(p-1)/2} I$ gives f-th power $\eta_q(m) I$. In $\mu_{\mathrm{ord}_\ell(p)}$, we choose β with β^f equal to the central sign. Thus, β times the operator has the given f-th power and $\mathcal{H}_\ell$-fixed coefficients.

At common cuspidal rank s^2, with $s \geq 2$, specialize Zhang [208, Proposition 7.26, Steps 2–3] at $m = y = 0$. Thus, the blocks have sizes $1, 3, \ldots, 4s-1$, with multiplicity 1. Each constituent admits a pair (U_N, θ_N) of total multiplicity 1 in the regular parent. Since $\rho|_{\Omega(W)}$ has both constituents in one orbit $\mathcal{O}_\rho$ in the regular overgroup, the pair (U_N, θ_N) for either has $\dim \mathrm{Hom}_{U_N}(\theta_N, \rho) = 1$, with $U_N \leq \Omega(W)$.

By Zhang [208, Lemma 7.23(ii),(iv)], take block forms for (U_N, θ_N) with reference lines over $\mathbb{F}_p$. On a multiplicity line, $-F_q$ gives the opposite discriminant to F_q. Indeed, for $z^q = -z \neq 0$, the fixed line has form z^2 times the reference form, with z^2 nonsquare in $\mathbb{F}_q^\times$. The diagonal product w of the selected signs satisfies $wN = Nw$. Thus, on the wF_q-fixed space, the prime-field map F_* fixes θ_N and satisfies $F_*^f = w^{-1}$. Carry the form and (U_N, θ_N) by the same isometry. Then, F_* is in the field coset of F_0. If α is its eigenvalue on the θ_N-line, then $\alpha^f \in \{1, -1\}$, since $w^2 = 1$. Put $k = \mathrm{ord}_\ell(p)$. Since d is even, $(\mu_k)^f = \mu_d \supseteq \{1, -1\}$. Thus, the twist by $\mathrm{Irr}(C_f)$ gives $\alpha \in \mu_k$.

Let $\sigma(\zeta_p) = \zeta_p^{a_\sigma}$. In Lemma 6.252, the half-grading h_σ at a_σ or its inverse gives $\theta_N^{h_\sigma} = \theta_N^{\sigma}$. Since h_σ is defined over $\mathbb{F}_p$ and $h_\sigma w = w h_\sigma$, we have that $h_\sigma F_* = F_* h_\sigma$. Consequently, F_* has eigenvalue α on the θ_N^σ-line. In the conjugate extension, F_* has eigenvalue $\sigma(\alpha) = \alpha$, since $k \mid \ell - 1$ and $\sigma \in \mathcal{H}_\ell$. Thus, the twist at F_* is trivial. Hence, the twist is 1 on C_f. Therefore, the cuspidal extension of ρ is $\mathcal{H}_\ell$-fixed. The multiplicity-one projection for U_1 or U_{η_q} is $\mathcal{H}_\ell$-invariant. Moreover, ρ, ρ_U may have distinct cuspidal ranks.

On U, tensor the extension of ρ_U with the active extension on b from Lemma 6.61. The active relation is $b^f = v^{-1}$ with v the product of paired rotations. Moreover, b acts by a scalar in $\mu_{\mathrm{ord}_\ell(p)}$ and the active extension is $\mathcal{H}_\ell$-fixed. Thus, on b, the local tensor extension has twist λ_σ, also when $\lambda_\sigma \neq 1$ at odd d. The action of b on U is F_0. For the spin lift of v, if $x^{-1} F_q(x) = v$, then $x F_0(x)^{-1}$ is F_q-fixed since $F_0(v) = v$. Thus, on J, the automorphisms b, F_0 differ by an inner action, so b fixes the F_0-fixed global constituents. Locally, b has scalar active operator. On U, it fixes the idempotents for $\rho_U|_{\Omega(U)}$.

If the symbol of V is nondegenerate, take self-twist group $\langle \nu \rangle$ for both parents, with spinor character ν. Thus, the global sign is g_σ^a by Proposition 6.253, with $g_\sigma = \eta_q(a_\sigma)$ for $\sigma(\zeta_p) = \zeta_p^{a_\sigma}$. On the active factor, $\widehat{J}_A = J_A/\sqrt{\kappa^u}$, with $\kappa = \tau \eta_q(-1)^e$. Indeed, $\widehat{J}_A$ has sign g_σ^{eu} on individual blocks by Lemma 6.250. On U, the residual idempotent difference has sign g_σ^r by Theorem 6.254. Thus, we have that $g_\sigma^{eu} g_\sigma^r = g_\sigma^a$, since $a = eu + r$. Both ordinary isometric inertia groups are $\ker \nu$. Thus, conjugate one pair by $g \in B_Q$ with $\nu(g) = -1$ for the second pair. If m is nonsquare, its action takes $\langle \nu \rangle$ to $\langle \nu \det \rangle$. However, above a fixed constituent of J, parents differ by a twist from B/J, so both self-twist groups are $\langle \nu \rangle$. Since $\nu, \det$ are rational, mixed action with multiplier m would change the orientation, from $\langle \nu \rangle$ to $\langle \nu \det \rangle$.

Suppose that $L_- = L_+$ is degenerate. Both L_- and the nonempty cores on U have equal rows. Thus, ρ, ρ_U are principal-series characters. The weight-u quotient is half-turn fixed. Thus, $2 \mid u$. The maximal flags on V, U give extensions with scalar action $zI \mapsto 1$. On the active torus, take $\widehat{\theta}$ from Proposition 6.241, with exponents 1 and -1

on equally many blocks. Thus, $\widehat{\theta}(zI) = 1$. Induction from the inertia of $\widehat{\theta}$ gives the two-colour parent on B_Q with twist ν on the marked complement. If $\ell \mid q-1$, the block argument still holds since $\widehat{\theta}$ satisfies $\widehat{\theta}^2 = 1$, so $\widehat{\theta}|_{T_{\mathrm{odd}}} = 1$ for the odd-order subgroup of T. Proper similarities fix the det-intertwiner. Moreover, the extension is over $\mathbb{Q}(\zeta_{\ell-1})$ with the field root for b.

By Lemmas 6.61 and 6.64, the active involutions D_A, H_A for $\det, \nu$ satisfy $D_A H_A = H_A D_A$ and are fixed by $b, \mathcal{H}_\ell$. Let D_U, H_U be the signed differences on U. Thus, the local involutions are $D_A \otimes D_U, H_A \otimes H_U$. The restriction to $B_Q \cap J$ has squared norm 4. By Proposition 6.251, $B_Q/(B_Q \cap J)$ acts transitively on the 4 joint eigenspaces. Moreover, the flag on V gives 4 global constituents fixed by $b, \mathcal{H}_\ell$.

On each parent, write D, H for the involutions for $\det, \nu$. Choose a nonsquare multiplier m and an element h in the torus normalizer of multiplier m, proper on each factor. Indeed, m lifts diagonally on split blocks. Otherwise, m is a field norm. Thus, $hDh^{-1} = D$, while $hHh^{-1} = \pm DH$. Indeed, hHh^{-1} and DH implement $\nu^h = \nu \det$ in a 1-dimensional intertwining space. Both have square 1, so their ratio is in $\{1, -1\}$. Replace D by $-D$ if the signs differ. Hence, the D, H-eigenvalue correspondence is equivariant for isometries, h, the field lift b and coefficient automorphisms σ.

In both cases, the multiplicity of χ is 1. Thus, V_χ affords an extension $\widehat{\chi}$ to the stabilizer of χ. If σ acts on coefficients, we have that $\widehat{\chi}^\sigma = \widehat{\chi^\sigma}\lambda_\sigma$, with $\lambda_\sigma = 1$ for invariant extensions. Isometric conjugation by g acts by $V_\chi \mapsto V_{\chi^g}$. Moreover, for an isometric correction g of b, we have that $\lambda_\sigma(gb) = \lambda_\sigma(b)$ by inflation. For unequal rows, take s as the common section over $(\mathbb{F}_q^\times)^2$ in semilinear isometries. Thus, s has errors in $\{I, -I\}$. By Theorem 6.254, both scalars at $-I$ coincide, with matched quadratic colours on Q. For equal rows, scalars act by 1. Thus, the two factor sets from s agree by Theorem 6.233. The common section s gives equal centralizer scalars. Moreover, s gives equal correcting functions. The kernel $\ker(X \to \mathrm{SO}(V))$ acts by 1 in both models.

Suppose that t also has eigenvalues $i, -i$. Keep the partition at i. If $L_- = L_+$, transpose both cores by det on U. The Gram determinants d_-, d_+ are equal in compatible bases. Thus, $d_- d_+$ is square. Given $i, -i$, the residue of dimension $2c_0$ has type $\eta_q(-1)^{c_0}$ by Zhang [208, (7.17)]. Hence, $\eta_q(\det B_{i,-i}) = 1$ for the Gram matrix $B_{i,-i}$ there. If $\dim U = 2m$, choose $z \in \mathbb{F}_q^\times$ with $z^2 = \det B_U$, where B_U is the Gram matrix on U. For a semilinear similarity $x = A\varphi$ of multiplier μ, we have that $A^T B_U A = \mu\varphi(B_U)$. Put $E_U(x) = \frac{\det(A)z}{\mu^m \varphi(z)}$. Thus, taking determinants gives $E_U(x)^2 = 1$. If $y = C\psi$ has multiplier μ_y, then $xy = A\varphi(C)\varphi\psi$ has multiplier $\mu\varphi(\mu_y)$. Therefore, $E_U(xy) = E_U(x)\varphi(E_U(y)) = E_U(x)E_U(y)$, since φ fixes signs. The rational character E_U is independent of the sign of z. Its restriction to $\mathrm{O}(U)$ is det, and $E_U(cI) = 1$ on scalar similarities. If $U = 0$, put $E_U = 1$. The normalizer of $\overline{Q}$ acts on $U = V^{\overline{Q}}$, so the normalizer has the character E_U via this action. The E_U-twist transposes both cores on U. Moreover, it has value 1 on scalar errors and mixed correction 1. Thus, with fixed active orientation, order U-cores to recover the V-symbol. For a core L_0 absent or with equal rows, det fixes L_0.

Suppose that L_-, L_+ are distinguishable. Fix ρ_U and its semilinear extension $\widehat{\rho}_U$. Thus, the cores of L_-, L_+ take their row orders from ρ_U. Row transposition of $L_\pm$ transposes the core on U and half-turns the quotient. Thus, U-cores control the orders, with twist ν on its one-colour complement. At i, keep the partition and quotient. Thus, L_-, L_+ and the partition at i follow by Proposition 6.260. Consequently, only the v_j-scalars vary with runner order. Given v_j, the field roots lie in $\mu_{\mathrm{ord}_\ell(p)}$ with powers given by Section 6.3. On Q, the character is fixed. If one $L_\pm$ alone has equal rows, take that $L_\pm$ with the quadratic colour. For nonsquare m, the quotient half-turn acts. If L_-, L_+ both have equal rows, m fixes their quotients.

The twist by det transposes both L_-, L_+, the twist by ν exchanges them, and multiplier m nonsquare transposes L_- alone. The label at i is fixed by both twists and m. Thus, distinguishable L_-, L_+ have self-twist group disjoint from $\{\nu, \nu\det\}$, even if the cores coincide. If L_-, L_+ are nondegenerate, a nonsquare multiplier m exchanges the orientations. If one of L_-, L_+ alone is degenerate, the restriction has 1 branch. If both $L_\pm$ are degenerate, there are 2 branches given by the eigenspaces of D. Thus, the branches on B_Q match via the extended packet at U and the signed inverse. If one $L_\pm$ is degenerate, the extension at U is given by Proposition 6.94.

Let Y have rank 4, and consider characters trivial on Z_Y. Thus, for involutions t_Y with both signs, Zhang [208, Lemma 7.38(i)] gives $I(1,3)$ and $I(2,2)$. Given $I(2,2)$, the paired flag has the D_4 central node and the signs $1, -1$ have rational rank-one complements. If t_Y has $1, -1, i, -i$, the type in Zhang [208, Lemma 7.38(i)] is $Q(1,2)$. Let t_Y be its rational lift. Thus, write $\mathbf{R} = \mathrm{Spin}(Y)$ and $R = \mathbf{R}^F$, with projected Sylow d-torus $\mathbf{S}_Y$. If $\ell \nmid |R|$, then $\mathbf{S}_Y = 1$ and we fix $\chi \mapsto \chi$ on R.

Suppose that $\ell \mid |R|$. Write $\epsilon_\pm$ for the sign-plane types and $\kappa = \eta_q(-1)$. Thus, the plane factors have orders $q - \epsilon_\pm$.

The remaining factor is $\mathrm{GL}_2(q)$ for $\kappa = 1$ and $\mathrm{GU}_2(q)$ for $\kappa = -1$, with p'-part $(q-\kappa)(q^2-1)$. The isogeny kernels and $A(s_Y)$ have 2-power orders. Hence, a character of ℓ'-degree in the series has $\ell \mid q - \tau$ for a unique $\tau \in \{1, -1\}$. With $A = v_\ell(q - \tau)$, we let k count planes of type τ. The fourth-root space has type $+1$, so R has type $\epsilon_+\epsilon_-$. Put $m = 4$ in positive type and $m = 3$ in negative type. Given $s_Y \in R^*$, the image of t_Y, the valuations are $v_\ell(|R|) = mA + v_\ell(m!)$ and $v_\ell(|C_{R^*}(s_Y)|) = (k + 1 + \delta_{\tau,\kappa})A$. Indeed, the split order has degrees $2, 4, 4, 6$, and in negative type one $q^4 - 1$ becomes $q^4 + 1$. If $m = 4$, we have that $k \in \{0, 2\}$, whereas for $m = 3$ we have that $k = 1$. Thus, the valuations coincide exactly when $\tau = \kappa$, $k = m - 2$, and $\ell \geq 5$. Then, the connected labels $1, \mathrm{St}$ have degrees $1, q$, respectively.

Let Q_Y be the ℓ-part of the marked torus for s_Y. Thus, by Lemmas 6.122 and 6.149, $N_R(Q_Y) = N_{\mathbf{R}}(\mathbf{S}_Y)^F$ for each type of R. In positive type, the mark of s_Y is from Lemma 6.115. Thus, m_μ in Proposition 6.117 pairs $1, \mathrm{St}$ by its sign, and J_I, m_I pair the orientations above $l/4$ and its Weyl conjugates. Given R of negative type, the weight is $(-1, i, i, 1)$, with the distinguished plane at coordinate 4. Thus, by Proposition 6.157, the sign on (23) pairs $1, \mathrm{St}$, and the sign on v pairs the orientations. With either weight ω, we have that ω^{-1} is Weyl-conjugate to ω. Hence, all characters above the ω-orbit are paired.

On Y, the ambient Levi action ι is by semilinear similarities fixing $Z_Y = \langle z_0 \rangle$. Both projective models on Y are composed with ι. Thus, their factor sets compose with ι. Mixed correcting functions also compose with ι, including for $\sigma(i) = -i$. In negative type for R, their operators satisfy the relation for C_{2f}. By Lemma 6.27, both operators at z_0 are 1. Thus, the scalar condition at Z_Y of Proposition 6.207 holds. Thus, for lift m_0, both factors have inner action m_0^2.

Let $X = \mathrm{Spin}^\eta_{2n}(q)$, with Witt type η and q odd. Fix an odd prime $\ell \nmid q$, and write Z_X for the spin kernel. Suppose that $\chi \in \mathcal{E}(X, s)$ satisfies $Z_X \leq \ker \chi$, with both signs in a lift t of s to $\mathrm{SO}(V)$. By Zhang [208, Proposition 7.3, Step 1], we obtain the lifts $t, -t$. Moreover, the full centralizer lifts to $\{g \in \mathrm{SO}(V) : gtg^{-1} \in \{t, -t\}\}$. Thus, for rational s, $F^*(t) \in \{t, -t\}$.

Let V_u be the u-eigenspace of t, and write $V_\pm = V_{\pm 1}$. By Zhang [208, proof of Proposition 7.3, Step 1], the lifted torus $\widetilde{\mathbf{Z}}_s$ fixes $V_+ \perp V_-$. If t and $-t$ are conjugate in $\mathrm{SO}(V)$, the torus has equal scalars on V_u, V_{-u}. By Proposition 6.190, the fixed space Y is $V_+ \perp V_-$ in the independent case, and $V_+ \perp V_- \perp (V_i \oplus V_{-i})$ in the paired case, with $i^2 = -1$. Since $\mathbf{L}$ is proper, $\dim Y < 2n$. If $n \geq 5$, take the natural semilinear action on Y. In rank 4, it fixes $Z_Y = \ker(\mathrm{Spin}(Y) \to \mathrm{SO}(Y))$. Thus, both forms of Y have the local weights in $Q(1, 2)$.

For the projection π of Lemma 6.10, the fibre over $[\pi(s)]$ is $\{[s], [ks]\}$, where k generates $\ker \pi$. For independent blocks, Lemma 6.11 gives $s^\sharp = ks$ equivariantly. Thus, apply Proposition 6.13 on $\{[s], [ks]\}$. If the blocks balance, $[s] = [ks]$. Thus, for $L, I = \mathrm{N}_{\mathbf{L}}(\mathbf{S})^F$, Proposition 6.207 gives the tensor extensions above $[s]$. If $F^*(t) = t$, apply Corollary 6.276 for both inductions. If $F^*(t) = -t$, Theorem 6.37 gives it.

Let $\mathbf{G}$ be simply connected of natural type D_n, $n \geq 5$, with Frobenius F of either Witt type over $\mathbb{F}_q$. Write $q = r^f$ with r odd, and fix an odd prime $p \neq r$. Thus, for F-stable $\mathbf{A}$, we put $A = \mathbf{A}^F$ and $A^* = (\mathbf{A}^*)^{F^*}$. Take semisimple $s \in G^*$ with $p \nmid [G^* : \mathrm{C}_{G^*}(s)]$. The order $|s|$ may be divisible by p. Thus, for $P^* \in \mathrm{Syl}_p(G^*)$, we can conjugate s so that $P^* \leq \mathrm{C}_{G^*}(s)$. Write $[s]_{A^*}$ for its A^*-class.

Moreover, let Q^* be the characteristic abelian base of P^*, with corresponding Sylow d-torus $\mathbf{S}^*$ for $d = \mathrm{ord}_p(q)$. In the block description, $\mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(Q^*) = \mathrm{C}_{\mathbf{G}^*}(\mathbf{S}^*)$, with $Q^* \in \mathrm{Syl}_p(K^*)$ and $Q^* \leq \mathrm{Z}(K^*)$. The orthogonal factor of K^* has order prime to p. Since $\mathbf{S}^*$ is the d-part of $\mathrm{Z}(\mathbf{K}^*)^\circ$, every normalizer of P^* normalizes $\mathbf{S}^*$, also for nonabelian $\mathbf{K}^*$. Let $\mathbf{S}$ be the primal torus. Thus, $D = \mathrm{N}_{\mathbf{G}}(\mathbf{S})^F$ is local for all eligible s.

Put $\mathbf{C}_s^* = \mathrm{C}_{\mathbf{G}^*}(s)$, and let t_0 be a lift of s. If s is non-quasi-isolated, Zhang [208, Proposition 7.3, Step 1] gives $\mathbf{Z}_s \leq \mathrm{Z}(\mathbf{C}_s^*)^\circ$ with $\mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathbf{Z}_s)$. For a graph automorphism δ that returns the class of s, the corrected action fixes $\mathbf{Z}_s$ pointwise. Two lifts of δ differ by $c \in \mathbf{C}_s^*$, so they agree on $\mathrm{Z}(\mathbf{C}_s^*)^\circ$. For the quartic exception s, see Proposition 6.16. Given $m = \mathrm{lcm}(2, |t_0|)$, take $j \in (\mathbb{Z}/m\mathbb{Z})^\times$ for the coefficient action on μ_m. Since $\langle s^j \rangle = \langle s \rangle$, the full dual centralizers coincide. Moreover, $xt_0^j x^{-1} = -t_0^j$ if and only if $xt_0x^{-1} = -t_0$, by the odd inverse exponent modulo m. Thus, t_0, t_0^j give the same $\mathbf{Z}_s$ and graph corrections. Under conjugation by x, we have $\mathbf{Z}_{s^x} = (\mathbf{Z}_s)^x$. Consequently, $s \mapsto \mathbf{Z}_s$ is jointly equivariant.

Since $\mathbf{Z}_s$ centralizes Q^*, we have that $\mathbf{Z}_s \leq \mathbf{K}^*$. Hence, $\mathbf{S}^* \leq \mathbf{L}^*$. Put $\mathbf{J}^* = \mathrm{C}_{\mathbf{K}^*}(\mathbf{Z}_s) = \mathbf{K}^* \cap \mathbf{L}^* = \mathrm{C}_{\mathbf{L}^*}(\mathbf{S}^*)$. The full centralizer $\mathbf{C}_s^*$ lies in $\mathbf{L}^*$, so $\mathrm{C}_{\mathbf{K}^*}(s) \leq \mathbf{J}^*$. A rational maximal torus $\mathbf{T}^*$ of $\mathrm{C}_{\mathbf{J}^*}(s)^\circ$ contains $s, \mathbf{Z}_s, \mathbf{S}^*$. Thus, duality on $\mathbf{T}^*$ gives $\mathbf{S} \leq \mathbf{L}$, $\mathbf{K} = \mathrm{C}_{\mathbf{G}}(\mathbf{S})$, and $\mathbf{J} = \mathbf{K} \cap \mathbf{L} = \mathrm{C}_{\mathbf{L}}(\mathbf{S})$. Let Y_K, Y_L be the cocharacter subspaces for $\mathbf{K}, \mathbf{L}$. Indeed, each root α of $\mathbf{J}$ has $\alpha|_{Y_K} = \alpha|_{Y_L} = 0$. Put $I = \mathrm{N}_{\mathbf{L}}(\mathbf{S})^F$. For signed Lusztig induction, write $\mathcal{R}_{G,L} : \mathcal{E}(L, s) \to \mathcal{E}(G, s)$ and $\mathcal{R}_{K,J} : b = \mathcal{E}(J, s) \to B = \mathcal{E}(K, s)$. Thus, $\mathbf{C}_s^* \leq \mathbf{L}^*$ and $\mathrm{C}_{\mathbf{K}^*}(s) \leq \mathbf{J}^*$ give bijections

by Geck–Malle [72, Theorem 3.3.22], with multipliers $[G:L]_{r'}$ and $[K:J]_{r'}$. Since $P^* \le L^*$ and $Q^* \le J^*$, both multipliers are prime to p. Thus, both maps restrict to p'-degree characters.

Let $\mathcal{B}_L = \mathcal{E}(L,s)$ and $\chi \in \mathcal{E}(G,s)$ have p'-degree. If $\beta = \mathcal{R}_{G,L}^{-1}(\chi)$, we have that $\chi(1) = [G:L]_{r'}\beta(1)$, so $p \nmid [G:L]$. Take $P \in \mathrm{Syl}_p(G)$ and conjugate the marked pair $(\mathbf{L}, \mathcal{B}_L)$ so that $P \le L$. By Zhang [208, (7.4)], precisely L stabilizes the marked pair. Whenever $P \le L'$, the transporter can be chosen in $\mathrm{N}_G(P)$ by Sylow conjugacy in L' Isaacs [97, Theorem 1.12]. Hence, the orbit under $\mathrm{N}_G(P)$ has stabilizer $\mathrm{N}_L(P)$. If a joint operation returns χ, its dual conjugator transports $\mathbf{Z}_s$. By Zhang [208, Proposition 7.3, Step 2], take the conjugate pair $(\mathbf{L}', \mathcal{B}_{L'})$. Its Sylow correction lies in $\mathrm{N}_G(P)$. Since $\mathcal{R}_{G,L}$ is equivariant for the correction, β is fixed. Corrections differ by $\mathrm{N}_L(P)$, whose inner conjugation fixes β. Thus, the Levi action on β is well-defined.

Lemma 6.7. *Let $I_b \le I$ and $D_B \le D$ stabilize b and B, respectively. Suppose that $\mathcal{R}_{K,J}(\gamma) = \alpha$. Then, we obtain that $D_B = KI_b$, $K \cap I_b = J$, $D_\alpha = KI_\gamma$, and $K \cap I_\gamma = J$. In the given rational G^*-class, the K^*-classes in orbits of p'-size form one orbit under $D^* = \mathrm{N}_{\mathbf{G}^*}(\mathbf{S}^*)^{F^*}$. In the chosen L^*-class, the J^*-classes in orbits of p'-size form one orbit under $I^* = \mathrm{N}_{\mathbf{L}^*}(\mathbf{S}^*)^{F^*}$.*

Proof. Suppose that $x \in D^*$ fixes $[s]_{K^*}$. In K^*, choose k with $kx \in \mathrm{C}_{G^*}(s)$. Since $\mathrm{C}_{G^*}(s) \le L^*$, $kx \in L^* \cap D^* = I^*$, fixing $[s]_{J^*}$. Conversely, K^* and the I^*-stabilizer of $[s]_{J^*}$ fix $[s]_{K^*}$. Their intersection is J^*. If $l \in L^*$ and $k \in K^*$ satisfy $s^l = s^k$, the quotient centralizes s and lies in L^*, so $k \in J^*$. By [208, Proposition 7.3, Step 2], the actions on b, B correspond to those on $[s]_{J^*}, [s]_{K^*}$. Write $W_{\mathbf{A}}(\mathbf{S})$ for the stabilizer in $W(\mathbf{A})$ of the subspace of $\mathbf{S}$. Thus, the inclusion $W_{\mathbf{L}}(\mathbf{S})/W(\mathbf{J}) \to W_{\mathbf{G}}(\mathbf{S})/W(\mathbf{K})$ is injective. Indeed, $W(\mathbf{L}) \cap W(\mathbf{K}) = W(\mathbf{J})$, since for central subspaces U_L, U_K, the intersection fixes $U_L + U_K$ pointwise. The inclusion is F-equivariant. By [114], connectedness of $\mathbf{J}, \mathbf{K}$ gives rational representatives of the F-fixed cosets. Thus, $D_B = KI_b$ and $K \cap I_b = J$. Since $\mathcal{R}_{K,J}$ is I_b-equivariant and injective, $D_\alpha = KI_\gamma$ and $K \cap I_\gamma = J$.

Given $t \in K^*$, its class stabilizer in D^* is $K^*\mathrm{C}_{D^*}(t)$. Since the central Sylow subgroup Q^* of K^* lies in $\mathrm{C}_{D^*}(t)$, the orbit has p'-size exactly when t centralizes a Sylow p-subgroup of D^*. This criterion also holds for J^* and I^*, since $Q^* \le J^*$ is central and Sylow there. By [132, Proposition 5.11], D^* controls G^*-fusion in K^*. The same proposition applies in $\mathbf{L}^*$, where $\mathbf{S}^*$ is still a Sylow d-torus, so I^* controls L^*-fusion in J^*. Both orbits through the classes of s have p'-size, since P^* centralizes s and is Sylow in both D^* and I^*. □

Proposition 6.8. *Consequently, we find a bijection $\mathcal{C}_{D,I}$ from $\mathrm{Irr}(I)$ above the I-orbit of b to $\mathrm{Irr}(D)$ above the D-orbit of B. Given $\nu \longleftrightarrow \phi$, the degree ratio is $\frac{\phi(1)}{\nu(1)} = \frac{[D:I]}{[K:J]_r}$. Thus, we have that $\mathcal{C}_{D,I}$ restricts to the p'-degree subsets.*

Proof. We fix a parabolic $\mathbf{P}$ with Levi $\mathbf{L}$ and radical $\mathbf{U}$. On the roots of $\mathbf{K}$, the cocharacter gives radical $\mathbf{V} = \mathbf{U} \cap \mathbf{K}$ with Levi $\mathbf{J}$. Since $I_\gamma \le L$ normalizes $\mathbf{K}$, it fixes $\mathbf{V}$. With $H_\gamma = (KI_\gamma) \times_{I_\gamma/J} I_\gamma$, we put $N_0 = K \times J$. With trivial field quotient, H_γ acts on $\mathcal{X}_{\mathbf{V}}$ by [208, proof of Lemma 4.19(b)]. More precisely, (g,h) acts by $x\mathbf{V} \mapsto (gh^{-1})(hxh^{-1})\mathbf{V}$. Thus, $(k,j) \in N_0$ acts by $x\mathbf{V} \mapsto kxj^{-1}\mathbf{V}$. The alternating cohomology character Ξ of $\mathcal{X}_{\mathbf{V}}$ has multiplicity ± 1 at $\vartheta = \alpha \boxtimes \gamma^\vee$ by signed induction. Thus, we choose Ξ with multiplicity $+1$.

By Proposition 2.4, the virtual determinant of Ξ gives an ordinary extension E of ϑ to H_γ. For $N_0 \le H_1 \le H_\gamma$, the same proposition gives $E|_{H_1}$ from $\Xi|_{H_1}$. Let P_γ be associated with γ on I_γ, on a space V_γ. Thus, by Corollary 2.5, E pairs it with P_α for α on KI_γ, on a space V_α. The common factor is c_1. An irreducible module T_1 of factor c_1^{-1} is common to both quotients via $KI_\gamma/K \simeq I_\gamma/J$ (Lemma 6.7). Thus, [95, Theorem 6.11] (Clifford correspondence) induces $P_\gamma \otimes T_1$ to I and $P_\alpha \otimes T_1$ to D, giving the paired characters under $\mathcal{C}_{D,I}$. A common cochain ζ on the models is cancelled by ζ^{-1} on T_1.

Since $K \cap I_\gamma = J$, the ratio of induction indices is $[D:I]/[K:J]$. The ratio of source degrees is $\alpha(1)/\gamma(1) = [K:J]_{r'}$, since the common dimension $\dim T_1$ cancels. Thus, $\phi(1)/\nu(1) = [D:I]/[K:J]_r$ as in the degree formula. Moreover, I^* contains P^*, so $[D^*:I^*]$ is prime to p. Since $|K| = |K^*|$, $|J| = |J^*|$ and the normalizer quotients have $|D/K| = |D^*/K^*|$ and $|I/J| = |I^*/J^*|$, $[D:I] = [D^*:I^*]$. Thus, $p \nmid \phi(1)$ if and only if $p \nmid \nu(1)$. If $\mathbf{J} = \mathbf{K}$, then $\mathbf{V} = 1$ and ordinary Clifford correspondence gives $\mathcal{C}_{D,I}$ on the orbit of b. □

We extend $\mathcal{C}_{D,I}$ of Proposition 6.8 to inertia groups in $G \rtimes \mathrm{Aut}(G)_P$. Thus, a joint operation (z,σ) fixing a character χ can have z moving its rational class. Write $q = r^f$ and $F = \delta^e F_r^f$ with $e \in \{0,1\}$ and diagram involution δ. Thus, we put $D_0 = \mathbf{G}_{\mathrm{ad}}^F \rtimes \langle\delta\rangle$ and $\widetilde{\mathcal{A}} = D_0 \rtimes \mathbb{Z}$, where $1 \in \mathbb{Z}$ acts by F_r. By Zhang [208, Proposition 7.3, Step 4], the natural automorphism group for $n \ge 5$ is $\mathcal{A} = \widetilde{\mathcal{A}}/\langle(\delta^e, f)\rangle$. Indeed, we have that (δ^e, f) acts by F, which centralizes

D_0, F_r. Thus, an action in $\mathcal{A}$ lifts uniquely at degree j if j mod f is its field class. Over G, lift through its action in $\mathcal{A}$. In particular, the identity of $\mathrm{Aut}(G)$ has degree-zero lift 1 on $\mathbf{G}$.

Let $\mathcal{A}_0$ be the ordinary stabilizer, with kernel B_1 of the field degree modulo f. Given $b \in B_1$, its lift of degree 0 is K_b. If $\mathcal{A}_0/B_1$ has order $m > 1$, choose $a \in \mathcal{A}_0$ of degree f/m mod f and lift it to φ of degree f/m. Thus, by the lift identities of Zhang [208, Proposition 7.3, Step 4], $\varphi^m = K_{a^m} F$ and $\varphi K_b = K_{aba^{-1}} \varphi$. If $\mathcal{A}_0 = B_1$, we set $m = 1, a = 1, \varphi = F$. When $\mathcal{A}_0$ stabilizes $\mathbf{K}, \mathbf{J}$, restrict the lifts to both. Thus, $\mathbf{K}$ has its action from $\mathbf{G}$ even for elements trivial on K.

Let z normalize $\mathcal{A}_0, B_1, \mathbf{S}$, with (z, σ) returning both rational series. Take Z of degree $j_0 \geq 0$ above z. Since $\mathcal{A}_0/B_1$ comes from the abelian degree quotient, $h_j = a^j z a^{-j} z^{-1} \in B_1$. In the product of Zhang [208, Lemma 4.19], set $Z_j = K_{h_j} Z$ for $0 \leq j < m$. On geometric points we have that $Z_j = \varphi^j Z \varphi^{-j}$, by the action and degree. Thus, Z_j is a morphism by $Z_j = K_{h_j} Z$. Set $\sigma_0 = K_{a^m}$. Thus, the adjacent maps are φ and the wrap map is $\sigma_0^{-1}\varphi$. We have that $Z_j \varphi = \varphi Z_{j-1}$ $(1 \leq j < m)$ and $Z \sigma_0^{-1} \varphi = \sigma_0^{-1} \varphi Z_{m-1}$. Indeed, the wrap follows from $\varphi^m = \sigma_0 F$ and $ZF = FZ$, including the diagram factor for $e = 1$. Write $Z_\bullet = (Z_0, \ldots, Z_{m-1})$ on $\mathbf{G}^m$. For the coordinate shift τ, conjugation by $Z_\bullet$ gives $D(zaz^{-1}a^{-1})\tau$, where $D(c)_j = K_{a^j c a^{-j}}$. Thus, the finite actions match under $Z_\bullet$, including the relation at a^m.

Suppose that B_1 fixes $\mathbf{U}$. Since z normalizes B_1, B_1 also fixes $Z\mathbf{U}$. Via $Z_\bullet$, the two étale sites are equivalent. $D(B_1), \tau$ fix $\mathbf{U}^m, (Z\mathbf{U})^m$ and satisfy $u\mathcal{F} = \mathcal{F}u$ for $u \in D(B_1) \cup \{\tau\}$. By Lemma 7.168 and Section 7.1, the e_s-parts agree. Before projection, Ξ_m takes values in $\mathbb{Z}$ by Deligne–Lusztig [53, Proposition 3.3]. Hence, $\Xi_m^\sigma = \Xi_m$ and $e_s \mapsto e_s^\sigma$. Since (z, σ) fixes ϑ, the normalized ϑ-part is also invariant. By Proposition 2.4, $(E^\sharp)^{(z,\sigma)} \simeq E^\sharp$.

On H_γ, compare $(\mathbf{V})_j$ and $(\varphi^j \mathbf{V})_j$ for the product of $\mathbf{K}$. For $i \in I_\gamma$, conjugation on the second tuple is $\varphi^j \mathrm{Int}(i) \varphi^{-j}$ on coordinate j. On the first tuple, $a^j i a^{-j} \in I_\gamma$ fixes $\mathbf{V}$. Thus, the $K \times J$-actions agree. By Zhang [208, proof of Lemma 4.19(b)], projection to coordinate 0 relates the second product to $\mathcal{X}_\mathbf{V}$ by purely inseparable maps. For i, equivariance follows from the representatives $\varphi^j(x)$ over the representative x at coordinate 0. Consequently, on H_γ, both constructions give the same normalized ϑ-part. By Proposition 2.4, $E^\sharp|_{H_\gamma} = E$ in Proposition 6.8.

Suppose that $D \rtimes \mathrm{Aut}(G)_P$ stabilizes $\mathbf{S}$. In its product with $\mathcal{H}_p$, we let $V^\#$ stabilize $\mathbf{L}, \mathcal{E}(L, s), b$ jointly. Then, the stabilizer of B is $KV^\#$, with $V^\# \cap D = I_b$. Indeed, if (z, σ) fixes $[s]_{K^*}$, we choose a correction in K^* taking the powered element back to s. Hence, we have that $\mathbf{Z}_s, \mathbf{L}^*$ are fixed by construction, and $\mathbf{J}^*$ is fixed since $\mathbf{S}^*$ is fixed. Take $\mathbf{T}_s^*$ rational and maximally split in $\mathrm{C}_{\mathbf{J}^*}(s)^\circ = \mathrm{C}_{\mathbf{K}^*}(s)^\circ$. Since $\gcd(j, |s|) = 1$, the connected centralizers at s^j, s coincide. Thus, rational torus conjugacy in $\mathrm{C}_{\mathbf{J}^*}(s)^{\circ F^*}$ returns $\mathbf{T}_s^*$. Its primal class returns up to K-conjugacy. The Weyl coset of $\mathbf{T}_s^*$ lifts to K by Zhang [208, Proposition 7.3, Step 2]. Hence, the correction belongs to $V^\#$. Conversely, we have that $V^\#$ fixes B by $\mathcal{R}_{K,J}$. The stabilizer in G of $(\mathbf{L}, \mathcal{E}(L, s))$ is L by Zhang [208, (7.4)]. Since $D \cap L = I$, the additional stabilizer of b is I_b. If the operation exchanges $[s], [ks]$, it carries the marks of $[s], [ks]$ as in Section 6.1.

Proposition 6.9. *Suppose that $f_{L,I}$ pairs $\beta \in \mathcal{E}(L, s)$ with $\nu = f_{L,I}(\beta)$ over the I-orbit of b. Assume that β, ν have common central character ω. Assume mixed tensor-extension comparison on $A_{\beta,\nu}$, their common stabilizer. Each ambient mixed stabilizer operation (z, σ) has a correction in D then K to the chosen series. On $\mathcal{X}_\mathbf{U}, \mathcal{X}_\mathbf{V}$, the ordinary field-free groups fix $\mathbf{U}, \mathbf{V}$, respectively. If such an operation exchanges $[s], [ks]$, take simultaneous corrections for $[s], [ks]$. Then, we have that $f_{G,D} = \mathcal{C}_{D,I} \circ f_{L,I} \circ \mathcal{R}_{G,L}^{-1}$ gives the mixed comparison on G, D. The maps are $\mathcal{R}_{G,L}$ and $\mathcal{C}_{D,I}$ of* Proposition 6.8.

Proof. With fixed $P \in \mathrm{Syl}_p(G)$, in $D \rtimes \mathrm{Aut}(G)_P$, let V_0 be the ordinary stabilizer of $\mathbf{L}, \mathcal{E}(L, s), b$. Thus, correction in D selects the eligible K-series by Lemma 6.7, and correction in K selects its class. In $\nu|_J$, the constituents from b form one I_b-orbit by Clifford theory. Thus, for $W = (V_0)_{\nu,\gamma}$, $(V_0)_\nu = I_b W$ and $(V_0)_\nu / I_b \simeq W/I_\gamma$. If the joint operation is (z, σ), correct first on b and then on γ. Consequently, under $\mathcal{R}_{G,L}$, the character χ has unique preimage β in $\mathcal{E}(L, s)$.

Over W, the γ-isotypic action is $P_\gamma \otimes T_1$. The product for K, J pairs P_γ, P_α with common correcting functions. Thus, on KW, take $P_\alpha \otimes T_1$ with the displayed multiplicity module T_1. The intertwiners agree on T_1, so its factors cancel on W/I_γ. Induce to the inertia groups over I, D. On a representative g, the paired matrices have factor on W/I_γ in common. Since $E^\sharp|_{H_\gamma} = E$, the restrictions are ν, ϕ. If g changes by a source element h, we choose the matrix at h as basis change. Thus, g is immaterial to the comparison.

On G, L, the product compares χ, β. Thus, compose the comparisons for $(\chi, \beta), (\beta, \nu), (\nu, \phi)$ with one model at each of β, ν. If either model is multiplied by ζ, the adjacent cochains change by ζ, ζ^{-1}. On pairs centralizing G, the ambient action on $\mathbf{G}$ is trivial. Thus, β, ν give equal scalars there. Consequently, $f_{G,D}$ satisfies the central comparison on G, D. Finally, for a p'-degree character ϕ of D above B, Clifford theory gives $p \nmid [D : D_B]$. The analogous I-index

for ν is prime to p. Thus, all eligible ν, ϕ above b, B follow by Lemma 6.7 and Proposition 6.8. If $[s] \neq [ks]$, combine the corrected comparisons on $\{[s], [ks]\}$. □

Take G, L, K, J, D, I and $P \in \mathrm{Syl}_p(G)$ with $P \leq I$ from Section 6.1. Suppose that $V_\pm \neq 0$, and put $Y_0 = V^{\widetilde{\mathbf{Z}}_s}$ for the natural module V. Given any nonzero pair $\{\lambda, -\lambda\}$, choose one representative in Λ. Thus, $\mathbf{A}_0 = \prod_{\lambda \in \Lambda} \mathrm{GL}(U_\lambda)$. Moreover, Frobenius permutes the pairs $\{U_\lambda, U_{-\lambda}\}$, allowing $F^*(U_\lambda) = U_{-\mu}$. On an independent scalar block, $t|_{U_\lambda} = cI$ and $t|_{U_{-\lambda}} = c^{-1}I$. If U_λ is balanced, its eigenvalues $c, -c$ have multiplicity $d_\lambda/2$, where $d_\lambda = \dim U_\lambda$.

Lemma 6.10. *For the proper Levi $\mathbf{L}^*$ with factors from $Y_0, \mathbf{A}_0$, the factor projection π has kernel $\langle k \rangle$ of order 2. Its nonempty fibres are $\{[s], [ks]\}$. If all U_λ are balanced, we find $[s] = [ks]$.*

Proof. The matrices on $U_\lambda, U_{-\lambda}$ are $a_\lambda, a_\lambda^{-T}$. Thus, the lifted torus centralizer is $\mathrm{SO}(Y_0) \times \mathbf{A}_0$. We put $-1_A = (-I_{U_\lambda})_{\lambda \in \Lambda}$. Passage to the adjoint quotient gives

$$\pi : \big(\mathrm{SO}(Y_0) \times \mathbf{A}_0\big)/\langle(-1_{Y_0}, -1_A)\rangle \longrightarrow \mathrm{PSO}(Y_0) \times (\mathbf{A}_0/\langle -1_A \rangle),$$
$$[(y, a)] \longmapsto (\bar{y}, \bar{a}).$$

Thus, $\ker \pi = \{1, k\}$ with $k = [(1, -1_A)]$ rational. A rational $[(y, a)]$ can have $F^*(y, a) = (-y, -a)$.

Suppose h conjugates $\pi(s_1)$ to $\pi(s_2)$ over $\mathbb{F}_q$. Take a lift g_0 of h, and a rational maximal torus $\mathbf{T} \leq \mathbf{L}^*$ containing s_1. Thus, $g_0^{-1}F^*(g_0) \in \langle k \rangle \leq \mathbf{T}$. By [114, corollary to Theorem 1], choose $t \in \mathbf{T}$ with $t^{-1}F^*(t) = (g_0^{-1}F^*(g_0))^{-1}$. Then, $g = g_0 t$ is rational and $g s_1 g^{-1} \in \{s_2, k s_2\}$. This gives the fibre $\{[s_2], [ks_2]\}$. If the blocks are balanced, write $s = [(y, a)]$ and $\mathcal{N}_a^- = \{u \in \mathbf{A}_0 : uau^{-1} = -a\}$. Exchanging the two eigenspaces on each U_λ gives a point of $\mathcal{N}_a^-$. The set $\mathcal{N}_a^-$ is a torsor under $\mathrm{C}_{\mathbf{A}_0}(a)$, a product of GL-factors. Since $F^*(a) = \pm a$ and $F^*(-1_A) = -1_A$, this torsor is F^*-stable. By [114, Theorem 2], take $u \in (\mathcal{N}_a^-)^{F^*}$. Hence, $[(1, u)]$ conjugates s to ks. □

Lemma 6.11. *Let $t \in \mathrm{SO}_{2n}$ lift s geometrically. Thus, put $Y = \ker(t^2 - 1)$, $E = \mathrm{im}(t^2 - 1)$, and $j_t = 1_Y \perp (-1_E)$. Write $s^\sharp$ for the image of $j_t t$. Then, $s \mapsto s^\sharp$ is an involution on rational s. Let u be a composite of natural isometries g, similarities h, field operations F_r^i, or permitted coefficient powers b. We have that $u(s) = s_1$ if and only if $u(s^\sharp) = s_1^\sharp$. For independent scalar blocks, $s^\sharp = ks$ in* Section 6.1. *Moreover, $s, s^\sharp$ have common full centralizer $\mathbf{C}_s^*$ and Levi $\mathbf{L}^*$.*

Proof. By semisimplicity, the natural space is $Y \perp E$ for the isometry t. The eigenvalues on E are paired as u, u^{-1} with $u \neq u^{-1}$. Thus, $\dim E$ is even and $j_t \in \mathrm{SO}_{2n}$. $j_t t = t j_t$, $j_{-t} = j_t$, and $(j_t t)^2 = t^2$. Thus, either lift $t, -t$ gives the projective map of square 1. If $F^*(t) = \pm t$, the rational decomposition gives $F^*(j_t) = j_t$. Hence, $s^\sharp$ is rational. For semilinear similarities, Y, E are transported by the maps for u. Given a coefficient operation on $\mu_{2|t|}$, take exponent b with $\gcd(b, 2|t|) = 1$. Thus, b is odd and $(u^b)^2 = 1$ exactly when $u^2 = 1$. Hence, $j_{t^b} = j_t = j_t^b$, giving $(s^b)^\sharp = (s^\sharp)^b$. The converse follows from $(s^\sharp)^\sharp = s$. For geometric conjugation, the stabilizers give $\mathrm{C}_{\mathbf{G}^*}(s) = \mathrm{C}_{\mathbf{G}^*}(s^\sharp)$, and $\mathrm{C}_{G^*}(s) = \mathrm{C}_{G^*}(s^\sharp)$. For independent blocks, $Y_0 = Y$, so j_t acts by -1_A on all chosen spaces. Thus, $s^\sharp = ks$ with k from Lemma 6.10. Their tori are $\mathbf{Z}_s = \mathbf{Z}_{s^\sharp}$, the common connected centre. □

Corollary 6.12. *Suppose that $F^*(t) = -t$, both $V_\pm$ are nonzero, and $E \neq 0$. Assume that no $g \in \mathrm{SO}_{2n}$ conjugates t to $-t$. The two rational lifts s, ks in* Lemma 6.10 *are conjugate exactly when V_+ is split over $\mathbb{F}_{q^2}$. If V_+ is nonsplit, $[s]_{G^*} \neq [ks]_{G^*}$.*

Proof. With $k_0 = \mathbb{F}_q$, we write $k_2 = \mathbb{F}_{q^2}$. Since F^* exchanges V_+, V_-, both have even dimension m. In $Y(k_0)$, the coordinates are $v, F^*(v)$ for $v \in V_+(k_2)$. Thus, Y is the restriction of scalars of $V_+(k_2)$, with form $\mathrm{Tr}_{k_2/k_0}(B_2)$ for the form B_2 on V_+. An exchange T of the sign spaces satisfies $T(cv) = c^q T(v)$ for $c \in k_2$. Since $B_2(v, w)$ is determined by the traces $\mathrm{Tr}_{k_2/k_0}(cB_2(v, w))$, an isometric T satisfies $B_2(Tv, Tw) = B_2(v, w)^q$.

With $B_2 = \mathrm{diag}(1, \ldots, 1, b)$, write $T = A\sigma$, where $\sigma(x) = x^{(q)}$ on k_2^m. Then, $A^{\mathsf{T}} B_2 A = B_2^{(q)}$ gives $\det A = \pm b^{(q-1)/2}$. The matrix $A = \mathrm{diag}(1, \ldots, 1, b^{(q-1)/2})$ gives an exchange. By [21, Ch. III, §9.4, Proposition 6], $\det_{k_0}(T) = (-1)^m N_{k_2/k_0}(\det A) = b^{(q^2-1)/2}$, since $\det_{k_0}(\sigma) = (-1)^m$ and m is even. Since -1 is square in k_2, B_2 is split exactly when $\det_{k_0}(T) = 1$.

Let g lift a projective conjugator $s \mapsto ks$. Thus, $g|_Y$ exchanges V_+, V_- and $g|_E$ centralizes $t|_E$, or $g|_Y$ fixes both signs and $g|_E$ acts by $t|_E \mapsto -t|_E$. In the latter case, join $g|_E$ to an exchange on Y and correct by a reflection in V_+. This gives a proper conjugator $t \mapsto -t$, contrary to hypothesis. Hence, $g|_E$ centralizes $t|_E$ and has determinant 1 by reciprocal pairing. Thus, $\det(g|_Y) = 1$. If $F^*(g) = -g$, the defect on Y is -1_Y. Since $-1_Y \in \mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$, Lang's

theorem corrects the defect on Y in that connected F^*-stable group. Thus, the correction fixes $V_\pm$ and has determinant 1. Conversely, a rational proper exchange on Y extends by 1_E to a rational Levi element h with $hsh^{-1} = ks$. □

Since $V_\pm \neq 0$, take $\mathbf{Z}_s = \mathrm{Z}(\mathrm{C}_{\mathbf{G}^*}(s))^\circ$ from Zhang [208, proof of Proposition 7.3, Step 1]. By Zhang [208, proof of Proposition 7.3, Step 2], a primal cocharacter $\lambda \in \mathfrak{z}_s$ defines $\mathbf{U}$ with Levi $\mathbf{L}$. Each corrected field-free action a fixes $\mathbf{U}$. Thus, $\mathbf{U} \cap \mathbf{K}$ is fixed with Levi $\mathbf{J}$. For field or mixed stabilizer operations (z, σ), Section 6.1 compares radicals on G, L and K, J.

Proposition 6.13. *For independent blocks, assume that $f_{L,I}$ bijects the eligible characters of L, I over the fibre of $[\pi(s)]$. Assume matching central characters at (β, ν) and equivariance under A_π, the joint stabilizer of $[\pi(s)]$. On the pair stabilizer $A_{\beta,\nu}$, assume central mixed comparison. Put $W = \mathrm{N}_G(\mathbf{L})/L$. Thus, we have that $f_{L,I}$ extends over the W-orbit of $[\pi(s)]$. The induced map $\bar{f}$ has the central mixed comparison on G, D. Moreover, for $\nu = f_{L,I}(\beta)$, classes u, v may differ.*

Proof. Set $M_1 = D \cap \mathrm{N}_G(\mathbf{L})$. Since $P \in \mathrm{Syl}_p(L)$, we have $\mathrm{N}_G(\mathbf{L}) = L\mathrm{N}_{\mathrm{N}_G(\mathbf{L})}(P)$ by [97, Lemma 1.13]. Since P determines its characteristic base and $\mathbf{S}$, $\mathrm{N}_{\mathrm{N}_G(\mathbf{L})}(P) \leq M_1$. Thus, $W = M_1/I \simeq \mathrm{N}_G(\mathbf{L})/L$, since $L \cap D = I$. By [208, Proposition 7.3, Step 2], W acts on rational L-series by dual normalizer cosets. On $\mathrm{Irr}(I)$, W acts through conjugation by M_1.

Thus, put $\mathcal{C} = W[s]_{L^*} \cup W[s^\sharp]_{L^*}$ by Lemma 6.10. Let $\mathcal{X}_L, \mathcal{X}_I$ be the eligible characters on L, I above $\mathcal{C}$. Since $\mathbf{C}_s^* \leq \mathbf{L}^*$ on each class, W acts freely on $\mathcal{C}$ by [208, (7.4)]. Given $\nu \in \mathcal{X}_I$, the constituents of $\nu|_J$ form one I-orbit, with class in $\mathcal{C}$. Conversely, each class selects one eligible I-orbit by Lemma 6.7. Given $\beta \in \mathcal{X}_L$, set $\chi = \mathcal{R}_{G,L}(\beta)$. The fibres on $\mathcal{X}_L$ are the W-orbits. Indeed, $\mathcal{R}_{G,L}$ is bijective on each class in $\mathcal{C}$. If two classes fuse in G^*, a rational conjugator takes $\mathbf{L}^*$ to the Levi $\mathbf{L}^*$ by Lemma 6.11. Thus, some $w \in W$ conjugates the classes and fixes χ after induction. Given $\nu \in \mathcal{X}_I$, we set $\phi = \mathcal{C}_{D,I}(\nu)$ via Proposition 6.8. If the images of ν_1, ν_2 are both ϕ, restriction to K gives one D-orbit. Hence, both L^*-classes lie in one G^*-class. Thus, a coset in W conjugates the two L^*-classes. Its representative $m_1 \in M_1$ transports the normalized extension E by conjugation and fixes ϕ. Thus, injectivity at the class of ν_2 gives $\nu_1^{m_1} = \nu_2$. Conversely, M_1-conjugation fixes the image on D. Hence, the fibres on $\mathcal{X}_I$ are the W-orbits. Thus, equivariance at $[\pi(s)]$ gives a well-defined extension of $f_{L,I}$ over W and a bijection $\bar{f} : \mathcal{X}_L/W \to \mathcal{X}_I/W$.

For stabilizers, we set $A = G \rtimes \mathrm{Aut}(G)_P$, $A_D = D \rtimes \mathrm{Aut}(G)_P$, and $V = \mathrm{N}_{A_D}(\mathbf{L})$. By equivariance of $f_{L,I}$ $V_\beta = V_\nu$. Since W acts freely and I acts by inner automorphisms on L, I, $V_\beta \cap G = I$. Moreover, $A_\chi = GV_\beta$ and $(A_D)_\phi = DV_\nu$. Indeed, an outer operation represented in $\mathrm{Aut}(G)_P$ that stabilizes χ, or ϕ locally, returns the ambient rational class of χ, or that of $\phi|_K$ locally. Thus, [208, Proposition 7.3, Step 2] gives a conjugator taking the image of $\mathbf{L}$ back to $\mathbf{L}$. Both Levis contain P, so a correction in L puts the conjugator in $\mathrm{N}_G(P)$, which normalizes $\mathbf{S}$. Thus, the corrected element lies in V. Its action on β or ν stays in the W-orbit, so M_1 gives the correction. For (z, σ), Lemma 6.11 gives equivariance of $s \mapsto s^\sharp$. Thus, $f_{L,I}$ gives one W-correction for both β, ν, and the ordinary quotient is V_ν/I.

Let u, v be the rational classes of β, ν, respectively. Thus, the paired extension at u compares χ, β. Locally, choose γ in $\nu|_J$ above v, with $\alpha = \mathcal{R}_{K,J}(\gamma)$. Thus, for $H = (V_\nu)_\gamma$, Clifford theory gives $V_\nu = IH$, $H \cap I = I_\gamma$ and $H/I_\gamma \simeq V_\nu/I$. Given (z, σ) in the mixed stabilizer, correct in I to return γ and leave β, ν fixed. Consequently, the paired extension at v compares (α, γ). By Section 6.1, its ordinary restriction is E for $\nu \mapsto \phi$. Thus, take the common module T_1 on H/I_γ. Induction in Proposition 6.9 compares ν, ϕ on V_ν/I.

Compose the comparisons at $(\chi, \beta), (\beta, \nu), (\nu, \phi)$ with one model for each of β, ν. If a model changes by ζ, the adjacent cochains change by ζ, ζ^{-1}. The centralizing scalars coincide for each $(\chi, \beta), (\beta, \nu), (\nu, \phi)$. Thus, $\bar{f}$ gives the central mixed comparison on $\mathcal{X}_L/W, \mathcal{X}_I/W$. The projectors on G, L are at u, and those on K, J are at v, also for $u \neq v$. □

Lemma 6.14. *Given $f_{L,I}$ of* Proposition 6.13*, suppose that $\dim Y_0 = 2a$ with $a \geq 4$, and that $Y_0^\perp \neq 0$. Put $\mathbf{R} = \mathrm{Spin}(Y_0)$. Thus, on the complement $Y_0^\perp$, we let $\mathbf{H} \leq \mathrm{Spin}(Y_0^\perp)$ cover the oriented groups $\prod_\lambda \mathrm{GL}(U_\lambda)$, including scalars. Write z_0 for the natural spin involution in $\mathbf{R}, \mathbf{H}$. Thus, multiplication gives a central isogeny $\pi_0 : \mathbf{R} \times \mathbf{H} \to \mathbf{L}$ with kernel $\langle (z_0, z_0) \rangle$. For Sylow d-tori $\mathbf{S}_R, \mathbf{S}_H$, we have that $\mathbf{S} = \pi_0(\mathbf{S}_R \times \mathbf{S}_H)$. Given $T \in \{R, H\}$, we put $I_T = \mathrm{N}_{\mathbf{T}}(\mathbf{S}_T)^F$ and $J_T = \mathrm{C}_{\mathbf{T}}(\mathbf{S}_T)^F$. Thus, we write $B_0 = \pi_0(R \times H)$, $B_I = \pi_0(I_R \times I_H)$, and $B_J = \pi_0(J_R \times J_H)$. Then, $B_I = I \cap B_0$ and $B_J = J \cap B_0$, with $J/B_J \simeq I/B_I \simeq L/B_0 \simeq C_2$ via inclusions.*

Proof. With $V_1 = Y_0$, we put $V_2 = Y_0^\perp$. Thus, by [137, Theorem 24.56(b)], $\mathrm{Cl}^0(V_1)$ and $\mathrm{Cl}^0(V_2)$ have $xy = yx$ for $x \in \mathrm{Cl}^0(V_1)$, $y \in \mathrm{Cl}^0(V_2)$ in $\mathrm{Cl}(V_1 \perp V_2)$. On V_1, V_2, the images give $\mathrm{SO}(V_1)$ and the block group on V_2. Thus, $\ker \pi_0$

consists of pairs of kernel scalars $1, z_0$ with product 1, so $\ker \pi_0 = \langle (z_0, z_0) \rangle$ for all blocks together. Since F fixes V_1 and permutes the other weight pairs, π_0 is rational and $F(z_0, z_0) = (z_0, z_0)$.

The connected preimage $\pi_0^{-1}(\mathbf{S})^\circ$ is a d-torus by the cocharacter criterion. Thus, $\mathbf{S}_R, \mathbf{S}_H$ are d-tori as well, so $\pi_0(\mathbf{S}_R \times \mathbf{S}_H) = \mathbf{S}$ by dimension, with Sylow projections. Thus, $\pi_0(r, h)$ normalizes $\mathbf{S}$ exactly when r, h normalize $\mathbf{S}_R, \mathbf{S}_H$. If $\pi_0(r, h)$ centralizes $\mathbf{S}$, then the commutator of (r, h) with $\mathbf{S}_R \times \mathbf{S}_H$ takes values in $\ker \pi_0$. Since the torus is connected, $[(r, h), \mathbf{S}_R \times \mathbf{S}_H] = 1$. The converse follows by π_0, so $B_I = I \cap B_0$ and $B_J = J \cap B_0$. The connecting map on L has kernel B_0 and target $H^1(F, \langle (z_0, z_0) \rangle) = C_2$. Take $\mathbf{T}$ rational maximal containing $\mathbf{S}$. Since $\pi_0^{-1}(\mathbf{T})$ is connected, Lang gives surjectivity on $\mathbf{T}^F$. Thus, $\mathbf{T}^F \leq J \leq I \leq L$ gives the three quotient isomorphisms with C_2. $\square$

Take the determinant cover $\mathbf{B} = \mathbf{H}$, allowing $\mathbf{B} = \mathbf{L}$ for central c in its dual. For a Sylow d-torus $\mathbf{S}_B$, put $B = \mathbf{B}^F$, $C_B = \mathrm{C}_{\mathbf{B}}(\mathbf{S}_B)^F$, and $N_B = \mathrm{N}_{\mathbf{B}}(\mathbf{S}_B)^F$. In the independent case, the tuple t is scalar on each U_λ, so its image c is central in $\mathbf{H}^* = \mathbf{A}_0^*/\langle -1_A \rangle$. This c is rational even if $F^*(t) = -t$ on the blocks. If $\mathbf{B} = \mathbf{H}$, the determinant covers are isomorphisms on $[\mathbf{H}, \mathbf{H}]$ onto the derived block groups. If $\mathbf{B} = \mathbf{L} \leq \mathbf{G}$, $\mathbf{G}$ is simply connected. Thus, $[\mathbf{B}, \mathbf{B}]$ is simply connected, and $\mathrm{Z}(\mathbf{B}^*)$ is dual to $\mathbf{B}/[\mathbf{B}, \mathbf{B}]$. By Malle–Testerman [133, Proposition 23.2], $B \to (\mathbf{B}/[\mathbf{B}, \mathbf{B}])^F$ is onto. Hence, c defines the character α_c of B. On each rational maximal torus $\mathbf{T}$, the restriction $\alpha_c|_{\mathbf{T}^F}$ is dual to c. By Geck–Malle [72, Proposition 2.5.21], $\mathcal{E}(B, c) = \alpha_c \mathcal{E}(B, 1)$ and $\mathcal{E}(C_B, c) = (\alpha_c|_{C_B})\mathcal{E}(C_B, 1)$ since c lies in the dual Levi. Since c is central, its singleton class is N_B-fixed. Thus, distinct c give disjoint N_B-characters above the C_B-series.

Take f_1 on $\mathbf{B}_{\mathrm{ad}}$, via $\pi_{\mathrm{ad}} : \mathbf{B} \to \mathbf{B}_{\mathrm{ad}}$. Thus, an orbit of F of length h gives a factor over q^h. By Lemma 6.191, the block torus is $\mathbf{S}_B$, with normalizer N_B. Moreover, the toral centre contributes 1. On each classical factor B_i, take the adjoint pair (χ_i, ψ_i) by Ruhstorfer et al. [156, §8.1 and proof of Theorem D]. Their restrictions to B_i^{der} and its torus normalizer N_i^{der} are irreducible. Thus, both pullbacks to B, N_B are irreducible. If the Sylow torus $\mathbf{S}_i$ of B_i is central, take the map $\chi_i \mapsto \chi_i$.

Under π_{ad}, the unipotent series $\mathcal{E}(C_B, 1)$ corresponds to $\mathcal{E}(C_{\mathrm{ad}}, 1)$ for the adjoint centralizer C_{ad}. Since $\mathbf{S}_B$ is central in $\mathrm{C}_{\mathbf{B}}(\mathbf{S}_B)$, every unipotent source λ of C_B is d-cuspidal. Under π_{ad}, both geometric quotients are $W_B = \mathrm{N}_{\mathbf{B}}(\mathbf{S}_B)/\mathrm{C}_{\mathbf{B}}(\mathbf{S}_B)$. By Lang's theorem, we choose W_B^F at both levels. Moreover, for central $\mathbf{T}_0$, the finite normalizer and centralizer of $\mathbf{S}_B\mathbf{T}_0$ are N_B, C_B. Thus, we have that λ has the adjoint stabilizer in W_B^F. Take its adjoint extension $\widehat{\lambda}$ and relative labels η_λ. Clifford induction with $\widehat{\lambda}$ and η_λ gives all characters above λ with matching degrees. Thus, the p'-degree characters over $\mathcal{E}(C_B, 1)$ are all covered.

The field powers F_r^j, inverse transpose $g \mapsto (g^{-1})^T$, and adjoint conjugations act on B_i. If they permute a factor orbit $\mathcal{O}$, we choose one pair (ρ_i, ρ_i') at B_i and transport it along $\mathcal{O}$. By Proposition 6.194, tensor factors over $\mathcal{O}$ give equal cocycle products for g and equal mixed functions at g and its inner powers g^m. Define the correspondence f_c for c by $\alpha_c\chi_0 \longleftrightarrow (\alpha_c|_{N_B})\psi_0$, with $\psi_0 = f_1(\chi_0)$. Thus, the local series on C_B is $(\alpha_c|_{C_B})\mathcal{E}(C_B, 1)$ for any scalar values of α_c. Any operation stabilizing a paired character and fixing c also fixes α_c. By conjugacy of Sylow d-tori $\mathbf{S}_B$, choose common outer representatives s_a with $\mathbf{S}_B^{s_a} = \mathbf{S}_B$. Thus, set $P_c(xs_a) = \alpha_c(x)$ for $x \in B$. Under $P_c(g)P_c(h) = \gamma(g, h)P_c(gh)$, its contribution is $\gamma(a, b) = \alpha_c(s_a s_b s_{ab}^{-1})^{-1}$. Since $s_a s_b s_{ab}^{-1} \in N_B$, this factor is common. If (g, σ) returns the character, $g(s_a)s_{a^g}^{-1} \in N_B$ for the transported coset a^g. Both models evaluate α_c on this correction. Moreover, σ carries α_c to α_{c^σ}. Thus, both mixed functions for (g, σ) coincide, also for an inner power g^m. On a centralizing element x, both linear factors have value $\alpha_c(x)$. Thus, f_c gives the central mixed comparison.

If z_0 has scalar 1, let A_0 be the product of the blocks A_i of linear or unitary type. For η_i quadratic, put $\eta = \prod_i (\eta_i \circ \det_i)$. Thus, H maps onto $\ker \eta$. Indeed, on a rational block A_i, the determinant-root cover has kernel μ_2 and includes the derived group of A_i. By Lang's theorem, the image in A_i has index 2 also for unitary A_i. Hence, the signs multiply to η. Let N_A be the torus normalizer product. Take $\mathbf{T}_A$ rational maximal containing the Sylow torus $\mathbf{S}_A$. Thus, $T_A \leq N_A$. Since the inverse image of $\mathbf{T}_A$ is connected, $T_A \to H^1(F, \mu_2)$ is onto by Lang's theorem. Consequently, η is onto on N_A. The η-twist multiplies the natural scalar tuple $\widetilde{c}$ by -1_A. Similarly, on N_A, the twist acts by $\widetilde{c} \mapsto -\widetilde{c}$ in the dual centralizer. Thus, $\widetilde{c}, -\widetilde{c}$ give distinct series on A_0 and distinct orbits on N_A. Thus, by Clifford theory, restriction to $\ker \eta$ or $N_A \cap \ker \eta$ is irreducible. If a stabilizing operation includes the η-twist, both signs are $\eta(s_a)$.

For a general central character, f_c has α_c on H. The unipotent factors are trivial on z_0, so both paired scalars are $\alpha_c(z_0)$. If c lifts rationally to $\mathbf{A}_0^*$, the determinant character product gives α_c. Hence, $\alpha_c(z_0) = 1$. Conversely, if $\alpha_c(z_0) = 1$, take a rational maximal torus $\mathbf{T}_H$ of $\mathbf{H}$ with image $\mathbf{T}_A$. Then, $\alpha_c|_{T_H}$ factors through the image in T_A. Since T_A is abelian, extend the character to T_A. Its dual element lifts c rationally. Thus, when F^* exchanges the two geometric lifts $\widetilde{c}, -\widetilde{c}$, both paired scalars are -1.

Proposition 6.15. *Suppose that f_R, f_H compare the projected classes on R, I_R and H, I_H, respectively. Assume matching central characters for f_R, f_H and matched J_R, J_H-orbits. The normalizer action Γ_{norm} includes I-conjugation, block permutations and transport of $\pi(s)$. Assume that f_R, f_H give central mixed comparisons for Γ_{norm}. Then, we have that f_R, f_H give the middle correspondence $f_{L,I}$ of* Proposition 6.13. *If $\pi^{-1}([\pi(s)]) = \{[s]\}$, the comparison pairs $\mathcal{E}(L, s)$ with its eligible I-orbit. If s has independent blocks, we set $f_H = \mathrm{f_c}$, including the central character of χ_H and its eligible I_H-orbit. If s has balanced blocks,* Proposition 6.194 *gives f_H by simultaneous restriction to H, I_H.*

Proof. Take $\chi_R \in \mathrm{Irr}(R)$, $\chi_H \in \mathrm{Irr}(H)$, with partners $\psi_R \in \mathrm{Irr}(I_R)$, $\psi_H \in \mathrm{Irr}(I_H)$. Given $j \in \{R, H\}$, we write $\omega_j(z_0) = \chi_j(z_0)/\chi_j(1) \in \{1, -1\}$. Then, $\chi_R \boxtimes \chi_H$ descends to B_0 exactly when $\omega_R(z_0) = \omega_H(z_0)$. Assume $\omega_R(z_0) = \omega_H(z_0)$. Then, $\psi_j(z_0)/\psi_j(1) = \omega_j(z_0)$, so $\psi_R \boxtimes \psi_H$ descends to B_I. Thus, we let β_0, β_I be the descended characters on B_0, B_I. Let $\Gamma \leq \Gamma_{\mathrm{norm}}$ be the ordinary stabilizer of $\chi_R, \chi_H, \psi_R, \psi_H$.

With $X_R = R$, we put $X_H = H$. If ρ_j, ρ_j' afford χ_j, ψ_j for $j = R, H$, take projective models $\mathcal{P}_j, \mathcal{P}_j'$ on Γ, with $\mathcal{P}_j(g)\rho_j(x)\mathcal{P}_j(g)^{-1} = \rho_j(g(x))$ and $\mathcal{P}_j'(g)\rho_j'(y)\mathcal{P}_j'(g)^{-1} = \rho_j'(g(y))$ for $g \in \Gamma$, $x \in X_j$ and $y \in I_j$. Thus, the comparison on Γ gives $\alpha_j = \alpha_j'$. Thus, the tensor models have $\alpha_R\alpha_H = \alpha_R'\alpha_H'$. Given (g, σ) in the mixed stabilizer, we have that their functions are $\mu_{R,g,\sigma}\mu_{H,g,\sigma} = \mu'_{R,g,\sigma}\mu'_{H,g,\sigma}$ by the corresponding factor comparisons.

Given $i_j \in I_j$, [95, Corollary 1.6] gives $\mathsf{N}_j = c_j\rho_j(i_j)$ and $\mathsf{N}_j' = c_j'\rho_j'(i_j)$ implementing conjugation by i_j. Thus, by the central comparison, $c_j = c_j'$. Thus, the two models at $b = \pi_0(i_R, i_H)$ have common scalar c_Rc_H relative to the inner operators. Since $\omega_R(z_0)\omega_H(z_0) = 1$, the kernel $\langle(z_0, z_0)\rangle$ acts trivially, so both lifts of b give equal operators. If $x \in I \setminus B_I$, $x^2 \in B_I$. If $\pi_0(i_R, i_H) = x^2$, the two actions of x^2 are $\rho_R(i_R) \otimes \rho_H(i_H)$ and $\rho_R'(i_R) \otimes \rho_H'(i_H)$.

By Lemma 6.14, write $C_{\mathrm{sh}} = L/B_0$ and take $h \in J \setminus B_J$. Thus, h represents the nonidentity coset and $h^2 \in B_J$. Let χ_0, χ_I be the paired characters of L, I above β_0, β_I. If $\beta_0^h \neq \beta_0$, then $\beta_I^h \neq \beta_I$, and the inertia groups are B_0, B_I, respectively. By [95, Theorem 6.11], $\chi_0 = \mathrm{Ind}_{B_0}^L \beta_0$ and $\chi_I = \mathrm{Ind}_{B_I}^I \beta_I$. Thus, $P_j(h)$ swaps β_j, β_j^h on $\chi_j|_{B_j}$ for $j = 0, I$. Thus, the mixed comparison holds for χ_0, χ_I.

Suppose that $\beta_0^h = \beta_0$ and $\beta_I^h = \beta_I$. With ρ_j affording β_j for $j \in \{0, I\}$, choose $P_j(h)$ for conjugation by h. By tensor comparison, $P_j(h)^2 = c\rho_j(h^2)$ for a common $c \in \mathbb{C}^\times$. Thus, we choose $a \in \mathbb{C}^\times$ with $a^2 = c^{-1}$. Then, $aP_j(h)$ gives the extension $\widetilde{\beta}_j^+$. The other extension is $\widetilde{\beta}_j^- = \widetilde{\beta}_j^+\lambda_{\mathrm{sh}}$ for the nonidentity character λ_{sh} of C_{sh} inflated to L, I. Given (g, σ) in the mixed stabilizer, both functions $\gamma_{j,g,\sigma}$ become $\gamma_{j,g,\sigma}/u_{g,\sigma}$ with common divisor $u_{g,\sigma}$. Thus, $\gamma'_{0,g,\sigma} = \gamma_{0,g,\sigma}/u_{g,\sigma} = \gamma_{I,g,\sigma}/u_{g,\sigma} = \gamma'_{I,g,\sigma}$. Thus, (g, σ) permutes $\widetilde{\beta}_j^\pm$ with common sign ε for both $j = 0, I$. Hence, χ_0, χ_I have equal stabilizers. The models of χ_0, χ_I are paired. If $j \in \{0, I\}$, $\chi_j(1)/\beta_j(1) \in \{1, 2\}$, so the p'-degree condition is equivalent for β_j, χ_j.

Write $\bar{s} = \pi(s)$. By central isotypy [72, 2.3.14–2.3.16 and Remark 2.5.9], restriction from $\mathcal{E}(L, t)$ projects to $[\pi(t)]$. Thus, a constituent β_0 has class $[\bar{s}]$ exactly when $[\pi(t)] = [\bar{s}]$. Thus, Clifford restriction from I to J gives the local union. Corrections in I_R, I_H have image in B_I. Let $\overline{P}^*$ be a projected Sylow p-subgroup. Since $\ker\pi = \langle k\rangle$ and the rational image of π has index at most 2, the group $\overline{P}^*$ lifts isomorphically to a Sylow p-subgroup P^*. If $\pi(t)$ centralizes $\overline{P}^*$, then $[t, u] \in \langle k\rangle$ for $u \in P^*$. This defines a homomorphism $P^* \to \langle k\rangle$, trivial since p is odd. Thus, t centralizes P^*, and the index is prime to p. By Lemma 6.7, t selects one eligible I^*-orbit of rational J^*-classes. If $\pi^{-1}([\bar{s}]) = \{[s]\}$, then the correspondence is on $\mathcal{E}(L, s)$ and its eligible I-orbit. □

On the two-class fibre $\{[s], [ks]\}$, Proposition 6.13 gives the comparison on $\mathcal{E}(L, s) \cup \mathcal{E}(L, ks)$. Let λ be the nontrivial character of L/B_0. Thus, its restrictions to I, J are nontrivial by Lemma 6.14. Moreover, λ is dual to k. By Geck–Malle [72, Proposition 2.5.21], $\lambda\mathcal{E}(L, s) = \mathcal{E}(L, ks)$. Suppose that $\lambda(z) = -1$ for $z \in \mathrm{Z}(\mathbf{L})^F$. Choose a rational torus $\mathbf{T} \leq \mathbf{J}$ dual to one containing s, with $z \in T$. Thus, the torus character θ_s is dual to s. Given $\eta \in \mathrm{Irr}(L) \cup \mathrm{Irr}(I)$, we put $\omega_\eta(z) = \eta(z)/\eta(1)$. If $\chi \in \mathcal{E}(L, s)$ has partner ψ, then $\omega_\chi(z) = \theta_s(z) = \omega_\psi(z)$. In $\mathcal{E}(L, ks)$, the scalar is $-\theta_s(z)$. Both models have equal z-scalars, including for $z \notin B_0$. Thus, $f_{L,I}$ pairs the respective local partners of $\mathcal{E}(L, s)$ and $\mathcal{E}(L, ks)$.

6.2 Exchanged signs and quartic series

Proposition 6.16. *Let $X = \mathrm{Spin}_{2n}^\epsilon(q)$, with odd $n \geq 5$, $\epsilon \in \{1, -1\}$, and $q = r^f$ for an odd prime r. Fix an odd prime $p \neq r$ dividing $|X|$. Let $\widetilde{s} \in \mathrm{SO}_{2n}$ lift $s \in (\mathbf{X}^*)^F$, with eigenvalues $i, -i$ of multiplicity n and $i^2 = -1$. Thus, we have that $\mathcal{E}(X, s)$ is the exceptional case of* Zhang [208, Proposition 7.3]. *Take L, A, H from* Zhang [208, Propositions 7.6

and 7.13] *and put* $e = \mathrm{ord}_p(\epsilon q)$ *and* $d = \mathrm{ord}_p(q)$. *The set* $\mathcal{E}(X, s)$ *contains characters of* p'*-degree if and only if* e *is odd. Then, we choose* $\mathbf{S}$ *as an ambient Sylow* d*-torus in* $\mathbf{L}$ *dual to* $\mathrm{C}_{\mathbf{X}^*}(s)$, *and put* $M = \mathrm{N}_{\mathbf{X}}(\mathbf{S})^F$. *Then, a bijection* f_s *pairs the* p'*-part of* $\mathcal{E}(X, s)$ *with* $\mathcal{X}_M(s)$ *of* p'*-degree over the eligible orbit of* s. *Under* f_s, *central characters match at* $\chi_\lambda, f_s(\chi_\lambda)$. *Moreover, on the natural overgroups,* f_s *gives mixed comparison for* X, M. *The labels on* X, L *are both* $\lambda \vdash n$ *for every* $m = n \bmod e$.

By Zhang [208, Proposition 7.6, Step 1 and Lemma 7.10], we have that $\mathrm{C}_{\mathbf{X}^*}(s)$ is the connected Levi $\mathbf{L}^*$ with roots $e_i - e_j$. Hence, the geometric class of s contains one rational class. Although $F^*(\widetilde{s})$ can equal $-\widetilde{s}$, the series has the rational label s in $\mathbf{X}^*$. Write $T_\epsilon = \mathbb{F}_q^\times$ if $\epsilon = 1$, and $T_\epsilon = \mu_{q+1}$ if $\epsilon = -1$. On T_ϵ, we write η for its quadratic character. The groups $\mathrm{GL}_n^\epsilon(q)$ are $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$, respectively. Thus, by Zhang [208, Proposition 7.6 and Lemma 7.10],

$$L = \{(A, z) \in \mathrm{GL}_n^\epsilon(q) \times T_\epsilon : \det A = z^2\},$$
$$\beta_0(A, z) = \eta(z),\ \beta_\lambda(A, z) = \eta(z) u_\lambda(A),\ \chi_\lambda = \varepsilon_{\mathbf{X}} \varepsilon_{\mathbf{L}} R_{\mathbf{L}}^{\mathbf{X}}(\beta_\lambda).$$

Given $\lambda \vdash n$, the unipotent character is u_λ. Moreover, the rank signs are $\varepsilon_{\mathbf{X}}, \varepsilon_{\mathbf{L}}$. The character β_0 is dual to $s \in \mathrm{Z}(\mathbf{L}^*)$. Since the full centralizer is $\mathbf{L}^*$, signed induction gives the bijection $\beta_\lambda \mapsto \chi_\lambda$ onto $\mathcal{E}(X, s)$ by Geck–Malle [72, Theorem 3.3.22]. The map $\pi_L : (A, z) \mapsto A$ has $|\ker \pi_L| = [\mathrm{GL}_n^\epsilon(q) : \mathrm{im}\, \pi_L] = 2$. Thus, we find $\chi_\lambda(1)/\beta_\lambda(1) = [X : L]_{r'} = \prod_{j=1}^{n-1}(q^j + \epsilon^j)$. If e is odd, we have that $(\epsilon q)^j \not\equiv -1 \bmod p$ for all j, so the degree cut is $p \nmid u_\lambda(1)$, including $\lambda = (n)$. If $e = 2b$, odd n gives $p \nmid q^n - \epsilon$. Thus, since $p \mid |X|$, some $q^{2j} - 1$ with $j < n$ is divisible by p. Then, we obtain that $b \mid j$, and $p \mid q^b + \epsilon^b$ excludes all λ.

Lemma 6.17. *Suppose that* $d = 2e$ *with* e *odd. In* Zhang [208, Lemma 7.7], *take a rational-valued* $\psi \in \mathrm{Irr}(Y)$ *invariant under* b *acting with order dividing* $2f$. *Suppose its intertwiner* V *in the rational central simple algebra* $\mathcal{A}_\psi$ *satisfies* $[V^*V] = [\widehat{r}]^h$ *and* $V^{2f} = (-1)^h (V^*V)^f$. *Then, we have that* ψ *extends to* $Y \rtimes \langle b \rangle$, *with* $|b| = 2f$, *by an* $\mathcal{H}_p$*-fixed character* $\widehat{\psi}$.

Proof. Scale V rationally so that $V^*V = r^h$. With $a = v_2(f)$, we choose ζ of order 2^{a+2}. If h is even, we set $c = r^{-h/2}$. If h is odd, set $c = \zeta r^{-h/2}$. Since $\zeta^{2f} = -1$, both choices give $(cV)^{2f} = 1$ and an extension with b acting by cV. By Section 2, $v_2(\mathrm{ord}_p(r)) = a + 1$ and $\sigma(\zeta)/\zeta = \sigma(\sqrt{r})/\sqrt{r}$ for $\sigma \in \mathcal{H}_p$. Thus, c is $\mathcal{H}_p$-fixed also for odd h. For $\rho : Y \to \mathcal{A}_\psi^\times$, the trace of $\rho(y)V^k$ is its rational reduced trace. Hence, $\widehat{\psi}(yb^k) = c^k \,\mathrm{Trd}_{\mathcal{A}_\psi}(\rho(y)V^k)$ is fixed by $\mathcal{H}_p$, for any Schur index. □

Lemma 6.18. *If* $2 \nmid e$, *the pair on the overgroups* A, H *of* Zhang [208, Propositions 7.6 and 7.13] *is* $(\chi_\lambda, \beta_\lambda)$. *It has matching factors* α, *mixed functions* $\mu_{g,\sigma}$ *and centralizing scalars* c_z.

Proof. Put $h = n(n-1)/2$. If $\epsilon = 1$, take the opposite parabolics $\mathbf{P}_+, \mathbf{P}_-$, their function spaces W_+, W_-, and the sum maps S_+, S_- of [208, Proposition 7.6, Step 4]. By that step, $S_+S_- = q^h I_{W_-}$ and $S_-S_+ = q^h I_{W_+}$, so $\widehat{S}_\pm = q^{-h/2} S_\pm$ are inverse. Apply Q_λ on values. Thus, for an exchange $x \in H$, compose with $\widehat{S}_\pm$. The factors and scalars on H match those of Q_λ. The additional coefficient scalar is $\sigma(q^{-h/2})/q^{-h/2}$ on an exchange. If f is even, $\sqrt{q}$ is rational. If f is odd, the odd order e gives $(r/p) = (q/p) = 1$. Hence, $\mathcal{H}_p$ fixes $\sqrt{q}$ by its quadratic conductor. Thus, $\sigma(q^{-h/2})/q^{-h/2} = 1$ and the comparison has the label λ on both sides.

If $\epsilon = -1$, $d = 2e$. Take $G, C, \widetilde{L}$ as before [208, Lemma 7.10]. Thus, $A = XH$, $X \cap H = L$, and $D_0 = \mathrm{C}_A(X) \le H$. With $\Delta = A \times_{A/X} H$, we put $Y = (G \times_{G/X} \widetilde{L})/\mathrm{diag}(C)$. Thus, $\Delta/\mathrm{diag}(D_0) = Y \rtimes \langle b \rangle$, with $b^{2f} = 1$ and $b^f = (j, j)$ for the exchange j of the two halves. Take $\mu = \lambda$ in [208, Lemma 7.10], with indicator $\nu(u_\lambda) = (-1)^{s_\lambda}$. Thus, the rational ψ_λ restricts irreducibly to $\chi_\lambda \boxtimes \beta_\lambda^\vee$, with norm class $[r]^{h+2s_\lambda} = [r]^h$. Moreover, $\nu(\psi_\lambda) = (-1)^h$, and both j-twisted indicators are 1 by [208, Lemma 7.12]. Thus, the irreducible restriction gives $\nu_{b^f}(\psi_\lambda) = 1 \cdot 1$. Thus, the sign in [208, Lemma 7.7(iv)] is $(-1)^h$. By Lemma 6.17, we obtain $\widehat{\psi}_\lambda$ invariant under $\mathcal{H}_p$. Inflate $\widehat{\psi}_\lambda$ to Δ, trivial on $\mathrm{diag}(D_0)$. The tensor occurs with multiplicity 1, so Corollary 2.5 gives the comparison. □

Write $n = ew + m$, with $0 \le m < e$. Since the order condition on p implies $e \mid j$ for some $1 \le j \le n$ when e is odd, we have that $w > 0$. The factors $q^{2j} - 1$ and $q^n - \epsilon$ give multiplicity w for $\Phi_d(q)$ by that test. Thus, the w blocks of length e in L give an ambient Sylow d-torus $\mathbf{S}$. With $K = \mathrm{C}_{\mathbf{X}}(\mathbf{S})^F$, we put $J = \mathrm{C}_{\mathbf{L}}(\mathbf{S})^F$ and $I = \mathrm{N}_{\mathbf{L}}(\mathbf{S})^F$ and $H' = \mathrm{N}_H(\mathbf{S})$. For H in Lemma 6.18, conjugacy in L gives $H = LH'$ and $\mathrm{N}_A(\mathbf{S}) = MH'$, with $H' \cap X = I$ and $\mathrm{C}_A(X) \le H'$.

Lemma 6.19. *Thus, the β_0-translate of the unipotent comparison on L, I is a central mixed correspondence $f_{L,I}$. Thus, its image $\mathcal{X}_I(s)$ consists of the p'-characters over $\mathcal{E}(J,s)$. Moreover, we have that $f_{I,M} : \mathcal{X}_I(s) \to \mathcal{X}_M(s)$ is an $\mathcal{H}_p$-equivariant bijection. If $m \leq 1$, we have that $f_{I,M}$ gives mixed comparison on H', MH'.*

Proof. The centre of $\mathbf{L}$ has equation $a^n = z^2$ and is connected since n is odd. By Lang, we obtain $L \twoheadrightarrow L_{\mathrm{ad}}$. Under π_{ad}, take the quotient comparison for f_c. The adjoint centralizer preimage has central d-part $\mathbf{S}$, including the central d-part of $\mathbf{L}$. Thus, the comparison of [156, Theorem D and its proof] on L_{ad} gives the unipotent comparison on L, I. The local p'-set is over $\mathcal{E}(J,1)$, as for f_c.

Thus, the action on $\mathbf{L}_{\mathrm{ad}}$ gives the ordinary models. Put $\tau_0(x,y) = \beta_0(xy^{-1})$ on the fibre product. Since β_0 is invariant, τ_0 is linear and gives twists $\beta_0, (\beta_0|_I)^\vee$. It is trivial on $\mathrm{diag}(\mathrm{C}_A(X))$. In $\mathbf{K}^*$, the full centralizer of s is $\mathbf{J}^* = \mathbf{L}^* \cap \mathbf{K}^*$. Thus, signed induction $\mathcal{R}_{K,J}$ pairs $\gamma \in \mathcal{E}(J,s)$ with $\alpha \in \mathcal{E}(K,s)$. By Lemma 6.7, $M_{\mathcal{E}(K,s)} = KI$ and $M_\alpha = KI_\gamma$. Since $I_\gamma \leq L$, it fixes the radical $\mathbf{U} \cap \mathbf{K}$. By Proposition 6.8, we obtain the extension E on K, J. The coefficient action in the stabilizer fixes s of order 2 and the alternating character Ξ, which is integer-valued. Thus, E is fixed. Induction with T_1 gives $f_{I,M}$. Its image is $\mathcal{X}_M(s)$ by Lemma 6.7.

If $m \leq 1$, the orthogonal residue has dimension ≤ 2, so $K = J = T$ is a torus. Its character is $\theta = \beta_0|_T$, with $M_\theta = I$ by the dual centralizer $\mathbf{L}^*$. Thus, $(MH')_\theta = H'$, since H' fixes θ. Given $\nu \in \mathcal{X}_I(s)$, we choose Q_ν on its label stabilizer $(H')_\nu$. Induce on the common cover. The M-restriction is irreducible by [95, Theorem 6.11]. If (g,σ) returns ν, its coset matrices have factor a on both sides. Moreover, $z \in \mathrm{C}_A(X)$ acts by the matched scalar c_z on each coset. Thus, $f_{I,M}$ gives mixed comparison on I, M. □

Lemma 6.20. *If $\epsilon = 1$ and $m \geq 2$,* Lemma 6.19 *gives mixed comparison for $f_{I,M}$.*

Proof. On $\mathbf{K}$, restrict both cocharacters from Lemma 6.18. Thus, $\mathbf{P}_\pm \cap \mathbf{K}$ are opposite with common Levi $\mathbf{J}$. The radicals $V_\pm$ have order q^b, with $b = m(m-1)/2$ by the residual roots. Moreover, $\{V_+, V_-\}$ is fixed by H'. On J, write $\gamma = (\beta_0|_J)u_\mu$, where u_μ has unipotent residual label μ and active labels 1. Thus, I, H' and $\mathcal{H}_p$ fix γ. Inflate γ to $\mathbf{P}_\pm \cap \mathbf{K}$ on rational points and induce to K, with function spaces $\mathcal{V}_+, \mathcal{V}_-$. Since the dual centralizer is $\mathbf{J}^*$, both afford α. Take the sum $S_+ : \mathcal{V}_+ \to \mathcal{V}_-$ and its adjoint $S_- = S_+^*$ in unitary coset models. In S_+, the $|V_+| = q^b$ unitary blocks occupy distinct columns since $V_+ \cap JV_- = 1$. Thus, by [208, Proposition 7.6, Step 4], $S_+S_- = q^b I_{\mathcal{V}_-}$ and $S_-S_+ = q^b I_{\mathcal{V}_+}$. Hence, $\widehat{S}_\pm = q^{-b/2} S_\pm$ are inverse.

Given $x \in H'$, the function operator acts by $v(y) \mapsto Q_\gamma(x) v(x^{-1}yx)$, composed with $\widehat{S}_\pm$ if x exchanges parabolics. On $J = H' \cap K$ it acts by right translation in K. Since $\widehat{S}_\pm$ are inverse, the factor on KH' is that of Q_γ. On values, C_σ transports $S_\pm$ to their corresponding sums. Moreover, $\mathcal{H}_p$ fixes $q^{-b/2}$ by the calculation for $\sqrt{q}$. On centralizing z the scalar is that of $Q_\gamma(z)$. Thus, the common multiplicity module on $I/J \simeq KI/K$ gives paired induced models over I, KI. Since $M_\alpha = KI$, the mixed comparison holds on I, M for every relative label over γ, α. □

In the negative case $d = 2e$ of Proposition 6.16, we also treat even m. Thus, we let e be odd and $m \geq 2$. On the spin group $\mathbf{R}$ in dimension $2m$, we put $F_R = \mathrm{Int}(j_m) F_q$ with $j_m(V_\pm) = V_\mp$ for the isotropic halves V_+, V_-. Then, we find $R = \mathbf{R}^{F_R} = \mathrm{Spin}_{2m}^{(-1)^m}(q)$. The oriented Levi $\mathbf{L}_R$ has $L_R = \{(A,z) \in \mathrm{GU}_m(q) \times \mu_{q+1} : \det A = z^2\}$. We choose the regular inclusion of $\mathbf{R}$ from Zhang [208, §7, before Lemma 7.10 and Proposition 7.36]. Thus, $\widetilde{R}, \widetilde{L}_R, Z_t$ are the rational points of $\widetilde{\mathbf{R}}, \widetilde{\mathbf{L}}_R, \mathbf{Z}_t$, respectively, with $\dim \mathbf{Z}_t = 2$. Hence, we find $D_t = \widetilde{R}/R = \widetilde{L}_R/L_R$.

Lemma 6.21. *Since $\widetilde{L}_R/\mathrm{SU}_m(q)$ is abelian, the linear character $(A,z) \mapsto \eta(z)$ of L_R extends to $\widetilde{\eta}$ on $\widetilde{L}_R$. Given $\rho \vdash m$, we set $\widetilde{\beta} = \widetilde{\eta} u_\rho$, where u_ρ is the unipotent character for ρ. Let Θ be the signed induction of $\widetilde{\beta}$ to $\widetilde{R}$. Since the full dual centralizer is $\widetilde{\mathbf{L}}_R^*$, $\Theta \in \mathrm{Irr}(\widetilde{R})$. We put $Y_t = (\widetilde{R} \times_{D_t} \widetilde{L}_R)/\mathrm{diag}(Z_t)$ and $\psi(x,y) = \Theta(x)\widetilde{\beta}(y^{-1})$. If m is even, we put $h_m = m(m-1)/2$ and $\delta = -\eta_q(-1)$, where η_q is quadratic on $\mathbb{F}_q^\times$. Write $\beta_\rho^\pm$ for the two extensions of $\eta(z)u_\rho(A)$ to $L_R\langle j_m\rangle$. Then, $\Theta|_R = \Theta_+ + \Theta_-$ with distinct $\Theta_\pm \in \mathrm{Irr}(R)$. The $\mathcal{H}_p$-actions on $\{\beta_\rho^+, \beta_\rho^-\}$ and $\{\Theta_+, \Theta_-\}$ both correspond to the action on $\{\kappa : \kappa^2 = \delta^{h_m}\}$. Moreover, on the stabilizer in $\mathcal{H}_p$, the coefficient ratios for $\Theta_\pm, \beta_\rho^\pm$ coincide on the field generator a_{fld} for suitable extensions.*

Proof. For Θ, the regular self-twist has order 2 and fixes ρ by [208, Proposition 7.36, Step 1]. Thus, $\Theta|_R = \Theta_+ + \Theta_-$, with multiplicity one. Let ζ_r be a primitive r-th root of unity, and write $\sigma(\zeta_r) = \zeta_r^{c_\sigma}$. First, suppose that ρ is a staircase 2-core of even size $k = b(b+1)/2 \geq 6$. Set $t = \lceil b/2 \rceil$ and $d_j = 2b - 1 - 4j$ for $0 \leq j < t$. Thus, $t \equiv 0 \bmod 2$. Moreover, the wave front partition is $(d_0, d_0, \ldots, d_{t-1}, d_{t-1})$. The total Gelfand–Graev multiplicity is $a_0 = 2^{t-1}$ for Θ by [208, Proposition 7.36, Step 3]. Thus, choose nonsplit planes U_j for $0 \leq j < t$. Since t is even, the space of

dimension $2k$ is split. Since d_j is odd, the rotations of U_j lift to $\mathbf{T}_0$ with character lattice $\Lambda = \mathbb{Z}^t + \mathbb{Z}(1/2, \dots, 1/2)$ and $F^* = -q$.

By [208, Proposition 7.36, Step 5], the strips at j have lengths $\{b-2j, b-2j-1\}$. Thus, the half-spin weights on $T_0 = \mathbf{T}_0^F$ are $\omega_\epsilon = \frac{q+1}{4}(\epsilon_0, \dots, \epsilon_{t-1}) \pmod{(q+1)\Lambda}$. The signs $\epsilon_j \in \{1, -1\}$ satisfy the fixed parity of $\prod_j \epsilon_j$. For two sign vectors ϵ, ϵ' of that parity, $(\epsilon - \epsilon')/4 \in \Lambda$ exactly when all entries lie in $\mathbb{Z}$ or all lie in $\mathbb{Z} + 1/2$. Hence, $\omega_\epsilon = \omega_{\epsilon'}$ exactly when $\epsilon' = \pm\epsilon$. Since $\epsilon/2 \in \Lambda$, $\omega_\epsilon^2 = 1$ as a character. The set $\Omega = \{\omega_\epsilon\}$ has size 2^{t-2}, also for $q+1 \equiv 2 \bmod 4$.

Write τ for the trace on T_0 up to sign. Given $\epsilon, -\epsilon$, both strip orders have multiplicity 1. Thus, $\tau = 2\sum_{\omega\in\Omega} \omega$ on regular points of T_0. If $q \equiv 3 \bmod 4$, [208, Proposition 7.36, Step 5] gives this trace on all of T_0. Given $q \equiv 1 \bmod 4$, we consider $\Delta_\tau = \tau - 2\sum_{\omega\in\Omega} \omega$. The bounds from [208, Proposition 7.36, Step 5] give at most $2a_0$ nonzero Fourier coefficients for Δ_τ. Moreover, its support in T_0 has proportion at most $2t^2/(q+1)$. Thus, it is zero for $q+1 > 4a_0t^2$. Since the multiplicities on T_0 are nonnegative, the common sign is $+1$. For smaller q, take the progression $a \equiv 1 \bmod M$ from [208, Lemma 7.27 and Proposition 7.30], with $2 \mid M$, as in [208, Proposition 7.36, Step 5]. Thus, for $a \equiv 1 \bmod M$, the source for ρ, the semisimple class for Θ and the component group actions for F^a are fixed. The norm $\mathbf{T}_0^{F^a} \to T_0$ multiplies weights by $(q^a+1)/(q+1)$. Thus, the weight formula has q^a+1 in place of $q+1$, with value $\omega_\epsilon^a = \omega_\epsilon$ on T_0. The trace sequences agree for large a and so at $a = 1$. Consequently, the chosen source accounts for a_0, with multiplicity $a_0/2$ for each of Θ_+, Θ_-.

The even inversions E_{inv} act transitively on Ω. Thus, the stabilizer at $\omega \in \Omega$ in the inversion quotient is $\langle -1 \rangle$, where -1 inverts all U_j. We denote the spin lift by w_0. On U_j, take reflections with Clifford lifts over $\mathbb{F}_r$ as in [208, Proposition 7.36, Steps 4–5]. Their product is w_0. Thus, $w_0 x = x w_0$ for even products x of plane lifts. The product w_0 has k orthogonal unit-vector factors, so $w_0^2 = z_0^{h_k}$ for $h_k = k(k-1)/2$ and spin kernel $\langle z_0 \rangle$. The central character takes value δ on z_0. Hence, the two extensions of ω have values $\kappa, -\kappa$ at w_0, where $\kappa^2 = \delta^{h_k}$. The rotation r_0 of nonsquare spinor norm in [208, Proposition 7.36, Steps 4–5] induces $\kappa \mapsto -\kappa$ on the weight line for ω. Hence, its lift acts by $\Theta_\pm \mapsto \Theta_\mp$. The induced source for $\pm\kappa$ has dimension $|\Omega| = a_0/2$. Thus, its multiplicity in the $\Theta_\pm$-space is 1.

On both blocks of size $d_j = 2l_j + 1$, put the weights $l_j, l_j - 1, \dots, -l_j$. The positive weight sum is $2\sum_j l_j(l_j+1)/2$, so the grading lifts to λ_{gr} in the spin group $\mathbf{R}$. For the source character ξ, conjugation by $\lambda_{\mathrm{gr}}(c_\sigma)$ gives ξ^σ. The element $g_\sigma = \lambda_{\mathrm{gr}}(c_\sigma)$ lies in $\mathbf{R}(\mathbb{F}_r)$ and satisfies $g_\sigma x = x g_\sigma$ for $x \in T_0$. Since $\omega \in \Omega$ is rational, the source action on κ has ratio $\sigma(\kappa)/\kappa$. In the model with Frobenius $\mathrm{Int}(w_0)F_q$, we let $\mathcal{M}_\kappa$ be the induced source affording R_0. On the cosets for $\mathcal{M}_\kappa$, we let P_0 be the field permutation. Since w_0 is fixed over $\mathbb{F}_r$, the coset action of P_0^f is that of $R_0(w_0)^{-1}$, with scalar κ on the source space. Thus, $P_0^f = \kappa R_0(w_0)^{-1}$. Choose $\alpha_0^f = \kappa^{-1}$, with normalized map $\alpha_0 P_0$. Let A_σ be the action of $\lambda_{\mathrm{gr}}(c_\sigma)$. Since this element lies in $\mathbf{R}(\mathbb{F}_r)$, we have $A_\sigma P_0 = P_0 A_\sigma$. Thus, if $\sigma(\kappa) = \kappa$, the ratio on $\mathcal{M}_\kappa$ is $\sigma(\alpha_0)/\alpha_0$. By multiplicity 1, it is also the ratio for $\Theta_\pm$.

In the unitary prime range, $2f_2 \mid p-1$ for $f_2 = 2^{v_2(f)}$. The odd part of f is coprime to $4f_2$. Thus, choose α_0 of order dividing $2f_2$ for $\kappa = \pm 1$ and dividing $4f_2$ for $\kappa = \pm i$. If $\kappa = \pm 1$, $\sigma(\alpha_0) = \alpha_0$. If $\kappa = \pm i$, $\delta = -1$ and h_k is odd. If f is even, then $p \equiv 1 \bmod 4$. The map $x \mapsto x^p$ acts by $\alpha_0 \mapsto -\alpha_0$ precisely when $v_2(p-1) = v_2(f) + 1$. By Lemma 6.17, for a p^j-power action, quadratic reciprocity gives $\sigma(\alpha_0)/\alpha_0 = (r/p)^j = (p/r)^j = (c_\sigma/r)$. If f is odd and $\sigma(\kappa) = \kappa = \pm i$, $\sigma(\alpha_0) = \alpha_0$. Moreover, $\delta = -1$ implies $r \equiv 1 \bmod 4$ for odd f, so $(c_\sigma/r) = 1$ for σ in the stabilizer. Thus, the ratio $\sigma(\alpha_0)/\alpha_0$ is $(c_\sigma/r)^{h_k}$ for $\delta = -1$ and 1 for $\delta = 1$.

For arbitrary ρ, apply the rim 2-hook matching of [208, Proposition 7.21] until the base. At rank $a+2$, the mixed plane has central cocharacter $e_1 - e_2$ with quartic value 1. The quotient Levi has subgroup $D_a = (\mathrm{SL}_2(q) \times \mathrm{Spin}_{2a}^+(q))/\langle (-1, z_0) \rangle$ on its derived factors. On $\mathrm{SL}_2(q)$ and $\mathrm{Spin}_{2a}^+(q)$, the other coset gD_a acts by both diagonal involutions. At $-1, z_0$, the signs are δ. One fixed $\mathrm{SL}_2(q)$ branch paired with both $\mathrm{Spin}_{2a}^+(q)$ branches gives two characters for rank $a+2$. Their signs under σ are products of the factors' signs ± 1. The sign for $\mathrm{SL}_2(q)$ is $\eta_q(c_\sigma)$ by Lemma 4.12. In this range $\eta_q(c_\sigma) = \sigma(i)/i$ for $\delta = -1$ and 1 otherwise. If f is even, $\eta_q(c_\sigma) = 1$ and $p \equiv 1 \bmod 4$. If f is odd, its value on $x \mapsto x^p$ is (p/r). Quadratic reciprocity with $\mathrm{ord}_p(q) = 2e$ gives the two signs. Since $h_{a+2} \equiv h_a + 1 \bmod 2$, the permutation at rank $a+2$ has the claimed form. If the 2-core is empty, the base is $\mathrm{Spin}_4^+(q) = \mathrm{SL}_2(q)^2$. Pair its cuspidal $\mathrm{SL}_2(q)$ characters as in Lemma 4.12 with 1 or St on the other factor $\mathrm{SL}_2(q)$.

If σ fixes both branches, the ratios at a_{fld} multiply by Lemma 4.12. If σ swaps both factor branches, then f is odd, $\delta = -1$ and h_a is odd. On the subgroup of $\mathcal{H}_p$ fixing both branches, the ratio at a_{fld} is 1. The fixed extensions form a torsor $\mathcal{T}$ under a subgroup of $\mathrm{Irr}(C_f)$, with f odd. Hence, C_2 fixes a point of $\mathcal{T}$ by Lemma 4.34. For the Levi module V, the summand has multiplicity 1 and inherits the extension of V. Therefore, at rank m the ratio is $(c_\sigma/r)^{h_m}$ for

$\delta = -1$ and 1 for $\delta = 1$ on the stabilizer of $\Theta_\pm$.

For u_ρ on $\mathrm{PGU}_m(q)$, the norm class is $[r]^{s_\rho}$, with ordinary indicator $(-1)^{s_\rho}$ and inverse transpose twisted indicator $+1$ by [208, Lemmas 7.7, 7.9 and proof of Lemma 7.12]. Lemma 6.17 with $h = s_\rho$ gives an $\mathcal{H}_p$-fixed extension U_ρ to the field group $\langle b\rangle$ of order $2f$. Since $j_m^2 = z_0^{h_m}$, the matrix for j_m on the Levi branch is $\kappa U_\rho(b^f)$ for $\kappa^2 = \delta^{h_m}$. For the pure field lift $a_{\mathrm{fld}}^f = j_m^{-1}$, the matrix is $\alpha_0 U_\rho(b)$ with $\alpha_0^f = \kappa^{-1}$. Thus, the branch ratio is $\sigma(\kappa)/\kappa$, and the pure field ratio is $\sigma(\alpha_0)/\alpha_0$, equal to that for $\Theta_\pm$. □

Proposition 6.22. *Given $m \geq 2$, the character ψ is irreducible and has an $\mathcal{H}_p$-fixed ordinary extension Ψ to $Y_t \rtimes \langle b\rangle$. The common prime field map b has order $2f$ and satisfies $b^f = (j_m, j_m)$.*

Proof. If $m \geq 5$ is odd, choose the pair (ρ, ρ) in Lemma 6.18. Thus, $\psi = \psi_{\rho,\rho}$ of [208, Lemma 7.10], because a twist $\nu \in \mathrm{Irr}(D_t)$ on $\widetilde{\beta}$ occurs also on Θ and cancels on Y_t. If $m = 3$, $R = \mathrm{Spin}_6^-(q) = \mathrm{SU}_4(q)$. The projective label in type A_3 is $\mathrm{diag}(1, 1, 1, -1)$. Since the multiplicities $3, 1$ differ, no scalar $z \neq 1$ fixes the eigenvalue multiset $\{1, 1, 1, -1\}$. Thus, the full dual centralizer is $\mathbf{L}_R^*$. The tensor ψ restricts irreducibly to $(R \times L_R)/\mathrm{diag}(\mathrm{Z}(R))$. The calculation in [208, Lemma 7.10] with $h = 3$ gives norm class $[r]^3$ and indicator $\nu(\psi) = -1$. Moreover, $\nu_{j_m} = 1$ on both factors by the rank 3 cases in [208, proof of Lemma 7.12]. Thus, $\nu_{b^f}(\psi) = 1$, and [208, Lemma 7.7(iv)] with Lemma 6.17 gives Ψ.

Suppose that m is even and put $h = h_m$. The actions on Z_t on the two factors of ψ cancel. Write $H_t = \widetilde{R} \times \widetilde{L}_R$, $P_t = \widetilde{R} \times_{D_t} \widetilde{L}_R$ and $\pi_t : H_t \to D_t$ for the quotient $(x, y) \mapsto xR(yL_R)^{-1}$. Since $\widetilde{\beta}|_{L_R}$ is irreducible, $\widetilde{\beta}\nu = \widetilde{\beta}$ implies $\nu = 1$ by [95, Corollary 6.17]. Thus, D_t has no nontrivial character fixing $\Theta \boxtimes \widetilde{\beta}^\vee$ via π_t. Since D_t is abelian, the restriction formula gives $\langle \psi, \psi\rangle = 1$. Given a coefficient or field operation τ, the restriction $\widetilde{\beta}^\tau|_{L_R} = \eta u_\rho$ is fixed. Thus, $\widetilde{\beta}^\tau = \widetilde{\beta}\nu_\tau$ with $\nu_\tau \in \mathrm{Irr}(D_t)$. By equivariance of signed induction, $\Theta^\tau = \Theta\nu_\tau$. Therefore, $\psi^\tau(x, y) = \psi(x, y)$ on Y_t and ψ is rational-valued.

To compute V^*V for the intertwiner V of [208, Lemma 7.7], let $\mathbf{V}$ be the radical with roots $e_i + e_j$ and dimension h. Thus, its opposite radical is $F_R(\mathbf{V})$. Write $\widetilde{Z} = \{x\mathbf{V} : x^{-1}F_R(x) \in \mathbf{V}F_R(\mathbf{V})\}$ in the regular group $\widetilde{\mathbf{R}}$. The image of x in $\widetilde{\mathbf{R}}/\mathbf{R}$ is F_R-fixed, so $\widetilde{Z}$ has fibres indexed by D_t. The identity fibre Z is the spin variety. Under $(g, d) : x\mathbf{V} \mapsto gxd^{-1}\mathbf{V}$ for $(g, d) \in H_t$, the fibre index s becomes $\pi_t(g, d)s$. Thus, P_t fixes Z and its diagonal Z_t acts trivially.

Let e_ρ be the idempotent for ηu_ρ in $\mathbb{Q}L_R$. Over $\mathbb{C}$, this idempotent is $e_\rho = \sum_{\nu \in \mathrm{Irr}(D_t)} e_{\widetilde{\beta}\nu}$ by [95, Corollary 6.17]. Since the dual label of $\widetilde{\beta}\nu$ is central in $\widetilde{\mathbf{L}}_R^*$, its full centralizer is $\widetilde{\mathbf{L}}_R^*$. Thus, by [18, Theorems 11.7 and B′], the projected cohomology of $\widetilde{Z}$ is concentrated in degree h. Moreover, $H_c^h(\widetilde{Z}, \mathbb{Q}_\ell)e_\rho \cong H^h(\widetilde{Z}, \mathbb{Q}_\ell)e_\rho$ via the canonical map. Thus, the H_t-character is $\sum_{\nu \in \mathrm{Irr}(D_t)} (\Theta\nu) \boxtimes (\widetilde{\beta}\nu)^\vee$ by signed induction. On P_t, each summand is the inflation of ψ. Moreover, H_t is transitive on the $|D_t|$ fibres. Thus, the projected cohomology of Z has dimension $\psi(1)$. Thus, $H_c^h(Z, \mathbb{Q}_\ell)e_\rho$ affords ψ and $H_c^h(Z, \mathbb{Q}_\ell)e_\rho \cong H^h(Z, \mathbb{Q}_\ell)e_\rho$. Since e_ρ is self-dual, the Poincaré pairing B restricts to $H_c^h(Z, \mathbb{Q}_\ell)e_\rho$. The inverse Frobenius Φ gives $B(\Phi v, \Phi w) = r^{-h}B(v, w)$ in degree h by [208, Lemmas 7.7–7.10]. The symmetry of B is $(-1)^h$. Hence, $[V^*V] = [r]^h$. We choose V with $V^*V = r^h$ and write $V^{2f} = \varepsilon r^{hf}$.

Put $u = 0$ for $\varepsilon = 1$ and $u = 1$ otherwise. The scalar $c = \zeta^u r^{-h/2}$ in Lemma 6.17 gives an ordinary extension Ψ_0. For $\sigma \in \mathcal{H}_p$ acting by p^j on the relevant roots, we write $\Psi_0^\sigma = \Psi_0\xi_\sigma$. Then, $\xi_\sigma|_{Y_t} = 1$ and $\xi_\sigma(b) = (r/p)^{j(h+u)}$. If $(r/p) = 1$, this gives $\xi_\sigma = 1$. Otherwise, suppose that $\xi_\sigma(b) = -1$. Put $K_1 = (R \times L_R\langle j_m\rangle)/\mathrm{diag}(\mathrm{Z}(R))$. On K_1, the extension Ψ_0 pairs Θ_+, Θ_- with distinct $\beta_\rho^+, \beta_\rho^-$, with multiplicity 1. Indeed, the regular diagonal action on $\Theta_\pm$ induces $\beta_\rho^\pm \mapsto \beta_\rho^\mp$. On the unitary half of dimension m, we choose a diagonal representative A_0 with nonsquare determinant $d_0 \in \mu_{q+1}$. By inverse transpose, its commutator with j_m has determinant d_0^2 on that half and central coordinate d_0. Thus, the commutator has quadratic value $\eta(d_0) = -1$. By Lemma 6.21, $(\Psi_0|_{K_1})^\sigma = \Psi_0|_{K_1}$ for either pairing. If f is odd, $\xi_\sigma(b^f) = -1$. Since $b^f = (j_m, j_m)$, the restriction $\xi_\sigma|_{K_1}$ has the action $\beta_\rho^\pm \mapsto \beta_\rho^\mp$ with $\Theta_\pm$ fixed. Thus, $\Psi_0|_{K_1}\,\xi_\sigma|_{K_1} \neq \Psi_0|_{K_1}$, a contradiction.

If f is even, then $p \equiv 1 \bmod 4$. Hence, $\sigma \in \mathcal{H}_p$ fixes each $\Theta_\pm, \beta_\rho^\pm$ by Lemma 6.21. Given a paired constituent $\Theta_\pm \boxtimes (\beta_\rho^\pm)^\vee$, choose pure field extensions $\widehat{\Theta}, \widehat{\beta}$. By the lemma, their ratios at a_{fld} agree. Thus, $\widehat{\Theta} \boxtimes \widehat{\beta}^\vee$ has ratio 1. The diagonal $\mathrm{Z}(R)$ acts trivially and $\widehat{\Theta} \boxtimes \widehat{\beta}^\vee$ satisfies $b^f = (j_m, j_m)$. Hence, on $K_1\langle b\rangle$, the paired summand of Ψ_0 is $(\widehat{\Theta} \boxtimes \widehat{\beta}^\vee)\xi$ with $\xi \in \mathrm{Irr}(K_1\langle b\rangle/K_1) = \mathrm{Irr}(C_f)$. Put $f_2 = 2^{v_2(f)}$. Since $2f_2 \mid p - 1$, σ fixes the 2-primary part of $\xi(b)$. Thus, $\xi^\sigma(b)/\xi(b)$ has odd order and cannot be -1. This contradicts $\xi_\sigma(b) = -1$. Consequently, $\Psi = \Psi_0$ is fixed under $\mathcal{H}_p$. If $m = 2$, $h = 1$ and Θ_+, Θ_- are the branches for $\mathrm{SL}_2(q)^2$ in Lemma 6.21. If $m = 4$, the actions come from b and from orthogonal maps on $\mathbb{F}_q^8$. □

Suppose that $\epsilon = -1$ and $2 \le m < e$ in Proposition 6.16. The w unitary cycles have odd length e. Over $\overline{\mathbb{F}}_q$, the active space is $V_A = U_A \oplus U_A^\vee$, with ew oriented lines in U_A. Thus, on the torus coordinates (x_i) of U_A, the map F cyclically permutes the powers x_i^{-q}. Since e is odd, each cycle has sign -1. Since $n = ew + m$ is odd, the residual $2m$-space has sign $(-1)^m$. Thus, we fix $\mathbf{R}, \mathbf{L}_R$ of Section 6.2. Given an F-stable group $\mathbf{H}$, we write $H = \mathbf{H}^F$.

Lemma 6.23. *Let $\mathbf{T}_A$ be the spin torus on V_A, with spin kernels $\langle z_A \rangle$ on V_A and $\langle z_R \rangle$ on the residual space V_R. Then, a connected central torus $\mathbf{T}_0 \le \mathbf{K}$ is stable under H'. The quotients of $\mathbf{K}$ and $\mathbf{J}$ by $\mathbf{T}_0$ are $\mathbf{C} = (\mathbb{G}_m \times \mathbf{R})/\langle (-1, z_R) \rangle$ and $\mathbf{B} = (\mathbb{G}_m \times \mathbf{L}_R)/\langle (-1, z_R) \rangle$, respectively. Moreover, we have that $F(z) = z^{-q}$ on the scalar coordinate z, and $B = \mathrm{GU}_m(q) \times \mu_{q+1}$. Let γ in the translated unipotent series of J have residual partition ρ, and let α be its signed induction to K. Then, we obtain that $\gamma = \gamma_c \circ q_J$ and $\alpha = \alpha_c \circ q_K$ for the quotient maps $q_J : J \to B$ and $q_K : K \to C$, where $\gamma_c(A, z) = \eta(z) u_\rho(A)$ and $\alpha_c = \varepsilon_{\mathbf{C}} \varepsilon_{\mathbf{B}} R_{\mathbf{B}}^{\mathbf{C}}(\gamma_c)$.*

Proof. On V_A, the spin projection is $\pi_A : \mathbf{T}_A \to \mathrm{SO}(V_A)$. Given $\mathbf{S}$, its centralizer in $\mathrm{SO}(V_A \oplus V_R)$ has factors $\pi_A(\mathbf{T}_A)$ and $\mathrm{SO}(V_R)$. Moreover, we have that $xy = yx$ in the even Clifford algebra for $x \in \mathrm{Cl}^0(V_A)$ and $y \in \mathrm{Cl}^0(V_R)$ by [137, Theorem 24.56]. Thus, the product on $\mathbf{T}_A \times \mathbf{R}$ has kernel $\langle (z_A, z_R) \rangle$. Consequently, $\mathbf{K} = (\mathbf{T}_A \times \mathbf{R})/\langle (z_A, z_R) \rangle$ and $\mathbf{J} = (\mathbf{T}_A \times \mathbf{L}_R)/\langle (z_A, z_R) \rangle$.

The character lattice of $\mathbf{T}_A$ is $\Lambda_A = \mathbb{Z}^{ew} + \mathbb{Z}(1/2, \dots, 1/2)$. Define $t : \mathbf{T}_A \to \mathbb{G}_m$ by the primitive weight $(1/2, \dots, 1/2)$. Then, $t(a)^2 = \det(\pi_A(a)|_{U_A})$ for $a \in \mathbf{T}_A$, and $t(z_A) = -1$. Thus, $\mathbf{T}_0 = \ker t$ is connected and embeds into $\mathbf{K}$. Since $t(F(a)) = t(a)^{-q}$, the quotient coordinate has $F(z) = z^{-q}$. Oriented permutations on U_A fix t^2, while exchanging $U_A, U_A^\vee$ gives $t^2 \mapsto t^{-2}$. Since Λ_A is torsion free, these actions give $t \mapsto t$ and $t \mapsto t^{-1}$, respectively. The diagonal action fixes $\mathbf{T}_A$ elementwise. Moreover, $t(a) \mapsto t(a)^r$ for the field map. Hence, H' stabilizes $\mathbf{T}_0$, and the quotient of $\mathbf{K}$ is $\mathbf{C}$ with Levi $\mathbf{B}$.

Since $\mathbf{T}_0$ is connected, $q_J : J \to B$ and $q_K : K \to C$ are onto by [133, Proposition 23.2]. On $\mathbf{L}_R$, we write (A, z') with $\det A = (z')^2$. Then, the map $\varrho : \mathbf{B} \to \mathrm{GL}_m \times \mathbb{G}_m$ acts by $[u, (A, z')] \mapsto (A, uz')$. Given any (A, z), choosing either root $(z')^2 = \det A$ gives the class $[z/z', (A, z')]$. Thus, ϱ is an isomorphism with $F\varrho = \varrho F$. Write U_R for the residual half on which A acts. Since $(t(a)z')^2 = \det(\pi_A(a)|_{U_A}) \det A$, the coordinate $z = t(a)z'$ is the half determinant on $U_A \oplus U_R$. The two signs at (z_A, z_R) cancel in $t(a)z'$. Therefore, the quadratic factor in γ is $\eta(z)$, as in [208, before Lemma 7.10], and $\gamma = \gamma_c \circ q_J$.

The quotient $\mathbf{K} \to \mathbf{C}$ is an F-equivariant isotypy with central kernel $\mathbf{T}_0$, carrying $\mathbf{J}$ to $\mathbf{B}$. The character $\gamma_c = \eta u_\rho$ is uniform because ρ is a unitary label and η is linear on z. Thus, by [72, 2.3.13–2.3.16], Deligne–Lusztig characters on $\mathbf{K}, \mathbf{C}$ agree along q_K. By transitivity through $\mathbf{B}$, $R_{\mathbf{J}}^{\mathbf{K}}(\gamma_c \circ q_J) = R_{\mathbf{B}}^{\mathbf{C}}(\gamma_c) \circ q_K$. The split ranks of $\mathbf{K}, \mathbf{J}$ exceed those of $\mathbf{C}, \mathbf{B}$ by the split rank of $\mathbf{T}_0$. Hence, the signs agree and $\alpha = \alpha_c \circ q_K$. □

Lemma 6.24. *Let $\mathbf{C}' = \mathbf{C} \times_{\mathbf{R}_{\mathrm{ad}}} \widetilde{\mathbf{R}}$ with $\widetilde{\mathbf{R}}$ as in* Section 6.2. *Write $\pi : \mathbf{C}' \to \mathbf{C}$ and $v : \mathbf{C}' \to \widetilde{\mathbf{R}}$ for the projections, and put $\mathbf{B}' = \pi^{-1}(\mathbf{B})$. Then, $C'/R = B'/L_R$ and a linear character λ_0 of this quotient satisfies $\gamma_c \circ \pi = \lambda_0(\widetilde{\beta} \circ v)$, $\quad \alpha_c \circ \pi = \lambda_0(\Theta \circ v)$.*

Proof. The kernel of π is the connected central torus $\mathbf{Z}_t$. Thus, $\mathbf{C}'$ is connected and $\pi : C' \to C$ is onto by [133, Proposition 23.2]. Now $\mathbf{C}'$ has derived group $\mathbf{R}$ diagonally embedded. Indeed, the image of $\mathbf{R}$ in $\mathbf{R}_{\mathrm{ad}}$ is onto, so each $c \in \mathbf{C}'(\overline{\mathbb{F}}_q)$ is a product rz with $r \in \mathbf{R}(\overline{\mathbb{F}}_q)$ and $z \in \mathrm{Z}(\mathbf{C}')(\overline{\mathbb{F}}_q)$. Moreover, $\mathbf{B}' = \mathbf{L}_R \, \mathrm{Z}(\mathbf{C}')$ and $v(\mathbf{B}') = \widetilde{\mathbf{L}}_R$. Thus, the inclusion gives $\mathbf{B}'/\mathbf{L}_R \cong \mathbf{C}'/\mathbf{R}$. Lang's theorem for $\mathbf{R}, \mathbf{L}_R$ then gives $B'/L_R \cong C'/R$.

On B', the unipotent factors of $\gamma_c \circ \pi$ and $\widetilde{\beta} \circ v$ both come from u_ρ on $\mathrm{PGU}_m(q)$. Their linear factors agree on L_R. Hence, their ratio λ_0 is linear on $B'/L_R = C'/R$. Thus, $\gamma_c \circ \pi = \lambda_0(\widetilde{\beta} \circ v)$. The maps π, v are central isotypies with $\pi|_{\mathbf{R}} = v|_{\mathbf{R}} = \mathrm{id}_{\mathbf{R}}$. Hence, for $\gamma_c, \widetilde{\beta}$ the formulas of [72, 2.3.13–2.3.16] give $R_{\mathbf{B}}^{\mathbf{C}}(\gamma_c) \circ \pi = R_{\mathbf{B}'}^{\mathbf{C}'}(\gamma_c \circ \pi)$, $\quad R_{\widetilde{\mathbf{L}}_R}^{\widetilde{\mathbf{R}}}(\widetilde{\beta}) \circ v = R_{\mathbf{B}'}^{\mathbf{C}'}(\widetilde{\beta} \circ v)$. On each torus $\mathbf{T} \le \mathbf{B}'$, multiplication by $\lambda_0|_{\mathbf{T}^F}$ gives the corresponding twist by λ_0 on C'. Thus, $R_{\mathbf{B}'}^{\mathbf{C}'}(\lambda_0 \chi) = \lambda_0 R_{\mathbf{B}'}^{\mathbf{C}'}(\chi)$ for the Levi character $\chi = \widetilde{\beta} \circ v$. Moreover, the kernels of π, v are central in $\mathbf{B}'$. Hence, the split rank differences for $(\mathbf{C}, \mathbf{B})$, $(\widetilde{\mathbf{R}}, \widetilde{\mathbf{L}}_R)$ and $(\mathbf{C}', \mathbf{B}')$ agree. Consequently, $\alpha_c \circ \pi = \lambda_0(\Theta \circ v)$ on C'. □

Proposition 6.25. *Given K, J of* Lemma 6.23, *let $\Delta_{K,J} = (KH') \times_{H'/J} H'$. Thus, the character Ψ of* Proposition 6.22 *gives an $\mathcal{H}_p$-fixed extension $\widehat{\Xi}$ of $\alpha \otimes \gamma^\vee$ to $\Delta_{K,J}$, with* $\mathrm{diag}(\mathrm{C}_A(X))$ *in its kernel.*

Proof. Let $\mathbf{L}_{\mathrm{a}}$ be the image of $\mathbf{L}_R$ in $\mathbf{R}_{\mathrm{ad}}$. Put $\Gamma = L_{\mathrm{a}} \rtimes \langle \varphi \rangle$, with prime field map φ of order $2f$. Moreover, the diagonal action of L_{a} on $\mathbf{R}$ fixes the scalar coordinate z, and $\varphi(z) = z^r$. On $\mathbf{C}$, write $J_m([z, x]) = [z^{-1}, j_m x j_m^{-1}]$. Since

j_m^2 is central, $J_m^2 = 1$. Moreover, $F = J_m\varphi^f$, so $\varphi^f|_C = J_m|_C$. Let $\widehat{\rho}_t$ afford Ψ on $V \otimes W^\vee$ for models ρ_Θ of Θ on V and $\rho_{\widetilde{\beta}}$ of $\widetilde{\beta}$ on W. By Lemma 6.24, the representations $\lambda_0(\rho_\Theta \circ v)$ and $\lambda_0(\rho_{\widetilde{\beta}} \circ v)$ factor through the maps $\pi : C' \to C$ and $\pi : B' \to B$, respectively. Thus, we write ρ_C, ρ_B for the representations of C, B. Thus, $\rho_C \boxtimes \rho_B^\vee$ represents $C \times B$ on $V \otimes W^\vee$ with character $\alpha_c \otimes \gamma_c^\vee$. An element $l \in \widetilde{L}_R$ acts diagonally by $\widehat{\rho}_t(l, l)$, and φ acts by $\widehat{\rho}_t(b)$. Since $\widehat{\rho}_t$ is trivial on $\mathrm{diag}(Z_t)$, these matrices give a representation of Γ by $\widetilde{L}_R/Z_t = L_{\mathrm{a}}$ and $b^{2f} = 1$.

Diagonal conjugation by l fixes λ_0 because C'/R is abelian. Thus, $\widehat{\rho}_t(l, l)$ acts on the matrices of ρ_C, ρ_B by conjugation. If $u = \varphi$ or $u = \sigma \in \mathcal{H}_p$, we write $\nu_u \in \mathrm{Irr}(D_t)$ for the twist with $\widetilde{\beta}^u = \widetilde{\beta}\nu_u$ and $\Theta^u = \Theta\nu_u$, by Proposition 6.22. Thus, T_u intertwines the actions on Y_t because the two twists cancel in ψ. If $u = \varphi$, $T_\varphi \in \mathbb{C}^\times \widehat{\rho}_t(b)$ by [95, Corollary 1.6]. Moreover, $\gamma_c^\varphi = \gamma_c$ since η and u_ρ are fixed under φ. Thus, $\lambda_0^\varphi(\nu_\varphi \circ v) = \lambda_0$ on B' and on C' by $B'/L_R = C'/R$. Therefore, the conjugation by $\widehat{\rho}_t(b)$ gives φ on $\rho_C \boxtimes \rho_B^\vee$, including independent arguments in $C \times B$. $\mathcal{P}$ is a representation of $\mathcal{G} = (C \times B) \rtimes \Gamma$. If $u = \sigma \in \mathcal{H}_p$, $\Psi^\sigma = \Psi$ by Proposition 6.22. Thus, T_σ also intertwines $\widehat{\rho}_t^\sigma$ and $\widehat{\rho}_t$ at b. Since $\gamma_c^\sigma = \gamma_c$, $\lambda_0^\sigma(\nu_\sigma \circ v) = \lambda_0$ on B' and C'. Therefore, T_σ is an intertwiner for $\mathcal{P}^\sigma, \mathcal{P}$, and the character Ξ of $\mathcal{P}$ is $\mathcal{H}_p$-fixed. Given $y \in B$, we choose $y' \in B'$ with $\pi(y') = y$. Its image $y_{\mathrm{a}} \in L_{\mathrm{a}}$ acts by $\rho_C(y) \otimes \rho_B(y)^\vee$, since the two factors of $\lambda_0(y')$ cancel on this tensor.

In [208, before Lemma 7.10], $H = \langle \widetilde{L}, \phi_r, j \rangle$, with prime field map ϕ_r and exchange j. Since H' stabilizes $\mathbf{S}$, its action on V_R has the three forms from H. By Lemma 6.23, permutations of blocks in U_A fix t. Moreover, exchanging $U_A, U_A^\vee$ acts by $t \mapsto t^{-1}$. On the residual Levi, j gives $A \mapsto A^{-T}$ up to conjugation in $\mathbf{L}_{\mathrm{a}}$. Since $\varphi^f = J_m$ on C, an element $y \in H'$ induces an action on C of the form $a\varphi^j$. Since the action is rational, $F(a) = a$ in the adjoint residual group, so $a \in L_{\mathrm{a}}$.

The action $\Gamma \to \mathrm{Aut}(C)$ is faithful. Indeed, φ^j fixes the scalar subgroup μ_{q+1} elementwise only if $(q+1) \mid (r^j - 1)$. Since the order of r mod $(q+1)$ divides $2f$ and $0 < r^j - 1 < q + 1$ for $1 \leq j \leq f$, this order is $2f$. It remains to consider $a \in L_{\mathrm{a}}$ acting trivially on R. Given $m \geq 3$, the natural highest weight e_1 is q-restricted. Thus, by Steinberg's Theorems 7.4 and 9.3, V_R is irreducible for R over $\overline{\mathbb{F}}_q$. If $m = 2$, $R = \mathrm{SL}_2(q)^2$ and $V_R = \overline{\mathbb{F}}_q^2 \boxtimes \overline{\mathbb{F}}_q^2$ is irreducible. Thus, a has a scalar representative on V_R, and $a = 1$ in $\mathbf{R}_{\mathrm{ad}}$. Hence, the action on C gives a homomorphism $\zeta : H' \to \Gamma$, $y \mapsto g_y$, with $\mathrm{C}_A(X) \leq \ker \zeta$.

Put $\bar{k} = q_K(k)$ for $k \in K$ and $\bar{j} = q_J(j)$ for $j \in J$. Define $\Phi : \Delta_{K,J} \to \mathcal{G}$ by $\Phi(x, y) = ((\overline{xy^{-1}}, 1), g_y)$. Since $xy^{-1} \in K$ and g_y acts on C by conjugation by y, Φ is a homomorphism. For $(k, j) \in K \times J$, $\rho_C(\bar{k}\bar{j}^{-1})\rho_C(\bar{j}) = \rho_C(\bar{k})$ on V, and the matrix on $W^\vee$ is $\rho_B(\bar{j})^\vee$. Thus, $\Xi \circ \Phi$ restricts to $\alpha \otimes \gamma^\vee$. Given $y \in \mathrm{C}_A(X)$, $\Phi(y, y) = 1$. Consequently, $\widehat{\Xi} = \Xi \circ \Phi$ is $\mathcal{H}_p$-fixed and trivial on $\mathrm{diag}(\mathrm{C}_A(X))$. □

Proof of Proposition 6.16. The eligible range is e odd, since $\mathcal{E}(X, s)$ has p'-degree characters precisely for odd e. Let $f_{X,L} : \chi_\lambda \mapsto \beta_\lambda$ be from Lemma 6.18. Next, compose with $f_{L,I}$ of Lemma 6.19. If $m = 0, 1$, that lemma gives $f_{I,M}$ as well. If $\epsilon = 1$ and $m \geq 2$, Lemma 6.20 gives $f_{I,M}$. If $\epsilon = -1$ and $m \geq 2$, by Proposition 6.25, we obtain $\widehat{\Xi}$ with base tensor $\alpha \otimes \gamma^\vee$. Thus, Corollary 2.5 gives associated models for α, γ with equal factors and (h, σ)-ratios. Since $\widehat{\Xi}$ extends $\alpha \otimes \gamma^\vee$, $\gamma^h = \gamma$ and $\alpha^h = \alpha$ for $h \in H'$. Since $M_\alpha = KI_\gamma$ and $I \leq H'$, the inertia groups for α in M, MH' are KI, KH', respectively. Given any relative label η, the multiplicity model $\mathcal{Q}_\eta$ on $I/J = (KI)/K$ has inverse factor set c^{-1} for the common factor set c. On I and on KI we choose $\mathcal{Q}_\eta$, so the stabilizer intertwiners on $\mathcal{Q}_\eta$ agree. Let $D \to MH'$ be the common central cover. Given $\phi \in \mathrm{Irr}(M \mid \alpha)$, the induction of Proposition 6.9 on D gives models on $(MH')_\phi$. Thus, their factors and scalars agree. Ind_{KI}^M gives all characters above α by [43]. Moreover, the s-orbit of Lemma 6.19 accounts for $\mathcal{X}_M(s)$. At a shared character ϑ, choose one associated model P_ϑ. Thus, a change by ζ gives adjacent cochains ζ, ζ^{-1} as in Proposition 6.9. Consequently, $f_s = f_{I,M} \circ f_{L,I} \circ f_{X,L}$ gives the central mixed comparison on X, M. □

Corollary 6.26. *Let q be odd, $n \geq 5$, and $X = \mathrm{Spin}_{2n}^\epsilon(q)$. Suppose that $p \nmid q$ is odd and $s \in \mathbf{X}^{*F}$ is eligible with natural lift $\widetilde{s}$ having no eigenvalues in $\{1, -1\}$. Let $\mathbf{S}$ be the Sylow d-torus for the correspondence and set $M = \mathrm{N}_{\mathbf{X}}(\mathbf{S})^F$. Then, the correspondence on $\mathcal{E}(X, s)$ and $\mathcal{X}_M(s)$ satisfies $\succeq_c$ for the natural overgroups.*

Proof. If $\widetilde{s}$ has spectrum $\{i, -i\}$, Proposition 6.16 gives the comparison for odd n. If $2 \mid n$, it follows from Lemma 6.36. Otherwise, let $\mathbf{L}^*$ be the proper Levi for s in [208, Proposition 7.3, Step 1]. If the scalars on pairs $\{x, x^{-1}\}$ are independent, $\mathbf{L}^*$ has only oriented blocks. Thus, for s, Corollary 6.195 gives $\succeq_c$. With the fixed torus of s, the action of $-\widetilde{s}$ is $\{x, x^{-1}\} \mapsto \{-x, -x^{-1}\}$. Each orbit $\mathcal{O}_j$ of size 2 gives an oriented half of dimension $2b_j$ in the balanced block. If $\{x, x^{-1}\}$ is fixed, $x = -x^{-1}$. Hence, the residual orthogonal space Y has spectrum in $\{i, -i\}$ for $\widetilde{s}$ and rank $a = n - 2\sum_j b_j < n$. If $a \leq 3$, then Theorem 6.199 gives the comparison on $\mathbf{L}^*$ and its image in $\mathbf{X}^*$, including

$D_3 = A_3$. If $a \geq 4$ and $p \nmid |\operatorname{Spin}(Y)^F|$, take the identity comparison on $\operatorname{Spin}(Y)^F$. Otherwise, the comparison for odd a is Proposition 6.16, and for even a it is Lemma 6.36. The map $\operatorname{Spin}(Y)^F \to \operatorname{PSO}(Y)^F$ has kernel order 2^c and image index $2^{c'}$ for integers $c, c' \geq 0$. Thus, the odd-p degree condition passes to $\operatorname{Spin}(Y)^F$.

On the determinant cover for V_j, the values at z_0 agree by Proposition 6.194. Thus, the V_j-tensors on the quotient by $\langle z_0 \rangle$ are paired by Proposition 6.207. On each $V_j \neq 0$, choose u_j with $(u_j \widetilde{s} u_j^{-1})|_{V_j} = -\widetilde{s}|_{V_j}$. On the balanced V_j, the product centralizer of $\widetilde{s}$ is connected. By Lang's theorem, $(u_j)_j$ has a rational correction u. Thus, for the factor projection π_{fac} of Lemma 6.10, the fibre is $\pi_{\mathrm{fac}}^{-1}(\pi_{\mathrm{fac}}([s])) = \{[s]\}$. For the transporter and radical at $[s]$, take Corollary 6.195. Thus, on X, M, the two inductions give $\succeq_c$ by Proposition 6.9. □

With $X = \operatorname{Spin}_{2n}^{\epsilon}(q)$ and q odd, we let z_0 generate the kernel of $\pi_{\mathrm{nat}} : X \to \operatorname{SO}_{2n}^{\epsilon}(q)$. Suppose that $p \mid |X|$ is odd and $p \nmid q$. Put $d = \operatorname{ord}_p(q)$, $e = \operatorname{ord}_p(q^2) = d/(2,d)$, and $\tau = 1$ for odd d and $\tau = -1$ for even d. Thus, $n = ew + R$ for w active blocks of dimension $2e$ and sign τ, with $R < e$ or $R = e$ with residual sign $-\tau$. Given $v_p(|X|)$, Lemma 6.96 takes the factors $q^n - \epsilon$ and $q^{2j} - 1$ for $1 \leq j < n$, also valid for odd q. Since $|\ker \pi_{\mathrm{nat}}| = 2$, the block Sylow group lifts through π_{nat}. In the Sylow group P, let Q be the toral base. Thus, $\mathbf{S}$ is the product of the central d-parts of the block tori. The Q-weights other than 1 determine the active tori. Hence, $K = \mathrm{C}_X(Q) = \mathrm{C}_{\mathbf{X}}(\mathbf{S})^F$ and $M = \mathrm{N}_X(Q) = \mathrm{N}_{\mathbf{X}}(\mathbf{S})^F$. Write T_{act} for the finite active torus. Thus, $U_Q = O_{2'}(T_{\mathrm{act}})$. In particular, U_Q contains the ℓ-part of T_{act} for each odd prime $\ell \neq p$.

Lemma 6.27. *Let s label a rational series of X or an eligible series of K below a character of M. Take a natural lift $t \in \operatorname{SO}_{2n}(\overline{\mathbb{F}}_q)$ of s. Then, the central character κ has $\kappa(z_0) = -1$ exactly when $F^*(t) = -t$. In that case, $F^*(V_+) = V_-$ and $F^*(V_-) = V_+$ for the eigenspaces $V_\pm = \ker(t \mp 1)$. For an active eigenvalue coordinate ξ, we have that $\xi^{q^e - \tau} = (-1)^e$. Moreover, we have that U_Q acts trivially exactly when $\xi_{2'} = 1$ for each ξ, with odd part $\xi_{2'}$.*

Proof. We choose a rational maximal torus $\mathbf{T}_{\mathrm{ad}}^*$ containing s, with inverse image $\mathbf{T}_{\mathrm{so}}^*$. By [72, Proposition 2.2.20, §§2.4.4–2.4.5 and Lemma 2.5.7], we have that $\kappa(z_0) = 1$ exactly when s has a lift in $(\mathbf{T}_{\mathrm{so}}^*)^{F^*}$. Since $F^*(-t) = -F^*(t)$, either lift is rational exactly when $F^*(t) = t$. This also gives the result for K on a maximal torus of K. If $\chi \in \operatorname{Irr}(M)$ lies above $\theta \in \operatorname{Irr}(K)$, $\chi(z_0)/\chi(1) = \theta(z_0)/\theta(1)$. Moreover, for $m \in M$, the constituents θ, θ^m give the ambient rational class $[s]$.

Suppose that $F^*(t) = -t$. If $tv = \pm v$, $tF^*(v) = \mp F^*(v)$. Thus, $F^*(V_\pm) = V_\mp$. On an active cycle of length e, we write $\xi_{j+1} = -\xi_j^{\delta_j q}$ for $0 \leq j < e$, with $\xi_e = \xi_0 = \xi$ and $\prod_j \delta_j = \tau$. Since q is odd and each $\delta_j = \pm 1$, $\xi = (-1)^e \xi^{\tau q^e}$. This gives the formula for ξ. The central isogenies of tori have 2^b as kernel order and $2^{b'}$ as image index for integers $b, b' \geq 0$. Hence, the maps give isomorphisms on U_Q and its dual U_Q^*. Moreover, we have that $F^*(t_{2'}) = t_{2'}$ since $F^*(t) = -t$. Thus, the character on U_Q pairs with the tuple $(\xi_{b,2'})_{b=1}^w$. By perfect duality on U_Q, the action is trivial exactly when $\xi_{b,2'} = 1$ for every b. □

Suppose that $F^*(t) = -t$. If e is even, we find $\tau = -1$ and $v_2(q^e + 1) = 1$. Thus, when U_Q acts trivially, the formula for ξ gives $\xi = \pm 1$. The Frobenius F^* exchanges their eigenspaces on a block W, so $\dim(W \cap V_+) = \dim(W \cap V_-) = e$. If e is odd, we find $\xi^{q^e - \tau} = -1$. Thus, if $\xi^4 = 1$, we find $v_2(q^e - \tau) = 1$ and $\xi = \pm i$. Conversely, if $v_2(q^e - \tau) = 1$ and U_Q acts trivially, then $\xi = \pm i$ for each pair $\{\xi, \xi^{-1}\}$. On the residual space V_0, we write $t_0 = t|_{V_0}$. Consequently, we have that $t^4 = 1$ exactly when U_Q acts trivially, $v_2(q^e - \tau) = 1$ for odd e, and $t_0^4 = 1$.

Write $X_0 = \operatorname{Spin}(V_0)^F$. The central product map to K on the active torus and X_0 gives $t \mapsto t_0$ on dual tori. Thus, restriction to X_0 gives the label t_0. For either lift t and every isometry conjugate t_0' of t_0, the number $c_0 = \frac{1}{2}\dim\ker(t_0^2 + 1) = \frac{1}{2}\dim\ker((t_0')^2 + 1)$ is fixed. At $R = 0$, we put $c_0 = 0$. If $t^4 = 1$ and both V_+, V_- occur, then $\dim V_+ = \dim V_-$ by $F^*(V_+) = V_-$. Also, we have that $1 = \det t = (-1)^{\dim V_-}$, since each quartic pair contributes 1. Thus, we put $\dim V_+ = \dim V_- = 2a$ and $n = 2a + c$ for c quartic pairs. Consequently, the shapes are $\mathrm{I}(a, a)$ for $c = 0$ and $\mathrm{Q}(a, c)$ for $c > 0$ by Zhang [208, Lemma 7.15].

Lemma 6.28. *Let $B = B_A H$ be the central product of B_A and the residual factor H. Put $M_0 = \ker a_B \cap \ker d_B$ for the determinant bit a_B and rational boundary bit d_B. Thus, we find $J = H \cap M_0$. Suppose that B_A meets the sign classes in B/M_0 and $\vartheta_A \lambda|_{B_A} = \vartheta_A$ for the corresponding $\lambda \in \operatorname{Irr}(B/M_0)$. For a constituent $\nu \in \operatorname{Irr}(M_0)$ of the full parent, we put $\mathcal{O}_\nu = \{\theta \in \operatorname{Irr}(J) : \langle \nu|_J, \theta \rangle_J > 0\}$. Then, we have that $\mathcal{O}_\nu$ is one H-orbit and determines the parent on H up to $\lambda \in \operatorname{Irr}(B/M_0)$.*

Proof. Given $h \in H$, choose $b_A \in B_A$ with inverse quotient signs, so $b_A h \in M_0$. Since b_A centralizes H, conjugation by $b_A h$ on J equals that by h. Hence, $\mathcal{O}_\nu$ is one H-orbit by [43]. If $\theta, \theta' \in \operatorname{Irr}(H)$ share a constituent on J, $\langle \theta|_J, \theta'|_J \rangle_J > 0$.

By reciprocity for $J \le H$, the inner product is $\sum_{\lambda \in \mathrm{Irr}(H/J)} \langle \theta, \lambda\theta' \rangle_H$. Thus, $\theta = \lambda\theta'$ for some $\lambda \in \mathrm{Irr}(H/J)$. The subgroup H/J of the abelian B/M_0 gives an extension $\widetilde{\lambda} \in \mathrm{Irr}(B/M_0)$ of λ. As a character of B, it has $\widetilde{\lambda}|_{M_0} = 1$ and $\vartheta_A \widetilde{\lambda}|_{B_A} = \vartheta_A$. Hence, the twist from B/M_0 fixes $\nu|_J$ and gives $\lambda\theta'$ on H. In the full parent V_ϑ, the constituents on M_0 label primitive central idempotents of $\mathrm{End}_{M_0}(V_\vartheta)$, with one per M_0-isotypic summand. □

Proposition 6.29. *Suppose that $n \ge 5$, $a > 0$ and $t^4 = 1$ with $F^*(t) = -t$ as in* Lemma 6.27. *Thus, both V_+, V_- are nonzero. If $2 \mid e$, take the model of* Proposition 6.32. *If $2 \nmid e$, take that of* Corollary 6.52 *on $\mathcal{D}$. The degree and core tests give (Λ, λ) in the eligible set. Its image $\chi_M \in \mathrm{Irr}(M)$ determines a, c and the rational class of s. Moreover, its cover labels $(\mathcal{O}_\nu, \eta)$ for $\nu \in \mathrm{Irr}(M_0)$ determine Λ and $\lambda \vdash c$. Thus, the paired idempotents on M_0 select the individual constituents.*

Proof. Put $A = v_p(q^e - \tau)$. For b active blocks and their permutations in $\mathfrak{S}_b$, by Lemma 6.96, we obtain the valuation $f(b) = bA + v_p(b!)$. Since $A > 0$, f is strictly increasing. Moreover, $f(u+v) - f(u) - f(v) = v_p\binom{u+v}{u} \ge 0$. On V_+ over q^2, the block periods are e if e is odd and j if $e = 2j$. Thus, on $V_+ \oplus V_-$ the ranks are $2e, e$, respectively. For the rank-c factor A_c of sign τ_A, the period $\mathrm{ord}_p(\tau_A q)$ is e if e is odd and $\tau_A = \tau$, and $2e$ otherwise. The valuation for b blocks in either factor is $f(b)$, by the factors of $q^{2e} - 1$. A block of rank $2e$ has sign τ^2 and valuation $f(1) < f(2)$. Thus, a block of rank $2e$ prevents the inclusion of P. By Lemma 6.96, rank R contributes no block for $R < e$ or $R = e$ with sign $-\tau$.

If $e = 2j$, then $d = 4j$ and the active blocks lie in $V_+ \oplus V_-$. Thus, $a_0 = a - jw$, $c_0 = c$ and $R = 2a_0 + c$, with split residual form at $R = e$. If e is odd, then $\tau_A = \tau$ and the active blocks lie in A_c, by Lemma 6.45. In that case $a_0 = a$, $c_0 = c - ew$ and $R = 2a + c_0$, with sign $-\tau$ at $R = e$. Hence, c_0 determines c and $a = (n-c)/2$. In particular, $c = 0$ occurs only for even e.

Let b be the available block count and w the weight of η. If $e = 2j$, the cohook degree formula of [40, §13.8] gives $v_p(u(1)) = (b-w)A + v_p(b!/w!) + v_p(\eta(1))$ for the unipotent label u. If $2 \nmid e$, the formula is Lemma 6.45. Since $w \le b$ and $A > 0$, the valuation is zero exactly when $w = b$ and $p \nmid \eta(1)$. Thus, with R and the stated sign, the labels η are exactly those with $p \nmid \eta(1)$. The other factor has p'-order. In the even case $e = 2j$, the label is $\eta \in \mathrm{Irr}(C_{2j} \wr \mathfrak{S}_w)$ and the core Λ_0 is the j-cocore. If $c > 0$, $a_0 < j$. If $c = 0$ and $a_0 = j$, the residual sign is $+1$ and the half-defect h_0 is even. A further j-cohook would give rank 0 with half-defect $h_0 \pm 1$. Since the rank is at least $(h_0 \pm 1)^2 \ge 1$, no such move exists. Hence, Λ_0 is a j-cocore.

Write the ordered symbol as $\Lambda = (L_0, L_1)$. Shift both rows to total length $2m$ with $m = a + 2j(w+1)$. For normalized length $2m'$, one row C has $0 \notin C$, so rank a is at least $h^2 + m' - |h|$ for half-defect h. Thus, $m' \le a$ and each row gains at least $2j(w+1)$ initial beads. Given $r \bmod j$, the runners are $R_{r,\ell} = \{u \ge 0 : r + ju \in L_{\ell + u \bmod 2}\}$ $(\ell = 0, 1)$. By Bessenrodt–Gramain–Olsson, Theorem 5.4, the $(j,1)$-twist maps a j-cohook to the move $u \mapsto u - 1$ on $R_{r,\ell}$. Thus, the packed runners of Λ_0 and the gap partitions on $R_{r,\ell}$ recover Λ. The row exchange maps $R_{r,0}$ to $R_{r,1}$, so the colour $k \in \mathbb{Z}/(2j)$ becomes $k + j$. Hence, η becomes $\eta\varepsilon_W$ for the linear character ε_W with value -1 on each cyclic generator and 1 on permutations.

If Λ_0 has unequal rows, fix parent P_0 for an order, and P_1 for the other. The orbit $\mathcal{O}_\nu$ from Lemma 6.28 gives $\{(\eta, P_0), (\eta\varepsilon_W, P_1)\}$. In the orientation P_0 we read the quotient η, changing colours by $k \mapsto k + j$ if the representative is $(\eta\varepsilon_W, P_1)$. Thus, (Λ_0, η) in order P_0 determines Λ. If $\eta\varepsilon_W \ne \eta$, the other orbit $\{(\eta\varepsilon_W, P_0), (\eta, P_1)\}$ gives a different Λ, since row exchange reorders Λ_0. If the rows of Λ_0 are equal, the orbit $\{\eta, \eta\varepsilon_W\}$ determines Λ up to row order, also for $\Lambda_0 = \varnothing$. Then, $\eta = \eta\varepsilon_W$ exactly when Λ has equal rows. Since $c = c_0$, we recover $\lambda \vdash c$ directly.

At $R = 0$, the active group is restricted to $M_\emptyset = \ker d_A \cap \ker a_A$ for the rational boundary d_A and parity a_A. Let V_η afford the active parent of degree $2^w\eta(1)$. Thus, the self-twist operators S, T in Proposition 6.32 satisfy $S^2 = T^2 = 1$ and $ST = (-1)^w TS$. The quotient inner product gives $\dim \mathcal{C}_{M_\emptyset} = 4$ for $\mathcal{C}_{M_\emptyset} = \mathrm{End}_{M_\emptyset}(V_\eta)$, with basis $1, S, T, ST$ in the four twist spaces. If w is even, $(1 \pm S)(1 \pm T)/4$ are its primitive central idempotents. Thus, $V_\eta|_{M_\emptyset}$ has four constituents. If w is odd, $\mathcal{C}_{M_\emptyset} \cong M_2(\mathbb{C})$. Consequently, $V_\eta|_{M_\emptyset}$ has one constituent of multiplicity 2, with central idempotent $1_{\mathcal{C}_{M_\emptyset}}$. If $R > 0$, the determinant and volume parity is $h \equiv h_0 + w \pmod 2$. The active field exponent is $jw = a - a_0$.

If $2 \nmid e$, the orthogonal factor has label Λ. Since $|\rho| = c_0 < e$, the partition ρ is an e-core. The ordered e-quotient of $\eta \in \mathrm{Irr}(C_e \wr \mathfrak{S}_w)$ and the core ρ determine λ at fixed charge. Moreover, η is independent of the residual orientation P_0, P_1. For the connected torus $\mathbf{T}_b$ on block b, Lang's theorem gives $j_b \in \mathbf{T}_b$ with the chosen boundary over $-I$. Thus, $j_b j_c = j_c j_b$ for every pair b, c. In corresponding signed bases take $j_A = \prod_b j_b$. Its boundary is the volume ω_A,

and $j_A^2 \in B_A$. Thus, whenever $\pi \in \mathfrak{S}_w$, $\pi j_A \pi^{-1} = j_A$. On $C_e \wr \mathfrak{S}_w$, conjugation by j_A is trivial. Thus, the weight extension $\widehat{\theta}_A$ changes by a linear $\nu_A \in \mathrm{Irr}(C_e \wr \mathfrak{S}_w)$, with $\nu_A|_{\mathfrak{S}_w} = 1$. Since $j_A^2 \in B_A$, its conjugation fixes ϑ_A. Thus, Clifford induction for ϑ_A gives $\nu_A^2 = 1$. Moreover, $\mathrm{ord}(\nu_A|_{C_e}) \mid e$. Since e is odd, $\nu_A|_{C_e} = 1$ and so $\nu_A = 1$. Thus, ω_A with the residual lift fixes η, also for $\rho = \varnothing$. Both orientations of nondegenerate Λ occur over η. If Λ is degenerate, its orbit includes χ_+, χ_- on the spin group.

Put $\epsilon = (-1)^v$, $r_0 = (q-1)/2 \bmod 2$, $l_2 = (q^2-1)/8 \bmod 2$, and $t_1 = v + cr_0 \bmod 2$. By Lemma 6.27, the faithful case has $y = 1$ in [208, (7.16) and (7.25)]. If $t_1 = 1$, the two cocycles $x = 0, 1$ represent one rational class by [208, (7.17)]. If $t_1 = 0$, the classes $[s_0], [s_1]$ are distinct. For $t_1 = 0$, the scalar at ω_V is $\kappa_x(\omega_V) = (-1)^{x+ar_0+cl_2} i^c$ by [208, (7.25)]. Thus, $\kappa_0(\omega_V) = -\kappa_1(\omega_V)$, independently of (Λ, λ). Since the characters on X, M agree at ω_V, the values of a, c and $\kappa_x(\omega_V)$ determine x. Moreover, if $\sigma(i) = i^b$ and $t_1 = 0$, then σ acts by $x \mapsto x + c(b-1)/2 \bmod 2$. Finally, on $\mathcal{O}_\nu$ from Lemma 6.28, the idempotents of $\mathrm{End}_{M_0}(V_\vartheta)$ project onto the constituents of $\vartheta|_{M_0}$. □

Lemma 6.30. *We set $d = 4j$ as in* Proposition 6.29. *Thus, we write $q = r^f$ with r prime and field map $F_r(x) = x^r$ on coordinates. For a primitive r-th root ζ_r and $\sigma \in \mathcal{H}_p$, we put $\sigma(\zeta_r) = \zeta_r^b$ and $\varepsilon_\sigma = (b/r)$. Since $4j \mid p-1$, $\mathcal{H}_p$ fixes μ_{4j}. Moreover, we have that ε_σ is determined by the action on ζ_r. Given a degenerate symbol Λ, induction of $\Delta \boxtimes \Delta$ on the two blocks gives Δ_4 on the determinant cover C_4. The field multiplier of Δ_4 under $\mathcal{H}_p$ is 1, and its scalar at $(-I_4, 1)$ is 1. Moreover, we have that Δ_4 extends across the volume lift h_4, with $\rho_4(h_4) = I$ and field multiplier 1.*

Proof. On an isotropic m-space, we put $C_m = \{(A,t) : A \in \mathrm{GL}_m(q),\ t \in \overline{\mathbb{F}}_r^\times,\ t^2 = \det A\}$. The projection $\pi_m : C_m \to \mathrm{GL}_m(q)$ has kernel $\langle z \rangle$ with $z = (I, -1)$. Since $\mathrm{diag}(A, A^{-T})$ has Clifford norm $\det A$, the lift in C_m has coordinate t with $t^2 = \det A$. Write $A = ts$ on C_2, with $s \in \mathrm{SL}_2(\overline{\mathbb{F}}_r)$. Thus, $F_q(s) = t^{1-q}s$ since $F_q(A) = A$. Take Δ on C_2 from the cuspidal Weil source W_{Weil} with its quartic torus character. Write $k = (r-1)/2 \bmod 2$ and $(-1)^{z'} = -\eta_q(-1)$ with $z' \in \{0, 1\}$. Thus, $z' + kf \equiv 1 \pmod 2$.

On the Weil parent W_{Weil} of Lemma 4.12, the sign is J and the field map is B. Let T_σ give the semilinear equivalence for σ. Thus, the relations are $J^2 = 1$, $BJ = JB$ and $B^f = J^{z'}$. Moreover, $T_\sigma B T_\sigma^{-1} = \varepsilon_\sigma^{z'} B$ and $T_\sigma J T_\sigma^{-1} = \varepsilon_\sigma^f J$. Under $F_r(t) = t^r$, the quartic character gains the factor $\delta(t)^k$, where $\delta(t) = t^{q-1}$. Since t, s both give $\delta(t)$, the field map on Δ is $B_C = J^k B$. Consequently, $B_C^f = J$ and $B_C^{2f} = 1$, with $T_\sigma B_C T_\sigma^{-1} = \varepsilon_\sigma B_C$. The factor J^k also occurs when $\varepsilon_\sigma^{z'} = 1$.

In C_4, the block Levi has cover $(C_2 \times C_2)/\langle (z,z) \rangle$ with root $t = t_1 t_2$. Thus, the cover acts on $\Delta \boxtimes \Delta$. On U, the lift $(u, 1)$ acts by I. After induction to C_4, we obtain $\mathcal{I}_4$. Choose $\kappa \in \mathrm{Irr}(T_{\mathrm{sq}})$, where $T_{\mathrm{sq}} = \mu_{2(q-1)}$ and $\kappa(-1) = -1$. Then, $(A,t) \mapsto \kappa(t)^{-1} \chi_\Delta(A,t)$ gives a character π of $\mathrm{GL}_2(q)$. Since $\Delta^{U_2} = 0$ for $U_2 = \{(u,1) : u \in \mathrm{GL}_2(q) \text{ upper unitriangular}\}$, the character π is cuspidal. By [84], $\pi \boxtimes \pi$ gives two Harish–Chandra constituents on $\mathrm{GL}_4(q)$, each of multiplicity 1, with labels $(2), (1,1)$. Their degrees have ratio q^2, the ratio of degrees of St and 1 in $\mathrm{GL}_2(q^2)$. Hence, the smaller constituent $\chi_{\min}$ has label (2), and Δ_4 has character $(A,t) \mapsto \kappa(t)\chi_{\min}(A)$. Indeed, the source space of $\mathcal{I}_4$ carries the common twist $\kappa(t_1 t_2)$.

On $\mathcal{I}_4$, the field map acts by F_r on cosets and $B_C \otimes B_C$ on $\Delta \boxtimes \Delta$. Under $T_\sigma \otimes T_\sigma$, the multiplier is $\varepsilon_\sigma^2 = 1$. Since the two summands of $\mathcal{I}_4$ have different degrees and multiplicity 1, the smaller summand Δ_4 is F_r-stable. Write B_4 for the field map on Δ_4 and ρ_4 for its representation. Thus, $B_4^{2f} = 1$. Since $F_q(A,t) = (A, \eta_q(\det A)t)$ and $\rho_4(z) = -I$, $B_4^f \rho_4(A,t) B_4^{-f} = \eta_q(\det A)\rho_4(A,t)$. Given $x \in \mathbb{F}_q^\times$, the two scalar actions on $\Delta \boxtimes \Delta$ give $\rho_4(xI_4, x^2) = \eta_q(x)^2 I = I$. In particular, $\rho_4(-I_4, 1) = I$.

Choose $\gamma^{q-1} = -1$ and put $h_4 = (\gamma I_4, \gamma^2)$. The element h_4 centralizes C_4, and $F_q(h_4)h_4^{-1} = (-I_4, 1)$. Moreover, $\gamma^2, \gamma^{r-1} \in \mathbb{F}_q^\times$, with $h_4^2 = (\gamma^2 I_4, (\gamma^2)^2)$ and $F_r(h_4)h_4^{-1} = (\gamma^{r-1} I_4, (\gamma^{r-1})^2)$. Since both elements act by I on Δ_4, we set $\rho_4(h_4) = I$ for the extension with field multiplier 1. In the ambient Levi choose h_R with residual boundary ω_R. Thus, the diagonal lift is $h_4 h_R$ with boundary $(-I_4, 1)\omega_R$. □

For the degenerate flag of type I$(2,2)$, we choose a maximal isotropic space $W_{\max}$ over $\mathbb{F}_r$ in the designated family. By Lemma 6.30, Δ_4 has label 1 on the remaining $\mathrm{GL}_2(q^2)$. Thus, L_{sp} is the rational spin Levi. Since B_4^f gives the determinant twist, the spaces $E_\pm = \ker(B_4^f \mp I)$ afford the two constituents on L_{sp}. Thus, a lift g of nonsquare determinant satisfies $gE_\pm = E_\mp$. By Zhang [208, Proposition 7.41], proper-isometry induction for $E_\pm$ gives $\chi_\pm$, each of multiplicity 1. The proper parent χ_{pr} has spin restriction $\chi_+ + \chi_-$ by transitivity. Moreover, ω_V acts by $+1$ and h_4 fixes $W_{\max}$.

The graph automorphism g_{gr} exchanges the two families at $W_{\max}$ and satisfies $g_{\mathrm{gr}}(\omega_V) = z_0\omega_V$. Hence, $\chi_{\mathrm{pr}}^{g_{\mathrm{gr}}}$ has scalar -1 at ω_V. By Clifford [43], induction of χ_{pr} across g_{gr} is irreducible. Its spin constituents are $\chi_+, \chi_-, \chi_+^{g_{\mathrm{gr}}}, \chi_-^{g_{\mathrm{gr}}}$. Each preceding pair of boxes has $R > 0$, so the incidence actions of Zhang [208, Lemma 7.19 and Proposition 7.20] carry B_4, h_4 and the common scalar at z_0. Thus, each pair adds 0 to the field exponent.

Lemma 6.31. *Let $\Lambda = (B_\xi, B_\xi)$ have rank $a > 0$. Then, the active parent ϑ_A for $e = 2j$ extends across the volume diagonal d'_A in its stabilizer, with coefficient multiplier* 1 *at d'_A.*

Proof. Since the rows of Λ are the beta sets B_ξ, by Proposition 6.29, $R_{k,0} = R_{k,1}$ for $0 \le k < j$. Thus, the core is (B_{ξ_0}, B_{ξ_0}) for $\xi_0 = \mathrm{core}_j(\xi)$. The quotient repeats $\mathrm{quot}_j(\xi)$ twice. Put $u = (|\xi| - |\xi_0|)/j$. Hence, $w = 2u$ and $\eta\varepsilon_W = \eta$. Given $\theta \in \mathrm{Irr}(C_j \wr \mathfrak{S}_u)$ with label $\mathrm{quot}_j(\xi)$, wreath induction gives $\eta(1) = \binom{2u}{u}\theta(1)^2$. Thus, the degree test $p \nmid \eta(1)$ for the fixed labels is $p \nmid \binom{2u}{u}\theta(1)$.

Let $\theta_A \in \mathrm{Irr}(T_A)$ be the active weight and I_A its inertia group. The alternating diagonal d'_A has boundary ω_A, the active volume. Since jw is even, $(d'_A)^2 = 1$, and $F_r(d'_A) = d'_A$. For the exponents n_b of rotations in I_A, the parity is common. Thus, $\sum_b n_b \equiv 0 \bmod 2$ since $w = 2u$. Moreover, $d'_A I_A (d'_A)^{-1} = I_A$. Given any squared rotation r_b^2, conjugation by d'_A contributes the sign of θ_A on block b. Moreover, $d'_A x (d'_A)^{-1} = x$ for $x \in T_A \cup \mathfrak{S}_w$. On $r_1 \cdots r_w$, it contributes ω_A with $\theta_A(\omega_A) = 1$ from the w blocks. Thus, the quotient twist is ε_W and the square root on $r_1 \cdots r_w$ is fixed.

Since $\eta\varepsilon_W = \eta$, the colours in $W = C_{2j} \wr \mathfrak{S}_w$ $t, t + j$ carry equal partitions. Let $w_{\mathrm{col}} \in \mathfrak{S}_w$ exchange the blocks for $t, t + j$, with $w_{\mathrm{col}}^2 = 1$. Let S_{col} exchange their Specht factors. Thus, for f in the induced model of η, choose $X_A f(g) = \varepsilon_W(g) S_{\mathrm{col}} f(w_{\mathrm{col}}^{-1} g)$. This gives the twist $\eta\varepsilon_W$, and $X_A^2 = 1$ since $\varepsilon_W(w_{\mathrm{col}}) = 1$. The coefficients of X_A lie in $\mathbb{Q}(\zeta_{2j})$, so they are fixed under $\mathcal{H}_p$. On the source space for I_A, we have that the field operator P_A is scalar on the wreath factor V_η. Thus, $X_A P_A = P_A X_A$, with the square on $r_1 \cdots r_w$ fixed. Since d'_A acts by X_A, induction in stages extends ϑ_A. For the boundary sign operator S_A and volume operator U_A, its relations are $X_A^2 = 1$, $X_A S_A X_A^{-1} = U_A S_A$ and $X_A U_A X_A^{-1} = U_A$. Thus, the boundary action of d'_A multiplies S_A by U_A. □

Proposition 6.32. *Let $q = r^f$ be odd and $N = \mathrm{Spin}_{2n}^{\epsilon}(q)$, with $n = 2a + c \ge 5$ and $a > 0$. Suppose that $p \ne r$ is an odd divisor of $|N|$ and $\mathrm{ord}_p(q) = 4j$. In the faithful series $Q(a, c)$ for $c > 0$ or $\mathrm{I}(a, a)$ for $c = 0$, we choose the eligible (Λ, λ) of* Proposition 6.29. *Then, for $M = N_N(Q)$, we have that the marked pairs (χ, ψ) at N, M give the relation $\succeq_c$ on their natural triples. Conversely, (Λ, λ) at N is recovered by* Proposition 6.29 *from ψ.*

Proof. If $c > 0$, put $b_2 = (c - |\mathrm{core}_2(\lambda)|)/2$, and otherwise put $b_2 = 0$. Thus, if $c > 0$, the partition λ has size c and its 2-core has size $c - 2b_2$. With $d_r = \mathrm{ord}_p(r)$, we put $f_2 = 2^{v_2(f)}$. Since $d_r / \gcd(d_r, f) = 4j$, $v_2(d_r) = v_2(f) + 2 + v_2(j)$ and $4f_2 \mid d_r \mid p - 1$. Thus, we have that $\mathcal{H}_p$ fixes μ_{4f_2}. Since $(\mu_{4f_2})^f = \mu_4$, each element of μ_4 has an f-th root fixed by $\mathcal{H}_p$.

Write $b = |\mathrm{core}_2(\lambda)|$, so $b = 0$ for $\mathrm{I}(s^2, s^2)$. For cuspidal rank s^2, $s \ge 2$, take the staircase construction of V_{st} after Lemma 6.41. Thus, by [208, Proposition 7.26, Steps 3–5], (q, s, b, F) determine the finite source. Its space V_{st} has multiplicity 1 and is irreducible on B_Δ. Moreover, its reductive source is over $\mathbb{Q}(i)$. For the form lift w, write $w^2 = z_0^{Q_b}$ and $\tau = s + b \bmod 2$. The line stabilizer lift is $w_B = w u_0^\tau$ for a perpendicular unit vector u_0, with scalar κ satisfying $\kappa^2 = (-1)^{Q_b + \tau}$. Thus, choose $\alpha, \beta \in \mu_{4f_2}$ with $\alpha^f = \kappa^{-1}$ on the field map and $\beta^f = i^\tau$ for the Pin model. Since $\alpha, \beta \in \mu_{4f_2}$, both are fixed by $\mathcal{H}_p$ for each Q_b.

If $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$, the half-grading lift x_σ gives $\theta^{x_\sigma} = \theta^\sigma$ and fixes the source on V_{st}. On the odd Jordan blocks, $F_r(x_\sigma) x_\sigma^{-1} = z_0^{s(1-\varepsilon_\sigma)/2}$ so the boundary is in $\langle z_0 \rangle$, and induction absorbs conjugation by x_σ. If σ exchanges the extensions at w_B, the volume coset conjugates w_B on these extensions. Since z_0 acts by -1, the coefficient character ν_σ has $\nu_\sigma|_{P_{\mathrm{pin}}} = 1$ on the finite Pin group P_{pin} and $\nu_\sigma(b_F) = \varepsilon_\sigma^s$ on the field generator b_F. On the base parent, $b_F^{2f} = w^{-2}$ and $(b_F^f w)^2 = 1$. Thus, the formulas hold for $b = 0$ and $b > 0$.

For cuspidal ranks s^2, u^2 over $\mathbb{F}_{q^2}$ on the two sides, suppose that $s, u \ge 2$. Put $c_2 = b$, so the j-cohook weight is $w = (s^2 - u^2)/j$. Along the rim domino chain from c_2 to c, the rank $2u^2 + c'$ has constant sign. If $2u^2 + c = 2j$ is an allowed boundary, then $2u^2 + c' < 2j$ for $c' < c$. Since both quotients have label η, Proposition 6.29 gives eligible labels at c'. If $c_2 = 0$, the initial series are $\mathrm{I}(s^2, s^2)$ and $\mathrm{I}(u^2, u^2)$, with positive residue u^2.

For one domino, take $W_+ = \langle e_1, f_2 \rangle$ and $W_- = \langle f_1, e_2 \rangle$ for dual bases e_i, f_i of the quartic summand V_c. In the unitary case, e_1, e_2 form a Hermitian hyperbolic pair. Thus, $W_\pm$ are rational by [208, Proposition 7.21, Step 1]. Both

$\mathbf{S}$ and Q fix $W_\pm$ pointwise. The residual parabolic P_{flag} has radical $U_0 = \mathrm{C}_U(Q)$ for ambient radical U. Thus, Harish–Chandra induction from P_{flag} keeps the active tensor factor V_A. By [208, Proposition 7.21, Step 1], a simultaneous mixed stabilizer operation (g, σ) admits a rational correction at $W_\pm$. If this moves P to P', Sylow conjugacy gives $P' \mapsto P$ while fixing $W_\pm$.

The quartic coordinates ξ_1, ξ_2 give $\xi_1 \xi_2^{-1} = 1$ on the central torus of W_+. Thus, the quotient source has group $(\widehat{S}_1 \times P_R)/\langle(-I, z_0)\rangle$. The Pin parent P_R belongs to the preceding partition λ^-. We choose $\widehat{S}_1$ from Lemma 6.44. Thus, the even Levi L^+ has residual parity 0 and boundary condition $d_S + d_R = 0$. The two factors of $\Theta \boxtimes \Phi_R$ have scalar -1 at $-I, z_0$, so $\langle(-I, z_0)\rangle$ acts trivially. On both sides Θ is quadratic principal for $q \equiv 3 \bmod 4$ and quadratic cuspidal for $q \equiv 1 \bmod 4$. Thus, Lemma 6.44 gives its field operator B with $B^{2f} = 1$, boundary operator B^f and multiplier ε_σ.

In both rational forms the domino $\lambda^+ \setminus \lambda^-$ has coefficient 1 by [208, Proposition 7.21, Step 2]. At $W_\pm$, the twists agree on $\lambda^\pm$. Thus, Step 2 gives their matching permutation π_D. On the cosets of P_{flag}, the field matrix is $B \otimes B_R$, with B_R acting at λ^-. Keep the summands selected by π_D. If the norm and volume matrices S, U_ω satisfy $SU_\omega = -U_\omega S$, keep both summands. On $B \otimes B_R$, the multiplier of Lemma 6.44 is ε_σ times that on B_R. For central plane volume -1, the normalized $i^n \rho(\omega)$ gains $i^2 = -1$. If $2 \nmid c$, the form parities at c' are constant. Thus, the volume gains -1 on the graded tensor. Modulo $\langle(-I, z_0)\rangle$, we pair the volume eigenspaces. At c_2 the exponents are s, u by the staircase computation. Thus, after b_2 dominoes they are $s + b_2, u + b_2$. Since $s^2 - u^2 = jw$, we have that $s + b_2 \equiv jw + u + b_2 \pmod 2$, where the active contribution is jw. The source tensor has the same B^f and action of w. Thus, we find $(b_F^f w)^2 = 1$ for the chosen w.

For the pure partition source with $c > 0$ even, assume that $\mathrm{core}_2(\lambda) = \varnothing$. Stop at $c' = 2$, with type A sign $t_A = -\eta_q(-1)$ and η_q quadratic on $\mathbb{F}_q^\times$. Put $S_1 = \mathrm{SL}_2(q)$. Thus, we obtain that $\mathrm{Spin}_4^+(q) = S_1 \times S_1$ and $z_0 = (-I, -I)$. On the natural module $V_1 \otimes V_2$, we have that the lift t has opposite eigenvalues with multiplicity 2. Thus, the adjoint label is $(s_1, 1)$ with s_1 a toral involution of $\mathrm{PGL}_2(q)$. Its torus has order $q - t_A$, and the other factor carries $V = 1$ for (2) or $V = \mathrm{St}$ for $(1, 1)$. The two orders are related by γ_{gr}. On the full proper cover $\widehat{S}_1 \times \widehat{S}_1$, we choose $\Theta \boxtimes V$ in the first order.

On $\mathbb{P}^1(\mathbb{F}_q)$, the constants give $V = 1$ and the augmentation space gives $V = \mathrm{St}$. Thus, the projective action of $\widehat{S}_1$ extends V, with scalar 1 on $-I$ and field permutation of order dividing f. Write $P^+ = \{(g_1, g_2) \in \widehat{S}_1^2 : d(g_1) = d(g_2)\}$. On $S_1 \times S_1$, the constituents of $\Theta \boxtimes V$ are $\theta_+ \boxtimes V, \theta_- \boxtimes V$ for the Weil pair $\theta_\pm$. The simultaneous nonsquare diagonal exchanges $\theta_\pm$, so the parent on P^+ is irreducible. Its conjugate under γ_{gr} has faithful scalar at $-I$ in the other factor, also for $q = 3$. Thus, induction across γ_{gr} is irreducible by [43]. Choose the F_r-fixed lift g_{gr} of γ_{gr}. On $P^+ g_{\mathrm{gr}}$, the square acts by $\rho_{P^+}(g_{\mathrm{gr}}^2)$. Thus, on both cosets of P^+, the field ratio is that of $\Theta \boxtimes V$. Consequently, the parent has 4 distinct spin constituents, with finite ratio 1 and field ratio ε_σ.

If $c' > 2$, take the steps of [208, Proposition 7.21]. If $c' = 4$, we choose γ_{gr} on the planes W_+, W_-. On $C_2 = \{(A, t) : A \in \mathrm{GL}_2(q),\ t^2 = \det A\}$, the source is $(A, t) \mapsto \Theta(t^{-1}A)$. Thus, (xI_2, x) acts trivially for $x \in \mathbb{F}_q^\times$. If $\gamma^{q-1} = -1$, we have that the lift $h_2 = (\gamma I_2, \gamma)$ has boundary $(-I_2, -1)$. Its square and $F_r(h_2)h_2^{-1}$ have scalar entries $\gamma^2, \gamma^{r-1} \in \mathbb{F}_q^\times$, respectively. Hence, $h_2 \mapsto I$ satisfies both relations. Combining h_2 with the residual lift h_R gives the diagonal $h_2 h_R$. Thus, the finite ratio is 1 and each plane adds a factor ε_σ. The final exponent is $1 + (c - 2)/2 = c/2 = b_2$.

Given a positive staircase $\delta_m = (m, m-1, \ldots, 1)$, put $b = m(m+1)/2$. First, suppose that $m \geq 2$ and set $\mathbf{G}_b = \mathrm{Spin}_{2b}$, with centre $Z = \mathrm{Z}(\mathbf{G}_b)$. Put $t = \lceil m/2 \rceil$ and $d_k = 2m - 1 - 4k$ for $0 \leq k < t$. Let γ, ν be the alternate rows of δ_m in that order. Thus, $\gamma' + \nu' = \delta_m'$ and $n(\delta_m) = 2n(\gamma) + 2n(\nu) + |\nu|$, with $|\nu| < |\gamma|$. By [208, Lemma 7.11, Step 1 and Proposition 7.36, Step 3], the Weyl label $\{\gamma, \nu\}$ has multiplicity 1 under truncated induction. Thus, for $\widetilde{\chi}$ the partition is $(d_0^2, \ldots, d_{t-1}^2)$ in both parities of b.

Given a pair (u, θ) in that orbit, the component group $A(u)$ has determinant signs $(\epsilon_k)_{k=0}^{t-1}$ with $\prod_k \epsilon_k = 1$ by [208, Lemma 7.24]. The element $-I$ has sign 1 on each multiplicity plane U_k. Thus, $|A(u)| = 2^{t-1}$ and F fixes each ϵ_k. Both degree factors of $\widetilde{\chi}(1)$ have lowest coefficient of absolute value 1. They are the type A degree and its index factor. Hence, $n_{\widetilde{\chi}} = 1$, and $\sum_{[u]} \langle \Gamma_u, \widetilde{\chi} \rangle = 2^{t-1}$ by [208, Lemma 7.24] and [188, Proposition 15.4], over rational classes $[u]$.

Choose a rational pair with $V_{\mathrm{src}} \neq 0$. On each plane U_k, choose orthogonal lines with determinant bits $(0, 0)$ if $\epsilon_k = 1$ and $(0, 1)$ if $\epsilon_k = -1$. Let $\Delta \in \mathbb{F}_2^{2t}$ be their determinant vector and put $\mathbf{1} = (1, \ldots, 1)$. Thus, the even volume group B_{vol} has exponent space $H_\Delta = \{v \in \mathbb{F}_2^{2t} : \mathbf{1} \cdot v = \Delta \cdot v = 0\}$ with commutator pairing $v \cdot v'$. If b is even, $\Delta \neq 0$ and $\mathbf{1} \cdot \Delta = 0$ in the faithful case of [208, Proposition 7.36, Step 4]. Thus, $\mathrm{rad}(H_\Delta) = \langle \mathbf{1}, \Delta \rangle$. With a fixed radical

character taking -1 on z_0, the volume degree is 2^{t-2}. Thus, the multiplicity of each $\chi_\pm$ is 2^{t-2}. Take the volume product $g = e_v$ with $\mathbf{1} \cdot v = 0$ and $\Delta \cdot v = 1$. Thus, g changes the scalar on $e_\Delta \in B_{\rm vol}$ and fixes that on Z. Moreover, the graph $g_{\rm gr}$ changes the geometric class of s. Consequently, the four conjugates of χ_+ lie under one parent $\Phi_{\rm st}$ by induction.

If b is odd, exchanging the quartic spaces U_+, U_- has determinant -1. Thus, $\mathrm{C}_{\mathbf{G}_b^*}(s)$ is connected and $\widetilde{\chi}$ restricts to a single χ. Since $\mathbf{1} \cdot \Delta = 1$, the form $v \cdot v'$ on H_Δ is nondegenerate of dimension $2t-2$. Therefore, $\dim V_{\rm src} \geq 2^{t-1}$ for each nonzero source, including $t = 1$. The lifted rotation torus $\mathbf{T}_{\rm rot}$ has lattice $\{v \in \mathbb{Z}^t : \sum_k d_k v_k \equiv 0 \bmod 2\}$. It is connected and contains Z. Thus, by Lang's theorem for $\mathbf{T}_{\rm rot}$, the spin and regular orbits of (u, θ) agree by [208, (7.37)]. By total multiplicity 2^{t-1}, there is one source orbit and $V_{\rm src}$ is irreducible on $B_{\rm vol}$.

In either parity, the reductive group C of the source has r'-order. If σ fixes i, ζ_r, it fixes the labels $\delta_m, [s]$ and the volume character on $V_{\rm src}$. Therefore, its C-character has values in $\mathbb{Q}(\zeta_r, i) \cap \mathbb{Q}(\zeta_{|C|}) \subseteq \mathbb{Q}(i)$. The Clifford model of $B_{\rm vol}$ is over $\mathbb{Q}(i)$. Thus, the irreducible $B_{\rm vol}$-source gives ρ_C over $\mathbb{Q}(i)$ with unnormalized field matrix I as in the staircase model. Since d_k occurs twice, the exponent is 0 mod 2. Moreover, we have that $\alpha, \beta \in \mu_{4f_2}$ are fixed by $\mathcal{H}_p$. If $2 \nmid b$, the lift ω conjugates both extensions at w_B. Thus, its spin restriction is 2χ.

If $b = 1$, the spin torus has order $q + \eta_q(-1) \equiv 2 \bmod 4$. Its quadratic character is -1 on z_0. For the unit reflection u over $\mathbb{F}_r$, we have that the graph lift uu_0 has square z_0. Hence, the extension scalars $\pm i$ are exchanged by ω. With the field scalar fixed by $\mathcal{H}_p$, the parent is unique for $b = 1$ and has field ratio 1. The normalization above gives ratio 1 for $b > 1$ as well. If $2 \mid |\lambda^\pm|$, we have that the permutation $\pi_D \in \mathfrak{S}_4$ is given by [208, Proposition 7.21]. If $2 \nmid |\lambda^\pm|$, the centralizers are connected, including sizes $3, 1$. Thus, the coefficient 1 at (λ^+, λ^-) selects $\chi_{\lambda^+}, \chi_{\lambda^-}$. On the cover, both extensions at w_B occur. Each plane adds ε_σ. Hence, the field ratio for λ is $\varepsilon_\sigma^{b_2}$, also for $b = 0 < c$.

Given an exchanged torus over $\mathbb{F}_{q^2}$, first take $c_2 = m(m+1)/2 > 0$. If $m \geq 2$, [208, (7.34), Proposition 7.35] gives singleton lengths $a, a+4$ and doubled lengths d_k. The exponent of length d is $h(d) = (d^2-1)/8$. Since $h(a+4) - h(a) = a + 2 \equiv 1 \bmod 2$, the two singleton contributions sum to 1 mod 2, while doubled lengths give 0 mod 2. Thus, for $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$, the grading lift x_σ has $F_r(x_\sigma)x_\sigma^{-1} = z_0^{(1-\varepsilon_\sigma)/2}$. Since the rational boundary is in $\langle z_0 \rangle$, x_σ fixes the induced finite character $\Phi_{\rm tor}$. Hence, at b_F the scalar of z_0 gives ratio ε_σ. If $m = 1$, the blocks have lengths $5, 1$ and $h(5) = 3$. Thus, the parity is 1 mod 2. The Whittaker line is distinguished at z_0 by Lemma 6.41.

Since $(\mu_{d_r})^f = \mu_{4j}$ contains μ_4, choose $\alpha_{\rm tor} \in \mu_{d_r}$ with $\alpha_{\rm tor}^f = \kappa_{\rm tor}^{-1}$ for the form scalar $\kappa_{\rm tor} \in \mu_4$. Thus, $d_r \mid p-1$ implies that $\mathcal{H}_p$ fixes $\alpha_{\rm tor}$, i and the model over $\mathbb{Q}(i)$. If x_σ exchanges extensions at w_B, conjugation by the boundary lift ω restores the extension. Hence, induction to $P_{\rm pin}$ gives $\Phi_{\rm tor}$. The normalized Pin volume U_ω is $\mathcal{H}_p$-fixed, so the ratio at b_F is ε_σ. If $c_2 = 0$ and the torus is negative, then $\mathrm{Spin}_4^-(q) = \mathrm{SL}_2(q^2)$. In the odd Weil model, $P_r \delta_x = \delta_{x^r}$ as in Lemma 4.12. For $h^2 = b_\sigma$ in $\mathbb{F}_{r^2}$, the dilation D_h gives the coefficient equivalence and $h^r/h = \varepsilon_\sigma$. Since $D_{-h} = -D_h$ on odd functions, $P_r D_h P_r^{-1} = D_{h^r} = \varepsilon_\sigma D_h$. Hence, the scalar $\alpha_{\rm tor}$ on P_r is $\mathcal{H}_p$-fixed.

For the positive torus, choose Θ_1, Θ_2 of opposite Weil parities on $S_1 \times S_1$. By Lemma 4.12, their scalars on $-I$ are opposite. Thus, take the signs J_1, J_2 and field maps B_1, B_2 from Lemma 6.44. On $E_{\rm bd} = \{(g_1, g_2) \in \widehat{S}_1^2 : d(g_1) = d(g_2)\}$, the tensor $J_1 \otimes J_2$ is an intertwiner. Thus, the eigenspaces $V_\pm$ restrict to distinct orbits on $S_1 \times S_1$, each of size 2 under the simultaneous nonsquare diagonal. Hence, $V_\pm$ are irreducible. Since $B_i J_i = J_i B_i$ and $J_1 \otimes J_2$ has ratio $\varepsilon_\sigma^{2f} = 1$, the field tensor $B_1 \otimes B_2$ acts on each $V_\pm$ with ratio $\varepsilon_\sigma \cdot 1$ from the two factors. Across $g_{\rm gr}$, induction adds $\Theta_2 \boxtimes \Theta_1$ with ratio ε_σ. Thus, at $q = 3$ the odd Weil constituent has degree 1.

For Λ_x of rank x with unequal rows, take the 1-hook of [208, Lemma 7.17] with a nondegenerate successor Λ_{x-1}. Thus, take separate chains for Λ and its j-cocore Λ_0. Ordinary induction allows each cohook weight w_x in those chains. The cocharacter $e_1 + e_{x+1}$ gives a plane with lines U_+, U_- at the eigenvalues $1, -1$. Thus, $F(U_\pm) = U_\mp$, and the torus $T_{\rm hook}$ has character η_q. By Proposition 6.49, choose Θ, the cuspidal Weil parent on $\widehat{S}_1$ from Lemma 6.44. On its cover, write $A = ts$, with $t^2 = \det A$ and $s \in \mathrm{SL}_2(\overline{\mathbb{F}}_q)$. Since $F_q(A) = A$, the boundary of s is t^{1-q}, the inverse of the sign t^{q-1}. Take a quartic ω_T on $\mu_{2(q-1)}$ with $\omega_T|_{\mathbb{F}_q^\times} = \eta_q$, and put $\Delta(t, s) = \omega_T(t)\Theta(s)$. Thus, $\Delta(-1, -I) = \eta_q(-1)[-\eta_q(-1)]I = -I$. The residual source Φ_R has $\Phi_R(z_0) = -I$, so $\langle((-1, -I), z_0)\rangle$ lies in the tensor kernel. At $(-1, I)$, its scalar is $\eta_q(-1)$, so the normalized volume gains $i^2\eta_q(-1) = -\eta_q(-1)$. Thus, by Lemma 6.30, the plane matrix $B_C = J^k B$ on the plane cover has $B_C^f = J$, $B_C^{2f} = 1$ and coefficient ratio ε_σ.

With the marked 1-hook, [208, Proposition 7.18, Steps 3–4] gives a permutation $\pi_{\rm hook}$ on the spin constituents. It pairs Λ_x, Λ_{x-1} with common central twist. Thus, $V_{\rm hook}$ has field action from $B_C \otimes B_R$ on the induction cosets. The operator B_R acts at Λ_{x-1}. Conversely, we have that $\pi_{\rm hook}^{-1}$ gives Λ_{x-1} with its row order. At $\mathrm{I}(2, 2)$, we choose the plane of $e_1 + e_3$ in [208, (7.45), Proposition 7.41] and the inherited graph $g_{\rm gr}$. Thus, at x the flags lie in spin rank

$2x + c$ at stage x.

For absolute half-defect $h \geq 2$, the number of 1-hooks from the cuspidal endpoint is $x - h^2$. Thus, the field exponent is $h + b_2 + x - h^2 \equiv x + b_2 \bmod 2$. If $h = 1$, the torus is nonsplit. If $h = 0$, it is split. Both start at 1 by the torus calculation. With b_2 from the quartic dominoes, the $x - 1$ hooks give $1 + b_2 + (x - 1) = x + b_2$. If Λ_x has equal rows, x is even. If $c > 0$, the pure partition parent has exponent b_2 and each paired box adds 0 by Lemma 6.30 and [208, Proposition 7.20]. If $c = 0$, the endpoint is the marked $\mathrm{I}(2, 2)$. Thus, the exponents at N and the nonzero residue are $a + b_2$ and $a_0 + b_2$, respectively.

If $e = 2j$, we choose the line stabilizer cover P_A on the blocks, with rational subgroup $B_A = \ker d_A$. By Lemma 4.24, the two characters $\phi_\eta^\pm$ of B_A above $\eta \in \mathrm{Irr}(C_{2j} \wr \mathfrak{S}_w)$ have degree $2^{w-1}\eta(1)$. The nonsquare diagonal exchanges ϕ_η^+, ϕ_η^- via the scalar at c_0, the simultaneous rotation. Thus, both induce $\Phi_{A,\eta} \in \mathrm{Irr}(P_A)$. On the source space, the lift v_A acts by $\kappa_A \nu_\eta$, with $\kappa_A = \zeta_8^{w(w+2j)}$ and $\nu_\eta^2 = \eta(d_0)/\eta(1)$ for the wreath diagonal d_0. Choose $\alpha_A^f = (\kappa_A \nu_\eta)^{-1}$. By Lemma 4.24, the ratios are $\sigma(\kappa_A)/\kappa_A = \varepsilon_\sigma^{fjw}$ and $\sigma(\alpha_A)/\alpha_A = \varepsilon_\sigma^{jw}$, also when σ exchanges $\phi_\eta^\pm$.

Let $\widehat{\phi}_\eta$ extend the source character to $B_A\langle b_A \rangle$ with $b_A^f v_A$ acting by 1. Write ξ_σ for the field character with $\xi_\sigma|_{B_A} = 1$ and $\xi_\sigma(b_A) = \varepsilon_\sigma^{jw}$. If $\varepsilon_\sigma^{fjw} = -1$, w is odd. Since $F(\omega_A) = z_0^w \omega_A$, the active volume ω_A lies in $P_A \setminus B_A$. It is F_r-fixed by the block formula $(-i_0)^{2j} E_l = (-1)^j E_l$ in Lemma 4.9, with $F_r(E_l) = E_l$. Thus, ω_A changes the scalar on c_0. Its conjugation fixes the even rotations k_l^2, permutations and b_A. At $b_A^f v_A$ the conjugation gives $-1 = \xi_\sigma(b_A)^f$. Thus, $\widehat{\phi}_\eta^\sigma = \widehat{\phi}_\eta^{\omega_A} \xi_\sigma$. If $\varepsilon_\sigma^{fjw} = 1$, $\widehat{\phi}_\eta^\sigma = \widehat{\phi}_\eta \xi_\sigma$. Thus, after induction to P_A, both conjugates give $\Phi_{A,\eta}$. Its finite ratio is 1 and the ratio at b_A is ε_σ^{jw}.

If $R > 0$, put $h_0 = |\operatorname{def}(\Lambda_0)|/2$ and $h = |\operatorname{def}(\Lambda)|/2$. Thus, we put $t_A = w \bmod 2$, $t_R = h_0 + c \bmod 2$ and $t = t_A + t_R = h + c \bmod 2$. The Pin lift comparison at $t_A, t_R \in \{0, 1\}$ gives models R_A, R_R and parity maps a_A, a_R. On the graded product, $R(xy) = R_A(x) U_A^{a_R(y)} \otimes R_R(y)$, with $U_A = i^{2jw} R_A(\omega_A)$. If $a_A(x) = a_R(y) = 1$, the order contributes -1 through U_A. Moreover, the actions agree at z_0. With $v = v_A v_R$, the coupled group $\widetilde{M}$ giving M is the kernel of $a_A + a_R$ and $d_A + d_R + t_R a_A + t_A a_R$ over $\mathbb{F}_2$. Thus, the second map is the $\mathrm{Int}(v)F_q$-boundary. Conversely, $g \in M$ satisfies $gV^Q = V^Q$ and permutes the active blocks V_l. Hence, g has a lift in $\widetilde{M}$.

Take the line stabilizer matrices B'_A, B'_R and put $L = t_A + t_R + t_A t_R \bmod 2$. If R_R is in Pin coordinates, invert via α, β from the staircase construction. Thus, at v the square gains $(-1)^{t_A + t_R}$ from the two conversions. Moreover, $v_R v_A = z_0^{t_A t_R} v_A v_R$, contributing $(-1)^{t_A t_R}$. Thus, the Pin square carries $(-1)^L$ relative to the line stabilizers. In μ_{4f_2}, choose β_{AR} with $\beta_{AR}^f = i^L$. Then, $B_{AR} = \beta_{AR}(B'_A \otimes B'_R)$ satisfies $B_{AR}^{2f} = R(v)^{-2}$. The field action fixes U_A, U_R. Since $a - a_0 = jw$, its local ratio is $\varepsilon_\sigma^{jw + a_0 + b_2} = \varepsilon_\sigma^{a + b_2}$. If $a_0 = c = 0$, $t = t_A = w \bmod 2$ and $M = \ker d_A \cap \ker a_A$. Thus, the active exponent is $jw = a$, with norm S and volume U_A satisfying $SU_A = (-1)^w U_A S$.

Suppose that Λ has equal rows and $c > 0$. The volume boundary of h_U is ω, and h_U fixes the finite parent Φ. Take Φ on the smaller boundary group H_0 and b_F with $b_F^{2f} = v^{-2}$. Thus, the two extensions have matrices X_U at h_U and B at b_F. Thus, with $t_U = F_r(h_U) h_U^{-1}$ and semilinear intertwiner T_σ of Φ, Schur's lemma gives

$$B X_U B^{-1} = s\Phi(t_U) X_U,\ T_\sigma X_U T_\sigma^{-1} = k_\sigma X_U,\ T_\sigma B T_\sigma^{-1} = l_\sigma B.$$

Since $F_r(h_U^2) = (t_U h_U)^2$, we have $s^2 = 1$. By $X_U^2 = \Phi(h_U^2)$, we have $k_\sigma^2 = 1$. The pure field extension has $l_\sigma = \varepsilon_\sigma^{b_2}$. Map $y h_U^\ell b_F^u$ to $\Phi(y) X_U^\ell B^u$, with $\ell \in \{0, 1\}$. Thus, the next volume bit ℓ' contributes the factor $s^{u\ell'}$. The central double cover $\widehat{H}_U$ for the cocycle $(-1)^{u\ell'}$ acts with central involution s. Hence, the matrices are ordinary on $\widehat{H}_U$, and their coefficient function is $k_\sigma^\ell l_\sigma^u$. Since both sides share u, ℓ', they share $\widehat{H}_U$.

Given any Δ_4 in Lemma 6.30, take $1 \otimes X_U$ and $B_4 \otimes B$. Thus, each step gives s, k_σ, l_σ. On the residue take h_U coupled to d'_A from Lemma 6.31. The graded action on xy is $\rho_A(x) U_A^{a_R(y)} \otimes \rho_R(y)$, with residual parity $a_R(y)$. Since $a_A(d'_A) = 0$, its matrix has $X_A U_A = U_A X_A$. Both $(d'_A)^2$ and $F_r(d'_A)(d'_A)^{-1}$ have parity 0. Since v_A fixes U_A, the product form lift has square $\rho_A(v_A)^2 \otimes \rho_R(v_R)^2$. Equal rows give $w \equiv 0 \bmod 2$. Hence, $t_A = 0$. Thus, the Pin product correction is 1. The relation for B^{2f} and scalars s, k_σ, l_σ agree at both ends. If $a = 2$ and $c > 0$, the last four-space plane $W_{\max}$ distinguishes the marked family of [208, Proposition 7.20, case (5)].

If $c = 0$, the parent from Δ_4 by graph induction has multiplier 1 on h_4 by Lemma 6.30. d'_A has multiplier 1 on B_A by Lemma 6.31. If Λ has unequal rows, the larger diagonal g exchanges φ_0, φ_1. For these parents, if Λ_0 has unequal rows, then $P_0 \neq P_1$ on the residue. If Λ_0 has equal rows, Proposition 6.29 gives $\eta \neq \eta\varepsilon_W$. Thus, the parents differ on B_A. Since the smaller quotient twists fix φ_i, their restrictions have inner product 0 by Lemma 6.28. Thus, the models for φ_1 are conjugates of those for φ_0 by g.

For restriction to N, M, take the cover $\widehat{H}$ and set $D = b_F^f v$. The global form lift w_G and the local v have parity

t. Thus, the argument of Lemma 6.38 gives x in the connected spin group with $F_q(x) = w_G^{-1}xv$. Conjugation by x matches ω and changes the field lift by $k_F = xF_r(x)^{-1}$, rational and even. Thus, $\nu_\sigma(k_F) = 1$ for twists trivial there. Take $b_F^{4f} = v^{-4}$ on $\widehat{H}$. The Lang lift g of Lemma 6.38 is even with boundary ω. Its active part is in T_A, with the residual part in $\mathrm{Spin}(V_R)$ if present. Thus, $gDg^{-1} = \omega^{-1}D$ and $gD^2g^{-1} = z_0^{n+t}D^2$. Thus, z_0^{n+t} is the square ratio.

With parents R_G, R_M on V_G, V_M, put $S = R(D)$ and $U = i^n R(\omega)$ for either model R. Since $F(\omega) = z_0^t\omega$, the relations are $S^2 = U^2 = 1$ and $SU = (-1)^t US$. By the restriction argument of Proposition 6.39, $1, S, U, SU$ span each commutant $\mathcal{A}_G = \mathrm{End}_N(V_G)$ and $\mathcal{A}_M = \mathrm{End}_M(V_M)$ via the quotient self-twists. If $SU = US$, the four joint idempotents are $(1\pm S)(1\pm U)/4$. The labels $(\pm1, \pm1)$ change by conjugation on D, ω and the shared coefficient action σ. This includes $D \mapsto \omega^{-1}D$ for g. Thus, at each joint idempotent e, the stabilizer ratios agree.

If $SU = -US$, both commutants are $M_2(\mathbb{C})$. On $E = \mathbb{C}^2$, take common matrices for S, U, and write $V_G = W_G \otimes E$ and $V_M = W_M \otimes E$. For g in the common quotient $\mathcal{Q}$, its two actions on $M_2(\mathbb{C})$ agree. Indeed, the relations at D, ω and scalar of $\widehat{H}$ determine the images of S, U. Thus, g acts by a signed permutation of S, U, iSU. By Skolem–Noether over $\mathbb{Q}(\zeta_8)$, we choose common implementers L_g over that field, with $L_1 = I$.

Given a lifted element x with image $\bar{x} = g$, factor the parent matrices as $P_G(x) \otimes L_g$ and $P_M(x) \otimes L_g$. Define $\alpha(g, h)$ by $L_gL_h = \alpha(g, h)L_{gh}$. Thus, the factors on W_G, W_M are both α^{-1}. Given a centralizing element z, both parents act on E by a common matrix C_z. Thus, $P_G(z)$ and $P_M(z)$ have the scalar in $C_z = \lambda_z L_{\bar{z}}$. Given (g, σ) in the mixed stabilizer, take the common semilinear map $T_{g,\sigma}$ on E. Thus, the two twist functions on W_G, W_M are $\mu^P_{g,\sigma} = \mu^R_{g,\sigma}/\mu^L_{g,\sigma}$, where $\mu^L_{g,\sigma}$ accounts for $\sigma(L_g)$. Finally, we choose a common section $s_H : H \to \widehat{H}$. Both corrections lie in $Z_{\widehat{H}} = \ker(\widehat{H} \to H)$. Hence, their scalars on W_G, W_M agree and give $\succeq_c$ on N, M. □

Lemma 6.33. *Let V_0 be the orthogonal space of weight 0 in a proper Levi. If $\dim V_0 = 8$, the flags at M_4 give the marking for X_4. Moreover, the action on V_0 comprises isometries, diagonals and fields. Let $X_4 = \mathrm{Spin}_8^\epsilon(q)$, with $q = r^f$ odd and $p \neq r$ an odd divisor of $|X_4|$. If $d = \mathrm{ord}_p(q)$, we put $M_4 = N_{X_4}(\mathbf{S}_4)$ for a Sylow d-torus $\mathbf{S}_4$. Then, on the natural overgroups of X_4, M_4, the marked pairs (χ, ψ) in the eligible exchanged $\mathrm{I}(2, 2)$ series satisfy $\succeq_c$. Moreover, the centralizer label below ψ determines the rational class $[s]$.*

Proof. In the frame of [208, proof of Proposition 7.41], we write $\mathbf{R}_1, \dots, \mathbf{R}_4$ for the four A_1-factors. Thus, the first SO_4 factor contains $\mathbf{R}_1, \mathbf{R}_2$. In the cited action, $A_{\mathrm{fr}} = \langle k, g\rangle$ acts through $k = (12)(34)$, $g = (13)(24)$ and $kg = (14)(23)$. If $\epsilon = -1$, the frame permutations F_{fr} have cycle types $2\,1^2$ and 4. Since only type 4 acts by $\{1, 2\}$ to $\{3, 4\}$, we find one exchanged $[s]$.

Let $P_4 \in \mathrm{Syl}_p(X_4)$ be the marked subgroup. If $c = 0$, Proposition 6.29 gives that $4 \mid d$, so $d = 4j$. By Proposition 6.148, $d \in \{4, 8\}$ with one negative block V_A of rank $2j$. On this cycle, $\mathbf{S}_4$ has rank $\varphi(d)$, and $V_R = V^{P_4}$ has dimension $8-4j$. By [63, Theorems 3.2–3.3], $P_4 = (\mathbf{S}_4^F)_p$ and $N_{X_4}(P_4) \leq M_4$. Since P_4 is characteristic in $\mathbf{S}_4^F$, $Q = P_4$ and $M_4 = N_{X_4}(P_4)$.

In type 4, we have that the labels on $\mathrm{SL}_2(q^4)$ are $1, \mathrm{St}$. By [208, Proposition 7.41, Steps 1–2 and (7.43)], we choose the flag at $e_1 + e_3$. The Levi constituent β paired with χ has multiplicity 1. Thus, by [208, Lemma 7.17], one 1-hook gives Λ from the rank 1 nonsplit endpoint, with $\mathrm{rank}\,\Lambda = 2$. The odd Weil parent has ratio ε_σ. Also, $B_C = J^kB$ has $B_C^f = J$ and ratio ε_σ by Lemma 6.30. Thus, B_{gl} has ratio $\varepsilon_\sigma^2 = 1$. In both orientations, S, U are the matrices of Proposition 6.32.

Locally, $w = 1$ and $a_0 = 2 - j$. If $d = 4$, $a_0 = j = 1$. Hence, Proposition 6.29 gives the split core Λ_0, with the Λ_0-classes paired on E_{bd} by the torus calculation. Hence, $a_0 = 1$ and $jw = 1$ are the exponents. If $d = 8$, then $a_0 = 0$, the local group is $\ker d_A \cap \ker a_A$, and $jw = 2$. Thus, the local ratio is 1 in both cases. Put $D = b_F^f v$ and let ω be the volume element. With $t = 1$, the base orientation has $R(D)^2 = R(\omega)^2 = 1$ and $R(D)R(\omega) = -R(\omega)R(D)$. Thus, both models split as $E \otimes W_\chi, E \otimes W_\psi$, with $\dim E = 2$. The comparison on W_χ, W_ψ is as in Proposition 6.32. Since $b_F^{4f} = v^{-4}$ and $n + t = 5$, the diagonal gives $D^2 \mapsto z_0D^2$. Hence, the mixed functions for (χ, ψ) agree in both orientations.

With marked $\psi \in \mathrm{Irr}(M_4)$, the scalar $\psi(z_0)/\psi(1) = -1$ gives $F^*(t) = -t$ by Lemma 6.27. Since $U_Q \leq \ker\psi$ and $v_2(q^{2j} + 1) = 1$, the condition on ξ gives alternating $x_i = \pm1$ on V_A. Thus, for $d \in \{4, 8\}$, the residue has shape $\mathrm{I}(2, 2)$. Proposition 6.29 gives Λ from (Λ_0, η). Moreover, the primitive central idempotent at Λ gives χ under restriction. Thus, the marking for $\epsilon = -1$ is on M_4 of Proposition 6.148.

If $\epsilon = 1$, k fixes both minus SO_4 factors, whereas g, kg exchange the factors. Since $4 \mid d$, the contributing degrees in $\{2, 4, 4, 6\}$ are $4, 4$. Thus, $d = 4$, $w = 2$ and $P_4 = Q$. For the k-series, the labels on $\mathrm{SL}_2(q^2)$ are $1, \mathrm{St}$, with central

kernel $\langle z_0\rangle$. The residual and linear scalars at $-I$ are $-\eta_q(-1)$ and $\eta_q(-1)$, respectively. Thus, ω acts by -1 for both labels.

By Lemma 6.122, $T = C_{X_4}(P_4)$ is a maximal torus and $M_4 = N_{X_4}(P_4) = N_{X_4}(T)$. If the k-labels agree, the flag of $e_1 + e_3$ takes one 1-hook on each SO_4^- factor. Thus, J, H from Lemmas 6.56 and 6.57 satisfy $J^2 = H^2 = 1$. Moreover, their field operators satisfy $JB_J = B_J J$ and $HB_H = B_H H$. If $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$, we have that both ratios are $\eta_q(b_\sigma)$. Hence, ${}^\sigma(J \otimes H) = J \otimes H$. On the norm eigenspaces and their d_{ns}-translates, the actions of F_r, σ agree. On the d_{ns}-translate, the twist is $\delta\nu$, with determinant δ and norm ν.

If the labels are $1, \mathrm{St}$ in either order, the cocharacter is $2e_1$ or $2e_3$ by [208, (7.45)]. Thus, the singular line U has characters $1, \eta_q$. Since both flags on the split $\mathbb{F}_{q^2}$-factor have multiplicity 1, the model on U is that of Proposition 6.74. On U, the characters $1, \eta_q$ exclude the twists $\nu, \delta\nu$, and the residual model excludes δ. Thus, both restrictions to $\Omega(V_0)$ are irreducible. On U, $R(x)B = BR(x)$ for the inherited field matrix B.

Put $T_o = O_{2'}(T)$. Since $v_2(q^2+1) = 1$, $T = T_o \times Z(X_4)$. In the k-sector, the conditions $\theta(\omega) = -1$ and $\theta|_{T_o} = 1$ give the colours $1, \eta$ with quadratic η. Let $L, M' \in \mathrm{Irr}(C_4)$ be the rotation characters, with ν included in M'. Then, with quadratic ε, we put $L^\dagger = L\varepsilon$. The actions on (L, M') are $\delta : (L\varepsilon, M'\varepsilon)$, $\nu : (M', L)$, $d_{\mathrm{ns}} : (L, M'\varepsilon)$. Indeed, a signed rotation changes δ, the norm ν exchanges $1, \eta$, and d_{ns} shifts the rotation on η.

The labels $1, \mathrm{St}$ correspond to disjoint pairs $\{L, L^\dagger\}$ in $\mathrm{Irr}(C_4)$. Hence, for equal residues, (L, L) has twist group $\langle\nu\rangle$. The matrix J_T of the block exchange has $J_T^2 = -I$. Thus, $S_T = i^{-1}J_T$ has square I, and $\mathcal{H}_p$ fixes its coefficients since $p \equiv 1 \bmod 4$. In the other orientation, $(L, L^\dagger)$ has twist group $\langle\delta\nu\rangle$. Hence, each equal residue gives $2 + 2 = 4$ constituents. If the residues of (L, M') differ, the twist group is $\{1\}$. Thus, the orbit of (L, M') and its d_{ns}-translate give 2 constituents. The $4 + 4 + 2 = 10$ constituents exhaust the k-sector.

Moreover, on the source space, F_r fixes the cyclic complements C_4 and the colours $1, \eta$. The simultaneous rotation has scalar $\zeta = L(c_1)M'(c_2) \in \mu_4$ for the cyclic generators c_1, c_2. Since $D = \mathrm{ord}_p(r)$ satisfies $D/\gcd(D, f) = 4$, we choose $\alpha \in \mu_D$ with $\alpha^f = \zeta^{-1}$. Thus, $B_T = \alpha P_r$ has f-th power the inverse rotation. Since $D \mid p - 1$, $\mathcal{H}_p$ fixes the coefficients of B_T. For equal residues, $B_T S_T = S_T B_T$. Thus, on the inertia group in $\ker\nu$, the ratio is 1 and the scalar at ω is -1. If $\{L, L^\dagger\} \neq \{M', M'\varepsilon\}$, the full isometry inertia has ratio 1 and scalar -1 at ω. By Lang conjugation with the lift v, we pair the field operators B_T, B_{gl}. Hence, both have values $(1, -1)$, and conjugation by d_{ns} gives the other orientation for (χ, ψ).

The triality γ_{tr} permutes k, g, kg and their kernels Z_k, Z_g, Z_{kg} of order 2. Given a diagram representative d_0, we choose x with $(P_4^{d_0})^x = P_4$. If $d_0 x$ and $d_1 y$ both act by $Z_k \mapsto Z_\ell$, their quotient has diagram part in $\mathrm{Stab}_{\mathfrak{S}_3}(Z_\ell)$ of order 2. Thus, the comparison for k defines the markings for g, kg up to natural action. Moreover, $\mathrm{Irr}(Z(X_4))$ is rational, so both markings are $\mathcal{H}_p$-equivariant.

Since $T = C_{X_4}(P_4) = T_o \times Z(X_4)$, each conjugation by $d_0 x$ fixes T_o and $Z(X_4)$ setwise. The torus weight θ satisfies $\theta|_{T_o} = 1$, so $\theta|_{Z(X_4)}$ determines θ. If $Z_k = \langle z_0\rangle$, the two negative cycles have labels $\eta, 1$. Thus, the sign spaces $V^+(t), V^-(t)$ both have minus type and dimension 4. The label is k, so Proposition 6.123 gives g, kg for the conjugate weights. By conjugation, $d_0 x$ transports B_T, B_{gl} and the mixed functions of (χ, ψ) with their centralizer scalars on X_4, M_4. Since $\mathrm{Stab}_{\mathfrak{S}_3}(Z_k)$ accounts for the choices of x, the natural action on X_4 relates the comparisons. Hence, V_0 carries the indicated Levi action of X_4. □

Lemma 6.34. *The exchanged shape* $\mathrm{Q}(a, c)$ *with* $2a + c = 4$ *and* $a, c > 0$ *contributes only for* $X_4 = \mathrm{Spin}_8^-(q)$ *and* $d = 4$. *Hence, at* $X_4 = \mathrm{Spin}_8^-(q)$ *and* $d = 4$, *there is one eligible* $[s]$. *Let* $\mathbf{S}_4$ *be its Sylow* d*-torus and* L_4 *its central-node Levi. Thus, put* $K_4 = C_{X_4}(\mathbf{S}_4)$ *and* $M_4 = N_{X_4}(\mathbf{S}_4)$. *On* L_4, *put* $J_4 = C_{L_4}(\mathbf{S}_4)$ *and* $I_4 = N_{L_4}(\mathbf{S}_4)$. *For* $[s]_{J_4^*}$ *and* $[s]_{K_4^*}$, *the series are* $\{\delta\}$ *and* $\{\alpha_1, \alpha_{\mathrm{St}}\}$. *Moreover, whenever* $u = 1, \mathrm{St}$, *we find* $\langle R_{J_4}^{K_4}(\delta), \alpha_u\rangle = 1$.

Proof. Since $2a + c = 4$ with $a, c > 0$, $(a, c) = (1, 2)$. If e is odd, the blocks of P lie in rank $c = 2$. Hence, $ew \leq 2$ gives $e = 1$, but the residual sign rank is $2 > e$. If e is even, the blocks of P lie in rank $2a = 2$, so $(e, w, d) = (2, 1, 4)$. The residual space V_R has split rank 2. Since V_A is negative, the full space has $\epsilon = -1$. Moreover, $Q = P$, so $M_4 = N_{X_4}(P)$.

By [208, Proposition 7.40], take the D_4 frame with outer roots $\beta_1 = e_1 - e_2$, $\beta_2 = e_1 + e_2$, $\beta_3 = e_3 + e_4$ and $\mu = e_3 - e_4$. The representative is $x = \frac{1}{4}\sum_i \beta_i$. With $w_i = s_{\beta_i}$, the stabilizer is $W_s = \langle s_\mu\rangle \times V$, with $V = \langle w_1 w_2, w_1 w_3\rangle$ and $J' = w_1 w_2 w_3 = -s_\mu$. With $r_0 = (q-1)/2 \bmod 2$, we let γ exchange β_a, β_b. By [208, (7.44)], the rational structures are $F_v = \mathrm{Int}(n_v n_{J'}^{r_0})\gamma F_q$ for $v \in V/(\gamma - 1)V$, with split root μ. Since $p \mid q^2 + 1$, the p-part lies on the moving axes β_a, β_b. Thus, for the signed transposition s_{ab} on $\{\beta_a, \beta_b\}$, $s_{ab}^2 = -I$. Its sign product $\nu_{\mathrm{mov}}(v)$ is nontrivial on V and

trivial on $(\gamma-1)V$. Both signs change under J', so their product remains $\nu_{\mathrm{mov}}(v)$. Thus, $\nu_{\mathrm{mov}}(v)=-1$ selects one class in $V/(\gamma-1)V$ of order 2. Take $\mathbf{S}_4$ in the common maximal torus of L_4, X_4.

Let $\beta_{c'}$ be the fixed outer root. Since $\alpha=(\mu-\sum_i\beta_i)/2$, the reflections w_i, s_μ give all roots $(\pm\mu\pm\beta_1\pm\beta_2\pm\beta_3)/2$. Thus, the roots of $\mathbf{K}_4^*$ are $\pm\mu, \pm\beta_{c'}$, and those of $\mathbf{J}_4^*$ are $\pm\beta_{c'}$. An element of V fixing β_a, β_b pointwise is the identity. Therefore, $C_{\mathbf{K}_4^*}(s)$ is connected with roots $\pm\mu$, and $C_{\mathbf{J}_4^*}(s)$ is a connected torus. Thus, μ defines the oriented rational parabolic $P_\mu \le K_4$ with Levi J_4. Hence, in the regular overgroup of K_4, [72, Theorem 4.7.2] gives induction on the split μ-factor. Its action on $\mathbb{P}^1(\mathbb{F}_q)$ has character $1+\mathrm{St}$, with the torus label on $\beta_{c'}$ fixed. Since $C_{\mathbf{K}_4^*}(s)$ and $C_{\mathbf{J}_4^*}(s)$ are connected, both component stabilizers are 1. Hence, $\langle R_{J_4}^{K_4}(\delta), \alpha_u\rangle = 1$ for $u=1,\mathrm{St}$, and the Jordan degrees give $\alpha_{\mathrm{St}}(1)=q\alpha_1(1)$. □

Proposition 6.35. *With notation from* Lemma 6.34*, let $u\in\{1,\mathrm{St}\}$. There is a correspondence f_u between the characters of X_4 in the eligible row of label u and the characters of M_4 over the orbit of α_u. Moreover, the pairs $(\chi, f_u(\chi))$ satisfy $\succeq_c$ on the marked triples.*

Proof. At μ, write $\mathbf{L}_4$ for the Levi subgroup. The roots β_1,β_2,β_3 give three simply connected SL_2 factors. Since $\mu=2\alpha+\sum_i\beta_i$, the frame lattice has index 2 and $\mu(-1)=\prod_i\beta_i(-1)$. Thus, $\mathbf{L}_4=(\mathrm{SL}_2^3\times\mathbb{G}_m)/\langle(-I,-I,-I,-1)\rangle$, with central cocharacter μ. The isomorphism $[g_1,g_2,g_3,t]\mapsto(tg_1,tg_2,tg_3)$ gives triples of equal determinant. Since F_v exchanges two SL_2 factors, Lang's theorem on PGL_2^3 conjugates F_v to its permutation-field part with μ fixed. Thus, for $k=\mathbb{F}_q$ and $k_2=\mathbb{F}_{q^2}$, $\widetilde{L}=\mathrm{GL}_2(k_2)\times\mathrm{GL}_2(k)$ and $L_4=\{(A,B)\in\widetilde{L}:\det A=\det B\}$. Put $\pi_{\det}(A,B)=\det A/\det B$ and $\widetilde{I}=N_{\widetilde{L}}(\mathbf{S}_4)$. On the nonsplit torus, $\pi_{\det}$ is the norm $N_{\mathbb{F}_{q^4}/k_2}$. Thus, both images are $k_2^\times$. Hence, $\widetilde{I}/I_4\cong\widetilde{L}/L_4$.

On $\mathrm{GL}_2(k_2)$, the two eigenvalues have ratio -1 by the pairing with the moving β_a. Since $p\mid q^2+1$, the split regular centralizer of order $(q^2-1)^2$ has order prime to p. Thus, the eligible series on $\mathrm{GL}_2(k_2)$ is nonsplit. Since $p\nmid|\mathrm{GL}_2(k)|$, the inactive map is $\lambda\mapsto\lambda$. For the eigenvalues $a,-a$, a stabilizing twist multiplies by 1 or -1. Thus, the stabilizer is $\langle\eta_{k_2}\rangle$ with quadratic η_{k_2}. Since $k^\times\subseteq(k_2^\times)^2$, its restriction to $k^\times$ is 1. Let $T_A=\mathbb{F}_{q^4}^\times$ be the active torus. For regular $\vartheta\in\mathrm{Irr}(T_A)$, the local induction from T_A has orbit $\{\vartheta,\vartheta^{q^2}\}$. Since $\vartheta^{q^2}/\vartheta$ has order 2, it equals $\eta_{k_2}\circ N_{\mathbb{F}_{q^4}/k_2}$. Thus, η_{k_2} swaps $\vartheta,\vartheta^{q^2}$. Since $N_{\mathbb{F}_{q^4}/k_2}$ is surjective, a twist fixing ϑ is trivial. Hence, $\langle\eta_{k_2}\rangle$ stabilizes exactly this orbit.

Since $\mathrm{ord}_p(q^2)=2$ and the active orbit has size 2, the orbit fills rank 2. Thus, Proposition 3.20 gives the relative cochain c_{rel} for field prime r and field degree $2f$. Let u_F lift F_r in the torus normalizer. Then, $c_{\mathrm{rel}}(ju_F^{2f})=c_{\mathrm{rel}}(j)$ and $c_{\mathrm{rel}}|_{\langle\eta_{k_2}\rangle}=1$. Hence, $\alpha_{\mathrm{gl}}=\alpha_{\mathrm{loc}}$ after normalization, with equal matrices at u_F^{2f}. On $\mathrm{GL}_2(k)$, both sides carry the representation of λ with identical actions. Since $\pi_{\det}$ defines both quotients, Lemma 3.4 pairs the $S_{\det}$-eigenspaces equivariantly. With the common normalization $S_{\det}^2=I$, each of the eigenvalues ±1 gives one constituent of multiplicity 1. Moreover, both twists are $\xi\circ\pi_{\det}$ under a lift change with $\xi\in\mathrm{Irr}(k_2^\times)$. Thus, $f_{L_4,I_4}\colon\beta\mapsto\nu$ pairs the individual constituents.

Let h stabilize the flags. Thus, h fixes the orientation of μ and stabilizes $\{\beta_a,\beta_b\}$. The transposition of β_a,β_b induces the k-field automorphism on $\mathrm{GL}_2(k_2)$. Moreover, on $\widetilde{L}$, the parts of h are inner, diagonal and field. On μ, the scalar action is a given F_r-power. Hence, the lifts on $\widetilde{L},\widetilde{I}$ induce one action on $k_2^\times$ through $\pi_{\det}$. Given a common centralizer element z, both parents act by $\omega(z)$. Thus, the $S_{\det}$-comparison is equivariant under (h,σ).

For fixed u, the flag pairing $f_{X_4,L_4}\colon\chi\mapsto\beta$ of [208, Proposition 7.40] is given by $\langle\chi,R_{L_4}^{X_4}\beta\rangle=1$. Compatible opposite flags form one rational $C^\circ(s)^F$-orbit by the same result. Thus, $C^\circ(s)^F$ restores the flags, and L_4 restores $\mathbf{S}_4$ by conjugation. Let $I_\delta\le I_4$ be the inertia subgroup of δ. Then, $(M_4)_{\alpha_u}=K_4I_\delta$ and $K_4\cap I_\delta=J_4$. Indeed, a dual normalizer element x fixing $[s]_{K_4^*}$ has $kx\in C_{M_4^*}(s)$ for some $k\in K_4^*$. Since kx fixes s, it fixes the root pair $\{\mu,-\mu\}$. If $(kx)(\mu)=-\mu$, we multiply by a rational representative $\dot{s}_\mu\in C_{K_4^*}(s)$ of s_μ from the split μ-factor. The corrected element fixes μ, so it lies in L_4^* and normalizes $\mathbf{S}_4^*$. Conversely, an element of I_4^* fixing $[s]_{J_4^*}$ fixes $[s]_{K_4^*}$. By Lemma 6.7, Lang's theorem on $\mathbf{K}_4,\mathbf{J}_4$ pairs rational relative cosets for X_4,X_4^*. In the J_4-series, δ is unique. Since I_δ stabilizes the parabolic P_μ and $\alpha_{\mathrm{St}}(1)=q\alpha_1(1)$, it fixes both α_u.

Let U_μ be the unipotent radical of P_μ. If x,y induce the same relative action, then (x,y) acts on $\mathcal{B}_\mu=\mathbb{C}[K_4/U_\mu]$ by $gU_\mu\mapsto xgy^{-1}U_\mu$. By Lemma 6.34, $\alpha_u\otimes\delta^\vee$ occurs with multiplicity 1 over $K_4\times J_4$. Thus, the isotypic summand $\widehat{\Theta}_u$ extends to the inertia of (α_u,δ). Since $\mathcal{B}_\mu$ is rational, an action fixing (α_u,δ) fixes the character of $\widehat{\Theta}_u$. For diagonal (z,z) centralizing K_4, we have that $zgz^{-1}U_\mu=gU_\mu$. Similarly, let U be the radical of the X_4-parabolic. Then, $\mathbb{C}[X_4/U]$ gives the paired extension at (χ,β). After the corrections in $C^\circ(s)^F$ and L_4, the actions stabilize U,U_μ with the same inner factors. Moreover, $\mathbf{S}_4$ is fixed.

The quotient $I_\delta/J_4 \cong (K_4 I_\delta)/K_4$ carries the common inverse-factor representation $\mathcal{Q}_\nu$. Hence, Clifford induction with $\mathcal{Q}_\nu$ at both ends gives the bijection f_{I_4,M_4} over the orbits of δ, α_u. Since $\alpha_{\mathrm{St}}(1) = q\alpha_1(1)$, the two M_4-orbits are disjoint. If $s_1^g = s_2$ are eligible, a correction in $C_{X_4^*}(s_2)$ puts the transporter in $N_{X_4^*}(\mathbf{S}_4^*)$. Hence, the local orbit over $[s]$ is unique. Since p is odd and $p \mid q^2+1$, the rank-one indices divide $[q(q^2-1)]^m$, and component indices divide 2^m for some m. Hence, $p \nmid \chi(1)$ is equivalent to $p \nmid f_u(\chi)(1)$. Finally, we choose the models P_β, P_ν at the shared characters β, ν. Thus, c_β, c_ν cancel in the composition $f_u = f_{I_4,M_4} \circ f_{L_4,I_4} \circ f_{X_4,L_4}$. Hence, the marked triples satisfy $\succeq_c$. □

Lemma 6.36. *If $2 \mid m$, we let $\mathrm{U}(m)$ be the shape with spectrum $\{i,-i\}$ of* Zhang [208, Lemma 7.15]. *The derived group $C^\circ(s)_{\mathrm{der}}$ has type A_{m-1}. Thus, we have that the sign κ is 1 for $\mathrm{GL}_m(q)$ and -1 for $\mathrm{GU}_m(q)$. With $m \geq 4$ even, we let $p \neq r$ be an odd prime dividing $|X|$, where $X = \mathrm{Spin}_{2m}^+(q)$ and $q = r^f$ is odd. Put $d = \mathrm{ord}_p(q)$, $e = d/\gcd(d,2)$ and $q^e \equiv \tau \pmod p$ for $\tau \in \{1,-1\}$. Then, the shape $\mathrm{U}(m)$ contributes precisely for odd e and $\kappa = \tau$. For either κ, the two spin constituents $\chi_{\lambda,\pm}$ of each eligible partition λ have marked partners on $M = N_X(\mathbf{S})$ and satisfy $\succeq_c$ on the natural overgroups of X, M.*

Proof. Since the space of $\mathrm{U}(m)$ is split, put $b = \lfloor m/e \rfloor$ and $A = v_p(q^e - \tau)$. If e is odd and $\kappa = \tau$, the A_{m-1}-factor has b blocks. The ambient count is also b since $m = eb$ implies $2 \mid b$. Otherwise, $\mathrm{GL}_m^\kappa(q)$ has period $2e$, with $\lfloor b/2 \rfloor$ blocks. The ambient count is b unless $m = eb$, $\tau = -1$ and b is odd. Then, the count is $b-1$. Since $p \mid |X|$, this exception has $b \geq 3$ and $\lfloor b/2 \rfloor < b-1$. Thus, $uA + v_p(u!)$ strictly increases in u and excludes period $2e$ on the Levi. For odd e and $\kappa = \tau$, $p \nmid [X^* : C_{X^*}(s)]$. Hence, λ is eligible exactly when $p \nmid u_\lambda(1)$. By [208, proof of Lemma 7.15(iii)], the component group C_2 exchanges the eigenspaces W_+, W_-. On the factor of type A_{m-1} the action is $A \mapsto A^{-\mathsf{T}}$ up to conjugation. By [72, Theorem 4.5.11], this fixes λ. Thus, $\widetilde{\chi}_\lambda$ restricts to $\chi_{\lambda,+} + \chi_{\lambda,-}$ on X.

Write $X = \mathbf{X}^F$ and take the isotropic halves W_+, W_-. The prime-field Clifford lift $j = j_m$ exchanges W_+, W_-. By Section 6.2, $F = F_q$ if $\kappa = 1$, and $F = \mathrm{Int}(j)F_q$ if $\kappa = -1$. Since m is even, $j \in X$ in both cases. Let $T = \mathbb{F}_q^\times$ for $\kappa = 1$ and $T = \mu_{q+1}$ for $\kappa = -1$. Put $L = \{(A,z) \in \mathrm{GL}_m^\kappa(q) \times T : \det A = z^2\}$ and $L^+ = L\langle j \rangle$. Thus, $|L^+/L| = 2$. For quadratic $\eta \in \mathrm{Irr}(T)$, set $\delta = \eta(-1)$ and $h_m = \binom{m}{2}$. Since j inverts z and $j^2 = z_0^{h_m}$, the character $(A,z) \mapsto \eta(z)$ has extensions ω_ξ with $\omega_\xi(j) = \xi$ and $\xi^2 = \delta^{h_m}$. Given $\lambda \vdash m$, the result of [156, proof of Theorem D] gives the adjoint extension U_λ with field and diagram actions. Put $\iota(A) = A^{-\mathsf{T}}$. If $\kappa = -1$, then $\iota = b_0^f$ with b_0 the prime-field generator of order $2f$. Thus, we choose U_λ of Lemma 6.21. The adjoint image of j is ι. Therefore, its representation on L^+, multiplied by ω_ξ, affords $\beta_{\lambda,\xi}$ over $(A,z) \mapsto \eta(z)u_\lambda(A)$.

If $\kappa = -1$, $d = 2e$. Thus, in Proposition 6.22, take $R = X$ and label λ. Since the even-rank proof sets $d = 2e$ and the regular unitary Levi $\widetilde{L}$, it gives the extension on $(X \times L^+)/\mathrm{diag}(\mathrm{Z}(X))$ with each $\chi_{\lambda,\xi} \otimes \beta_{\lambda,\xi}^\vee$ occurring with multiplicity 1. This determines $\chi_{\lambda,\xi}$. Indeed, $b^f = (j,j)$ and a regular diagonal exchanges $\xi, -\xi$ on both sides. Since $\mathcal{H}_p$ fixes the full character, each summand $\chi_{\lambda,\xi} \otimes \beta_{\lambda,\xi}^\vee$ extends on its stabilizer H_ξ with mixed invariance. Write h for a diagonal-field product centralizing X in the overgroup with $|b| = 2f$. The field image of h is trivial modulo $\langle b^f \rangle$, so its field exponent lies in $f\mathbb{Z}$. Modulo b^f with its inner representative, h is a regular scalar. Thus, (h,h) has trivial scalar action in Y_t after that correction. The remaining relation $b^f = (j,j)$ gives the given diagonal action, since $j^{-1} = j(j^2)^{-1}$ and $j^2 \in \mathrm{Z}(X)$. Moreover, $\widetilde{X} = X\widetilde{L}$. Hence, Corollary 2.5 gives the comparison on X, L^+.

If $\kappa = 1$, $d = e$. Put $P_\pm = LU_\pm$ for the two opposite parabolics and $\beta = \beta_{\lambda,\xi}|_L$. Given a regular dual lift $\widetilde{s}$, the connected centralizer has roots $e_i - e_j$ for $i \neq j$. Thus, its unipotent translate $\widetilde{\beta}$ extends β to $\widetilde{L}$. Since $C_{\widetilde{\mathbf{X}}^*}(\widetilde{s}) \leq \widetilde{\mathbf{L}}^*$ is connected, both regular inductions of $\widetilde{\beta}$ are irreducible. Since $\widetilde{X} = X\widetilde{L}$, their restrictions give the induced spaces $\mathcal{F}_\pm$ with character $\chi_{\lambda,+} + \chi_{\lambda,-}$.

By [208, proof of Proposition 4.16], take $S_\pm : \mathcal{F}_\pm \to \mathcal{F}_\mp$ with $(S_\pm\varphi)(g) = \sum_{u \in U_\mp} \varphi(ug)$. On $\widetilde{X}$, the coset inner products give $S_\mp = S_\pm^*$. Since $U_\mp \cap P_\pm = 1$, the unitary blocks lie in q^{h_m} distinct columns per row. Thus, by [208, Proposition 7.6, Step 4], $S_\mp S_\pm = q^{h_m} I$. The restriction to X matches the cosets and $U_\mp$-sums, so the identity holds on $\mathcal{F}_\pm$. Hence, $\widehat{S}_\pm = q^{-h_m/2} S_\pm$ are inverse on each $\chi_{\lambda,\xi}$. Moreover, $\mathcal{H}_p$ fixes $\sqrt{q}$ since $d = e$ is odd, by Lemma 6.18.

With $A_0 = \widetilde{X} \rtimes \langle F_r \rangle$, we put $H = \langle \widetilde{L}, j, F_r \rangle$. Thus, $A_0 = XH$ and $X \cap H = L^+$. Since H fixes β, we let B_y implement $y \in H$ on its space. Let T_y act by $\varphi(g) \mapsto B_y \varphi(y^{-1}gy)$, with $\widehat{S}_\pm$ if y exchanges P_+, P_-. Since y permutes the $U_\pm$-sums under conjugation, T_y has the factor of B_y. With right translation π by X, the fibre product $A_0 \times_{A_0/X} H$ has ordinary representation $\mathcal{T}_\lambda(x,y) = \pi(xy^{-1})T_y \otimes B_y^\vee$. The scalar of B_y cancels in $\mathcal{T}_\lambda$. Since β is rational and $q^{-h_m/2}$

is fixed by $\mathcal{H}_p$, the character of $\mathcal{T}_\lambda$ is $\mathcal{H}_p$-invariant. If y centralizes X, its argument action is trivial and B_y is scalar. Thus, $\mathcal{T}_\lambda(y,y) = I$.

Choose B_j from $\beta_{\lambda,\xi}$. Then, $A_\xi = \pi(j^{-1})T_j$ centralizes $\pi(X)$ and has square I, since $B_j^2 = \pi(j^2) = \delta^{h_m} I$. The coset kernel of A_ξ has support on $P_+ j P_+$, disjoint from P_+. Thus, $\mathrm{tr}(A_\xi) = 0$. Since $\mathcal{F}_+$ has two constituents, each A_ξ-eigenspace gives one. On each, $(1,j)$ acts by $\pm B_j^\vee$. Thus, the second factors afford $\beta^\vee_{\lambda,\pm\xi}$. Hence, $\mathcal{T}_\lambda$ on either summand stabilizer gives $\succeq_c$, as for $\kappa = -1$. For both signs κ, denote the map by $f_{X,L^+} \colon \chi_{\lambda,\xi} \mapsto \beta_{\lambda,\xi}$.

Take $h = m - eb$, the residual rank. Since $2 \nmid e$, choose a self-dual basis $\mathcal{B}$ for the trace of $\mathbb{F}_{q^e}/\mathbb{F}_q$ from [13, Theorem 5.6]. If $\kappa = -1$, we extend the basis to $\mathbb{F}_{q^2}$. Thus, the multiplication matrix A_x satisfies $A_x^{\mathsf{T}} = A_x$, so $\iota(A_x) = A_x^{-1}$. Hence, the Sylow d-torus $\mathbf{S}$ of $\mathbf{L}$ is j-stable and is Sylow in $\mathbf{X}$ by the block count. Put $I = N_L(\mathbf{S})$, $I^+ = I\langle j\rangle$, $J = C_L(\mathbf{S})$, $K = C_X(\mathbf{S})$ and $M = N_X(\mathbf{S})$.

The adjoint normalizer partner of U_λ is $\widehat{U}_\lambda$ from [156, §8.1 and proof of Theorem D]. Since equivariance holds for $\mathcal{H}_p$ and the quotient characters, $U_\lambda, \widehat{U}_\lambda$ have equal coefficient ratios. Moreover, $\widehat{U}_\lambda$ is irreducible along $I \to I_{\mathrm{ad}}$ by the scalar twist. From ω_ξ, we obtain the twist $\gamma_{\lambda,\xi} = \omega_\xi|_{I^+}\, \widehat{U}_\lambda|_{I^+}$, $D_{\gamma_{\lambda,\xi}}(j) = \xi\, D_{\widehat{U}_\lambda}(\iota)$, where D_γ denotes a representation of γ. Since $\iota^2 = 1$, the square on j is the scalar δ^{h_m} of j^2. Thus, the two choices of ξ give exactly the extensions to I^+ of the translated partner on I.

Let $a = lF_r$ normalize $\mathbf{S}$ with $l \in L$, and put $l_f = lF_r(l)\cdots F_r^{f-1}(l)$. Thus, $a^f = l_f F_q$, with F_q acting on X as 1 for $\kappa = 1$ and $\mathrm{Int}(j^{-1})$ for $\kappa = -1$. If $\kappa = -1$, the scalar c_ξ of Lemma 6.21 has $c_\xi^f = \xi^{-1}$. Thus, both matrices at a^f include the matrix of l_f and its scalar $\eta(z(l_f))$. Moreover, conjugacy in L gives representatives that normalize $\mathbf{S}$. If the product defect is $i \in I^+$, the scalar twist gives $\omega_\xi(i)$ for both factors. By [156, Theorem D], $U_\lambda, \widehat{U}_\lambda$ acquire one quotient character ζ_σ under a coefficient action σ. Even if $\zeta_\sigma(\iota)$ cancels the change of ξ, the correcting functions for $\beta_{\lambda,\xi}, \gamma_{\lambda,\xi}$ agree. Hence, $f_{L^+,I^+} \colon \beta_{\lambda,\xi} \mapsto \gamma_{\lambda,\xi}$ gives the comparison on L^+, I^+.

The e-core $\rho \vdash h$ and numbered quotient of λ come from the local label. On J, it lies over $\gamma_0 = \eta(z)u_\rho$, with $u_\rho = 1$ if $h = 0$. The coordinates of J satisfy $\prod_{k=1}^b N_e(x_k)\det D = z^2$, with $D \in \mathrm{GL}_h^\kappa(q)$ and $N_e(x_k)$ the e-slot determinant. If $h = 0$, the factor D is trivial. Since $p \mid |X|$, $b > 0$. Write $U_A, U_A^\vee$ for the oriented isotropic halves on V_A. Since e is odd, on each slot of U_A no weights form a pair $\lambda, -\lambda$. Moreover, the different slots in U_A give independent weights. Thus, $g^* \in \mathbf{K}^*$ cannot exchange $U_A, U_A^\vee$, since it fixes the weights on $\mathbf{S}^*$. Hence, $C_{\mathbf{K}^*}(s) \le \mathbf{J}^*$ and the signed induction $\alpha = \mathcal{R}_{K,J}(\gamma_0)$ is irreducible.

Then, $M_\alpha = KI^+$ and $K \cap I^+ = J$. Indeed, a stabilizer in M^* of $[s]_{K^*}$ admits a correction in K^* to fix s. By [208, proof of Lemma 7.15(iii)], it fixes or exchanges W_+, W_-. Oriented rotations and permutations give I/J, while the exchange gives jJ on the primal torus. Since e is odd, jJ lies outside I/J. By Lemma 6.7, the primal quotient is KI^+/K. Both actions fix ρ and $\eta(z)$, so they fix γ_0, α. Put $V = N_H(\mathbf{S})$, with $H = \langle \widetilde{L}, j, F_r\rangle$. Then, $V \cap X = I^+$ and $V \cap K = J$. Since H normalizes the oriented Levi L, conjugacy of Sylow tori gives $H = LV$. Thus, $N_{XH}(\mathbf{S}) = MV$. After correction in M, both paths have representatives in V.

For $\kappa = -1$ and $h \ge 2$, the active slots have sign $(-1)^b$. Since $m = eb + h$ is even, V_R has sign $(-1)^h$. By Lemma 6.23, we obtain the quotients C, B in residual rank h. On the half determinant z, oriented operations fix z, whereas j gives $z \mapsto z^{-1}$ and acts on V_R by j_h. On C, j_h acts as F_r^f. By Proposition 6.25, the fibre product $(KV) \times_{V/J} V$ has an extension above $\alpha \otimes \gamma_0^\vee$, also for $j \in X$. By Lemma 6.24, the scalar change gives Y. Thus, Y is fixed by $\mathcal{H}_p$ and trivial on the diagonal of $C_V(X)$. For $\kappa = 1$ and $h \ge 2$, $K \cap P_\pm$ have Levi J and radicals of order $q^{\binom{h}{2}}$. Since both inductions afford α, the sums of Lemma 6.20 on $K \cap U_\pm$ give the extension on $(KV) \times_{V/J} V$. Indeed, V stabilizes $\{K \cap U_+, K \cap U_-\}$ and $\mathcal{H}_p$ fixes $\sqrt{q}$. Given $h \in \{0,1\}$, the groups $K = J$ are tori for both κ, so the extension is $(x,y) \mapsto \gamma_0(xy^{-1})$.

The Clifford multiplicity module $E_{\lambda,\xi}$ belongs to $\gamma_{\lambda,\xi}$ over γ_0. Via $KI^+/K \cong I^+/J$, take $E_{\lambda,\xi}$ with its inverse factors over α. Thus, the jJ-operator is the one in $\gamma_{\lambda,\xi}$. Since $\xi \mapsto -\xi$ twists both modules by the I^+/I sign, Clifford induction from KI^+ to M gives $f_{I^+,M}(\gamma_{\lambda,\xi})$ for both ξ. By [95, Theorem 6.11(b)], we obtain $\mathrm{Irr}(M \mid \alpha)$ as the induction image of $\mathrm{Irr}(KI^+ \mid \alpha)$. Moreover, the orbit over $[s]_{X^*}$ is unique by Lemma 6.7. A Sylow p-subgroup of K lies in the active tori of J. Moreover, I has a Sylow p-subgroup of X since the Levi has all b blocks with their permutation p-part. The determinant cover has 2-power kernel and cokernel. Thus, $[M : KI^+]$ and the degree multiplier are p'-numbers. By Proposition 6.16, induction from KI^+ on the common cover gives matching $\mu_{g,\sigma}$ and c_z.

Thus, the paths meet at $\beta_{\lambda,\xi}$ and $\gamma_{\lambda,\xi}$. If either model changes by c, its two cochains gain c, c^{-1}, respectively. Hence, $f_{\mathrm{U}(m)} = f_{I^+,M} \circ f_{L^+,I^+} \circ f_{X,L^+}$ gives the relation $\succeq_c$, including mixed stabilizer operations with $\xi \mapsto -\xi$. The

graph automorphism exchanges the two $\mathrm{U}(m)$ classes. Since $s^2 = 1$, odd powers fix s. Thus, for the other class, conjugate $f_{\mathrm{U}(m)}$ by the graph automorphism. Moreover, for $m = 4$, the Levi action on X stabilizes s by field and diagonal actions on X. □

Theorem 6.37. *Let $X = \mathrm{Spin}^{\epsilon}_{2n}(q)$, with q odd, $n \geq 5$ and $\epsilon = \pm 1$. Let p be an odd prime with $p \mid |X|$ and $p \nmid q$. Thus, we put $d = \mathrm{ord}_p(q)$, take the marked Sylow d-torus $\mathbf{S}$, and set $M = N_{\mathbf{X}}(\mathbf{S})^F$. On the natural overgroups of X, M, there is an equivariant bijection f_- between the characters in $\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M)$ with central sign -1 at z_0. Moreover, by* Definition 2.1*, each pair $(\chi, f_-(\chi))$ satisfies $\succeq_c$ on its stabilizer.*

Proof. Let χ be in the domain of f_-, with rational label s and natural lift t. By Lemma 6.27, $F^*(t) = -t$. Thus, the sign spaces V_+, V_- have equal dimension. If $V_+ = V_- = 0$, Corollary 6.26 gives the comparison on M. Given $\mathrm{U}(m)$ with even m, the labels are $\gamma_{\lambda,\xi}$ of the construction on I^+. Suppose that $V_\pm \neq 0$. If $t^4 = 1$, put $2a = \dim V_+$ and $c = \dim V_i$. By [208, Lemma 7.15], the shapes are $\mathrm{I}(a, a)$ for $c = 0$ and $\mathrm{Q}(a, c)$ for $c > 0$. In both shapes $\mathrm{Z}(C_{\mathbf{X}^*}(s))^\circ = 1$. If $2 \mid e$, Proposition 6.32 gives $\succeq_c$, with field actions in multiplicity dimension 2. If $2 \nmid e$, Proposition 6.29 places the active blocks in $V_i \oplus V_{-i}$, of half-rank c. Since $p \mid |X|$, the shape with $c = 0$ has no p'-degree character. If $c > 0$, then $2 \leq 2a + c_0 \leq e$, so odd e is at least 3. By Proposition 6.29, the sign of V_A is the given one. Thus, at $N_X(Q_0)$, Corollary 6.52 gives $\succeq_c$. Given a cuspidal label of rank $a = h^2$ with $2 \mid h$, the exponent in Proposition 6.47 is $h/2$ mod 2 at λ and its core ρ. Conversely, the rank and scalar of ψ give $[s]$ by Proposition 6.29.

Assume now that $t^4 \neq 1$. By [208, Proposition 7.3, Step 1 and the proof of Lemma 7.15], the torus $\mathbf{Z}_s = \mathrm{Z}(C_{\mathbf{X}^*}(s))^\circ$ gives the proper Levi $\mathbf{L}^* = C_{\mathbf{X}^*}(\mathbf{Z}_s)$. Then, we have that Y is the zero-weight space of $\mathbf{Z}_s$, and $t_Y = t|_Y$ has $F^*(t_Y) = -t_Y$. If s is central on $\mathbf{L}^*$ or $\dim Y \leq 6$, Theorem 6.199 gives the comparison. Otherwise, put $\mathbf{R} = \mathrm{Spin}(Y)$ and $R = \mathbf{R}^F$. Let f_R be the residual comparison on R, and take H from Lemma 6.14. Since s is eligible, fix $s \in C_{\mathbf{X}^*}(P^*)$ for the marked Sylow subgroup P^*. Then, $\mathbf{Z}_s$ centralizes the characteristic base Q^* of P^*, so $\mathbf{Z}_s \leq C_{\mathbf{X}^*}(\mathbf{S}^*)$. Thus, on the common torus $\mathbf{T}^*$, duality gives $\mathbf{S} \leq \mathbf{L}$ by Section 6.1. By Lemma 6.206, $R \times H \to L$ has kernel and cokernel of order 2. Thus, the degree condition passes to R, H at odd p.

Suppose that $j \in \mathrm{SO}(V)$ acts by $t \mapsto -t$. Given any nonquartic reciprocal pair $\{x, x^{-1}\}$, the map j gives its negative $\{-x, -x^{-1}\}$. Thus, for nonzero weights λ of $\mathbf{Z}_s$, Proposition 6.190 gives balanced blocks U_λ. Moreover, on Y, the nonsign spectrum lies in $\{i, -i\}$, so t_Y has exchanged shape I or Q. For the determinant cover H, Proposition 6.194 gives $\succeq_c$. Since $C_{\mathbf{A}_0}(t|_{Y^\perp})$ is connected, Lang gives an F^*-fixed exchange by Lemma 6.10. Thus, $\{[s]\}$ is the fibre in $\mathbf{L}^*$. If t and $-t$ are not conjugate in $\mathrm{SO}(V)$, each pair has independent scalars. Hence, $Y = V_+ \oplus V_-$ and t_Y has shape $\mathrm{I}(a, a)$. On H, take the scalar comparison with α_c. Since F^* exchanges its two scalar lifts, $\alpha_c(z_0) = -1$ at both ends. By Corollary 6.12, if V_+ is split over $\mathbb{F}_{q^2}$, the fibre is $\{[s]\}$. Otherwise, take both $[s], [s^\sharp]$, distinct in X^*.

For the projected Sylow torus $\mathbf{S}_R$, we put $I_R = N_{\mathbf{R}}(\mathbf{S}_R)^F$ and $J_R = C_{\mathbf{R}}(\mathbf{S}_R)^F$. If $p \nmid |R|$, then $I_R = J_R = R$ and f_R is the identity. If $p \mid |R|$ and $\dim Y \geq 10$, choose the above maps on R, I_R for I, Q. By Proposition 6.29, we obtain the J_R-orbit. If $\dim Y = 8$, the shape I follows from Lemma 6.33 and Q follows from Lemma 6.34 and Proposition 6.35. Given Q, choose α_u on J_R for $u = 1, \mathrm{St}$. After restoration of $\mathbf{L}$, the action on Y is by semilinear similitudes h with $B(hv, hw) = c_h B(v, w)^{\varphi_h}$ for the form B. Thus, they fix the vector kernel $\langle z_0 \rangle$, and f_R is equivariant even for split D_4.

With $L = \mathbf{L}^F$, we put $K = C_{\mathbf{X}}(\mathbf{S})^F$ and $J = K \cap L$ and $I = M \cap L$. In both paths, z_0 acts by -1 on both projected representations. Thus, (z_0, z_0) acts trivially on the tensor $V_R \otimes V_H$. By Lemma 6.14, J, I, L have a common quotient of order 2. By Proposition 6.15, we obtain $f_{L,I} : \beta \mapsto \nu$ on the projected fibre for $\beta \in \mathrm{Irr}_{p'}(L)$ and $\nu \in \mathrm{Irr}_{p'}(I)$. If β has a self-twist, on $\bar{h}$ the common square P_h^2 gives both extensions and their $\mu_{g,\sigma}$. Moreover, $f_{L,I}$ is equivariant on the stabilizer A_π of the projected class $[\pi(s)]$, including block permutations in A_π.

Given a singleton fibre $\{[s_L]\}$, Lemma 6.7 gives one eligible M-orbit of K-series over $[s]$. Thus, a correction in M restores the chosen K-series. Since (g, σ) returns the K-series, take its correction (g', σ) from Section 6.1. Thus, (g', σ) restores $\mathbf{L}$ and both induction marks. Moreover, I aligns the J-constituent by a correction fixing β, ν. If the fibre is $\{[s_L], [s_L^\sharp]\}$, we set $W = (M \cap N_X(\mathbf{L}))/I$. By Proposition 6.13, W acts freely on $[s_L], [s_L^\sharp]$ and their W-orbits. Since $f_{L,I}$ is equivariant, the W-coset returning β also returns ν. By Lemma 6.11, $(s^\sharp)^{g,\sigma} = (s^{g,\sigma})^\sharp$. Thus, one W-coset serves both inductions even for $e_{[s_L]} \neq e_{[s_L^\sharp]}$.

By [208, Proposition 7.3, Steps 1–2], the lift of $\mathbf{Z}_s$ fixes $V_\pm$ pointwise. Thus, the cocharacter λ_s gives a radical $\mathbf{U}$ with Levi $\mathbf{L}$, fixed by stabilizers of field degree 0. On $\mathbf{K} = C_{\mathbf{X}}(\mathbf{S})$, the same λ_s gives the stable radical $\mathbf{U} \cap \mathbf{K}$ with Levi $\mathbf{J} = \mathbf{K} \cap \mathbf{L}$. Moreover, for the field lift φ, take the relations for φ. Given a mixed action (g, σ), take the change of

U from Section 6.1. Hence, on $\{[s]\}$, by Proposition 6.9, we obtain $\succeq_c$ for the composition through $f_{L,I}$. On $\{[s],[s^\sharp]\}$, both classes have scalar -1 at z_0, so the composition gives $\succeq_c$ by Proposition 6.13. Finally, every $\psi \in \mathrm{Irr}_{p'}(M)$ restricts to an eligible M-orbit of rational K-series. Since the K-constituents are M-conjugate, each torus pair $(\mathbf{T}^*, s)$ gives one ambient class $[s]_{X^*}$ for the orbit. Moreover, the condition $\psi(z_0) = -\psi(1)$ is the sign $F^*(t) = -t$. The inverse on each fibre of $[s]$ gives the preimage of ψ. Thus, the stated target is $\mathrm{im}\, f_-$. □

Let $q = r^f$ be odd, with primes r, p and $p \neq r, 2$. Thus, in Zhang [208, Lemma 7.15], the shape $Q(a,c)$ has sign factors of rank a and a factor of type A_{c-1}. For the exchanged $Q(a,c)$, take the labels of $\mathrm{SO}^\eta_{2a}(q^2)$, with $n = 2a+c \geq 5$. Moreover, the series is faithful on $\langle z_0\rangle$. Put $N = \mathrm{Spin}^\epsilon_{2n}(q)$, and for $P \in \mathrm{Syl}_p(N)$ let Q_0 be its characteristic abelian base. Thus, $M = N_N(Q_0)$ contains $N_N(P)$, including when P has a permutation part.

Put $e = \mathrm{ord}_p(q^2)$, with $e \geq 3$ odd, and $d = \mathrm{ord}_p(q)$. Thus, we put $t = 1$ for $d = e$ and $t = -1$ for $d = 2e$. In the eligible domain, the A_{c-1}-factor has sign $t = -\eta_q(-1)$, for quadratic $\eta_q \in \mathrm{Irr}(\mathbb{F}_q^\times)$. Moreover, we choose a primitive r-th root ζ_r and write $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$ for $\sigma \in \mathcal{H}_p$. Put $\varepsilon_\sigma = (b_\sigma/r) = (-1)^{h_\sigma}$. Then, $\sigma(i)/i = \varepsilon_\sigma^f$. Indeed, we have that $q^e \equiv t \pmod p$ and odd e give $(q/p) = t^{(p-1)/2}$. By quadratic reciprocity, $(p/r)^f = (t\eta_q(-1))^{(p-1)/2} = (-1)^{(p-1)/2}$. Thus, we find that the p-power map gives $\sigma(i)/i = \varepsilon_\sigma^f$ by iteration for each σ.

Lemma 6.38. *Take prime-field models of determinant 1 for N and the blocks V_A, V_R. Suppose that the Pin lifts $w, v = v_A v_R$ both have parity $\tau = h + \tau_R \bmod 2$, where h is the active rank. Let ω be the prime-field volume with $\omega^2 = z_0^n$. Then, the common cover A of both parents has $b^{4f} = v^{-4}$, $D = b^f v$ and $D^4 = 1$. Moreover, the even representative j of the remaining diagonal normalizes Q_0 and satisfies $jDj^{-1} = \omega^{-1}D$ and $jD^2j^{-1} = z_0^{c+\tau}D^2$.*

Proof. In the algebraic Pin group, put $F = \mathrm{Int}(v)F_q$, $Z = \langle z_0, \omega\rangle$ and $\partial_F(x) = F(x)x^{-1}$. Define $K = \{x : \partial_F(x) \in Z\}$ and $K_0 = \{x \in K : \partial_F(x) \in \langle z_0\rangle\}$. Thus, $F(\omega) = z_0^\tau\omega$, and every $x \in K$ satisfies $F^4(x) = x$. Hence, K is finite. On K_0, the parity $a(x)$ and exponent $d(x)$ in $\partial_F(x) = z_0^{d(x)}$ define $\pi_{\mathrm{bd}} = (a,d) : K_0 \to \mathbb{F}_2^2$ with kernel N. By Lang's theorem on **Spin**, we obtain the even boundaries $1, z_0$. A rational reflection of norm 1 gives odd parity, so $K_0/N \cong C_2^2$. The remaining coset has boundary in $\omega\langle z_0\rangle$. Let $A = \langle K, b\rangle$, with b acting by F_r and $b^{4f} = v^{-4}$. Then, D acts by F and $D^4 = 1$ in A, whereas the base parent matrices have $R(D)^2 = I$.

Since $wv^{-1} \in \mathbf{Spin}$, Lang's theorem gives $x \in \mathbf{Spin}$ with $F_q(x) = w^{-1}xv$ for the twisted Frobenius F. Thus, $c_x : y \mapsto xyx^{-1}$ is a rational isomorphism with $c_x(\omega) = \omega$. The prime-field lifts differ by $xF_r(x)^{-1}$. Since $F_r(v) = v$ and $F_r(w) = w$, $xF_r(x)^{-1}$ is rational and even. Moreover, c_x matches D and the relation for b^{4f}. Thus, any twist ν trivial on the finite Pin group K takes equal values on the two F_r-lifts.

On the blocks V_A, V_R, write a_A, a_R for parities and d_A, d_R for boundary exponents. For homogeneous x, y on distinct blocks, by [137, Theorem 24.56], $xy = (-1)^{a(x)a(y)}yx$. Hence, membership in M has the conditions $a_A + a_R = 0$ and $d_A + d_R + \tau_R a_A + h a_R = 0$ in $\mathbb{F}_2$. The boundary signs from v_A, v_R on opposite blocks give $\tau_R a_A + h a_R$. Since an element of $N_K(Q_0)$ stabilizes V_R and permutes the Q_0-blocks, the conditions characterize M. Thus, $B = N_K(Q_0)\langle b\rangle$ has $N \cap B = M$.

Given any $g \in A$, Sylow conjugacy in N restores P^g to P. Hence, it also restores Q_0^g to Q_0 by characteristicity. Thus, $A = NB$. Moreover, $C_A(N) = \langle z_0, \omega, D\rangle$. Indeed, $x \in C_A(N)$ has field exponent fu for some u. Then, xD^{-u} acts linearly on the natural space V and centralizes N. Thus, its image on V is scalar. Moreover, ω lifts the scalar $-I$. Thus, $C_A(N)$ fixes Q_0 pointwise. Since the active torus $\mathbf{T}_A$ and residual group $\mathbf{Spin}(V_R)$ centralize Q_0, Lang's theorem there gives even j with $\partial_F(j) = \omega$. Since $F(j) = \omega j$, $jDj^{-1} = \omega^{-1}D$. Moreover, $F(\omega) = z_0^\tau\omega$ and $\omega^{-2} = z_0^n = z_0^c$ give $jD^2j^{-1} = z_0^{c+\tau}D^2$. □

Proposition 6.39. *On the cover A of* Lemma 6.38*, take parents R_G, R_M on $K_0\langle b\rangle$ and $B_0 = (K_0 \cap N_K(Q_0))\langle b\rangle$. Suppose that $\mathcal{H}_p$ fixes both finite restrictions. On b, both coefficient ratios are ε_σ^E, and the finite parts have ratio 1. For either $R = R_G, R_M$, the independent twists have operators $S = R(D)$ and $U = i^n R(\omega)$. If j exchanges the orientations of R_G and of R_M, the pairs (χ, ψ) in both orientations satisfy $\succeq_c$.*

Proof. Given R_G, R_M, write V_G, V_M for their respective spaces. The quotients K_0/N and $(K_0 \cap N_K(Q_0))/M$ have order 4. Since the generators fix R, [95, Lemma 5.2 and Problem 5.3] gives dimension 4 for each commutant $\mathcal{A}_G = \mathrm{End}_N(V_G)$ and $\mathcal{A}_M = \mathrm{End}_M(V_M)$. In both commutants, the operators $1, S, U, SU$ span distinct quotient lines. Thus, in A $S^2 = U^2 = I$ and $SU = (-1)^\tau US$. If $\tau = 0$, the restrictions have 4 distinct constituents of multiplicity 1. If $\tau = 1$, the commutant is $\mathrm{M}_2(\mathbb{C})$, so one constituent occurs twice.

If $\tau = 0$, pair the eigenspaces $V_{1,\kappa}$ with D-eigenvalue 1 and ω-eigenvalue κ, where $\kappa^2 = (-1)^n$. Then, (a, d) gives the other pairs. Given $\sigma(i)$, the ratio formula gives signs on D, ω are ε_σ^{Ef} and ε_σ^{cf}, since $n \equiv c \bmod 2$. Conjugation by g gives signs $(-1)^{d(g)}, (-1)^{a(g)}$. Thus, a mixed operation stabilizing $V_{1,\kappa}$ has $a(g) = cfh_\sigma$ and $d(g) = Efh_\sigma \bmod 2$ on each side. The conditions for M give both quotient coordinates in B_0. Ordinary inertia and mixed intertwiners $T_{g,\sigma}$ act on $V_{1,\kappa}$. Thus, its factor is 1, and the mixed function at $y = kb^u$ is ε_σ^{Eu} on both sides. Moreover, $C_A(N)$ acts by the matched scalars.

If $\tau = 1$, let $V_{\mathrm{mult}} = \mathbb{C}^2$ be the common multiplicity space. If $y = kb^u$, take (a, d, u) modulo N or M, respectively. Thus, we put $Z_1 = \mathrm{diag}(1, -1)$, $X_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\zeta = 1$ for even n or $\zeta = -i$ for odd n. Choose bases with $R(D) = 1 \otimes Z_1$ and $R(\omega) = 1 \otimes \zeta X_1$. Thus, $L(a, d, u) = Z_1^a(\zeta X_1)^d$ has factor $\alpha(x, y) = (-1)^{d_x a_y + n d_x d_y}$. Thus, $R_G = P_G \otimes L$ and $R_M = P_M \otimes L$ give the factor α^{-1} for P_G, P_M. On V_{mult}, the intertwiner $C_\sigma = Z_1^{nfh_\sigma} X_1^{Efh_\sigma}$ compensates $\sigma(\zeta)/\zeta$ and acts by $Z_1 \mapsto \varepsilon_\sigma^{Ef} Z_1$. Hence, the remaining coefficient function is $\varepsilon_\sigma^{E(u+fa)}$. Since conjugation by g contributes $\alpha(g, y)/\alpha(y, g)$, both groups have $\mu_{g,\sigma}(y) = \varepsilon_\sigma^{E(u_y + f a_y)} (-1)^{d_g a_y + d_y a_g}$. The exponent $n d_g d_y$ cancels. Moreover, D, ω have the indicated matrices and z_0 acts by -1. Thus, the scalars on $C_A(N)$ match. Since σ fixes each finite parent and j exchanges them, the pair (j, σ) exchanges them. Thus, the conjugates R_G^j, R_M^j give the second orientation, with jDj^{-1} and jD^2j^{-1} given by Lemma 6.38. □

Lemma 6.40. *Given $Q(1, c)$, the rational orthogonal torus has sign $\eta = (-1)^{c+\tau}$, with $\tau \in \mathbb{F}_2$. Thus, we compare the square w^2 on the source with v_R^2 on the residue. Let $\delta_m = (m, m-1, \dots, 1)$, with $m \geq 2$ and $c = m(m+1)/2$. By* Zhang [208, Proposition 7.35, Step 2]*, let w be the selected lift. Thus, write $w^2 = z_0^{Q_w}$. If $\eta = 1$, then $Q_w = \binom{c}{2} \bmod 2$. If $\eta = -1$, the orientations of w and ωw give opposite values of Q_w.*

Proof. Put $k = \lceil m/2 \rceil$. By [208, (7.34) and Proposition 7.35, Step 2], the 2 singleton blocks accompany $k-1$ doubled planes. Hence, diagonalization gives $2k$ odd blocks of sizes $2m - 1 \bmod 4$. Their Clifford volumes e_i satisfy $e_i^2 = z_0^{m-1}$ and $e_i e_j = z_0 e_j e_i$ for $i \neq j$. For the determinant vector Δ, we put $d = |\Delta|$ and $w = e_\Delta$, the ordered volume product. Thus, $\tau = d \bmod 2$ and $Q_w = \binom{d}{2} + (m-1)d \bmod 2$. Since $(\omega w)^2 = z_0^{c+\tau} w^2$, the orientations for $\eta = -1$ have opposite bits. If $\eta = 1$, $\tau = c = k \bmod 2$. Suppose that a doubled plane E has square determinant. Since E represents both classes, its bases admit $(0, 0)$ and $(1, 1)$. Thus, d varies by 2 on the fixed source. By [208, Proposition 7.35, Step 2], B_Δ covers $H_\Delta = \{u \in \mathbb{F}_2^{2k} : |u| \equiv 0 \bmod 2,\ \Delta \cdot u = 0\}$, with $e_u^2 = z_0^{|u|/2}$.

If $2 \mid k$, $\tau = 0$, and [208, Proposition 7.35, Step 2] excludes constant Δ. Thus, the commutator radical on H_Δ is $\langle \mathbf{1}, \Delta \rangle$. For the scalar ξ_Δ on e_Δ, $\xi_\Delta^2 = (-1)^{d/2}$. Let σ fix ζ_r and act by $i \mapsto -i$. Thus, σ fixes the additive character θ and, since $2 \mid c$, the central character ω_χ. The faithful character ϑ_Δ is zero off $\mathrm{Z}(B_\Delta)$. Hence, $\vartheta_\Delta^\sigma = \vartheta_\Delta$ exactly when $\xi_\Delta \in \mathbb{R}$. Thus, the two bases give $\chi^\sigma = \chi$ and $\chi^\sigma \neq \chi$ by [208, Proposition 7.35, Step 2].

If $2 \nmid k$, d is odd. Thus, H_Δ has nondegenerate commutator form and B_Δ is extraspecial. Its unique faithful character ϑ_Δ has degree 2^{k-1} and rational values. Let C_{src} be the reductive source, with character ψ_{src}. If $\eta_q(b) = -1$, we multiply the grading lift by ω for the additive change by b. Since $F(\omega) = z_0 \omega$, this lift is rational and fixes ϑ_Δ. Thus, the bijection of [208, Proposition 7.35, Step 2] gives $\chi^\sigma = \chi$ under coefficients. By [208, Proposition 7.35], ψ_{src} has values in $\mathbb{Q}(\zeta_r)$. Since $r \nmid |C_{\mathrm{src}}|$, it has values in $\mathbb{Q}(\zeta_r) \cap \mathbb{Q}(\zeta_{|C_{\mathrm{src}}|}) = \mathbb{Q}$. Hence, its invariant form B_{src} is symmetric or alternating. Since $\psi_{\mathrm{src}}|_{B_\Delta}$ is irreducible, its indicator is $\nu(\psi_{\mathrm{src}})$ in both bases, given by B_{src}. The indicator on B_Δ is $2^{-(k-1)} \sum_{u \in H_\Delta} (-1)^{|u|/2} = (-1)^{(k-d)/2}$. Indeed, the even-weight sum $\mathrm{Re}(1+i)^{2k} = 0$ since k is odd. The corresponding Δ-weighted sum is $\mathrm{Re}((1+i)^{2k-d}(1-i)^d) = 2^k (-1)^{(k-d)/2}$. Their average gives the sum on H_Δ. Thus, the indicator of ϑ_Δ changes sign under $d \mapsto d+2$, contrary to the fixed symmetry of B_{src}. Consequently, each of the $k-1$ planes has bits $0, 1$. If $m = 2$, there are no doubled planes. Since $d \equiv k \bmod 2$, exactly one singleton is nonsquare and $d = k$. Substitution of $m = 2k-1$ and $m = 2k$ gives $Q_w = \binom{k}{2}$ and $Q_w = \binom{k+1}{2}$, respectively, modulo 2. Since the corresponding values of c are $k(2k-1)$ and $k(2k+1)$, both give $Q_w = \binom{c}{2} \bmod 2$. □

Lemma 6.41. *For the staircase source of exchanged $Q(1, c)$, the faithful Pin parent R has ratio $\varepsilon_\sigma^{Q_w+1}$ on b and ratio 1 on the finite part. Moreover, sizes $1, 0$ give the same ratios. In each, $Q_w = 0$ at $\eta = 1$, and $Q_w \in \{0, 1\}$ at $\eta = -1$.*

Proof. Take u_0 perpendicular to the natural space V, with $Q(u_0) = 1$. Thus, the odd-spin lift $w_B = w u_0^\tau$ has $w_B^2 = z_0^{Q_w + \tau}$, and the field generator has f-th power w_B^{-1}. Given $m \geq 3$, [208, Proposition 7.35, Step 2] gives B_0 and its irreducible module $V_{\mathrm{src}}|_{B_0}$. Moreover, the matrix pairs Z_1, X_1 of Proposition 6.39 give B_0-matrices over $\mathbb{Q}(i)$ with the given central scalars. Since $\dim E_j \leq 2$ on the multiplicity spaces, the stabilizer C has $r \nmid |C|$.

If σ fixes ζ_r, i, it fixes the additive source and $V_{\mathrm{src}}|_{B_0}$. Thus, by [208, Proposition 7.35, Step 2], uniqueness gives the field $\mathbb{Q}(\zeta_r, i)$ for the values on V_{src}. Since the restriction to C also has values in $\mathbb{Q}(\zeta_{|C|})$, its values lie in $\mathbb{Q}(i)$. If σ fixes $\mathbb{Q}(i)$, both C-models have equal characters. An intertwiner T_σ centralizes the B_0-matrices, so it is scalar. Thus, σ fixes all matrices on C. The pinned F_r fixes the additive character and B_0-matrices. Hence, the source is F_r-stable by [208, Proposition 7.35, Step 2]. Since the B_0-source is irreducible, its intertwiner on C is scalar. Thus, take $B_{\mathrm{src}} = I$ before field normalization. If $m = 2$, we choose the source line ℓ_2 from [208, Proposition 7.35, Step 2] on the finite centre.

Let κ be the scalar of w_B on V_{src}. If $\tau = 1$, the graph extension at w_B fixes ψ_{src} because the f-th field power acts by w_B^{-1}. Put $f_2 = 2^{v_2(f)}$ and, normalize the field map by $\alpha \in \mu_{4f_2}$ with $\alpha^f = \kappa^{-1}$. Thus, $\kappa^2 = (-1)^{Q_w+\tau}$. If $2 \mid f$, then $t = -1$, $q \equiv 1 \bmod 4$ and $v_2(\mathrm{ord}_p(r)) = 1 + v_2(f)$. In particular, $p \equiv 1 \bmod 2f_2$. Given a primitive $\zeta \in \mu_{4f_2}$, the ratio is $\zeta^p/\zeta = (-1)^{(p-1)/(2f_2)} = (r/p) = (p/r)$, by the 2-part $2f_2$ of $\mathrm{ord}_p(r)$. Thus, on α the sign has exponent $Q_w + \tau \bmod 2$. If f is odd, we choose $\alpha \in \mu_4$. The ratio on i gives exponent $Q_w + \tau$ including the scalar on w_B.

For odd f with $\tau = 0$, a coefficient action with $\zeta_r \mapsto \zeta_r,\ i \mapsto -i$ is corrected by e_u with $|u| \equiv c$ and $\Delta \cdot u = Q_w \bmod 2$. The equations are independent if Δ is nonconstant. If $\Delta \in \{0, \mathbf{1}\}$, the formula for Q_w in Lemma 6.40 gives consistency. If $\tau = 1$, the reductive source is rational, and $\omega^{Q_w+\tau}$ corrects the graph scalar. Both e_u and $\omega^{Q_w+\tau}$ are F_r-fixed. For the block sizes d_i of [208, (7.34)], the half-grading gives $\sum_i (d_i^2 - 1)/8 \equiv 1 \bmod 2$. Thus, the additive ratio is ε_σ. The finite corrections in induction are $e_u, \omega^{Q_w+\tau}$. Hence, the odd-spin ratio is $\varepsilon_\sigma^{Q_w+\tau+1}$. The Pin model multiplies by U on odd matrices. Thus, it adds τ to the exponent, giving $\varepsilon_\sigma^{Q_w+1}$.

For residual size 1, the Jordan blocks have sizes $5, 1$ and volumes with square 1. Moreover, the Whittaker line ℓ_1 has multiplicity 1 in the overgroup $\widetilde{N}$ with connected centre. Diagonal translates of ℓ_1 distinguish the spin constituents χ, with source lines indexed by ω_χ. If $\eta = 1$, $|\Delta| = 1$. If $\eta = -1$, $|\Delta| \in \{0, 2\}$. This gives the stated values of Q_w. The half-grading exponent is $(5^2 - 1)/8 \equiv 1 \bmod 2$.

If the residual size is 0 and $\eta = 1$, the spin group is $\mathrm{SL}_2(q)^2$. Take W_+, W_- of Lemma 6.57. Thus, on $W_+ \boxtimes W_-$, $(-I, -I)$ acts by -1. In trace coordinates, ℓ_{src} is spanned by $(\delta_1 + \delta_{-1}) \otimes (\delta_1 - \delta_{-1})$. Thus, on U_1, the weights for distinct classes in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ are disjoint away from 0. A nonsquare diagonal h changes both classes. Moreover, the graph γ swaps the factors of $\mathrm{SL}_2(q)^2$. Thus, changing one class in $W_+ \boxtimes W_-$ gives the other parent. For sizes $3, 1$, both determinant vectors $00, 11$ have $Q_w = 0$.

If the residual size is 0 and $\eta = -1$, take the odd Weil space of $\mathrm{SL}_2(q^2)$ in either trace class. Choosing block 3 or 1 gives $Q_w \in \{0, 1\}$. On the point masses, $P_r \delta_x = \delta_{x^r}$. Since $a^2 = b_\sigma$, dilation by a implements σ. On the odd space W_-, the dilations by $a, -a$ differ by -1. Hence, the commutator with P_r is ε_σ. In μ_{4f_2}, choose α with $\alpha^f = i^{Q_w+1}$. Then, $(\alpha P_r)^{2f} = (-1)^{Q_w+1}$, the scalar on w_B^{-2}. Since $\tau = 1$, the ratio on αP_r has exponent Q_w before the Pin lift comparison. If $R = e = 3$, the core has size 1 and $\eta = -1$, so the source has blocks $5, 1$. □

Let $s \geq 2$, with staircase $\delta_m = (m, m-1, \dots, 1)$ of size $b = m(m+1)/2$. Put $n_b = 2s^2 + b$. Thus, the empty staircase has shape $\mathrm{I}(s^2, s^2)$, and the exchanged form has $\tau = s + b \bmod 2$ by Zhang [208, (7.24)]. Let d_i be the odd block sizes. Given E_j, diagonalization gives determinant bits $\Delta_i \in \mathbb{F}_2$. Thus, in the model of determinant 1, the prime-field unit volumes e_i give $w = e_\Delta = \prod_{\Delta_i = 1} e_i$ in increasing index order. Since d_i is odd, conjugation by w gives determinant bit Δ_i on block i. Thus, the marked form has Frobenius $F = \mathrm{Int}(w) F_q$. Moreover, $w^2 = z_0^{Q_b}$, where $Q_b = \binom{|\Delta|}{2} + \sum_i \frac{d_i - 1}{2} \Delta_i \bmod 2$.

With the chosen χ, let (U, θ) be the source pair and e_θ its isotypic projector. Thus, on $W = e_\theta V_\chi$, Zhang [208, Proposition 7.26, Steps 3–5] gives an irreducible module for $U \rtimes C$ with multiplicity 1 in V_χ. Moreover, W restricts irreducibly to $B_\Delta = \{\pm e_v : |v| \equiv 0,\ \Delta \cdot v = 0 \bmod 2\}$. In the orbit of χ under the regular overgroup, the N-orbit of (U, θ) and the B_Δ-character determine χ. Step 4 includes $m > 2s$ since F exchanges the orthogonal factors. Since $\dim E_j \leq 2$ for each plane or line, $r \nmid |C|$. By Step 5, the C-character has values in $\mathbb{Q}(\zeta_r, i) \cap \mathbb{Q}(\zeta_{|C|}) \subseteq \mathbb{Q}(i)$. By Lemma 6.41, choose B_Δ-matrices over $\mathbb{Q}(i)$. Hence, if $\sigma(i) = i$, a coefficient intertwiner centralizes B_Δ. By Schur, it is scalar. Thus, ρ_C is over $\mathbb{Q}(i)$. The map F_r fixes e_i and the absolute-trace character θ. The cuspidal labels and δ_m are fixed. Moreover, if $\tau = 0$, Zhang [208, (7.25)] distinguishes the classes by $\omega_\chi(\omega)$, where $F_r(\omega) = \omega$. Thus, F_r fixes the regular orbit of χ and its marked source W, including the character on C. The intertwiner B_{st} centralizes B_Δ, so take $B_{\mathrm{st}} = I$ on W.

Given $b' \in \mathbb{F}_r^\times$, the half-grading lift $d_{b'}$ gives $\theta \mapsto \theta^{b'}$. Since the block has dimension $2u + 1$, the weights $u, u - 1, \dots, -u$ give spinor norm $(b')^{u(u+1)/2}$. The exponents have sum $s \bmod 2$. Indeed, in Zhang [208, (7.27)], leading pairs $(d, d-4)$ give 1 mod 2 each, as do pairs $4j - 1, 4j - 3$. In Zhang [208, (7.28)], the $2s$ leading sizes give $s \bmod 2$,

and doubled sizes give 0 mod 2. Thus, $F_r(d_{b'})d_{b'}^{-1} = z_0^{s(1-(b'/r))/2}$. The grading torus $\mathbf{T}_{\mathrm{gr}}$ centralizes the isometries on all blocks V_i. Given a block lift g, $[\widetilde{\mathbf{T}}_{\mathrm{gr}}, g] \subseteq \langle z_0 \rangle$. Since the spin lift $\widetilde{\mathbf{T}}_{\mathrm{gr}}$ is connected, the commutator is 1. Hence, $d_{b'}$ centralizes C, w.

If $\tau = 1$, then $F(\omega) = z_0\omega$. Thus, a power of ω times $d_{b'}$ lies in N and restores θ. Since B_Δ is extraspecial, its faithful character ϑ_Δ is unique. By Zhang [208, Proposition 7.26, Step 5], χ^σ lies in the regular orbit of χ. Thus, the source gives $\chi^\sigma = \chi$. Hence, on C, the values lie in $\mathbb{Q}(\zeta_r) \cap \mathbb{Q}(\zeta_{|C|}) = \mathbb{Q}$. Take a perpendicular unit vector u_0. On its line stabilizer, the lift $w_B = wu_0^\tau$ has $w_B^2 = z_0^{Q_b+\tau}$ and w_B centralizes B_Δ. If $\tau = 0$, then $w_B \in N$. If $\tau = 1$, its action on N is the inverse of F_r^f, so it fixes χ. Hence, extend χ to $N\langle w_B \rangle$, with quotient of order 2. Thus, the scalar κ on W distinguishes the two choices, with $\kappa^2 = (-1)^{Q_b+\tau}$ in both cases. If $f_2 = 2^{v_2(f)}$, choose $\alpha \in \mu_{4f_2}$ with $\alpha^f = \kappa^{-1}$. Then, on W, the field relation is $a^f = w_B^{-1}$. By Isaacs [95, Lemma 5.2], χ occurs with multiplicity 1 in the induction. Since a stabilizes V_χ, the induced operator extends χ.

For the finite parent $\mathcal{P}$, take coordinates (a, d) from Lemma 6.38. If $\tau = 0$, the odd graph gives a factor -1 at ω. Moreover, by Zhang [208, Proposition 7.26, Step 1], the even element j_0 of boundary z_0 acts freely on the χ-orbit. Thus, $|\chi^{\mathcal{P}}| = 4$ for the orbit $\chi^{\mathcal{P}}$. If $\tau = 1$, the boundary kernel is $N\langle w_B \rangle$. Since ω centralizes N and anticommutes with w_B, the graph extensions are ω-conjugate. Thus, induction from N or $N\langle w_B \rangle$, respectively, gives the irreducible parent Φ. Moreover, its extension has $a^{2f} = w_B^{-2}$. The diagonal j in Lemma 6.38 of boundary ω gives the other parent. Indeed, for $\tau = 0$, the orbit under $\ker a$ has size 2 at fixed ω-scalar. If $\tau = 1$, it fixes the constituent χ in a regular orbit of size 2.

The additive change by d_{b_σ} contributes ε_σ^s by its commutator with F_r. If f is even, the formula for $\sigma(i)$ gives $\sigma(i) = i$. If f is odd, we choose $\alpha \in \mu_4$. For conjugation of i fixing ζ_r at $\tau = 0$, a word e_v with $|v| \equiv b$, $\Delta \cdot v = Q_b \bmod 2$ restores the two scalars on ω, w. If Δ is nonconstant, they are independent. At $\Delta = 0$, $Q_b = 0$, and at $\Delta = \mathbf{1}$, $Q_b = b \bmod 2$. Moreover, the classes of ω, e_Δ generate the radical on $B_\Delta/\langle z_0 \rangle$. Hence, their scalars recover the B_Δ-character and the constituent χ with source W. If $\tau = 1$, the C-character is rational, and $\omega^{Q_b+\tau}$ restores κ. The words $e_v, \omega^{Q_b+\tau}$ are F_r-fixed and lie in $\mathcal{P}$, with $e_v u_0$ for odd e_v. Thus, they act on $\Phi|_{\mathcal{P}}$. On a, Lemma 6.41 gives the remaining scalar $\sigma(\alpha)/\alpha = \varepsilon_\sigma^{Q_b+\tau}$. Hence, the ratio on a is $\varepsilon_\sigma^{s+Q_b+\tau}$.

Put $V_\omega = i^{n_b}\Phi(\omega)$. Thus, for x of Pin parity $a(x)$, the matrix is $R(x) = \Phi(xu_0^{a(x)})V_\omega^{a(x)}$. The equation $V_\omega\Phi(y) = (-1)^{a(y)}\Phi(y)V_\omega$ cancels the sign in $\Phi(xu_0^{a(x)})\Phi(yu_0^{a(y)})$. Moreover, we choose $\beta \in \mu_{4f_2}$ with $\beta^f = i^\tau$. Thus, $\beta^{2f} = (-1)^\tau$ compensates the ratio between w^2 and w_B^2 in the field power. On $D = a^f w$, the Pin matrix is $i^\tau\Phi(a^f w_B)V_\omega^\tau$. Since $\Phi(a^f w_B)^2 = I$ and $\Phi(a^f w_B)V_\omega\Phi(a^f w_B)^{-1} = (-1)^\tau V_\omega$, $R(D)^2 = I$. Moreover, $\sigma(\beta)/\beta = \varepsilon_\sigma^\tau$. If $\sigma(i)/i = (-1)^j$, the operator $V_\omega^{n_b j}$ corrects the finite intertwiner at V_ω. Since it centralizes a, the finite ratio is 1 and the field exponent is $E_b^0 = s + Q_b \bmod 2$.

Let $s \geq 2$ be even and $b = m(m+1)/2$, including $m = 0$. Thus, we put $t_0 = \lceil m/2 \rceil$ and $h_0 = \max(t_0 - s, 0)$. By Zhang [208, (7.27)–(7.28)], the $2s$ single Jordan blocks accompany h_0 multiplicity planes. Thus, the blocks V_u are in increasing order. The vector $x \in \mathbb{F}_2^{2s}$ records their determinant bits. Moreover, in $\widetilde{N}$, a nonsquare diagonal gives $x \mapsto x + \mathbf{1}$ and fixes the determinants on E_j. $w \mapsto \omega w$ changes its square by $z_0^{b+\tau} = z_0^s = 1$. Thus, Q_b is determined by $x \bmod \langle \mathbf{1} \rangle$ and the plane determinants.

If $h_0 > 0$, suppose that a plane E has square determinant. On E, the two represented classes give bits 00 and 11. With $L = |\Delta|$, we put $t = t_0$. Then, the $2t$ odd blocks have sizes $2m - 1 \bmod 4$ and the change of basis gives $L \mapsto L + 2$, with (U, θ) and W fixed. If b is even, then $\tau = 0$, L is even, and Zhang [208, Proposition 7.26, Step 4] gives nonconstant Δ in both bases. On e_Δ, the scalar has square $(-1)^{L/2}$ for fixed ω-scalar. An automorphism with $\zeta_r \mapsto \zeta_r$, $i \mapsto -i$ fixes θ and the central character because $\omega^2 = 1$. Hence, by the source bijection, $\chi^\sigma = \chi$ exactly when $L/2$ is even. This contradicts $L \mapsto L + 2$.

If b is odd, then t, L are odd and $\tau = 1$. Thus, the rational C-character ψ_W has irreducible restriction to both B_Δ. Hence, its indicator on either B_Δ equals $\nu(\psi_W)$, the form symmetry on W. If $H = \{v : |v| \equiv 0,\ \Delta \cdot v = 0 \bmod 2\}$, $e_v^2 = z_0^{|v|/2}$. The indicator calculation in Lemma 6.40 gives $2^{1-t}\sum_{v\in H}(-1)^{|v|/2} = (-1)^{(t-L)/2}$. Indeed, average the real parts of $(1+i)^{2t}$ and $(1+i)^{2t-L}(1-i)^L$, with the first zero for odd t. Thus, $L \mapsto L + 2$ changes the indicator on W. Consequently, each plane E has nonsquare determinant.

To determine x, read the class of ι_A from $\widetilde{\chi}$ over χ. Let $I = \{0, \ldots, 2s-1\}$, $A_0 = \{u \in I : 2 \mid u\}$ and $j_u = u \bmod 2$. Thus, for $A \subset I$ with $|A| = s$, we put $B = I \setminus A$ and $M_A = \mathbf{1}_{A \triangle A_0}$. Thus, index each increasing row from 0. When $m < 2s - 1$, put $f(u) = 2u + \max(m - 2s + 1 + u, 0)$, $U_A = f(A)$ and $U_B = f(B)$. Given $m \geq 2s - 1$, we put

$T = \max(s, t_0)$, $\delta = m - (2T - 1) \in \{0, 1\}$ and $h = T - s$. Thus, the rows are

$$U_A = \{4i + \delta : 0 \le i < h\} \cup \{4h + \delta + 3u : u \in A\},\ U_B = \{4i + 1 + \delta : 0 \le i < h\} \cup \{4h + \delta + 3u : u \in B\}.$$

The multiset $U_A \sqcup U_B$ is independent of A. Thus, $A = A_0$ gives alternating entries. In type D, the result of Juteau et al. [104, §5.2, (5.3)] gives the rows $U_{A,i} - 2i, U_{B,i} - 2i$. For $u = 2i, 2i + 1$, they are the special bipartition π_0 of Zhang [208, (7.26)], with leading zeros allowed. Thus, A_0 gives the constant local system. Changing A reassigns the intervals $\{u\}$. Thus, prefix intervals still have length 2. By Shoji [172, §§3.4, 3.7 and Theorem 3.8], ι_A has character M_A on V_u and trivial character on E_j. Since $2 \mid |M_A|$, it descends to the component group $A_{\widetilde{\mathbf{N}}}(X)$. Moreover, the Springer fibre $\mathcal{B}_X$ is independent of the cover.

Let x_{ref} record Shoji's split representative. Normalize $Y_A(x_{\mathrm{ref}}) = 1$. Thus, Shoji [172, Theorems 4.2 and 4.3(ii)] gives ordinary F in split type, with $\epsilon = 1$, and the preferred Weyl extension for $\epsilon = -1$. On the selected class, $Y_A(x) = (-1)^{M_A \cdot (x - x_{\mathrm{ref}})}$. For the weighted Gelfand–Graev character Γ_A, F fixes the component group $A_{\widetilde{\mathbf{N}}}(X)$ by Zhang [208, Lemma 7.24]. Since Zhang [208, Proposition 7.26, Steps 3–4] gives one class $[x]$ of nonzero multiplicity, the ratio $\mathcal{R}_A = \langle \Gamma_A, \widetilde{\chi} \rangle / \langle \Gamma_{A_0}, \widetilde{\chi} \rangle$ is $Y_A(x)$.

For θ on U, the opposite nilpotent is X, with Springer map $v \mapsto 2(v - 1)(v + 1)^{-1}$. In the Fourier formula, $-X$ gives the same forms since $2 \mid d_i - 1$. On the derived Lie algebra, j_{op} exchanges each hyperbolic pair $(v_i, v_i^\vee)$ over $\overline{\mathbb{F}}_q$. Thus, j_{op} centralizes split F_q and the last swap in F. It inverts the diagonal torus $\mathbf{T}$ and acts by $\mathfrak{g}_\alpha \mapsto \mathfrak{g}_{-\alpha}$ as in Taylor [188, §5]. The isometry j_{op} leaves the forms B_i isometric. If its determinant is -1, the reflection on V_u centralizes (U, θ) and corrects the determinant. Hence, x also labels the opposite nilpotent in $\widetilde{N}$. A scalar a on the trace pairing contributes $a^{d_i - 1}$, a square since $2 \mid d_i - 1$.

By Digne et al. [59, Proposition 3.11 and Corollary 3.14(i)], for a principal ι_A, write $\langle \Gamma_A, \widetilde{\chi} \rangle = c_A \langle E_A \otimes \mathrm{sgn}, E_{\widetilde{\chi}} \rangle_{\mathcal{W}_s}$. $\mathcal{W}_s$ is the dual centralizer coset, E_A is the Springer character on $\mathcal{W}_s$, and $c_A = |A_{\widetilde{\mathbf{N}}}(X)|$. Since odd r is very good for D, it is acceptable by Taylor [188, Lemma 6.3(i)]. Thus, for $\widetilde{N}$ with connected centre, Taylor [188, Corollary 13.6] applies at every q. The expansion of Γ_A in Taylor [188, Theorem 11.13 and Lemma 11.16] has ordinary Deligne–Lusztig summands $R_{\mathbf{T}}^{\mathbf{G}}$. Projection to the series of $\widetilde{\chi}$ restricts Weyl functions to $\mathcal{W}_s$ by Digne et al. [59, (3.12)]. Since the support of ι' is larger, the result of Digne et al. [59, Proposition 3.13] gives scalar product 0, and the diagonal coefficient is 1. Since $c_A = c_{A_0}$, these factors cancel in $\mathcal{R}_A$.

Put $z = 1$ in the linear form and $z = -1$ in the unitary form. Thus, F_s exchanges the D_{s^2}-blocks and acts by z on the b-letter block. Since $2 \mid s$, the square F_s^2 has split type on D_{s^2}. In the coset $\mathcal{W}_s$, take graph parts γ_1, γ_2 modulo inner Weyl factors. Since $\gamma_1 \gamma_2 = 1$, we deduce that $\gamma_1 = \gamma_2$. Then, conjugation on one block by γ_1 cancels both graph parts. Since E_A extends to $W(B_{n_b})$, conjugation fixes the values of E_A.

Let E be the character on V_E and let T_{fl} be the tensor flip on $V_E \otimes V_E$. Then, the trace at $(g_1, g_2) T_{\mathrm{fl}}$ is $E(g_1 g_2)$. The average over $W(D_{s^2})$ has weight $1 + \epsilon_{\mathrm{base}}$ on $W(B_{s^2})$, with $\epsilon_{\mathrm{base}}(g) = \prod_i \epsilon_i(g)$. Thus, E has both orders $(A', B'), (B', A')$. A cycle of length a and sign ξ in $g_1 g_2$ gives length $2a$ and sign ξ. On the b-letter block, a cycle of length a acquires sign z^a. Thus, for rows A', B', we put $\alpha_i = A'_i - i$, $\gamma_i = B'_i - i$. For the ambient rows put $\phi_i = U_{A,i} - 2i$, $\psi_i = U_{B,i} - 2i$. Thus, the signed characteristic map gives $\langle E_A \otimes \mathrm{sgn}, E_{\widetilde{\chi}} \rangle_{\mathcal{W}_s}$ by the coefficient of $s_\phi(X) s_\psi(Y)$ in $s_\alpha(X^2) s_\gamma(Y^2) s_{\delta_m}(X, zY)$, for each ordered extension. $X^2 = (x_i^2)_i$, and similarly for Y^2. Since $\delta_m = \delta_m^{\mathrm{t}}$, both scalar factors are independent of A, so they cancel in $\mathcal{R}_A$.

Denote the coefficient by $\mathfrak{c}_{A;A',B'}(z)$. Take $T = \max(s, t_0)$ variables in each alphabet and $h = T - s$. Next, pad A' to $A'^h = \{0, \ldots, h - 1\} \cup (h + A')$, and set $B'^h = \{0, \ldots, h - 1\} \cup (h + B')$. The alternant extracts powers $U_{A,i} - i, U_{B,i} - i$ from $\frac{\det(x_i^{2A'^h_j}) \det(y_i^{2B'^h_j}) s_{\delta_m}(X, zY)}{\prod_{i<j}(x_i + x_j) \prod_{i<j}(y_i + y_j)}$. If $m < 2s - 1$, the least entry has target 0. If its variable is 0, its row must contain bead 0. Delete x_i or y_i and subtract 1 from each remaining bead u. Thus, each determinant power falls by 2, with 1 restored by the row denominator. If $f_N(u) = 2u + \max(m - N + 1 + u, 0)$, the new targets follow from $f_N(u + 1) - 2 = f_{N-1}(u)$. Each deletion has sign 1. Thus, continue to $N = m + 1$ variables.

At $N = m + 1$, or initially when $m \ge 2s - 1$, the staircase alternant cancels the denominators D_X, D_Y. Thus, the cross product is $\prod_{i,j}(x_i + z y_j)$, with the sign from within Y and monomial $(\prod x_i \prod z y_j)^\delta$ for $m = 2T$. Order the variables by their targets $U_A \sqcup U_B$. With padding, the first $2h$ alternate between X, Y, with powers $0, 0, 2, 2, \ldots, 2h - 2, 2h - 2$ in the determinants. A later u needs power $2(h + u)$. Without padding, the powers are $0, 2, \ldots, 2(N - 1)$. Thus, the target at v after the common monomial is $d(v) + c(v)$, where $d(v)$ is its determinant power. If $v \in X$, then $c(v)$ counts earlier Y variables. If $v \in Y$, then $c(v)$ counts earlier X variables. Thus, a prefix P has target

$\sum_{v\in P} d(v) + c(P)$, where $c(P)$ counts the cross factors internal to P. The sum $\sum_{v\in P} d(v)$ takes the smallest powers. Thus, at P, the target places each bead in A or B. The determinant order is fixed, and $x_i + zy_j$ contributes its later variable. Hence, only $A' = A$ contributes for ordered rows, with $\mathfrak{c}_{A;A',B'}(1) = 1$. Therefore, one ordered extension contributes on $W(D_{s^2})$. Outside this family E, the cuspidal coefficient is 0 by Pan [149, Proposition 3.12].

If $z = -1$, a Y variable contributes one sign per earlier X, and the within-Y factors give one per earlier Y. In the deletion case, we put $L_0 = 2s - m - 1$. Then, the exponent is $\sum_{u\in B,\ u\geq L_0}(u - L_0)$. With padding, the A-dependent exponent is $\sum_{u\in B} u$. The monomial and counts h, N give a common sign c_0. Thus, we put $\ell_u = \max(m - 2s + 1 + u, 0) \bmod 2$. Hence, the relative unitary sign is $(-1)^{M_A\cdot z_m}$, where $z_m = \ell$ if $t_0 \leq s$ and $z_m = j$ if $t_0 > s$. For $m = 2s - 1, 2s$, the vectors ℓ, j agree modulo $\mathbf{1}$ and $M_A \cdot \mathbf{1} = 0$. Thus, $(\varnothing, I)$ moves the entries of A_0 alongside $I \setminus A_0$. By Pan [149, (2.13), (3.13)], the Fourier sign is $(-1)^{j\cdot M_A}$. Moreover, the Weyl sign transposes ϕ, ψ. In the s-bead rectangle, the row swap gives $u \mapsto 2s - 1 - u$. Thus, j becomes $j + \mathbf{1}$ with $M_A \cdot \mathbf{1} = 0$. This is the factor over $\mathbb{F}_{q^2}$, with flip T_{fl}.

Write $\nu = 0$ for $q \equiv 1 \bmod 4$ and $\nu = 1$ otherwise. Let $\upsilon = 1$ in the unitary form and $\upsilon = 0$ in the linear form, so $\nu + \upsilon = 1$. For split type, Shoji [172, (2.9.1)] gives endpoint pairing $(-1)^{(d_i-1)/2+i}$ on odd block i, indexed increasingly from 0. In Zhang [208, Lemma 7.23(iv)], the block of determinant 1 has pairing 1. Thus, the form on V_u is its endpoint pairing. Put $a_u = (d_u - 1)/2 \bmod 2$. If $t_0 \leq s$, $d_u = 2u + 1 + 2\max(m - 2s + 1 + u, 0)$, so $a = j + \ell$ and $x_{\mathrm{ref}} = \nu\ell$. If $t_0 > s$, the prefix has even length, so $a_u = m - 1 \bmod 2$ is constant and $x_{\mathrm{ref}} = \nu j \bmod \langle \mathbf{1}\rangle$.

By Zhang [208, (7.24)] and the discriminant in rank n_b, the nonsplit form occurs when b is odd and $\upsilon = 1$. By Shoji [172, (2.10.1)], the twist is on $V_{\min}$ of smallest odd size. If $h = 0$, it adds $\mathbf{e}_0$ to x_{ref} at $u = 0$. The least noncommon entry is in U_B for the preferred extension. Given $U_{A,i} - i, U_{B,i} - i$, that entry has $u = 0$. Thus, on $\mathcal{W}_s$ the two extensions have ratio -1. Thus, the ratio for A, A_0 gains $(-1)^{M_A(0)}$, cancelling the reference twist. If $h > 0$, the twist is on a plane E_j with component character 1. The least noncommon entry $u_{\min}$ lies in the prefix of U_A, U_B. Hence, its sign is independent of A and cancels in $\mathcal{R}_A$.

Combining both evaluations of $\mathcal{R}_A$ gives $M_A \cdot x = M_A \cdot (\nu\ell + j + \upsilon\ell)$ for $t_0 \leq s$, and $M_A \cdot x = M_A \cdot (\nu j + j + \upsilon j)$ for $t_0 > s$. The vectors M_A span the even-weight subspace of $\mathbb{F}_2^I$. Indeed, exchanging $u \in A$ and $v \notin A$ gives each $\mathbf{e}_u + \mathbf{e}_v$. Since $\nu + \upsilon = 1$, the selected x is a modulo $\mathbf{1}$ if $t_0 \leq s$, and is 0 modulo $\mathbf{1}$ otherwise. Given $t_0 \leq s$, $a = j + \ell$ has weight $s - t_0$ for odd m and $s + t_0$ for even m. Thus, the Clifford formula gives $Q_b = \binom{|a|+1}{2} \bmod 2$. If $t_0 > s$, each plane has bits $0, 1$. Hence, we find $Q_b = \binom{t_0-s}{2} + (m-1)(t_0 - s) \bmod 2$. Since $m = 2t_0 - 1$ gives $b = t_0(2t_0 - 1) \equiv t_0 \bmod 4$, and $m = 2t_0$ gives $b = t_0(2t_0 + 1) \equiv -t_0 \bmod 4$, substitution in both cases gives $Q_b = s/2 + \binom{b}{2} \bmod 2$.

Lemma 6.42. *Let $N \lhd A = NH$, with $P = N \cap H$ and $U \lhd H$ contained in P. Put $L = P/U$. Suppose that $\chi \in \mathrm{Irr}(N)$ and $\beta \in \mathrm{Irr}(L)$ satisfy $\langle \chi, \mathrm{Ind}_P^N \mathrm{Inf}_L^P \beta\rangle = 1$. Suppose that χ, β have equal stabilizers on H and in $H \times \mathcal{H}_p$. Thus, we choose models with factors α and functions $\mu_{g,\sigma}$ equal on $J = A_\chi/N$ and scalars equal on $C_A(N) \cap H$.*

Proof. With $J = A_\chi/N = H_\beta/P$, we put $\Delta = A_\chi \times_J (H_\beta/U)$. Thus, we have that Δ acts on $\mathcal{V} = \mathbb{C}[N/U]$ by $(a, hU)(nU) = anh^{-1}U$. Since $aN = hN$, the image belongs to N/U, and replacing h by hu for $u \in U$ gives the same coset. At U, we have that the stabilizer in $N \times L$ is the graph of $P \to L$. By [95, Lemma 5.2], we obtain multiplicity 1 for $\chi \boxtimes \beta^\vee$ in $\mathcal{V}$. Given (g, σ) in the mixed stabilizer, the source normalizer permutes N/U, and σ fixes the matrices of $\mathcal{V}$. Thus, the character at $\chi \boxtimes \beta^\vee$ is invariant. If $z \in C_A(N) \cap H$, $(z, zU)(nU) = nU$. By Proposition 2.4, the extension on Δ is invariant and acts by $(z, zU) \mapsto I$. Hence, Corollary 2.5 gives equal $\alpha, \mu_{g,\sigma}$ on J and equal scalars on $C_A(N) \cap H$. Moreover, for an inner field power (a, hU), take the coset permutation $nU \mapsto anh^{-1}U$. Thus, inflation from H_β/U gives the models on H_β. □

Lemma 6.43. *Let $\mathbf{Y} = \mathrm{Spin}(V)$ be simply connected, with $Z = \mathrm{Z}(\mathbf{Y})$. Thus, take the connected central torus $\mathbf{T}$ and its map $\iota : Z \hookrightarrow \mathbf{T}$ for the regular inclusion. For F on both groups, we put $\widehat{Y} = \{g : F(g)g^{-1} \in Z\}$ and $T^\sharp = \{t : F(t)t^{-1} \in \iota(Z)\}$, with $\partial_Y(g) = F(g)g^{-1}$ and $\partial_T(t) = \iota^{-1}(F(t)t^{-1})$ in Z. Thus, we put $\Delta Z = \{(z, \iota(z)^{-1}) : z \in Z\}$ and $Y = \mathbf{Y}^F$. Then, the rational regular group $\widetilde{Y}$ and its quotient satisfy $\widetilde{Y} = \{(g, t) : \partial_Y(g)\partial_T(t) = 1\}/\Delta Z$, $\widetilde{Y}/Y \cong T^\sharp/\iota(Z)$. The elements of $\mathrm{Irr}(\widetilde{Y})$ are represented by pairs $(\Theta, \gamma) \in \mathrm{Irr}(\widehat{Y}) \times \mathrm{Irr}(T^\sharp)$ with central character $\kappa_\Theta = \gamma \circ \iota$ on Z. The equivalence twists both entries by $\nu \circ \partial_Y, \nu \circ \partial_T$ for $\nu \in \mathrm{Irr}(Z)$. Moreover, for connected $\mathbf{L} \supseteq Z$, the result holds on $\mathbf{L}$ and on the normalizer of Q_0.*

Proof. Since Z is central, ∂_Y, ∂_T are homomorphisms. By [114, Theorem 1], both are onto Z, with finite kernels $Y, \mathbf{T}^F$. In the central product, $(g, t)\Delta Z$ is F-fixed exactly when $\partial_Y(g)\partial_T(t) = 1$. Thus, we put $E = \ker(\partial_Y\partial_T)$. Since

$E \to T^\sharp$ is onto with kernel $Y \times 1$, the quotient is $T^\sharp/\iota(Z)$. Moreover, we have that t acts trivially on Q_0. The product $\widehat{Y} \times T^\sharp$ equals $E(1 \times T^\sharp)$, with $1 \times T^\sharp$ central. Hence, restriction to E is irreducible. Conversely, for $\vartheta \in \mathrm{Irr}(E)$, extend its scalar on $1 \times \mathbf{T}^F$ to $\gamma \in \mathrm{Irr}(T^\sharp)$. This extends ϑ to $\widehat{Y} \times T^\sharp$. By [95, Corollary 6.17], the quotient Z gives the extension twists $(\nu \circ \partial_Y, \nu \circ \partial_T)$. Descent to $E/\Delta Z$ is equivalent to $\kappa_\Theta = \gamma \circ \iota$. Since ∂_T is onto, twists on (Θ, γ) act freely. For connected $\mathbf{L}$, [114, Theorem 1] gives $\partial_Y(\widehat{Y} \cap \mathbf{L}) = Z$. Given $N_{\widehat{Y}}(Q_0)$, Sylow conjugacy and characteristicity give $\widehat{Y} = Y N_{\widehat{Y}}(Q_0)$. Thus, the extension argument is on $T^\sharp$. Given (g, σ) in the joint stabilizer, both sides take γ over κ_Θ. The tensor models on E descend to $E/\Delta Z$ with the same scalars from γ. Moreover, a commutator with $(y, 1)$ has torus coordinate 1, so a commutator in ΔZ equals 1. Thus, for $(g, t) \in E$, the F-difference is $(\partial_Y(g), \iota(\partial_Y(g))^{-1})$. Hence, $F = \delta F_0^f$ gives $F_0^f = \delta^{-1}$ on $E/\Delta Z$. □

For matched orbits on $Y, N_Y(Q_0)$, take associated models P, P' for their boundary overgroups. Let T_0 be irreducible with inverse factor α^{-1} on the common quotient J. Thus, Clifford induction with $P \otimes T_0, P' \otimes T_0$ gives the characters on $\widehat{Y}, N_{\widehat{Y}}(Q_0)$ over each orbit with multiplicity 1. A twist ν gives $T_0\nu$ on both sides. Moreover, a joint stabilizer element (g, σ) gives $\mu_{g,\sigma}$ on T_0. Given $z \in \mathrm{Z}(\widehat{Y})$, we have that both scalars from P, P' agree, and T_0 gives the same operator at z. Hence, Proposition 7.6 gives the comparison on $\widehat{Y}, N_{\widehat{Y}}(Q_0)$. Thus, for $C_2^2 \leq Z$, this includes the joint stabilizer actions on Z of Lemma 6.43.

Lemma 6.44. *Let $S_1 = \mathrm{SL}_2(q)$, $z = -I$ and $\widehat{S}_1 = \{g \in \mathrm{SL}_2(\overline{\mathbb{F}}_r) : F_q(g) = z^{d(g)}g,\ d(g) \in \mathbb{F}_2\}$. Put $\Theta = \mathrm{Ind}_{S_1}^{\widehat{S}_1} \chi_1$ for a quadratic pair (χ_1, χ_2) of principal or cuspidal type. Then, Θ has a prime-field operator B. If z acts by -1, B^f implements $(-1)^d$, with $B^{2f} = 1$ and coefficient ratio ε_σ on B. If z acts by 1, then $B^f = 1$ and a separate involution J implements $(-1)^d$. The coefficient ratios on J, B are $\eta_q(b_\sigma), 1$.*

Proof. An element of boundary z induces the nonsquare diagonal on S_1 and exchanges the two constituents. Thus, Θ is irreducible with restriction $\chi_1 + \chi_2$ to S_1 and value 0 off S_1. By [119, §2.2, (2.1)], χ_1, χ_2 have additive classes in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$, which Galois permutes. Hence, Θ is rational. On the group U_1, choose the absolute-trace character $\theta(x)$ of coefficient 1. In a parity space $W_\pm$, the nonzero weights lie in one $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$-class, with one line per $\{u, -u\}$. Thus, θ occurs in exactly one of χ_1, χ_2, with multiplicity 1. Hence, Θ occurs with multiplicity 1 in $\mathcal{I}_\theta = \{v : v(ug) = \theta(u)v(g)\}$. The operator $Bv(g) = v(F_r^{-1}(g))$ fixes $\mathcal{I}_\theta$ and its unique Θ-summand V_Θ.

Choose $s^2 = b_\sigma$ in $\mathbb{F}_{r^2}$, and put $d_b = \mathrm{diag}(s, s^{-1})$. Then, $F_r(d_b) = z^j d_b$ with $(-1)^j = \varepsilon_\sigma$. The map $T_\sigma v(g) = \sigma(v(d_b^{-1}g))$ fixes $\mathcal{I}_\theta$ setwise and centralizes right translations R_g. Since Θ is rational, it fixes V_Θ. Its commutator with B acts there by z^j. Moreover, $B^f R_g B^{-f} = R_{z^{d(g)}g}$, and $B^{2f} = 1$. If z acts by -1, the twist is $(-1)^d$ and the ratio is ε_σ. If z acts by 1, the factor $z^{d(g)}$ is trivial on V_Θ, so $B^f = 1$. Put $Jv(g) = (-1)^{d(g)}v(g)$. Since the boundary twist fixes Θ, the operator J fixes V_Θ. Moreover, we have that $JB = BJ$ and the ratio of J under T_σ is $(-1)^{d(d_b)} = \eta_q(b_\sigma)$. □

Let ϕ stabilize $H \leq G_0$ and satisfy $\phi^f = \mathrm{Int}(v)$ for $v \in H$. Take a source model β with intertwiner $B_H^f = \beta(v)$, and let $J_H^2 = I$ with $J_H B_H = B_H J_H$. Suppose that J_H implements the restriction to H of a rational ϕ-invariant quadratic character ν of G_0. On $\mathcal{V} = \mathrm{Ind}_H^{G_0} \beta$, set $(B_V F)(g) = B_H F(\phi^{-1}g)$ and $(J_V F)(g) = \nu(g) J_H F(g)$. Moreover, set $(T_V F)(g) = T_H(F(g))$, where T_H is semilinear and centralizes $\beta(H)$. Since $\nu(h)^2 = 1$, the two factors cancel in $(J_V F)(hg) = \beta(h)(J_V F)(g)$. Moreover, we have that $(B_V^f F)(g) = \beta(v)F(v^{-1}gv) = F(gv)$. The commutators of T_H with B_H, J_H give the same scalars on T_V with B_V, J_V. Thus, on a stable $\mathcal{V}_0$ of multiplicity 1, the identities for B_V, J_V, T_V hold. If distinct base summands V_ξ have multiplicity 1 and form one parent orbit Ω, we have that their sum W is irreducible. Indeed, a nonzero $W_0 \leq W$ contains V_ξ for some ξ. Hence, it contains all summands for Ω.

Lemma 6.45. *Take q, p, e, t from* Section 6.2. *Let η be the orthogonal sign over $\mathbb{F}_{q^2}$, and let u_λ be the unipotent character of $\mathrm{GL}_c^t(q)$ with label $\lambda \vdash c$. Put $\rho = \mathrm{core}_e(\lambda)$, $c_0 = |\rho|$ and $c - c_0 = ew$. Thus, its quotient character is $\eta_\lambda \in \mathrm{Irr}(C_e \wr \mathfrak{S}_w)$. For exchanged $Q(a, c)$, take odd e and $p \mid |N|$, and let t_A denote the type A sign. Then, the p'-degree domain $\mathcal{D}$ has $t_A = t$, $w > 0$, $c_0 = c \bmod e$, and $p \nmid \eta_\lambda(1)$. Moreover, for $R = 2a + c_0$, either $R < e$ or $R = e$ with $\eta t^{c_0} = -t$. We fix $c \bmod e \in \{0, \ldots, e-1\}$. Thus, on $\mathcal{D}$ the orthogonal label is arbitrary in $\mathrm{SO}_{2a}^\eta(q^2)$.*

Proof. Put $A_0 = v_p(q^e - t)$. Thus, for k blocks of rank e, the factors $q^{2ej} - 1$ and $q^n - \epsilon$ give $A_0 k + v_p(k!)$ as in Lemma 6.96. Thus, for $R < e$, the residue contributes 0, and for $R = e$ it contributes 0 precisely at sign $-t$. At odd e, a block of $\mathrm{SO}_{2a}^\eta(q^2)$ takes ambient rank $2e$. $t_A = -t$ gives period $2e$. A rank-$2e$ block has combined sign $+1$, so it contributes 1 block to the centralizer and 2 to N. Thus, $v_p(|N|)$ is larger. If there are 0 blocks, the centralizer has p'-order and $p \mid |N|$. Thus, all active blocks lie in $\mathrm{GL}_c^t(q)$ at equal valuations, with $t_A = t$ and $v = \lfloor c/e \rfloor > 0$.

If $r_c = c \bmod e$, the residue has rank $2a + r_c$ and sign ηt^{r_c}. Thus, $\mathrm{SO}^{\eta}_{2a}(q^2)$ and the residue have p'-order. Hence, only $\mathrm{GL}^t_c(q)$ contributes the factorial $v!$. Central and restriction indices are 2-powers. Given tq, by [84] and [190] the valuation is $A_0 v + v_p(v!)$. Thus, for hooks h with $e \mid h$, the quotient hooks h/e have product $w!/\eta_\lambda(1)$ by Lemma 3.2. Hence, $v_p(u_\lambda(1)) = A_0(v-w) + v_p(v!/w!) + v_p(\eta_\lambda(1))$. Since $A_0 > 0$ and $w \le v$, the valuation is 0 exactly when $w = v$ and $p \nmid \eta_\lambda(1)$. Thus, $c_0 = r_c$. □

Proposition 6.46. *Take exchanged $Q(s^2, c)$ with cuspidal label Λ. If $s = 1$, take either torus sign η. Thus, on the staircase parent Φ_R, choose the boundary and volume operators S_R, U_R. If $\lambda \mapsto \mu$ is a downward 2-hook step, the parent over λ has $E_\lambda = E_\mu + 1 \bmod 2$ and finite Pin ratio 1. For rank 4, the graph action on D_4 is the natural involution γ.*

Proof. By [208, Proposition 7.21, Step 1], choose dual coordinates e_j, f_j on the $i, -i$ eigenspaces and the plane $W_{\mathrm{dom}} = \langle e_1, f_2 \rangle$. For the unitary form, e_1, e_2 are a hyperbolic pair, so W_{dom} is rational. The cocharacter $e_1 - e_2$ lies in the coroot lattice $Q^\vee$. Moreover, the component groups $A_{\mathbf{L}^*}(s), A_{\mathbf{N}^*}(s)$ are F-equivariantly isomorphic. Thus, by [208, Proposition 7.21, Step 2], on $\widetilde{s}$ both self-twist groups coincide. For μ and plane eigenvalues $\{\zeta, -\zeta\}$, the restriction matrix $(m_{\chi,\beta})$ is a permutation matrix. For (g, σ) and central twist ξ, Step 3 gives a rational correction g' in the dual Levi with twist ξ. Thus, g' and σ fix λ, μ and their hook. Hence, the entry $m_{\chi,\beta} = 1$ matches the mixed stabilizers. For residual D_4, choose the natural parabolic $P_{W_{\mathrm{dom}}}$.

Let $T \simeq \mathbb{G}_m$ be central in the Levi. Since $i/i = 1$, the semisimple element is trivial on T. The algebraic Levi is $(\mathbb{G}_m \times \mathrm{SL}_2 \times \mathrm{Spin}_{2(n-2)})/\langle (-1, -I, z_0) \rangle$. Thus, modulo T, the derived factors share $(-I, z_0)$. Since T is connected, Lang gives $L^F \to (L/T)^F$ onto. For the residual Pin parent P_R with boundary in $\langle z_0 \rangle$, the source group is $L_{\mathrm{src}} = (\widehat{S}_1 \times P_R)/\langle (-I, z_0) \rangle$. By Lemma 6.44, the parity condition is $a_R = 0$, and rationality gives $d(g) + d_R = 0$. Indeed, the differences $(-I)^{d(g)}, z_0^{d_R}$ match in the common kernel. Conversely, $d(g) = d_R$ gives a point of L_{src}, and Lang's theorem on T lifts it rationally. The torus T fixes the residual space V_R pointwise. Hence, a scalar from T on the natural space must be $+1$. Since T acts faithfully on W_{dom}, $T \cap \mathrm{Z}(N) = 1$. With active volume $T(-1)$, the image of ω is ω_R. Thus, the normalizations i^n, i^{n-2} have ratio -1.

The pair for $\widehat{S}_1$ is quadratic principal if $q \equiv -1 \bmod 4$ and cuspidal if $q \equiv 1 \bmod 4$. Since $-I$ and z_0 both act by -1 on their factors, $\Theta \otimes \Phi_R$ descends to L_{src}. Take $B \otimes B_R$ on the field generator, with $2f$-th power $1 \otimes \Phi_R(v_R^{-2})$. Thus, its rational Frobenius operator is $B^f \otimes S_R$, and its volume operator is $-I \otimes U_R$. Both have square 1 and commutator $(-1)^\tau$. Thus, by Lemma 6.44, $B \otimes B_R$ has ratio $\varepsilon_\sigma^{1+E_\mu}$.

Induce $\Theta \otimes \Phi_R$ through the parabolic normalizer and take V_λ at label λ. If $\tau = 0$, the 4 source constituents have multiplicity 1, and the 4 partners form one parent orbit. If $\tau = 1$, the spin constituent has multiplicity 2, and the two boundary operators generate $\mathrm{M}_2(\mathbb{C})$ on its multiplicity space. Thus, V_λ is irreducible. Moreover, by Lemma 6.42, $B \otimes B_R$ acts on each summand. If z is central, the diagonal (z, z) acts by I. A quotient twist ξ multiplies the coset at g by $\xi(g)$. Thus, by Lemma 6.43, the operators on V_λ after descent are $B^f \otimes S_R, -I \otimes U_R$. Thus, its field ratio is $\varepsilon_\sigma^{1+E_\mu}$ for every 2-hook successor μ of λ. □

Proposition 6.47. *Let the exchanged series $Q(s^2, c)$ have cuspidal orthogonal label Λ, with either η allowed for $s = 1$. Suppose that $\lambda \in \mathcal{D}$. With $\rho = \mathrm{core}_e(\lambda)$, we put $k = |\mathrm{core}_2(\lambda)|$ and $l = |\mathrm{core}_2(\rho)|$. Then, we find a comparison f_{cusp} with $\succeq_c$ at $M = N_N(Q_0)$. If $2 \mid s$, the chosen staircase exponents satisfy $E^0_k - \binom{k}{2} \equiv E^0_l - \binom{l}{2} \equiv s/2 \pmod 2$.*

Proof. By Proposition 6.46, each 2-hook adds 1 to the exponent E. Thus, the k, l staircase parents have transfers of lengths $(c-k)/2, (c_0-l)/2$, respectively. For $s = 1, \eta = 1$, Lemmas 6.40 and 6.41 give $E_\lambda = 1 + \binom{c}{2} \bmod 2$ because $\binom{c+2}{2} - \binom{c}{2}$ is odd. Given $s \ge 2$, the higher staircase calculation gives $E^0_k = s + Q_k$ and $E^0_l = s + Q_l$ modulo 2. If s is odd and $\eta = -1$, we let w_0 be a lift at staircase size b, with parity $\tau = s + b \bmod 2$. On rank $2s^2 + b$, the volume ω has $\omega^2 = z_0^b$ and $w_0 \omega w_0^{-1} = z_0^\tau \omega$. Thus, $(\omega w_0)^2 = z_0^{b+\tau} w_0^2 = z_0 w_0^2$. Since $\mathrm{Int}(\omega)|_N = 1$, both lifts give F on N. By Lemma 6.38, the diagonal j_ω of boundary ω acts by $D^2 \mapsto z_0 D^2$. Hence, orientation by j_ω changes Q_b by 1. The 2-hook transfers add $(c-k)/2, (c_0-l)/2$ at both orientations. Thus, j_ω changes each of E_λ, E_ρ by 1.

For the active parent, we put $h = ew$ and $K_h = h(h+1)/2$. Thus, with residual rank 0, take Proposition 4.37 for $t = 1$ and Proposition 4.10 for $t = -1$. The norm kernel has two characters $\theta_{A,+}, \theta_{A,-}$ above $\eta_\lambda \in \mathrm{Irr}(C_e \wr \mathfrak{S}_w)$, differing at the product $\prod_{l=1}^w r_l$ of block reflections r_l. Moreover, σ permutes the pair with sign $(\sigma(i)/i)^{K_h}$. The boundary g_0 exchanges them with η_λ fixed. Indeed, $g_0 = (-i_0)^e E_1$ is a norm-one volume with $g_0^2 = z_0$. $i_0^2 = -1$ in

$\overline{\mathbb{F}}_r$ and $E_1 = \prod_{j=1}^{e} T_j(-1)$ on the first block. On $\mathbf{T}_A$, the toral commutators of g_0 equal those of the diagonals. Since $F_r(E_1) = E_1$, $F_r(g_0) = z_0^j g_0$, where $j = (r-1)/2 \bmod 2$.

Let Φ_{A,η_λ} be the induction across g_0. The cosets $1, g_0$ give the two source summands $V_{A,0}, V_{A,1}$. If $V_{A,0}$ has field matrix B, $V_{A,1}$ has $(-1)^j B$, since z_0 acts by -1. In the linear case, $q \equiv 3 \bmod 4$ gives odd f and $j = 1$. By Proposition 4.35, the scalar on B is $\mathcal{H}_p$-fixed. Thus, the coset exchange contributes $(\sigma(i)/i)^{K_h} = \varepsilon_\sigma^{K_h}$ on B. If $t = -1$ and $2 \mid f$, the two characters are fixed since $\sigma(i) = i$ by the ratio on i. By Proposition 4.10, the field scalar α on the source space has $\sigma(\alpha)/\alpha = \varepsilon_\sigma^{K_h}$. For odd f in this case, $q \equiv 1 \bmod 4$ gives $j = 0$, so the cosets both have matrix B also under the exchange by σ. The rotation and permutation lifts satisfy $F_r(u_l) = u_l$ and $F_r(\pi) = \pi$. Moreover, their quotient matrices lie in $\mathbb{Q}(\zeta_{2e})$, fixed by $\mathcal{H}_p$. Thus, for both t, the pair $\{\theta_{A,+}, \theta_{A,-}\}$ is coefficient-stable. Hence, Φ_{A,η_λ} has finite ratio 1 and field ratio $\varepsilon_\sigma^{K_h}$.

The graded product has scalar $\omega_A = (-i_0)^h \prod_l E_l$. Choose the field action fixing ω_A. If $r \equiv 3 \bmod 4$ and h is odd, $2 \nmid w$. On the first block, set $g = c_1^e$ for $t = 1$ by Proposition 4.37, and $g = u_1^e$ for $t = -1$ by Lemma 4.9. Thus, $g \in N(\mathbf{T}_A)$ has $F_r(g) = g$ and $g\omega_A g^{-1} = -\omega_A$. Moreover, $[g, v_A] = 1$, since $2 \mid w - 1$ and $[u_l, u_m] = z_0$ on the negative blocks. Thus, replace b_A, v_A by $gb_A, v_A g^{-f}$. From $(gb_A)^f v_A g^{-f} = b_A^f v_A$, we obtain the rational Frobenius. Moreover, gb_A fixes ω_A, and since g acts in the finite part of Φ_{A,η_λ}, the ratio remains $\varepsilon_\sigma^{K_h}$.

Let R_A, R_R act on V_A, V_R, respectively, with $U_A = i^h R_A(\omega_A)$. By [137, Theorem 24.56], the graded product has $R(xy) = R_A(x) U_A^{a_R(y)} \otimes R_R(y)$ on block elements x, y. Thus, U_A gives the sign $(-1)^{a_A(x) a_R(y)}$. The kernel $\langle (z_0, z_0) \rangle$ acts by I. Moreover, the matrix algebras generate $\mathrm{End}(V_A \otimes V_R)$. Given M, we apply the equations on a_A, a_R, d_A, d_R. Thus, we put $h_1 = h \bmod 2$, $h_2 = \tau_R = \tau + h \bmod 2$ and $f_2 = 2^{v_2(f)}$. For the product field operator take $\beta \in \mu_{4f_2}$ with $\beta^f = i^{h_1 + h_2 + h_1 h_2 \bmod 2}$. Then, β^{2f} compensates the sign in $R(v)^2$, and Lemma 6.41 gives ratio $\varepsilon_\sigma^{h_1 + h_2 + h_1 h_2}$. The single-block conversions contribute h_1, h_2. Thus, since $K_h + h \equiv \binom{h}{2} \bmod 2$, the mixed contribution is $h\tau_R$. The field ratios agree if $E_\lambda \equiv E_\rho + \binom{h}{2} + h\tau_R \pmod 2$. Moreover, if the ratios agree for $\varepsilon_\sigma = -1$, this congruence follows.

Substitute the transfers $E_\lambda = E_k^0 + (c-k)/2$ and $E_\rho = E_l^0 + (c_0 - l)/2$ modulo 2. Thus, we need $E_k^0 - E_l^0 \equiv \binom{h}{2} + h\tau_R + \frac{c_0 - l}{2} - \frac{c-k}{2} \pmod 2$. If $s = 1, \eta = 1$, $\tau_R = c_0 \bmod 2$. Thus, $c = h + c_0$ gives $\binom{c}{2} - \binom{c_0}{2} = \binom{h}{2} + hc_0$. If s is odd and $\eta = -1$, we fix Q_l by the congruence. The diagonal j_ω gives the other pair. If $2 \mid s$, the calculation of Q_b gives $E_b^0 = s/2 + \binom{b}{2} \bmod 2$. Thus, the 2-hook steps give $E_\lambda = s/2 + \binom{c}{2}$ and $E_\rho = s/2 + \binom{c_0}{2}$ modulo 2, by the parity change on adding 2. Since $\tau_R = c_0 \bmod 2$, their difference is $\binom{h}{2} + hc_0 \bmod 2$. Hence, both congruences hold for k, l.

If $\sigma(i)/i = (-1)^j$, we multiply the odd-spin tensor $T_A \otimes T_R$ by $U_A^{hj} \otimes U_R^{(h+R)j}$. This cancels the parity signs on $R(xy)$ and fixes B_A, B_R. Thus, the finite ratios are 1. At $\eta = 1$, the source orbits of $R_R^\pm$ differ, including $l = 0, 1$ by Lemma 6.41. At $\eta = -1$, take the two orientations under j_ω. Since each quotient twist fixes $R_R^\pm$, Clifford theory gives disjoint restriction constituents $\psi^\pm$. The diagonal j_ω of boundary ω conjugates the full models $R^\pm$. Hence, (χ, ψ) satisfies $\succeq_c$ on M by Lemma 6.38 and Proposition 6.39. The scalars c_z agree on z_0, ω, D. The triple $(\rho, \eta_\lambda, \text{orientation})$ recovers the original label λ. Since $p \nmid \eta_\lambda(1)$ and the residue has p'-order, the local character lies in $\mathrm{Irr}_{p'}(M)$. □

6.3 Orthogonal symbols

Lemma 6.48. *Let $\Lambda = (A, B)$ be in $\mathcal{D}$. We fix V_A, Q_0 and the labels $\lambda, \rho, \eta_\lambda$. Thus,* Corollary 6.52 *treats both $A \neq B$ and $A = B$. For the residual spin group G_0, the cohomological induction is via f_{G_0} from* Section 6.2. *If $\mathrm{rk}(G_0)$ is even, we choose $\xi \in \Xi$, with $|\Xi| = 2$, from* Lemma 6.36. *Write $V = V_A \perp V_0$, with $V_0 = V^{Q_0}$ and $\dim V_0 = 2R$. Let $\mathcal{F}$ be a hook flag in V_0, with parabolic $P = LU$. Then, we find $P_p = Q_0 \rtimes P_{\mathrm{perm}} \in \mathrm{Syl}_p(P) \cap \mathrm{Syl}_p(N)$. If $M = N_N(Q_0)$, we put $L_M = N_L(Q_0)$, $U_M = C_U(Q_0)$ and $P_M = M \cap P$. Thus, we find $P_M = L_M U_M$. Suppose that $\chi \in \mathrm{Irr}(N)$ and $\psi \in \mathrm{Irr}(M)$ have Levi sources $\beta \in \mathrm{Irr}(L)$, $\beta_M \in \mathrm{Irr}(L_M)$ with multiplicities $\langle \chi, \mathrm{Ind}_P^N \mathrm{Inf}_L^P \beta \rangle = \langle \psi, \mathrm{Ind}_{P_M}^M \mathrm{Inf}_{L_M}^{P_M} \beta_M \rangle = 1$. Suppose that $(\chi, \beta), (\psi, \beta_M)$ are equivariant under mixed stabilizer operations (g, σ) and quotient twists ζ. Then, a central comparison $\beta \longleftrightarrow \beta_M$ gives a central comparison $\chi \longleftrightarrow \psi$.*

Proof. Write $V_A = \perp_j V_j$ and $Q_0 = \prod_j Q_j$, with Q_j cyclic on V_j, trivial on V_l for $l \neq j$, and $V_j^{Q_j} = 0$. Thus, $V^{Q_0} = V_0$. The permutation Sylow subgroup P_{perm} acts on the blocks V_j. Since $Q_0, P_{\mathrm{perm}} \leq L$, by Lemma 6.45, $P_p = Q_0 \rtimes P_{\mathrm{perm}} \in \mathrm{Syl}_p(N)$ with $P_p \leq P$. Moreover, $M(V_0) = V_0$. If $ul \in N_P(Q_0)$, projection to L gives $l \in L_M$, so $u \in N_U(Q_0)$. Since $[u, Q_0] \leq U \cap Q_0 = 1$, $u \in U_M$. Conversely, we have that $L_M U_M \leq N_P(Q_0)$, so $P_M = L_M U_M$. Given $F_i \in \mathcal{F}$, the inclusion $F_i \leq V_0$ gives $\mathrm{Hom}_{Q_0}(F_i, V_A) = 0$. An element of U_M stabilizes V_A and acts trivially on $F_{\max}^\perp / F_{\max}$, so it is the identity on V_A. Conversely, if $u|_{V_A} = I$, $u \in U_M$. Thus, U_M is the unipotent radical on V_0.

Take the finite cover $N \lhd A = NH$ for the source orbit $(\mathcal{F}, \beta)$. Then, $N \cap H = P$, $P \lhd H$ and $U \lhd H$. Put $H_Q = N_H(Q_0)$. Since $P_p \in \mathrm{Syl}_p(P)$, for $h \in H$ there is $x \in P$ with $P_p^{hx} = P_p$. Hence, $Q_0^{hx} = Q_0$ by characteristicity. Thus, $H = PH_Q$, $A = NH_Q$, and $N_A(Q_0) = MH_Q$. Moreover, $H_Q \cap P = P_M$. With $\mathcal{Q}_{\mathrm{src}} = H/P$, the maps $\iota_H, \iota_A, \iota_M$ on H_Q/P_M act by $hP_M \mapsto \big(hP, hN, hM\big)$, respectively. Thus, we have that $\iota_H, \iota_A, \iota_M$ are onto. Since $H_Q \cap P = H_Q \cap N = H_Q \cap M = P_M$, each kernel is 1.

Thus, for a mixed stabilizer operation (a, σ) with $a \in N_A(Q_0)$ and $\sigma \in \mathcal{H}_p$, we choose $n \in N$ with $an \in H$. By Sylow conjugacy in P, we choose $x \in P$ with $anx \in H_Q$. Thus, $nx \in N \cap N_A(Q_0) = M$. Put $\beta_P = \mathrm{Inf}_L^P \beta$. Since $x \in P$, $\beta_P^x = \beta_P$. Moreover, $aN = anxN$ and $aM = anxM$, so ι_H compares both mixed actions on $\mathcal{Q}_{\mathrm{src}}$. Given a flag $\mathcal{F}^n$, the associated quadruple is (P^n, L^n, U^n, β^n), and equivariance gives the pairing (χ^n, β^n).

On $J \leq \mathcal{Q}_{\mathrm{src}}$, take Levi models $\mathcal{P}_L, \mathcal{P}_{L_M}$, with $\alpha_{\mathcal{P}_L} = \alpha_{\mathcal{P}_{L_M}} = \alpha$ and $m_{\mathcal{P}_L}(g, \sigma; j) = m_{\mathcal{P}_{L_M}}(g, \sigma; j)$ at each mixed stabilizer operation and $j \in J$. Moreover, the maps π_N, π_M on $A_\chi, N_A(Q_0)_\psi$ are induced by $\iota_H \iota_A^{-1}, \iota_H \iota_M^{-1}$, respectively. Thus, set $\mathcal{V} = \mathbb{C}[N/U]$ and $\mathcal{W} = \mathbb{C}[M/U_M]$. The multiplicity hypotheses give lines

$$\ell_{\chi,\beta} = \mathrm{Hom}_{N \times L}(V_\chi \otimes V_\beta^\vee, \mathcal{V}),\ \ell_{\psi,\beta_M} = \mathrm{Hom}_{M \times L_M}(V_\psi \otimes V_{\beta_M}^\vee, \mathcal{W}).$$

Thus, Lemma 6.42 gives scalar corrections $u_N, u_M : J \to \mathbb{C}^\times$ for the two ambient models. With $\widetilde{\mathcal{P}}_N = (u_N \circ \pi_N)\mathcal{P}_N$ and $\widetilde{\mathcal{P}}_M = (u_M \circ \pi_M)\mathcal{P}_M$, both factors are α. Moreover, the mixed functions are $m_{\mathcal{P}_L} = m_{\mathcal{P}_{L_M}}$ on J.

Since $C_A(N)$ fixes $(\mathcal{F}, \beta, Q_0)$, it lies in H_Q. Given $z \in C_A(N)$, the pairs $(z, zU), (z, zU_M)$ act by I on $\mathcal{V}, \mathcal{W}$. By Lemma 6.42, $\kappa_N(z) = \kappa_L(z) = \kappa_{L_M}(z) = \kappa_M(z)$. In the regular gradings $\mathcal{V} = \bigoplus_{\xi \in \mathcal{I}} \mathcal{V}_\xi$ and $\mathcal{W} = \bigoplus_{\xi \in \mathcal{I}} \mathcal{W}_\xi$, the quotient-twist hypothesis pairs $\mathcal{V}_\xi, \mathcal{W}_\xi$. Take the corrections $u_N(j), u_M(j)$ on $\mathcal{V}_\xi, \mathcal{W}_\xi$. Thus, $\chi \longleftrightarrow \psi$ is a central comparison on N, M. □

Proposition 6.49. *Let f_{cusp} be as in* Proposition 6.47. *Given $\Lambda \in \mathcal{D}$ with $A \neq B$, we find a central comparison f_{nd} at $M = N_N(Q_0)$, including $R = e$. Moreover, $f_{\mathrm{nd}}|_{\mathrm{Dom}(f_{\mathrm{cusp}})} = f_{\mathrm{cusp}}$.*

Proof. Take a noncuspidal $\Lambda = (A, B)$, with $A \neq B$ and $a = \mathrm{rk}(\Lambda) \geq 2$. Thus, the hook set is $H_1(\Lambda) = \{(C, x) : C \in \{A, B\},\ x \in C,\ x > 0,\ x - 1 \notin C\}$. By [208, Lemma 7.17(i)], we choose $(C, x) \in H_1(\Lambda)$ with nondegenerate successor Λ^-. For the other row D, we put $\Lambda^- = (C^-, D)$ with $C^- = (C \setminus \{x\}) \cup \{x - 1\}$. Thus, $\mathrm{rk}(\Lambda^-) = a - 1$ and $|C^-| - |D| = |C| - |D|$, so $\mathrm{type}(\Lambda^-) = \eta$. Since F exchanges the factors, one hook on Λ over $\mathbb{F}_{q^2}$ gives (Λ^-, Λ^-). With $(\lambda, \rho, \eta_\lambda)$ fixed, the residual rank is $R^- = R - 2 < e$. Hence, $(a - 1, c, \lambda)$ satisfies $\mathcal{D}$ also when $R = e$ initially.

By [208, Proposition 7.18, Step 1], the extracted Levi L has cocharacter $\nu = e_1 + e_{a+1}$. The coordinates $(s_1, s_2) = (1, -1)$ have $s_1 s_2 = -1$, giving character η_q on T_{hook} globally and locally. Thus, the lines E_1, E_{-1} satisfy $F(E_1) = E_{-1}$ and $F(E_{-1}) = E_1$. Moreover, on G_{hook}, take the quadratic cuspidal τ_{hook}. By Lemma 6.44, $\tau_{\mathrm{hook}}(z_{\mathrm{hook}})/\tau_{\mathrm{hook}}(1) = -\eta_q(-1)$. For the residual model ρ_0, $\rho_0(z_0) = -I$. Thus, on the shared kernel K, the scalar is $\eta_q(-1)(-\eta_q(-1))(-1) = 1$. If $q \equiv 1 \bmod 4$, the extracted central sign is -1, and if $q \equiv -1 \bmod 4$, it is 1.

By Lemma 6.43, take the factors $G_{\mathrm{hook}}, G_0, T_{\mathrm{hook}}$. Thus, we let $\mathcal{P}_{\mathrm{h}}$ model $\eta_q \boxtimes \tau_{\mathrm{hook}}$, and take $\mathcal{P}_0, \mathcal{P}_0'$ from f_{Λ^-}. On $\mathcal{Q}_{\mathrm{src}}$ the residual factors and functions are α_0, m_0, and those of $\mathcal{P}_{\mathrm{h}}$ are β, n. Thus, the tensor models are $\mathcal{P}_L = \mathcal{P}_{\mathrm{h}} \otimes \mathcal{P}_0$ and $\mathcal{P}_{L_M} = \mathcal{P}_{\mathrm{h}} \otimes \mathcal{P}_0'$, where $L_M = N_L(Q_0)$. Their factors are $\alpha_L = \beta\alpha_0 = \alpha_{L_M}$, and their mixed functions are $m_L = nm_0 = m_{L_M}$. Since K acts trivially, both models descend to L, L_M. By [137, §24.h], $xy = yx$ for $x \in \mathrm{Pin}(V_A)$ and $y \in \mathrm{Cl}^0(V_{\mathrm{hook}})$. Thus, in the comparison f_{Λ^-}, the active operator is $I_{V_A^{\mathrm{rep}}}$.

By [208, Proposition 7.18, Step 3], take the ordinary branching matrix B_{glob} for the hook (C, x). On the residual factor G_0, the local branching matrix is B_{loc}. Thus, $B_{\mathrm{glob}}(\Lambda, \Lambda^-) = B_{\mathrm{loc}}(\Lambda, \Lambda^-) = 1$ gives the constituents χ, ψ. For ordinary hook labels Ξ, the two entries are $B_{\mathrm{glob}}(\Lambda, \Xi) = B_{\mathrm{loc}}(\Lambda, \Xi) \in \{0, 1\}$. Take component representatives d with $d\mathcal{F} = \mathcal{F}$. Since both symbols are nondegenerate, $d\Lambda = \Lambda$ holds exactly when $d\Lambda^- = \Lambda^-$. On the regular gradings $V = \bigoplus_{\xi \in \mathcal{I}} V_\xi$ and $W = \bigoplus_{\xi \in \mathcal{I}} W_\xi$, $d(V_\xi) = V_{\xi^d}$ and $d(W_\xi) = W_{\xi^d}$.

Thus, for (g, σ, ζ), [208, Proposition 7.18, Step 4] pairs the flags at power k_σ, group action g and central twist ζ. The source characters are θ_L, θ_{L_M} with twists $\theta_L\zeta, \theta_{L_M}\zeta|_{L_M}$. Thus, the marked label $(\Lambda, (C, x))$ stays in the pairing. If $\Lambda_1^- = \Lambda_2^-$ but $\Lambda_1 \neq \Lambda_2$, the marked labels $(\Lambda_i, (C_i, x_i))$ remain distinct. The local label of ψ is $(\Lambda, \rho, \eta_\lambda)$. Hence, $S_{\mathrm{glob}} = (\mathcal{Q}_{\mathrm{src}} \times \mathcal{H}_p)_\chi$ equals $S_{\mathrm{loc}} = (\mathcal{Q}_{\mathrm{src}} \times \mathcal{H}_p)_\psi$. At residual rank 4, the Levi has type A_1^3. Thus, γ fixes G_{hook} and exchanges $G_{0,1}, G_{0,2}$.

Since $\mathrm{rk}(\Lambda^-) = a - 1$, the chain $\Lambda^{(0)} = \Lambda, \ldots, \Lambda^{(m)}$ reaches $H_1(\Lambda^{(m)}) = \varnothing$ or rank 1. By [208, proof of Lemma 7.17], an endpoint without a 1-hook is cuspidal, of rank $(|A_m| - |B_m|)^2/4$. In defect 0, the endpoint has rank 1 and $\eta = 1$.

Thus, for $\eta \in \{1,-1\}$ at rank 1, Proposition 6.47 gives $f_{\rm cusp}$. Put $f_{\Lambda^{(m)}} = f_{\rm cusp}$ on this packet. Thus, Lemma 6.48 gives $f_{\Lambda^{(m-1)}}, \dots, f_{\Lambda^{(0)}}$ at $M = N_N(Q_0)$. The union of f_Λ on the nondegenerate packets is $f_{\rm nd}$. □

Lemma 6.50. *Suppose that $c = ew + c_0$, $w > 0$ and $4 + c_0 < e$ in $\mathcal{D}$. Let $Y = \mathrm{Spin}^{t^c}_{2c}(q)$ be the derived factor of L_4. Given a pure quartic point $s \in Y^*$, we find a central mixed comparison $f_{\rm pure}$ at $M_Y = N_Y(Q_0)$ for the eligible labels $\lambda \vdash c$ in the faithful series $\mathcal{E}(Y,s)$. Moreover, we have that $f_{\rm pure}$ extends to $\widehat{Y}, \widehat{M}$. If c is even, each geometric class $s^{\mathbf{Y}^*}$ has two branches $\chi_{\lambda,\xi}$. If c is odd, the branch is χ_λ.*

Proof. Since the four-plane W_4 lies in V^{Q_0}, $Q_0 \le Y$. Write $V_Y = (\perp_{j=1}^w V_j) \perp V_{\rm rem}$, with $\dim V_j = 2e$ and $\dim V_{\rm rem} = 2c_0$. With $Y = \mathbf{Y}^F$, let $\pi : \mathbf{Y} \to \mathrm{SO}(V_Y)$ have kernel $\langle z_0 \rangle$ of order 2. Thus, the Sylow d-torus $\mathbf{S}$, with $d = \mathrm{ord}_p(q)$, acts on the spaces V_j. We write $Q_0 = \prod_{j=1}^w Q_j$, with independent cyclic groups Q_j of order $(q^e - t)_p$. Since $c_0 < e$, $p \nmid |\mathrm{Spin}(V_{\rm rem})^F|$ and $Q_0 = (\mathbf{S}^F)_p$.

Let ϑ_j generate $\mathrm{Irr}(Q_j)$, so that the weights on V_j are $\vartheta_j^{\pm q^k}$ for $0 \le k < e$. If $\mathrm{ord}_p(q) = e$, odd e implies that the residues $\pm q^k \bmod p$ are distinct. If $\mathrm{ord}_p(q) = 2e$, then $q^e \equiv -1 \bmod p$ also implies that there are $2e$ distinct residues. Moreover, $\vartheta_j|_{Q_i} = 1$ for $i \ne j$. Hence, each weight ω of $\pi(\mathbf{S})$ has $(V_Y)_{\omega|_{\pi(Q_0)}} = (V_Y)_\omega$, with $(V_Y)_0 = V_{\rm rem}$. Thus, $C_{\mathrm{SO}(V_Y)}(\pi(Q_0)) = C_{\mathrm{SO}(V_Y)}(\pi(\mathbf{S}))$. Given $g \in \pi^{-1}(C_{\mathrm{SO}(V_Y)}(\pi(Q_0)))$, the commutator $x \mapsto [g,x]$ is a homomorphism $Q_0 \to \langle z_0 \rangle$. Since p is odd, $[g, Q_0] = 1$. Similarly, for $g \in \pi^{-1}(C_{\mathrm{SO}(V_Y)}(\pi(\mathbf{S})))$, the morphism $x \mapsto [g,x]$ is trivial since $\mathbf{S}$ is connected. Hence, $\mathbf{C} = C_{\mathbf{Y}}(Q_0) = C_{\mathbf{Y}}(\mathbf{S})$. Since $\mathbf{S} = Z^\circ(\mathbf{C})_{\Phi_d}$, the largest torus of type Φ_d in $Z^\circ(\mathbf{C})$, $N_Y(Q_0) = N_Y(\mathbf{S}) = M_Y$. In the finite cover $\mathcal{A}_Y$, a field or boundary representative a has $a(Q_0) = Q_0$ exactly when $a(\mathbf{S}) = \mathbf{S}$, by the formulas for $Q_0, \mathbf{S}$. Thus, $N_{\mathcal{A}_Y}(Q_0) = N_{\mathcal{A}_Y}(\mathbf{S})$.

Since $w > 0$ and $4 + c_0 < e$, $c \ge e \ge 5$. If c is odd, Proposition 6.16 gives that $\mathcal{X}_{M_Y}(s,\lambda) = \{\psi_\lambda\}$ for each eligible λ. Thus, $f_{\rm pure}(\chi_\lambda) = \psi_\lambda$. Let j_c be the Clifford lift of $\bar{j}_c$, with $\bar{j}_c(E_\pm) = E_\mp$ on the halves $E_\pm$ of V_Y. If c is odd, we choose j_c from Proposition 6.16. If c is even, choose j_c from Lemma 6.36. Since $\det(\bar{j}_c) = (-1)^c$, the action $\mathrm{Int}(j_c)|_Y$ is outer for odd c.

If c is even, put $T = \mathbb{F}_q^\times$ for $t = 1$ and $T = \mu_{q+1}$ for $t = -1$. For the quadratic $\eta_T \in \mathrm{Irr}(T)$, set $\delta = \eta_T(-1)$ and $\Xi_c = \{\xi \in \mathbb{C}^\times : \xi^2 = \delta^{\binom{c}{2}}\}$. By Lemma 6.36, the characters $\gamma_{\lambda,\xi}$ give that $f_{\rm pure}(\chi_{\lambda,\xi}) = \psi_{\lambda,\xi}$ for $\xi \in \Xi_c$. Moreover, for the graph automorphism τ, $f_{\rm pure}(\chi^\tau_{\lambda,\xi}) = \psi^\tau_{\lambda,\xi}$ on $(s^\tau)^{\mathbf{Y}^*}$. A field representative $a = lF_r$ with $l \in L_Y$ has $a^f = l_f F_q$, where $l_f = lF_r(l) \cdots F_r^{f-1}(l)$. Thus, $F_q|_Y = 1$ for $t = 1$ and $F_q|_Y = \mathrm{Int}(j_c^{-1})$ for $t = -1$, as in Lemma 6.36. Hence, both models $\mathcal{P}, \mathcal{P}'$ carry the field power $l_f F_q$.

In Lemma 6.43, we let $\widehat{Y}$ be the boundary cover and $\widehat{M} = N_{\widehat{Y}}(Q_0)$. Since $\widehat{Y} = Y\widehat{M}$ by Lemma 6.43, the map $\jmath : \widehat{M}/M_Y \to \widehat{Y}/Y$ given by $xM_Y \mapsto xY$ is an isomorphism. If $\psi = f_{\rm pure}(\chi)$, we set $I = \widehat{Y}_\chi$, $I_M = \widehat{M}_\psi$ and $J = I/Y$. Since $f_{\rm pure}$ is equivariant, $I_M/M_Y = \jmath^{-1}(J)$. Thus, write $\pi_I : I \to J$ and $\pi_{I_M} : I_M \to J$ for the quotient maps through I/Y and $\jmath$, respectively. The models $\mathcal{P}, \mathcal{P}'$ on I, I_M have common factor α and mixed functions $m_{\mathcal{P}} = m_{\mathcal{P}'}$. By [43] and Proposition 7.6, the irreducible projective T_0 on J of factor α^{-1} gives each pair (Θ, Ψ) with multiplicity 1 via the models $\mathcal{R}_\Theta = \mathrm{Ind}_I^{\widehat{Y}}(\mathcal{P} \otimes (T_0 \circ \pi_I))$, $\mathcal{R}_\Psi = \mathrm{Ind}_{I_M}^{\widehat{M}}(\mathcal{P}' \otimes (T_0 \circ \pi_{I_M}))$. Moreover, a boundary character $\nu \in \mathrm{Irr}(Z)$ with $Z = Z(\mathbf{Y})$ acts by $T_0 \mapsto T_0 \otimes \eta_\nu$ on both sides, where $\eta_\nu(gY) = \nu(\partial_Y(g))$. Since $m_{\mathcal{P}} = m_{\mathcal{P}'}$, the pairing $\Theta \mapsto \Psi$ is equivariant under the joint actions (g, σ, ν).

By Lemma 6.43, write $E = \ker(\partial_Y \partial_T)$ in $\widehat{Y} \times T^\sharp$, and $E_M = E \cap (\widehat{M} \times T^\sharp)$. Put $\Delta Z = \{(z, \iota(z)^{-1}) : z \in Z\}$. Since $\kappa_\Theta = \kappa_\Psi$ on Z, we choose $\gamma \in \mathrm{Irr}(T^\sharp)$ with $\gamma \circ \iota = \kappa_\Theta$. Thus, the models $\mathcal{R}_\Theta \boxtimes \gamma, \mathcal{R}_\Psi \boxtimes \gamma$ give the paired characters on $E/\Delta Z, E_M/\Delta Z$. By Lemma 6.43, the common twist $\nu(\partial_Y(g))\nu(\partial_T(t)) = 1$ on E, and ΔZ acts by 1 in both models. Hence, on $E/\Delta Z, E_M/\Delta Z$, the paired characters have $\kappa_\Theta = \kappa_\Psi$ and $m_{\mathcal{P}} = m_{\mathcal{P}'}$. □

Proposition 6.51. *Let $\Lambda = (A, A)$ be a degenerate symbol in $\mathcal{D}$, with eligible $\lambda \vdash c$. Then, a central comparison $f_{\rm deg}$ exists at $M = N_N(Q_0)$ for both characters D^+_Λ, D^-_Λ, with rational flags $\mathcal{F}_+, \mathcal{F}_-$.*

Proof. If $\Lambda = (A, A)$, $\eta = 1$ and $a = 2u$. Write $\xi \vdash u$ for the partition of Λ. If $R = e$, $c_0 = e - 2a$ is odd, so $\eta t^{c_0} = t$. Therefore, the condition $\eta t^{c_0} = -t$ in $\mathcal{D}$ gives $R = 2a + c_0 < e$. Suppose that $a \ge 4$. By [208, Lemma 7.19 and Proposition 7.20, Step 2], we choose $b \in \xi$ with $\xi^- = \xi \setminus \{b\} \vdash u - 1$. The symbol Λ^- has rank $a^- = a - 2$ and $\eta^- = 1$. Thus, $R^- = R - 4 < e$, so $(a^-, c, \lambda) \in \mathcal{D}$ with the labels $(\lambda, \rho, \eta_\lambda)$ of type A fixed. By [208, Proposition 7.20, Step 2], choose $E_2 \le V_+$ with $\dim E_2 = 2$ and $Q_V|_{E_2} = 0$. The rational space $W_4 = E_2 \oplus F(E_2)$ has cocharacter $\nu = e_1 + e_2 + e_{a+1} + e_{a+2} \in Q^\vee(D_n)$. Thus, the factor $G_{\rm ext}$ has type $\mathrm{GL}_4(q)$, and $C^*_{\rm ext}$ has type $\mathrm{GL}_2(q^2)$ with label $1_{C^*_{\rm ext}}$. Since $e > 2a + c_0 \ge 8$, $p \nmid |G_{\rm ext}|$.

Take B from [208, Lemma 7.19 and Proposition 7.20, Step 2], with rows $D^{\varepsilon}_{\Lambda}$ and columns $D^{\varepsilon'}_{\Lambda^-}$ for $\varepsilon,\varepsilon' \in \{+,-\}$. Since F exchanges the factors, the matrix B is taken over $\mathbb{F}_{q^2}$ on (Λ,Λ). Thus, $B_{\mathrm{glob}} = B_{\mathrm{res}} = B$ on V, V_0 and $B_{\mathrm{loc}} = I_{V_A^{\mathrm{rep}}} \otimes B$ on the local model. Given any ε, the unique ε' has $B_{\varepsilon,\varepsilon'} = 1$ and $B_{\varepsilon,\delta} = 0$ for $\delta \neq \varepsilon'$. Since $\xi = \xi^- \cup \{b\}$ for the marked choice (ξ, b), B^{-1} gives $D^{\varepsilon'}_{\Lambda^-} \mapsto D^{\varepsilon}_{\Lambda}$. By [208, Proposition 7.20, Step 2], the exchanged rational forms $\mathfrak{c}_1, \mathfrak{c}_2$ satisfy $\mathfrak{c}_1 \neq \mathfrak{c}_2$ for even c and $\mathfrak{c}_1 = \mathfrak{c}_2$ for odd c. Moreover, on the flags $\mathcal{F}$, each mixed stabilizer operation (g,σ,ζ) has permutations $\Pi_{\mathrm{src}}, \Pi_{\mathrm{tar}}$ on $D^{\varepsilon'}_{\Lambda^-}, D^{\varepsilon}_{\Lambda}$. By the equivariance of B, $B\Pi_{\mathrm{src}} = \Pi_{\mathrm{tar}} B$.

Let $N^- = \mathrm{Spin}^{\epsilon}_{2(n-4)}(q)$ and put $M^- = N_{N^-}(Q_0)$. If f_{Λ^-} is the comparison at M^-, take models $\mathcal{P}_0, \mathcal{P}'_0$ with $\alpha_{\mathcal{P}_0} = \alpha_{\mathcal{P}'_0} = \alpha_0$ and $m_{\mathcal{P}_0} = m_{\mathcal{P}'_0} = m_0$. Let $\mathcal{P}_{\mathrm{ext}}$ be the model for $1_{C^*_{\mathrm{ext}}}$. Thus, $\mathcal{P}_L = \mathcal{P}_{\mathrm{ext}} \otimes \mathcal{P}_0$ and $\mathcal{P}_{L_M} = \mathcal{P}_{\mathrm{ext}} \otimes \mathcal{P}'_0$ have $\alpha_L = \alpha_{\mathcal{P}_{\mathrm{ext}}}\alpha_0 = \alpha_{L_M}$ and $m_L = m_{\mathcal{P}_{\mathrm{ext}}} m_0 = m_{L_M}$. For the central product L of Proposition 6.49, take the common $\gamma \in \mathrm{Irr}(T^\sharp)$ as in Lemma 6.43. For z in the diagonal kernel K_{ext}, both models have $\mathcal{P}_L(z) = \mathcal{P}_{L_M}(z) = I$. By Lemma 6.48, we obtain f_Λ at M for the matched signs $\varepsilon, \varepsilon'$. Since $a^- = a - 2$, the reductions end at $a = 2$.

Since $R < e$, at $a = 2$ $4 + c_0 < e$ and $c \geq e \geq 5$. By [208, Proposition 7.20, Step 3, case (5)], the flags $\mathcal{F}_+, \mathcal{F}_-$ of D^+_Λ, D^-_Λ give the rational space W_4 and its Levi L_4. The orthogonal factor is $Y = \mathrm{Spin}^{t^c}_{2c}(q)$, with $t = 1$ for $q \equiv -1 \pmod 4$ and $t = -1$ for $q \equiv 1 \pmod 4$. By Lemma 6.50, we obtain f_{pure} at $N_Y(Q_0)$ on the class $s_Y^{\mathbf{Y}^*}$ marked by $\mathcal{F}_\varepsilon$. If c is even, its branches $\chi_{\lambda,\xi_{\mathrm{br}}}$ for $\xi_{\mathrm{br}} \in \Xi_c$ have models $D_{\gamma_{\lambda},\xi_{\mathrm{br}}}$ with $D_{\gamma_{\lambda},-\xi_{\mathrm{br}}}(j_c) = -D_{\gamma_{\lambda},\xi_{\mathrm{br}}}(j_c)$ at the exchange representative j_c. If $c \equiv 1 \pmod 2$, then only $\mathcal{O}_\lambda$ occurs. Let G_{ext} have derived rank 3, with model $\mathcal{P}_{\mathrm{ext}}$. For $\mathcal{P}_Y, \mathcal{P}'_Y$ from f_{pure}, take a common $\gamma \in \mathrm{Irr}(T^\sharp)$ for the sources $L_4, L_{4,M}$. Thus, $\mathcal{P}_{L_4} = \mathcal{P}_{\mathrm{ext}} \otimes \mathcal{P}_Y \otimes \gamma$ and $\mathcal{P}_{L_{4,M}} = \mathcal{P}_{\mathrm{ext}} \otimes \mathcal{P}'_Y \otimes \gamma$. By Lemma 6.43, the models on E, E_M descend to $E/\Delta Z, E_M/\Delta Z$.

Given $k \in \{c, c_0\}$, let s_k be the dual point in $\mathbf{G}^*_k$ of rank $4 + k$. Put $A(s_k) = C_{\mathbf{G}^*_k}(s_k)/C^\circ_{\mathbf{G}^*_k}(s_k)$ and $J_k = \mathrm{Stab}_{A(s_k)}(\mathbf{d}_k)$, with $\mathbf{d}_k = (x,y) \in \mathbb{F}_2^2$ the two degenerate labels. Moreover, by [208, Proposition 7.20, Steps 2 and 3], $A(s_k) = \langle \kappa_k, g_k \rangle$ acts by $\kappa_k(x,y) = (x+1, y+1)$ and $g_k(x,y) = (y, x+\bar{k})$ with $\bar{k} = k \bmod 2$. Thus, the cited stabilizers have $|J_k| = 2$ for even k and $|J_k| = 1$ for odd k. Hence, $|J_c| \neq |J_{c_0}|$ if $c \not\equiv c_0 \pmod 2$. Let $\widehat{\Psi}_\lambda$ be the full local parent on $\widehat{M}_Y = N_{\widehat{Y}}(Q_0)$ from f_{pure}. Moreover, its quotient model is V_{η_λ}. Given an ordinary model R_λ of $\widehat{\Psi}_\lambda$, put $I_\lambda = (N_{\mathcal{A}_Y}(Q_0))_{\widehat{\Psi}_\lambda}$. Thus, $D_g R_\lambda(x) D_g^{-1} = R_\lambda(gxg^{-1})$ for $g \in I_\lambda$ and $x \in \widehat{M}_Y$. Hence, the comparison at c, c_0 is on V_{η_λ} in $\widehat{\Psi}_\lambda$.

By [208, Proposition 7.20, Step 3, case (5)], take the matrices B_4 on V and $B_{4,0}$ on V_0. Then, $B_{4,\mathrm{loc}} = I_{V_A^{\mathrm{rep}}} \otimes B_{4,0}$. Thus, the family $\mathcal{F}_\varepsilon$ has coefficient 1 and $\mathcal{F}_{-\varepsilon}$ has coefficient 0. If $c_0 = 0$, $R = 4$. In $W(D_2) = \langle u_0, v_0 \rangle \simeq C_2^2$, the cited characters δ_+, δ_- have values $(\delta_+(u_0), \delta_+(v_0)) = (1,-1)$ and $(\delta_-(u_0), \delta_-(v_0)) = (-1,1)$. Thus, for $H_+ = \langle u_0 \rangle$ and $H_- = \langle v_0 \rangle$, the coefficients $b_{\varepsilon,\varepsilon'} = \langle \delta_\varepsilon|_{H_{\varepsilon'}}, 1_{H_{\varepsilon'}} \rangle$ give the matrix I_2. Moreover, the graph involution τ has $\delta^\tau_\varepsilon = \delta_{-\varepsilon}$ and $\mathcal{F}^\tau_\varepsilon = \mathcal{F}_{-\varepsilon}$.

Still with $c_0 = 0$, take a residual representative r_0 of τ with $\det(r_0) = -1$. Since $w > 0$, take an active block V_j and its inversion v_j, with $\det(v_j) = (-1)^e = -1$. Let $T_j \leq \mathrm{SO}(V_j)$ be the torus of order $q^e - t$. By Section 6.3, the spinor norm sn maps T_j onto $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ for $t = \pm 1$. Thus, we choose $h_j \in T_j$ with $\mathrm{sn}(h_j) = \mathrm{sn}(r_0 v_j)^{-1}$. Then, $\det(r_0 h_j v_j) = \mathrm{sn}(r_0 h_j v_j) = 1$. By [137, §24.h], the spin map $\pi : N \to \Omega(V)$ gives a lift $\widetilde{r}$ of $r_0 h_j v_j$. Since $r_0 h_j v_j$ normalizes $\pi(Q_0)$ and $\ker \pi$ has order 2, the odd-order subgroup Q_0 is characteristic in $\pi^{-1}(\pi(Q_0))$. Thus, $\widetilde{r} \in M$.

If $c_0 = 1$, $R = 5$. By [208, Proposition 7.20, Step 3, case (5)], the Levi L_5 has simple roots $\Delta(D_5) \setminus \{\alpha_4, \alpha_5\}$, with $[L_5, L_5]$ of type A_3 and central torus T_{res} of type SO_2. Thus, at $\mathcal{F}_\varepsilon$ in L_5, take the coefficient $b_{\varepsilon,\varepsilon'}$. Given any c_0, equivariance of $B_4, B_{4,0}$ gives that the flag $\mathcal{F}_\varepsilon \subseteq V_0$ determines D^ε_Λ under (g,σ,ζ). By Lemma 6.48, we obtain f_{deg} on M. □

Corollary 6.52. *Let $q = r^f$ for an odd prime r, with $a, c \geq 1$, $n = 2a + c \geq 5$ and $N = \mathrm{Spin}^{\epsilon}_{2n}(q)$. Suppose that p is an odd prime dividing $|N|$ with $p \neq r$ and $e = \mathrm{ord}_p(q^2)$ odd. Then, for the exchanged series $\mathcal{E}(N,s)$ of type $Q(a,c)$, there is a central comparison $f_{Q(a,c)}$ at $M = N_N(Q_0)$ for the natural overgroups $\mathcal{A}$ and every eligible (Λ,λ) in $\mathcal{D}$.*

Proof. By Lemma 6.45, if $e = 1$, then $c_0 = 0$ and $R = 2a > e$. Thus, whenever $e = 1$, $\mathcal{D} = \varnothing$. Suppose that $e \geq 3$ is odd and $(\Lambda,\lambda) \in \mathcal{D}$. If $\Lambda = (A,B)$ with $A \neq B$, then we choose f_{nd} from Proposition 6.49. At the step (C,x), the rank is $a^- = a - 1$ and both sides have labels $(\lambda, \rho, \eta_\lambda)$. Thus, the endpoint has $H_1(\Lambda) = \varnothing$ or $\mathrm{rk}(\Lambda) = 1$. At rank 1, Lemma 6.41 and Proposition 6.46 cover both signs $\eta \in \{1,-1\}$. If the cuspidal endpoint has $a = s_0^2$ with $s_0 \geq 2$, then by Proposition 6.47, we obtain f_{cusp} for λ and $\rho = \mathrm{core}_e(\lambda)$. If $\Lambda = (A,A)$, then by Proposition 6.51, we obtain f_{deg} from f_{pure} at $N_Y(Q_0)$, with $a^- = a - 2$ at each pair (b,b) in (ξ,ξ).

Thus, put $f = f_{Q(a,c)} = f_{\mathrm{nd}} \cup f_{\mathrm{deg}}$, with $\mathrm{Dom}(f_{\mathrm{nd}}) \cap \mathrm{Dom}(f_{\mathrm{deg}}) = \varnothing$. For the inverse $g = g_{Q(a,c)}$, follow the

recovery in Proposition 6.29 on M with the constructed models of f. The class of s_M determines $a, c, \mathfrak{c}(s)$, where $\mathfrak{c}(s)$ is the rational class of s. Moreover, the residual triple $(s_M, \Lambda_0, \eta_\lambda)$ determines Λ, λ. The paired idempotents $e_\varepsilon^N, e_\varepsilon^M$ select $\chi_\varepsilon, \psi_\varepsilon$. If $\Lambda = (A, A)$, the flag $\mathcal{F}_\varepsilon$ also determines D_Λ^ε. Thus, $gf = \mathrm{id}_{\mathrm{Dom}(f)}$ and $fg = \mathrm{id}_{\mathrm{Dom}(g)}$. Put $\mathcal{B} = N_{\mathcal{A}}(Q_0)$. By Lemma 6.38, $\mathcal{A} = N\mathcal{B}$ and $N \cap \mathcal{B} = M$. Moreover, $f_{\mathrm{nd}}, f_{\mathrm{deg}}$ give that $\alpha_N = \alpha_M$ and $m_N = m_M$.

With $N = \mathbf{N}^F$, we put $G_R = \mathbf{G}_R^F = \mathrm{Spin}_{2R}^{\eta t^{c_0}}(q)$. By Lemma 6.45, $p \nmid |G_R|$ also holds when $R = e$ and $\eta t^{c_0} = -t$. Thus, an F-stable torus $\mathbf{T} \leq \mathbf{G}_R$ of type Φ_d, with $d = \mathrm{ord}_p(q)$, has $|\mathbf{T}^F| = \Phi_d(q)^m$ with $m = 0$ since $p \mid \Phi_d(q)$. Hence, $\mathbf{T} = 1$. Let $\mathbf{S} \leq \mathbf{N}$ be the Sylow d-torus on the active spaces V_j. The weights $\vartheta_j^{\pm q^k}$ of Q_0 on V_j distinguish the weight spaces of $\mathbf{S}$ as in Lemma 6.50. Since $\mathbf{G}_R$ has no Φ_d-torus, $\mathbf{S} = Z^\circ(C_{\mathbf{N}}(Q_0))_{\Phi_d}$. Moreover, $Q_0 = (\mathbf{S}^F)_p$, so $M = N_N(Q_0) = N_N(\mathbf{S})$. Finally, for $P_p \in \mathrm{Syl}_p(N)$ with characteristic base Q_0, $N_N(P_p) \leq M$. □

Suppose that F fixes the orthogonal factors of $Q(s^2, c)$. Then, the spin kernel $\langle z_0 \rangle$ acts trivially. We fix O, SO, Ω on the natural space. With $q = r^f$ and r prime, we let ν be the quadratic spinor norm on O. If $s \geq 2$, the cuspidal factors have rational signs $\eta_1 = \eta_2 = (-1)^s$. If $s = 1$, the torus signs η_1, η_2 come from the rational class of $Q(1, c)$. Thus, we have that $\eta_1 \neq \eta_2$ is possible. Given $\lambda \vdash c$, we put $j_\lambda = (c - |\mathrm{core}_2(\lambda)|)/2$.

Lemma 6.53. *Let ζ_r have order r, and put $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$ with $b_\sigma \in \mathbb{F}_r^\times$. If $s \geq 2$ or $\eta_1 = \eta_2$ at $s = 1$, then in $\langle \det, \nu \rangle$ the twists fixing the parent of λ form $\langle \nu \rangle$. Moreover, its field extension to $O \rtimes \langle \phi \rangle$ is fixed by σ. The normalized operator H_λ for ν has coefficient ratio $\eta_q(b_\sigma)^{s+j_\lambda}$. If $s = 1$ and $\eta_1 \neq \eta_2$, then the extension to $O \rtimes \langle \phi \rangle$ is rational, and only $1 \in \langle \det, \nu \rangle$ fixes the parent for λ.*

Proof. First, put $\lambda = \delta_m = (m, m-1, \ldots, 1)$, with $\delta_0 = \varnothing$. Let $O = O(V)$ and $\Omega = \ker \det \cap \ker \nu$ in O, and take the character χ on Ω with module V_χ. For the nilpotent X and cocharacter γ, we fix $U, \theta = \theta_X$ from [208, Lemma 7.23]. Thus, θ is the character of U defined by the absolute trace. With $C_O = C_O(X, \gamma)$, we put $C = C_O \cap \Omega$. Thus, on $W_\chi = e_\theta V_\chi$, the projector e_θ gives the action of U by θ. We normalize both C and the field generator ϕ on W_χ.

Suppose that $\eta_1 = \eta_2$. Given $s \geq 2$, by [208, Proposition 7.26, Step 3, and Proposition 7.35, Steps 4–6], we obtain the pair (X, γ) with $W_\chi \neq 0$. There are $2s$ odd sizes of multiplicity 1 and $h = \max(0, \lceil m/2 \rceil - s)$ of multiplicity 2. If $h = 0$, then $\dim W_\chi = 1$ and all singleton determinant classes d_i are equal. If $h > 0$, the rotation tori T_j on the multiplicity planes E_j have $4 \mid |T_j|$. Put $T = \prod_j T_j$ and take $\psi_j^2 = \eta_j$, with η_j the quadratic character of T_j. On $T_0 = T \cap \Omega = \ker \prod_j \eta_j$, the weights are $(\prod_j \psi_j^{\varepsilon_j})|_{T_0}$ for $\varepsilon_j \in \{1, -1\}$ modulo $(\varepsilon_j)_j \sim (-\varepsilon_j)_j$. Moreover, each weight of T_0 has multiplicity 1. If $s = 1$ and $m \geq 2$, [208, Proposition 7.35, Steps 2–4] describes W_χ. Thus, the Ω-orbit of (X, γ) and the irreducible C-module W_χ determine χ within its packet.

Let u_i act by -1 on singleton i and by 1 on the other blocks. If $h > 0$ and $\nu(u_i) \neq \nu(u_l)$, we choose $v \in T$ with $\nu(v) = -1$. Then, $u_i u_l v \in C$ fixes every T_0-weight line. Put $\lambda_0 = (\prod_j \psi_j)|_{T_0}$. Since $\lambda_0^2 = 1$, the cited inversion $j \in C$ fixes its line ℓ_{λ_0}. But their commutator acts by $\lambda_0(v^{\pm 2}) = \prod_j \eta_j(v) = -1$ on that line. Thus, $\nu(u_i) = \nu(u_l)$ for every i, l. If $h = 0$, the constant classes d_i in [208, Lemma 7.23] give equal $\nu(u_i)$. A nonsquare $b_0 \in \mathbb{F}_q^\times$ scales the forms and acts by $\nu(u_i) \mapsto -\nu(u_i)$ for each i. Thus, a similitude on V allows $\nu(u_i) = 1$ for all i.

Over $\mathbb{F}_r$, we choose a reflection r_j on E_j with $\nu(r_j) = 1$. Indeed, the form on E_j represents both square classes and d_j is odd. Write w_j for its product with u_1, and put $E = \langle u_i u_l \rangle$ and $W = \langle w_j \rangle$. Thus, $C = (T_0 \rtimes W) \times E$. Indeed, the w_j are the inversions with $\det(w_j) = 1$. The other conditions are $\det = 1$ on the singletons and $\nu = 1$ on T. In $O(V)$, the w_j are involutions with $w_i w_j = w_j w_i$. If $h > 0$, a subset of the w_j fixes λ_0 exactly when the product of its η_j is trivial on T_0. Thus, $W_{\lambda_0} = \langle \prod_j w_j \rangle$. On the line ℓ_{λ_0} in W_χ, the character of C_{λ_0} extends λ_0 by signs on $\prod_j w_j$ and E. By [43], $W_\chi \cong \mathrm{Ind}_{C_{\lambda_0}}^C \ell_{\lambda_0}$, with signed permutation matrices. If $h = 0$, we choose the linear character of E on W_χ. Thus, all scalars on E are those of W_χ.

Take the singleton forms over $\mathbb{F}_r$ with $\nu(u_i) = 1$. Take E_j split over $\mathbb{F}_r$ if $q \equiv 1 \bmod 4$ and nonsplit if $q \equiv -1 \bmod 4$ with f odd. By [208, Lemma 7.23(iv)], the forms over $\mathbb{F}_q$ give the given orbit of (X, γ). Thus, $g \in O$ gives each Ω-orbit in it, with lift ϕ^g for $(X, \gamma)^g$. Over $\mathbb{F}_r$, the automorphism ϕ fixes u_i, w_j, θ. Moreover, ϕ acts on T_j by a common odd power r or $-r$, so every weight of T_0 is fixed. Thus, $B_C = 1$ implements ϕ on W_χ. The reflection u_1 centralizes C and acts by 1 in the chosen extension of W_χ. By the cited source correspondence, both ϕ and u_1 fix χ. Hence, normalize the operators on V_χ by those on W_χ. Since defects are scalar on V_χ by Schur's lemma, evaluate them on W_χ. Thus, χ has an extension $\widehat{\chi}$ to $(\ker \nu) \rtimes \langle \phi \rangle$.

If $b = b_\sigma$, we choose the isometry D_b of diagonal entries $b^{(d-1)/2-j}$ on the vectors $j = 0, \ldots, d-1$ of each block of size d. Choose the action convention with $\theta^{D_b} = \theta^\sigma$. Then, D_b centralizes C_O and ϕ. Moreover, $\nu(D_b) = \eta_q(b)^K$

for $K = \sum_d (d^2-1)/8$ with multiplicities of d. Indeed, for a hyperbolic pair, $\mathrm{diag}(a, a^{-1})$ has norm class $a(\mathbb{F}_q^\times)^2$ by Section 6.3. Doubled blocks add 0 mod 2. Pairing the remaining parts in [208, (7.27)–(7.28)] gives $K \equiv s \bmod 2$, since pairs at distance 4 and tail pairs $(4i-1, 4i-3)$ each contribute 1 mod 2. If $s = 1$, the parts of [208, (7.34)] give $K \equiv 1 \bmod 2$.

The rational model on W_χ and the source correspondence give $\chi^\sigma = \chi^{D_b}$. Moreover, normalization gives value 1 on u_1, ϕ for the quotient twist. Thus, $\widehat{\chi}^\sigma = \widehat{\chi}^{D_b}$. The regular diagonal coset with $\nu = -1$ in $\mathrm{SO}(V)$ exchanges χ, χ^D, where D represents the coset. Hence, $\mathrm{Ind}_{(\ker\nu)\rtimes\langle\phi\rangle}^{O\rtimes\langle\phi\rangle} \widehat{\chi}$ extends the parent on O rationally. On the cosets of $\ker\nu$, multiplication by ν gives H_λ, with $H_\lambda B_\lambda = B_\lambda H_\lambda$ and coefficient ratio $\nu(D_b) = \eta_q(b)^s$. The twist by det changes the scalar on u_1 in $\widehat{\chi}$. Since the coset $\nu = -1$ changes χ, both det and $\det\nu$ move the induced parent.

Suppose that $s = 1$ and $\eta_1 \neq \eta_2$, so $\nu(u_1) = 1$ and $\nu(u_2) = -1$ after ordering u_1, u_2. Thus, their images generate O/Ω. On E_j, take $\mathrm{Ind}_{T_j}^{O(E_j)} \psi_j$. Tensor the $O(E_j)$-modules with signs on u_1, u_2. Since each plane module factors through D_8 of order 8, the module on C_O is rational. Each nonidentity twist in $\mathrm{Irr}(O/\Omega)$ changes a sign on u_1, u_2. Thus, by [43], the restriction to C is irreducible. By [208, Proposition 7.35, Step 4], it affords W_χ. On each T_0-weight space, we let J represent full inversion and P represent u_1u_2v with $\nu(v) = -1$. Then, $JP = -PJ$, and J^2, P^2 give the given scalars. Thus, $\mathbb{C}\langle J, P\rangle \cong \mathrm{M}_2(\mathbb{C})$. In C, partial inversions permute the T_0-weight spaces. If $h = 0$, $\dim W_\chi = 1$. The module on C_O extends the source to UC_O. Thus, normalization on W_χ extends χ to $O = \Omega C_O$.

If $2 \nmid f$, take the singleton forms and E_j over $\mathbb{F}_r$. Since ϕ fixes the weights of T_0 and the reflections r_j, take $B_C = 1$. If f is even, start over $\mathbb{F}_r$ with square singleton norms and split E_j. Under the descent u_2F_q, scale singleton 2 by x with $x^q = -x$. Thus, x^2 is nonsquare in $\mathbb{F}_q$, giving $\nu(u_2) = -1$. The field lift has $\phi^f = u_2$. Choose scalar 1 at u_2. If $r \equiv 3 \bmod 4$, take the reflection matrices on the E_j-factors for ϕ. If $r \equiv 1 \bmod 4$, set $B_C = 1$. The operator B_C is rational with $B_C^f = 1$, so the relation $\phi^f = u_2$ holds on W_χ and the normalized extension. Moreover, D_b satisfies $D_bC_O = C_OD_b$, $D_b\phi = \phi D_b$ and belongs to O. Thus, we have that D_b fixes the extension to $O \rtimes \langle\phi\rangle$, which is rational. Since $\nu(-I) = -1$, either scalar on $-I$ is available. If $2 \mid f$, take the twist $\nu\det^j$ with $(\nu\det^j)(u_2) = 1$ and value 1 at ϕ. If $2 \nmid f$, choose ν.

If $s = m = 1$, the Jordan sizes are $5, 1$ and the regular dual centralizer of s is a torus. Thus, its character $\widetilde{\chi}$ has multiplicity 1 at θ by [72, §3.4]. The orbit $(U, \theta)^\Omega$ has diagonal stabilizer 1 because $u_1u_2 = -I$. By [208, Proposition 7.35, Step 1], the diagonal quotient indexes both $(U, \theta)^\Omega$ and the constituents χ. Thus, each χ has its marked line W_χ, with $K = 3$ in the singleton argument. If $s = 1$ and $m = 0$, Lemma 6.57 gives the field model for $\eta_1 = \eta_2$. If $\eta_1 \neq \eta_2$, take the model of Lemma 6.88 on O.

Given a two-hook $\lambda \mapsto \mu$, choose the mixed plane W_{dom} from [208, Proposition 7.21]. Its torus T has weight 1 by Proposition 6.46. Moreover, $\langle z_0\rangle$ acts by $+1$ on both derived factors. Thus, $\mathrm{PGL}_2(q) \times O_R$ is the parent quotient. The conditions $\det_R = 1$ and $\nu_1\nu_R = 1$ give the spin image, with ν_1 the determinant square-class character of $\mathrm{PGL}_2(q)$. The scalar on the extracted $-I_4$ is 1, so ω has the scalar of ω_R. On $\mathrm{PGL}_2(q)$, take the quadratic principal pair for $q \equiv 1 \bmod 4$ and the quadratic cuspidal pair for $q \equiv -1 \bmod 4$. By Lemma 6.44, the parent of sign $+1$ has norm operator J with ratio $\eta_q(b_\sigma)$, and fixed field operator B_1.

If the residual self-twists form $\langle\nu_R\rangle$, then only $1, \nu_1\nu_R$ fix the tensor on $\mathrm{PGL}_2(q) \times O_R$. Thus, by [43], its restriction has norm 2, and $J \otimes H_\mu$ has irreducible eigenspaces of signs ± 1. The orthogonal parabolic maps onto O/Ω. Hence, [95, Problem 5.2] on Ω gives the inductions from $J \otimes H_\mu$-eigenspaces. By [208, Proposition 7.21], the selected Ω-constituents $\chi^\pm$ each have coefficient 1. Since the coset $\nu = -1$ exchanges χ^+, χ^-, their sum gives an irreducible O-parent with quotient self-twists $\langle\nu\rangle$. On Ind in Section 6.2, the tensor operators give B_λ, H_λ. Thus, the ratio for H_λ is that for H_μ times $\eta_q(b_\sigma)$, while the ratio for B_λ is 1. By Proposition 6.46, the flag $\mathcal{F}$ gives the selected pair for mixed stabilizer operations (g, σ). Since the staircase has exponent s, the j_λ steps give exponent $s + j_\lambda$. If $s = 1$ and $\eta_1 = \eta_2$, the staircase exponent is 1 by $K \equiv 1 \bmod 2$.

If $s = 1$ and $\eta_1 \neq \eta_2$, only 1 fixes the staircase parent at δ_m. Thus, any twist fixing the O_R-factor equals 1. Hence, the tensor restriction to Ω is irreducible. By [208, Proposition 7.21], the selected Ω-constituent has multiplicity 1. Thus, the transferred field operator B_λ has coefficient ratio 1. Moreover, the ambient $-I$ has the residual scalar. At rank 4, the plane W_{dom} gives rational flags $\mathcal{F}, \mathcal{F}^\tau$. □

Lemma 6.54. *Suppose that the orthogonal rank is* 0 *and the partition rank is* $c > 0$. *Let the* $\mathrm{SO}_{2c}(q)$*-character have connected dual centralizer* $C(s)$*, with rational points* $\mathrm{GL}_c(q)$ *for* $q \equiv 1 \bmod 4$ *and* $\mathrm{GU}_c(q)$ *for* $q \equiv -1 \bmod 4$. *Then, its orthogonal parent* $\Phi_\lambda^{\mathrm{pure}}$ *has an involution* H_λ *for the norm character* ν*, with coefficient ratio* $\eta_q(b_\sigma)^{j_\lambda}$. *The rational field extension has operator* B_λ *with* $B_\lambda H_\lambda = H_\lambda B_\lambda$. *Moreover, at* $-I$*, we obtain scalar* $(-1)^{c(q^2-1)/8}$.

Proof. Write $O = O(V)$, $G = \mathrm{SO}(V)$ and let s be the dual point for G. Since the full orthogonal centralizer of s lies in G, the O-conjugate G-series are distinct. Thus, $\Phi_\lambda^{\mathrm{pure}}$ is induced from G. Its twists by det and ν fix $\Phi_\lambda^{\mathrm{pure}}$. Multiplication of s by $-I$ fixes each G-class for even c and exchanges the two G-classes for odd c. First, let $\lambda = \delta_m$, and put $h = \lceil m/2 \rceil$ and $d_j = 2m - 1 - 4j$ for $0 \le j < h$. If $m = 1$, the source is $\mathrm{Ind}_{T_0}^{O(E_0)} \psi_0$ for the quartic rotation character ψ_0 of T_0. If $m \ge 2$, Proposition 6.32 gives Jordan size d_j with multiplicity 2 by truncated induction from δ_m. Since the Weyl label is δ_m in both parities of c, the calculation covers every c. In the regular overgroup of G, the component group on the h planes has order 2^{h-1}. Indeed, the plane determinants multiply to 1, and each plane scalar aI_{E_j} has determinant 1. The lowest degree coefficient for δ_m has absolute value 1. Thus, [208, Lemma 7.24] gives total multiplicity 2^{h-1} at the pairs (X, γ). Since the dual centralizer of s in G^* is connected, the character $\widetilde{\chi}$ restricts irreducibly to G.

On the h planes E_j, take rotation tori T_j, quartic characters ψ_j, and $T = \prod_j T_j$. Let W_λ be the θ_X-space in one G-series. If $z \in T$ is regular, then $z_j \notin \{\pm 1\}$ and $z_i \ne z_j^{\pm 1}$ for $i \ne j$. The Gelfand–Graev average on $C_G(z)$ gives $\mathrm{Tr}(z|W_\lambda)$ by [208, Proposition 7.36, Step 5]. When $q \equiv 1 \bmod 4$, take reference element $+1$ in the signed permutation group. If $q \equiv -1 \bmod 4$, take reference element -1, in the Frobenius coset for odd c, and in $W(D_c)$ for even c. Thus, on $\prod_j \mathfrak{S}_{d_j}$ the split-rank exponent for $q \equiv 1 \bmod 4$ counts all cycles. If $q \equiv -1 \bmod 4$, it counts even cycles. Thus, at $w \in \prod_j \mathfrak{S}_{d_j}$, the factor is $\mathrm{sgn}(w)$ times a common sign.

If $l_j + k_j = d_j$, the coefficient up to sign is $[s_{\delta_m}] \prod_j e_{l_j} e_{k_j}$, with e_i the elementary symmetric function of degree i. Moreover, the parity of $\sum_j k_j$ selects the G-class. By [208, Proposition 7.36, Step 5], vertical Pieri gives $\{l_0, k_0\} = \{m, m-1\}$ and remaining shape δ_{m-2}. Thus, $\{l_j, k_j\} = \{m - 2j, m - 2j - 1\}$ and the coefficient is 1. On T, the characters $\prod_j \psi_j^{l_j - k_j}$ are distinct even for opposite tuples $(k_j, l_j)_j$. Hence, 2^{h-1} occur for each parity of $\sum_j k_j$.

Let F_T be the difference between $\mathrm{Tr}(z|W_\lambda)$ and the signed sum of $\prod_j \psi_j^{l_j - k_j}(z)$. Then, $|\operatorname{supp} \widehat{F}_T| \le 2^h$ since $\dim W_\lambda \le 2^{h-1}$. Over $\mathbb{F}_{q^b}$ with odd $b \to \infty$, the exceptional tuples $z_i = \pm 1$ or $z_i = z_j^{\pm 1}$ have density tending to 0. By the Fourier bound in [208, Proposition 7.36, Step 5], if $F_T \ne 0$, then $|\operatorname{supp} F_T| \, |\operatorname{supp} \widehat{F}_T| \ge |T|$. Thus, $F_T = 0$ for sufficiently large b, with sign $+1$ by nonnegative weight multiplicities.

Next, for $z \in T$ over $\mathbb{F}_q$, take the trace with index $u \in C_U(z)$ in the regular overgroup. Since z lies in a Levi torus of L, the algebraic group $C_U(z)$ is connected by [58, Theorem 2.3.1(v)]. Choose $b \equiv 1 \bmod M$ with $4r \mid M$ and $|T_j| \mid M$ for all j. By [208, Proposition 7.36, Step 5], include in M the orders of aF_A for component elements $a \in A$ and of wF for $w \in W(D_c)$. Thus, the forms remain over $\mathbb{F}_q$, and the extensions of θ_X, ψ_j agree on $\mathbb{F}_q$ with their original values. By [208, Proposition 7.30(a) and Corollary 7.33(a)], $F_T(z)$ is a quotient of sums $\sum_i a_i \alpha_i^b$ with denominator $\ne 0$. Since $F_T(z) = 0$ for large b in the progression, [208, Lemma 7.27(ii)] gives zero at $b = 1$. The trace at $z = 1$ reaches the multiplicity 2^{h-1}. The two graph-conjugate G-series give opposite parities of $\sum_j k_j$, exchanged by a reflection on E_j. Thus, for $C_O = T \rtimes C_2^h$, the θ_X-space of $\Phi_{\delta_m}^{\mathrm{pure}}$ affords $\mathrm{Ind}_T^{C_O}(\prod_j \psi_j)$ with multiplicity 1 on UC_O.

Choose reflections r_j over $\mathbb{F}_r$ with $\nu(r_j) = 1$. Each r_j swaps the two weights ψ_j, ψ_j^{-1}. Hence, the product J_C of the swaps implements ν, with $J_C^2 = 1$ and entries in $\mathbb{Q}$. Given $q \equiv 1 \bmod 4$, the field operator is $J_C^{(r-1)/2}$ and its f-th power is 1 since $q \equiv 1 \bmod 4$. If $q \equiv -1 \bmod 4$, the exponent on T is $-r \equiv 1 \bmod 4$, so the field operator is 1. Conjugation on i is represented by J_C. The half-grading D_{b_σ} centralizes C_O and has norm 1 since each d_j occurs twice. Moreover, the actions σ, ϕ act by $s \mapsto \pm s$ with δ_m fixed. Thus, both fix $\Phi_{\delta_m}^{\mathrm{pure}}$. On its multiplicity-one space W_λ for UC_O, normalize the field operator and H_λ by J_C and its field counterpart on W_λ. By Schur's lemma, the relations on W_λ give those on $\Phi_{\delta_m}^{\mathrm{pure}}$. In particular, $-I$ acts by $\prod_j \psi_j(-1) = (-1)^{c(q^2-1)/8}$, since $h \equiv c \bmod 2$. If $m = 1$, these operators on E_0 give the rank-one model.

If $c \ge 3$, take a two-hook step $\lambda \mapsto \mu$. By [208, Proposition 7.21], the coefficient is 1 on each G-series after the graph action. Thus, $\Phi_\lambda^{\mathrm{pure}}$ occurs with multiplicity 1. The induced operator H_λ comes from $J \otimes H_\mu$ by Lemma 6.53. Hence, the ratio for H_λ is the ratio for H_μ times $\eta_q(b_\sigma)$. Thus, induction on O gives the ratio for odd c. If $\mathrm{core}_2(\lambda) = \varnothing$, the first positive rank is $c = 2$. Put $A_1 = \mathrm{PGL}_2(q)$, $A_0 = \mathrm{PSL}_2(q)$ and $D_0 = \{(x, y) \in A_1^2 : \nu_1(x) = \nu_1(y)\}$. Then, $D_0 \cong \mathrm{SO}_4^+(q)/\langle -I \rangle$. Take the rank-one parent Θ and put $u = 1$ for $\lambda = (2)$ and $u = \mathrm{St}$ for $\lambda = (1, 1)$. Since $u\nu_1 \ne u$, [43] gives that $(\Theta \otimes u)|_{D_0}$ is irreducible. Let $\xi_\pm$ be the constituents of $\Theta|_{A_0}$. On A_0^2, the constituents $\xi_\pm \otimes u$ are distinct from $u \otimes \xi_\pm$ because $\xi_\pm \ne u|_{A_0}$. Thus, induction to $D_0 \rtimes C_2$ is irreducible. On $\mathrm{M}_2(\mathbb{F}_q)$ with its determinant form, $\gamma_{\mathrm{ad}} : X \mapsto \mathrm{adj}(X)$ exchanges the factors over $\mathbb{F}_r$. Moreover, $\det(\gamma_{\mathrm{ad}}) = -1$ and $\nu(\gamma_{\mathrm{ad}}) = 1$. In each D_0-coset, choose J on Θ and the flag operator on u. Thus, the extension to $\langle \phi \rangle$ is rational, H_λ has ratio $\eta_q(b_\sigma)$, and $-I$ acts by

1. This gives $j_\lambda = 1$ at $c = 2$. Each hyperbolic four-space gives $c \mapsto c - 2$, so the scalar at $-I$ is constant. Finally, at $c = 0$, the residual group is 1. □

Proposition 6.55. *Let $N = \mathrm{Spin}^\epsilon_{2n}(q)$ with $q = r^f$ odd, $n = 2s^2 + c \geq 5$ and $s, c \geq 1$. Let $p \neq r$ be an odd prime dividing $|N|$, with characteristic toral base Q_0 in $P_p \in \mathrm{Syl}_p(N)$. In the separately rational type $Q(s^2, c)$, suppose that the labels Λ_1, Λ_2 are cuspidal. If $s = 1$, take torus labels for η_1, η_2. Then, for each eligible $\lambda \vdash c$, there is an equivariant comparison f_{sep} at $M = N_N(Q_0)$ with relation $\succeq_c$ on the stabilizers of χ, ψ in* Definition 2.1. *Moreover, we have that $\psi = f_{\mathrm{sep}}(\chi)$ determines the rational series of χ and λ, including η_1, η_2 at $s = 1$.*

Proof. Put $d = \mathrm{ord}_p(q)$, $e = \mathrm{ord}_p(q^2)$, and $q^e \equiv t \bmod p$ with $t \in \{1, -1\}$. Thus, $(e, t) = (d, 1)$ when $2 \nmid d$ and $(e, t) = (d/2, -1)$ when $2 \mid d$. Given type A period e, we write $\rho = \mathrm{core}_e(\lambda)$, $c_0 = |\rho|$ and $c = c_0 + ew$, with quotient character η_λ. If $s \geq 2$, we apply Lemma 6.96 to $k\, v_p(q^{2e} - 1) + v_p(k!)$ and Lemma 6.59 to the symbol degree $u_\Lambda(1)$.

(1) Suppose that $q \equiv 1 \bmod 4$ and $d = e$ is odd. Since Λ_i has no ordinary hooks and each $q^k + 1$ is a p-unit, both factors have p'-order. Hence, $w > 0$, $2s^2 + c_0 < e$ and $p \nmid \eta_\lambda(1)$. If $2s^2 + c_0 = e$, the residual sign $+1$ gives an ambient block of rank e.

(2) Suppose that $q \equiv -1 \bmod 4$, $d = 2e$ and e is odd. Put $r_0 = \mathrm{dist}(s, e\mathbb{Z})$ and $u = (s^2 - r_0^2)/e$, with cohook quotient η_O of size u. Thus, the conditions are $2u + w > 0$, $2r_0^2 + c_0 < e$ and $p \nmid \frac{(2u+w)!}{u!^2 w!} \eta_O(1)^2 \eta_\lambda(1)$.
If $\Lambda_i = (\{0, \ldots, 2s - 1\}, \varnothing)$, by [115, §3.3], we obtain $2s - k$ cohooks of length $k < 2s$. Since $\mathrm{ord}_p(q) = 2e$, the contributing lengths are $k = ej$ with odd j. By Lemma 6.59, the runners have size $u = \sum_{j>0,\ j\ \mathrm{odd}} (2s - ej)_+$, where $x_+ = \max(x, 0)$. If $2s = 2be + l$ with $0 \leq l < 2e$, the packed lengths are $be + \min(l, e)$ and $be + (l - e)_+$. Thus, their difference is $\min(l, 2e - l) = 2r_0$, so the core $\Lambda_{i,0}$ has rank r_0^2 and is empty at $r_0 = 0$. Moreover, the runner formula gives $\eta_O(1) = u! / \prod_{j>0,\ j\ \mathrm{odd}} j^{(2s-ej)_+}$. Given $\mathrm{GL}^t_c(q)$, the valuation is $v_p(u_\lambda(1))$ of Lemma 6.45. Thus, at $2u + w$ blocks leaves the degrees $\eta_O(1)^2, \eta_\lambda(1)$ and the index $(2u + w)!/(u!^2 w!)$. If $2r_0^2 + c_0 = e$, c_0 is odd and the sign -1 admits another block of rank e.

(3) Suppose that $d = 2e$ with $q \equiv 1 \bmod 4$ or $2 \mid e$. A type A block has rank $2e$ and takes two ambient ranks $e + e$. Thus, $\lambda \vdash c$ lies on the residue and $w = 0$. With r_0, u as in (2), the conditions are $u > 0$, $2r_0^2 + c \leq e$ and $p \nmid \binom{2u}{u} \eta_O(1)^2$. If $2r_0^2 + c = e$, the residual sign is $+1$, including the unitary case since $2 \mid e, c$. Since $c \leq e < 2e$, every $\lambda \vdash c$ is eligible at period $2e$.

With active $\mathrm{GL}^t_c(q)$-blocks, $c_0 = c \bmod e$ by the degree comparison. If $s = 1$ and $e \geq 3$ is odd, the type A sign is t and $w > 0$ with $p \nmid \eta_\lambda(1)$. Put $R = 2 + c_0$. Then, $R < e$ or $R = e$ with $\eta_1 \eta_2 t^{c_0} = -t$, so both rank-one tori have p'-order. If $2 \mid e$, the rank-one factors and $\mathrm{GL}^t_c(q)$ cannot give the blocks of rank e. If $e = 1$, the type A period is 1. Let k count the signs $\eta_i = t$. Then, $k \in \{1, 2\}$ and $p \nmid ((c + k)!/c!)\, f^\lambda$. Indeed, $(c + k)!/c!$ counts the k separate torus letters and f^λ is the Specht degree for λ.

In (1), (2) and the permitted cases of $s = 1$, $t = \eta_q(-1)$. Thus, quadratic reciprocity gives $(p/r)^f = (t\eta_q(-1))^{(p-1)/2} = 1$. Hence, $\eta_q(b_\sigma) = 1$ for $\sigma \in \mathcal{H}_p$, also in (2) when $w = 0$. Thus, Lemmas 6.53 and 6.54 give ratio 1. The normalizer takes η_λ and the residual parent Θ_R. If $e = 1$, the field action on S^λ is rational. Moreover, $\det_A \det_R = 1$ and $\nu_A \nu_R = 1$ are the kernel conditions.

If (3) holds, λ is a label of Θ_R. If $r_0 > 0$, Lemma 6.53 gives its norm sign $\eta_q(b_\sigma)^{r_0 + j_\lambda}$. If $r_0 = 0$, Lemma 6.54 gives ratio $\eta_q(b_\sigma)^{j_\lambda}$. On the negative cycles of Π_A, the scalar has square $(-\eta_q(-1))^{s-r_0}$. Thus, the reciprocity choice gives factor $\eta_q(b_\sigma)^{s-r_0}$, giving global ratio $\eta_q(b_\sigma)^{s+j_\lambda}$. The form twist scalar is $z_v \in \mu_{2e}$. Since $d = 2e$, the f-th powers of $\mu_{\mathrm{ord}_p(r)}$ form μ_{2e}. Thus, the field matrix has an $\mathcal{H}_p$-fixed scalar with power z_v^{-1}. Since $\det_A$ moves Π_A, its counterpart on Θ_R adds no twist fixing $\Pi_A \otimes \Theta_R$. Take a rotation v_σ with norm $\eta_q(b_\sigma)^{s+j_\lambda}$ on the eigenspaces. Thus, $m_N = m_M$ on the ordinary inertias.

Write $O_A = N_{O(V_A)}(\pi(Q_0))$ on the active space and $O_R = O(V_R)$ on $V_R = V^{Q_0}$ for the spin map π. Since the natural normalizer permutes signed Q_0-weights and fixes V_R, it lies in $O_A \times O_R$. Conversely, Section 6.3 gives O_A by rotations and signed permutations. Since Q_0 is the odd-order subgroup over $\pi(Q_0)$, the image $M/\langle z_0 \rangle$ is the joint kernel of $\det_A \det_R, \nu_A \nu_R$ in $O_A \times O_R$. Write $\mathcal{Q}_{\det,\nu} = (O_A \times O_R)/(M/\langle z_0 \rangle)$. A rotation has $(\det, \nu) = (1, -1)$, while a signed reflection has $\det = -1$. Thus, $|\mathcal{Q}_{\det,\nu}| = 4$.

For equal cores of rank r_0^2, Lemma 6.53 gives Θ_R with $r_0 = s$ if $t = 1$. Take partition ρ for active $\mathrm{GL}^t_c(q)$ and λ otherwise. If $r_0 = 0$ with positive partition rank, take Θ_R from Lemma 6.54. If $V_R = 0$, set $\Theta_R = 1$. If $w = 0$, Lemma 6.61 gives Π_A on $2u$ blocks with η_O on trivial colours and $\eta_O\, \nu_A$ on quadratic colours. Then, $J_A^2 = \kappa^u$, where a block transposition has norm κ. If $w > 0$, Lemma 6.62 adds w quartic colours with label η_λ and gives $J_A^2 = 1$. Thus,

both constructions give Π_A on signed cycles. If $e = 1$ with two active sign planes, take $u = 1$ and $w = c$. Both sign planes have label 1, normalized by ν_A on the plane of colour $\nu_A|_T$. Moreover, the quartic label is λ of $\mathfrak{S}_c$.

The self-twist group of $\Pi_A \otimes \Theta_R$ in $\mathrm{Irr}(\mathcal{Q}_{\det,\nu})$ is $\langle \nu_A \nu_R \rangle$. If $r_0 > 0$, the other two twists move Θ_R. If $r_0 = 0$, then $\det_A$ moves Π_A. Indeed, equal runner labels for Λ_i reconstruct equal rows by Lemma 6.59. At $e = 1$ with two sign planes, $\det_A$ changes their cyclic labels 1. Thus, by [43], the constituents on $M/\langle z_0 \rangle$ are the eigenspaces of $H_{\mathrm{loc}} = (J_A \otimes H_R)/\sqrt{J_A^2}$, with $H_R = 1$ if $V_R = 0$. With that root of J_A^2, the field matrix $B_A \otimes B_R$ fixes $\ker(H_{\mathrm{loc}} \mp I)$. Moreover, Lemmas 6.61 and 6.62 give $(B_A \otimes B_R)^f$.

If $\sigma \in \mathcal{H}_p$, both normalized involutions have ratio $\eta_q(b_\sigma)^{s+j_\lambda}$. Thus, we choose v_σ with $\det(v_\sigma) = 1$ and $\nu(v_\sigma) = \eta_q(b_\sigma)^{s+j_\lambda}$. On the ordinary inertias, $\widetilde{\chi}^\sigma = \widetilde{\chi}^{v_\sigma}$ and $\widetilde{\psi}^\sigma = \widetilde{\psi}^{v_\sigma}$. Since σ fixes both parent extensions, the cosets $g\langle\phi\rangle$ satisfy these identities. If a similitude y has nonsquare multiplier, then $\nu^y = \nu \det$ by the formula on reflections r_j. Thus, the inertias $\ker \nu, \ker(\nu \det)$ distinguish the two orientations. Conjugate $\widetilde{\chi}, \widetilde{\psi}$ together by that similitude.

Suppose that $s = 1$ and $\eta_1 \neq \eta_2$. If $e \geq 3$, only the w quartic slots are active. Only 1 in $\langle \det_R, \nu_R \rangle$ fixes Θ_R by Lemma 6.53. Thus, $\Pi_A \otimes \Theta_R$ restricts irreducibly to $M/\langle z_0 \rangle$. Its other orientation is distinct on Ω_R, with Lemma 6.88 for $c_0 = 0$. Moreover, $\mathcal{H}_p$ fixes both field extensions. If $e = 1$, we let V_1 be the sign plane of colour 1. The other c active planes have quartic colours ψ_j, and V_R has type $-t$. Moreover, take cyclic label 1 on V_1 and label λ on $\mathfrak{S}_c$. On V_R, the character $\nu_R \det_R^j$ has quadratic torus colour. Thus, $\Pi_A(1) = 2^c(c+1)f^\lambda$. The twist $\det_A$ changes the cyclic label of V_1, and ν_A changes its colour 1. Thus, $(\Pi_A \otimes \Theta_R)|_{M/\langle z_0 \rangle}$ is irreducible. By Lemma 6.62, inversion of ψ_j implements σ, ϕ on i and fix V_1's label. A nonsquare similitude gives $\Theta_R \det_R$. The compensating twist $\det_A \det_R$ changes the cyclic label on V_1, so the $e = 1$ branches are distinct.

Given b, choose the lift over $\mathbb{F}_r$ with cycles R_i from Lemmas 6.61 and 6.62. Its twist v has a spin lift over $\mathbb{F}_r$. If $2 \mid f$ and the forms over $\mathbb{F}_r$ differ, correct the field model by $a \in O(V)$ over $\mathbb{F}_r$ of order 2, with $\det(a) = -1$ and $\nu(a) = 1$. If the residue has staircase δ_m, set $a = u_i$ on singleton i. If it has pure partition ρ, set $a = r_j$ on a plane. At rank 2, set $a = \gamma_{\mathrm{ad}}$ by Lemma 6.54. On each restored plane E, put $a|_E = I$. If V_R has neither source, we choose a of norm 1 on a positive block of colour 1. For opposite types at $e = 1$, we set $a = a_A v_R$ for the active reflection a_A and twist v_R with $(\nu_R \det_R^j)(v_R) = 1$. Let A_ϕ be the field matrix and A_a the matrix of a. Since $\phi(a) = a$ and $av = va$, $(A_\phi A_a)^f = A_\phi^f$. Moreover, $\nu(a) = 1$ fixes both eigenspaces. If f is odd, descent to $\mathbb{F}_r$ gives equal square classes.

If $2 \mid f$ and $\eta_1 \neq \eta_2$, both form twists include u_i with $\nu(u_i) = 1$ over $\mathbb{F}_r$. Thus, an isomorphism of the $\mathbb{F}_r$-spaces gives a relative twist in $\pi(\mathrm{Spin}(V)(\mathbb{F}_r))$. By Lang's theorem, we choose x as in Lemma 6.38 with $\mathrm{Int}(x)F = F' \mathrm{Int}(x)$. Then, the change from ϕ is $x\phi(x)^{-1}$. Since ϕ fixes both twists, $F'(x\phi(x)^{-1}) = x\phi(x)^{-1}$. Thus, the change is inner on N. A lift ambiguity in $\langle z_0 \rangle$ acts by 1.

If $\eta_1 = \eta_2$, the active scalar at $-I$ is κ^u for two colours and $(-1)^{w(q^e-t)/4}$ for three colours on Π_A. The product with the Θ_R-scalar gives [208, (7.25)]. Indeed, for $t = -1$, $eu = s^2 - r_0^2$ and $u \equiv s - r_0 \bmod 2$. In active partition period, odd e gives $q^e \equiv q \bmod 8$. If $s = 1$ and $\eta_1 = \eta_2$, take the scalar at $-I$ from Lemma 6.57 for each sign η_1. If $\eta_1 \neq \eta_2$, Lemma 6.53 permits matching scalars at $-I$. If $e = 1$, the quadratic residual colour has scalar -1. Moreover, all characters have scalar 1 at z_0.

A nonsquare multiplier moves each $\mathcal{H}_p$-orbit. For a similitude g of square multiplier $\mu(g)$, the isometry model is unique modulo $\{\pm I\}$. Indeed, Ω acts absolutely irreducibly on V, giving $C_{O(V)}(\Omega) = \{\pm I\}$. If a field power centralizes Ω, its field outer class is 1, with inner power b^f. On the cover of Ω, take a common section s_Ω and inflate over $\langle z_0 \rangle$. Its defects have equal scalars at $\pm I, z_0$. Hence, v_σ gives $m_N = m_M$ on both stabilizers $\ker \nu, \ker(\nu \det)$.

Finally, the quartic quotient gives the e-quotient of λ, or λ itself at $e = 1$. Moreover, inverse packing on $\rho, \Lambda_{i,0}$ and their runners gives λ, Λ_i. By Proposition 6.260, the active colours and V_R determine s, η_1, η_2 and the rational class of s_M. Changing the lift twists by $\nu_A \nu_R$, exchanging the sign labels ± 1. If two constituents occur, ϵ_ψ determines ψ. Thus, f_{sep} is bijective onto the marked M-set. □

Let $q = r^f$ for an odd prime r, and let η_q be the quadratic character of $\mathbb{F}_q^\times$, with $\delta = \eta_q(-1)$. In the separately rational type $Q(a, c)$, both orthogonal factors have nondegenerate label Λ. Moreover, the remaining rank c has label $\lambda \vdash c$ and $n = 2a + c$. If $c = 0$, we have that the type is $I(a, a)$. Keep $-I$ in $O(V)$. Moreover, at z_0 the scalar is 1. Let ζ_r have order r. Thus, $\sigma(\zeta_r) = \zeta_r^b$ defines $b \in \mathbb{F}_r^\times$ and $g_\sigma = \eta_q(b)$.

Lemma 6.56. *Suppose that the two occurrences of Λ both undergo the 1-hook step at h. Then, Θ_{pr} on the extracted $\mathrm{GL}_2(q)$ has rational field operator B and normalized intertwiner J for $\eta_q \circ \det$. Moreover, we have that $J^2 = I$, $BJ = JB$, and $J^\sigma = g_\sigma J$. The scalar at xI is $\eta_q(x)$.*

Proof. Take the functions F on $\mathbb{F}_q^2 \setminus \{0\}$ with $F(xv) = \eta_q(x)F(v)$. Thus, in a basis indexed by $\mathbb{P}^1(\mathbb{F}_q)$, we have that the action $(R(g)F)(v) = F(g^{-1}v)$ has signed permutation matrices. At $[v]$, its stabilizer acts by η_q on the first diagonal entry a_{11}. Hence, the induced principal series Θ_{pr} has torus character $(\eta_q, 1)$ for eigenvalues $-1, 1$ on the extracted plane. If $w \neq 1$ in the Weyl group, w moves $(\eta_q, 1)$. By [95, Lemma 5.2 and Problem 5.6], Θ_{pr} has norm 1.

Put $\eta_q(0) = 0$. Define $(AF)(v) = \sum_{[w]\in\mathbb{P}^1(\mathbb{F}_q)} \eta_q(\det(v,w))F(w)$. Since scaling w by x contributes $\eta_q(x)^2 = 1$, the operator A is well defined. Substitution $w = gu$ gives that $AR(g) = \eta_q(\det g)R(g)A$. The diagonal coefficient of A^2 at $[v]$ is $q\delta$, since each $[w] \neq [v]$ contributes $\eta_q(-\det(v,w)^2) = \delta$. If $[z] \neq [v]$, we choose basis v, z and take $w = v + tz$. Then, $\det(v,w)\det(w,z) = t\det(v,z)^2$, so the contributions are $\eta_q(t)$. The line $w = z$ has coefficient 0. Thus, $\sum_{t\in\mathbb{F}_q} \eta_q(t) = 0$ gives that $A^2 = \delta q I$.

For the absolute-trace character of $\mathbb{F}_q$, we let γ_q be the quadratic Gauss sum. By [44, Corollary 2.3], $\gamma_q^2 = \delta q$. Moreover, changing the additive argument by b gives that $\gamma_q^\sigma = g_\sigma \gamma_q$. Thus, $J = A/\gamma_q$ has $J^2 = I$. Take $(BF)(v) = F(F_r^{-1}v)$, so $B^f = I$. Since F_r permutes the lines and fixes η_q, $BA = AB$. Both B and $R(g)$ are rational. Hence, $BJ = JB$ and $J^\sigma = g_\sigma J$, and $R(xI) = \eta_q(x)I$. □

Lemma 6.57. *Given $I(1,1)$, suppose that $\eta_1 = \eta_2$. Then, the orthogonal parent Φ has self-twist group $\langle\nu\rangle$ in $\langle\det, \nu\rangle$. Its field extension $\widehat{\Phi}$ is Galois invariant, and H for ν has coefficient ratio g_σ. Moreover, the scalar on $-I_4$ is δ in the split case and $-\delta$ in the nonsplit case.*

Proof. On $\mathrm{M}_2(\mathbb{F}_q)$, put $Q(X) = \det X$. Thus, the action $(x,y)X = xXy^{-1}$ gives $\Omega = \Omega_4^+(q) \cong \mathrm{SL}_2(q)^2/\langle(-I,-I)\rangle$. By [119, §2.2], the Weil space W_τ for parity $\tau \in \{1,-1\}$ has dimension $(q+\tau)/2$, and $h(a)$ acts by $F(t) \mapsto \eta_q(a)F(at)$. Thus, the scalar of $-I$ is $\tau\delta$. The tensor $W_\tau \otimes W_\tau$ has scalar 1 at $(-I,-I)$, so it is an Ω-module. The image of $(-I, I)$ has scalar $\tau\delta$.

Set $gX = \mathrm{adj}(X)$, so $g^2 = 1$ and g exchanges the two $\mathrm{SL}_2(q)$-factors. The negative space $\ker(g+I)$ has orthogonal basis $\mathrm{diag}(1,-1), E_{12}+E_{21}, E_{12}-E_{21}$ of norms $-1,-1,1$. Thus, $\det(g) = -1$ and $\nu(g) = 1$. Put $K = \ker\nu = \Omega \rtimes \langle g\rangle$ in $O_4^+(q)$. Thus, the tensor extends to K by the flip at g. On W_τ, let P act by $F(t) \mapsto F(F_r^{-1}t)$. Under F_r, we have that the additive character θ satisfies $\theta(t^r) = \theta(t)$, so the Fourier kernel $\theta(tu)$ is fixed under simultaneous Frobenius. Also, $\eta_q(a^r) = \eta_q(a)$. Thus, $B_0 = P \otimes P$ implements F_r, fixes the flip and satisfies $B_0^f = I$.

For b from σ, take $D_b = \mathrm{diag}(b,1)$ and $d_b(X) = D_b X D_b^{-1}$. On $\langle E_{12}, E_{21}\rangle$, the action is $\mathrm{diag}(b, b^{-1})$, so $\det(d_b) = 1$ and $\nu(d_b) = \eta_q(b)$. Moreover, $d_b g = g d_b$ and $F_r(d_b) = d_b$. By [119, §2.2], conjugation by D_b acts by $u(t) \mapsto u(bt)$. Thus, σ conjugates the additive values of $u(t)$. The diagonal operator at $h(a)$ is fixed by σ. On the Weyl generator w, the Fourier argument gains b and the Gauss ratio is $\eta_q(b)$, from $h(b)$ in the conjugate of w. Hence, the extension model β has $\beta(x)^\sigma = \beta(d_b x d_b^{-1})$ for $x \in K$ and for the field generator F_r. Indeed, the matrices of g and B_0 are rational.

Given $t \neq 0$, the trace of $u(t)$ on W_τ is $(\eta_q(t)\gamma_q + \tau)/2$, by the projection $(I + \tau P_-)/2$, where $P_- F(t) = F(-t)$. If $a \in \mathbb{F}_q^\times$ has $\eta_q(a) = -1$, the trace changes under $t \mapsto at$. The difference on $(u(t), 1)$ remains after scalar multiplication by $\dim W_\tau$, including $q = 3$. Thus, the $\nu = -1$ coset moves the tensor character on Ω. By [43], induction of β from K to $O_4^+(q)$ is irreducible with two distinct Ω-constituents of multiplicity 1. Moreover, the det-twist gives the flip at g with factor -1. By Schur's lemma on Ω, this gives a different extension on K. Since the other K-coset changes the Ω-character, both det and $\det\nu$ move Φ.

On the induced functions $F(kx) = \beta(k)F(x)$, we set $(HF)(x) = \nu(x)F(x)$ and $(BF)(x) = B_0 F(F_r^{-1}x)$. Moreover, $(T_\sigma F)(x) = \sigma\big(F(d_b^{-1}x)\big)$. Since $\beta(x)^\sigma = \beta(d_b x d_b^{-1})$, the formula for T_σ acts on $\mathrm{Ind}_K^{O_4^+(q)}\beta$. In particular, T_σ centralizes right translation and satisfies $T_\sigma B = B T_\sigma$. Moreover, $T_\sigma H = g_\sigma H T_\sigma$ since $\nu(d_b) = g_\sigma$. Thus, $\widehat{\Phi}^\sigma = \widehat{\Phi}$ and the norm ratio is g_σ. □

Proposition 6.58. *Let Λ have rank a and absolute defect $2s$. Its parent $\Phi_{\Lambda,\lambda}$ in the norm orientation extends with B, H satisfying $T_\sigma H = g_\sigma^{a+j_\lambda} H T_\sigma$ and $BH = HB$, where $j_\lambda = (c - |\mathrm{core}_2(\lambda)|)/2$. Moreover, its quotient self-twists are $1, \nu$. With $r_q = (q-1)/2 \bmod 2$, we put $l_2 = (q^2-1)/8 \bmod 2$. Then, we obtain the central scalar $\kappa_\Lambda(-I) = (-1)^{s + a r_q + c l_2}$.*

Proof. By [208, Lemma 7.17(i)], a noncuspidal $\Lambda = (X, Y)$ of rank at least 2 has a one-hook h with nondegenerate successor Λ^-. Thus, the paired steps reach rank 1 or cuspidal rank s^2. If $s = 0$, the endpoint has rows $\{1\}, \{0\}$ up to shift. At cuspidal defect 2, the endpoint has rows $\{0,1\}, \varnothing$. If $c = 0$ at rank 1, Lemma 6.57 gives parent Φ for each torus type. If $c > 0$, the staircase source of Lemma 6.53 gives (X, γ) for torus signs η_1, η_2 at rank 1. Moreover, it gives (X, γ) at $s \geq 2$ and the operator for λ.

Given h, we let the Levi be $L = \mathrm{GL}_2(q) \times O_R$. On the dual plane, $g \in \mathrm{GL}_2(q)$ acts by $g^{-\mathsf{T}}$. Thus, on L, $\det = \det_R$ and $\nu = (\eta_q \circ \det_{\mathrm{GL}_2})\nu_R$. For the $\mathrm{GL}_2(q)$-factor, choose Θ_{pr} of Lemma 6.56. Since the residual parent Θ_R has self-twists $1, \nu_R$, the tensor $\Theta_{\mathrm{pr}} \otimes \Theta_R$ has self-twists $1, \nu$, with intertwiner $J \otimes H_R$. Hence, [43] gives that its restriction to $\ker\det \cap \ker\nu$ has norm 2 and two constituents of multiplicity 1.

By [208, Proposition 7.18, Step 1], opposite lines in V_+, V_- give opposite planes E, E'. Thus, the quartic label stays λ and the two coefficients give $1 \cdot 1 = 1$. By [208, Proposition 7.18, Steps 2–3], the graph component has representative $r_+ r_-$ on the complements, and the swap $V_+ \leftrightarrow V_-$ fixes the paired hook h. Thus, the labels $(\Lambda, \lambda), (\Lambda^-, \lambda)$ have matched stabilizers on N, L. Their central commutators agree. At x in the component group, conjugacy of (E, E') gives a correction. Hence, $\mathcal{B}_h$ is a permutation matrix by [208, Lemma 7.16 and Proposition 7.18]. If $I(2,2)$ occurs, the four A_1-labels are all 1 or all St. Then, by [208, (7.45) and Proposition 7.41, Step 2], take $e_1 + e_3$ and $\mathcal{B}_h$. Thus, the graph fixes the $e_1 + e_3$-flag. The selected constituents χ^+, χ^- form one orbit under $O(V)$, so [43] gives the irreducible $\mathcal{V}_{\Lambda,\lambda}$ with multiplicity 1 at $\chi^\pm$. The restriction has norm 2 and the norm operator is H.

By Lemma 6.42, induction on $\mathcal{V}_{\Lambda,\lambda}$ gives the source relations for B, H, including B^f. Moreover, the ratio gains g_σ from J. The source volume is $(-I_2, -I_R)$, so the scalar gains δ per hook h. At rank 1, Lemma 6.57 and [208, (7.25)] give scalar $(-1)^{s+r_q+cl_2}$. Thus, the initial norm exponent $1 + j_\lambda$ gains $a - 1$. If $s \geq 2$, the initial exponent $s + j_\lambda$ gains $a - s^2$ and the initial scalar is $(-1)^{s+s^2 r_q + cl_2}$. Since $s^2 \equiv s \bmod 2$, both chains give $g_\sigma^{a+j_\lambda}$ and $\kappa_\Lambda(-I) = (-1)^{s+ar_q+cl_2}$. □

If a similitude h_m has nonsquare multiplier m, it acts by $\nu \mapsto \nu\det$. Indeed, every reflection r_v contributes the factor $\eta_q(m)$, and their determinants multiply to det. At rank 1, let ξ generate $\mathbb{F}_q^\times$. Left multiplication by $\mathrm{diag}(\xi, 1)$ changes the additive class on W_τ in the first factor by ξ. Thus, the choices $(1,1)$ and $(\xi,1)$ give distinct $O_4^+(q)$-orbits. Under $\mathcal{B}_h$, the image orbits are distinct. Put $a' = \xi^{(1-r)/2}$ and $L_{a'}X = \mathrm{diag}(a', a'^{-1})X$. Then, the conjugate field action is $\mathrm{Int}(L_{a'})F_r$, since the ratio to $\mathrm{diag}(\xi^{1-r}, 1)$ is central. In the field cover, $(L_{a'}F_r)^f = L_{(a')^{1+r+\cdots+r^{f-1}}} = -I_4$. Thus, the transported generator has f-th power scalar $\kappa_\Lambda(-I)$.

Let $p \neq r$ be an odd prime. Put $d = \mathrm{ord}_p(q)$, $e = d/\gcd(d,2)$, $t = (-1)^{d+1}$ and $h_p = v_p(q^e - t)$. Thus, the block of rank e has torus order $q^e - t$. Let $m(a,\epsilon)$ count such blocks at rank a and sign ϵ. Then, we have that $m(a,\epsilon) = \lfloor a/e \rfloor$, with one fewer if $a > 0$, $e \mid a$ and $\epsilon \neq t^{a/e}$. Indeed, a residue of rank $R > 0$ has either sign. If $R = 0$, we have that its sign is $+1$, so $m(0,+1) = 0$. By the orthogonal order formula (Carter [40]), $v_p(|\mathrm{SO}^\epsilon_{2a}(q)|) = mh_p + v_p(m!)$ for $m = m(a,\epsilon)$. Moreover, $q^{2i} - 1$ contributes $h_p + v_p(i/e)$ when $e \mid i$ for $i < a$. Thus, the last contribution comes from $q^a - \epsilon$ if $e \mid a$ and $\epsilon = t^{a/e}$.

Lemma 6.59. *Let Λ be an ordered symbol of rank a. Thus, packing the $2e$ runners gives the core Λ_0 of rank a_0, with e-hooks for $t = 1$ and e-cohooks for $t = -1$. Let u be the runner partition size and $\eta_O \in \mathrm{Irr}(C_{2e} \wr \mathfrak{S}_u)$ its character. Then, we obtain that $a = a_0 + eu$, and the unipotent degree $u_\Lambda(1)$ satisfies $v_p(u_\Lambda(1)) = (m-u)h_p + v_p(m!/u!) + v_p(\eta_O(1))$, where $m = m(a, \epsilon(\Lambda)) = u + m(a_0, \epsilon(\Lambda_0))$. If Λ is degenerate, the formula holds for u_Λ^+, u_Λ^-.*

Proof. Number the rows by $0, 1$. For $0 \leq i < e$ and $j \in \{0,1\}$, we let B_{ij} contain $k \geq 0$ if $i + ek$ lies in row j for $t = 1$, and in row $j + k \bmod 2$ for $t = -1$. If $B = \{b_1 > \cdots > b_l\}$, the partition has parts $b_i - (l - i)$. A move $k \mapsto k - 1$ into a vacant place is an e-hook if $t = 1$ and an e-cohook if $t = -1$. Thus, packing all B_{ij} takes u moves without changing their cardinalities, and $a - a_0 = eu$. If $t = 1$, each move leaves the defect of Λ fixed. If $t = -1$ and $\Lambda = (X, Y)$, each move changes $(|X| - |Y|)/2$ by 1 or -1. For absolute defects $2s, 2s_0$, this gives $s - s_0 \equiv u \bmod 2$. Hence, $\epsilon(\Lambda) = t^u \epsilon(\Lambda_0)$, and the endpoint condition for m gives $m(a, \epsilon(\Lambda)) = u + m(a_0, \epsilon(\Lambda_0))$.

By [115, §3.3], at $t = 1$ the contributing factors have hook length eh. If $t = -1$, length eh gives a cohook for odd h and a hook for even h. Thus, the pair lies on B_{ij} at distance h, a hook length of its partition λ_{ij} by Lemma 3.2. Hence, its valuation is $h_p + v_p(h)$ by the valuation formula. Write $H = \prod_{ij} \prod_{h \in \mathrm{Hook}(\lambda_{ij})} h$. The Specht hook formula gives degree $|\lambda_{ij}|! / \prod_{h \in \mathrm{Hook}(\lambda_{ij})} h$ on λ_{ij}. Thus, the Young subgroup index gives $\eta_O(1) = u!/H$. Hence, the denominator has valuation $uh_p + v_p(H)$. The other factors have valuation 0, including 2 at degenerate Λ. Thus, subtracting from $mh_p + v_p(m!)$ gives $v_p(u_\Lambda(1))$. In particular, we have that $v_p(u_\Lambda(1)) = 0$ exactly when the core group has p'-order and $p \nmid \eta_O(1)$. □

A row exchange acts by $B_{i0} \leftrightarrow B_{i1}$ for both choices of t. If the packed rows agree, then $|B_{i0}| = |B_{i1}|$ for every i, since the two indicators at $i + ek$ agree. Conversely, equal original B_{i0}, B_{i1} give equal packed rows. With $|B_{ij}|$ fixed, the beta-set formula gives B_{ij} from λ_{ij}. Thus, for degenerate Λ_0, the symbol Λ is degenerate exactly if $\lambda_{i0} = \lambda_{i1}$ for all i. Number the colours by $i + ej$. Then, the twist trivial on $\mathfrak{S}_u$ and equal to -1 on each cyclic generator

gives $k \mapsto k + e \bmod 2e$ on colours. Thus, for degenerate Λ_0, fixed labels have $\lambda_{i0} = \lambda_{i1}$ and give degenerate Λ. If $\Lambda_0 = (X_0, Y_0)$ has $X_0 \neq Y_0$, recover Λ with that row order.

Proposition 6.60. *Suppose that $p \mid |N|$. In separately rational type $Q(a,c)$, the partition factor has sign $\tau = \delta$, with $\tau = 1$ for the split form and $\tau = -1$ for the unitary form. If e is odd and $\tau = t$, put $\rho = \mathrm{core}_e(\lambda)$, $c_0 = |\rho|$, $c = c_0 + ew$ and let η_A be the quotient character. Otherwise, put $\rho = \lambda$, $c_0 = c$, $w = 0$ and $\eta_A = 1$. Take a_0, u, η_O from* Lemma 6.59*, and let $K = 2u + w$ and $R = 2a_0 + c_0$. Then, the degree is prime to p exactly when $K > 0$, $p \nmid \frac{K!}{u!^2 w!}\, \eta_O(1)^2 \eta_A(1)$, and either $R < e$ or $R = e$ with $\tau^{c_0} = -t$. Moreover, if the partition period is e, we find $R < e$.*

Proof. Given $\chi(1)$, the Jordan formula [40] cancels the r'-orders in the numerators of $u_\Lambda(1), u_\lambda(1)$. Since isogeny and component indices are 2-powers, the numerator contributes $v_p(|N|)$. For the orthogonal label Λ, Lemma 6.59 gives denominator valuation $uh_p + v_p(u!) - v_p(\eta_O(1))$. Given λ, put $h = \mathrm{ord}_p(\tau q)$, so $h = e$ if e is odd and $\tau = t$, and $h = 2e$ otherwise. A hook of length hj has valuation $h_p + v_p(j)$ since $v_p((\tau q)^h - 1) = h_p$. By Lemma 3.2, the weight w', core size $c' = c - hw'$, and quotient character η' give denominator valuation $w'h_p + v_p(w'!) - v_p(\eta'(1))$. Thus, with ambient count m_*, the valuation is $(m_* - 2u - w')h_p + v_p\big(\frac{m_*!}{u!^2 w'!}\big) + 2v_p(\eta_O(1)) + v_p(\eta'(1))$. If $h = e$, the three factors give $2u + w'$ blocks of rank e and sign t. The orthogonal residual signs multiply to 1, and the partition contributes $\tau^{c'}$. If $h = 2e$, each partition block has rank $2e$ and sign $\tau^{2e} = 1 = t^2$. Thus, the ambient count has $m_* \geq 2u + 2w'$ if $h = 2e$ and $m_* \geq 2u + w'$ always. Hence, every valuation contribution is ≥ 0. If $h = 2e$, a positive w' gives positive valuation. If $w' = 0$, the core is λ itself. Conversely, the stated bound on R gives $c \leq e < 2e$, so $w' = 0$ in the period $2e$ case. In both surviving cases, $m_* = K + m(R, \tau^{c_0})$. The cut for $Q(a,c)$ holds on the residue exactly when $m(R, \tau^{c_0}) = 0$. If $h = e$ and $R = e$, c_0 is odd and $\tau^{c_0} = t$, which contradicts the boundary condition. Thus, $R < e$ if $h = e$. Since p divides the ambient order, $m_* = K > 0$. Thus, the remaining valuation is $v_p(K!/(u!^2 w!)) + 2v_p(\eta_O(1)) + v_p(\eta_A(1))$. □

In the degree domain for $Q(a,c)$, we put $T = (C_{q^e - t})^K$, $W_k = C_{2e} \wr \mathfrak{S}_k$ and $O_A = T \rtimes W_K$ on the active space. If $t = 1$, we choose $E \oplus E^\vee$ with $E = \mathbb{F}_{q^e}$ and the trace pairing. The q-Frobenius on E and the swap of $E, E^\vee$ centralize each other. Thus, their product R_i has order $2e$ since e is odd. If $t = -1$, take the signed e-cycle R_i in a common cyclic orientation.

Let ν_A be the spinor norm on O_A. On each torus coordinate T_i, the restriction $\nu_A|_{T_i}$ is quadratic. If $t = 1$, we have that the class $\nu_A|_{E^\times}$ is given by $N_{E/\mathbb{F}_q}$ in the hyperbolic model. If $t = -1$, the covering torus has kernel μ_2. By Lang's theorem, we deduce that $H^1(F, \widehat{\mathbf{T}}_i) = 1$. Thus, the image of $\widehat{\mathbf{T}}_i^F$ in T_i has index 2. Thus, we have that $\nu_A|_{T_i}$ is quadratic. Given a block transposition s_{ij}, we set $\kappa = \nu_A(s_{ij})$. Its negative space $\{(v, -v) : v \in V_i\}$ has the form $2Q$. Since $\dim V_i = 2e$, the determinant class of $2Q$ is that of Q. Thus, we find $\kappa = \delta$ for split V_i with $t = 1$ and $\kappa = -\delta^e$ for nonsplit V_i with $t = -1$. If $t = 1$, the field permutation on E has determinant $(-1)^{e-1} = 1$ in a normal basis $(b_i)_i$. Moreover, the matrix $C = (b_i^{q^j})_{j,i}$ has $F_q(\det C) = \det C$ since the row cycle has sign $+1$. Thus, the trace determinant $(\det C)^2$ is square in $\mathbb{F}_q^\times$. The swap $E \leftrightarrow E^\vee$ has norm $\delta^e = \delta$. Hence, we find $\nu_A(R_i) = \delta$ for $t = 1$. If $t = -1$, we write $d_R = \nu_A(R_i)$ for the chosen negative block.

Lemma 6.61. *If $w = 0$, we choose u trivial coordinates and u quadratic coordinates. With $\alpha = \nu_A|_{W_u}$, its label pair is $\eta_O, \eta_O\alpha$ on $W_u \times W_u$. Then, the induced parent Π_A has a norm intertwiner J_A over $K_0 = \mathbb{Q}(\zeta_{2e})$ with $J_A^2 = \kappa^u I$ and $\Pi_A(-I) = \kappa^u I$. Moreover, we have that Π_A extends to b with $b^f = v^{-1}$ by a scalar fixed by $\mathcal{H}_p$.*

Proof. Given the character θ of T, the inertia is $I = T \rtimes (W_u \times W_u)$. Thus, extend θ trivially on $W_u \times W_u$ and tensor with $\eta_O \boxtimes (\eta_O\alpha)$ to obtain ρ. On the second W_u, the matrices are those of η_O multiplied by α. In particular, α on block permutations has value κ at a transposition. By [43], $\Pi_A = \mathrm{Ind}_I^{O_A} \rho$ is irreducible. Moreover, the labels on C_{2e} give a K_0-model with rational Specht matrices.

Let s_A exchange the two sets of u coordinates in order, and let F be the tensor flip. Thus, $\theta^{s_A} = \theta\, \nu_A|_T$. On the complement the flip gives α on both factors. Hence, for $h \in I$, $F\rho(s_A h s_A^{-1}) = \nu_A(h)\rho(h)F$. In the induced model $f(hx) = \rho(h)f(x)$, take $(J_A f)(x) = \nu_A(x)Ff(s_A x)$. Thus, by covariance, J_A acts on Π_A. Right translation gives the twist by ν_A and $J_A^2 = \nu_A(s_A)I = \kappa^u I$. Only quadratic colours contribute at $-I$, each with value $(-1)^{(q^e - t)/2} = \kappa$, with e odd if $t = 1$. Thus, $\Pi_A(-I) = \kappa^u I$.

If $t = 1$, we let v be the product of the field e-cycles c_i. Each c_i has determinant 1, with hyperbolic lift $\widetilde{c}_i$ over $\mathbb{F}_r$. If $t = -1$, we put $v = \prod_{i=1}^{2u} R_i$. A Clifford lift of R_i is $(e_1 + f_1)\widetilde{c}_i$ for a hyperbolic pair e_1, f_1 and the lift $\widetilde{c}_i$ of the e-cycle c_i. Its parity is 1 and its norm is $(-1)^{e-1}$. Hence, the $2u$ lifts give even parity and norm 1, with representatives

over $\mathbb{F}_r$. In the twisted form, take $b^f = v^{-1}$. Since F_r fixes both colours and W_K, begin with I on Π_A before the scalar for v^{-1}.

Let z_O be the scalar on η_O of the product of the positive e-cycles if $t = 1$, or negative cycles R_i if $t = -1$. This product is central in W_u. Since v centralizes the coset representatives in W_K, its scalar on Π_A is the product on $W_u \times W_u$. Thus, $\Pi_A(v) = z_v I$, where $z_v = z_O^2$ for $t = 1$ and $z_v = z_O^2 d_R^u$ for $t = -1$. Indeed, the second label includes α on the cyclic element, which is 1 on odd-order cycles and d_R^u on the negative product.

Put $D_r = \mathrm{ord}_p(r)$. Then, $x \mapsto x^f$ on μ_{D_r} has image of order $D_r/\gcd(D_r, f) = d$. Since $z_v \in \mu_e$ for $t = 1$ and $z_v \in \mu_{2e}$ for $t = -1$, choose $\alpha_f \in \mu_{D_r}$ with $\alpha_f^f = z_v^{-1}$. Thus, $B_A = \alpha_f I$ represents the field generator, with $B_A J_A = J_A B_A$. Both D_r and $2e$ divide $p - 1$, with $2e \mid p - 1$ for odd $e = d$ since p is odd. Hence, $\mathcal{H}_p$ fixes K_0 and α_f. □

Lemma 6.62. *Suppose that $w > 0$, so e is odd and $t = \delta$. Then, we have that O_A has an irreducible parent Π_A of degree $2^w \frac{K!}{u!^2 w!} \eta_O(1)^2 \eta_A(1)$, with norm involution J_A over K_0. Moreover, in the extension to b, each $\sigma \in \mathcal{H}_p$ has a semilinear operator T_σ with $T_\sigma J_A = J_A T_\sigma$ and $T_\sigma B_A = B_A T_\sigma$.*

Proof. On a torus coordinate T_i, let ψ have order 4. Give the sets of sizes u, u, w the colours $1, \psi^2, \psi$, respectively. If $t = 1$, the generators of W_K have norm 1 by the calculation of ν_A above. If $t = -1$, we choose R_i with its first reflection of norm 1. The e-cycle has norm 1 for odd e and a transposition has norm $-\delta^e = 1$. Thus, $\nu_A|_{W_K} = 1$. Since R_i inverts ψ, its stabilizing powers form $\langle R_i^2 \rangle$. Hence, the complement in the inertia I is $W_u \times W_u \times (C_e \wr \mathfrak{S}_w)$. Extend θ trivially on the complement of I and tensor with $\eta_O \boxtimes \eta_O \boxtimes \eta_A$. By [43], the induction Π_A is irreducible, with index $[O_A : I] = 2^w K!/(u!^2 w!)$. If $u = 0$, the first two factors have degree 1.

Let s_A swap the two sets of size u and invert each of the w remaining coordinates. Thus, s_A normalizes I and acts by $\theta \mapsto \theta\, \nu_A|_T$. The tensor flip F exchanges the two η_O-factors, and s_A centralizes $C_e \wr \mathfrak{S}_w$. By Lemma 6.61, $(J_A f)(x) = \nu_A(x) F f(s_A x)$, with $J_A^2 = I$ since $\nu_A(s_A) = 1$. Since the representatives and $s_A x$ lie in W_K, the entries of J_A lie in K_0.

With $j = \prod_{i=1}^K R_i^e$, we put $S = \Pi_A(j)$. Then, j centralizes W_K and inverts T. Thus, S is over K_0 and $S^2 = I$. Since $\nu_A(j) = 1$, $SJ_A = J_A S$. Write $\sigma(i) = (-1)^{a_\sigma} i$ with $i = \sqrt{-1}$, and set $a_r = (r-1)/2 \bmod 2$. On each torus weight, coefficient conjugation C_σ inverts the values precisely when $a_\sigma = 1$ on ψ. Moreover, the matrices on W_K are fixed by σ. Thus, $T_\sigma = S^{a_\sigma} C_\sigma$ centralizes Π_A semilinearly, and $B_0 = S^{a_r}$ implements the r-power map. Conjugation by j inverts every T_i and fixes all permutations in W_K, including permutations of the w quartic positions. Hence, T_σ, B_0 give the stated actions on Π_A, with $T_\sigma J_A = J_A T_\sigma$ and $B_0 J_A = J_A B_0$.

If $t = 1$, then $q \equiv 1 \bmod 4$ and $f a_r \equiv 0 \bmod 2$. Let z_O, z_A be the diagonal cycle scalars on η_O, η_A. By Lemma 6.61, $\Pi_A(v) = z_O^2 z_A I$. Thus, take $B_A = \alpha_f B_0$, with $\alpha_f \in \mu_{D_r}$ and $\alpha_f^f = (z_O^2 z_A)^{-1}$. Since $B_0^f = I$, this gives $B_A^f = \Pi_A(v^{-1})$. If $t = -1$, then $q \equiv -1 \bmod 4$, so $r \equiv -1 \bmod 4$ and f is odd. Put $v = \prod_i R_i$, with z_O the diagonal R_i-scalar and z_A the diagonal R_i^2-scalar on the quotient modules. Then, $vj^{-1} = \prod_i R_i^{1-e}$ fixes θ and centralizes W_K. Its scalar is $\gamma = z_O^{2(1-e)} z_A^{(1-e)/2} = z_O^2 z_A^{(1-e)/2} \in \mu_e$. Hence, $\Pi_A(v) = \gamma S$. In μ_{D_r}, we choose α_f with $\alpha_f^f = \gamma^{-1}$, so $B_A = \alpha_f S$ has $B_A^f = \Pi_A(v^{-1})$ since f is odd. Both roots α_f exist and are fixed by $\mathcal{H}_p$ by Lemma 6.61. Thus, $T_\sigma B_A = B_A T_\sigma$. For the negative field form, put $b' = jb$ and $v' = vj^{-f}$. Then, $(b')^f = (v')^{-1}$. Since e, f are odd, the lift of R_i^{1-ef} has parity 0 and norm 1. Thus, the lift in $\mathrm{Spin}(V)$ is fixed by the adjusted F_r. The central sign z_0 acts trivially. On Π_A, the operators at b', v' are $SB_A = \alpha_f I$ and γI, so $(b')^f = (v')^{-1}$ also holds for odd K. □

Theorem 6.63. *With $N = \mathrm{Spin}_{2n}^\epsilon(q)$ and $q = r^f$ odd, we let $p \neq r$ be an odd prime dividing $|N|$. Suppose that $Q(a, c)$ is separately rational, with $n = 2a + c \geq 5$, $c \geq 1$ and labels $(\Lambda, \Lambda, \lambda)$. Take the core $\Lambda_0 = (X_0, Y_0)$ of* Lemma 6.59 *with $X_0 \neq Y_0$. Then, in the* degree domain for $Q(a, c)$, *the packet has the central $\mathcal{H}_p$-comparison of* Definition 2.1 *at $N_N(Q_0)$ for each natural overgroup $\mathcal{A}$. The toral base Q_0 is characteristic in P_p. Both twists $\nu, \nu\det$ occur. Moreover, the core Λ_0 and quotient η_O allow independent defects at Λ, Λ_0, including defect 0.*

Proof. Take the orientation of ν. Since Λ_0 has unequal rows, its rank is $a_0 \geq 1$ and $\Lambda = (X, Y)$ has $X \neq Y$ by Lemma 6.59. Thus, the parents for $(\Lambda, \Lambda, \lambda)$ and $(\Lambda_0, \Lambda_0, \rho)$ come from Proposition 6.58, with Θ_0 on O_R. If $c_0 = 0$, take the $I(a_0, a_0)$ parent with $\rho = \varnothing$ and $j_\rho = 0$. If $c_0 = 0$ and $a_0 = 1$, we choose Θ_0 from Lemma 6.57. If $s_0^2 \geq 4$, take the cuspidal source of $I(s_0^2, s_0^2)$ and its ordinary paired hooks.

The K active blocks have distinct Q_0-weights up to duality, and $V_R = V^{Q_0}$. By Proposition 6.55, $N_{O(V)}(\pi(Q_0)) = O_A \times O_R$ and $\overline{M} = \ker(\det_A \det_R) \cap \ker(\nu_A \nu_R)$. The group O_A has generators from T and W_K. Under the spin map

π, the inverse image of $\pi(Q_0)$ has a central kernel of order 2. Thus, Q_0 is the unique Sylow p-subgroup of that inverse image and $\overline{M} = \pi(N_N(Q_0))$. Since Q_0 is characteristic in $P_p \in \mathrm{Syl}_p(N)$, $N_N(P_p) \le N_N(Q_0)$.

If $w = 0$, choose Π_A from Lemma 6.61. If $w > 0$, we choose Π_A from Lemma 6.62. Then, $\Pi = \Pi_A \otimes \Theta_0$ is irreducible, with only $1, \nu_A\nu_R$ fixing it in $\langle \det_A \det_R, \nu_A\nu_R\rangle$. Indeed, a twist must fix Π_A, Θ_0 separately, and Θ_0 has group $\langle \nu_R\rangle$. By [43], $\langle \Pi|_{\overline{M}}, \Pi|_{\overline{M}}\rangle = 2$, with both constituents of multiplicity 1. An element of norm -1 exchanges the eigenspaces of J. Thus, each has finite inertia $\ker(\nu_A\nu_R)$. Moreover, the field matrix $B_A \otimes B_R$ acts on $\ker(J \mp I)$ since it centralizes J.

First, suppose that $w = 0$, and take $\zeta^2 = \kappa^u$. For the residual involution H_0, put $J = (J_A \otimes H_0)/\zeta$. Since $\rho = \lambda$, its residual ratio is $g_\sigma^{a_0+j_\lambda}$, with extra factor $\sigma(\zeta)/\zeta$. If $t = 1$, q has odd order modulo p. Thus, $\mathcal{H}_p$ fixes $\sqrt{q}$ by [156, Lemma 2.3(d)]. Since $\gamma_q^2 = \delta q$, the ratio on $\sqrt{\delta} = \gamma_q/\sqrt{q}$ is g_σ. Thus, $\sigma(\zeta)/\zeta = g_\sigma^u = g_\sigma^{eu}$. If $t = -1$ and e is odd, $-q$ has odd order modulo p. Hence, $\mathcal{H}_p$ fixes $\sqrt{-q}$ by [156, Lemma 2.3(d)], and the ratio on $\sqrt{-\delta} = \gamma_q/\sqrt{-q}$ is g_σ. If $t = -1$ and e is even, $4 \mid p-1$ and $\sqrt{\kappa} = i$ is fixed, while $g_\sigma^e = 1$. Therefore, $\sigma(\zeta)/\zeta = g_\sigma^{eu}$ in each case and $T_\sigma J = g_\sigma^{a+j_\lambda} J T_\sigma$, as for the global extension.

If $t = 1$, the absolute half-defects have $s = s_0$. If $t = -1$, by Lemma 6.59, $s - s_0 \equiv u \bmod 2$, so the cuspidal ranks s^2, s_0^2 may differ. At $-I$, Proposition 6.58 gives ratio $(-1)^{s-s_0+eur_q}$. It equals δ^u for $t = 1$ and $(-\delta^e)^u$ for $t = -1$. Thus, the active quadratic coordinates give κ^u. Suppose that $w > 0$. Then, $t = \delta$ and e is odd. The preceding computation gives $g_\sigma = 1$ by $\gamma_q^2 = q$ if $t = 1$ and $\gamma_q^2 = -q$ if $t = -1$. Hence, both $g_\sigma^{j_\lambda}$ and $g_\sigma^{j_\rho}$ are 1. Take $J = J_A \otimes H_0$, so $J^2 = I$. By Lemma 6.62, T_σ, B_A centralize J_A. Since $4 \mid q^e - t$, the quadratic colours have scalar 1 at $-I$. The quartic colours have scalar $(-1)^{w(q^e-t)/4} = (-1)^{(c-c_0)l_2}$, since $q^e \equiv q \bmod 8$ and $(q-t)/4 \equiv l_2 \bmod 2$. The orthogonal scalar ratio is 1, by $s = s_0, \delta = 1$ if $t = 1$, and $s - s_0 \equiv u \bmod 2$, odd e and $\delta = -1$ if $t = -1$. Thus, the central scalars agree at $-I$ and z_0.

Given $\sigma \in \mathcal{H}_p$, choose d_σ in an active torus of norm $g_\sigma^{a+j_\lambda}$, possible since $K > 0$ and ν_A is quadratic there. Then, $\det(d_\sigma) = 1$. On the local extension, σ, d_σ both act by $\ker(J - \varepsilon I) \mapsto \ker(J - \varepsilon g_\sigma^{a+j_\lambda} I)$. Since T_σ centralizes the field operator, $\widehat{\psi}^\sigma = \widehat{\psi}^{d_\sigma}$ holds on $\ker(\nu_A\nu_R)\langle b\rangle$. The global operator H gives $\widehat{\chi}^\sigma = \widehat{\chi}^{d_\sigma}$ with ratio $g_\sigma^{a+j_\lambda}$. Thus, $\alpha_N = \alpha_M = 1$ and $m_N = m_M = 1$ on these ordinary covers.

If the $\mathbb{F}_r$-forms differ, take the residual reflection g of Proposition 6.55. At rank 1, $gX = \mathrm{adj}(X)$ by Lemma 6.57, extended by I on the added hyperbolic space. Then, $\det(g) = -1$ and $\nu(g) = 1$. If f is even, $g^f = 1$, and $\nu(g) = 1$ fixes both J-eigenspaces. By Lemmas 6.61 and 6.62, we choose b, with b' if $t = -1$ and $w > 0$. Since $F_r(v) = v$, the Lang argument in Proposition 6.55 gives $x^{-1}F_q(x) = v$, so $F_q(xF_r(x)^{-1}) = xF_r(x)^{-1}$. This also holds with adjusted F_r. Thus, the change on N is inner, with the power of b or b'.

The finite inertia is $\ker\nu$. If h_m has nonsquare multiplier m, its conjugate inertia is $\ker(\nu\det)$. Thus, h_m gives a different orientation under every σ. If $m = a^2$, the isometry $a^{-1}h_m$ is unique up to $-I$. On the common cover, $\widehat{\chi}, \widehat{\psi}$ have equal scalars at $-I, z_0$. Since Q_0 is characteristic in P_p, Sylow conjugacy gives representatives in $N_N(Q_0)$ for each returning coset $\bar{g}$. Thus, restriction to these stabilizers yields $\succeq_c$. The cores on O_R and quotients on O_A give Λ, λ and the rational class s_M by Proposition 6.260 and Lemma 6.59. Finally, $\mathrm{Int}(h_m)$ transports both sides for nonsquare m. At rank 1, the first Weil factor W_τ changes additive class under h_m, as after Proposition 6.58. Thus, $\nu^{h_m} = \nu\det$. If the transported b changes by central scalar c', both models acquire c'. Thus, m_N, m_M acquire $\sigma(c')/c'$. Thus, the transported stabilizers give $\succeq_c$. □

For example, we choose $s_0 \ge 3$, odd $e > 2s_0^2$ and $q \equiv 3 \bmod 4$ with $\mathrm{ord}_p(q) = 2e$. In $\Lambda_0 = (\{0, \dots, 2s_0 - 1\}, \varnothing)$, move the last bead up by e into the opposite row and set $\lambda = (e)$. Thus, we obtain that $u = w = 1$, $c_0 = 0$, $a_0 = s_0^2$ and $s = s_0 - 1$. We have $2s_0^2 < e$, both quotient degrees are 1 and the multinomial is 6, prime to p since $2e \mid p - 1$. Thus, both quotients have weight 1, and $s \ne s_0$. If $q \equiv 1 \bmod 4$ and $\mathrm{ord}_p(q) = e > 2s_0^2$, move the bead up by e in its original row. Then, $u = w = 1$, $R = 2s_0^2$ and the multinomial is again 6, while $s = s_0$.

Let ε be the character of W_u with value -1 on its cyclic generators and 1 on $\mathfrak{S}_u$. The block transposition s_{ij} exchanges even-dimensional spaces, so $\det_A(s_{ij}) = 1$. If $t = 1$, the generator R_i swaps $E, E^\vee$ of odd dimension, so $\det_A(R_i) = -1$. If $t = -1$, $\det_A(R_i) = -1$ from the reflection in R_i. Thus, we find $\det_A|_{W_u} = \varepsilon$. On $C_e \wr \mathfrak{S}_w$, we have that the generators are R_i^2, s_{ij}, so $\det_A$ is trivial.

Lemma 6.64. *Let Π_A be the parent of* Lemmas 6.61–6.62. *Then, $\Pi_A \det_A = \Pi_A$ exactly when $\eta_O\varepsilon = \eta_O$. In this case, its determinant involution D_A is over K_0 and satisfies $D_AJ_A = (-1)^w J_AD_A$. If $w > 0$, the positive or unadjusted negative generator b has $D_AB_A = (-1)^{wa_r} B_AD_A$, and $D_AT_\sigma = (-1)^{wa_\sigma} T_\sigma D_A$. If $w = 0$, both signs are 1.*

Proof. The twist by $\det_A$ fixes the torus weight θ and multiplies the two W_u-labels by ε. By [43], $\Pi_A \det_A = \Pi_A$ if both labels are fixed. If $w = 0$, they are $\eta_O, \eta_O\alpha$, while for $w > 0$ they are η_O, η_O. Hence, the condition is $\eta_O\varepsilon = \eta_O$. If $\eta_O\varepsilon = \eta_O$, the paired partitions satisfy $\lambda_i = \lambda_{i+e}$ by Lemma 6.59. Let w_0 exchange their ordered coordinates, and F_0 their Specht factors. Thus, $(Lf)(x) = \varepsilon(x)F_0 f(w_0 x)$ implements ε on η_O, since w_0 acts on weights by $\theta_i \mapsto \theta_i\varepsilon$. Moreover, $w_0^2 = 1$ and $\varepsilon(w_0) = 1$ give $L^2 = I$. The entries of L lie in K_0, and that matrix also acts on $\eta_O\alpha$ since α is scalar. On Π_A, put $(D_A f)(x) = \det_A(x)(L \otimes L \otimes 1)f(x)$, with $L \otimes L$ if $w = 0$. Since $L \otimes L$ centralizes the flip in J_A, the two composites at x differ by $\det_A(s_A) = (-1)^w$. Indeed, s_A acts on each quartic T_i by inversion. If $w > 0$, the inversion j of all coordinates has $\det_A(j) = (-1)^{2u+w} = (-1)^w$, so $D_A S = (-1)^w S D_A$. Since C_σ fixes D_A, the factors S^{a_r} and $S^{a_\sigma}C_\sigma$ give the signs with B_A, T_σ. If $b' = jb$, its scalar matrix $SB_A = \alpha_f I$ centralizes D_A. □

Lemma 6.65. *The* $\mathrm{GL}_4(q)$*-module* τ *induced from the two plane labels* $(\eta_q \circ \det, 1)$ *is irreducible, with scalar* 1 *at* aI_4*. Moreover, its rational norm involution* J_4 *satisfies* $B_4J_4 = J_4B_4$ *for field operator* B_4*.*

Proof. For full-rank matrices $A \in \mathrm{M}_{4,2}(\mathbb{F}_q)$, we choose functions with $F(AM) = \eta_q(\det M)F(A)$. Then, $R(g)F(A) = F(g^{-1}A)$ gives the induction on frames $[A]$ with rational matrices. If E, F' are distinct planes, we write $E = \ell \oplus E_0$ with $E \cap F' \le E_0$. Scaling ℓ by a nonsquare while fixing E_0, F' and a complement to $E + F'$ gives g in the common stabilizer with $\eta_q(\det(g|_E)) = -1$ and $\eta_q(\det(g|_{F'})) = 1$. Hence, only $E = F'$ contributes. By [95, Lemma 5.2 and Problem 5.6], $\dim \mathrm{End}_{\mathrm{GL}_4(q)}(\tau) = 1$. Moreover, aI_4 acts by $\eta_q(a^2) = 1$.

Choose one frame B for each plane $[B]$, and put $(A_4F)(A) = \sum_{[B]} \eta_q(\det[A\ B])F(B)$. If the frame is changed by M, the two factors have product multiplier $\eta_q(\det M)^2 = 1$. Substitution $B = gC$ gives $A_4R(g) = \eta_q(\det g)R(g)A_4$. Since τ is irreducible, A_4^2 is scalar. Its diagonal at $[A]$ counts complementary planes. There are q^4, given by graphs of linear maps $\mathbb{F}_q^2 \to \mathbb{F}_q^2$. Swapping the column blocks in $[A\ B]$ has sign $+1$, so every complementary plane contributes 1. Thus, $J_4 = A_4/q^2$ is rational with $J_4^2 = I$. The action $B_4F(A) = F(F_r^{-1}A)$ fixes the kernel $\eta_q(\det[A\ B])$ under simultaneous Frobenius. Hence, $B_4J_4 = J_4B_4$. □

Proposition 6.66. *If* s *is separately rational, take* $(\Lambda, \Lambda, \lambda)$*, where* Λ *is degenerate and* $\lambda \vdash c$*. On* $V_\pm$*, take both characters* D_Λ^+, D_Λ^-*. Then, the orthogonal parent* Θ *has operators* D, J *for* $\det, \nu$ *satisfying* $D^2 = J^2 = I$*,* $DJ = (-1)^c JD$*,* $BJ = JB$*. Moreover, we have that* $T_\sigma D = (-1)^{ca_\sigma} DT_\sigma$*,* $T_\sigma J = g_\sigma^{j_\lambda} JT_\sigma$*,* $BD = (-1)^\beta DB$*. In the split* $\mathbb{F}_r$*-form,* $\beta = ca_r \bmod 2$*. Otherwise, we have that* $\beta = 0$*. Moreover, at* $-I$*, we obtain scalar* $(-1)^{cl_2}$*. These relations include rank* $c > 0$ *with pure partition. If* $\dim V = 0$*, its character is* 1*.*

Proof. First, take orthogonal rank 0 with $c > 0$. By Lemma 6.54, the parent Θ is induced from one of χ^+, χ^- on SO, exchanged by an isometry of determinant -1. Let D have signs $+1, -1$ on the two cosets. The eigenvalues $i, -i$ of the dual point s each have multiplicity c, so the eigenspace exchange j_c has determinant $(-1)^c$. Thus, the norm twist acts by $s \mapsto -s$ and swaps the two SO-classes if c is odd and fixes each if c is even. Hence, $DJ = (-1)^c JD$. A coefficient action inverts i exactly for $a_\sigma = 1$ and fixes λ, giving $T_\sigma D = (-1)^{ca_\sigma} DT_\sigma$. By Lemma 6.54, J has ratio $g_\sigma^{j_\lambda}$ and the scalar at $-I$ is $(-1)^{cl_2}$. If the planes are split, r-power on the weights gives $BD = (-1)^{ca_r} DB$. If T_i is nonsplit, its exponent is $-r \equiv 1 \bmod 4$, so $BD = DB$.

Let $O = O(V)$ and take the natural dual point s with centralizer $C = O(V_+) \times O(V_-) \times C_0$, where C_0 carries λ, on the positive-dimensional factors. Let $\xi \vdash h$ give the degenerate symbol on $V_\pm$. Then, its character U_ξ on $O(V_\pm)$ has special restriction $D_\xi^+ + D_\xi^-$, and the graph exchanges D_ξ^+, D_ξ^-. By [149, proof of Corollary 4.13], U_ξ is uniform, in the span of R_T from rational maximal tori T. Moreover, u_λ is uniform. Thus, by [149, §6.1, (6.1) and (6.9)], $\Theta \in \mathrm{Irr}(O)$ is the unique uniform character for $u = U_\xi \otimes U_\xi \otimes u_\lambda$ over s. If $h = 0$, it equals $\Phi_\lambda^{\mathrm{pure}}$.

Given a box $\xi \mapsto \xi'$, we choose totally singular planes in V_+, V_- with sum E of dimension 4. Put $L = \mathrm{GL}(E) \times O(E^\perp/E)$ and its dual Levi with s. Then, we choose V_L of character $\tau \otimes \Theta_R$, where τ is from Lemma 6.65 and Θ_R carries (ξ', ξ', λ). Moreover, τ, Θ_R are uniform. On $\mathrm{GL}(E)$, the two scalar eigenvalues of s have multiplicity 2. Since τ is induced from $(\eta_q \circ \det) \otimes 1$ on the two blocks, the corresponding u_L on $C_L(s)$ has trivial unipotent characters on both GL_2-factors.

If $C_L = C_L(s)$, parabolic induction gives $\langle \Theta, R_L^O(\tau \otimes \Theta_R)\rangle_O = \langle u, R_{C_L}^C u_L\rangle_C$. Indeed, the uniform expansion on L is in the characters $R_{T,s}^L$, and transitivity gives $R_{T,s}^O$. On C_L, C, the signs of [149, §6.1] cancel, since L, C_L have the same split ranks as O, C, respectively. Given a disconnected factor C_1, by [149, (3.4)], $R_T^{C_1} = \mathrm{Ind}_{C_1^\circ}^{C_1} R_T^{C_1^\circ}$. Thus, ordinary induction through P° gives transitivity for $P \le C_1^\circ$ also at a maximal plane.

Since C_0 contributes 1, we multiply the coefficients on $O(V_+), O(V_-)$. If $h > 1$, the plane stabilizer P contains a residual reflection r_0. Thus, $O(V_\pm) = \mathrm{SO}(V_\pm)P$ gives one Mackey double coset. Given D^+_ξ, the coefficient at $1_{\mathrm{GL}_2} \otimes (D^+_{\xi'} + D^-_{\xi'})$ is 1 by [208, Lemma 7.19] and reciprocity. If $h = 1$, then $P \leq \mathrm{SO}(V_\pm)$ with Levi Weyl group $W(L_P)$ a factor of $W(D_2) = C_2^2$. By [208, Proposition 7.20, Step 3], the two characters of $U_\xi|_{\mathrm{SO}}$ have values $(1,-1)$ and $(-1,1)$, giving coefficient 1 for either plane family. If $c = 0$, the final Levi is $\mathrm{GL}_4(q)$. By [208, (7.45) and Proposition 7.41], its cocharacter is $(e_1 \pm e_2) + (e_3 \pm e_4)$, with one St-label per pair of A_1-factors. For the families $\mathcal{F}_\pm$, the sign products are ± 1.

Thus, Θ has multiplicity 1 in $R^O_L V_L$. On each torus T, $\det|_T = 1$ and $\nu|_T$ corresponds to $-I$ by [149, §8.2]. Hence, ν acts by $s \mapsto -s$, acts by $V_+ \leftrightarrow V_-$ and inverts $i, -i$ with λ fixed. If the quartic space V_{q} is split, an isometry exchanges E_+, E_-. If it is nonsplit, the map $x \mapsto x^q$ in an orthogonal Hermitian basis inverts i and fixes the form Q. Thus, the uniform character correspondence gives $\Theta\nu = \Theta$. Moreover, U_ξ, u_λ have rational Weyl coefficients. s^{2j+1} fix the sign labels and give $i \mapsto \pm i$. Hence, $\Theta^\sigma = \Theta$ and $\Theta^\phi = \Theta$ for the marked actions. Thus, on $R^O_L V_L$, the operators stabilize the Θ-summand.

On L, the characters $\det, \nu$ restrict to $\det_R, (\eta_q \circ \det_{\mathrm{GL}_4})\nu_R$. Thus, take $1 \otimes D_R, J_4 \otimes J_R$. Given a quotient twist ζ with operator A_L, the induced operator is $f(x) \mapsto \zeta(x) A_L f(x)$. Squares and commutators are those of A_L. Since J_4 is rational with $J_4 B_4 = B_4 J_4$ by Lemma 6.65, and $\tau(-I_4) = \tau(1)$, the relations on Θ_R give those on Θ at $\xi \mapsto \xi'$. If $\dim V_R = 0$, take its identity operator. Then, D acts by $f(x) \mapsto \det(x) f(x)$ on the cosets of $\mathrm{SO}(V)$, distinguishing the two plane families $\mathcal{F}_\pm$.

Finally, for $h > 0$, the quotient O/Ω has characters $1, \det, \nu, \det\nu$. By [43], the endomorphism algebra of $\Theta|_\Omega$ has basis $1, D, J, DJ$, of dimension 4. If c is even, the algebra is $\mathbb{C}^4$ with four constituents of multiplicity 1. If c is odd, it is $\mathrm{M}_2(\mathbb{C})$ and one constituent has multiplicity 2. Inflation along $\mathrm{Spin}(V) \to \Omega$ has the same projected s, and the sign branching selects D^+_ξ, D^-_ξ. The graph and swap on $D^\pm_\xi$ follow [208, Proposition 7.20, Steps 2–3]. Thus, B, J act on the multiplicity space for $\Theta|_\Omega$. □

Corollary 6.67. *For equal nondegenerate symbols Λ, the comparison of* Theorem 6.63 *holds for $Q(a,c)$ in the* degree domain *for every Λ_0.*

Proof. Suppose that Λ_0 is degenerate. Since Λ is nondegenerate, Lemma 6.59 gives $\eta_O \neq \eta_O\varepsilon$. By Lemma 6.64, $\Pi_A \det_A \neq \Pi_A$. Since $\Pi_A \nu_A = \Pi_A$, the twist $\det_A \nu_A$ also moves Π_A. Thus, on the product $\Pi_A \otimes \Theta_R$, only $1, \nu_A\nu_R$ are self-twists, also when Θ_R has all four. If $V_R = 0$, we set $\Theta_R = 1$. By [43], restriction to $\overline{M} = \ker(\det_A \det_R) \cap \ker(\nu_A\nu_R)$ gives two constituents of multiplicity 1.

The core has $s_0 = 0$ and even a_0. If $a_0 = 0 < c_0$, Lemma 6.54 gives the field extension at ρ, with norm exponent j_ρ and volume scalar $(-1)^{c_0 l_2}$. If $V_R = 0$, both signs are 1. If $a_0 > 0$, Proposition 6.66 gives those scalars since $a_0 + j_\rho \equiv j_\rho \bmod 2$. Thus, on $\ker\nu$, Theorem 6.63 gives ratio $g_\sigma^{a+j_\lambda}$ with the same square normalization for J. The field correction g has $\nu(g) = 1$, so it centralizes the residual operator J_R also for equal rows of Λ_0.

The two orientations have inertias $\ker\nu, \ker(\nu\det)$, also if $V_R = 0$. Since O/Ω is abelian and $\ker\nu \neq \ker(\nu\det)$, the orbits $\mathcal{O}_1, \mathcal{O}_2$ differ. Finally, order $(\eta_O, \eta_O\varepsilon)$ at Λ_0. Changing their order twists Π_A by $\det_A$, while $\Theta_R \det_R = \Theta_R$ if $V_R \neq 0$. Thus, we have $(\det_A \det_R)|_{\overline{M}} = 1$. The order names the eigenspaces of J, and $\mathcal{H}_p$ fixes their labels since it fixes K_0. □

If Λ is degenerate, by Proposition 6.66, we obtain D, J on the global parent. By Lemma 6.59, $\eta_O\varepsilon = \eta_O$ and $2 \mid u$ by paired partitions. Put $D = D_A \otimes D_R$ and $J = J_A \otimes J_R$ locally with $D^2 = J^2 = I$, with $D = D_A$, $J = J_A$ if $V_R = 0$. Thus, we have that $DJ = (-1)^{w+c_0} JD = (-1)^c JD$, since e is odd when $w > 0$. The coefficient sign on D is $(-1)^{(w+c_0)a_\sigma} = (-1)^{ca_\sigma}$. If $w = 0$, then $2 \mid u$ and $\rho = \lambda$ give the global sign on J. If $w > 0$, we find $g_\sigma = 1$. For split $\mathbb{F}_r$-form, the field sign on D is $(-1)^{ca_r}$. In nonsplit type, we have that b' acts by a scalar on Π_A when $w > 0$, and the sign on D_R is 1. Thus, the signs on D, J agree. A nonsquare multiplier can fix Θ and gives $\nu \mapsto \nu\det$. Hence, compare the action on $\mathbb{C}\langle D, J\rangle$.

Lemma 6.68. *Let V_b be a plane with multiplier μ, split for $q \equiv 1 \bmod 4$ and nonsplit for $q \equiv -1 \bmod 4$. Then, its quartic parent extends to $V(x)$ on $\langle \mathrm{GO}(V_b), \phi\rangle$ with scalar character λ_b. Moreover, there are characters α_σ of $\mathbb{F}_q^\times$ and a scalar $k(h)$ for proper torus elements h. With $m = \mu(h)$ and $(-1)^{a(m)} = \eta_q(m)$, we have $hDh^{-1} = D$ and $hJh^{-1} = k(h)D^{a(m)}J$, where $k(h)^2 = \eta_q(m)$. Moreover, $T_\sigma V(x) = \alpha_\sigma(\mu(x))V(x)T_\sigma$, with $\lambda_b^\sigma/\lambda_b = \alpha_\sigma^2$ and $\lambda_b(-1) = (-1)^{l_2}$. If V_b is split, we set $h = (m, 1)$.*

Proof. If V_b is split, the proper torus is $C = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ with $\mu(a,b) = ab$. The norm-one reflection swaps a, b. Choose ψ of order 4 on $\mathbb{F}_q^\times$. If $r \equiv 1 \bmod 4$, set $\xi = 1$. Otherwise, f is even and $8 \mid q-1$, so choose ξ of order 8 with $\xi^{r-1} = \psi^{-1}$ since $r - 1 \equiv 2 \bmod 4$. Induce $\theta(a,b) = \xi(a)\xi(b)\psi(b)^{-1}$ from C. The weights on (t, t^{-1}) are ψ, ψ^{-1}, so the induction is irreducible on $O(V_b)$ by [43]. On its weight coordinates, take $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. If $h = (m, 1)$, the diagonal ratio is $k(h) = \psi(m)$. Thus, $hJh^{-1} = k(h)D^{a(m)}J$ since $k(h)^2 = \eta_q(m)$. The scalar character on $zI = (z,z)$ is $\lambda_b = \xi^2\psi^{-1}$. Moreover, $h^2 = (m,m)(m, m^{-1})$, with scalar part mI. The r-power action swaps the weights of θ exactly for $a_r = 1$, by the choice of ξ. Thus, its matrix J^{a_r} has f-th power I. Put $\alpha_\sigma = (\xi^\sigma/\xi)\psi^{a_\sigma}$. The ratio of θ^σ to the weight after a_σ swaps is $\alpha_\sigma(ab)$. Thus, $T_\sigma = J^{a_\sigma}C_\sigma$ gives the multiplier and centralizes J^{a_r}.

If V_b is nonsplit, we set $C = \mathbb{F}_{q^2}^\times$, with reflection $a \mapsto a^q$ and $\mu(a) = a^{q+1}$. Induce θ of order 8 across the reflection. Since $v_2(q-1) = 1$, the restriction to $\ker\mu$ has order 4. For multiplication by a, the ratio is $k(h) = \theta(a^{1-q})$. Its square on a generator of C is -1, so $k(h)^2 = \eta_q(a^{q+1})$. Hence, h has the stated action on D, J and $\lambda_b = \theta|_{\mathbb{F}_q^\times}$. The field action on C is $a \mapsto a^{rq}$. Since $r \equiv -1 \bmod 4$ and f is odd, we have $rq \equiv 1 \bmod 8$. Thus, θ is fixed and the operator is I. If $\sigma(\zeta_8) = \zeta_8^{b_{\sigma,8}}$, we put $k_\sigma = (b_{\sigma,8} - q^{a_\sigma})/4 \bmod 2$ and $\alpha_\sigma = \eta_q^{k_\sigma}$. Since $b_{\sigma,8} \equiv q^{a_\sigma} \bmod 4$ and $\theta^4 = \eta_q \circ \mu$ on a generator of C, the operator $T_\sigma = J^{a_\sigma}C_\sigma$ gives the multiplier. Also, $h^2 = \mu(h)I \cdot (a/a^q)$ in C. Thus, $\mu(zI) = z^2$ gives $\lambda_b^\sigma/\lambda_b = \alpha_\sigma^2$. If V_b is split, $\xi^2(-1)\psi(-1)^{-1} = (-1)^{(q-1)/4} = (-1)^{l_2}$. If it is nonsplit, $\theta(-1) = (-1)^{(q^2-1)/8}$. □

Proposition 6.69. *Take Θ from* Proposition 6.66 *and its local parent $\Pi_A \otimes \Theta_R$. Put $\mathcal{G} = \langle \mathrm{GO}(V), \phi\rangle$ and $\mathcal{L} = N_{\mathcal{G}}(\overline{Q}_0)$, with $\overline{Q}_0$ the image of Q_0. Then, the parents extend to $\mathcal{G}, \mathcal{L}$ respectively, with scalar character λ_b^c and multiplier α_σ^c on $\mathcal{L}$. For a proper $h \in \mathcal{L}$ of multiplier m generating $\mathbb{F}_q^\times$, orient D so that both actions are $D \mapsto D$ and $J \mapsto k(h)^c D^{a(m)}J$, with $k(h)$ from* Lemma 6.68.

Proof. First, take $\Phi^{\mathrm{pure}}_{\delta_m}$ of size c. On E_j in Lemma 6.54, the Jordan sizes are $d_j = 2m - 1 - 4j > 0$ and $\sum_j d_j = c$. On E_j, we choose θ^{d_j} from Lemma 6.68, with quartic weight in $\{\psi_j, \psi_j^{-1}\}$. Thus, tensor the models over μ. Thus, the scalar and multiplier are $\lambda_b^c, \alpha_\sigma^c$. Moreover, $\bigotimes_j D_j^{a(\mu)} = D^{a(\mu)}$ and the scalar is $k(h)^{\sum_j d_j} = k(h)^c$. The similitudes on E_j centralize $(X, \gamma), \theta_X, D_{b_\sigma}$. By Proposition 6.66, $\Phi^{\mathrm{pure}}_{\delta_m}$ is uniform. The argument on $\mathbf{T}$ below gives $(\Phi^{\mathrm{pure}}_{\delta_m})^h = \Phi^{\mathrm{pure}}_{\delta_m}$ for proper h. Hence, on W_λ, we choose the intertwiner $V(h)$. Since the relations hold on W_λ, Schur's lemma gives them on $\Phi^{\mathrm{pure}}_{\delta_m}$, including the semilinear relation with D_{b_σ}.

If $\mathrm{core}_2(\lambda) = \varnothing$, start at $c = 2$ with Lemma 6.54. Let $A = \mathrm{PGL}_2(q)$ and take the quadratic parent Θ_1 of Lemma 6.44. Thus, induce $\Theta_1 \otimes u$ to $(A \times A) \rtimes \langle \mathrm{swap}\rangle$, with $u = 1$ for $\lambda = (2)$ and $u = \mathrm{St}$ for $\lambda = (1,1)$. Since $\Theta_1 \neq u$, this induction is irreducible by [43]. On $\mathrm{M}_2(\mathbb{F}_q)$, the left and right actions give $\mathrm{GO}_4^+(q)/Z$, where Z is scalar. After restriction to $O_4^+(q)$, we obtain $\Phi^{\mathrm{pure}}_\lambda$. Inflate and twist by $\lambda_b \circ \mu$ to obtain scalar λ_b^2 and multiplier α_σ^2. The signs of D distinguish the swap cosets, and J acts on Θ_1 in each. If $\mu(h)$ is nonsquare, h gives $J \mapsto -J$ on one coset. Thus, orient D so that $J \mapsto -DJ = k(h)^2 DJ$.

Given the subspace E with $\dim E = d$, we choose the decomposition $E \oplus W \oplus E^\vee$. The similitude Levi consists of $\mathrm{diag}(mA, h_W, A^{-\mathsf{T}})$ with $A \in \mathrm{GL}(E)$ and $\mu(h_W) = m$. If $W = 0$, take just the coordinate m. For the extracted representation ρ_d and the residual extension R_W, we choose $\rho_d(A) \otimes R_W(h_W)\lambda_b(m)^j$ with $j = 1$ for a two-hook and $j = 0$ for a paired box. At zI, we have $m = z^2$, $A = z^{-1}I_d$ and $\rho_d(A) = I$. Thus, the scalar gains $\lambda_b(z)^{2j}$ and the multiplier gains $\alpha_\sigma(m)^{2j}$. Take the field operator on ρ_2 from Lemma 6.44 and on ρ_4 from Lemma 6.65. Since $\lambda_b^\phi = \lambda_b$ by Lemma 6.68, the Levi extends to ϕ.

Inflate across U and induce to $\mathrm{GO}(V)$. Since the parabolic multiplier is onto $\mathbb{F}_q^\times$, [95, Problem 5.2] gives one induced module on $O(V)$. A Levi similitude h acts by $f(x) \mapsto H_h f(h^{-1}xh)$, with source matrix H_h. If $j = 1$, we set $D = -D_R$, giving the extra factor $(-1)^{a(m)} = k(h)^2$ when c increases by 2. If $j = 0$, set $D = D_R$. Thus, in both cases $\nu(h^{-1}xh) = \nu(x)\det(x)^{a(m)}$ gives the action on J. At $I(2,2)$, $D_R = 1$ by Proposition 6.66.

Moreover, the uniform parent Θ has multiplicity 1 by Lemma 6.54 or Proposition 6.66. For rational $\mathbf{T} \leq \mathrm{SO}(V)$, the product with the scalar torus $\mathbf{Z}$ has multiplier kernel $\mathbf{T}$. Since $\mathbf{T}$ is connected, Lang's theorem gives $\mu((\mathbf{TZ})^F) = \mathbb{F}_q^\times$. For proper h, divide by its multiplier representative in $(\mathbf{TZ})^F$ to obtain $g \in \mathrm{SO}(V)$. Thus, h acts by g-conjugacy on each torus pair $(\mathbf{T}, s)$ and fixes its uniform character $R_{\mathbf{T},s}$. Hence, the Θ-space is invariant under $\mathrm{GO}(V)$, giving an extension also when $\Theta|_\Omega$ has multiplicity 2. Moreover, σ, ϕ fix the Θ-space, so the multiplier is α_σ^c. On the residue, it is $\alpha_\sigma^{c_0}$.

Take m generating $\mathbb{F}_q^\times$. If $t = 1$, on $E \oplus E^\vee$ choose $h = (mI, I)$. If $t = -1$, take multiplication by $a \in \mathbb{F}_{q^{2e}}^\times$ with $a^{1+q^e} = m$ by norm surjectivity. Choose identical h on like blocks. If $t = -1$ and e is odd, take $a \in \mathbb{F}_{q^2}^\times$ with

$a^{1+q} = m$. Then, h centralizes T_i and block permutations, and $hR_ih^{-1} = t_iR_i$. Thus, the quadratic character η_T has $\eta_T(t_i) = \eta_q(m)$ by norm covariance. If m is nonsquare, the second η_O gains ε. By Lemma 6.64, $1 \otimes L$ implements it and centralizes $L \otimes L$. On the flip F, this conjugation gives $(L \otimes L)F$. Hence, evaluation at $h^{-1}xh$ fixes D_A and gives $D_A^{a(m)}$ in the norm action.

Let I be the colour inertia in O_A. Thus, its preimage is $\langle I, h\rangle$ with quotient of order $q-1$. The orthogonal source operator $1 \otimes L$ has $(q-1)$-st power I. Since $2 \mid u$, the u equal quadratic values on h^{q-1} have product 1. Similarly, the products at $h^2/(mI)$ and zI are 1. Thus, the orthogonal scalar factor is 1 and L is over K_0. If $t = 1$, compose the split character with $N_{\mathbb{F}_{q^e}/\mathbb{F}_q}$, so its multiplier formula has exponent e. If $t = -1$, we choose $\mathbb{F}_{q^2} \subseteq \mathbb{F}_{q^{2e}}$ and compose θ with $N_{\mathbb{F}_{q^{2e}}/\mathbb{F}_{q^2}}$. Since e is odd, on $a \in \mathbb{F}_{q^2}^\times$ it has value $\theta(a)^e$ and multiplier a^{q+1}. Thus, choose the quartic weight ψ_i. Given R_i^2, the corrections are inverse on T_i if $t = 1$. If $t = -1$, their exponent lies in $(1+q)\mathbb{Z} \subseteq 4\mathbb{Z}$. Thus, η_A is fixed. The plane operators and $1 \otimes L$ give scalar and multiplier exponents ew and factor $k(h)^{ew}$ on J_A. Hence, the torus relation holds at h^{q-1}.

Induce from $\langle I, h\rangle$ to the active extension. On O_A, we choose B_A, T_σ of Lemmas 6.61 and 6.62. Since at h the correction has equal η_T-values on the u like blocks, their product is 1. The simultaneous inversion has ε-value $(-1)^{eu} = 1$, so it centralizes L. Given any quartic T_i, the split action or $a \mapsto a^{rq}$ gives exponent e by Lemma 6.68. Thus, induction gives exponent ew. Moreover, B_A^f has the given value at v^{-1} or $(v')^{-1}$. The linear part of $\mathcal{L}$ fixes $W = V^{\overline{Q}_0}$ and $W^\perp$. By the distinct kernels in $\overline{Q}_0$, it is the product over μ of the active normalizer and $\mathrm{GO}(W)$. Thus, choose $\Pi_A \otimes \Theta_R$ if $W \neq 0$ and Π_A if $W = 0$. Their exponents add to $ew + c_0 = c$. Since $\overline{Q}_0$ is characteristic in P_p, Sylow conjugacy gives $\mathcal{G} = \Omega\mathcal{L}$. Hence, $\mathcal{L}/(\mathcal{L} \cap \Omega) \cong \mathcal{G}/\Omega$.

By Theorem 6.63, the spin transporter for v or v' gives the field correction in Ω. If f is odd, isometry over $\mathbb{F}_q$ gives isometry over $\mathbb{F}_r$ by discriminant descent. If f is even, $q \equiv 1 \bmod 4$. Over $\mathbb{F}_r$, both forms are hyperbolic from E_j, the determinant form on M_2 and the added blocks $E \oplus E^\vee$. Similarly, the active form before v is hyperbolic. If $t = 1$, we choose the hyperbolic model for E. If $t = -1$, then $w = 0$ and the $2u$ blocks give the lift v of Lemma 6.61. Thus, the $\mathbb{F}_r$-forms are isometric. The correction in Ω fixes D, J and has multiplier 1. Hence, the field generators in $\mathcal{G}$ have the powers $b^f, (b')^f$.

Finally, choose $h \in \mathcal{L}$. Any global proper representative of m differs from h by a proper isometry t_0. Thus, replacing it fixes D and changes the coefficient of J by $\nu(t_0)$. Since m is nonsquare, $D \mapsto -D$ adjusts the factor $\nu(t_0)$ while the signs of B, T_σ are fixed. Hence, the actions at h agree on $\mathcal{G}/\Omega$. For either R, $R(h)^2 = \lambda_b(m)^c R((mI)^{-1}h^2)$ since $(mI)^{-1}h^2$ lies in $\mathrm{SO}(V)$. Thus, both scalars are $\lambda_b(m)^c$. □

Theorem 6.70. *With $N = \mathrm{Spin}_{2n}^\epsilon(q)$ and q odd, we let $p \nmid q$ be an odd prime dividing $|N|$. In separately rational type $Q(a, c)$ with $n = 2a + c \geq 5$ and $c \geq 1$, take equal symbols Λ, Λ in the* degree domain. *Then, $\succeq_c$ gives the central $\mathcal{H}_p$-comparison at $N_N(Q_0)$ on the natural overgroups. For degenerate Λ, it includes the labels D_Λ^+, D_Λ^- and their component orbits.*

Proof. If Λ is nondegenerate, apply Corollary 6.67. For degenerate Λ, both multiplicity algebras have $DJ = (-1)^c JD$ and the common actions of B, T_σ on D, J. By Proposition 6.69, the action of h and scalar λ_b^c agree. If c is even, pair the idempotents $(1 \pm D)(1 \pm J)/4$ by their signs $(\pm 1, \pm 1)$. Since the actions on D, J match, the permutations of $(\pm 1, \pm 1)$ agree. Each U has multiplicity 1. Thus, the parent acts on U over I_U and extends its character. At U, the multiplier for the stabilizer action is $\alpha_\sigma(\mu(x))^c$. If c is odd, each restriction has module $\mathbb{C}^2 \otimes U$ with irreducible U. Take the rational matrices D, J of Lemma 6.68 on $\mathbb{C}^2$. For an ordinary x acting by signs $(-1)^{a(x)}, (-1)^{b(x)}$ on D, J, we set $Q_x = J^{a(x)}D^{b(x)}$. Thus, $Q_xQ_y = (-1)^{b(x)a(y)}Q_{xy}$ for sign elements. The factor on U is also $(-1)^{b(x)a(y)}$. For semilinear signs with exponents d_σ, n_σ on D, J, the multiplicity operator is $J^{d_\sigma}D^{n_\sigma}C_\sigma$. Hence, the mixed sign is $(-1)^{n_\sigma a(x) + d_\sigma b(x)}$.

For the proper similitude h, we set $Q_h = (1+D)/2 + k(h)^{-c}(1-D)/2$. Then, $Q_hDQ_h^{-1} = D$ and $Q_hJQ_h^{-1} = k(h)^c D^{a(\mu(h))}J$. If $\mu(h)$ is nonsquare, $Q_h^2 = D$. Moreover, the parent has scalar $\lambda_b(m)^c$ in the relation for h^2 by Proposition 6.69. On $\mathcal{G}/\Omega$, choose common matrices Q_x for all products. Given R, the operator $Q_x^{-1}R(x)$ centralizes $\mathrm{M}_2(\mathbb{C})$ on the multiplicity space. Thus, $R(x) = Q_x \otimes P(x)$. Since R is ordinary, the scalar of $P(x)P(y)P(xy)^{-1}$ is inverse to that of $Q_xQ_yQ_{xy}^{-1}$. Hence, $\alpha_N = \alpha_M$.

Put $S_\sigma = J^{d_\sigma}D^{n_\sigma}$. Since the parent multiplier is $\alpha_\sigma(\mu(x))^c$, we define $s_\sigma(x)$ by $S_\sigma\sigma(Q_x)S_\sigma^{-1} = s_\sigma(x)Q_x$. Thus, the constituent multiplier is $\alpha_\sigma(\mu(x))^c/s_\sigma(x)$, with multiplier 1 at the field generator ϕ. In Q_h, σ acts on $k(h)^{-c}$. For nonsquare $\mu(h)$, $k(h) = \pm i$ and $d_\sigma = a_\sigma$ since c is odd. If $\sigma(i) = -i$, the matrix S_σ swaps the two entries of $\sigma(Q_h)$,

giving $s_\sigma(h) = k(h)^c$. If $\sigma(i) = i$, $s_\sigma(h) = 1$. Thus, $m_N = m_M$ on $\mathcal{G}/\Omega$ from Proposition 6.69.

If a semilinear element gF_r^j centralizes Ω, take a split root group $I + tA$. At $t = 1$ its conjugation fixes A, and at all $t \in \mathbb{F}_q$ $t^{r^j} = t$. Thus, $F_r^j = 1$. Since $V|_\Omega$ is absolutely irreducible, $C_\mathcal{G}(\Omega)$ consists of scalars zI. At zI, we choose $Q_{zI} = I$, so both constituents have scalar $\lambda_b(z)^c$. Thus, the inflated character on N has scalar $\lambda_b(z)^c$ and kernel containing $\langle z_0 \rangle$. On the Sylow stabilizers, we have $\succeq_c$. By Lemma 6.59, recover Λ from the core and quotient. If c is even, the idempotents determine U. □

For example, take odd $e \geq 5$, $\Lambda = (\{e\}, \{e\})$ and $\lambda = (e)$. If $q \equiv 1 \bmod 4$, take period e with $t = 1$. If $q \equiv -1 \bmod 4$, take order $2e$ with $t = -1$. Packing gives $\Lambda_0 = (\{0\}, \{0\})$ and the paired partitions $(1), (1)$. Thus, we obtain that $u = 2$, $\eta_O(1) = 2$, $w = 1$, $K = 5$ and $R = 0$. The multinomial $5!/(2!^2) = 30$ is prime to p since $2e \mid p - 1$. The ambient rank is $5e$ and $c = e$ is odd. Thus, on Π_A alone, $DJ = -JD$ and h acts with nonsquare multiplier $\mu(h)$.

In separately rational type $Q(a, c)$, let V_i have symbol Λ_i of rank a and orthogonal types ϵ_i, for $i \in \{+, -\}$. The eigenvalues mark V_+, V_-, with separate types ϵ_+, ϵ_-. Take d, e, t, h_p and $\tau = \eta_q(-1)$ as in Proposition 6.60. Thus, packing Λ_i by Lemma 6.59 gives core rank b_i, weight u_i and wreath character η_i, with $a = b_i + eu_i$. Take (ρ, c_0, w, η_A) from Proposition 6.60, including $w = 0$ in period $2e$. Thus, we put $k = u_+ + u_- + w$, $R = b_+ + b_- + c_0$ and $\epsilon_{i0} = \epsilon_i t^{u_i}$.

Proposition 6.71. *Suppose that p is odd, $p \nmid q$ and $p \mid |N|$. Then, the degree is prime to p exactly when $k > 0$, $p \nmid \frac{k!}{u_+!u_-!w!}\eta_+(1)\eta_-(1)\eta_A(1)$, and either $R < e$ or $R = e$ with $\epsilon_{+0}\epsilon_{-0}\tau^{c_0} = -t$. If $\epsilon_+ = \epsilon_-$, we find $b_+ = b_-$ and $u_+ = u_-$. If the types are opposite and $b_+ \neq b_-$, we have that $\{b_+, b_-\} = \{0, e\}$, $c_0 = 0$ and $|u_+ - u_-| = 1$.*

Proof. By Proposition 6.60, the denominator at Λ_i has valuation $u_i h_p + v_p(u_i!) - v_p(\eta_i(1))$. If the partition period is $2e$, each move takes rank $2e$ for one denominator factor. Thus, p'-degree gives $2e$-weight 0. The ambient count is $k + m(R, \epsilon_{+0}\epsilon_{-0}\tau^{c_0})$. Hence, $v_p(\chi(1)) = 0$ exactly when $m(R, \epsilon_{+0}\epsilon_{-0}\tau^{c_0}) = 0$ and $p \nmid \frac{k!}{u_+!u_-!w!}\eta_+(1)\eta_-(1)\eta_A(1)$. Since $p \mid |N|$, we require $k > 0$. Thus, the conditions give $v_p(\chi(1)) = 0$ in both directions.

Since $a = b_i + eu_i$, $b_+ - b_- = e(u_- - u_+)$. If the ranks differ, then $e \leq |b_+ - b_-| \leq R \leq e$. Thus, $\{b_+, b_-\} = \{0, e\}$, $c_0 = 0$ and $|u_+ - u_-| = 1$. The sum $u_+ + u_-$ is odd, so the sign at $R = e$ is $\epsilon_+\epsilon_- t$. Hence, the boundary requires $\epsilon_+\epsilon_- = -1$. Since $c \geq 1$ and $c_0 = 0$, the partition is active, so e is odd and $\tau = t$. If the types agree, then $k = 2u + w$ and $R = 2b + c_0$, with boundary sign $\tau^{c_0} = -t$. Thus, with $w > 0$, Proposition 6.60 gives $R < e$. □

For example, we set $a = c = e$ with odd $e \geq 3$ and $\lambda = (e)$. If $t = \tau = 1$, take $(\{e\}, \{0\})$ and $(\{0, e\}, \varnothing)$. The counts of e-hooks are $1, 0$. If $t = \tau = -1$, take $(\{0, e\}, \varnothing)$ and $(\{0\}, \{e\})$, so only the first has an e-cohook. Thus, we obtain that $b_+ = 0$, $b_- = e$, $\epsilon_{-0} = -t$ and $c_0 = 0$ in both cases. The degrees $\eta_+(1), \eta_-(1), \eta_A(1)$ are 1 and the multinomial is 2. Hence, both examples have p'-degree at $R = e$.

Lemma 6.72. *If $t = -1$, order the rows $\Lambda = (X, Y)$ and put $h(\Lambda) = (|X| - |Y|)/2$. On B_{ij} of* Lemma 6.59*, we let m_{ij} be the bead count and ν_{ij} the quotient partition. Put $C(\nu) = \sum_{(l,b)\in\nu}(-1)^{l+b}$ with indices $l, b \geq 1$. Then, $h(\Lambda) - h(\Lambda_0) = \sum_{i,j}(-1)^{j+m_{ij}}C(\operatorname{core}_2 \nu_{ij})$.*

Proof. Take a runner of length m. Thus, its descending heights are $\nu_l + m - l$, including $\nu_l = 0$. At bead l, the contributions to $|X| - |Y|$ before and after packing are $(-1)^{j+\nu_l+m-l}$ and $(-1)^{j+m-l}$. Thus, half their difference is 0 for even ν_l and $-(-1)^{j+m-l}$ for odd ν_l. The row sum in $C(\nu)$ at l is 0 or $(-1)^{l+1}$, respectively. Hence, the contribution to $h(\Lambda) - h(\Lambda_0)$ is $(-1)^{j+m}C(\nu)$. Each rim domino contributes 0, so $C(\nu) = C(\operatorname{core}_2 \nu)$. For $(v, \dots, 1)$, the odd rows give $C = (-1)^{v+1}\lceil v/2 \rceil$. Moreover, $h(\Lambda) - h(\Lambda_0)$ has parity u and absolute value at most u. Under a common shift, residues $i < e - 1$ advance by 1. At the wrap from $i = e - 1$ to $i = 0$, the runner index changes to $1 - j$ and its beta set becomes $\{0\} \cup (B + 1)$. Thus, m increases by 1 and $j + m \bmod 2$ and the partition are fixed. Now $X \leftrightarrow Y$ gives $(h(\Lambda), h(\Lambda_0)) \mapsto (-h(\Lambda), -h(\Lambda_0))$. Hence, shifts and row swaps fix $|h(\Lambda)|, |h(\Lambda_0)|$. □

If $|h(\Lambda_+)| = |h(\Lambda_-)|$, the residual values $|h(\Lambda_{i0})|$ can differ. For lengths $m + h, m - h$, the initial intervals attain minimum rank h^2. If $b_+ = b_- = b$, we have that $b \leq \lfloor e/2 \rfloor$. Thus, if $\epsilon_{i0} = 1$ and $b < 4$, only $h = 0$ occurs. If $\epsilon_{i0} = -1$ and $b < 9$, only $|h| = 1$ occurs. Hence, at equal type $\epsilon_{+0} = \epsilon_{-0}$, nondegenerate cores have $|h(\Lambda_{+0})| = |h(\Lambda_{-0})|$ if $e \leq 7$, or if $e \leq 17$ with $\epsilon_{+0} = \epsilon_{-0} = -1$.

If Λ is degenerate, a corner box b gives the partition $\xi' = \xi \setminus \{b\}$ and an isotropic plane E. If Λ is nondegenerate, a one-hook replaces $x \in D$ by $x - 1 \notin D$ in its row D, with $\dim E = 1$. Thus, $L_E = \mathrm{GL}(E) \times \mathrm{O}(E^\perp/E)$, with $\dim(E^\perp/E) = \dim V - 2\dim E$. In both cases, ϵ on E remains fixed through induction, also if $a_+ = a_-$.

Proposition 6.73. *Take distinct nonempty partitions ξ_+, ξ_- on the degenerate symbols, and let $\lambda \vdash c$ label the quartic factor. If $c = 0$, the series is $I(2|\xi_+|, 2|\xi_-|)$. Then, the full parent χ has self-twists $1, \det$ in $\langle \det, \nu \rangle$. Its determinant operator D satisfies $T_\sigma D = (-1)^{ca_\sigma} D T_\sigma$ and $BD = (-1)^\beta DB$, with $\beta = ca_r \bmod 2$ for split $\mathbb{F}_r$-form and $\beta = 0$ otherwise. Moreover, a proper h has $hDh^{-1} = D$. The scalar and multiplier characters are $\lambda_b^c, \alpha_\sigma^c$ from* Lemma 6.68.

Proof. Put $\xi_* = \xi_+ \cap \xi_-$, which contains the box $(1,1)$. A maximal box b outside ξ_* has no successor in ξ_* since ξ_* is an order ideal. Thus, iterate $\xi_i' \mapsto \xi_i' \setminus \{b\}$ until both diagrams are ξ_*. At each shape ξ_i' in the sequence, the ranks are $2|\xi_i'|$. If $\xi_+' = \xi_-'$, both are contained in ξ_*. Thus, the first equal pair is (ξ_*, ξ_*), with operators from Proposition 6.69 also for $c = 0$. For the upward step, choose E with $\dim E = 2$. With $W = E^\perp/E$, we put $L = \mathrm{GL}(E) \times O(W)$. If the eigenvalue on E is $+1$, then $\gamma = 1$ on $\mathrm{GL}(E)$. If it is -1, $\gamma = \eta_q \circ \det$. Thus, $\gamma(aI_2) = 1$ and γ is rational. On the similitude factor $\mathrm{diag}(mI_2, I_2)$, extend by 1. Thus, at zI the factors have $m = z^2$ and $A = z^{-1}I_2$. The restrictions of $\det, \nu$ are $\det_R, (\eta_q \circ \det)\nu_R$. The residual parent Θ_R is fixed by $\det_R$. If a twist contains ν, then it changes γ at $\det(g)$ nonsquare. Hence, by [43], the restriction to $\ker\det \cap \ker\nu$ has norm 2, with multiplicity-one eigenspaces of $1 \otimes D_R$.

By Proposition 6.66, U_{ξ_+}, U_{ξ_-} are uniform, as are Θ_R, χ. Thus, on L its torus calculation gives coefficients in $C_L(t)$. On GL_2, the label at $\epsilon \in \{1, -1\}$ is 1. For its two residual signs, [208, Lemma 7.19] gives $a_+ + a_- = 1$. Since every other factor has coefficient $\langle \theta, \theta \rangle = 1$ at the fixed character θ, the summand χ occurs with multiplicity 1, also with ranks $a_+ \neq a_-$. Its dual label records ϵ on the changed factor. The twist ν acts by $t \mapsto -t$ and exchanges ξ_+, ξ_-. Since these labels differ, $\chi\nu \neq \chi$, and uniformity gives $\chi \det = \chi$. Thus, by [43], $\chi|_{\ker\det \cap \ker\nu}$ has two constituents of multiplicity 1. By Propositions 6.66 and 6.69, χ is fixed by h, ϕ, σ. On that summand, D is the induced operator $f(x) \mapsto \det(x) D_R f(x)$. Thus, its relations with B, T_σ, h are those of D_R. Since γ is rational and $\gamma(zI_2) = 1$, the characters remain $\lambda_b^c, \alpha_\sigma^c$. At $-I$, the scalar is $\lambda_b(-1)^c = (-1)^{cl_2}$. □

Proposition 6.74. *For separately rational V_+, V_-, take distinct symbols $\Lambda_+ \neq \Lambda_-$, both nondegenerate with absolute defect $2s$. Then, each parent Φ_i restricts irreducibly to Ω and has field extension fixed by $\mathcal{H}_p$. If proper h has nonsquare $\mu(h)$, it exchanges $\Phi_1|_\Omega, \Phi_2|_\Omega$. If both spaces have rank a, their scalar at $-I$ is $(-1)^{s + ar_q + cl_2}$.*

Proof. By Proposition 6.58, the descent on Λ_i stays nondegenerate. Its endpoint has rank s^2 if $s > 0$ and is split of rank 1 if $s = 0$. Descend on V_+ and then on V_- to the first equal pair $\Lambda_+ = \Lambda_-$ of rank b. Thus, the two parents are Φ, Φ^h from Proposition 6.58 and proper similitude h. Their SO-restrictions are irreducible since $\Phi \det \neq \Phi$ and $\Phi^h \det \neq \Phi^h$. If the restrictions to SO agreed, [43] would give $\Phi^h \in \{\Phi, \Phi \det\}$. But their self-twist groups are $\langle \nu_R \rangle, \langle \det_R \nu_R \rangle$. Thus, $\Phi|_{\mathrm{SO}} \neq \Phi^h|_{\mathrm{SO}}$.

At an inverse step, we let V be the ambient space and W the complement to the restored plane $E \oplus E^\vee$. Put $K = \mathrm{SO}(V)$ with line Levi $L = \mathrm{GL}_1(q) \times \mathrm{SO}(W)$. If the eigenvalue on E is ϵ, we set $\gamma = 1$ for $\epsilon = 1$ and $\gamma = \eta_q$ for $\epsilon = -1$. Thus, the two Levi sources are $\beta_j = \gamma \boxtimes \theta_j$ with residual special characters θ_1, θ_2. Let $J = \mathrm{GSO}(V)$. Its line Levi J_L consists of $\mathrm{diag}(a, h, \mu(h)a^{-1})$ with $h \in \mathrm{GSO}(W)$. Thus, $J = KJ_L$ and $A = J/K = J_L/L \cong \mathbb{F}_q^\times$. Since $\ker(\mathrm{GSpin} \to \mathrm{SO})$ is connected, Lang's theorem gives $\widetilde{t}$ in the rational dual Levi. Take ρ, ψ in the series of $\widetilde{t}$. By [72, Theorem 4.7.2], $\langle \rho, R_{J_L}^J \psi \rangle$ multiplies the dual coefficients. Thus, on V_i the hook coefficient is 1 by [208, Lemma 7.17(i)], and on every fixed label θ it is $\langle \theta, \theta \rangle = 1$. Thus, $\langle \rho, R_{J_L}^J \psi \rangle = 1$.

Given $\xi \in \mathrm{Irr}(A)$, we put $c_\xi = \langle \rho\xi, R_{J_L}^J \psi \rangle$. The centralizer $C(t)$ stabilizes V_+, V_-. Moreover, their special groups and C_0 lie in $C(t)^\circ$. Since both residual sign spaces in W are nonzero, we choose one reflection r_i on each. Then, $r_+ r_-$ represents the other component of $C(t)$. The Clifford commutator with $\widetilde{t}$ is -1 by eigenvalues $1, -1$ on the reflection vectors. Thus, by [208, Lemma 7.16 and the subsequent discussion], $r_+ r_-$ gives the quadratic $\xi_2 \in \mathrm{Irr}(A)$ for both groups. Since Λ_i are graph invariant, ξ_2 fixes ρ, ψ. Other twists move $\widetilde{t}$ to distinct rational classes. Hence, $c_1 = c_{\xi_2} = 1$ and $\sum_{\xi \in \mathrm{Irr}(A)} c_\xi = 2$.

By the quotient formula, $\rho|_K, \psi|_L$ each have two constituents of multiplicity 1. The constituents on L are β_1, β_2, since γ is fixed and θ_1, θ_2 occur on $\mathrm{SO}(W)$. Changing $\widetilde{t}$ twists by $\xi \in \mathrm{Irr}(A)$, with $\xi|_L = 1$. Put $\rho|_K = \varphi_1 + \varphi_2$. Since $J = KJ_L$, [95, Lemma 5.2 and Problems 5.2–5.3] gives $\sum_{i,j=1}^2 \langle \varphi_i, R_L^K \beta_j \rangle = \sum_{\xi \in \mathrm{Irr}(A)} \langle \rho\xi, R_{J_L}^J \psi \rangle = 2$. Since J_L is transitive on $\{\varphi_i\}$ and $\{\beta_j\}$, each row and column sums to 1. Hence, the matrix is I_2 after indexing by β_i. The choice of φ_i is determined by θ_i, ϵ.

On K, a nonidentity twist of K/Ω acts by $t \mapsto -t$. Then, the sign labels (Λ_+, Λ_-) are exchanged. Since $\Lambda_+ \neq \Lambda_-$ at each step, the twist of φ_i lies outside their packet. Thus, by [43], $\varphi_1|_\Omega, \varphi_2|_\Omega$ are irreducible and distinct. On $O(V)$, induce from P with source $\gamma \boxtimes \Phi_{W,j}$, where $\Phi_{W,j}$ is the residual isometry parent over θ_j. Since P contains g_W on W of determinant -1, [95, Problem 5.2] gives $R_L^K \beta_j$ on K. On quartic eigenvalues $i, -i$, inversion fixes λ. A reflection r_i

on V_i corrects the determinant, and Λ_i is graph invariant. Thus, φ_1, φ_2 are stable under $O(V), \phi, \sigma$ by [208, discussion following Lemma 7.16]. Since $R_L^K \beta_j$ has only φ_j of this pair with multiplicity 1, the operators act on the φ_j-space. Choose ϕ fixing E, W. On the induction it acts by $f(x) \mapsto B_W f(\phi^{-1}x)$, with the field power of B_W and of ϕ on the argument. Moreover, γ is rational, so T_σ acts on that space. Since proper $h \in J_L$ of nonsquare multiplier exchanges β_1, β_2, it exchanges φ_1, φ_2. If both original ranks are a, there are $a-b$ negative lines with scalar δ at $-I$. The positive lines have scalar 1. Thus, $\delta^{a-b}(-1)^{s+br_q+cl_2} = (-1)^{s+ar_q+cl_2}$ in both orientations, since h fixes $-I$. □

If $\epsilon_+ = \epsilon_-$, by Proposition 6.71, we obtain that $b_+ = b_- = b$ and $u_+ = u_- = u$. On T of Section 6.3, take $1, \eta_T, \psi$ on u, u, w coordinates. Give the inertia weight θ the complement labels $\eta_+, \eta_-\alpha, \eta_A$, where $\alpha = \nu_A|_{W_u}$. If $w > 0$, we find $\alpha = 1$. If $w = 0$, we have that the η_A-factor is trivial. Write Π for the induction. The twist ν_A swaps $1, \eta_T$ and acts by $\psi \mapsto \psi^{-1}$. Thus, in the order $1, \eta_T$, we have that ν_A acts by $(\eta_+, \eta_-\alpha) \mapsto (\eta_-, \eta_+\alpha)$. Hence, we have that $\Pi\nu_A = \Pi$ exactly when $\eta_+ = \eta_-$. The twist $\det_A$ fixes the colours and takes $\eta_i \mapsto \eta_i\varepsilon$, so $\Pi \det_A = \Pi$ exactly when both $\eta_i\varepsilon = \eta_i$.

Lemma 6.75. *The parent Π extends to b with $B_A^f = \Pi(v^{-1})$, and $\mathcal{H}_p$ fixes this extension. If $t = -1$ and $w > 0$, we let z_+, z_- be the R_i-scalars on η_+, η_- and z_A the R_i^2-scalar on η_A, on the diagonal cycles of W_K. Then, for $j = \prod_i R_i^e$ and $S = \Pi(j)$, we have $\Pi(v) = \gamma S$, where $\gamma = (z_+z_-)^{1-e} z_A^{(1-e)/2} \in \mu_e$.*

Proof. If $t = 1$, take z_+, z_- for the positive e-cycles of $\eta_\pm$. By Lemma 6.61, α is trivial on the e-cycles. Hence, whenever $w = 0$, $\Pi(v) = z_+z_-I$. If $w > 0$, write z_A for the positive diagonal e-cycle scalar on η_A. Then, $\Pi(v) = z_+z_-z_AI$. If $t = -1$ and $w = 0$, the value of α is d_R^u with $d_R = \nu_A(R_i)$, so $\Pi(v) = z_+z_-d_R^uI$. If $w = 0$, the field map fixes θ, and the complement matrices lie in $K_0 = \mathbb{Q}(\zeta_{2e})$.

If $t = -1$ and $w > 0$, e is odd and $vj^{-1} = \prod_i R_i^{1-e}$ fixes θ and centralizes W_K. Its scalars on η_+, η_-, η_A are $z_+^{1-e}, z_-^{1-e}, z_A^{(1-e)/2}$. Thus, all cosets in W_K give the product γ. Since $1-e$ is even, $\gamma \in \mu_e$. If $z_+ = z_- = z_O$, $(z_O^2)^{1-e} = z_O^2$. If $\eta_i\varepsilon = \eta_i$, colours $j, j+e$ have equal sizes by Lemma 6.59. Thus, $z_i^e = (-1)^{u/2}$ for odd e. Therefore, if both labels are fixed, $(z_+z_-)^e = 1$. For unrestricted labels the exponent is $1-e$.

By Lemma 6.61, the f-th power on $\mu_{\mathrm{ord}_p(r)}$ has image μ_d, and both groups are fixed by $\mathcal{H}_p$. If $t = -1, w > 0$, take $\xi^f = \gamma^{-1}$ in $\mu_{\mathrm{ord}_p(r)}$. Then, $r \equiv -1 \bmod 4$ and f is odd, so $B_A = \xi S$ gives $B_A^f = \Pi(v^{-1})$. If $t = 1, w > 0$, we set $B_A = \xi S^{a_r}$ with $\xi^f = (z_+z_-z_A)^{-1}$ and $S^{fa_r} = I$. If $w = 0$, take the scalar root alone. If $w > 0$, Lemma 6.62 gives $T_\sigma = S^{a_\sigma}C_\sigma$ on T, W_K, centralizing Π, B_A. If $t = -1, w > 0$, jb acts by ξI with carried element vj^{-f} by Lemma 6.62. Thus, the lift of jb has kernel $\langle z_0 \rangle$. □

Theorem 6.76. *Take $N, p, Q(a, c)$ from* Theorem 6.70*, with degenerate Λ_+, Λ_-. Then, $\succeq_c$ holds for $\mathcal{H}_p$ at $N_N(Q_0)$ on the* degree domain *for all ξ_+, ξ_- and both special orthogonal labels $D_{\xi_i}^\pm$.*

Proof. If $\xi_+ = \xi_-$, we apply Theorem 6.70. If $\xi_+ \neq \xi_-$, both cores are degenerate and $\eta_i\varepsilon = \eta_i$. Thus, the paired sizes give $2 \mid u$. Also $b_+ - b_- \in 2e\mathbb{Z}$ and $|b_+ - b_-| \leq e$ give $b_+ = b_- = b$. If $b > 0$, we choose Θ_R from Proposition 6.66 when $\Lambda_{+0} = \Lambda_{-0}$, and from Proposition 6.73 otherwise. If $b = 0 < c_0$, set $\Theta_R = \Phi_\rho^{\mathrm{pure}}$. If $b = c_0 = 0$, we set $\Theta_R = 1$. Then, $\Theta_R \det_R = \Theta_R$, and $\Theta_R\nu_R = \Theta_R$ exactly when $\Lambda_{+0} = \Lambda_{-0}$, including $b = 0$. Thus, $(\Pi \otimes \Theta_R)\nu = \Pi \otimes \Theta_R$ would give $\Lambda_{+0} = \Lambda_{-0}$ and $\eta_+ = \eta_-$. Since $\eta_i\varepsilon = \eta_i$, inverse packing would give $\Lambda_+ = \Lambda_-$. Hence, the self-twist group of $\Pi \otimes \Theta_R$ is $\langle \det_A \det_R \rangle$. By Clifford theory [43], the restriction to $\overline{M}$ has eigenspaces $\ker(D-I), \ker(D+I)$ of multiplicity 1.

Next, on the active source, D_A has operator $L_+ \otimes L_-$, with L_i from Lemma 6.64 for η_i. The proper nonsquare h acts by $1 \otimes L_-$ and the quartic factors for η_A in Proposition 6.69. Thus, h fixes $L_+ \otimes L_-$. Since $2 \mid u$, the quadratic contribution to h^2 is 1. Hence, the active extension of Π has characters $\lambda_b^{ew}, \alpha_\sigma^{ew}$. The inversion j gives signs $(-1)^{wa_r}, (-1)^{wa_\sigma}$ on D_A before adjustment. If $w = 0$, both are 1. Moreover, for jb, the field sign is 1. Thus, B_A is the operator of Lemma 6.75.

On $\Pi \otimes \Theta_R$, the exponent is $ew + c_0 = c$, and the D-signs are as in Proposition 6.73. Since $2 \mid u$, the quadratic scalar at $-I$ is 1. The scalars at $-I$ for η_A, Θ_R are $(-1)^{(c-c_0)l_2}, (-1)^{c_0l_2}$. Thus, the scalars at $-I, z_0$ agree. Pair the constituents on $\ker(D-I), \ker(D+I)$ by their signs in $\{1, -1\}$. Moreover, mixed stabilizer operations permute the D-signs identically on both sides. On $\ker(D \mp I)$, we restrict to its inertia in $\mathcal{G}$. Thus, the common coset correction gives α_σ^c on each sign's inertia.

Take $\mathcal{L} \leq \mathcal{G}$ from Proposition 6.69. The spaces $V_\pm$ are split. Moreover, Proposition 6.73 adds split 4-spaces over $\mathbb{F}_r$. Thus, Theorem 6.70 gives field operators on $V_\pm$ and the centralizer in $\mathcal{G}/\Omega$. Given $\psi \in \mathrm{Irr}(B_Q^0)$, we choose

$\widehat{\psi} \in \mathrm{Irr}(B_Q)$ above ψ. By Clifford theory [43], the possible $\widehat{\psi}$ form an orbit under $\mathrm{Irr}(B_Q/B_Q^0)$. Moreover, the twist $\det$ fixes η_i, and ν acts by $(\Lambda_{+0}, \eta_+) \leftrightarrow (\Lambda_{-0}, \eta_-)$. Hence, inverse packing recovers Λ_+, Λ_- up to $t \mapsto -t$. The D-sign determines the constituent at $N_N(Q_0)$. □

Theorem 6.77. *In type $Q(a,c)$, suppose that $\Lambda_+ \neq \Lambda_-$ are nondegenerate of absolute defect $2s$. Suppose that the cores $\Lambda_{+0}, \Lambda_{-0}$ are nondegenerate of absolute defect $2s_0$. On the* degree domain for $Q(a,c)$*, both orientations give an $\mathcal{H}_p$-comparison $\succeq_c$ at $N_N(Q_0)$ onto the ordinary packet of B_Q^0. Its characters are restrictions to $\overline{M}$ of the full parents on B_Q with ordered symbols Λ_+, Λ_-. If $t = 1$, then $s_0 = s$. If $t = -1$, then s_0 may differ from s.*

Proof. Since $\epsilon_+ = \epsilon_-$, Proposition 6.71 gives $b_+ = b_- = b$ and $u_+ = u_- = u$. If $\Lambda_{+0} \neq \Lambda_{-0}$, choose Θ_R from Proposition 6.74. Otherwise, we choose Θ_R on Q or I in norm orientation by Proposition 6.58. Thus, $\Theta_R \det_R \neq \Theta_R$ and $\Theta_R \det_R \nu_R \neq \Theta_R$. A norm self-twist of $\Pi \otimes \Theta_R$ would give $\Lambda_{+0} = \Lambda_{-0}$ and $\eta_+ = \eta_-$. Since inverse packing would give $\Lambda_+ = \Lambda_-$, the self-twist group of $\Pi \otimes \Theta_R$ is trivial. Hence, by [43], $(\Pi \otimes \Theta_R)|_{\overline{M}}$ is irreducible. By Lemma 6.75, B_A is the field operator on Π. Thus, on the inertia under $O(V)$ and ϕ, the tensor extension of $\Pi \otimes \Theta_R$ restricts to an extension of $(\Pi \otimes \Theta_R)|_{\overline{M}}$.

At $-I$, the scalar ratio for $O(V), O_R$ is $(-1)^{s-s_0+(a-b)r_q+(c-c_0)l_2}$. If $t = 1$, the orthogonal factor is δ^u, since $s = s_0$, $a - b = eu$ and e is odd. If $t = -1$, Lemma 6.72 gives $s - s_0 \equiv u \bmod 2$, so the factor is $(-\delta^e)^u$. Thus, the quadratic coordinates of Π give the orthogonal factor at $-I$. If $w > 0$, the quartic coordinates give $(-1)^{(c-c_0)l_2}$. Moreover, z_0 acts by 1 in both models.

If $\Lambda_{+0} \neq \Lambda_{-0}$, the residual orientations have distinct Ω_R-restrictions. If $\Lambda_{+0} = \Lambda_{-0}$, the residual self-twist groups of Θ_R differ. Thus, no twist by $\mathrm{Irr}(B_Q/\overline{M})$ pairs the two tensor parents on B_Q. Hence, their restrictions to $\overline{M}$ are distinct. By Proposition 6.74, $h \in N_{GO(V)}(Q_0)$ has nonsquare multiplier and exchanges $\Lambda_-, \Lambda_-^{\mathrm{tr}}$. Since $\mathcal{H}_p$ fixes both ordinary extensions, $h\sigma$ exchanges them for every $\sigma \in \mathcal{H}_p$.

Take $E = \langle O(V), \phi \rangle$ with the prime-field semilinear isometry ϕ, and put $E_0 = N_E(\overline{Q}_0)$. Since $\overline{Q}_0$ is characteristic in the Sylow p-subgroup of Ω, Sylow conjugacy gives $E = \Omega E_0$. Thus, $E/\Omega \cong E_0/\overline{M}$ with $\overline{M} = E_0 \cap \Omega$. Let b_A be the lift of Lemma 6.75 and b_R the lift on V_R. Match the forms over $\mathbb{F}_q$. Since $b_A \oplus b_R$ and the global lift induce ϕ on scalars in $\mathbb{F}_q$, their quotient lies in $O(V)$. Thus, the extensions on $O(V)$ and B_Q evaluate on $b_A \oplus b_R$ with its power in E. On the ordinary covers, $\alpha_N = \alpha_M = 1$ and $m_N = m_M = 1$. For the induced automorphisms of N, take a common section of E/Ω. Moreover, $\{\pm I\}$ acts by the volume character and z_0 acts by 1. Hence, the descended factors for $\succeq_c$ coincide.

Write $B_Q = O_A \times O_R$ and $B_Q^0 = \overline{M}$ as in Theorem 6.259. Let $\psi \in \mathrm{Irr}_{p'}(B_Q^0)$ have the chosen labels at B_Q^0. By Clifford theory [43], any $\widehat{\psi} \in \mathrm{Irr}(B_Q)$ above ψ has p'-degree since $[B_Q : B_Q^0]$ is a power of 2. By Proposition 6.260, the test on $\widehat{\psi}$ reads the torus weights of Π and the label of Θ_R. The inverse bijection on B_Q gives the ordered cores $\Lambda_{+0}, \Lambda_{-0}$ from Θ_R. Then, over $\Lambda_{\pm 0}$, unpack $\eta_\pm$ after dividing the negative label by α. If $w > 0$, unpack the quartic quotient over ρ. Thus, the tests for $Q(a,c)$ give the stated degree domain.

Fix Θ_R and order $\Lambda_\pm$ by the cores $\Lambda_{\pm 0}$. Since each $\Lambda_{\pm 0}$ has distinct rows, the order of $\Lambda_\pm$ is unique. Over $SO(V_R)$, the two O_R-extensions differ by $\det_R$. Since V_R has square Gram determinant, $\det_R$ extends over ϕ by Section 6.1. Thus, both orders on O_R extend over ϕ. In Lemma 6.75, η_+, η_-, η_A give the scalars z_+, z_-, z_A. By [149, Corollary 8.3] and Section 6.1, h transposes Λ_-, Λ_{-0} and gives $j \mapsto j + e$ on the negative cyclic labels. Moreover, h fixes η_A. Thus, the full-parent inverse at B_Q has both orientations of Θ_R.

By Clifford theory [43], the lifts of ψ form an orbit under $\{1, \det, \nu, \det\nu\}$. On the inverse labels, $\det$ transposes the rows of Λ_+, Λ_- and ν gives $(\Lambda_+, \Lambda_-) \mapsto (\Lambda_-, \Lambda_+)$. Since the Ω-restrictions are irreducible and distinct, ψ determines the reconstructed character independently of $\widehat{\psi}$. Moreover, the packed parent restricts to ψ. Thus, the comparison is bijective at $N_N(Q_0)$. □

In Proposition 6.74, suppose that Λ_+, Λ_- have equal absolute defect $2s$ and are nondegenerate. The core Λ_{-0} is degenerate and Λ_{+0} is nondegenerate. In the balanced case $t = -1$, both cores are split of rank $b > 0$ with $2 \mid b$. Thus, we put $\Lambda_{\pm 0}$ on $V_\pm$ for the dual involution t. For $-t$, the sign spaces are V_+, V_-. Thus, the two orientations of the $\mathrm{SO}(V)$-parent differ by ν, which is trivial on $\Omega(V)$. Hence, we keep the order V_-, V_+ until restriction to $\Omega(V)$. Over $\mathbb{F}_q$, we write $J = \mathrm{GSO}(V)$, $K = \mathrm{SO}(V)$, $X = \Omega(V)$, $O = O(V)$ and $E = \mathrm{GO}(V)$. By Section 6.1, the regular parent on J and the ordered parent on O have equal restrictions to K. Thus, in the flag with partition 2ξ, the folded formula at 2ξ gives their common constituent on X.

Lemma 6.78. *Let $X \lhd A$ and suppose that $\chi \in \mathrm{Irr}(X)$ occurs in an ordinary A-module Y with $\chi(1) > \dim(Y)/2$. Then, the χ-isotypic space affords an extension $\widehat{\chi}$ to A. Given a coefficient automorphism σ with $Y^\sigma \cong Y$, we have that $\widehat{\chi}^\sigma = \widehat{\chi}$. Now on $V = \bigwedge^2 \mathbb{F}_q^4$, take the flag module Y for singular planes $U \le V$ with character $\eta_q(\det|_U)$. Thus, for $A = E\langle F\rangle$, we have that the module Y has such a χ, with $\chi(1)/\dim(Y) = q/(q+1)$ and $\widehat{\chi}(zI) = \chi(1)$ for $z \in \mathbb{F}_q^\times$.*

Proof. Since $\chi(1) > \dim(Y)/2$, only χ has this degree in $Y|_X$, with multiplicity 1. By Clifford theory [43], the χ-space is an A-submodule. Since χ is unique of degree $\chi(1)$, if $Y^\sigma \cong Y$ then $\chi^\sigma = \chi$ and $\widehat{\chi}^\sigma = \widehat{\chi}$. Thus, for Y over $\mathbb{Q}$, the central idempotent $e_\chi \in \mathbb{Q}X$ gives the extension on $e_\chi Y$ over $\mathbb{Q}$. Put $W = \mathbb{F}_q^4$ and equip $\bigwedge^2 W$ with the wedge-product form $u \wedge v \in \bigwedge^4 W \cong \mathbb{F}_q$. Two independent singular bivectors spanning U give 2-spaces in W whose intersection is a line L and whose sum is a hyperplane H. Thus, $U = L \wedge H$. Conversely, each $L \subset H$ gives exactly one such plane U. The image X of $\mathrm{SL}_4(q)$ is transitive on $L \subset H$. Hence, choose Y induced from $\eta_q(\det|_U)$ on the stabilizer of U, with value 1 on F for a flag $L \subset H$ over the prime field. Since $\dim(U) = 2$, the scalar zI has value $\eta_q(z^2) = 1$, and Y is a signed permutation module over $\mathbb{Q}$.

A Levi element $\mathrm{diag}(a, B, d)$ acts on $L \wedge H$ by aB. Thus, its η_q-value is $\eta_q(\det B)$. By transitivity of $\mathrm{SL}_4(q)$, the inflation of $Y|_X$ to $\mathrm{SL}_4(q)$ is the restriction of the $\mathrm{GL}_4(q)$-flag induction. The two trivial $\mathrm{GL}_1(q)$-factors give the permutation character $1 + \mathrm{St}_2$ on $\mathbb{P}^1(\mathbb{F}_q)$. Hence, Green's product rule [84, Theorems 12–14] with the $\eta_q \circ \det$ factor gives two irreducibles on $\mathrm{GL}_4(q)$, of degrees C, qC with Grassmannian index $C = (q^2+1)(q^2+q+1)$. The character of degree qC has partitions $(1,1),(2)$ at $1, \eta_q$. Thus, a nontrivial twist fixing $\{1, \eta_q\}$ is $\eta_q \circ \det$, which exchanges $(1,1),(2)$. Hence, by [43], the restriction to $\mathrm{SL}_4(q)$ is irreducible, and so is Y's constituent χ on X. Moreover, $qC/\dim(Y) = q/(q+1) > 1/2$. Finally, on the split torus of $\mathrm{SO}_6(q)$, the plane character $\eta_q(t_1 t_2)$ determines the series of Y. □

Lemma 6.79. *For the dual involution t, we let V_- be split of rank 2. Choose the maximal-plane family at $1_{\mathrm{GL}_2(q)}$ with nonzero coefficient. Thus, P_J denotes the corresponding plane parabolic, β its Levi source and $\varphi \in \mathrm{Irr}(J)$ the selected constituent. Then, we find $\varphi(1)/([J : P_J]\beta(1)) = q/(q+1)$. In particular, for $J = \mathrm{GSO}_8^+(q)$ and split V_+ of rank 2, the X-isotypic space for $\varphi|_X$ occupies the fraction $q/(q+1)$ of the coupled flag module.*

Proof. On V_-, the connected dual centralizer has type $D_2 = A_1A_1$. Thus, the chosen planes form $\mathbb{P}^1(\mathbb{F}_q)$ of size $q+1$. The source has unipotent degree 1 and the selected label is $1 \boxtimes \mathrm{St}$ or $\mathrm{St} \boxtimes 1$, of degree q. Thus, the V_+-label H contributes equally to $\varphi(1)$ and $\beta(1)$. By [58, Proposition 11.5.6], the r'-order ratio for J and the Levi is $[J : P_J]$, and the dual centralizer ratio is $1/(q+1)$. Hence, $\varphi(1)/([J : P_J]\beta(1)) = q/(q+1)$ for the chosen plane in V_-.

Now take $J = \mathrm{GSO}_8^+(q)$ and $H = 1$ or St on the positive split 4-space V_+. Then, for the building Δ of $\mathrm{SO}(V_+)$, the Solomon–Tits theorem gives St on $\widetilde{H}_1(\Delta, \mathbb{Q})$. Thus, F and the similitudes of V_+ act on $\widetilde{H}_1(\Delta, \mathbb{Q})$. Every simplex of Δ is fixed by zI. Hence, both choices of H extend rationally and trivially on zI. On the plane parabolic P_J, the linear factor is $\eta_q \circ \det_{\mathrm{GL}_2(q)}$. Moreover, $\mathrm{GL}_2(q)$ gives ν, and the similitudes on V_+ give det and all multipliers. The generator F fixes the plane U over the prime field. Thus, $N_{E\langle F\rangle}(U)X = E\langle F\rangle$. The induction on $E\langle F\rangle$ and $\mathrm{Ind}_{P_J}^J \beta$ have equal restrictions to X.

The dual element above $t = (-I_4, +I_4)$ is $z\,\mathrm{vol}(V_-)$ for central z. Thus, by Section 6.1, only $\eta_q \circ m$ is nontrivial among the J/K-twists in the series of φ. On V_-, $\eta_q \circ m$ exchanges the degenerate labels. Hence, $\varphi|_K$ is irreducible by Clifford theory [43]. The ν-twist takes t to $-t$. Since exchanging V_-, V_+ would pair degree q with degree 1 or q^2, the ν-twist moves $\varphi|_K$. Thus, $\varphi|_X$ is irreducible. Since its fraction is $q/(q+1) > 1/2$, Lemma 6.78 gives the extension of $\varphi|_X$. Moreover, O/K induces the natural graph on V. □

Proposition 6.80. *Let V_+ be split of rank $b \ge 2$ with nondegenerate unipotent label H. Suppose that H on the adjoint group extends to $\mathrm{GO}(V_+)\langle F\rangle$ trivially on scalars and $\mathcal{H}_p$ fixes that extension. For each $\xi \ne \varnothing$ degenerate on V_-, the natural parent extends to $E\langle F\rangle$, is trivial on scalars zI and restricts irreducibly to X. If the graph extension of H has values in $\mathbb{Q}$, so does the extension on O.*

Proof. On the orthogonal quotient V_+, we choose the extension $\widehat{H}$ of H. On the linear factor take $(\eta_q \circ \det)U_{2\xi}$. After induction to $E\langle F\rangle$, we obtain a module Y. By Witt transitivity of O on the flags, $Y|_O = F_{2\xi}(h)$ with $h = \widehat{H}|_{O(V_+)}$. By the folded formula at 2ξ, the constituent χ_B on X has multiplicity 1. Moreover, its K-parent is the common restriction from J by Section 6.1. Since $E\langle F\rangle$ fixes χ_B, the unique constituent of Y above χ_B is its extension by [208, proof of Lemma 3.13]. At zI, the quadratic value is $\eta_q(z^{2|\xi|}) = 1$, and $U_{2\xi}$ is unipotent. Thus, zI acts trivially on Y.

On J, the χ_B-space affords a regular parent φ. By Section 6.1, the other parent is $\varphi(\eta_q \circ m)$. Since F fixes η_q, the value at semilinear g is $\eta_q(m(g))$. This character is trivial on O and on scalars zI, for which $m(zI) = z^2$. Thus, multiplier twists of $\widehat{H}$ trivial on scalars are characters of $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. The field extension of $U_{2\xi}$ is rational. Hence, the χ_B-space in Y is $\mathcal{H}_p$-invariant. If h has values in $\mathbb{Q}$, the sole parent of χ_B in $Y|_O$ has values in $\mathbb{Q}$. □

In the residue below $2 \mid b$. Thus, for $b = 2$, we set $H = 1$ or St by Lemma 6.79. If $b \geq 4$ is even and $t = -1$, then on split adjoint D_b, the result of Ruhstorfer et al. [156, §§7–8] gives graph and field operators for H in the even-order case. Since H is nondegenerate, the graph from $O(V_+)$ fixes H. Inflate the extension to $\mathrm{GO}(V_+)\langle F\rangle$. By Dudas–Malle [61, Theorem 3], the graph extension of H on $O(V_+)$ is rational. Thus, the extension over F is $\mathcal{H}_p$-invariant and the character on $O(V_+)$ is rational.

Proposition 6.81. *Let $q \equiv 1 \bmod 4$ and $V = T \oplus W \oplus T'$, with dual singular spaces T, T' of dimension $l > 0$. Take the hyperbolic pairs over $\mathbb{F}_r$. Thus, on W, we choose Θ_R from* Proposition 6.80*, with $\Theta_R(zI) = \Theta_R(1)$. Its field extension is $\mathcal{H}_p$-invariant. Take E, J, K, X from* Section 6.3*. Thus, in the stabilizer P of T, the Levi coordinates on T, W, T' are $g, y, \mu(y)g^{-\mathsf{T}}$. Let U be unipotent for $\mathrm{GL}_l(q)$ and take $\psi, \xi, \lambda_b, \alpha_\sigma$ from* Lemma 6.68*. With Θ_R from* Proposition 6.80*, the quartic parent on E restricts irreducibly to X. Moreover, its scalar character is λ_b^l. Thus, its field extension has correction $\alpha_\sigma(\mu(\,\cdot\,))^l$ on E and value 1 at the field generator ϕ.*

Proof. On the Levi of P, take the two sources

$$\beta_+(g,y) = \psi(\det g)(\xi\psi^{-1})(\mu(y))^l U(g) \otimes \Theta_R(y), \;\; \beta_-(g,y) = \psi(\det g)^{-1}\xi(\mu(y))^l U(g) \otimes \Theta_R(y).$$

After inflation to P, we write $I_\pm = \mathrm{Ind}_P^E \beta_\pm$. Since $\mu(zI) = z^2$, the scalar factors of $\beta_\pm(zI)$ are $\lambda_b(z)^l = (\xi^2\psi^{-1})(z)^l$. Both U, Θ_R act trivially on zI.

The Levi on W gives $\mu, \det$, and $\mathrm{GL}_l(q)$ gives ν. Thus, J, K, X each have one orbit on E/P. The dual element s_J has ordered spaces for $i, -i$ and ordered sign spaces on W. Hence, the dual Levi contains $C_{J^*}(s_J)^\circ$. By [72, §4.7], the relevant irreducibles are $I_\pm|_J$. Moreover, on $i, -i$, the connected factor maps trivially into $\langle z_0\rangle$. Thus, among J/K-twists only the dual scalar -1 keeps the series of $I_\pm|_J$. By Section 6.1, -1 exchanges the degenerate choices on W. On K, the twist by ν exchanges the symbols on W_+, W_-, of distinct degeneracy. Hence, by [43], $I_\pm|_K, I_\pm|_X$ are irreducible.

Let w exchange T, T' in the chosen coordinates and fix W pointwise. Then, $w^2 = 1$ over $\mathbb{F}_r$. On the Levi, g goes to $\mu(y)g^{-\mathsf{T}}$, with determinant $\mu(y)^l(\det g)^{-1}$. Thus, conjugation by w exchanges the scalar factors of $\beta_\pm$. Take S_U for inverse transpose γ, with $S_U^2 = I$ and the field operator F_U. If U has partition λ, the rational flag module for λ has $K_{\lambda\lambda} = 1$. Thus, U and F_U have a model over $\mathbb{Q}$. By [156, proof of Theorem D], the adjoint type A character extends across $\langle\gamma, \phi\rangle$. Thus, $S_U F_U = F_U S_U$, independently of their scalar choices. If $l = 1$, set $S_U = I$. If $l = 2$, take the inverse transpose representative in $\mathrm{PGL}_2(r)$. If $l > 2$, the pinned graph correction for S_U is over $\mathbb{F}_r$, with central square acting trivially on U. If $r \equiv 3 \bmod 4$, q is a square. Hence, on $\mathrm{PGL}_l(q)$, [61, Theorem 3] gives a rational graph extension $\widehat{U}$. Induce U over $\mathbb{Q}$ to the graph group A. Thus, the central idempotent $e_{\widehat{U}}$ projects onto a summand of type $\widehat{U}$. Thus, S_U is rational when $r \equiv 3 \bmod 4$.

Write U_P for the unipotent radical of P. In left-covariant induction define $A_\pm : I_\pm \to I_\mp$ by $(A_\pm f)(x) = \sum_{v \in U_P}(S_U \otimes 1)f(wvx)$. Hence, conjugation by w on the Levi and reindexing U_P give covariance under P. Thus, $A_\pm$ intertwine right translations. By [95, Corollary 1.6], $A_\mp A_\pm$ are scalar. Evaluate at 1 on a function supported on P. Since $wU_Pw^{-1} \cap P = 1$, the condition $wv'wv \in P$ gives $v' = 1$. Hence, $S_U^2 = 1$ gives $A_\mp A_\pm = |U_P|I = q^M I$, where the radical U_P has $l \dim W$ cross coordinates and $l(l-1)/2$ skew coordinates, so $M = l \dim W + l(l-1)/2$. Thus, the operator B on $I_+ \oplus I_-$ with blocks $q^{-M/2}A_\pm$ satisfies $B^2 = I$.

Put $a_r = (r-1)/2 \bmod 2$. On $I_+ \oplus I_-$, let F act by inverse Frobenius on x and by the field operators of U, Θ_R on values. Then, $F^f = I$, and F exchanges $I_\pm$ when $a_r = 1$ by $\xi^{r-1} = \psi^{-1}$. Since $F(w) = w$, $F(U_P) = U_P$ and $S_U F_U = F_U S_U$, $BF = FB$. Hence, $\Phi = B^{a_r}F$ acts on I_+, and $\Phi^f = I$ since $2 \mid a_r f$. Given $\sigma \in \mathcal{H}_p$, we let $\sigma(i) = i^{(-1)^{a_\sigma}}$ and put $\alpha_\sigma = (\xi^\sigma/\xi)\psi^{a_\sigma}$. Thus, coefficient action on $\beta_\pm$ exchanges the sources when $a_\sigma = 1$ and contributes $\alpha_\sigma^l \circ \mu$. Take C given by coefficient action on U, Θ_R and multiplication by $\alpha_\sigma(\mu(x))^{-l}$ on the argument x. The scalar corrects covariance at P, and $CR(y) = \alpha_\sigma(\mu(y))^l R(y)C$ for right translation $R(y)$. Since α_σ is fixed by the r-power map, $CF = FC$. If $a_r = 1$, $q^{M/2}$ and S_U are rational and $\mu(wv) = 1$ for $v \in U_P$. Hence, $CB = BC$ when $a_r = 1$. Thus, $B^{a_\sigma}C$ acts on I_+, satisfies $B^{a_\sigma}C\,\Phi = \Phi\,B^{a_\sigma}C$ and has correction $\alpha_\sigma^l \circ \mu$ on E. If $a_r = 0$, this follows from $CF = FC$ and $BF = FB$ with $\Phi = F$. □

Lemma 6.82. *Let $q \equiv 3 \bmod 4$, so $r \equiv 3 \bmod 4$ and $2 \nmid f$. On the Hermitian space V_4 of dimension l over $\mathbb{F}_{q^2}$, take the trace form. Then, t acts on V_4 by multiplication by i. Thus, Frobenius exchanges the geometric $i, -i$ eigenspaces. Thus, on the separate factor W, take the same degenerate choice on W_-. On W, take the parent Θ_R of* Proposition 6.80. *For V_4 and any unitary partition ρ, the oriented character χ_K restricts irreducibly to X. Moreover, both extensions to O have values in $\mathbb{Q}$ and are J-invariant. With prime-field generator ϕ, each extends rationally to $O\langle\phi\rangle$. At the natural volume ω, the scalar is $(-1)^{ll_2}$.*

Proof. The dual unitary factor for J is connected, so its map into $\langle z_0\rangle$ is trivial. The possible nontrivial J/K-twist exchanges the degenerate choices on W_- by Section 6.1. On K, the ν-twist exchanges the sign spaces of W, with distinct symbols. Thus, χ_K and $\chi_K|_X$ are irreducible by [43]. Next, choose s to be a graph reflection on W_+ with $s|_{V_4} = I$. Its Clifford lift centralizes $\mathrm{vol}(W_-)$ and the even lift on V_4. The nondegenerate symbol on W_+ is s-invariant. Moreover, $\phi(s) = s$ since s is over $\mathbb{F}_r$. Hence, the J-parent is s-invariant and extends to E across E/J of order 2. Its restriction to O is an extension of χ_K invariant under J. Its determinant twist on O is also J-invariant.

An odd coefficient power gives $t \mapsto t$ or t^{-1}. Coordinate conjugation on V_4 inverts multiplication by i and has determinant $(-1)^l$. If l is odd, compose it with s to get determinant 1. This isometry fixes the labels on W and the partition on V_4. The lift $\widetilde{t}$ to the dual of J may acquire a central scalar z. Thus, the multiplier twist for z is trivial on K. By [169, Theorem 1.1] on J, the character χ_K takes values in $\mathbb{Q}$. Moreover, the prime-field model of V_4 and W gives $\chi_K^{\phi} = \chi_K$.

In $\mathbf{K}$, we let $\mathbf{L}$ stabilize the opposite geometric l-spaces T, T' in V_4. Let $\mathbf{U}$ be the unipotent radical at T. Then, the Levi $\mathbf{L}$ is rational and $M = \mathbf{L}^{F_q} = \mathrm{U}_l(q) \times \mathrm{SO}(W)$. The reflection s normalizes $\mathbf{U}, M$, and $\phi(s) = s$ and $s^2 = 1$. Hence, $M \rtimes \langle s\rangle = \mathrm{U}_l(q) \times O(W)$. Take $\widehat{\Theta}_R$ on $O(W)$ from Proposition 6.80 with values in $\mathbb{Q}$. On $\mathrm{U}_l(q)$, twist the unipotent character for ρ by the quartic determinant character τ. Thus, the tensor $\widehat{\beta}$ has values in $\mathbb{Q}(i)$.

On the Deligne–Lusztig variety $Y_{\mathbf{U}}$, we have $k(ym) = (ky)m$ for $k \in K$, $m \in M$ and $y \in Y_{\mathbf{U}}$. After conjugation by s, we obtain an involution of $Y_{\mathbf{U}}$ that induces s on both actions. Hence, on $H_c^J(Y_{\mathbf{U}}) \otimes_M \widehat{\beta}|_M$, pair this involution with s on $\widehat{\beta}$. This gives a virtual O-character Z by the alternating sum. To evaluate $Z(xs^{\epsilon})$, apply the averaging idempotent for M. Thus, each product has one $\widehat{\beta}$-trace on Ms^{ϵ} and one Lefschetz number on $Y_{\mathbf{U}}$. Since the action is by $(K \times M) \rtimes \langle s\rangle$, the Lefschetz number on $Y_{\mathbf{U}}$ is integral by [53, Proposition 3.3]. Consequently, Z has values in $\mathbb{Q}(i)$ by [208, proof of Proposition 4.16]. By Wonenburger's factorization theorem, $g \in O$ is conjugate to g^{-1} in O. Thus, $Z(g) \in \mathbb{Q}(i) \cap \mathbb{R} = \mathbb{Q}$.

The restriction $Z|_K$ is the signed Lusztig induction of $\widehat{\beta}|_M$. Since $i, -i$ are distinct, the dual centralizer of t fixes T, T' individually. Its paired odd reflections on W lie in the Levi factor $\mathrm{SO}(W)$. Thus, $C_{\mathbf{K}^*}(t) \leq \mathbf{L}^*$, and [72, Theorem 3.3.22] gives $Z|_K = \pm\chi$ for $\chi \in \mathrm{Irr}(K)$. On the Levi of J, lift both factors with multiplier μ. The character τ of $\ker N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ extends to $\mathbb{F}_{q^2}^{\times}$. Since $N(\mathbb{F}_{q^2}^{\times}) = \mathbb{F}_q^{\times}$, the similitudes of V_4 are generated by scalars and $\mathrm{U}_l(q)$. Thus, the unipotent character inflates from $\mathrm{PU}_l(q)$. Tensor with the parent on $\mathrm{GSO}(W)$. By Jordan decomposition on J, we obtain the marked labels on W and partition ρ on V_4. By [72, Corollary 3.3.25], the signed induction $Z|_K$ has the J-Levi labels. The J-parent restricts irreducibly to K, so $\chi = \chi_K$.

Let n_+, n_- be the coefficients in Z of the two extensions of χ_K to O. Since $Z|_K = \pm\chi_K$, $n_+ + n_- = \pm 1$. If an automorphism of $\overline{\mathbb{Q}}$ exchanged the extensions, rationality of Z would give $n_+ = n_-$. Hence, both extensions have values in $\mathbb{Q}$. Since f is odd, both extensions are ϕ-invariant. On $O\langle\phi\rangle$, [143, Lemma 2.1 and Corollary 2.2] gives unique real extensions over O, with values in $\mathbb{Q}$. Take the split torus coordinates on W_+, W_- with characters $1, \eta_q$, and l anisotropic coordinates of order $q+1$ on V_4. By [165, Proposition 2.7(i)], the scalar on ω is the torus value at $-I$. Since the rank of W_- is even, its value is 1. Hence, the quartic coordinates give $(-1)^{l(q+1)/4} = (-1)^{ll_2}$. □

If ρ has empty 2-core, rational parabolic induction on E gives an alternative construction. Starting from W, add one rim 2-hook at each step on V_4. A Hermitian pair gives U with linear Levi $\mathrm{GL}_2(q)$. Thus, the quadratic cuspidal source on $\mathrm{PGL}_2(q)$ from Lemma 6.44 inflates to $\mathrm{GL}_2(q)$. Indeed, the quartic character of $\mathbb{F}_{q^2}^{\times}$ is trivial on the ground field $(\mathbb{F}_{q^2}^{\times})^{q+1}$. Since f is odd, the Hermitian pair over $\mathbb{F}_{r^2}$ gives opposite planes U, U' over $\mathbb{F}_r$. On $\mathrm{GL}_1(q^2) \times \mathrm{U}_{l-2}(q)$, the chosen 2-hook coefficient is 1 by Zhang [208, Proposition 7.21, Step 2]. On W_-, the quadratic multiplier twist exchanges D_{ξ}^+, D_{ξ}^- and has coefficient 0. Thus, the selected K-character has multiplicity 1. Its isotypic space on $E\langle\phi\rangle$ inherits the action of ϕ and is trivial on zI. Moreover, we have that Z gives the sign on $O \setminus K$ also when ρ has nonempty 2-core.

Theorem 6.83. *In the separately rational family $Q(a,c)$, we put* $\mathrm{ord}_p(q) = 2e$. *Suppose that $\Lambda_+ \neq \Lambda_-$ are nondegenerate of equal absolute defect $2s$, and exactly one of $\Lambda_{+0}, \Lambda_{-0}$ is degenerate. Given $Q(a,c)$, we choose the* degree domain at p *and the marked parents Θ, Θ_R of* Propositions 6.74 and 6.80. *Take the $\succeq_c$ cover of* Theorem 6.63. *Then, the $\mathcal{H}_p$-comparison $\succeq_c$ holds at $N_N(Q_0)$ for each allowed ρ in either quartic form.*

Proof. Balance gives $b = b_+ = b_- > 0$ with $2 \mid b$ and $u = u_+ = u_-$. Orient $\Lambda_{\pm 0}$ with Λ_{-0} degenerate. Hence, $\Lambda_{+0}, \Lambda_{-0}$ are split. Thus, Λ_{-0} belongs to the η_T-side as in Section 6.3. If $c_0 = 0$, we choose the involution parent Θ_R. If $c_0 > 0$ and $q \equiv 1 \bmod 4$, we choose Θ_R from Proposition 6.81. If $c_0 > 0$ and $q \equiv 3 \bmod 4$, we choose Θ_R from Lemma 6.82. Thus, $\Theta_R|_{\Omega_R}$ is irreducible and Θ_R has trivial self-twists. On O_R, proper similitudes fix $\Theta_R|_{O_R}$. Moreover, on $O_R\langle\phi\rangle$, $\widehat{\Theta}_R^{\sigma} = \widehat{\Theta}_R$ for $\sigma \in \mathcal{H}_p$.

Tensor Θ_R with Π from Lemma 6.75. Since Θ_R has trivial self-twists, so does $\Pi \otimes \Theta_R$. Hence, by [43], the restriction to B_Q^0 is irreducible. Let h be a proper similitude centralizing T with nonsquare multiplier. By Proposition 6.69, h acts by $\eta_- \mapsto \eta_-\varepsilon$ and fixes η_+, η_A. Since Λ_{-0} is degenerate and Λ_- is nondegenerate, the quotient labels satisfy $\eta_-\varepsilon \neq \eta_-$. If h fixed the restriction to B_Q^0, Clifford theory would give $(\Pi \otimes \Theta_R)^h = (\Pi \otimes \Theta_R)\omega$ for $\omega \in \mathrm{Irr}(B_Q/B_Q^0)$. Since $\Theta_R(\omega|_{O_R}) = \Theta_R$, the twist is trivial. This contradicts $\eta_-\varepsilon \neq \eta_-$. Since $\mathcal{H}_p$ fixes the isometry and field extensions, $h\sigma$ exchanges the branches for all $\sigma \in \mathcal{H}_p$. Thus, the stabilizers agree on N and $N_N(Q_0)$.

The residual volume on Θ_R has scalar $(-1)^{c_0 l_2}$. By Lemma 6.72 at the degenerate core, $s \equiv u \bmod 2$. Moreover, $a = b + eu$ with $2 \mid b$. Thus, at $-I$, the scalar ratio of Proposition 6.74 is $(-\delta^e)^u(-1)^{(c-c_0)l_2}$. The quadratic and quartic factors of Π give this product and z_0 acts by 1. Thus, on the inertia in $O(V)\langle\phi\rangle$, the ordinary covers have $\alpha_N = \alpha_M = 1$ and $m_N = m_M = 1$. If $m(g) = z^2$, we have that the isometry $z^{-1}g$ is determined up to $\{\pm I\}$. Since the scalars at $-I$ agree, a common section gives $\alpha_N = \alpha_M$ and $m_N = m_M$ on the quotient. Hence, $\succeq_c$ holds at $N_N(Q_0)$. Moreover, h gives the other orientation at $N_N(Q_0)$ with field lift $h\phi h^{-1}$. □

Let $2s_+, 2s_-$ be the absolute defects of Λ_+, Λ_-. On the negative space W_- of rank b, the quadratic torus character at $-I$ is $(-1)^{br_q+s_-}$. Indeed, each split plane gives $\delta = (-1)^{r_q}$, and an anisotropic plane gives $(-1)^{r_q+1}$. Thus, we choose s_- mod 2 anisotropic planes in W_-. On W_+, at $-I$, we obtain scalar 1. Hence, with l quartic coordinates, the scalar at ω is $(-1)^{br_q+s_-+ll_2}$. By Lemma 6.82, the torus value at $-I$ gives this scalar also for $s_+ \neq s_-$.

Proposition 6.84. *Suppose that $s_+ \neq s_-$ have the same parity and Λ_+, Λ_- are nondegenerate. If $s > 0$, take the cuspidal endpoint of rank s^2. If $s = 0$, take the split endpoint of rank* 1. *For (s_+, s_-), by* Zhang [208, Proposition 7.25(i),(ii)]*, we obtain the marked source Θ with rational orientations on O and constituent permutations at each L. Thus, the two O-parents have values in $\mathbb{Q}$. Their restrictions to Ω are irreducible and distinct. Given $\lambda \vdash l$, the quartic block of rank l also gives rational parents on O with irreducible distinct Ω-restrictions. If $2 \mid \mathrm{ord}_p(q)$, the extension to $O\langle b_0\rangle$ can be chosen $\mathcal{H}_p$-invariant.*

Proof. At $s = 0, 1$, the endpoint rank $n(s)$ is 1. Since $s_+ \equiv s_- \bmod 2$ and $s_+ \neq s_-$, $n(s_+) \neq n(s_-)$. Thus, $\max\{n(s_+), n(s_-)\} \geq 4$, so $n(s_+) + n(s_-) \geq 5$ along the paths. On W, the ordinary source has eigenvalues $+1, -1$. At rank 1, the torus is split for $s = 0$ and anisotropic for $s = 1$. Moreover, by [165, Theorems A and 10.6–10.8], a nonsquare similitude h exchanges the split endpoint branches on K. The sources on O are ordinary isometry characters.

When $d = \mathrm{ord}_p(q)$ is even, normalize b_0 with $b_0^f = v^{-1}$. If v^{-1} acts by -1, take an f-th root in $\mu_{\mathrm{ord}_p(r)}$, whose f-th powers are μ_d. Hence, $\mathcal{H}_p$ fixes the chosen root. The space $W = W_+ \perp W_-$ is split. Given $v \in \mathbf{SO}(W)^{F_r}$, [58, §4.2] gives x with $x^{-1}F_q(x) = v$. Then, $xF_r(x)^{-1}$ is F_q-fixed. Thus, the field lift $xF_r(x)^{-1}F_r$ has f-th power $xv^{-1}x^{-1}$, also for two nonsplit spaces W_+, W_-.

Choose ordinary hook paths to Λ_+, Λ_- with every symbol nondegenerate. At $+1, -1$, the linear labels are $1, \eta_q$, respectively. Moreover, two paired reflections fix Λ_+, Λ_- at each step. The defects $s_+ \neq s_-$ distinguish the labels at $+1, -1$. Thus, at each marked line L, the connected one-hook coefficient is 1. By [208, Proposition 7.18, Steps 2–4], the constituent matrix M_L has total sum 2. It is invariant under simultaneous permutation $(1,2)$ of rows and columns. Hence, M_L has row and column sums 1. For the Levi at L, Proposition 6.74 gives irreducible restriction to Ω in that Levi. Thus, M_L selects one target with multiplicity 1. Its isotypic space gives the rational O-character and inherits b_0. Since $s_+ \neq s_-$, the marked labels exclude the ν-twist. Hence, the restrictions to Ω are distinct. A nonsquare similitude h exchanges the source columns of M_L, and so the target rows.

If $l > 0$, we choose the oriented character on $\mathrm{SO}(W)$. On $\mathrm{GL}_l(q)$ or $\mathrm{U}_l(q)$, take $U(\psi \circ \det)$ with ψ of order 4. Since $i, -i$ differ, the dual centralizer of t fixes T, T' separately. Moreover, the paired reflections on W lie in $\mathrm{SO}(W)$.

Thus, [72, Theorem 3.3.22] gives distinct signed irreducibles $\pm\chi_K^{\pm}$. By [72, Corollary 3.3.25], restriction to K gives $\chi_K^{\pm}$ from the J-Levi labels.

If $q \equiv 3 \bmod 4$, we choose Θ_W rational on $O(W)$. The tensor in Lemma 6.82 gives Z with $Z|_K = \pm\chi_K$ for the chosen orientation χ_K. Thus, both O-extensions have values in $\mathbb{Q}$. Since $2 \nmid f$, the field group fixes both K-orientations and their O-extensions. Thus, on $O\langle b_0\rangle$, On the residual factor Θ_W, the opposite operator is I by Proposition 6.81. Thus, it pairs the O-inductions. In split form, choose Θ_W on $O(W)$ and induce $\psi(\det)^{\pm 1}U \otimes \Theta_W$ to O. Then, both restrictions to K equal χ_K. On the residual factor Θ_W, the opposite operator is I by Proposition 6.81. Thus, it pairs both O-inductions. Hence, the O-character is rational. Moreover, on O, the operators $B^{a_r}F, B^{a_\sigma}C$ give the field correction of Θ_W. For nonsquare m, we extend the similitude of W by $\mathrm{diag}(mI, I)$ on the split block or by zI with norm m in unitary form. Thus, on O, conjugation exchanges the orientations on W and their induced characters on O. Consequently, the two Ω-restrictions stay distinct by $s_+ \neq s_-$. □

Lemma 6.85. *Suppose that $\Lambda_+ \neq \Lambda_-$ are nondegenerate and $\epsilon_+ = \epsilon_-$. On the* degree domain of $Q(a, c)$*, suppose that $\Lambda_{+0}, \Lambda_{-0}$ are degenerate. Then, the local parent $\Pi \otimes \Theta_R$ restricts irreducibly to B_Q^0 and has an extension to $\langle B_Q, \phi\rangle$ fixed by $\mathcal{H}_p$. The nonsquare conjugate under h has a distinct restriction to B_Q^0. This includes $b = 0$ with any partition residue c_0. If $s_+ = s_-$, both branches give the $\mathcal{H}_p$-comparison $\succeq_c$ at $N_N(Q_0)$.*

Proof. By balance, $b_+ = b_- = b$ and $u_+ = u_- = u > 0$. Shift $\Lambda_\pm$ to a common bead count n. Then, for $\Lambda_\pm$, order the rows and take P, M. Write $x' = x\varepsilon$. At $\Lambda_{\pm 0}$, paired runner lengths agree. Hence, for $x' = x\varepsilon$, inverse packing exchanges the rows of $\Lambda_\pm$, including on alternating runners. Thus, $P' \neq P$ and $M' \neq M$. If $\Lambda_{+0} = \Lambda_{-0}$, then $P = M$ or $P = M'$ would give $\Lambda_+ = \Lambda_-$ at bead count n.

For labels $P, M\alpha$, write $D = \det$ and $V = \nu$ for the two twists. Let h of nonsquare multiplier centralize T. Thus, $(P, M)^D = (P', M')$ and $(P, M)^V = (M, P)$. Moreover, $(P, M)^{DV} = (M', P')$ and $(P, M)^h = (P, M')$. Indeed, D shifts both rotation labels by ε. Under V, the colour exchange cancels the α-factors, and inversion leaves η_A fixed. By Proposition 6.69, h acts by $R_i \mapsto t_i R_i$. Thus, the nonsquare sign affects η_T alone, and h fixes η_A.

The active parent Π has no D-self-twist. A self-twist by V would give $P = M$, and by DV would give $P = M'$. If $\Lambda_{+0} = \Lambda_{-0}$, both relations are excluded. If $b > 0$ and the cores differ, Proposition 6.73 excludes V, DV on Θ_R. Thus, $\Pi \otimes \Theta_R$ has trivial self-twists for all b, c_0. Moreover, (P, M') differs from $(P, M), (P', M')$ since $P' \neq P$ and $M' \neq M$. If V or DV matched the pairs, $P = M = M'$. Hence, by [43], the restriction of $\Pi \otimes \Theta_R$ to B_Q^0 is irreducible and differs from its h-conjugate.

Take the ordinary extension Θ_R of the two degenerate labels. If $b = 0 < c_0$, we choose the pure partition extension Θ_R. If $R = 0$, we set $\Theta_R = 1$. The residual correction is trivial on O_R. Thus, tensor Θ_R with the field extension of Π from Lemma 6.75. Since $\Theta_R \det_R = \Theta_R$, simultaneous row exchange on P, M gives the total twist D. Moreover, exchange on η_T alone gives h, which exchanges the restrictions to B_Q^0. Since $2 \mid b$, the residual volume ω_R acts by $(-1)^{c_0 l_2}$. If $t = 1$, $s_- = 0$, and the scalar at ω has orthogonal ratio δ^{eu}. If $t = -1$, $s_- \equiv u \bmod 2$ gives the ratio $(-\delta^e)^u$. For either t, the quartic ratio is $(-1)^{(c-c_0)l_2}$. Hence, the torus weights of Π give the ratio at ω also for $R = 0$.

If $s_+ = s_-$, we choose the global field extensions $\widehat{\chi}_\pm$ of Proposition 6.74. The scalars at ω, z_0 agree for $\widehat{\chi}_\pm$ and the local models on B_Q. Moreover, their inertia images agree in $O(V)\langle\phi\rangle/\Omega(V)$. By Theorem 6.77, we obtain $\succeq_c$ with its common section. Given P, M, Proposition 6.260 reads row orders at $\Lambda_{\pm 0}$. The action on (P, M) gives simultaneous exchange by D and negative exchange by h. Moreover, the full-parent inverse is V-equivariant. Thus, the comparison at $N_N(Q_0)$ is bijective by inverse packing. □

Lemma 6.86. *Let C_f be a field quotient with $2 \mid f$. Given a rational character θ, an extension U may satisfy $U^\sigma = Uk_\sigma$ with $k_\sigma \neq 1$ in $\mathrm{Irr}(C_f)$. Thus, for $t = 1$, take a common extension U_* for ρ, ρ_0 on the smallest space W_* of signed defects (d_+, d_-). Hence, compare the field action on $O(V)$ before restriction to $\Omega(V)$. Let $U_* \in \mathrm{Irr}(\widehat{G}_*)$ extend the rational field-stable $\theta_* \in \mathrm{Irr}(G_*)$, with $\widehat{G}_*/G_* \cong C_f$. Thus, for $\sigma \in \mathcal{H}_p$, we put $U_*^\sigma = U_* k_\sigma$ with $k_\sigma \in \mathrm{Irr}(C_f)$. Next, for a line or plane L_i, take the marked induction R_i with rational field-stable linear label λ_i. Suppose that χ_i is stable of multiplicity 1 and $\chi_i^\sigma = \chi_i$. Then, its field extension U_i satisfies $U_i^\sigma = U_i k_\sigma$ with k_σ inflated from C_f. The split quartic induction R_{sp} of* Proposition 6.81 *also gives k_σ.*

Suppose that f is even and χ_1, χ_2 are full isometry parents over s_i with $s_i^4 = 1$ as in Proposition 6.260*. For ordered rows X, Y, take signed defect $|X| - |Y|$. If $d_{1,+} = d_{2,+}$ and $d_{1,-} = d_{2,-}$, we have that χ_1, χ_2 have extensions U_1, U_2 with the same k_σ on C_f, for any quartic partitions λ_1, λ_2. If $2 \mid d = \mathrm{ord}_p(q)$, each χ_i has an extension $\widehat{\chi}_i$ fixed by $\mathcal{H}_p$ for any signed defects $d_{i,+}, d_{i,-}$.*

Proof. By Clifford theory [43], the extensions of θ_* are $U_*\lambda$ for $\lambda \in \mathrm{Irr}(C_f)$, each with multiplicity 1. Thus, k_σ is unique. If f is odd, take the unique real U_* as in Lemma 6.82. Then, $k_\sigma = 1$. Let $\widehat{H}_i \le \widehat{G}_i$ be the source subgroup. For k_σ inflated to $\widehat{G}_i$, induction gives $R_i(U \otimes k_\sigma|_{\widehat{H}_i}) = R_i(U) \otimes k_\sigma$. Moreover, $\sigma(e_{\chi_i}) = e_{\chi_i}$ and $k_\sigma|_{G_i} = 1$. Thus, the selected extension satisfies $U_i^\sigma = U_i k_\sigma$. The selection of Lemma 6.78 gives $U_i^\sigma = U_i k_\sigma$ as well.

Given a singular line L, let U_γ be parabolic induction with $\gamma \in \{1, \eta_q\}$ with adjoint D_γ. Thus, in the involution series $\mathcal{E}_{\mathrm{inv}}$ of $O(V)$, we write (λ_1, λ_2) for the row partitions of Θ_Λ. By [149, §3.4, (6.11) and Theorem 8.12], a successor Λ' adds one box to λ_i and fixes the other sign symbol Ξ. Conversely, D_γ deletes one box from λ_i at a predecessor. By Proposition 6.4, $\langle U_\gamma(\Theta_\Lambda), \Theta_{\Lambda'}\rangle = 1$, including equal rows. Moreover, $|s_1 - s_2| = 2$ for support sizes s_1, s_2, with Mackey contributions m_1, m_2 from the two line orbits. If $O(0) = 1$, the first split plane has torus T_1 of index 2. Thus, the induction of γ gives the distinct extensions γ_+, γ_-.

Let Θ have rational values on $O(V)$. For the quadratic form Q, the field conjugate $Q^{(r)}$ has $[\mathrm{disc}(Q^{(r)})] = [\mathrm{disc}(Q)]$ in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. Hence, we obtain a semilinear isometry ϕ with $\phi(av) = a^r\phi(v)$. Write $x = su = us$ in $O(V)$. Choose j prime to $|x|$ with $j \equiv r \bmod |s|$ and $j \equiv 1 \bmod |u|$. Then, $\phi x \phi^{-1}$ and $x^j = s^r u$ share elementary divisors. Given any eigenvalue λ, both have block sizes $\mathcal{J}_\lambda$ at λ^r. Moreover, on the primary spaces $V_\pm$ at ± 1, $x^j|_{V_\pm} = x|_{V_\pm}$. Given $\epsilon \in \{1, -1\}$, the forms $B_\epsilon, B_\epsilon^{(r)}$ have equal dimensions and discriminants. Hence, $\phi x \phi^{-1}$ is conjugate to x^j in $O(V)$ by [203, §2.6, pp. 38–39]. Since Θ is rational, $\Theta(\phi x \phi^{-1}) = \Theta(x^j) = \Theta(x)$. Thus, $\Theta^\phi = \Theta$ for nonsplit Q and for $\phi^f \ne 1$.

At signed defect d_0, subtract $0, 1, \ldots$ from the increasing rows X, Y to get λ_X, λ_Y. Thus, $\lambda_X = \lambda_Y = \varnothing$ at the smallest symbol Λ_{d_0}, of rank $d_0^2/4$. Simultaneous shifts give all representatives of Λ_{d_0}. Hence, d_+, d_- give one parent Θ_* on the smallest sign spaces W_*, with quartic rank 0. By Lemma 6.245, Θ_* is rational. Hence, $\Theta_*^\phi = \Theta_*$. Take U_* on the field group of W_*. If $W_* = 0$, take $U_* = 1$. Given $\epsilon \in \{+, -\}$, let $\lambda_{\epsilon,1}, \lambda_{\epsilon,2}$ be the row partitions of Λ_ϵ. Thus, recover Λ_ϵ by line induction. At step i, the character Θ_i is rational and $\Theta_i^\phi = \Theta_i$. Moreover, by Proposition 6.4, its multiplicity is 1. For the added plane H, we extend ϕ_W by coordinate Frobenius on H. If $\phi_W^f = a$, $\phi_{W\oplus H}^f = a \oplus 1_H$. Thus, for either form of W, both paths carry k_σ from U_*.

Now suppose that d is even. Choose U_* with $k_\sigma = 1$. Put $S_* = \mathrm{SO}(W_*)$. If $W_* \ne 0$, some $\Lambda_\pm$ has nonzero defect. The twist by det transposes each symbol $\Lambda_\pm$. Hence, $\chi_* = \Theta_*|_{S_*}$ is irreducible by [43]. Suppose that χ_* has nonzero Harish–Chandra restriction to $P < S_*$. Since $s^2 = 1$, a cuspidal Levi for χ_* has a split line factor T_ℓ with label 1 or η_q by [165, Lemma 11.2]. Put the line ℓ first in the flag. Its stabilizers P_ℓ, P_ℓ^O in $S_*, O(W_*)$ have radical U_ℓ and line torus T_ℓ. Thus, the U_ℓ-invariants of $\Theta_*|_{P_\ell^O}$ have a nonzero part for 1 or η_q on T_ℓ. By adjunction with D_γ, a sign symbol Λ_ϵ would have a predecessor. Since $\lambda_{\epsilon,1} = \lambda_{\epsilon,2} = \varnothing$ at Λ_ϵ, χ_* is cuspidal.

If $\dim W_* > 2$, then $s^2 = 1$ and s is quasi-isolated by [165, §10.1]. Given S_*, take the generalized Gelfand–Graev character Γ from [165, proof of Theorem 10.8], with $\langle \Gamma, \chi_*\rangle = 1$. By [165, proof of Proposition 10.4], each part λ_j is odd. Hence, the nilpotent element X has an even grading. Let P_X be its parabolic and U the radical with weights at least 2. Take a nontrivial additive character ψ of $\mathbb{F}_r$ and put $c(u) = 2(u-1)(u+1)^{-1}$. Thus, the linear character is $\theta_X(u) = \psi(\mathrm{Tr}_{q/r}\,\mathrm{tr}(Xc(u)))$. By the multiplicity in $\Gamma = \mathrm{Ind}_U^{S_*}\theta_X$, the space $\mathcal{L} = \mathrm{Hom}_U(\theta_X, \Theta_*)$ has dimension 1. Thus, $\mathcal{L}$ is as in Lemma 6.238.

On W_* of either form, take a block V_h of length $2h+1$ with basis $x_0, \ldots, x_{2h}$. Thus, on V_h, the shift is $Xx_j = x_{j+1}$ for $j < 2h$, with $Xx_{2h} = 0$. The antidiagonal entries are $(-1)^j$. Next, tensor V_h with the multiplicity space M_h. Take a diagonal reference form Q_h on M_h over $\mathbb{F}_r$. Thus, for the other discriminant over $\mathbb{F}_q$, twist F_q by a sign ϵ_h on e_h. On that coordinate, we have that ze_h is fixed for $z^q = -z$, and z^2 is nonsquare in $\mathbb{F}_q$. Thus, dimension and discriminant determine M_h over $\mathbb{F}_q$. The matching spaces M_h give a rational isometry α of graded pairs, also for repeated $2h+1$. Moreover, α exists for a single sign space W_*. The full isometry α suffices since χ_* extends to $O(W_*)$.

With $w = \prod_h \epsilon_h$, let b act by coordinate r-Frobenius. Thus, $bw = wb$ and $bX = Xb$. Moreover, b fixes $\psi \circ \mathrm{Tr}_{q/r}$. On the wF_q-fixed space, $b^f = w^{-1}$, and w^{-1} acts on $\mathcal{L}$ by $\omega \in \{1, -1\}$. Put $k = \mathrm{ord}_p(r)$. Since $d = k/\gcd(k, f)$ is even, $(\mu_k)^f = \mu_d$ contains ω. In μ_k, we choose A with $A^f = \omega$. In any extension of Θ_*, the b-eigenvalue on $\mathcal{L}$ has f-th power ω. Thus, a twist from $\mathrm{Irr}(C_f)$ gives the eigenvalue A.

Given $\sigma \in \mathcal{H}_p$, we write $\sigma(\zeta_r) = \zeta_r^a$ with $a \in \mathbb{F}_r^\times$. The isometry $g_a : x_j \mapsto a^{j-h}x_j$ satisfies $g_a b = bg_a$, $g_a w = wg_a$ and $g_a X g_a^{-1} = aX$. Hence, $\theta_X^\sigma(u) = \theta_X(g_a^{-1}ug_a)$, and g_a takes $\mathcal{L}$ to $\mathrm{Hom}_U(\theta_X^\sigma, \Theta_*)$. The b-eigenvalue there is A. In U_*^σ it is $A^\sigma = A$, since $k \mid p-1$. Moreover, $U_*^\sigma|_{O(W_*)} = U_*|_{O(W_*)}$. Thus, the quotient character has $k_\sigma(b) = 1$, giving $U_*^\sigma = U_*$. Given an anisotropic plane E, take the norm form on $\mathbb{F}_{q^2}$. Thus, Θ_* extends the torus character ξ with $\xi^2 = 1$ to $O_2^-(q)$. The lift $b : z \mapsto z^r$ has $b^f : z \mapsto z^q$. Since $\Theta_*(b^f) \in \{1, -1\}$, take its f-th root in μ_k. If $W_* = 0$, we

set $U_* = 1$. Thus, the paths for $\lambda_{\epsilon,j}$ give fixed extensions of Θ_Λ, including equal rows of Λ.

Now take a quartic partition ρ of size $l > 0$. Since f is even, $i \in \mathbb{F}_q$. Let P_L stabilize a singular l-space L, and put $Z_\pm = (\psi^{\pm 1} \circ \det) U_\rho$, where ψ has order 4 and U_ρ is unipotent for $\mathrm{GL}_l(q)$. Thus, we choose Y on W from the involution construction. Let $R_\pm$ be the inductions of $Z_\pm \boxtimes Y$ from P_L to $O(V)$. If $W \neq 0$, P_L contains an improper isometry of W, so $R_\pm|_{\mathrm{SO}(V)}$ is ordinary induction from $Z_\pm \boxtimes Y|_{\mathrm{SO}(W)}$. The dual centralizer $C^*(t)$ fixes E_i, E_{-i} separately and lies in L^*. Hence, by [208, Proposition 7.3], each constituent η of $Y|_{\mathrm{SO}(W)}$ gives a distinct θ_η with quartic partition ρ. If there are two constituents η_1, η_2, an improper isometry r_W satisfies $\eta_1^{r_W} = \eta_2$ and $\theta_{\eta_1}^{r_W} = \theta_{\eta_2}$. Thus, $R_\pm$ are irreducible by Clifford theory [43].

If $W = 0$, $P_L \leq \mathrm{SO}(V)$. The maximal singular spaces have two $\mathrm{SO}(V)$-orbits $\mathcal{O}_1, \mathcal{O}_2$. The dual elements t_1, t_2 are properly nonconjugate. Since $x \in C^*(t_i)$ has $\det x = 1$, the improper isometry exchanges their $\mathrm{SO}(V)$-characters. Thus, $O(V)$ has one induced irreducible χ. The complete flag module M_{flag} contains U_ρ. At each step on E_i, E_{-i}, the signs of Λ_+, Λ_- are $+, -$ by [149, (6.11)]. Hence, the ordered signs of $R_\pm$ are those of Y.

Take the opposite operator T on R_+, R_- from Proposition 6.81. Its map on W is 1_W, so T is defined for isometries. Thus, $T^2 = q^{l \dim W + l(l-1)/2} I$ on both summands. On $R_\pm$, we choose the field representative φ. On each new pair H, it acts by coordinate Frobenius. Thus, φ^f acts by right translation by a_W, the old power on W. Moreover, the normalized B has $B\varphi = \varphi B$ and $B^2 = 1$. Since f is even, the correction $B^{a_r f}$ is I. The operators C, F satisfy $CF = k_\sigma(F_r)FC$ by the residual relation. If $a_r = 1$, the normalization ν of B and graph matrix G are rational. Thus, $CB = BC$ in this case. If $a_r = 0$, the field operator is F. Hence, $B^{a_\sigma} C$ has ratio $k_\sigma(F_r)$ on $\Phi = B^{a_r} F$. The multiplier m_{sim} is 1 on $O(V)$. Finally, for any inner correction $g \in O(V)$ to φ, $k_\sigma(g) = 1$. Thus, isomorphism of $\widehat{G}_1, \widehat{G}_2$ matches the carried power P_{fld}. □

Lemma 6.87. *Suppose that H_* is split cuspidal unipotent of rank $b = s^2 \geq 4$, where $s > 0$ is even. Let $U_{*,\gamma}$ be its rational graph extension from G_{ad}, inflated trivially on Z_{sc}. Then, take a compatible field extension U_*. For the global $I(1,b)$-parent χ, take the split negative torus T_- of rank* 1. *In* Proposition 6.80*, marked lines from H_* give U_{loc}. Thus, for χ, U_{loc}, the extensions have equal k_σ in* Lemma 6.86.

Proof. Take $U_{*,\gamma}$ from [61, Theorem 3]. By [156, §§7–8], U_* extends across $\langle \gamma, \phi \rangle$. If the restrictions differ on $\langle G_{\mathrm{ad}}, \gamma \rangle$, twist U_* by τ with $\tau(\gamma) = -1$, $\tau(\phi) = 1$ and $\tau|_{G_{\mathrm{ad}}} = 1$. Thus, both inductions start from U_*. Write $N = L \oplus L'$ for the split negative plane. On $P_J = \mathrm{Stab}_J(L)$, take $\vartheta_J(x,y) = \eta_q(x) U_*(y)$. Thus, the blocks on $L, N^\perp, L'$ are $x, y, \mu(y)/x$. For the dual volume v_- of N, the centralizer C_- is connected and lies in $L^\vee$. The reflections r_-, r_+ on $N, N^\perp$ satisfy $v_-^{r_- r_+} = -v_-$, and $r_- r_+$ fixes the unipotent labels λ_+, λ_-. Thus, $\chi = \mathrm{Ind}_{P_J}^J \vartheta_J$ is irreducible and $\chi\epsilon = \chi$ for the quadratic multiplier character ϵ. By [208, Proposition 7.25, Step 1], its K-constituents χ_+, χ_- each have multiplicity 1. Since $\dim W_+ \neq \dim W_-$, the spinor twist ϵ_{sp} cannot exchange the sign spaces. Hence, $\chi_\pm|_\Omega$ are distinct.

With $W = N^\perp$, put $P_O = \mathrm{Stab}_O(L)$ and $P_K = P_O \cap K$. Take ϑ_O inflated from $\eta_q \boxtimes (U_*|_{O(W)})$, and put $\mathcal{U} = \mathrm{Ind}_{P_O}^O \vartheta_O$. Since $O = KP_O$, $\mathcal{U}|_K = \mathrm{Ind}_{P_K}^K(\vartheta_O|_{P_K})$. In O, we choose w with $w(L) = L'$, $w(L') = L$ and $w|_W = 1$. Given $f \in \mathcal{U}$, we put $(\mathcal{T}f)(x) = \sum_{v \in U_{P_O}} f(wvx)$. As in Proposition 6.81, this is in $\mathrm{End}_O(\mathcal{U})$ since $\eta_q^{-1} = \eta_q$. For f supported on P_O with $f(1) \neq 0$, we have that $(\mathcal{T}f)(1) = 0$ and $(\mathcal{T}f)(w) = f(1)$, as $wvw \in P_O$ gives $v = 1$. Thus, $\mathcal{T}$ is nonscalar and $\dim \mathrm{End}_O(\mathcal{U}) = \dim \mathrm{End}_K(\mathcal{U}) = 2$. Consequently, $\chi_\pm$ are individually O-stable. By [165, Theorem A(i),(ii)], $\mathbb{Q}(\chi_\pm) = \mathbb{Q}$ and $\chi_\pm^\phi = \chi_\pm$. By [208, Lemma 3.13], the multiplicity 1 gives extensions $U_\pm$ to $O\langle\phi\rangle$. Their coefficient twists satisfy $U_\pm^\sigma = U_\pm k_\sigma$ by Lemma 6.86. If $g \in J$ has nonsquare multiplier, then $\chi_+^g = \chi_-$ on K.

Given U_{loc}, take a path $H_0 = H_*, \dots, H_m = H$ of ordinary 1-hooks, with $U_0 = U_*$. The source on P_i is $1_{\mathrm{GL}_1} \otimes U_{i-1}$. Since H_i is nondegenerate, $H_i^\gamma = H_i$. Moreover, rationality gives $\chi_{H_i}^\phi = \chi_{H_i}$. The 1-hook coefficient at χ_{H_i} is 1 by [208, Lemma 7.17]. By [208, Lemma 3.13], induction gives the extension U_i on proper similitudes. Thus, $U_i^\sigma = U_i k_\sigma$ and $U_i(z) = U_i(1)$ at $z \in Z_{\mathrm{sc}}$. Let R_{flag} be induction in Proposition 6.80. Then, χ_{mix} occurs with multiplicity 1 in $R_{\mathrm{flag}}(U_m)$. Its extension has $U_{\mathrm{mix}}^\sigma = U_{\mathrm{mix}} k_\sigma$. For the split extension, Lemma 6.86 gives $U_{\mathrm{split}}^\sigma = U_{\mathrm{split}} k_\sigma$. In unitary type $2 \nmid f$, so take the real U_*. Then, $U_{\mathrm{iso}}, U_{\mathrm{fld}}$ are rational by Lemma 6.82, with $k_\sigma = 1$. □

Let W be split and positive, and let Λ be nondegenerate of defect 0. Write $J_W = \mathrm{GO}(W)$ with field operator ϕ. Then, χ_Λ has a rational module V_Λ on $J_W\langle\phi\rangle$, with trivial action of Z_{sc}. At Λ_0, the trivial character 1_T on the split rank-one torus T has such a module. Then, take $\Lambda_0, \dots, \Lambda_m = \Lambda$ by 1-hooks through nondegenerate symbols. At step i, choose P_i over $\mathbb{F}_r$, with linear character 1_{GL_1}. Since $J_i = P_i \mathrm{SO}(W_i)$, restriction to $\mathrm{SO}(W_i)$ is ordinary induction. Thus, at Λ_i, Proposition 6.4 gives multiplicity 1.

The split pinning $\mathcal{P}$ and the double cosets BwB are fixed by ϕ. Moreover, a proper diagonal automorphism δ fixes every Weyl label λ. The graph γ exchanges the end nodes of D_b. If $\lambda \neq \mu$, restriction from $W(B_b)$ to $W(D_b)$ gives that γ fixes $\{\lambda, \mu\}$ by Pan [149, §§2.1, 3.3]. At rank 2, γ exchanges the two A_1-factors and fixes $(1,1)$ and $(\mathrm{St}, \mathrm{St})$. The source at i is $1_{\mathrm{GL}_1} \otimes V_{\Lambda_{i-1}}$ inflated to $P_i\langle\phi\rangle$, with flag induction $\mathcal{F}_i$. By Zhang [208, Lemma 6.8], $V_{\Lambda_i} = e_{\chi_{\Lambda_i}}\mathcal{F}_i$ is a rational extension to $J_i\langle\phi\rangle$. On $\mathrm{GSO}(W)$, it lies in unipotent induction. Hence, $V_\Lambda(z) = I$ for $z \in Z_{\mathrm{sc}}$. This applies at all ranks of W and all $d = \mathrm{ord}_p(q)$.

Lemma 6.88. *Write S for the spin image. Given $I(1,1)$ of opposite types, we let $\chi_1, \chi_2 \in \mathrm{Irr}(S)$ be the two orientations. Then, each χ_i has a rational isometry model M_i and a rational field model with $a^f = \gamma$. The graph γ has order 2 on the minus space V_- of dimension 4. Thus, for the negative split plane P_{sp}, the scalar at $-I$ is δ. On the anisotropic plane P_{an}, it is $(-1)^{(q+1)/2}$.*

Proof. Put $k = \mathbb{F}_{q^2}$ and take the Hermitian 2×2 matrix model H of V_- with form $\det H$. Then, $S = \mathrm{PSL}_2(k)$ acts by $H \mapsto xHx^{q,\mathsf{T}}$. Over k, take the rational function space $\mathcal{R}$ of Lemma 6.56 with the quadratic character η_k. Write π for the action on $\mathcal{R}$. Since $\eta_k(-1) = 1$, its restriction to $\mathrm{SL}_2(k)$ factors through S. A singular line L is spanned by $vv^{q,\mathsf{T}}$. The stabilizer acts on L with scalar $\mathrm{N}_{k/\mathbb{F}_q}(b)$ at diagonal entry b. Thus, its quadratic character is $\eta_k(b)$, the source for negative split L.

For the unnormalized kernel A over k, by Lemma 6.56, $A^2 = q^2 I$ and $A\pi(g) = \pi(g)A$ for $g \in \mathrm{SL}_2(k)$. Since A has zero diagonal, the rational eigenspaces $E_\pm = \ker(A \mp qI)$ have dimension $(q^2+1)/2$. Let B_k be a Borel subgroup of $\mathrm{SL}_2(k)$, with Weyl representative w. Since $\eta_k^{-1} = \eta_k$, [95, Problem 5.6] gives contribution 1 from each of $B_k, B_k w B_k$. Thus, $\dim \mathrm{End}_S(\mathcal{R}) = 2$ and E_+, E_- give distinct irreducibles. Moreover, the r-power permutation a satisfies $aA = Aa$. Given x of nonsquare determinant in $k^\times$, the determinant rule gives $\pi(x)A = -A\pi(x)$.

Thus, the transpose γ of H induces the q-field action on $\mathrm{PSL}_2(k)$. $\det(\gamma|_{V_-}) = -1$, and $-I$ represents the nontrivial SO/Ω coset. Hence, the orders give $O_4^-(q) = \big(\mathrm{PSL}_2(k) \rtimes \langle\gamma\rangle\big) \times \langle -I\rangle$. On $E_\pm$, take the r-power permutation for a and a^f for γ. For the ordered negative plane P_-, let η_{T_-} be quadratic on its torus T_-. Thus, at the volume, choose $\eta_{T_-}(-1) = \delta$ or $(-1)^{(q+1)/2}$ for P_{sp} or P_{an}, respectively. Thus, $E_\pm$ have isometry models with $a^f = \gamma$. The nonsquare determinant of x has nonsquare norm multiplier over $\mathbb{F}_q$. Thus, $\pi(x)E_+ = E_-$. At equal ranks $P_{\mathrm{sp}}, P_{\mathrm{an}}$ have distinct types over $\mathbb{F}_q$. □

Suppose that Λ_+, Λ_- are nondegenerate of opposite types. Let $\mathcal{P}_+, \mathcal{P}_-$ be their ordinary hook paths to H_+, H_- of ranks n_+, n_-. Then, $n_+ \neq n_-$ unless $\{s_+, s_-\} = \{0, 1\}$. If $n_+ \neq n_-$, the endpoint sources $S_{\mathrm{cusp}}, S_{\mathrm{tor}}$ are from Proposition 6.84. If $n_+ = n_- = 1$, take the sources of Lemma 6.88. On $\mathcal{M}$, the orientation ω has multiplicity 1. Indeed, V_+, V_- differ in rank or type. Thus, $[V_+] \neq [V_-]$ under $O(V)$. In the complements of L_+, L_-, take reflections r_+, r_- down to rank 1. On the connected regular group G°_{reg}, the coefficient is 1. Moreover, we have that m_1, m_2 are conjugate by a similitude g with nonsquare multiplier $\mu(g)$. Thus, we have that π_{graph} has rows and columns from m_1, m_2. The natural graph γ is the one on ranks $2, 3, 4$.

On the nonsplit block W_{ns} of the minus space V_-, choose the reflection γ over $\mathbb{F}_r$. The rational structure is $\mathrm{Int}(\gamma)F_q$, so the lift a has $a^f = \gamma$. If γ' has determinant -1, we find $\gamma'\gamma^{-1} \in \mathrm{SO}(V_-)$. By Digne–Michel [58, Theorem 4.2.9] for $F_\gamma = \mathrm{Int}(\gamma)F_q$, choose $h \in \mathrm{SO}(V_-)$ with $hF_\gamma(h)^{-1} = \gamma'\gamma^{-1}$. Transport a by $\mathrm{Int}(h)$. On the added split quartic space W_{quart}, we put $\gamma|_{W_{\mathrm{quart}}} = 1$. Thus, the operator F of Proposition 6.81 has $F^f = R(\gamma)$. Indeed, on an induced function u, left covariance gives $R_{\mathrm{res}}(\gamma)u(\gamma^{-1}x\gamma) = u(x\gamma)$. Since $w_0|_W = 1$, the opposite correction gives $(B^{a_r}F)^f = R(\gamma)$.

For the unitary space W_{unit}, take the integral-trace operator T_{tr} with γ from Lemma 6.82 on either form of $W \neq 0$. The field lift a acts on $\{\omega_1, \omega_2\}$ and a^f fixes both orientations. Thus, for odd f, we have that a fixes each ω_i. By Lemma 6.82, $\widetilde{\omega}_i$ are rational. Over C_f, Navarro–Tiep [143, Lemma 2.1] gives their unique real extensions $\widehat{\omega}_i$. Hence, $\widehat{\omega}_1, \widehat{\omega}_2$ carry the field characters $k_{\sigma,1}, k_{\sigma,2}$ on the minus model.

Proposition 6.89. *Suppose that Λ_+, Λ_- are nondegenerate with $\epsilon_+\epsilon_- = -1$. Their core ranks satisfy $n_+ \neq n_-$. Let $\mathcal{D}_{\mathrm{uneq}}$ be the* degree domain for $Q(a,c)$. *Then, with source $\mathcal{S}$ and transport $\mathcal{T}$, the comparison $\mathcal{C}$ holds on $\mathcal{D}_{\mathrm{uneq}}$, including $u_+ = 0$. The sole residual orthogonal label H is unipotent on W of rank e and type $-t$.*

Proof. By the common ν-twist [149, Corollary 8.14], we choose lifts $\widetilde{\chi}_{\mathrm{gl}}, \widetilde{\chi}_{\mathrm{loc}}$ on $N, N_N(Q_0)$ with $b_- = 0$. By Proposition 6.71, we have $b_- = 0$, $b_+ = e$ and $c_0 = 0$. Moreover, $u_- = v$ and $u_+ = v - 1$. We have $a = ev$ and $c = ew$ with $v, w \geq 1$. e is odd, $\tau = t$ and the sole core C has sign $-t$. Since e is odd, C is nondegenerate. By Lemma 6.75, Π has independent cyclic scalars z_+, z_-, z_A. If $t = 1$, the field scalar is $z_+z_-z_A$. If $t = -1$, we write $v_A = v$ from

Lemma 6.75. Then, we have that $v_A j^{-1}$ has even exponents at each i. Thus, these formulas apply at sizes $v-1, v, w$, with scalar 1 on an absent colour. On C_0, the norm character ν is trivial for both quartic forms Q_1, Q_2.

If $t=-1$, then $q \equiv 3 \bmod 4$ and H is split. Its graph extension $U_{H,\gamma}$ is rational on G_{ad} by [61, Theorem 3], with 1_T for rank one. Inflate to J trivially on Z_{sc}. Since $|C_f|$ is odd, ϕ fixes both choices of $U_{H,\gamma}$. Thus, by Lemma 6.82, the unique real U_H has values in $\mathbb{Q}$. If $t=1$, Λ_- has defect 0 and split torus endpoint T of rank 1. The symbols Λ_+, H share the odd half-defect s and endpoint H_*. If $s=1$, the endpoint label is $1_{T_{\mathrm{an}}}$ on the anisotropic torus T_{an}. On G_{ad}, choose a common extension U_* for graph and field. Moreover, $U_*|_{G_{\mathrm{ad}}}$ has values in $\mathbb{Q}$. In twisted type, $\phi^f = \gamma$. Thus, a character $\lambda \in \mathrm{Irr}(C_{2f})$ with $\lambda(\phi)^f = -1$ adjusts the sign at γ. Both paths inherit k_σ on $\widehat{J}$.

In Lemma 6.87, the line formula on $L \le N_- \le V_-$ gives the source $\eta_q \boxtimes U_*$. Since $g \in C^\circ$ fixes the dual volume v_-, stabilizing N_- gives $\det(g|_{N_-}) = 1$. Thus, the rational types $\epsilon_1 \neq \epsilon_2$ distinguish ω_1, ω_2 even for $n_1 = n_2$. For the two rank-one tori T_1, T_2, Lemma 6.88 gives $k_\sigma = 1$. Next, take the local inductions R_i on marked lines along $H_0 = H_*, H_1, \dots, H_m = H$. The coefficient $\langle R_i(\chi_{H_{i-1}}), \chi_{H_i}\rangle$ is 1 by Proposition 6.4. Each H_i is nondegenerate, so γ, ϕ fix χ_{H_i}, and Z_{sc} acts trivially. By Lemma 6.86, the projections e_i carry k_σ on $\widehat{J}$. Moreover, the active root ζ and correction h are coefficient-fixed.

Write Ξ for the local tensor with multiplicities $v-1, v$ on T. A quotient twist λ containing ν gives $v, v-1$. Thus, for g centralizing T, the weights of $\Xi^g, \Xi\lambda$ on T differ. Moreover, $\Theta_H|_{\mathrm{SO}(W)}$ is irreducible. Hence, $\Theta_H \det \neq \Theta_H$, and the restriction $\Xi_0 = \Xi|_{B_Q^0}$ is irreducible by [43]. Let g have nonsquare multiplier. Its action fixes Θ_H and gives $\eta_- \mapsto \eta_- \varepsilon$. Since Λ_- is nondegenerate over the empty core C_-, $\eta_-\varepsilon \neq \eta_-$. The tests for $\nu, \det$ exclude every quotient correction λ, so the local orientations ω_1, ω_2 are distinct as on N.

At the residual volume ω_R, the scalar is 1. If $t=1$, then $s_- = 0$ and the v quadratic colours contribute δ^v. If $t=-1$, then $s_- \equiv v \bmod 2$ and they contribute $(-\delta^e)^v$. The w quartic colours contribute $(-1)^{cl_2}$. Hence, at $-I$, the scalar ϵ_{loc} matches the global scalar at $-I$, including $-I \notin \Omega$. The local sign is $t^{2v-1+w}(-t) = -\tau^c$, equal to the global sign. Thus, T_{rat} and the section s_{cen} match k_σ and $\boldsymbol{\alpha}$. Moreover, the degree cut keeps $(2v-1+w)!/((v-1)!v!w!)$. □

Theorem 6.90. *Let $N = \mathrm{Spin}^\epsilon_{2n}(q)$ for odd q, with $n = 2a + c \ge 5$ and $c \ge 1$. Let p be odd with $p \nmid q$ and $p \mid |N|$. Suppose that χ has separately rational form $Q(a,c)$ with $z_0 \in \ker \chi$. Suppose that the original symbols Λ_+, Λ_- are nondegenerate. On the* degree domain $\mathcal{D}_{\mathrm{uneq}}$*, we have that the $\mathcal{H}_p$-comparison $\succeq_c$ holds at $N_N(Q_0)$ for either period ϵ_{tor}, every f and every admissible pair C_+, C_-.*

Proof. Write $q = r^f$ with r prime and $d = \mathrm{ord}_p(q)$. If $\Lambda_+ = \Lambda_-$, we apply Theorem 6.70. Suppose that $\Lambda_+ \neq \Lambda_-$ as unordered symbols. By Proposition 6.260, I pairs parents on $O(V)$ with parents on $B_Q = O_A \times O_R$. On (Λ_+, Λ_-), the twist by $\varepsilon_{\det}$ gives $(\Lambda_+^{\mathsf{t}}, \Lambda_-^{\mathsf{t}})$ and the twist by $\varepsilon_{\mathrm{sp}}$ gives (Λ_-, Λ_+) by [149, Corollaries 8.13 and 8.14]. Thus, Θ is moved by $\varepsilon_{\det}, \varepsilon_{\mathrm{sp}}, \varepsilon_{\det}\varepsilon_{\mathrm{sp}}$. By [43], the restrictions χ_G, χ_B to $\Omega(V), B_Q^0 = B_Q \cap \Omega(V)$ are irreducible. Hence, the inverse I descends to χ_G, χ_B. By [149, Corollary 8.3], a nonsquare similitude g gives $(\Lambda_+, \Lambda_-) \mapsto (\Lambda_+, \Lambda_-^{\mathsf{t}})$. On the negative torus weight ω_-, the action h shifts each cyclic label by e, and h acts on C_- as in Section 6.1. Since $\Lambda_\pm^{\mathsf{t}} \neq \Lambda_\pm$, the action of g moves both χ_G, χ_B. Thus, this includes $W_- = 0$ or $W_+ = 0$ and $u_+ \neq u_-$.

Let Θ be a full isometry parent on W. The argument $\mathcal{A}$ in the proof of Proposition 6.261 gives $\mathbb{Q}(\Theta) = \mathbb{Q}$ from its involution source Θ_* and ordinary quartic induction R_{quart}. If $W = 0$, set $\Theta = 1$. By Lemma 6.86, $\Theta^\phi = \Theta$ for each semilinear isometry ϕ that induces r-Frobenius. Since C_f is cyclic, Θ extends by [95, Corollary 11.22]. If $2 \nmid f$, [143, Lemma 2.1 and Corollary 2.2] gives the unique real extension U_{real} with values in $\mathbb{Q}$.

The active counts u_+, u_-, w include the boundary $\mathcal{B}_{\mathrm{uneq}}$ of Proposition 6.71. Thus, on $T = \prod_i T_i$, take the ordered weight ϑ of multiplicities u_+, u_-, w. On V_i^+, choose K_i and its trace dual $K_i^\vee$. On V_j^-, we choose L_j/K_j quadratic with $Q_j(x) = \mathrm{Tr}_{K_j/\mathbb{F}_q}(\mathrm{N}_{L_j/K_j}(x))$. Then, we let $b : x \mapsto x^r$ on the field coordinates. Thus, rotations ρ_i, split inversions ι_i and block permutations ϖ satisfy $b\rho_i = \rho_i b$, $b\iota_i = \iota_i b$ and $b\varpi = \varpi b$. On V_i, the power b^f is the q-rotation $r_{q,i}$ of order $d = e$ or $d = 2e$. Moreover, in O_A, we find stabilizer $I_{\mathrm{rel}} = T \rtimes W_\vartheta$ for ϑ. The sign colours have cycles C_{2e} and the quartic inertia has cycles C_e. The block permutations in W_ϑ fix the colours.

If b inverts the quartic weight ϑ_4, we choose j to invert each quartic coordinate T_i and act trivially on T_+, T_-. Otherwise, we set $j = 1$. Thus, jb fixes ϑ and $(jb)x = x(jb)$ for $x \in W_\vartheta$. The power $(jb)^f$ lies in $Z(W_\vartheta)$ with order dividing d. Indeed, in positive period $r_{q,i}$ has order e. If $j \neq 1$ in that period, $r \equiv 3 \bmod 4$ and $q \equiv 1 \bmod 4$. Hence, $j^f = 1$. If $t=-1$, the quartic inversion is $\iota_{\mathrm{quart}} = r_{q,\mathrm{quart}}^e$. Write $\rho((jb)^f) = sI$ for the relative representation ρ, including the complement character $\varepsilon_{\mathrm{sp}}$ on the negative colour. Since $s \in \mu_d$, the f-power map from $\mu_{\mathrm{ord}_p(r)}$ gives ξ with $\xi^f = s$. Thus, we extend ρ to $\widehat{\rho}$ with $\widehat{\rho}(jb) = \xi I$. Induction from $\langle I_{\mathrm{rel}}, jb\rangle$ to $\langle O_A, b\rangle$ gives $\widehat{\Pi}_{\mathrm{act}}|_{O_A} = \Pi_{\mathrm{act}}$,

irreducible on O_A by the inertia construction. Since $2e, \mathrm{ord}_p(r) \mid p-1$, the relative labels η and ξ are fixed by $\mathcal{H}_p$. If $\sigma \in \mathcal{H}_p$ inverts ϑ_4, take the quartic inversion j_σ, and otherwise take $j_\sigma = 1$. Then, $\vartheta^\sigma = \vartheta^{j_\sigma}$. Moreover, j_σ centralizes jb and W_ϑ. Thus, $\widehat{\rho}^\sigma \simeq \widehat{\rho}^{j_\sigma}$. Hence, $\widehat{\Pi}^\sigma_{\mathrm{act}} = \widehat{\Pi}_{\mathrm{act}}$ on $\langle O_A, b\rangle$.

Let U_G, U_R extend Θ_G, Θ_R to the semilinear groups $\widehat{G}, \widehat{R}$. The action on $\mathbb{F}_q$ gives an isomorphism $C_G \cong C_R$. If f is odd, take the rational extensions U_G, U_R as above. If f is even and $\epsilon_{\mathrm{tor}} = +$, packing moves beads within X, so $d_{G,+} = d_{R,+}$ and $d_{G,-} = d_{R,-}$. Thus, by Lemma 6.86, one Θ_* gives U_G, U_R with transition k_σ, including $\lambda_G \neq \lambda_R$. In negative period $d = 2e$, the same lemma gives $\mathcal{H}_p$-fixed extensions on both groups. Hence, $U_G^\sigma = U_G k_\sigma$, $U_R^\sigma = U_R k_\sigma$ ($\sigma \in \mathcal{H}_p$), with one $k_\sigma \in \mathrm{Irr}(C_f)$. Since $\widehat{\Pi}_{\mathrm{act}}$ is fixed by $\mathcal{H}_p$, its tensor with U_R has transition k_σ on the local field cosets $\mathcal{C}_{\mathrm{fld}}$. Given an isometry h, the lift $h\phi$ inherits the formulas since $k_\sigma(h) = 1$.

Take the cover E of $\Omega(V)$ from Theorem 6.77. Thus, for $E_0 = N_E(\overline{Q}_0)$, Sylow conjugacy gives $E_0/B_Q^0 \cong E/\Omega(V)$. The nonsquare multiplier $\mu(g)$ exchanges orientation o with o'. Moreover, field actions ϕ and coefficient actions $\sigma \in \mathcal{H}_p$ fix the restrictions χ_G, χ_B. Thus, the stabilizer of branch $\mathfrak{b}$ is represented in E. If $\mu(x) = z^2$, $z^{-1}x$ is an isometry determined up to $\{\pm I\}$. By Proposition 6.260, the scalar ϵ_{-I} agrees at $-I$, including the active torus contribution for ϵ_{tor}. Moreover, $\chi_{\mathrm{gl}}, \chi_{\mathrm{loc}}$ are trivial on z_0. Thus, a common section s gives $\alpha_G = \alpha_R$, $c_G = c_R$ and $\mu_{G,\sigma} = \mu_{R,\sigma}$, with transition k_σ on field cosets. Conjugation by g gives these identities for o'. □

Suppose that $\mathfrak{s}_-$ is degenerate and $\mathfrak{s}_+$ is nondegenerate. Given s_{lift}, we fix the ordering o on both sides. Thus, its packet $\mathcal{P}(s_{\mathrm{lift}})$ includes the degenerate choices. Let P, M be independent quotient modules for $1, \eta_q$. Then, packing $\mathfrak{s}_-$ gives a degenerate core C_-, with $M\varepsilon = M$ and $2 \mid u_-$. Thus, we find that a similitude g of nonsquare multiplier $\mu(g)$ can stabilize χ. On the common section s, keep the transition of $\widetilde{\chi}$ at g.

Lemma 6.91. *Take the complement labels $P, M\alpha, \eta_A$ of* Section 6.3 *with weights u_+, u_-. Then, we have that $\widetilde{\chi}$ extends to the full cover $\widehat{J}$ for similitudes and fields. On zI, the scalar character is λ_b^{ew}, and the Galois transition is $(\alpha_\sigma \circ \mu)^{ew}$. Both characters have value 1 at the field generator b and are trivial if $w = 0$.*

Proof. On M, take the K_0-rational involution L_M of Lemma 6.64. Since α is scalar, including permutation sign sgn, L_M implements ε on $M\alpha$ as well. On P, take the identity 1_P. If $t = 1$, set $h_m = \mathrm{diag}(mI_e, I_e)$ on each positive block V_i. Given $m, m' \in \mathbb{F}_q^\times$, $h_m h_{m'} = h_{mm'}$. The generator R_i acquires the torus correction $t_i(m)$, with quadratic value $\eta_q(m) = (-1)^{a(m)}$. Thus, on $P \otimes M$, put $D_m = 1 \otimes L_M^{a(m)}$. For a block permutation π stabilizing the colours, let T_π be its operator. Then, $D_m D_{m'} = D_{mm'}$ and $D_m T_\pi = T_\pi D_m$. The scalar zI differs from h_{z^2} by t_z of quadratic value $\eta_q(z)^{eu_-} = 1$. Let j invert all coordinates. Then, $jh_m j^{-1}$ differs from h_m by a torus scalar $t_{j,m}$ of total quadratic value 1 since $2 \mid u_-$. Moreover, $a(m^r) = a(m)$. Thus, the orthogonal corrections κ_{orth} are trivial on $\widehat{J}$. For the quartic coordinates V_{quart}, we choose V_{plane} from Lemma 6.68 through the norm N_i on each e-block. Hence, λ_b, α_σ gain exponent ew. Moreover, the diagonal cycle scalar is $z_+ z_- z_A$, with $z_A = 1$ if $w = 0$.

On a negative $2e$-space V_i, set $L = \mathbb{F}_{q^{2e}}$ and $T_1 = \ker \mathrm{N}_{L/\mathbb{F}_{q^e}}$. Define the cyclic group $C = \{x \in L^\times : x^{q^e+1} \in \mathbb{F}_q^\times\}$ of order $(q-1)(q^e+1)$. Thus, $m = x^{q^e+1}$ maps C onto $\mathbb{F}_q^\times$ with kernel T_1. A negative generator R_i acts on torus scalars by q-power or q^{-1}-power. Its quadratic commutator value is $\eta_{T_1}(x^{1-q}) = \eta_q(m)$. Indeed, for a generator x of C, x^{q-1} generates T_1 and m generates $\mathbb{F}_q^\times$. Thus, the value holds for every $x \in C$ and either orientation of R_i. On $P \otimes M$, take $1 \otimes L_M^{a(m)}$. For the diagonal action of $x \in T_1$, the torus value is $\eta_{T_1}(x)^{u_-} = 1$. Hence, the representation ρ extends to the product with C, and its induction extends $\widetilde{\chi}$ since C gives every multiplier. On zI the scalar is 1, with $\mu(zI) = z^2$. If $w = 0$, the scalar field operator B_A of Lemma 6.75 implements the r-power map on C. Its cycle commutator κ is trivial on T_1, including even e.

If $t = -1$ and $w > 0$, e is odd and $q \equiv 3 \bmod 4$. The diagonal subgroup $\mathbb{F}_{q^2}^\times \leq L^\times$ gives every $m = x^{q+1}$. Thus, on V_{quart}, we choose $\theta(x)^{ew}$ from Lemma 6.68. On M, take the same $L_M^{a(m)}$. Given $x \in T_1 \cap \mathbb{F}_{q^2}^\times$, the quartic block norm to $\mathbb{F}_{q^2}$ is x^e. Thus, $\theta(x)^{ew}$ gives the given torus weight. Since R_i^2 centralizes $\mathbb{F}_{q^2}$, the partition label η_A is unrestricted. The field operator Φ_A acts by rq-power on $\mathbb{F}_{q^2}$, fixing θ since $rq \equiv 1 \bmod 8$. By Lemma 6.75, ξ has the given power for independent η_i. For the semilinear correction j^{a_σ}, the action of j on $\mathbb{F}_{q^2}$ is q-power. Thus, the quartic ratio is $\alpha_\sigma(m)^{ew}$. Given a complement representative v, $xv = vtx$ for $t \in T$. The semilinear identity holds on t. Hence, the transition $(\alpha_\sigma \circ \mu)^{ew}$ holds on the whole induction $\widetilde{\chi}$. On zI the scalar character is λ_b^{ew}. □

Proposition 6.92. *Suppose that $q \equiv 3 \bmod 4$ and $J_W = \mathrm{GO}(W)$. Let the mixed involution extension $\Theta_W \in \mathrm{Irr}(J_W)$ have $\mathbb{Q}(\Theta_W) = \mathbb{Q}$, $Z_{\mathrm{sc}} \leq \ker \Theta_W$ and $\Theta_W^\phi = \Theta_W$ for the field actions ϕ. Suppose that $\Theta_W|_{X_W}$ is irreducible for the*

spin image X_W. Then, for a unitary quartic partition λ with $l > 0$, we have that the extension $\widetilde{R}$ is defined on $\widehat{J}$ for similitudes and fields. On zI, its scalar character is λ_b^l, and its transition is $(\alpha_\sigma \circ \mu)^l$ on all of $\widehat{J}$.

Proof. Let V_1, V_2 be the geometric quartic spaces and L_{qs} their proper-similitude Levi. Thus, for $(g, y) \in L_{\mathrm{qs}}$ with $\mu(g) = \mu(y)$, we put $\beta(g, y) = \theta(\det_{\mathbb{F}_{q^2}} g)U(g) \otimes \Theta_W(y)$ with θ of order 8. As in Lemma 6.82, U is inflated from $\mathrm{PGU}_l(q)$. Moreover, its partition λ is fixed by γ, ϕ, and $\mathbb{Q}(U) = \mathbb{Q}$. Since $\beta|_{L_{\mathrm{iso}}}$ is irreducible on the isometry Levi L_{iso}, β is irreducible. The dual centralizer $\mathbf{C}^*$ in J^* fixes each of V_1, V_2 setwise. Thus, $\mathbf{C}^* \leq \mathbf{L}^*$, and signed induction gives one irreducible R on J by [72, Theorem 3.3.22]. The dual centralizer for K also fixes both spaces setwise, so $\mathbf{C}_K^* \leq \mathbf{L}_K^*$. By [72, Corollary 3.3.25], $R|_K$ is the oriented constituent for (Λ_+, Λ_-). The twist by ν_K gives (Λ_-, Λ_+), so $R|_X$ is irreducible by [43]. Moreover, r_W normalizes P_{qs} and fixes β by the extension on W. Hence, R extends to E. For either form of W, Lemma 6.82 gives that $R_+|_O, R_-|_O$ are rational and distinct.

Let b_σ be the exponent of σ on μ_8 and put $j = a_\sigma$. Then, by Lemma 6.68, $\theta^{b_\sigma} = \theta^{q^j}(\alpha_\sigma \circ \mathrm{N}_{q^2/q})$. On the unitary similitude g, $\mathrm{N}_{q^2/q}(\det_{\mathbb{F}_{q^2}} g) = \mu(g)^l$. On an orthonormal Hermitian basis, w acts by q-conjugation. If l is odd, w also reflects W. Thus, $w \in K$ acts by $\det \mapsto \det^q$ and fixes U. Its action on Θ_W is inner on $O(W)$. Hence, on L_{qs} $\beta^\sigma = \beta^{w^j}(\alpha_\sigma \circ \mu)^l$.

For the geometric parabolic $\mathbf{P}$, $w(\mathbf{P}) = F_q(\mathbf{P})$. The map $g\mathbf{U} \mapsto F_q(g)F_q(\mathbf{U})$ is a universal homeomorphism. Since F_q fixes J, L_{qs}, it gives the bimodule isomorphism $\mathcal{B}_{\mathbf{U}} \cong \mathcal{B}_{F_q(\mathbf{U})}$. Thus, the signed induction $\mathcal{I}$ of β^{w^j} gives R. By Lemma 6.82, integral traces give $\mathcal{I}(\gamma^\sigma) = \mathcal{I}(\gamma)^\sigma$ for a class function γ on L_{qs}. Now put $\lambda_\sigma = \alpha_\sigma \circ \mu$. Moreover, on $Y_{\mathbf{U}}$, the relation $g^{-1}F_q(g) \in \mathbf{U}F_q(\mathbf{U})$ gives $\mu(g) \in \mathbb{F}_q^\times$. Thus, a nonzero trace of (x, y) requires $\mu(x) = \mu(y)$, by the action $g \mapsto xgy^{-1}$ on multiplier components. Hence, $\mathcal{I}(\beta(\lambda_\sigma|_{L_{\mathrm{qs}}})^l) = \mathcal{I}(\beta)\lambda_\sigma^l$, so $R^\sigma = R\lambda_\sigma^l$ on J. For either E-extension $R_\pm$, this identity on O follows from rationality and $\lambda_\sigma|_O = 1$. The nontrivial quotient character $\delta \in \mathrm{Irr}(E/J)$ has $\delta|_O \neq 1$. Thus, the identities on J, O determine the transition on E. On the induction bimodule $\mathcal{B}$, the left and right actions of zI coincide. Hence, its scalar on R is $\lambda_b^l(z)$.

On θ, the prime-field action is the r- or rq-power map. Since f is odd, $r \equiv q \bmod 8$ and $rq \equiv 1 \bmod 8$. Thus, the field powers act by $\theta \mapsto \theta^q$ or θ, and the preceding parabolic argument fixes R. Let a be the field lift, with $a^f \in E$. Since $E\langle a\rangle/E$ has odd order f, each E-extension of R is a-stable. Put $A_\sigma = (\alpha_\sigma \circ \mu)^l$ with $A_\sigma(a) = 1$. This is well-defined since $\alpha_\sigma^2 = 1$, $\alpha_\sigma^a = \alpha_\sigma$ and $\mu(a^f) = 1$.

Fix $\widehat{R} \in \mathrm{Irr}(E)$ above R and let $\mathcal{T}$ be its extensions to $E\langle a\rangle$. Write c for complex conjugation and put $\tau(Z) = Z^c A_{\mathrm{conj}}^{-1}$ on $\mathcal{T}$. Thus, the identity on E gives $\tau(\mathcal{T}) = \mathcal{T}$ and $\tau^2 = 1$. Moreover, we have that $\tau(Z\xi) = \tau(Z)\xi^{-1}$ for $\xi \in \mathrm{Irr}(E\langle a\rangle/E)$. Since f is odd, a unique $Y \in \mathcal{T}$ satisfies $\tau(Y) = Y$. Moreover, $c\sigma = \sigma c$ and A_σ is rational. Thus, $\tau(Y^\sigma A_\sigma^{-1}) = Y^\sigma A_\sigma^{-1}$, and uniqueness gives $Y^\sigma = YA_\sigma$ on $E\langle a\rangle$. Thus, $\widetilde{R} = Y$ has $\xi_{\mathrm{fld}} = 1$, and $Y|_E$ may have nonreal values. □

Lemma 6.93. *Let E be a similitude group with isometries $O \lhd E$. Suppose that $\widehat{E} = E\langle a\rangle$ has cyclic quotient $\widehat{E}/E$ of order f, with $a^f \in O$. Put $B = O\langle a\rangle$. Let $U \in \mathrm{Irr}(E)$ be rational and $U^a = U$, with $U|_O$ irreducible. Then, each extension V of $U|_O$ to B gives a unique extension $\widehat{U}$ of U to $\widehat{E}$ with $\widehat{U}|_B = V$. If $V^\sigma = Vk_\sigma$ for $k_\sigma \in \mathrm{Irr}(B/O)$, then $\widehat{U}^\sigma = \widehat{U}k_\sigma$ under $B/O \cong \widehat{E}/E$.*

Proof. By Clifford theory [43], U extends to U_0 over $\widehat{E}$. Put $V_0 = U_0|_B$. Since $a^f \in O$ and aE has order f, the map $B/O \to \widehat{E}/E$ is an isomorphism. Specifically, $a^iO \mapsto a^iE$ for $0 \leq i < f$. Hence, the unique $\lambda \in \mathrm{Irr}(B/O)$ with $V_0\lambda = V$ gives $\widehat{U} = U_0\lambda$ by inflation. Since $U^\sigma = U$, both $\widehat{U}^\sigma, \widehat{U}k_\sigma$ extend U and restrict to Vk_σ. Thus, uniqueness gives $\widehat{U}^\sigma = \widehat{U}k_\sigma$. □

For the involution s, we let W be positive. Next, we choose $h \in \mathrm{Irr}(\mathrm{O}(W))$ with nondegenerate unipotent symbol Λ. With $K_W = \mathrm{SO}(W)$, we write $J_W = \mathrm{GSO}(W)$ and $E_W = \mathrm{GO}(W)$. Since Λ is nondegenerate, $h|_{K_W}$ is irreducible. By the regular inclusion ι of Dudas–Malle [61, §2.2], its unique unipotent lift is H_0 on J_W. The quotient J_W/K_W is cyclic. Thus, $\lambda \neq 1$ as a multiplier character gives $H_0\lambda \notin \mathcal{E}_1$. On a scalar zI, we have that $H_0(zI) = H_0(1)$ as for $R_T^{J_W}(1)$. Since graph and field actions fix $\mathcal{E}_1, h|_{K_W}$, they fix H_0. Moreover, we have that h is rational by Lemma 6.245 and $\mathcal{E}_1$ is Galois-stable. Hence, we find $\mathbb{Q}(H_0) = \mathbb{Q}$.

Extend H_0 across E_W/J_W of order 2. The two extensions $U_\pm$ differ by the sign ε with $\varepsilon|_{\mathrm{O}(W)} = \det$. Thus, one U has $U|_{\mathrm{O}(W)} = h$. Field invariance of h follows from Lemma 6.86. Hence, uniqueness of U gives $U^\phi = U$ and $\mathbb{Q}(U) = \mathbb{Q}$. Given a plane W, $h|_{K_W} = 1$, so $U = 1$ or ε. If $W = 0$, we set $U = 1$. Thus, for $B = \mathrm{O}(W)\langle a\rangle$ and a chosen extension V of h, by Lemma 6.93, we obtain $\widehat{U}|_B = V$ with its transition k_σ.

Put the degenerate symbol (λ,λ) on the negative space, with $b=2|\lambda|>0$. Thus, the involution space is $H^b\perp W$, with H^b hyperbolic. Let P_L stabilize the marked singular b-space L. On the ordinary Levi L_P, the three blocks are $g, y, \mu(y)g^{-\mathsf{T}}$ for $g\in \mathrm{GL}_b(q)$, $y\in E_W$. Moreover, the field lift a_W extends by coordinate Frobenius on H^b. Thus, P_L meets every multiplier and field coset C for either form of W. Induce $\Xi=((\eta_q\circ\det)U_{2\lambda})\boxtimes\widehat{U}$ from P_L. The partial flag module $\mathcal{F}_{2\lambda}$ over $\mathbb{Q}$ contains $U_{2\lambda}$ with multiplicity 1. Thus, we have that $U_{2\lambda}$ has a field extension over $\mathbb{Q}$. Since $U_{2\lambda}, H_0$ are trivial on scalars and $\eta_q(z)^b=1$, we have that zI acts trivially on Ξ.

Put $B=\mathrm{O}(H^b\perp W)$, $J=\Omega(H^b\perp W)$ and $\Delta=\det|_B$. For the ordered parent Θ with label h, put $\chi=\Theta|_J$, irreducible by the folded formula on J. Next, for each partition τ of b, replace 2λ by τ in Ξ. By Witt transitivity, induction restricts on B to $F_\tau(h)$ of Section 6.1. Thus, each semilinear similarity g fixes $F_\tau(h)$. Since U_τ, h are rational, $F_\tau(h)$ has values in $\mathbb{Q}$.

Let $u=g\sigma$ with $\sigma\in\mathcal{H}_p$. Then, $\Delta^u=\Delta$. The torus pairing $\mathcal{T}$ before the folded formula on B gives $\langle F_\tau(h),(\Theta+\Theta\Delta)^u\rangle_B=\langle F_\tau(h),\Theta+\Theta\Delta\rangle_B=[s_\tau]s_\lambda^2$. Since the coefficient is 1 at $\tau=2\lambda$, choose the member Ψ occurring in $F_{2\lambda}(h)$. By Pan [149, (6.11), (6.14) and Theorem 8.12], quadratic-line induction fixes h up to row order. Thus, for Λ_-, the split principal-series label is (μ,ω). On $\mathfrak{S}_b$, the character of $W(B_b)$ for (μ,ω) restricts to the induced product of S^μ, S^ω. For equal rows, take the sum $\psi_+ + \psi_-$ of the two connected characters. Thus, $\langle F_\tau(h),\Psi+\Psi\Delta\rangle_B=[s_\tau]s_\mu s_\omega$. Since $\{\Psi,\Psi\Delta\}$ is the transformed pair, $s_\mu s_\omega = s_\lambda^2$. By Section 6.1, this Schur product gives $\mu=\omega=\lambda$. Hence, the ordered parents are $\Theta,\Theta\Delta$ and $\chi^u=\chi$, also for nonsquare $\mu(g)$. The folded coefficient is 1 at $\tau=2\lambda$. Thus, in the semilinear flag module $\mathcal{F}$, $V_\chi=e_\chi\mathcal{F}$ gives the extension to $\mathcal{G}$, with scalar 1 and transition k_σ by Lemma 6.6. On V_χ, the f-th power is the action of a^f, including improper a^f.

Proposition 6.94. *Let $N=\mathrm{Spin}^\epsilon_{2n}(q)$ for odd q, with $n=2a+c\geq 5$, $c\geq 1$, and let p be odd with $p\nmid q$ and $p\mid|N|$. Suppose that the packet $\mathcal{P}$ has separately rational form $Q(a,c)$ and is trivial on z_0. On the* degree domain $\mathcal{D}_{\mathrm{uneq}}$*, suppose that $\mathfrak{s}_-$ is the unique degenerate symbol. Then, $\mathcal{P}$ satisfies $\succeq_c$ for $\mathcal{H}_p$ at $N_N(Q_0)$. Moreover, on the common cover E, both scalar characters are λ_b^c, with coefficient transition $(\alpha_\sigma\circ\mu)^c k_\sigma$ for $\sigma\in\mathcal{H}_p$. k_σ is inflated from C_f, with $k_\sigma=1$ for $\epsilon_{\mathrm{tor}}=-$ or a unitary quartic block V_{unit}.*

Proof. Choose the isometry and field extensions U_G, U_R of the positive labels h_G, h_R. If $\epsilon_{\mathrm{tor}}=+$, each hook step fixes r_1-r_2 for the ordered row lengths r_1, r_2. Thus, h_G, h_R share the signed defect d and transition k_σ by Lemma 6.86. If $d=0$, the endpoint is 1 of rank 0 with $k_\sigma=1$. If $\epsilon_{\mathrm{tor}}=-$ and f is even, the lemma gives invariant U_G, U_R. If f is odd, take the unique real U_{real} in either period. Thus, the module $\mathcal{F}$ and Lemma 6.93 give extensions to $\widehat{J_G}, \widehat{J_R}$ after adjoining $\mathfrak{s}_-$.

For the quartic partition λ of size l on Θ, take $I_\pm$ from Proposition 6.81 in split form V_{sp}. On the residual factor V_R, the opposite operator B_{opp} acts by 1. Since $B_{\mathrm{opp}}F=FB_{\mathrm{opp}}$, the field correction is k_σ, with multiplier $(\alpha_\sigma\circ\mu)^l$. Moreover, since $B_{\mathrm{opp}}^{a_r f}=I$, the power $\widetilde{F}^f$ represents g when $F^f=g$. For unitary V_{unit}, $q\equiv 3 \bmod 4$ and f is odd. Thus, we have that $\mathcal{F}$ gives the rational ordinary extension U_{mix} of Proposition 6.92. The character β and spin-image character χ are rational. Hence, $(s_{\mathrm{mix}}, k_{\mathrm{mix}})=(1,1)$. Thus, either form has scalar λ_b^l, with $l=c$ globally.

For the local parent, set $l=c_0$. The core C_- has equal rows. If C_+ is nondegenerate, the construction gives U_R. If $b_-=0<b_+$, Proposition 6.71 gives $c_0=0$ and U_+ gives χ_{src}. A sole nonempty C_- would have rank e odd, whereas equal rows (κ,κ) give even rank. Thus, for degenerate C_+, C_-, $b_+=b_-$ also at $\mathcal{B}$. If $\lambda_+\neq\lambda_-$, choose U_R from Proposition 6.73. If $\lambda_+=\lambda_-$, apply Proposition 6.69. Include the pure quartic $\mathcal{R}_{\mathrm{quart}}$ and the empty case $\mathcal{R}_0$. In these cases with degenerate C_+, its transition is 1. Thus, the global k_σ is 1 by signed defect 0 in positive period.

With $X=\Omega(V)$, we write $\mathcal{G}$ for the semilinear similarities of V. Next, we put $\mathcal{L}=N_{\mathcal{G}}(\bar{Q}_0)$. By Proposition 6.69, the fixed space V_R and coordinate kernels on $V_R^\perp$ give two actions sharing μ,ϕ. Thus, on $\mathcal{L}$, tensor U_R with U_A from Lemma 6.91. Its active exponents a_{sc}, a_μ are ew, with field correction 1. Since $c=c_0+ew$, the tensor has scalar λ_b^c and transition $(\alpha_\sigma\circ\mu)^c k_\sigma$. Moreover, α_σ is field-invariant, so $\alpha_\sigma\circ\mu$ is a character on $\mathcal{L}$.

Let R_G, R_L be the representations of $\mathcal{G},\mathcal{L}$, with $\chi_G=R_G|_X$ and $\chi_L=R_L|_{N_X(\bar{Q}_0)}$. On Θ_G, determinant transposes $\mathfrak{s}_+$, and ν_{sp} acts by $(\mathfrak{s}_-,\mathfrak{s}_+)\mapsto(\mathfrak{s}_+,\mathfrak{s}_-)$, with degeneracy orders exchanged. Thus, Θ_G has trivial quotient self-twists, and Θ_L does too by Proposition 6.260. Hence, by [43], χ_G,χ_L are irreducible. Since R_G, R_L extend χ_G,χ_L, ordinary actions fix both characters. Moreover, $A_\sigma=(\alpha_\sigma\circ\mu)^c k_\sigma$ is trivial on $X, N_X(\bar{Q}_0)$. Thus, the actions of $\mathcal{H}_p$ and $g\sigma$ also fix the respective characters.

In $\mathrm{Syl}_p(N)$, choose P with characteristic toral base Q_0. Put $M=N_N(Q_0)$, $\Gamma=\mathrm{Aut}(N)_P$, and let $\bar{x}$ be the natural image of $x\in N$. Moreover, for $n\geq 5$, the semilinear similarities on X map onto Γ with kernel $Z=\mathbb{F}_q^\times I$ by Theorem 6.70. Since $\gamma\in\Gamma$ normalizes $\bar{Q}_0$, we choose $s(\gamma)\in\mathcal{L}$ with $s(1)=1$. Put $L(x,\gamma)=\bar{x}s(\gamma)$ on $N\rtimes\Gamma$.

Then, $L(M \rtimes \Gamma) \subseteq \mathcal{L}$. Thus, $R_G \circ L$ and $R_L \circ L$ have the common factor set $(\gamma, \delta) \longmapsto \lambda_b\big(s(\gamma)s(\delta)s(\gamma\delta)^{-1}\big)^c$, where $zI \in Z$ corresponds to $z \in \mathbb{F}_q^\times$. If (x, γ) centralizes N, $\gamma = \mathrm{Inn}(x^{-1})$. Hence, $x \in N_N(P) \leq M$ and $L(x, \gamma) \in Z$. Thus, both scalars are $\lambda_b(L(x, \gamma))^c$. Given $u, v \in M \rtimes \Gamma$, define $d(v, u) \in Z$ by $L(vuv^{-1}) = d(v, u)L(v)L(u)L(v)^{-1}$. After conjugation by $R_\bullet(L(v))$, we obtain the inner action $\mathrm{Inn}(v)$. Thus, after coefficient action by σ, the correcting function at u is $\sigma(\lambda_b(d(v, u))^c)A_\sigma(L(u))$ in both models. Since R_G, R_L are trivial on z_0, this gives $\succeq_c$. □

For $q = 3, p = 13, e = d = 3$, we set $\mathfrak{s}_- = (\{e\}, \{e\})$, $\mathfrak{s}_+ = (\{2e\}, \{0\})$ and $\lambda = (1)$. Thus, $C_+ = C_- = \varnothing$, $u_+ = u_- = 2$ and the quotient degrees are $2, 1$. Moreover, we have that $w = 0, k = 4, R = 1$ and the multinomial is 6. The 13-power map on μ_8 has exponent 5 and fixes i. Thus, we obtain that $\alpha_\sigma = \eta_3^{(5-1)/4} = \eta_3$ and $\chi^\sigma = \chi$. Given a nonsquare similitude g, we have that the transition is $\eta_3(\mu(g)) = -1$ in rank 13.

Corollary 6.95. *Let $N = \mathrm{Spin}^\epsilon_{2n}(q)$ for odd q, with $n = 2a + c \geq 5$, $c \geq 1$, and p odd with $p \nmid q$ and $p \mid |N|$. Suppose that $\mathcal{P}$ has separately rational form $Q(a, c)$ and is trivial on z_0. On the* degree domain $\mathcal{D}_{\mathrm{uneq}}$*, the comparison $\succeq_c$ for $\mathcal{H}_p$ at $N_N(Q_0)$ holds for every pair $\mathfrak{s}_+, \mathfrak{s}_-$, each eligible λ, both periods ϵ_{tor} and every field degree f.*

Proof. If $\mathfrak{s}_+ = \mathfrak{s}_-$, Theorem 6.70 applies. For distinct degenerate $\mathfrak{s}_\pm$, we apply Theorem 6.76. If $\mathfrak{s}_\pm$ are nondegenerate, Theorem 6.90 gives Φ. Otherwise, by [149, Corollary 8.14], ν allows the order with $\mathfrak{s}_-$ degenerate. Then, Proposition 6.94 gives the comparison Φ. Hence, each Φ gives its comparison at $N_N(Q_0)$. Moreover, the corresponding projective representations on N and $N_N(Q_0)$ have central character λ_{sc} on the spin centre. □

6.4 Even characteristic and triality

Let $X = \mathbf{X}^F$ have type D_n or ${}^2D_n$ over $\mathbb{F}_q$, with $q = 2^f$. Write $\epsilon = 1$ for D_n and $\epsilon = -1$ for ${}^2D_n$, and take an odd prime $p \mid |X|$. By Zhang [208, Lemma 8.1], the central isogenies from $\mathbf{G}_{\mathrm{sc}}$ to $\mathbf{G}_{\mathrm{so}}$ and $\mathbf{G}_{\mathrm{ad}}$ are bijective on $\overline{\mathbb{F}}_q$-points. Since both maps are F-equivariant, they induce bijections on $\mathbb{F}_q$-points. On the natural module V, by Taylor [189, Theorem 11.43 and the following paragraph], the Dickson invariant $D(g) = \mathrm{rk}(g - 1) \bmod 2$ has kernel $\mathrm{SO}(V)$ on $\mathrm{O}(V)$. We take the primal torus $\mathbf{T}$ and its dual $\mathbf{T}^*$.

Lemma 6.96. *With $d = \mathrm{ord}_p(q)$, we put $e = d/\gcd(d, 2)$. Next, we set $t = 1$ for odd d and $t = -1$ for even d. Write $n = ew + c$ with $0 \leq c < e$, and $a = v_p(q^e - t)$. Then, the block count b, residual rank c_0 and sign ϵ_0 satisfy $(b, c_0, \epsilon_0) = (w, c, \epsilon t^{-w})$ when $c > 0$ or $\epsilon = t^w$. Otherwise, $(b, c_0, \epsilon_0) = (w - 1, e, -t)$ when $c = 0$ and $\epsilon = -t^w$. There is $P = Q \rtimes U \in \mathrm{Syl}_p(X)$ with $Q = (C_{p^a})^b$ characteristic in P and $U \in \mathrm{Syl}_p(\mathfrak{S}_b)$. For the corresponding Sylow d-torus $\mathbf{S}$, we put $M = \mathrm{N}_{\mathbf{X}}(\mathbf{S})^F$. Then, $M = \mathrm{N}_X(Q)$, with $\mathrm{N}_X(P) \leq M$ and M stable under $\mathrm{Aut}(X)_P$. If X is simple, $M < X$. Moreover, in $K = \mathrm{C}_{\mathbf{X}}(\mathbf{S})^F$, the Sylow p-subgroup Q is central.*

Proof. By [189, p. 165], $|X|_{2'} = (q^n - \epsilon)\prod_{j=1}^{n-1}(q^{2j} - 1)$. By [63, Lemma 4.1] at $x = q^{2e}$, $v_p(q^{2j} - 1) = a + v_p(j/e)$ if $e \mid j$, and 0 otherwise by the order of q^2 modulo p. If $c = 0$ and $\epsilon = t^w$, then $q^n + \epsilon$ is prime to p, so the same formula gives $v_p(q^n - \epsilon) = a + v_p(w)$. In the other cases $p \nmid q^n - \epsilon$. Thus, $v_p(|X|) = ba + v_p(b!)$. By [130, §4.4], choose isometric $2e$-spaces V_i of type t, with tori T_i of order $q^e - t$. The residual space V_0 has sign ϵ_0, with isometry group of p'-order. At $c_0 = e$, its last odd factor is $q^e + t$. Since p is odd, the isometries from U and the p-parts of T_i lie in $\ker D$. Thus, $P \leq X$ has order $|X|_p$. By Lemma 4.3, Q is the largest abelian subgroup of $P = Q \rtimes U$. Its uniqueness makes Q characteristic in P.

By [130, §4.4], $K = \prod_i T_i \times \mathrm{SO}(V_0)^F$, with unique Sylow p-subgroup Q. Since $\mathrm{ord}_{p^a}(q) = d$, the Frobenius orbit of a primitive p^a-root and its inverses give $2e$ distinct weights on each V_i. Thus, Q separates the same weight spaces as $\mathbf{S}$, with V_0 the fixed space. Consequently, $\mathrm{C}_{\mathbf{X}}(Q) = \mathrm{C}_{\mathbf{X}}(\mathbf{S})$, and $\mathbf{S}$ is the connected central d-part of this group. Hence, $M = \mathrm{N}_X(Q)$. Since Q is characteristic in P, $\mathrm{N}_X(P) \leq M$ and $\mathrm{Aut}(X)_P \leq \mathrm{Aut}(X)_Q$. Finally, $Q \neq 1$, so simplicity gives $M < X$. □

Proposition 6.97. *Suppose that $n \geq 5$ or the rational type is ${}^2D_4$. Let $s \neq 1$ be eligible semisimple in $\mathbf{X}^{*F}$. Then, on $\mathcal{E}(X, s)$, the p'-characters admit $\succeq_c$ with the characters of M over $\mathcal{O}_s$.*

Proof. If $\mathcal{E}(X, s)$ contains a character of p'-degree, its Jordan degree formula [58, Proposition 11.5.6] gives $p \nmid [\mathbf{X}^{*F} : \mathrm{C}_{\mathbf{X}^{*F}}(s)]$. Conjugate s so that it centralizes P^*, hence also Q^*. By Lemma 6.96, s lies in $\mathbf{K}^* = \mathrm{C}_{\mathbf{X}^*}(\mathbf{S}^*)$. By [208, Lemma 8.1(i)–(ii)], the connected proper Levi $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(s)$ has the form $\mathrm{SO}(V_1) \times \prod_{\{x, x^{-1}\},\, x \neq 1} \mathrm{GL}(V_x)$, with nondegenerate V_1. Moreover, $\mathbf{J}^* = \mathrm{C}_{\mathbf{K}^*}(s) = \mathbf{L}^* \cap \mathbf{K}^* = \mathrm{C}_{\mathbf{L}^*}(\mathbf{S}^*)$. Since $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(\mathrm{Z}(\mathbf{L}^*)^\circ)$, the intersection is

the centralizer in $\mathbf{K}^*$ of $\mathrm{Z}(\mathbf{L}^*)^\circ \le \mathbf{T}^*$. Thus, $\mathbf{J}^*$ is a Levi subgroup of $\mathbf{K}^*$. Dualize on $\mathbf{T}, \mathbf{T}^*$ and put $L = \mathbf{L}^F$, $J = \mathrm{C}_{\mathbf{L}}(\mathbf{S})^F$, and $I = \mathrm{N}_{\mathbf{L}}(\mathbf{S})^F$. By [18, Theorem B′ and Remark 10.12] in characteristic 0, signed induction pairs $\mathcal{E}(L,s)$ with $\mathcal{E}(X,s)$ and $\mathcal{E}(J,s)$ with $\mathcal{E}(K,s)$. The degree ratios are $[X:L]_{2'}$ and $[K:J]_{2'}$. Since $P^* \le \mathbf{L}^{*F}$ and $Q^* \le \mathbf{J}^{*F}$, both ratios are prime to p.

As in [132, proof of Proposition 7.7], the M^*-stabilizer of the K^*-class of s is $K^*(\mathbf{L}^{*F} \cap M^*)$. Under rational Weyl duality on $\mathbf{T}, \mathbf{T}^*$, its image in M^*/K^* corresponds to $I/J \le M/K$ (see Lemma 6.7). Thus, $\mathrm{Stab}_M(\mathcal{E}(K,s)) = KI$ and $K \cap I = J$. Since $I \le L$ fixes the radical for $\mathbf{L}$, it fixes its intersection with $\mathbf{K}$. The signed induction map on $\mathcal{E}(J,s)$ is I-equivariant. Hence, paired $\gamma \in \mathcal{E}(J,s)$ and $\alpha \in \mathcal{E}(K,s)$ satisfy $M_\alpha = KI_\gamma$. The Euler character e_γ and determinant δ_γ of Proposition 6.8 give Φ_{loc}, with multiplicity 1. Thus, we apply that proposition to $\mathbf{L}^*, \mathbf{J}^*$ and their rational relative normalizers. Moreover, $\mathbf{U}$ is fixed by I_γ, and $[M:I]/[K:J]_2$ is a p-unit. Given a nonabelian factor K_i, its character χ_i enters via the signed correspondent ψ_i on J.

Next, let $\lambda_s \in \mathrm{Irr}(L)$ correspond to $s \in Z(\mathbf{L}^*)^F$. On each linear or unitary block V_i, its factor λ_i comes from the determinant. On V_1, it is 1. Now, choose a rational maximal torus $\mathbf{T}$ common to $\mathbf{J}, \mathbf{L}$. By [18, (11.5)], both torus characters for s coincide on $\mathbf{T}^F$. Thus, the linear character for J is $\lambda_s|_J$. By [18, §11.5], $\mathcal{E}(L,s) = \lambda_s \mathcal{E}(L,1)$ and $\mathcal{E}(J,s) = (\lambda_s|_J)\mathcal{E}(J,1)$. For active classical L_i, take adjoint extensions by [156, Theorem D and its proof]. Since $\ker \pi$ has 2-power order, π is bijective on points. For linear L_i, the unipotent character χ_i inflates from $\overline{L}_i$ and is trivial on scalars zI. If h factors form an F-cycle, the factor for F^h has period $d/\gcd(d,h)$. Indeed, eigenvectors extend through the h-cycle. For an F-eigenvalue ζ, the F^h-eigenvalue is ζ^h. Thus, the projected torus $\overline{\mathbf{S}}$ has the corresponding multiplicity and normalizer $\overline{N}$. On inactive L_i, take the identity. Thus, on $\overline{N}$, the d-Harish–Chandra labels lie over $\mathcal{E}(J,1)$, and J has central Sylow d-torus.

On isomorphic L_i, take identical models $\mathcal{V}_i$, with positional permutations on $\bigotimes_i \mathcal{V}_i$. Given an orbit of transported labels η, we choose η_0 and conjugate to the stabilizer I_{η_0}. Thus, for factor actions ρ_i, the product $\prod_i (f_i \circ \rho_i)$ agrees on both sides. On D, we multiply the paired extension by $(x,y) \mapsto \lambda_s(xy^{-1})$. Thus, ω_s is multiplicative on I_{ord} and carries the coefficient power on s. On ΔC, it is 1. Thus, ω_s gives the middle mixed comparison for $\mathcal{O}_s$.

For the varieties $\mathcal{X}, \mathcal{Y}$, the radical $\mathbf{U}$ must be fixed by ordinary g as in Proposition 6.9. By [208, Lemma 8.1 and Proposition 8.2], the cocharacter ν^* has weight 0 on V_1 and opposite weights on each pair $V_x, V_{x^{-1}}$. Thus, the dual cocharacter ν gives a radical $\mathbf{U}$ fixed by the ordinary representatives g. Moreover, inner corrections in L fix $\mathbf{U}$. On $\mathbf{K}$, the restricted ν gives $\mathbf{U} \cap \mathbf{K}$ with Levi $\mathbf{J}$. Moreover, on J, the argument takes the central element s.

For the field action take $F = \delta^\iota F_2^f$, with diagram involution δ. Thus, $\langle \delta \rangle$ gives the rational graph group, and π gives diagonal quotient 1. Thus, in the integer extension, (δ^ι, f) induces F. For degree f/h, choose the field lift φ. By Section 6.1, its power is $\varphi^h = K_{a_0^h} F$, for finite generator a_0 and degree-zero correction $K_{a_0^h}$. Thus, in $\varphi^h = K_{a_0^h} F$, the actions and degrees agree, including δ if $\iota = 1$. Moreover, on L, J, the commutators agree. For (g,σ), g moves the radicals and σ acts on coefficients. Since $\mathrm{C}_{\mathbf{X}^*}(s) = \mathbf{L}^*$ and $\mathrm{C}_{\mathbf{K}^*}(s) = \mathbf{J}^*$, [19, Theorem 7.2] gives bimodule isomorphisms on $\mathcal{E}(L,s), \mathcal{E}(J,s)$. On $\mathbf{U}, g(\mathbf{U})$ and their intersections with $\mathbf{K}$, apply Lemma 7.168 on $\mathcal{X}, \mathcal{Y}$. By [160, Lemma 3.1], this gives diagonal equivariance on $\mathcal{X}, \mathcal{Y}$ in the finite product models. Choose ℓ, a cohomology prime away from 2 and the group orders for $\mathcal{X}, \mathcal{Y}$. Thus, the paired extension over K, J has determinant δ_γ. Composing with ω_s gives the mixed comparison at s. By [132, Proposition 7.7 and proof of Theorem 7.8], each $\psi \in \mathrm{Irr}_{p'}(M)$ lies over a series $\mathcal{E}(K,t)$ for which $t \in K^*$ centralizes a Sylow p-subgroup of M^*. By Lemma 6.96, $\mathbf{S}$ is a Sylow d-torus, and X has type D_n or ${}^2D_n$, outside the exceptions of [132, Theorem 5.14]. Since $P^* \in \mathrm{Syl}_p(M^*)$, choose s in the M^*-orbit of t with $s \in \mathrm{C}_{K^*}(P^*)$. By [132, Proposition 5.11], the X^*-class of s determines the M^*-orbit of its K^*-class. Hence, the local images partition $\mathrm{Irr}_{p'}(M)$. At $s = 1$, take the unipotent map into M. □

Take $d = 4$ for the plus form of rank 6. Thus, the two negative 2-cycles have complement V_0 of plus type and dimension 4. In type $D_2 = A_1 \sqcup A_1$, the split covering has kernel μ_2. Since its geometric kernel is trivial, $\mathrm{SO}(V_0)^F \simeq H_1 \times H_2$, where $H_j = \mathrm{SL}_2(q)$. Let (s_1, s_2) be the residual dual pair. At $s_j = 1$, the unipotent labels of H_j are $1, \mathrm{St}$. At $s_j \ne 1$, its dual centralizer is the torus $\mathbf{C}_j$. By Bonnafé–Rouquier [18, Theorem 11.8], the label on $\mathbf{C}_j^F$ is 1. Thus, (s_1, s_2) gives the residual component in $\mathbf{J}^{*F}$.

On V_0, the diagram operation δ may exchange factors H_1, H_2, Levis L_1, L_2 and their labels. If δ admits a correction returning γ, Proposition 6.8 gives paired operators above γ, α. Indeed, I_γ/J has one projective action ρ in the paired bases. Thus, on I_γ/J, each constituent has equal signs at γ, α. If δ has no correction at γ, induce from each stabilizer. This includes $s_j = 1$ at exactly one j, with field or coefficient action u. Thus, cosets correspond by (γ^u, α^u).

Let $X = \Omega_8^+(q)$, with $q = 2^f > 2$ and diagram group $\mathfrak{S}_3$. The degrees $2, 4, 4, 6$ give $d \in \{1, 2, 3, 4, 6\}$. At $d = 3, 6$,

exactly one degree is divisible by d. Since $d \mid p-1$, we have $p \nmid |W(D_4)| = 2^6 3$. By Lemma 2.11, P is cyclic and $\mathrm{N}_X(P[p]) = \mathrm{N}_X(P)$. By Corollary 2.10, iMN$(X,p)$ holds on $\mathrm{N}_X(P)$. At $d = 1,2,4$, $K = T$ is a maximal torus. Next, for $d = 1,2$, its rational Weyl type is $1, -1$, respectively. At $d = 4$, the Weyl element $w = w_1 w_2$ has negative two-cycles w_1, w_2. Let H_s stabilize s with field component 1. When a common radical $\mathbf{U}$ is fixed by H_s, the proof of Proposition 6.97 constructs Φ_s from L and the stabilizer of $\lambda_s|_T$. By Section 6.1, we lift H_s with integer degrees in type D_4. A pinned diagram automorphism σ satisfies $\sigma F = F\sigma$, since $F = F_2^f$ has diagram component 1.

Lemma 6.98. *Given $d \in \{1,2,4\}$, we let $s \neq 1$ be eligible in T^*. Then, H_s fixes a cocharacter ν with $\mathbf{L}(\nu) = \mathbf{L}$, except possibly when $d = 1,2$ and s is the order 3 class of* Lemma 6.99.

Proof. At $d = 4$, $|s|$ divides q^2+1 and $\gcd(|s|, 6) = 1$. In $W(D_4) \rtimes \mathfrak{S}_3$, we write H for the stabilizer of s. Thus, $|H|$ divides 1152. The torus map $v \mapsto \prod_{h \in H} h(v)$ has image in $((\mathbf{T}^*)^H)^\circ$ and acts by $s \mapsto s^{|H|}$. Thus, $s \in ((\mathbf{T}^*)^H)^\circ$ since $\gcd(|s|, |H|) = 1$. Let E_0 be the H-fixed cocharacter space. If $\alpha(s) = 1$, the reflection r_α is in H and $\alpha|_{E_0} = 0$. Conversely, $\alpha|_{E_0} = 0$ gives $\alpha(s) = 1$. Hence, for generic integral $\nu \in E_0$, the parabolic $\mathbf{P}(\nu)$ has dual Levi $\mathrm{C}_{\mathbf{X}^*}(s)$. In particular, $\mathbf{L}(\nu)$ is rational.

At $d = 1,2$, we write $s = x_s + \Lambda$ with $\Lambda = \mathbb{Z}^4 + \mathbb{Z}(1/2,1/2,1/2,1/2)$ and s of odd order. Let Φ_{aux} have short roots $\pm e_i \pm e_j$ and long roots $\pm 2e_i, (\pm 1, \pm 1, \pm 1, \pm 1)$. Thus, Φ_{aux} has type F_4, with coroot lattice Λ and Weyl group $W(D_4) \rtimes \mathfrak{S}_3$. In a closed alcove, choose x_s. By [20, Propositions 3.11 and 3.13(a)], $H = \langle r_\alpha \mid \alpha(x_s) \in \mathbb{Z} \rangle$ in the finite Weyl group. Thus, E_0 is tangent to the face containing x_s. If $E_0 \neq 0$, take the Levi $\mathbf{L}_{\mathrm{aux}}$ with roots zero on E_0. The integral roots at x_s span $E_0^\perp$, so $\mathrm{C}_{\mathbf{L}_{\mathrm{aux}}}(s)$ has full semisimple rank in $\mathbf{L}_{\mathrm{aux}}$. The adjoint factors of $\mathbf{L}_{\mathrm{aux}}$ have types A, B, C. By [20, Theorem 5.1 and Table II], the image of s has order 1 or 2 in each factor. Thus, in $\mathbf{L}_{\mathrm{aux}}$, s is central. The short roots of $\mathbf{L}_{\mathrm{aux}}$ are exactly the D_4 roots with $\alpha(s) = 1$. Hence, a generic integral $\nu \in E_0$ gives $\mathbf{L}(\nu) = \mathbf{L}$. Finally, H_s fixes ν, and an inner correction in $\mathbf{L}$ fixes $\mathbf{U}(\nu)$. If $E_0 = 0$, [20, Theorem 5.1 and Table I] gives $|s| = 3$ for the nonidentity isolated class of F_4 in characteristic 2. Next, for Φ_{aux}, the simple roots are $e_2 - e_3, e_3 - e_4, 2e_4, e_1 - e_2 - e_3 - e_4$. The highest root $2e_1$ has coefficients $2,4,3,2$. Thus, the vertex at the coefficient 3 is $x_s = (1/2, 1/6, 1/6, 1/6)$, and $x_s + \Lambda = (0, -1/3, -1/3, -1/3) + \Lambda$. □

Lemma 6.99. *At the vertex s of* Lemma 6.98*, the three pairs (χ_i, ϕ_i) satisfy $\succeq_c$ for the mixed actions of $\mathcal{H}_p$ and $\mathrm{Aut}(X)_P$ with graph and field automorphisms.*

Proof. Put $t = 1$ at $d = 1$ and $t = -1$ at $d = 2$. On T^*, Frobenius acts by tq, so $s \in T^*$ is rational exactly when $3 \mid q - t$. In $W(D_4)$, the last three coordinates of s have stabilizer $\mathfrak{S}_3$. Indeed, $\Lambda \cap \frac{1}{3}\mathbb{Z}^4 = \mathbb{Z}^4$, and even sign parity fixes coordinate 1. Hence, the $W(D_4)$-orbit has size $192/6 = 32$. The integral short roots are $\pm(e_i - e_j)$ for $2 \leq i < j \leq 4$. By [20, Theorem 5.1(b) and Table I], the centralizer in F_4 has type $A_2\widetilde{A}_2$. Thus, $|H| = 36$. The diagram image has order $36/6 = 6$, so the extended orbit equals $W(D_4)s$. Since $-1 \in W(D_4)$ and $|s| = 3$, coefficient powers of s leave $\mathcal{E}(X, s)$ invariant.

By [208, Lemma 8.1(ii)], the two factors of L are $G_3(t)$ and a torus of order $q - t$. Put $G_3(1) = \mathrm{GL}_3(q)$ and $G_3(-1) = \mathrm{GU}_3(q)$. By [84, Theorem 14] and [190, Theorem 5.1], the unipotent degrees of $G_3(t)$ are $1, q(q+t), q^3$. Thus, [58, Proposition 11.5.6] gives $\chi_i(1)$ by multiplication by $[X : L]_{2'}$. For $I = M_{\lambda_s|_T}$, [18, (11.5)] gives $I/T = C_{W(D_4)}(s)$. Hence, $[M : I] = 32$. The local degrees of $\phi_i = \mathrm{Ind}_I^M((\lambda_s|_I)\eta_i)$ are $32, 64, 32$, where η_i runs through $\mathrm{Irr}(I/T) = \mathrm{Irr}(\mathfrak{S}_3)$ via inflation and λ_s is the linear character of L for s. Thus, the stabilizers fix each χ_i by its degree. On I, a restorer of $\lambda_s|_T$ fixes λ_s after correction in M. The action on $I/T \cong \mathfrak{S}_3$ is inner, so each η_i is fixed. By [63, Lemma 4.1], $v_3(|X|) = v_3(|L|) = 4v_3(q - t) + 1$, so $3 \nmid \chi_i(1)\phi_i(1)$. Since $p \mid q - t$ and s is eligible, the factors $1, q(q+t), q^3$ and the index 32 are p-units, as is $[X : L]_{2'}$.

Let Π act on X by left X, right M and $\mathrm{Aut}(X)_P$, as in Section 2. Since $Z(X) = 1$, the base is $B_0 = X \times M$, and Sylow conjugacy gives $\Pi/B_0 \cong \mathrm{Out}(X)$. Set $\psi = \chi \otimes \phi^\vee$ for one pair (χ, ϕ). Let Π_2 be the preimage of $C_2 \times \langle F_2 \rangle$, with diagram subgroup C_2. Thus, $H_2 = \langle r_{2e_1}, \mathfrak{S}_3 \rangle \leq H$, where $\mathfrak{S}_3$ permutes coordinates $2,3,4$. Hence, H_2 fixes $\nu = (0,1,1,1)$, whose zero roots in D_4 form A_2. With $\mathbf{U} = \mathbf{U}(\nu)$, the construction of Proposition 6.97 gives E_2 on Π_2. If $\sigma \in \mathcal{H}_p$ acts by $s \mapsto s^{-1}$, the rational lift of -1 in M restores s with diagram component 1. Thus, the mixed stabilizer of ψ fixes E_2.

Let $\Pi_3/B_0 \in \mathrm{Syl}_3(\Pi/B_0)$. Since $X = [X, X]$, $\det \psi|_X = 1$. On T, inversion by the lift of -1 gives $(\det \psi|_T)^2 = 1$. Since $|T|$ is odd, $\det \psi$ factors through $M/T = W(D_4)$. Thus, $(\det \psi)^2 = 1$ since $W(D_4)$ is generated by involutions. As Π_3/B_0 is a 3-group and $3 \nmid \psi(1)$, [95, Corollary 6.28] gives the unique extension E_3 with determinant of $3'$-order. Thus, the Galois stabilizer of ψ fixes E_3, including when $3 \mid f$.

Put $h = [\Pi : \Pi_3]$, so $3 \nmid h$ and $[\Pi : \Pi_2] = 3$. Take $c_2, c_3 \in \mathbb{Z}$ with $3c_2 + hc_3 = 1$. In Section 2, the virtual character is $\Xi = c_2 \operatorname{Ind}_{\Pi_2}^{\Pi}(E_2) + c_3 \operatorname{Ind}_{\Pi_3}^{\Pi}(E_3)$. By [95, Problem 5.6], $\Xi|_{B_0} = (3c_2 + hc_3)\psi = \psi$. Since $E_j^\sigma = E_j$ for σ fixing ψ, we have $\Xi^\sigma = \Xi$. On the fibre product B_I of Section 2, we fix $\Xi \circ \pi_I$ for $\pi_I : B_I \to \Pi$. Consequently, Corollary 2.5 gives paired models of χ, ϕ, with equal scalars on the kernel $\ker \pi_I$. □

Proof of Theorem 6.2. By [208, proof of Corollary 8.7], X is the universal cover of $X/Z(X)$, except at $D_4(2)$. At $D_4(2)$, [101, Theorem 1.1] covers $p = 2$, and [157, Lemma 4.1(2)] covers odd p. If $n \geq 5$ or X has type ${}^2D_4$, Lemma 6.96 gives M. Then, for $s \neq 1$, Proposition 6.97 gives the maps on $\mathcal{E}(X, s)$. Moreover, at $s = 1$, the result of [156, Theorem D] gives the comparison on M.

In split type D_4, the periods $d = 3, 6$ have the cyclic comparison on $\mathrm{N}_X(P)$. Next, for $d = 1, 2, 4$, Lemma 6.98 gives the ordinary radical $\mathbf{U}$, except at the vertex s of Lemma 6.99. At that vertex, Ξ gives $\succeq_c$ for diagram group $\mathfrak{S}_3$. Otherwise, on the stabilizer of (χ, ϕ), construct Φ_s as in Proposition 6.97. The Levi $\mathbf{L}$ has factors of types A_1, A_2, A_3, with H_s permuting isomorphic factors. Moreover, [156, Theorem D] treats $\mathcal{E}(X, 1)$ over 1_T. Thus, for each p we choose a single M. Hence, the orbits $\mathcal{O}_s$ partition $\mathrm{Irr}_{p'}(M)$ by Proposition 6.97. □

Let $S = {}^3D_4(q)$, with $q = s^f$ for a prime s. The Schur multiplier of S is trivial. Moreover, by Zhang [208, §9.2], $\mathrm{Out}(S)$ is cyclic of order $3f$, generated by the prime field Frobenius ϕ. Next, set $F = \tau F_s^f$, with diagram triality τ. In odd characteristic, the central kernel Z of the simply connected isogeny π is a Klein four group. Thus, F permutes the three elements of $Z \setminus \{1\}$ cyclically. In characteristic 2, Z has only the point 1. Hence, $Z^F = 1$ and $z \mapsto z^{-1}F(z)$ is onto Z on geometric points. By Geck–Malle [72, Proposition 1.4.13], $\pi : \mathbf{G}_{\mathrm{sc}}^F \to \mathbf{G}_{\mathrm{ad}}^F$ is an isomorphism, with $\pi(\mathbf{T}^F) = (\pi\mathbf{T})^F$ on rational tori. By Babai et al. [12, Theorem 3.1 and Table I], choose $T = \mathbf{T}^F$ cyclic of order $m = \Phi_{12}(q) = q^4 - q^2 + 1$. Moreover, $\gcd(m, |S : T|) = 1$ by Babai et al. [12, proof of Theorem 3.1]. Thus, $P = T_p \in \mathrm{Syl}_p(S)$ for $p \mid m$. By Babai et al. [12, Lemma 3.3], each prime $r \mid m$ has $\mathrm{ord}_r(q) = 12$.

Lemma 6.100. *Each $x \in T \setminus \{1\}$ has geometric centralizer $\mathbf{T}$ in $\mathbf{G}_{\mathrm{sc}}$. Similarly, each $x^* \in T^* \setminus \{1\}$ has geometric centralizer $\mathbf{T}^*$ in $\mathbf{G}_{\mathrm{ad}}$. In particular, $\mathrm{N}_S(P) = \mathrm{N}_S(\mathbf{T})$ and $\mathrm{C}_S(P) = T$.*

Proof. By [12, Theorem 3.1], $C_S(x) = T$ when $x \neq 1$. By [133, Theorem 14.16(a)], $\mathbf{C} = C_{\mathbf{G}_{\mathrm{sc}}}(x)$ is connected. If $\mathbf{C} > \mathbf{T}$, the Borel radical gives $s \mid |\mathbf{C}^F|$ by [133, Corollary 21.12 and Proposition 23.8]. Since $|\mathbf{C}^F| = |T|$, $\mathbf{C} = \mathbf{T}$. Each $x^* \in T^* \setminus \{1\}$ has a unique odd-order lift x by [97, Theorems 3.8 and 3.12]. Thus, x is F-fixed and lifts of $C_{\mathbf{G}_{\mathrm{ad}}}(x^*)$ fix x. Consequently, $C_{\mathbf{G}_{\mathrm{ad}}}(x^*) = \pi(\mathbf{T}) = \mathbf{T}^*$. Moreover, $C_{\mathbf{G}_{\mathrm{sc}}}(P) = \mathbf{T}$ and $C_S(P) = T$. Since P is characteristic in T, $\mathrm{N}_S(P) = \mathrm{N}_S(\mathbf{T})$. □

Proposition 6.101. *At $p \mid \Phi_{12}(q)$, the nonunipotent $\chi \in \mathrm{Irr}_{p'}(S)$ correspond bijectively to the $\psi \in \mathrm{Irr}_{p'}(M)$ above $\theta \neq 1_T$, with $M = \mathrm{N}_S(P)$. Moreover, the pairs (χ, ψ) satisfy $\succeq_c$. Thus, with* Ruhstorfer et al. [156, Theorem D] *on M,* iMN(S, p) *holds.*

Proof. Let $\chi \in \mathcal{E}(S, x) \cap \mathrm{Irr}_{p'}(S)$ with $x \neq 1$. By [58, Proposition 11.5.6], $[\mathbf{G}_{\mathrm{ad}}^F : C_{\mathbf{G}_{\mathrm{ad}}}(x)^F]$ is prime to p. Thus, some conjugate of x centralizes P^*. By Lemma 6.100, take $x \in T^*$ with $\mathrm{C}_{\mathbf{G}_{\mathrm{ad}}}(x) = \mathbf{T}^*$. For the dual weight θ, [55, Theorem 4.3] gives $\chi = \varepsilon R_{\mathbf{T}}^S(\theta)$ of degree $|S : T|_{s'}$. By [70, Table 3, torus 6], $M/T \cong C_4$. Since $\mathrm{C}_{\mathbf{G}_{\mathrm{ad}}}(x) = \mathbf{T}^*$, the stabilizer of θ in M/T is trivial. By [95, Theorem 6.11], $\psi = \operatorname{Ind}_T^M \theta$ is the unique irreducible above θ, with $\psi(1) = 4$. Thus, $\chi \mapsto \psi$ is indexed by the M/T-orbits of nontrivial θ. By [160, Lemma 2.23], $R_{\mathbf{T}}^S$ is equivariant under $\Gamma = \mathrm{Aut}(S)_P$. By [72, Corollary 3.3.14], $R_{\mathbf{T}}^S(\theta)^\sigma = R_{\mathbf{T}}^S(\theta^\sigma)$. Ordinary induction is equivariant under $\Gamma \times \mathcal{H}_p$, and ε depends only on $\mathbf{T}$. Thus, $\chi \mapsto \psi$ is equivariant. By [156, Theorem D and its proof], the unipotent p'-characters match inflations of $\mathrm{Irr}_{p'}(M/T)$, since 1_T is the sole unipotent character of T.

Set $A = \mathrm{Aut}(S)$. If the image of ϕ^k in A/S fixes χ, ϕ^k fixes the geometric class of x. Thus, for a conjugate $x_0 \in \mathbf{T}_0^*$ in the pinned torus, some $w \in W(D_4)$ satisfies $w(x_0) = x_0^{s^k}$, up to replacing ϕ^k by its inverse. Given a prime $r \mid |x|$, $\mathrm{ord}_r(s) \mid k|w|$. Since $\mathrm{ord}_r(s^f) = 12$, $v_2(\mathrm{ord}_r(s)) = v_2(f) + 2$. A signed permutation w with $8 \mid |w|$ has a negative 4-cycle. But $W(D_4)$ has even sign parity. Hence, $v_2(|w|) \leq 2$, so $2^{v_2(f)} \mid k$ and $[A_\chi : S]$ is odd.

By [70, §5, torus 6], $-1 \in M/T$ acts on T by inversion. Thus, θ^{-1} lies in the orbit of θ, so χ and ψ are real. Put $B = \mathrm{N}_A(P)$. By [97, Lemma 1.13], $A = SB$ and $S \cap B = M$. Thus, by equivariance, $B_\psi = B_\chi$ and $B_\psi/M \cong A_\chi/S$. By [143, Lemma 2.1], we obtain unique real extensions $\widehat{\chi}, \widehat{\psi}$ on A_χ, B_ψ. Moreover, $\sigma(\bar{z}) = \overline{\sigma(z)}$ on $\mathbb{Q}(\zeta_{|A|})$, so each mixed operation stabilizing (χ, ψ) fixes $\widehat{\chi}, \widehat{\psi}$. Hence, both associated models have $\alpha = \mu_{a,\sigma} = 1$. For $\succeq_c$ in Definition 2.1, let $m_0 : S \rtimes B \to A$ act by $(x, b) \mapsto xb$. We compose the two models with m_0 on $S \rtimes B_\chi$ and $M \rtimes B_\psi$, respectively. Since

$\mathrm{C}_A(S) = 1$, $\mathrm{C}_{S\rtimes B}(S) = \ker m_0 \le M \rtimes B$. On $\ker m_0$, both models are trivial. Finally, $M < S$ because $1 < T < S$ and S is simple, while B normalizes $M = \mathrm{N}_S(P)$. Thus, the pair has $\succeq_c$ on the universal cover S. □

Proposition 6.102. *Suppose that $S = {}^3D_4(q)$ with $q = s^f$, $F = \tau F_s^f$, and $d \in \{1,2,3,6\}$. Put $A = S \rtimes E$, where $E = \langle\phi\rangle$ has order $3f$. Each nonunipotent $\chi \in \mathrm{Irr}(S)$ has an extension $\widetilde{\chi}$ to A_χ fixed by $(A \times \mathcal{H}_p)_\chi$.*

Proof. By [208, Proposition 9.17(i)–(iii)], isolated-series characters extend to A over $\mathbb{Q}$. Thus, the extension is fixed by $(A \times \mathcal{H}_p)_\chi$. For a nonisolated series, [199, Theorem 1] gives that x and x^{-1} are conjugate in S, so χ is real. If $2 \nmid [A_\chi : S]$, [143, Lemma 2.1] gives a unique real extension $\widetilde{\chi}$, so $(A \times \mathcal{H}_p)_\chi$ fixes it. Otherwise, f is even and $j = \phi^{3f/2} \in A_\chi$. Put $F_0 = \tau^2 F_s^{f/2}$. Since $\tau^3 = 1$, $F_0^2 = F$. Since $\tau|_S = \phi^{-f}$, $F_0|_S = \phi^{-3f/2} = j$. Thus, in the adjoint form $S = \mathbf{G}_{\mathrm{ad}}^{F_0^2}$ and $S^j = \mathbf{G}_{\mathrm{ad}}^{F_0}$. By [105, Proposition 5.4.2], the nonunipotent $\chi = \chi^j$ has multiplicity 1 in $\mathrm{Ind}_{S^j}^S 1_{S^j}$. Since $j \in E$ and E is abelian, E normalizes S^j. Hence, the permutation module $\mathcal{V} = \mathbb{Q}[S/S^j]$ affords a rational character of A. By [208, proof of Lemma 3.13], the unique constituent of $\mathcal{V}|_{A_\chi}$ above χ is an extension $\widetilde{\chi}$. Since A/S is abelian, $A_\chi \lhd A$. Thus, the character of $\mathcal{V}|_{A_\chi}$ is $A \times \mathcal{H}_p$-invariant. Thus, each element of $A \times \mathcal{H}_p$ fixing χ fixes its unique constituent $\widetilde{\chi}$. □

Let p be odd with $p \nmid q$ and $p \mid |S|$. Thus, outside Proposition 6.101, we have that $d \in \{1,2,3,6\}$ from $|S|$. Let $\mathbf{G}$ be the simply connected D_4 group with pinned split torus $\mathbf{T}_0$, and put $W = W(D_4)$. Given any d, we work with $F' = \mathrm{Int}(n)\tau F_s^f$ and $S = \mathbf{G}^{F'}$. Moreover, F_s will fix the complement V of $T = \mathbf{T}_0^{F'}$ pointwise. At $d = 1, 2$, put $\mathbf{S}_d = (\mathbf{T}_0^\tau)^\circ$. Given a regular dominant integral cocharacter λ, the sum $\nu = \lambda + \tau\lambda + \tau^2\lambda$ is regular and belongs to $Y(\mathbf{S}_d)$. Hence, $\mathrm{C}_{\mathbf{G}}(\mathbf{S}_d) = \mathbf{T}_0$. In the pinned subgroup $G_2(s)$, take the complement V of the split torus by Galt–Staroletov [70, Theorem 1.1 and §3]. Thus, V is fixed pointwise by τ and F_s, with Weyl image $W^\tau = W(G_2)$. The central involution $v \in V$ lifts $-1 \in W^\tau$, since -1 acts on $Y(\mathbf{S}_d) \otimes \mathbb{Q}$ as the longest element. Let $n = 1$ at $d = 1$ and $n = v$ at $d = 2$. On $Y(\mathbf{S}_d)$, the scalars of F' are q at $d = 1$ and $-q$ at $d = 2$. Thus, $\mathbf{S}_d$ is the Sylow d-torus, by the multiplicity 2 of Φ_d in $|S|$. By Galt–Staroletov [70, Table 3, tori 1 and 7], the relative Weyl group is W^τ. The chosen lifts in V give $M = \mathrm{N}_{\mathbf{G}}(\mathbf{S}_d)^{F'} = T \rtimes V$.

Next, at $d = 3, 6$, take tori 4, 5 of Galt–Staroletov [70, §5 and Table 3]. The central root in Galt–Staroletov [70, §5] is r_2, and $r_{12} = r_1 + 2r_2 + r_3 + r_4$. Let n_i lift the reflection in r_i, and set $n_0 = n_1 n_3 n_4 n_{12}$ above -1. Thus, with $\mathrm{Int}(n)(g) = ngn^{-1}$, choose $n = (n_{12}n_2)^{-1}$ at $d = 3$ and $n = (n_0 n_{12} n_2)^{-1}$ at $d = 6$. Put $w = n\mathbf{T}_0 \in W$. The automorphism τ fixes the root groups at $\pm r_2, \pm r_{12}$, where its signs have order dividing 2. Thus, $\tau(n_2) = n_2$ and $\tau(n_{12}) = n_{12}$. Moreover, $[n_i, n_j] = 1$ for distinct $i, j \in \{1,3,4\}$, so $\tau(n_0) = n_0$. Hence, $\tau(n) = n$, and $F_s(n) = n$ by the prime field definition of n_i. By Galt–Staroletov [70, Table 3], the order polynomial of T is Φ_d^2, with $\mathbf{S}_d = \mathbf{T}_0$ and $M/T = C_W(w\tau) \cong \mathrm{SL}_2(3)$.

In the Tits group $\widehat{W}$, put $D = \ker(\widehat{W} \to W)$ and $h = \mathrm{Int}(n)\tau$. By Galt–Staroletov [70, §2], $D = \mathbf{T}_0[2]$ for odd s and $D = 1$ for $s = 2$. Since $\det(1 - w\tau) = \Phi_d(1)^2$ is odd, $D^h = 1$. For odd s, Galt–Staroletov [70, §5, tori 4 and 5] gives a complement in $\widehat{W}^h$. Thus, $V = \widehat{W}^h$ maps isomorphically onto $C_W(w\tau)$. If $s = 2$, the same isomorphism follows from $D = 1$. In either case, $M = T \rtimes V$, and $n \in V$ since $\tau(n) = n$. Moreover, the Tits lifts lie over $\mathbb{F}_s$, so $F_s|_V = 1$.

By Enguehard–Michel [63, Theorem 3.2(2)], M contains $P \in \mathrm{Syl}_p(S)$. By Enguehard–Michel [63, Theorem 3.3], $\mathbf{S}_d$ is the unique Sylow d-torus normalized by P, including $p = 3$. Hence, $\Gamma = \mathrm{Aut}(S)_P$ stabilizes $\mathbf{S}_d$. If $B = \mathrm{N}_A(\mathbf{S}_d)$, where $A = \mathrm{Aut}(S)$, the Frattini argument (Isaacs [97, Lemma 1.13]) gives $B = M\Gamma$. The group $M < S$ is solvable, since T is abelian and V is dihedral of order 12 or $\mathrm{SL}_2(3)$.

Given $\theta \in \mathrm{Irr}(T)$, we put $\theta^+(tv) = \theta(t)$ on $T \rtimes V_\theta$. The formula is equivariant for V, $\mathcal{H}_p$ and F_s, since F_s fixes each $v \in V$. If $x^{-1}F(x) = n$ transfers F' to F, then $F_s(n) = n$ gives $F_s(x)x^{-1} \in \mathbf{G}^F$. Thus, F_s induces $\phi S \in A/S$ after conjugation by x. In a regular overgroup, $\widetilde{S} = S \times Z_{\mathrm{add}}$, since $S \to \mathbf{G}_{\mathrm{ad}}^{F'}$ is an isomorphism. On $T \times Z_{\mathrm{add}}$, the extension of $\theta \otimes \lambda$ is $\theta^+ \otimes \lambda$. Thus, the result of Ruhstorfer et al. [156, Theorem B] gives the equivariant bijection on $\mathrm{Irr}_{p'}(S)$ and $\mathrm{Irr}_{p'}(M)$ over $1_{Z_{\mathrm{add}}}$. In the bijection, the rational label s^* is dual to the weight θ. In particular, $\mathcal{E}(S,1) \cap \mathrm{Irr}_{p'}(S)$ corresponds to 1_T, and Ruhstorfer et al. [156, Theorem D] gives $\succeq_c$ there.

Write $v \in V$ for the lift of -1 also at $d = 3, 6$. Since $\tau(n) = F_s(n) = n$, $(F')^3 = \mathrm{Int}(n^3)F_s^{3f}$. Thus, on S we have $F_s^{3f} = \mathrm{Int}(b^{-1})$, with $b = n^3$. By the Weyl representatives above and $V \cong M/T$, $b = 1$ for $d = 1, 3$ and $b = v$ for $d = 2, 6$. In particular, b is central in V, and $F_s^{6f} = 1$ on T. Let u have order $6f$ and act as F_s on T. Put

$B^\sharp = T \rtimes (V \times \langle u \rangle)$, with homomorphism to B given by $(t, w, u^i) \mapsto \mathrm{Int}(tw)F_s^i$. The image contains M and maps onto A/S of order $3f$, so the kernel has order 2. Thus, $k_0 = (b, u^{3f})$ generates the kernel since $F_s^{3f} = \mathrm{Int}(b^{-1})$.

Fix $\theta \neq 1_T$ and let $L = V_\theta$ and $\eta \in \mathrm{Irr}(L)$. Then, we find $\psi = \mathrm{Ind}_{T\rtimes L}^M(\theta^+\eta)$ by Clifford [43], with η inflated from L. Put $K = \mathrm{N}_V(L)_\eta$. At $d = 3, 6$, the unique involution $v \in V \cong \mathrm{SL}_2(3)$ inverts T. Since $|T| = \Phi_d(q)^2$ is odd, we deduce that $v \notin L$, so $|L| = 1$ or 3. If $|L| = 3$, then $\mathrm{N}_V(L) = L \times \langle v \rangle$ by the quotient $V/\langle v \rangle \cong A_4$, and we extend η trivially on v. At $L = 1$, take 1_K. Thus, we have that the extension η_0 has values in $\mathbb{Q}(\zeta_3)$, fixed by $\mathcal{H}_p$ since $p \equiv 1 \bmod 3$.

At $d = 1, 2$, we write $V = \mathfrak{S}_3 \times \langle v \rangle$. Thus, for the dual label s^* of θ, we put $\Phi_{s^*} = \{\alpha \mid \alpha(s^*) = 1\}$ in the simply connected model $\mathbf{G}_{\mathrm{sc}}$. By connectedness of $\mathrm{C}_{\mathbf{G}_{\mathrm{sc}}}(s^*)$, the full Weyl stabilizer is $W(\Phi_{s^*})$. Since $\tau(s^*)$ generates $\langle s^* \rangle$, we deduce that the positive roots $\Phi_{s^*} \cap \Phi^+$ are stable under τ. By Steinberg (Mem. AMS **80** (1968), 1.32), $L = W(\Phi_{s^*})^\tau$ is a Weyl group. Hence, $\eta \in \mathrm{Irr}(L)$ of degree 1 takes values in $\{1, -1\}$. Since $[K, K] \leq A_3 \times 1$, we have that the sign η is trivial on $L \cap [K, K]$. The 2-part of $K/[K, K]$ has exponent at most 2, so η extends to a sign η_0 of K. If η is nonlinear, then L projects onto $\mathfrak{S}_3$, since a cyclic subgroup of $\mathfrak{S}_3$ has abelian preimage in V. Thus, we obtain that $V = L$ or $V = L \times \langle v \rangle$. In either case $L \cong \mathfrak{S}_3$ or $\mathfrak{S}_3 \times C_2$, so η is rational valued. Hence, we set $\eta_0 = \eta$ when $K = L$, or extend trivially on v otherwise. We choose η_0 for representatives of the V-orbits of (L, η). This gives a V-equivariant choice since the stabilizer K acts on $\mathrm{Irr}(K)$ by inner automorphisms.

Let $J = \mathrm{Stab}_{V\times\langle u\rangle}(\theta, \eta)$. If $(w, u^i) \in J$, then $w \in K$, since u^i fixes V and acts on θ by a coprime power. Let ρ_0 afford η_0. On $T \rtimes J$, put $\rho(t(w, u^i)) = \theta(t)\rho_0(w)c_0^i$. If $\rho_0(b) = I$, set $c_0 = 1$. Otherwise, $d = 2$, $b = v$, and $\rho_0(b) = -I$. Take c_0 of order $2^{v_2(f)+1}$, so $c_0^{3f} = -1$. Since $s^f \equiv -1 \bmod p$, $v_2(\mathrm{ord}_p(s)) = v_2(f) + 1$, so c_0 has order dividing $p - 1$. Hence, $\mathcal{H}_p$ fixes c_0. Since k_0 acts trivially on T, V, $k_0 \in J$. Thus, $c_0^{6f} = 1$ and $\rho(k_0) = \rho_0(b)c_0^{3f} = I$, so ρ descends to the image of $T \rtimes J$ in B.

By Isaacs [95, Theorem 6.11], the inertia of ψ in $B^\sharp$ is $I^\sharp = M(T \rtimes J)$. Since $(T \rtimes J) \cap M = T \rtimes L$, Isaacs [95, Problem 5.2] gives $\big(\mathrm{Ind}_{T\rtimes J}^{I^\sharp} \mathrm{tr}\,\rho\big)|_M = \psi$. Thus, on B_ψ this induction gives the extension $\widetilde{\psi}$. For a mixed operation stabilizing ψ, conjugation by some $w \in V$ restores (θ, η) by Clifford correspondence. Since η_0 is chosen V-equivariantly and is $\mathcal{H}_p$-invariant, the corrected operation fixes η_0. Moreover, $\mathcal{H}_p$ fixes c_0, whose choice depends on $\rho_0(b)$ with $b \in \mathrm{Z}(V)$. Thus, the corrected operation fixes $\mathrm{tr}\,\rho$. Since $w \in M$, the original operation fixes $\widetilde{\psi}$. By Proposition 6.102, the global extension $\widetilde{\chi}$ is mixed invariant on A_χ. Thus, both associated models have $\alpha = \mu_{a,\sigma} = 1$. By the multiplication map $m_0 : S \rtimes \Gamma \to A$ as in Proposition 6.101, the pair satisfies $\succeq_c$.

6.5 Split rank-four Levi series

Lemma 6.103. *Take the factor L over $\mathbb{F}_Q$ from* Section 6.1. *Next, on the $2b$-space, we have that the eigenvalues $s, -s$ have multiplicity b each. We allow F to act as 1 or $(s, -s)$ on $\{s, -s\}$, and compare degrees at odd $\ell \nmid Q$. Let $u = \epsilon Q$ with $\epsilon = 1$ for* GL *and $\epsilon = -1$ for* GU*, and set $d = \mathrm{ord}_\ell(u)$. Suppose that $\ell \mid |L|$. If $F(s) = -s$, then $\ell \nmid \chi(1)$ gives $d = 2e$. If $d = 2e$, the Jordan character μ of $\mathrm{GL}_b(Q^2)$ has ℓ'-degree exactly when χ does. If F fixes $s, -s$, write $b = dk + c$, with $0 \leq c < d$. Let C^* be the dual centralizer. Then, $|L^* : C^*|$ is an ℓ-unit exactly when $2c < d$ and $\ell \nmid \binom{2k}{k}$.*

Proof. Put $a = v_\ell(u^d - 1)$ and $L(j) = ja + v_\ell(j!)$. By [132, Lemma 5.2], $v_\ell(u^{di} - 1) = a + v_\ell(i)$ for $i \geq 1$. Thus, the order product $\prod_{i=1}^m (u^i - 1)$ has valuation $L(\lfloor m/d \rfloor)$. If $F(s) = -s$, the dual centralizer is $C^* = \mathrm{GL}_b(Q^2)$. If $2 \nmid d$, we have that $\mathrm{ord}_\ell(Q^2) = d$ and $v_\ell((Q^2)^d - 1) = a$, since $u^d + 1$ is an ℓ-unit. Hence, $|L|$ and $|C^*|$ have valuations $L(\lfloor 2b/d \rfloor)$ and $L(\lfloor b/d \rfloor)$, respectively. Since $\ell \mid |L|$, $\lfloor 2b/d \rfloor > \lfloor b/d \rfloor$. Thus, $\ell \mid \chi(1)$ by Jordan degrees [58, Proposition 11.5.6]. At $d = 2e$, $\mathrm{ord}_\ell(Q^2) = e$ and $(Q^2)^e = u^d$. Consequently, both orders have valuation $L(\lfloor b/e \rfloor)$, and $v_\ell(\chi(1)) = v_\ell(\mu(1))$. If F fixes $s, -s$, the two factors of C^* have order valuation $L(k)$ each. Thus, $v_\ell(|L^* : C^*|) = L(2k + \lfloor 2c/d \rfloor) - 2L(k)$. If $2c < d$, this is $v_\ell\binom{2k}{k}$. Otherwise, the valuation increases by $a + v_\ell(2k + 1) > 0$. Finally, when $v_\ell(|L^* : C^*|) = 0$, $\ell \nmid \chi(1)$ exactly when $\ell \nmid \mu_1(1)\mu_2(1)$ for the Jordan factors μ_1, μ_2. □

Lemma 6.104. *Let $Q = r^2$, $G = \mathrm{GL}_m(Q)$ and $J \in \{F_r, \gamma F_r\}$, where $\gamma(g) = (g^{-1})^t$. Then, for $H = G^J$, we have that each $\chi \in \mathrm{Irr}(G)$ with $\chi^J = \bar{\chi}$ has multiplicity 1 in $\mathbb{Q}[G/H]$ over $\mathbb{C}$. On the constituent $\mathcal{V}_\chi$, transport by F_p, γ, σ with $p \mid r$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ and determinant twists $\delta|_H = 1_H$ gives the operators $T_{\alpha,\sigma}, D_\delta$ with their ordinary composition laws.*

Proof. Thus, H is $\mathrm{GL}_m(r)$ for $J = F_r$ and $\mathrm{GU}_m(r)$ for $J = \gamma F_r$. By [58, Theorem 11.7.3], each $\chi \in \mathrm{Irr}(G)$ is uniform. By [105, Theorem 3.1.3(ii)], $\langle \chi, \mathrm{Ind}_H^G 1_H \rangle = 1$ when $\chi^J = \bar{\chi}$. In the unitary case $J = \gamma F_r$, this condition is $\chi^{F_r} = \chi$,

since g^t is similar to g and $\chi^\gamma = \bar{\chi}$. Write L_x for left translation on G/H. If $\delta|_H = 1_H$, we put $D_\delta e_{gH} = \delta(g)e_{gH}$. Thus, $D_\delta L_x D_\delta^{-1} = \delta(x)L_x$ and $D_\delta D_\nu = D_{\delta\nu}$. In particular, $D_{\delta^{-1}}$ permutes the summands by $\chi \mapsto \chi\delta$. Each $\alpha \in \langle F_p, \gamma\rangle$ satisfies $\alpha(H) = H$. Thus, for σ on $\mathbb{Q}^{\mathrm{ab}}$, the map $T_{\alpha,\sigma}(ce_{gH}) = \sigma(c)e_{\alpha(g)H}$ satisfies $T_{\alpha,\sigma}D_\delta T_{\alpha,\sigma}^{-1} = D_{(\sigma\delta)\circ\alpha^{-1}}$. We restrict to the isotypic summands $\mathcal{V}_\chi$ over $\mathbb{Q}^{\mathrm{ab}}$. Each $\mathcal{V}_\chi$ is irreducible, so any composite fixing χ acts on $\mathcal{V}_\chi$ by the stated operators.

The image $\det(H)$ has order $r-1$ or $r+1$, and J acts on determinants by r or $-r$, respectively. Thus, in the cyclic group D of determinant characters, $\delta|_H = 1_H$ exactly when $\delta^J = \delta^{-1}$. Let $X = \{\nu \in D \mid \chi\nu = \chi\}$. Suppose that $\chi_1\delta = \chi$ with $\chi_1^J = \bar{\chi}_1$ and $\chi^J = \bar{\chi}$. Then, $\delta\delta^J \in X^J$ by applying J and complex conjugation to $\chi_1\delta = \chi$. Replacing δ by $\delta\nu$ for $\nu \in X$ changes the product by $\nu\nu^J$. Hence, a representative δ_0 trivial on H exists exactly when $\delta\delta^J$ is trivial in $X^J/\{\nu\nu^J \mid \nu \in X\}$. This cyclic quotient of X^J has order at most 2, since $\nu^2 = \nu\nu^J$ for $\nu \in X^J$. If $X = 1$, δ_0 is unique. Otherwise, we include D_ν for self twists $\nu \in X$ with $\nu|_H = 1_H$. □

Proposition 6.105. *Suppose that $Q = p^f = r^2$ is odd. Let G have rational dual eigenvalue a alone or eigenvalues $a, -a$ of equal multiplicity. Each eligible χ and its local character ψ admit mixed comparison for F_p, γ, σ and determinant twists δ. If η is the quadratic determinant character and $\chi\eta = \chi$, the comparison includes both eigenspaces of D_η.*

Proof. We twist by $\kappa^{-1}\circ\det$ to give scalar weight 1, or the weights $1, \eta$, where η is quadratic on $\mathbb{F}_Q^\times$. In the normalized label, F_p, γ, σ fix each (ξ, λ_ξ), since $\xi \in \{1, \eta\}$ has values in $\{1, -1\}$ and the unipotent character χ_{λ_ξ} is rational. Thus, with $J = F_r$, Lemma 6.104 gives the rational model $e_\chi\mathbb{Q}[G/H]$. A twist δ fixes $\{1, \eta\}$ exactly for $\delta \in \{1, \eta\}$. Since $r+1$ is even, $\eta|_{\mathbb{F}_r^\times} = 1$. Hence, $D_\eta^2 = 1$, and the conjugates of D_η by F_p, γ, σ are D_η. On labels, D_η acts by $(\lambda_1, \lambda_\eta) \mapsto (\lambda_\eta, \lambda_1)$. Thus, $\mathcal{V}_\chi$ has the involution D_η when $\lambda_1 = \lambda_\eta$. For residual cores κ_ξ, the lemma gives these operators on $\mathcal{R} = \mathbb{Q}[\mathrm{GL}_t(Q)/\mathrm{GL}_t(r)]$.

Next, write $m = dw + t$, with $0 \le t < d$, and put $K = \mathbb{F}_{Q^d}$. On the marked slots K^w and residue $R = \mathbb{F}_Q^t$, the normalizer is $M = ((K^\times \rtimes \langle v\rangle) \wr \mathfrak{S}_w) \times \mathrm{GL}_t(Q)$, with $v(x) = x^Q$. In particular, at $d = 1$, we have the monomial model with $v = 1$ and $t = 0$. If the Sylow projection is 1, choose the identity comparison on G. Given an active balanced G, Lemma 6.103 gives $w = 2k$ and $t = 2c$, where $b = dk + c$.

Next, for $\lambda \vdash dk + c$, take its ordered d-quotient $(\nu_j)_{j\in\mathbb{Z}/d}$, and put $s_\lambda = \sum_j |\nu_j|$. By Olsson (Essen Lectures **20** (1994), Theorem 3.3), h/d for hooks h of λ divisible by d runs through the hooks of the ν_j with multiplicity. Put $a = v_\ell(Q^d - 1)$. Then, Green's formula [84, Theorem 14] gives $v_\ell(\chi_\lambda(1)) = (k - s_\lambda)a + v_\ell(k!) - \sum_{j=0}^{d-1}\sum_{h\in\mathrm{Hook}(\nu_j)} v_\ell(h)$. By the hook formula, the product H_λ of the hooks of $(\nu_j)_j$ divides $\prod_j |\nu_j|!$. Hence, H_λ divides $s_\lambda!$. Thus, χ_λ of ℓ'-degree has $s_\lambda = k$ and core of size c. With two weights $\xi = 1, \eta$, the centralizer index contributes $v_\ell((2k)!) - 2v_\ell(k!)$. Hence, for $s_{\lambda_\xi} = k$, the degree valuation is that of $w!/\prod_\xi H_{\lambda_\xi}$ with $w = 2k$. With one weight ξ, we set $k = w$ and $c = t$. Therefore, λ_ξ is determined by its core κ_ξ and ordered quotient $(\nu_{\xi,j})_j$.

Let $W = \{1, \dots, w\}$ index the slots. Choose (ξ, j) with the order $0, \dots, d-1$ on $j \in \mathbb{Z}/d$ for each ξ. Next, we let $\mathcal{A}$ consist of partitions $W = \bigsqcup_{\xi,j} A_{\xi,j}$ with $|A_{\xi,j}| = |\nu_{\xi,j}|$. Put $\mathcal{B} = \bigoplus_{(A_{\xi,j})\in\mathcal{A}} \bigotimes_{\xi,j} S^{\nu_{\xi,j}}(A_{\xi,j})$, with rational Specht modules $S^{\nu_{\xi,j}}(A_{\xi,j})$. Thus, on a slot in $A_{\xi,j}$, a multiplier x acts by $\xi(N_{K/\mathbb{F}_Q}x)$, and v acts by ζ_d^j. A permutation π acts by $A_{\xi,j} \mapsto \pi A_{\xi,j}$ and transports $S^{\nu_{\xi,j}}(A_{\xi,j})$ along π. Thus, by [43], $\mathcal{B}$ is induced from the (ξ, j)-weight stabilizer. Choose $\mathcal{V}_{\mathrm{loc}} = \mathcal{B}\otimes\mathcal{C}$ with residual core model $\mathcal{C}$ on R. Moreover, $\mathcal{B}$ is defined over $\mathbb{Q}(\zeta_d)$, which $\mathcal{H}_\ell$ fixes since $d \mid \ell - 1$.

For the twist by $\eta\circ\det$, let E_η act by $A_{\xi,j} \mapsto A_{\xi\eta,j}$, with the associated tensor permutations. Then, $E_\eta^2 = 1$, and the multiplier x gives the twist factor $\eta(N_{K/\mathbb{F}_Q}x)$. In a normal basis of $K/\mathbb{F}_Q$, $\det v = (-1)^{d-1}$. Moreover, a slot transposition π has $\det\pi = (-1)^d$. Since Q is an odd square, $\eta(\det v) = \eta(\det\pi) = 1$. Hence, $E_\eta \otimes D_\eta$ gives the determinant twist on M, with D_η on $\mathcal{C}$.

Take the same basis $\mathbf{e}$ of $K/\mathbb{F}_Q$ in every slot. Let $u(x) = x^p$ on K and $u = F_p$ on R. Next, give K the trace pairing $\mathrm{Tr}_{K/\mathbb{F}_Q}(xy)$ and R the pairing $x^t y$. Then, the graph map $\widehat{\gamma}$ acts on multipliers by $x \mapsto x^{-1}$ and fixes v and each slot permutation π. Since $\xi = \xi^{-1}$, we choose the identity on $\mathcal{B}$ for $\widehat{\gamma}$. The field lift u fixes the slot weights (ξ, j), rotations v and permutations of W. The power u^f must act by the simultaneous rotation v_{all}, whose scalar on $\mathcal{B}$ is $\omega = \zeta_d^{\sum_{\xi,j} j|\nu_{\xi,j}|}$. Since $d = \mathrm{ord}_\ell(p^f)$, $d \mid (\ell-1)/\gcd(\ell-1, f)$. Thus, there is $\alpha \in \mu_{\ell-1}$ with $\alpha^f = \omega$. Under the swap $1 \leftrightarrow \eta$, the exponent $\sum_{\xi,j} j|\nu_{\xi,j}|$ has equal values before and after $1 \leftrightarrow \eta$, so both yield α. Moreover, $\mathcal{H}_\ell$ fixes α. We tensor the operators on $\mathcal{B}$ with F_p, γ on $\mathcal{C} \subset \mathcal{R}$. Hence, $u^f = v_{\mathrm{all}}$ and $\widehat{\gamma}^2 = [u, \widehat{\gamma}] = 1$ on $\mathcal{V}_{\mathrm{loc}}$.

On $\mathcal{V}_{\mathrm{gl}} = e_\chi\mathbb{Q}[G/H]$, write $u = AF_p$ and $\widehat{\gamma} = B^{-1}\gamma$, where B is the Gram matrix. Since $A^tBA = F_p(B)$ and $B^t = B$, $[u, \widehat{\gamma}] = 1$, $\widehat{\gamma}^2 = 1$ and $u^f = v_{\mathrm{all}}$. We write the operators as $L_A T_{F_p,1}$ and $L_{B^{-1}}T_{\gamma,1}$ by Lemma 6.104. In

the balanced case, $A = A_0^{\oplus 2k} \oplus I_R$ and $B = B_0^{\oplus 2k} \oplus I_R$. Thus, $\det A = (\det A_0)^{2k}$ and $\det B = (\det B_0)^{2k}$. Thus, $[u, D_\eta] = [\widehat{\gamma}, D_\eta] = 1$. An inner change by $g \in M$ contributes $\eta(\det g)$ on both models. Moreover, zI acts globally by $\eta(z)^b$. On $\mathcal{V}_{\mathrm{loc}}$, it acts by $\eta(z)^{dk}\eta(z)^c$ from k slots of weight η and the core of size c. Thus, $\eta(z)^b = \eta(z)^{dk+c}$. With one normalized weight 1, both scalars are 1.

Finally, we restore $\kappa \circ \det$ on G and its restriction to M. A mixed operation h acts by $\kappa \mapsto \kappa^h$ on both models. Thus, for $g \in M$, including v_{all}, the inner change contributes $\kappa(\det g)$ at both ends. The operators D_η and $E_\eta \otimes D_\eta$ act on the weights by $\xi \mapsto \xi\eta$. When $\chi\eta = \chi$, we pair the eigenspaces for $+1$ and for -1 separately. Consequently, on $\mathcal{V}_{\mathrm{gl}} \otimes \mathcal{V}_{\mathrm{loc}}^\vee$, a scalar pair (a, a) contributes $aa^{-1} = 1$. On isometric factors G_i, a permutation π of the factors acts by $\bigotimes_i v_i \mapsto \bigotimes_i v_{\pi^{-1}(i)}$, with the operators transported along π. Hence, we have $\succeq_c$ on the marked factor G of Section 6.1. □

With $Q = p^f$ odd, we let $\ell \neq p$ be odd. Next, take the factor L of Section 6.1 in the spin group X. On L and its normalizer $\mathcal{N}$, we choose unipotent sources with extensions $\widetilde{\chi}, \widetilde{\psi}$. After the adjoint maps π, the twist functions satisfy $\mu_{\widetilde{\chi}} = \mu_{\widetilde{\psi}}$. In particular, for a central label z, the proof of Theorem 6.199 gives $\succeq_c$ by Ruhstorfer et al. [156, Theorem D and §§7–8]. On isometric factors L_i, we transport the source maps $\mathcal{I}_i$ along the isometries.

Proposition 6.106. *Let $G = \mathrm{GL}_{2b}(Q)$ have an eligible dual label s_0 with rational eigenvalues $a, -a$, each of multiplicity b. Then, the sources on $L, \mathcal{N}$ give induced models ρ_G, ρ_M with equal mixed factors on G and its torus normalizer M. This includes a determinant self twist δ of (ρ_G, ρ_M).*

Proof. We normalize the weights to $1, \eta$ by a determinant twist, where η is quadratic on $\mathbb{F}_Q^\times$. Next, put $d = \mathrm{ord}_\ell(Q)$ and $b = dk + c$, with $0 \leq c < d$. By Lemma 6.103, an eligible pair has $2c < d$ and $v_\ell(|G : \mathrm{GL}_b(Q)^2|) = v_\ell\binom{2k}{k} = 0$. Hence, $\ell \nmid U_\lambda(1)U_\mu(1)$ for the unipotent sources U_λ, U_μ. Write $V = V_1 \oplus V_2$, with k torus blocks and a residual c-space R_i in each V_i. The stabilizers P_+, P_- of V_1, V_2 have common Levi $L = \mathrm{GL}_b(Q)^2$. If M_b is the marked normalizer on V_i, $K_\pm = M \cap P_\pm$ have Levi M_b^2. Indeed, a bulk permutation π fixing the slots in V_1 fixes those in V_2. Thus, the global radicals $U_{G,\pm}$ consist of off-diagonal maps between V_1, V_2, and the local radicals $U_{M,\pm}$ consist of such maps between R_1, R_2. Consequently, their orders are Q^{b^2} and Q^{c^2}, respectively.

We choose the Levi sources $U_\lambda \boxtimes ((\eta \circ \det)U_\mu)$ and $y_\lambda \boxtimes ((\eta \circ \det)y_\mu)$, where y_λ, y_μ are the local partners. On each b-space the determinant includes the rotations v and the permutations in $\mathfrak{S}_k$. Thus, $\eta \circ \det$ is on the full factor $\mathrm{GL}_b(Q)$ or M_b. We inflate the sources to $P_\pm, K_\pm$ and induce to G, M. The weights $1, \eta$ give dual centralizer $\mathbf{L}^*$. By [72, Theorem 3.3.22], we obtain the irreducible ρ_G with Jordan labels λ, μ. On the bulk of M, the stabilizer of (θ_1, θ_2) fixes each of its two k-sets. Thus, [43] gives irreducible bulk induction ρ_{bulk}. On the residue $\mathrm{GL}_{2c}(Q)$, the distinct weights $1, \eta$ give irreducible induction ρ_{res} by [72, Theorem 3.3.22]. We recover λ, μ from the bulk inertia at θ_1, θ_2 and the Jordan labels of ρ_{res}. Moreover, the bulk index is $\binom{2k}{k}$, and $2c < d$ makes the residual index an ℓ-unit.

Given $H \in \{G, M\}$, we let $\mathcal{F}_{H,\pm}$ denote functions induced from $P_\pm$ or $K_\pm$ with left covariance. Next, on $\mathcal{F}_{H,+}$, put $A_{H,-}f(g) = R_H^{-1} \sum_{u \in U_{H,-}} f(ug)$, with $R_H = \sqrt{|U_{H,-}|}$. Define $A_{H,+}$ from the average on $U_{H,+}$. The products $A_{H,+}A_{H,-}$ and $A_{H,-}A_{H,+}$ are scalar on ρ_H. By Lemma 3.6's calculation, for f supported on P_+ or K_+, u_-u_+ lies in the support only if $u_- = 1$. Thus, before division by R_H^2, evaluation at 1 gives $|U_{H,+}|f(1)$. Therefore, $A_{H,+}A_{H,-} = 1$ and $A_{H,-}A_{H,+} = 1$. We have $R_G = Q^{b^2/2}$ and $R_M = Q^{c^2/2}$, including $R_M = 1$ if $c = 0$. If d is even, $b - c = dk$ is even and $R_G/R_M \in \mathbb{Q}$. If d is odd, $\mathcal{H}_\ell$ fixes $\sqrt{Q}$ by [156, Lemma 2.3(d)]. Hence, it fixes R_G/R_M. In either case, $\sigma(R_G)/R_G = \sigma(R_M)/R_M$ for $\sigma \in \mathcal{H}_\ell$.

Next, take on V_2 the basis from V_1 and its trace form B_1. The swap $s(x, y) = (y, x)$ has $s^2 = 1$ and $\det s = (-1)^b$. Thus, $[s, \widehat{F}] = [s, \widehat{\gamma}] = 1$ for the lifts $\widehat{F}, \widehat{\gamma}$ of field and graph actions. On $P_\pm$, the lift $\widehat{F}$ fixes each subgroup and $\widehat{\gamma}$ acts by $P_\pm \mapsto P_\mp$. The twist by η acts by $(1, \eta) \mapsto (\eta, 1)$, and s returns the order with $sP_\pm s^{-1} = P_\mp$. Thus, a map into $\mathcal{F}_{H,\mp}$ is followed by $A_{H,\pm}$.

Given a transport T, write $TL_uT^{-1} = L_{\alpha(u)}$ for $u \in U_{H,\pm}$, where L_u is translation and α acts on $U_{H,\pm}$. For a twist, $\alpha = 1$ since $\det u = 1$. With coefficient action σ, we have $TA_{H,\pm}T^{-1} = (R_H/\sigma(R_H))A_{H,\alpha(\pm)}$, with $U_{H,\alpha(\pm)} = \alpha(U_{H,\pm})$. Thus, opposite averages cancel, and $R_H/\sigma(R_H)$ is independent of H. Moreover, a twist δ contributes $\delta(s)$ on both models. By hypothesis, $\mathcal{I}_G, \mathcal{I}_M$ give equal ordinary factors $\alpha_G = \alpha_M$ and twists $\mu_G = \mu_M$. Next, take $\mathcal{V}_G$ over $\mathbb{Q}$ and $\mathcal{V}_M$ over $\mathbb{Q}(\zeta_d)$, with σ acting on each matrix entry a_{ij}. The extension characters $\widetilde{\chi}, \widetilde{\psi}$ determine the twists. Consequently, $T_1 \cdots T_j$ has equal factors for G, M, including, after a twist δ, operations that return the paired models. On the torus blocks, $\widehat{F}^f = v_{\mathrm{all}}$ acts by the simultaneous Q-power. Finally, with $\kappa \circ \det$ restored, an inner correction g contributes $\kappa(\det g)$ in each model. Hence, for $\lambda = \mu$ and $\lambda \neq \mu$, the mixed factors agree. □

Proposition 6.107. *Let $G = \mathrm{GU}_{2b}(Q)$ have an eligible balanced label with $F(\theta) = \theta' \neq \theta$ and Sylow projection $P_L \neq 1$. Then, the Levi sources U, y on $L, \mathcal{N}$ give models ρ_G, ρ_M with equal factors and powers $\widehat{F}^{2f}$.*

Proof. By Lemma 6.103, the period $d = \mathrm{ord}_\ell(-Q)$ equals $2e$, with $e = \mathrm{ord}_\ell(Q^2)$. Next, write the hyperbolic space as $X \oplus Y$ over $k_0 = \mathbb{F}_{Q^2}$. The opposite parabolics P_X, P_Y have Levi $L = \{h_A \mid A \in \mathrm{GL}_b(k_0)\}$, where $h_A = \mathrm{diag}(A, (A^Q)^{-T})$. The source weights $\theta \circ \det$ and $\theta^{-Q} \circ \det$ are distinct. Moreover, we write $b = et + c$, with $0 \leq c < e$, and decompose X as $X_{\mathrm{bulk}} \oplus R_X$ with $X_{\mathrm{bulk}} = (\mathbb{F}_{Q^{2e}})^t$ and $\dim R_X = c$. We consider the dual decomposition in Y. Thus, on block i, the signed cycle w_i has period $2e$. Its rotation acts on multipliers by $-Q$, so that w_i^{2j} acts by Q^{2j}. Given M_b, the e-torus normalizer of $\mathrm{GL}_b(k_0)$, each $K_Z = M \cap P_Z$ for $Z = X, Y$ has Levi h_{M_b}. Thus, the local radical $U_{M,Z}$ consists of skew-Hermitian c-by-c matrices and has order Q^{c^2}. The global radical $U_{G,Z}$ has order Q^{b^2}.

Next, we choose B as the trace matrix on $\mathbb{F}_{Q^{2e}}$ and the identity on R_X. Write the blockwise p-power as $U = A_p F_p$ and $U^f = A_Q F_Q$, with F_p, F_Q acting on coordinates. Thus, $A_p^T B A_p = F_p(B)$, and $\gamma_B U = U \gamma_B$ on Levi matrices for $\gamma_B(A) = B^{-1} A^{-T} B$. Let w_0 exchange X, Y, and put $D = B^{-1} A_Q^{-T}$, $s = h_D w_0$ and $v = U^{2f}$. On L, we have that conjugation by s acts by $\gamma_B U^f$. Hence, s normalizes T and acts by $P_X \mapsto P_Y$.

On X, the upper block of s^2 is $D(D^Q)^{-T}$. Since $A_Q^T B A_Q = B^Q$, the block is $B^{-1} A_Q^{-T} B^Q A_Q^Q$, which equals $A_Q A_Q^Q = v$. Thus, $s^2 = h_v$, with v acting by Q^2-power on X_{bulk} and by 1 on R_X. The lift $\widehat{F} = h_{A_p} F_p$ satisfies $A_p F_p(D) = D(A_p^Q)^{-T}$, by $\gamma_B U = U \gamma_B$ and $U U^f = U^f U$. Hence, $[\widehat{F}, s] = 1$, and $\widehat{F}^{2f} = h_v$ by $U^{2f} = v$. In particular, h_v belongs to both L and h_{M_b}.

We inflate $(\theta \circ \det) U_\lambda$ to P_X and $(\theta \circ \det) y_\lambda$ to K_X and induce. The dual centralizer on G is $\mathbf{L}^*$, so [72, Theorem 3.3.22] gives irreducible ρ_G. On the bulk of M, an odd power of w_i acts by $\theta \mapsto \theta^{-Q}$, while w_i^{2j} fixes θ. Thus, on the bulk group M_{bulk}, the inertia of θ is $h_{M_b} \cap M_{\mathrm{bulk}}$. By [43], the induction ρ_{bulk} is irreducible. On $\mathrm{GU}_{2c}(Q)$, the residual induction ρ_{res} is irreducible by [72, Theorem 3.3.22]. Consequently, ρ_M is irreducible. The bulk orbit of θ and the Jordan label of ρ_{res} recover λ. The bulk index 2^t is an ℓ-unit. Moreover, $2c < d$ gives an ℓ-unit residual index. If $\delta = \delta_0 \circ \det$, its Levi value is $\delta(h_A) = \delta_0(\det(A)^{1-Q})$.

For opposites P_X, P_Y, take the averages $A_{G,X}, A_{G,Y}$ with $R_G = Q^{b^2/2}$, and on K_X, K_Y take $R_M = Q^{c^2/2}$. If e is even, $b - c = et$ is even. If e is odd, $Q^e \equiv 1 \bmod \ell$ since $(-Q)^e \equiv -1 \bmod \ell$. Thus, [156, Lemma 2.3(d)] gives $\mathcal{H}_\ell$-invariance of $\sqrt{Q}$. Thus, $\sigma(R_G)/R_G = \sigma(R_M)/R_M$ as in Proposition 6.106. Thus, $A_{G,X} A_{G,Y} = 1$ and $A_{M,X} A_{M,Y} = 1$, while $s^2 = h_v$ acts through U, y. Moreover, a twist δ contributes $\delta(s)$ on G, M. By Proposition 6.106, the coefficient factor is $R_G/\sigma(R_G) = R_M/\sigma(R_M)$. Given $g \in G$, inverse transpose satisfies $\gamma(g) = w_0 F_Q(g) w_0^{-1}$ by the hyperbolic Gram matrix w_0. Hence, the products in $s, \widehat{F}, \gamma, \sigma, \delta$ have equal mixed factors on ρ_G, ρ_M, with $s^2 = \widehat{F}^{2f} = h_v$. □

Let $Q = p^f$ be odd and $\ell \neq p$ be prime with $\ell > 2$. On the balanced factor G of rank $2b$ in Section 6.1, we let F be Frobenius. Thus, we find $F(\xi) = \xi'$ on the linear weights, and $F(\xi) = \xi$, $F(\xi') = \xi'$ for G unitary. Keep M and the labels $\lambda_{\mathrm{core}}, \boldsymbol{\nu}$ of Lemma 6.103 for characters of ℓ'-degree. If the Sylow ℓ-subgroup has trivial projection on G, we pair each χ with χ. With $D = \det(G)$, we let η, η_Q be quadratic on $D, \mathbb{F}_Q^\times$, respectively. Next, we put $\sigma(\zeta_p) = \zeta_p^a$ for $\sigma \in \mathcal{H}_\ell$ and $a \in \mathbb{F}_p^\times$. On finitely many roots ζ of ℓ'-order, we choose j with $\sigma(\zeta) = \zeta^{\ell^j}$.

Lemma 6.108. *Suppose that both geometric weights ξ, ξ' have partition $\lambda \vdash b$. Let χ be the character of G with weights ξ, ξ' and let χ_1, χ_2 be its constituents on $G^1 = \ker(\det : G \to D)$. If $\chi^\sigma = \chi(\delta \circ \det)$ for some $\delta \in \mathrm{Irr}(D)$, then σ has sign $\eta_Q(a)^b$ on $\{\chi_1, \chi_2\}$.*

Proof. The determinant twists fixing χ are 1 and $\eta \circ \det$, since the two weights are $\{\xi, \xi\eta\}$. For the linear weights, η denotes $\eta_Q \circ \mathrm{N}_{Q^2/Q}$. By [95, Corollary 11.22], each constituent of $\chi|_{G^1}$ extends to its inertia group in G. Thus, $\chi|_{G^1}$ is multiplicity free. By [95, Lemma 5.2], $\langle \chi|_{G^1}, \chi|_{G^1} \rangle = 2$ from the two determinant twists. Consequently, $\chi|_{G^1} = \chi_1 + \chi_2$, and $\chi_i^g = \chi_{3-i}$ precisely when $\eta(\det g) = -1$.

By [188, Theorem 14.10 and §§12.7–12.10], the wave front class $\mathcal{O}_\chi^*$ corresponds to $j_{\mathfrak{S}_b \times \mathfrak{S}_b}^{\mathfrak{S}_{2b}}(S^{\lambda'} \boxtimes S^{\lambda'})$, where λ' is the transpose of λ. The type A rule for j gives the Specht module $S^{2\lambda'}$ [40, §11.3]. Thus, the Jordan chains in $\mathcal{O}_\chi^*$ have lengths $D_i = 2\lambda_i'$. Since each D_i is even, the weights of ϖ on V are odd. Hence, $\mathrm{End}(V)$ has zero weight space in degree 1, and the grading subgroups satisfy $U_1 = U_2 = U$. We write ξ_U for the linear character on U with trace functional $\mathrm{Tr}(J \cdot)$ for the opposite Jordan matrix J and additive root ζ_p. Thus, by [188, Corollary 15.7], $\langle \chi|_U, \xi_U \rangle = 1$. Exactly one of χ_1, χ_2 contains ξ_U on U.

Under σ, the functional with J becomes the functional with aJ. Next, in the linear group G, diagonal matrices $g_i = \mathrm{diag}(1, a, \ldots, a^{D_i-1})$ on the chains act by $J \mapsto aJ$ by conjugation, or $\mathrm{Ad}(g_i^{-1})J = aJ$ in the opposite convention. Hence, for $g = \bigoplus_i g_i$, $\eta_Q(\det g) = \eta_Q(a)^E$, where $E = \sum_i \binom{D_i}{2} \equiv b \pmod 2$. In the unitary group G, we choose $v \in \mathbb{F}_{Q^2}^\times$ with $v^{Q+1} = a$. On chain i, we let $u_{i,r} = v^{1-D_i}a^r$ for $0 \le r < D_i$. Since $u_{i,r}u_{i,D_i-1-r}^Q = 1$, the matrix $v^{1-D_i}g_i$ is an isometry. Its determinant is $(v^{Q-1})^{\binom{D_i}{2}}$, with $\eta(v^{Q-1}) = a^{(Q-1)/2} = \eta_Q(a)$. Consequently, for the unitary sum $g = \bigoplus_i v^{1-D_i}g_i$, we again have that $\eta(\det g) = \eta_Q(a)^b$. For the opposite convention we fix g^{-1}, with the same quadratic value $\eta(\det g)$. Since the Cayley map c is g-equivariant, $\xi_U^\sigma = \xi_U^g$. Since $\delta \circ \det$ is trivial on G^1, the permutations of $\{\chi_1, \chi_2\}$ by σ and g coincide. □

Assume that $G = \mathrm{GL}_{2b}(Q)$ has $\ell \mid |G|$ and exchanged weights ξ, ξ'. By Lemma 6.103, $d = \mathrm{ord}_\ell(Q) = 2e$ and we write $b = et + c$ with $0 \le c < e$. In particular, $t > 0$ since $2b \ge d$. Let γ_2 generate $\mathrm{Irr}(\mathbb{F}_{Q^2}^\times)$, and define $\gamma \in \mathrm{Irr}(\mathbb{F}_Q^\times)$ by $\gamma \circ \mathrm{N}_{Q^2/Q} = \gamma_2^{Q+1}$. The weight θ has $\theta^{Q-1} = \eta_Q \circ \mathrm{N}_{Q^2/Q}$. Thus, its exponent in γ_2 is $(Q+1)/2 \bmod (Q+1)$, and a common determinant twist gives $\theta = \gamma_2^{(Q+1)/2}$. If σ acts on $\mu_{2(Q-1)}$ by exponent h, $\theta^\sigma = \theta(\delta \circ \mathrm{N}_{Q^2/Q})$, where $\delta = \gamma^{(h-1)/2}$. Consequently, we obtain that $\chi^\sigma = \chi(\delta \circ \det)$, with the unipotent label λ fixed.

Put $K = \mathbb{F}_{Q^{2e}}$ and $B = (K^\times \rtimes C_{2e}) \wr \mathfrak{S}_t$, with $v_i : x \mapsto x^Q$ as the rotation of slot i. The torus weight $\theta \circ \mathrm{N}_{K/\mathbb{F}_{Q^2}}$ on each slot has inertia group $L = (K^\times)^t \rtimes (C_e \wr \mathfrak{S}_t)$, with the even rotations C_e. Next, the extension $\widetilde{\theta}$ has value 1 on the section $C_e \wr \mathfrak{S}_t$. With Y of label $\boldsymbol{\nu}$ inflated to L, we write $\rho = \widetilde{\theta} \otimes Y$ on V_ρ. Thus, $\psi = \mathrm{Ind}_L^B(\mathrm{tr}\,\rho)$ is irreducible by Clifford [43]. The module Y is induced from products of S^ν over $\mathbb{Q}$ and linear characters of C_e. Since $2e \mid \ell - 1$, the group $\mathcal{H}_\ell$ fixes $\mathbb{Q}(\zeta_e)$. Moreover, the section $C_e \wr \mathfrak{S}_t$ has determinant 1. Indeed, the rotations v_i^2 and transpositions of $2e$-dimensional slots have determinant 1. Hence, we find $\rho^\sigma = \rho(\delta \circ \det)$ in the given bases and $\psi^\sigma = \psi(\delta \circ \det)$.

In the model for ψ, a function $F : B \to V_\rho$ satisfies $F(gx) = \rho(g)F(x)$ for $g \in L$. The action of $b_0 \in B$ is $F(x) \mapsto F(xb_0)$. Next, we put $s = v_1 \cdots v_t$. Write $\omega \in \mu_e$ be the scalar of s^2 on Y. A normal basis of $K/\mathbb{F}_Q$ gives $\det s = (-1)^t$. Since s centralizes the section $C_e \wr \mathfrak{S}_t$ and conjugates $\widetilde{\theta}$ to $\widetilde{\theta}(\eta_Q \circ \det)$, the quadratic intertwiner J_B has $J_B F(x) = \eta_Q(\det x)F(sx)$ and $J_B^2 = \eta_Q((-1)^t)\omega$. We divide J_B by a square root c_B of $\eta_Q((-1)^t)\omega$. Since σ fixes μ_{2e}, it fixes the root of ω in μ_{2e}. Next, the multiplier $F(x) \mapsto \delta(\det x)F(x)$ changes J_B by $\delta(\det s) = (-1)^{t(h-1)/2}$. In addition, if $Q \equiv 3 \pmod 4$ and t is odd, then c_B has a factor $i = \sqrt{-1}$ and $\sigma(i)/i = (-1)^{(h-1)/2}$. Thus, the sign ϵ_{bulk} on the two eigenspaces is $(-1)^{t(h-1)/2}$ for $Q \equiv 1 \pmod 4$, and 1 for $Q \equiv 3 \pmod 4$. By quadratic reciprocity, we deduce that $\left(\frac{\ell}{Q}\right)^e = \left(\frac{Q}{\ell}\right)^e (-1)^{e(\ell-1)(Q-1)/4}$. Since $Q^e \equiv -1 \pmod \ell$, we have $\left(\frac{Q}{\ell}\right)^e = (-1)^{(\ell-1)/2}$. Thus, for $Q \equiv 3 \pmod 4$, the signs cancel if e is odd, and $4 \mid \ell - 1$ if e is even. Given $Q \equiv 1 \pmod 4$, the sign is $(-1)^{(\ell-1)/2}$. With $a = \ell^j \bmod p$ and $h = \ell^j \bmod 2(Q-1)$, we obtain that $\eta_Q(a)^{et} = \epsilon_{\mathrm{bulk}}$.

If $c > 0$, the factor $\mathrm{GL}_{2c}(Q)$ has the exchanged weights and partition $\lambda_{\mathrm{core}} \vdash c$. By Lemma 6.108, its sign is $\eta_Q(a)^c$ with the correction $\delta \circ \det$. Next, the tensor J of normalized intertwiners implements $\eta_Q \circ \det$. Since $\mathrm{N}_{K/\mathbb{F}_Q}$ is surjective, each self-twist κ permutes the two weights on $K^\times$, so $\kappa \in \{1, \eta_Q\}$. Thus, the two eigenspaces of J are the two constituents on $M \cap G^1$, with permutation sign $\eta_Q(a)^{et+c} = \eta_Q(a)^b$.

Assume that $G = \mathrm{GU}_{2b}(Q)$ has $\ell \mid |G|$ and individually rational weights ξ, ξ'. We normalize ξ, ξ' to $1, \eta$ on $D = \mu_{Q+1}$ by a determinant twist. Next, put $d = \mathrm{ord}_\ell(-Q)$ and $b = dk + c$, with $0 \le c < d$. By Lemma 6.103, the degree condition $\ell \nmid \chi(1)$ gives $2c < d$. Consequently, $k > 0$ and the marked space has $2k$ slots of dimension d with residue of dimension $2c$. On each cyclic torus T_i of order $Q^d - (-1)^d$, the determinant has exponent $\sum_{r=0}^{d-1}(-Q)^r$, whose absolute value is $(Q^d - (-1)^d)/(Q+1)$. Thus, $\det(T_i) = D$.

When $\lambda = \mu$, the weights $1, \eta$ occupy sets W_1, W_2 of k slots, respectively. With H the normalizer on W_1, the bulk weight ϑ has inertia group $H \times H$ since $\det(T_i) = D$. Its source is $V \boxtimes ((\eta \circ \det)V)$, with V inflated from $C_d \wr \mathfrak{S}_k$ and afforded over $\mathbb{Q}(\zeta_d)$. We choose isometries f_i between paired slots with $f_i T_i f_i^{-1} = T_{i+k}$. Next, s_0 exchanges the slots via f_i, f_i^{-1}. Thus, $s_0^2 = 1$ and $\det s_0 = (-1)^{dk}$. With the tensor flip Π, the quadratic intertwiner J_B has $J_B F(x) = \eta(\det x)\Pi F(s_0 x)$ and $J_B^2 = \eta((-1)^{dk})$.

Since $d \mid \ell - 1$, $\mathcal{H}_\ell$ fixes the source matrices and J_B. Thus, dividing J_B by a square root of J_B^2 contributes $\sigma(i)/i$ if dk is odd and $Q \equiv 1 \pmod 4$, and 1 otherwise. If dk is odd, the odd order d of $-Q \bmod \ell$ gives $(Q/\ell) = (-1/\ell)$. By quadratic reciprocity, we deduce that $(\ell/Q) = (-1/\ell)$ for $Q \equiv 1 \pmod 4$, and 1 for $Q \equiv 3 \pmod 4$. Consequently, with $a = \ell^j \bmod p$, the sign is $\eta_Q(a)^{dk}$. If $c > 0$, the two core labels λ_{core} on $\mathrm{GU}_{2c}(Q)$ contribute $\eta_Q(a)^c$ by Lemma 6.108. Since $\det(T_i) = D$, the torus weights $1, \eta$ give precisely the twists $1, \eta \circ \det$, so the tensor intertwiner J has two eigenspaces on $M \cap G^1$ with sign $\eta_Q(a)^b$.

Proposition 6.109. *Suppose that the unitary weights ξ, ξ' satisfy $\xi^{-Q} = \xi$ and $(\xi')^{-Q} = \xi'$. Then, the normalized character χ and its local correspondent ψ have equal factors for (F, δ, σ), with field lift F, determinant character δ and $\sigma \in \mathcal{H}_\ell$. If $\lambda \neq \mu$, we fix K_χ and A_{u_J} from* Propositions 3.28 and 3.31*, respectively.*

Proof. First, let $\lambda = \mu$, and write $n = b$ on G and $n = c$ on $\mathrm{GU}_{2c}(Q)$. The weights are $1, \eta$. Thus, field and coefficient automorphisms fix χ, and the central scalar at zI_{2n} is $\eta(z)^n$. On the chains $D_i = 2\lambda'_i$ of Lemma 6.108, we choose $0 \neq C \in \mathbb{F}_{Q^2}$ with $C^Q = -C$. The Gram matrix is CA_0, with A_0 alternating over $\mathbb{F}_p$. Since ϖ and J are defined over $\mathbb{F}_p$, the coordinate map $\phi : x \mapsto x^p$ satisfies $\phi(G) = G$, $\phi(U) = U$ and $\xi_U \circ \phi = \xi_U$. Indeed, $\phi(CA_0) = C^{p-1}(CA_0)$ and the trace to $\mathbb{F}_p$ is invariant. On $\mathrm{Ind}_U^G \xi_U$, we put $P_\phi F(x) = F(\phi^{-1}x)$ and $J_\eta F(x) = \eta(\det x)F(x)$. By [188, Corollary 15.7], χ has multiplicity 1. Thus, $\mathcal{V}_\chi$ is invariant under both maps, with $[P_\phi, J_\eta] = 1$ and $P_\phi^{2f} = J_\eta^2 = 1$.

In $\mathbb{F}_{Q^2}^\times$, choose v with $v^{Q+1} = \ell \bmod p$, and write $L = |v|$. If $L > 1$, the action of σ on $\mu_{\ell^L - 1}$ determines $j \bmod L$. Hence, $v_\sigma = v^j$ satisfies $v_\sigma^{Q+1} = a$ and $v_{\sigma\tau} = v_\sigma v_\tau$. If $L = 1$, we set $v_\sigma = 1$. On a chain of length D_i, the diagonal entries of t_σ are $v_\sigma^{-1} a^{r+1-D_i/2}$ for $0 \leq r < D_i$. The exponents at $r, D_i - 1 - r$ sum to 1, so paired entries z_r, z_{D_i-1-r} satisfy $z_r z_{D_i-1-r}^Q = 1$. We order the chain so that $t_\sigma J t_\sigma^{-1} = aJ$. Consequently, on the induced functions, $S_\sigma F(x) = \sigma(F(t_\sigma x))$ has covariance $S_\sigma F(ux) = \xi_U(u) S_\sigma F(x)$ for $u \in U$. It has $S_\sigma R(g) = R(g) S_\sigma$ for the right translations $R(g)$ and $S_\sigma S_\tau = S_{\sigma\tau}$. Thus, $\mathcal{V}_\chi$ is invariant under S_σ. The ratio $\phi(t_\sigma)t_\sigma^{-1} = v_\sigma^{1-p} I$ has $\eta(v_\sigma^{1-p}) = a^{(1-p)/2} = (a/p)$. Thus, the ratio between $S_\sigma P_\phi$ and $P_\phi S_\sigma$ on $\mathcal{V}_\chi$ is $(a/p)^n$. By Lemma 6.108, the analogous ratio for J_η is $\eta_Q(a)^n$, since J_η has the two eigenspaces on G^1.

In $\mathbb{F}_{Q^2}^\times$, we choose w with $w^{Q+1} = C^{p-1}$ and put $K_0 = w\phi$. Thus, $\langle K_0 x, K_0 y \rangle = \langle x, y \rangle^p$, and K_0 induces ϕ on G. Moreover, $K_0^{2f} = \mathrm{N}_{\mathbb{F}_{Q^2}/\mathbb{F}_p}(w) I = C^{Q-1} I = -I$. Given a marked semilinear isometry U_0, we put $g = U_0 K_0^{-1} \in G$. Take $P_{U_0} = \beta_n R(g) P_\phi$, where $\beta_n^{2f} = \eta(-1)^n$. Then, $(R(g)P_\phi)^{2f} = R(U_0^{2f} K_0^{-2f})$. Hence, $P_{U_0}^{2f} = R(U_0^{2f})$ by the central scalar at $-I$. The factor of $S_\sigma P_{U_0}$ relative to $P_{U_0} S_\sigma$ is $(a/p)^n \sigma(\beta_n)/\beta_n$. The corresponding factor for J_η is $\eta(\det g)$.

Next, in orthonormal coordinates, we write $U_0 = A_U F_p$. We show that $\eta(\det g) = \eta(\det A_U)$. Indeed, choose $B^* B = CA_0$ and $t_0 = \det B$. Hence, $t_0^{Q+1} = C^{2n} \mathrm{Pf}(A_0)^2$ by the Pfaffian formula for $\det A_0$. The linear coefficient of K_0 has determinant $w^{2n} t_0^{1-p}$. Its $(Q+1)/2$-th power is 1, since $t_0^{(Q+1)/2} = \pm C^n \mathrm{Pf}(A_0)$, with $\mathrm{Pf}(A_0) \in \mathbb{F}_p^\times$. Thus, the linear coefficient of K_0 contributes 1 to $\eta(\det g)$.

On the d-space of T_i, we let U_i be the field lift and v_i the rotation. Thus, $[U_i, v_i] = 1$ and $U_i^{2f} = v_i^2$ by Proposition 3.30. If $2 \nmid d$, the maps on $\mathbb{F}_{Q^{2d}}$ are $U_i : x \mapsto x^p$ and $v_i : x \mapsto x^{Q^{d+1}}$. If $2 \mid d$, fix U_i, v_i on $X \oplus Y$ by Proposition 6.107. Next, on W_1, W_2, take identical lifts. On $\mathrm{GU}_{2c}(Q)$, we choose ϕ. The bulk coefficient A_{bulk} has $\det A_{\mathrm{bulk}} = (\det A_{W_1})^2$. Hence, the commutator with J_η has identical quadratic scalars on G and $\mathrm{GU}_{2c}(Q)$.

Put $U_{\mathrm{bulk}} = \bigoplus_i U_i$. Next, on the model for ψ, we put $P_0 F(x) = F(U_{\mathrm{bulk}}^{-1} x U_{\mathrm{bulk}})$. The lift U_{bulk} acts identically on W_{rel} and satisfies $U_{\mathrm{bulk}} s_0 = s_0 U_{\mathrm{bulk}}$. Moreover, $\eta(\det(U_{\mathrm{bulk}}^{-1} x U_{\mathrm{bulk}})) = \eta(\det x)$. Thus, $P_0 J_B = J_B P_0$ and P_0 satisfies $P_0\, \sigma = \sigma\, P_0$. Let $\omega_i \in \mu_d$ be the scalar of $\prod_{j \in W_i} v_j$ on Y_i for $i = 1, 2$. Put $v_{\mathrm{all}} = \prod_{j=1}^{2k} v_j$. Since $U_{\mathrm{bulk}}^{2f} = v_{\mathrm{all}}^2$, the representation ρ has $\rho(U_{\mathrm{bulk}}^{2f}) = (\omega_1\omega_2)^2 I$. The determinant on the negative space V_- contributes 1, because $v_{\mathrm{all}}^2|_{V_-}$ is a square. Hence, $P_0^{2f} = (\omega_1\omega_2)^{-2} R(U_{\mathrm{bulk}}^{2f})$, and the denominator is ω_1^4 if $\lambda = \mu$.

We choose $\alpha_{\mathrm{bulk}} \in \mu_{\ell-1}$ with $\alpha_{\mathrm{bulk}}^{2f} = (\omega_1\omega_2)^2$. Indeed, for $h_0 = \mathrm{ord}_\ell(p)$, the order of Q is $h_0/\gcd(h_0, f)$. At an odd prime $r \mid d$, the r-parts of the orders of Q and $-Q$ coincide. Thus, $v_r(\ell - 1) \geq v_r(d) + v_r(f)$. If $4 \mid d$, the bound $v_2(\ell - 1) \geq v_2(d) + v_2(f)$ also holds, and the 2-part of the order of $(\omega_1\omega_2)^2$ divides $d_2/2$. If $4 \nmid d$, $(\omega_1\omega_2)^2$ has odd order. Consequently, $\alpha_{\mathrm{bulk}} P_0$ has $2f$-th power $R(U_{\mathrm{bulk}}^{2f})$ and satisfies $\alpha_{\mathrm{bulk}} P_0\, \sigma = \sigma\, \alpha_{\mathrm{bulk}} P_0$.

Suppose next that $(\mathrm{ord}_\ell(Q))_2 = 2$, and put $f_2 = 2^{v_2(f)}$. Thus, $(h_0)_2 = 2f_2$. We choose a root β_* of ℓ'-order with $\beta_*^{2f} = (-1)^{(Q+1)/2}$ and $\sigma(\beta_*)/\beta_* = (a/p)$. If $Q \equiv 1 \bmod 4$, take β_* of order $4f_2$. Then, $\beta_*^{2f} = -1$, and $\beta_*^\ell = -\beta_*$ exactly when $(\ell - 1)_2 = 2f_2$, or equivalently $(p/\ell) = -1$. By quadratic reciprocity, $(p/\ell) = (\ell/p)$ because p, ℓ cannot both be $3 \bmod 4$ with $Q \equiv 1 \bmod 4$ and $(h_0)_2 = 2f_2$. Indeed, $p \equiv \ell \equiv 3 \bmod 4$ gives $f_2 = 1$ and $Q \equiv 3 \bmod 4$. If $Q \equiv 3 \bmod 4$, f is odd and $p \equiv 3 \bmod 4$. Thus, $(h_0)_2 = 2$ gives $(p/\ell) = 1$ for $\ell \equiv 1 \bmod 4$ and $(p/\ell) = -1$ for $\ell \equiv 3 \bmod 4$. Consequently, $(\ell/p) = 1$ in both cases and $\beta_* = 1$ suffices. Finally, $a = \ell^j \bmod p$ and $\sigma(\beta_*) = \beta_*^{\ell^j}$ give the coefficient factor for every σ.

Put $m = b - c = dk$. If m is even, we set $\beta_b = \beta_c$. Otherwise, d is odd, so $(\mathrm{ord}_\ell(Q))_2 = 2$, and we set $\beta_b = \beta_c \beta_*$. On an empty residue we set $\beta_c = 1$. Thus, the field maps P_b on G and $(\alpha_{\mathrm{bulk}} P_0) \otimes P_c$ on M have equal σ-commutators and the given $2f$-th powers. The commutators with $J_b, J_B \otimes J_c$ agree since $\eta(\det A_{\mathrm{bulk}}) = 1$. The coefficient commutators with $J_b, J_B \otimes J_c$ have sign $\eta_Q(a)^b$ by Lemma 6.108 and the calculation of J_B^2. Moreover, the common scalar at zI is $\eta(z)^b$, since $\dim V_- = dk + c = b$. The twist operator satisfies $J_\eta R(g) = \eta(\det g) R(g) J_\eta$ on both models. On G with

Gram matrix B_{form}, inverse transpose has $g^{-T} = B_{\mathrm{form}}^T g^Q (B_{\mathrm{form}}^T)^{-1}$. Thus, the products are ordered as $R(g)P_b^h J_b^i S_\sigma$ on G, with h reduced modulo $2f$ by $P_b^{2f} = R(U_0^{2f})$. The determinant twists fixing χ are $1, \eta \circ \det$. Hence, the factors for (F, δ, σ) coincide on χ, ψ.

Next, for arbitrary $\lambda, \mu \vdash b$, put $x_i = \lambda'_i$ and $y_i = \mu'_i$, appending zeros. Thus, $D_i = x_i + y_i$. On the even chains D_i, write $K_\chi = \eta^{h_\chi}$ by Proposition 3.28. Column i has character η^{y_i}. Moreover, a pair $i < j$ of opposite parities contributes η^{D_j} through the normalized Weil action, since $D_j \le D_i$. Thus, modulo 2, $h_\chi \equiv \sum_{2|D_i} y_i + \sum_{i<j} (1+D_i)D_j$. The first sum is $\sum_i x_i y_i \bmod 2$. Hence, the mixed products in x, y contribute $|\lambda||\mu| \bmod 2$. Given x, the remaining sum is $\sum_i (i-1)x_i + \sum_{i<j} x_i x_j$. With $\mathrm{cont}(\lambda) = \sum_{(r,s)\in[\lambda]} (s-r)$, $\mathrm{cont}(\lambda) = \sum_i ((i-1)x_i - \binom{x_i}{2})$. Thus, the x-sum is $\mathrm{cont}(\lambda) + \binom{|\lambda|}{2} \bmod 2$, with the corresponding formula for y, μ. Consequently, $h_\chi \equiv \mathrm{cont}(\lambda) + \mathrm{cont}(\mu) + \binom{|\lambda|+|\mu|}{2} \pmod 2$.

By Proposition 3.46, the condition $\ell \nmid \chi(1)$ gives d-quotients of size k for λ, μ when $d > 1$. If $d = 1$, then $k = b$ and $c = 0$. Thus, the cores $\lambda_0 = \mathrm{core}_d(\lambda)$ and $\mu_0 = \mathrm{core}_d(\mu)$ have size c. If $2 \mid d$, each rim d-hook has consecutive contents $t, \dots, t+d-1$, of sum $d/2 \bmod 2$. There are $2k$ such hooks, and $2b - 2c = 2dk \equiv 0 \bmod 4$. Consequently, $h_\chi \equiv h_{\chi_0} \pmod 2$ for the residual parent χ_0, including when $\lambda_0 \ne \mu_0$.

When $\lambda \ne \mu$, the maps I_σ have $I_\sigma I_\tau = I_{\sigma\tau}$ by Proposition 3.24. Thus, the factor with F is $K_\chi(b_\sigma^{p-1})$ with $b_\sigma^{Q+1} = a^{-1}$ by field conjugation. Hence, $\eta(b_\sigma^{p-1}) = (a/p)$, and the I_σ-commutator is $(a/p)^{h_\chi}$. Next, by the field power relation, F^{2f} has factor $K_\chi(-1)^{-1}$. Thus, the scalar β has $\beta^{2f} = \eta(-1)^{h_\chi}$. With the marked lift, multiplication by $R(g)$ leaves the I_σ-factor equal to $(a/p)^{h_\chi}$, since $I_\sigma R(g) = R(g) I_\sigma$.

If $2 \mid d$, the field roots on χ, χ_0 are equal because $h_\chi \equiv h_{\chi_0} \pmod 2$. If $2 \nmid d$, set their quotient to be 1 when $h_\chi - h_{\chi_0}$ is even and β_* otherwise. On the two distinct labels Y_1, Y_2, the choice of $\alpha_{\mathrm{bulk}} P_0$ above gives $2f$-th power $R(U_{\mathrm{bulk}}^{2f})$ and coefficient factor 1. Hence, on χ, ψ, we have equal F^{2f} relations and equal σ-commutators. Their central characters at zI both have value $\eta(z)^b$.

The twist $\eta \circ \det$ exchanges the different partitions λ, μ. Thus, the only determinant twist fixing χ is 1. On M the twist exchanges both quotients $\boldsymbol{\nu}_\lambda, \boldsymbol{\nu}_\mu$ and cores λ_0, μ_0. The core and quotient determine λ, μ, so the twist moves ψ. Consequently, by [95, Corollary 11.22] and reciprocity, $\chi|_{G^1}$ is irreducible. Each field F and coefficient σ fixes χ. Thus, the pairs (F, σ) exhaust the normalized stabilizer. Finally, for λ, μ, we tensor both models by $\delta \circ \det$, with scalar $\delta(\det g)$ at each inner lift g. Hence, the factors coincide on each $\chi(\delta \circ \det)$ and $\psi(\delta \circ \det)$. □

Proposition 6.110. *Suppose that the linear weights ξ, ξ' satisfy $F_Q(\xi) = \xi'$, $F_Q(\xi') = \xi$ and $d = 2e$. Then, the global parent χ and its local correspondent ψ have equal mixed factors for $(\delta, F, \gamma, \sigma)$ with determinant twist δ, field map F, graph map γ and coefficient automorphism σ. Moreover, the $\eta_Q \circ \det$-eigenspaces give an equivariant pairing on $G^1, M \cap G^1$.*

Proof. Given $X \le \mathrm{GL}_{2n}(Q)$, we put $\widehat{X} = \{(x, z) \in X \times \mathbb{F}_{Q^2}^\times \mid z^2 = \det x\}$. Thus, $z \in \widehat{D} = \mu_{2(Q-1)}$, and we choose a faithful $\tau \in \mathrm{Irr}(\widehat{D})$ with $\theta(u) = \tau(u^{(Q+1)/2})$ on $\mathbb{F}_{Q^2}^\times$. If ρ is the original parent, we put $\widehat{\rho}(x, z) = \tau(z)^{-1}\rho(x)$ on $\widehat{X}$. Then, $\widehat{\rho}$ is irreducible and $(1, -1)$ acts by -1. The global central scalar on xI_{2n} is $\theta(x)^n$ for $x \in \mathbb{F}_Q^\times$, because the exchanged weight θ has multiplicity n. Consequently, the lift (xI_{2n}, x^n) acts by $\eta_Q(x)^n$.

Let h denote the Galois exponent on $\widehat{D}$. By [180, Lemma 4.1 and Theorem 5.1], the weight θ changes to θ^h, with its unipotent label λ fixed. Thus, $\rho^\sigma(x) = \tau(\det x)^{(h-1)/2}\rho(x)$ on G. Since $z^2 = \det x$, the factor $\tau(z)^{h-1}$ cancels the twist, so the global $\widehat{\rho}$ is rational valued. Similarly, the powers $p, -1$ on the type A weights describe the coordinate p-power and inverse transpose. Hence, $\widehat{\rho}$ is invariant under $(x, z) \mapsto (x^{(p)}, z^p)$ and $(x, z) \mapsto (x^{-T}, z^{-1})$. Finally, $\kappa(x, z) = (x, z^Q)$ implements the twist $\eta_Q \circ \det$ because $z^{Q-1} = \eta_Q(\det x)$.

In $\widehat{G}$, the even-chain source ξ_U of Lemma 6.108 is on $\widehat{U} = \{(u, 1) \mid u \in U\}$. Thus, $\widehat{\rho}$ occurs with multiplicity 1 in $\mathrm{Ind}_{\widehat{U}}^{\widehat{G}} \xi_U$. Let F be the inverse pullback by $F_p : (x, z) \mapsto (x^{(p)}, z^p)$. Next, on chain i, take $A_0|_i$ antidiagonal with alternating signs. Thus, $A_0^T = -A_0$, $A_0^2 = -I$ and $\det A_0 = 1$. Put $h_0(x, z) = (A_0 x^{-T} A_0^{-1}, z^{-1})$, and let H be its inverse pullback. Since $J^T A_0 + A_0 J = 0$, the trace functional $\mathrm{Tr}(J \cdot)$ is invariant under $Y \mapsto -A_0 Y^T A_0^{-1}$. Thus, h_0 fixes ξ_U via the Cayley map on U, and F_p fixes ξ_U since J is defined over $\mathbb{F}_p$. Consequently, F, H act on the unique $\widehat{\rho}$-summand, with $FH = HF$ and $F^{2f} = H^2 = 1$.

Choose $s_* \in \mathbb{F}_{p^2}^\times$ with $s_*^2 = \ell \bmod p$, and put $s_\sigma = s_*^j$ for $a = \ell^j \bmod p$. As in Proposition 6.109, the action on $\mu_{\ell^{|s_*|}-1}$ determines $j \bmod |s_*|$. Thus, $s_\sigma^2 = a$ and $s_{\sigma\tau} = s_\sigma s_\tau$. On chain i, we put $T_a|_i = \mathrm{diag}(a^{r+1-D_i/2})_{0 \le r < D_i}$. Since $\det T_a = a^n$, $t_\sigma = (T_a, s_\sigma^n) \in \widehat{G}$. The choice of chain order gives $T_a J T_a^{-1} = aJ$. Hence, the semilinear map

$S_\sigma F_1(x) = \sigma(F_1(t_\sigma x))$ satisfies the ξ_U-covariance on $\widehat{U}$, with $S_\sigma R(g) = R(g)S_\sigma$ and $S_\sigma S_\tau = S_{\sigma\tau}$. Since $\widehat{\rho}$ is rational valued, S_σ acts on $\mathcal{V}_{\widehat{\rho}}$.

The quotients for F_p, h_0 are $F_p(t_\sigma)t_\sigma^{-1} = (1, s_\sigma^{n(p-1)})$ and $h_0(t_\sigma)t_\sigma^{-1} = (a^{-1}I, a^{-n})$. Both are central in $\widehat{G}$. Since $s_\sigma^{p-1} = a^{(p-1)/2}$, the first acts as $(a/p)^n$. The second acts as $\eta_Q(a)^n$ by the scalar on (xI, x^n). Thus, the factors of $S_\sigma F, S_\sigma H$ relative to FS_σ, HS_σ are $(a/p)^n, \eta_Q(a)^n$, respectively. On $\mathrm{GL}_{2c}(Q)$, we set $n = c$ and the exchanged core label λ_{core}.

Write $U = AF_p$ on the natural space V, with $U : x \mapsto x^p$ on each K and $U = F_p$ on the residue. Thus, $U^f = s$. The Gram matrix B_0 has blocks for $\mathrm{Tr}_{K/\mathbb{F}_Q}(xy)$ and the identity block I_{2c}. The marked graph is $x \mapsto B_0^{-1}x^{-T}B_0$. Next, we put $E = B_0^{-1}A_0^{-1}$ and choose $\alpha^2 = \det A$, $\beta^2 = \det E$. On the lift of $\widehat{\rho}$, the operators are $P = R(A, \alpha)F$ and $T = R(E, \beta)H$. In particular, on the residue, $A = B_0 = I$.

For the basis $(x_1, \ldots, x_{2e})$ of a slot K, write $M_i = (x_j^{Q^r})_{0\le r<2e,\, 1\le j\le 2e}$ and $\Delta = \prod_i \det M_i$. Indeed, the slot matrices have $M_iA_i = M_i^{(p)}$ and $B_{0,i} = M_i^T M_i$. Thus, $\det B_0 = \Delta^2$ and $\det A = \Delta^{p-1}$. The map Q-power cycles the $2e$ rows in each M_i, so $\Delta^Q = (-1)^t\Delta$. Hence, set $\beta = \Delta^{-1} \in \mathbb{F}_{Q^2}^\times$. From $A^T B_0 A = B_0^{(p)}$, $AE^{(p)} = E(A_0A^{-T}A_0^{-1})$. Moreover, the root ratio is $\alpha^2\beta^{p-1} = 1$. Thus, $PT = TP$.

On the global $\widehat{\rho}$, the field power is $P^{2f} = (-1)^{t(Q+1)/2}R(s^2, 1)$. Indeed, the root in P^{2f} is l^{Q+1}, where $l = \alpha^{(Q-1)/(p-1)}$ has $l^2 = \det s = (-1)^t$. For the graph, symmetry of B_0 and $A_0^T = -A_0$ give $E(A_0E^{-T}A_0^{-1}) = -I$. The root in T^2 is $\beta\beta^{-1} = 1$. Thus, $T^2 = R(-I, 1) = (-1)^{n(Q+1)/2}I$. On the residue, $P^{2f} = 1$ and the graph square is $(-1)^{c(Q+1)/2}I$. Moreover, S_σ satisfies $S_\sigma D = DS_\sigma$ for $D \in \{R(A, \alpha), R(E, \beta)\}$, so the factors with P, T are those with F, H, respectively.

On the bulk inertia group $L = (K^\times)^t \rtimes (C_e \wr \mathfrak{S}_t)$, take the cover $\widehat{L}$. For $(x, z) \in \widehat{L}$, we let x_i be the i-th torus coordinate of x. Next, we put $\Theta(x) = \prod_i \mathrm{N}_{K/\mathbb{F}_{Q^2}}(x_i)^{(Q+1)/2}$. Thus, the source module is $Y \otimes \xi$, with $\xi(x, z) = \tau(z^{-1}\Theta(x))$. Since $z^2 = \det x = \Theta(x)^2$, the root $z^{-1}\Theta(x)$ is in $\{\pm 1\}$. Moreover, $\Theta(x)$ is invariant under $C_e \wr \mathfrak{S}_t$, so ξ is linear. The group $\mathcal{H}_\ell$ fixes $Y \otimes \xi$ entrywise because Y is defined over $\mathbb{Q}(\zeta_e)$ and $\xi^2 = 1$.

Given $u \in K^\times$, the multiplication map m_u has adjoint $m_u^* = m_u$. The slot rotations v_i have $v_i^* = v_i^{-1}$. Thus, the marked maps fix $C_e \wr \mathfrak{S}_t$. On $(\Theta(x), z)$, the field acts by p-power and the graph by $(\Theta(x)^{-1}, z^{-1})$. Hence, the inverse pullbacks P_0, T_0 satisfy $P_0T_0 = T_0P_0$ and satisfy $P_0\sigma = \sigma P_0$, $T_0\sigma = \sigma T_0$ entrywise. Moreover, $T_0^2 = 1$ and $P_0^{2f} = \omega^{-1}R(s^2, 1)$, where $\omega \in \mu_e$ is the scalar of s^2 on Y. Indeed, P_0^{2f} has argument $(s^2, 1)^{-1}x(s^2, 1)$, and left covariance contributes ω^{-1}. Next, we let $\widehat{B}, \widehat{G}_c$ be the covers of $B, \mathrm{GL}_{2c}(Q)$. The map $\widehat{B} \times \widehat{G}_c \to \widehat{M}$ combines matrices blockwise and gives $(z_1, z_2) \mapsto z_1z_2$. Thus, its kernel is generated by $((1, -1), (1, -1))$, with scalar $(-1)(-1) = 1$. Thus, the tensor of $\mathrm{Ind}_{\widehat{L}}^{\widehat{B}}(Y \otimes \xi)$ with the model on $\widehat{G}_c$ descends to $\widehat{M}$.

Put $m = b - c = et$. Thus, we compare X_0P_0 and Y_0T_0 with P, T. The field conditions are $X_0^{2f} = \omega(-1)^{t(Q+1)/2}$ and $\sigma(X_0)/X_0 = (a/p)^m$. The graph conditions are $Y_0^2 = (-1)^{m(Q+1)/2}$ and $\sigma(Y_0)/Y_0 = \eta_Q(a)^m$. First, choose a $2f$-th root x_ω of ω in $\mu_{\ell-1}$. Indeed, $d = \mathrm{ord}_\ell(Q) = 2e$ gives $v_r(\ell - 1) \ge v_r(f) + v_r(2e)$ for primes $r \mid e$. If t is even, set $X_0 = x_\omega$. If t, e are odd, m is odd and $(\mathrm{ord}_\ell(Q))_2 = 2$. Thus, we set $X_0 = x_\omega\beta_*$ by Proposition 6.109. Finally, when t is odd and e is even, $(a/p)^m = 1$. Set $X_0 = x_\omega$ if $Q \equiv 3 \bmod 4$, and $X_0 = x_\omega\zeta$ otherwise, with ζ of order $4f_2$. Since $4 \mid 2e$, $4f_2 \mid \ell - 1$, so $\sigma(\zeta) = \zeta$ and $\zeta^{2f} = -1$. Given Y_0, we choose i^m if $Q \equiv 1 \bmod 4$ and 1 otherwise. Then, $Y_0^2 = (-1)^{m(Q+1)/2}$. If m is odd, e is odd, so $-Q$ has odd order modulo ℓ. By Proposition 6.109, (ℓ/Q) equals $(-1/\ell)$ if $Q \equiv 1 \bmod 4$ and 1 otherwise. Thus, $\sigma(Y_0)/Y_0 = \eta_Q(a)^m$. If m is even, the factors $\sigma(Y_0)/Y_0$ and $\eta_Q(a)^m$ are 1.

Given $x \in \mathbb{F}_Q^\times$, the norm $\mathrm{N}_{K/\mathbb{F}_{Q^2}}(x) = x^e$ on each K-slot gives the bulk scalar $\theta(x)^{et}$ apart from $\tau(z)^{-1}$. The residue contributes $\theta(x)^c$, so (xI, z) acts by $\tau(z)^{-1}\theta(x)^b$ on both covers. Thus, the products $R(g)P^uT^vS_\sigma$ have equal factors, by $PT = TP$, $T^2 = (-1)^{b(Q+1)/2}I$ and $P^{2f} = (-1)^{t(Q+1)/2}R(s^2, 1)$. Choose $\widehat{s} = (s, z_s)$ with $z_s^2 = (-1)^t$, and let $\widehat{U}$ lift U. Since $U^f = s$ and the root changes by Q-power, $\kappa = \mathrm{Int}(\widehat{s}^{-1})\widehat{U}^f$. Hence, $J_\kappa = R(\widehat{s})^{-1}P^f$ implements κ on $\widehat{G}, \widehat{M}$. Thus, $J_\kappa^2 = R(s^{-2}, z_s^{-1-Q})P^{2f}$. Since $z_s^{-1-Q} = (-1)^{t(Q+1)/2}$, the kernel sign cancels $(-1)^{t(Q+1)/2}$ in P^{2f}. Hence, $J_\kappa^2 = 1$ on $\widehat{G}, \widehat{M}$. Moreover, the product $R(\widehat{s})^{-1}P^f$ has equal factors with $R(g), P, T, S_\sigma$.

Finally, $\rho(x) = \tau(z)\widehat{\rho}(x, z)$. Let S be σ-semilinear with $SR_{\widehat{\rho}}(x, z)S^{-1} = R_{\widehat{\rho}}(\alpha_0(x), z^r)$, and let σ act on τ by h. Thus, $SR_\rho(x)S^{-1} = \tau(z)^{h-r}R_\rho(\alpha_0(x))$. Since $2 \mid h - r$, the multiplier is $\tau(\det x)^{(h-r)/2}$. In particular, for κ this character is $\eta_Q \circ \det$. Moreover, two determinant restorers δ_1, δ_2 have $\delta_1\delta_2^{-1} \in \{1, \eta_Q\}$ by the weights on G, M. A change $z \mapsto -z$ in an inner lift (x, z) gives the sign -1 on $\widehat{G}, \widehat{M}$. Consequently, the stabilizer of χ has equal factors on χ, ψ, and Lemma 3.4 pairs the corresponding $+1$ and -1 eigenspaces of J_κ. Given a determinant translate $\chi(\delta \circ \det)$, we tensor both models by $\delta \circ \det$. Between parents χ_1, χ_2 in the orbit of χ, compose the maps $\chi_1 \to \chi$ and $\chi \to \chi_2$

with an operator fixing χ. Thus, on $\mathcal{V}_\chi \otimes \mathcal{V}_\psi^\vee$, each pair of equal factors (c,c) gives $cc^{-1}=1$. □

By Propositions 6.106, 6.107, 6.109 and 6.110, the factors for $(\delta, F, \gamma, \sigma)$ coincide on G, M. Thus, Proposition 6.194 compares G, M in four cases. Moreover, transfer to $\mathrm{Spin}(V)$ has the conditions of Section 6.1 on G, M. If H is degenerate, we compare stabilizers in $\mathrm{SO}(V)$ by Section 6.1. On the common quotient J, the factor and mixed functions satisfy $c_G(y,z) = c_M(y,z)$, $\mu^G_{g,\sigma}(y) = \mu^M_{g,\sigma}(y)$ $(y,z \in J)$.

Proposition 6.111. *Let $X = \mathrm{Spin}_8^+(q)$, with $q = r^f$ odd. By* Zhang [208, proof of Proposition 7.41]*, the four A_1-factors in* I(2, 2) *have Frobenius permutation v in the group V of double transpositions. At $v = 1$, each A_1-factor is split over $\mathbb{F}_q$. Moreover, triality permutes the three classes $v \neq 1$ as in* Lemma 6.33*. Let $p \neq r$ be an odd divisor of $|X|$, and write $\mathcal{E}_1$ for the series with $v = 1$. Then, we have that $\mathcal{E}_1$ has characters of p'-degree exactly when $p \geq 5$ and $q \equiv t \bmod p$, with $t \in \{1, -1\}$. In that case, we have that $P \in \mathrm{Syl}_p(X)$ lies in a torus T_t of type tI, and $\mathrm{N}_X(P) = \mathrm{N}_X(T_t)$. Thus, on $N = X/\mathrm{Z}(X)$, the weight is $\theta_t = (\eta_{q-t} \circ l)|_{T_{N,t}}$, with $l = e_1 + e_2$. Its stabilizer is $W_\theta = W_c \rtimes V$, with $W_c \simeq C_2^4$ and $V \simeq C_2^2$. Moreover, the pinned group $\mathfrak{S}_3$ fixes θ_t and acts faithfully on $V \setminus \{1\}$.*

Proof. By the Jordan degree formula [58, Proposition 11.5.6], the p-part of $\chi(1)$ for $\chi \in \mathcal{E}_1$ is $|X^* : C_{X^*}(l/2)|_p$. Indeed, the four split factors $A_1(q)$ have unipotent degrees $1, q$, with central and component indices of 2-power order. Thus, if $p \nmid q^2-1$, then $p \mid \chi(1)$ for every $\chi \in \mathcal{E}_1$. If $p \mid q^2-1$, the index away from $2, r$ is $(q^2+1)^2(q^4+q^2+1)$, by cancellation of $(q^2-1)^4$ in $(q^2-1)(q^4-1)^2(q^6-1)$. At $q \equiv \pm 1 \bmod p$, $q^2+1 \equiv 2 \bmod p$ and $q^4+q^2+1 \equiv 3 \bmod p$. Hence, $p \nmid \chi(1)$ exactly when $p \geq 5$. Since $|W(D_4)| = 192$ and $|T_t| = (q-t)^4$, $P \in \mathrm{Syl}_p(T_t)$ is Sylow in X. Given any root α, we find a cocharacter ν with $\langle \alpha, \nu \rangle = \pm 1$. Thus, α is nontrivial on P, and the algebraic torus $\mathbf{T}$ is $\mathrm{C}_{\mathbf{X}}(P)^\circ$. Consequently, $\mathrm{N}_X(P) \leq \mathrm{N}_X(T_t)$. $\mathrm{N}_X(T_t) \leq \mathrm{N}_X(P)$ follows since P is the unique Sylow p-subgroup of T_t.

Let Q_{rt} be the root lattice and P_{wt} the weight lattice. On $T_{\mathrm{ad},t}$, we put $\widetilde{\theta}_t = \eta_{q-t} \circ l$, with η_{q-t} quadratic on μ_{q-t}. Thus, the restriction is $\theta_t = \widetilde{\theta}_t|_{T_{N,t}}$, where $T_{N,t} = T_t\mathrm{Z}(X)/\mathrm{Z}(X)$. The split Borel has θ_+ for both choices of t, dual to $l/2 \bmod P_{\mathrm{wt}}$. If $t = -1$, the torus changes by $-I = s_{l_1}s_{l_2}s_{l_3}s_{l_4} \in W_c$. Hence, it leaves $v = 1$. Moreover, $l \in Q_{\mathrm{rt}}$ gives the trivial character on $\mathrm{Z}(X)$.

Thus, W_θ is the stabilizer of $l/2 \bmod P_{\mathrm{wt}}$. By [208, proof of Proposition 7.41], $W_\theta = W_c \rtimes V$, with $W_c = \langle s_{l_i} \mid 1 \leq i \leq 4 \rangle$ for $l_1 = e_1 - e_2$, $l_2 = l$, $l_3 = e_3 - e_4$ and $l_4 = e_3 + e_4$. The generators k, g act on the roots by (12)(34) and (13)(24). Moreover, the cited triality action fixes l_2 and permutes l_1, l_3, l_4 as $\mathfrak{S}_3$. Hence, it acts faithfully on $V \setminus \{1\}$. □

Lemma 6.112. *Put $Y_G = \mathrm{Ind}_{B_N}^N \theta_+$, with θ_+ inflated to B_N. Let $Y_M = \mathrm{Ind}_{T_{N,t}}^{M_N} \theta_t$, where $M_N = \mathrm{N}_X(P)/\mathrm{Z}(X)$. Then, both endomorphism algebras over $\mathbb{Q}$ are isomorphic to $\mathbb{Q}[W_c \rtimes V]$. The isomorphisms match the actions of $\mathfrak{S}_3$, the field F_r and the adjoint diagonal quotient D_{ad}.*

Proof. The modules Y_G, Y_M are over $\mathbb{Q}$. Moreover, the adjoint inductions $\widetilde{Y}_G, \widetilde{Y}_M$ have source characters $\widetilde{\theta}_+, \widetilde{\theta}_t$, respectively. Thus, Lang's theorem for $\mathbf{T}$ [58, §4.2] gives $T_{\mathrm{ad},t} \twoheadrightarrow D_{\mathrm{ad}}$. Hence, restriction of functions gives Y_G, Y_M. In the split model Y_G, we write T_s^θ for the Bruhat operator from θ to ${}^s\theta$ with representative n_s. The rank-one relations give $(T_s^\theta)^2 = (q-1)T_s^\theta + q$ if $\theta \circ h_s = 1$. In the quadratic case $\theta \circ h_s \neq 1$, the open-cell sum $\sum_{u \in \mathbb{F}_q^\times} \theta(h_s(u))$ is 0. Thus, $T_s^{{}^s\theta} T_s^\theta = q\theta(n_s^2)$ on θ, with $n_s^2 = h_s(-1)$, cf. [122, §3]. Moreover, any reduced expression for w gives the operator T_w.

Next, the sign change k inverts the roots $e_2 \pm e_i$ for $i = 3, 4$. By the pinned $\mathfrak{S}_3$-action, every $v \in V \setminus \{1\}$ has $\ell(v) = 4$, with all inversions outside $\{\pm l_i\}$. Thus, we put $U_1 = 1$ and $U_v = q^{-2}T_v$ for $v \neq 1$. Given $v, v' \neq 1$, braid moves leave $(\ell(v) + \ell(v') - \ell(vv'))/2$ opposite crossings. This number is 4 if $v = v'$, and 2 otherwise. Since V fixes the chamber of W_c, each deleted crossing is outside $\{\pm l_i\}$ and has coroot sign $(-1)^{(q-1)/2}$. Hence, the signs have product 1, and the powers of q give $U_vU_{v'} = U_{vv'}$.

Given $i \in \{1, 3, 4\}$, we put $H_i = T_{s_i}$, where $s_i = s_{l_i}$. Define $H_2 = U_vH_iU_v^{-1}$ for $v(i) = 2$. If $i, vi \in \{1, 3, 4\}$, length addition in $vs_i = s_{vi}v$ gives $U_vH_i = H_{vi}U_v$. Consequently, H_2 is independent of i, and $\mathfrak{S}_3$ permutes the H_i as the l_i. Each pair $\{i, j\}$ has a V-translate in $\{1, 3, 4\}$. Thus, $[H_i, H_j] = 0$ and $(H_i - q)(H_i + 1) = 0$. In $\prod_{i \in I} H_i\, U_v$, each wall l_i for $i \in I$ has multiplicity 1. Consequently, each deleted wall α lies outside $\{\pm l_i\}$ and contributes $q\theta(h_\alpha(-1)) \neq 0$. Thus, distinct (I, v) give distinct Bruhat cosets $B_N(\prod_{i \in I} s_i)vB_N$. [95, Lemma 5.2 and Problem 5.6] gives $\dim_{\mathbb{Q}} \mathrm{End}_N(Y_G) = |W_\theta|$. Thus, $h_i = (2H_i - (q-1))/(q+1)$ and U_v give the group algebra $\mathbb{Q}[W_c \rtimes V]$.

Given Y_M, we put $F_t = \mathrm{Int}(n_t)F_q$, with $n_+ = 1$ and $n_- = n_0$, the positive Tits lift of $-I$. The longest braid relation gives $n_0n_s = n_sn_0$ for each simple s, so $F_t(n_s) = n_s$. On Y_M, we let $L_vu(x) = u(n_vx)$ for the positive Tits lift n_v of $v \in V$. Each opposite crossing has sign $(-1)^{(q-t)/2}$. Thus, $L_vL_{v'} = L_{vv'}$, since the crossing count is 2

or 4 when $v, v' \neq 1$. Since $\theta_t(h_{s_i}(-1)) = 1$, the outer operators $L_{n_{s_i}}$ square to 1 and satisfy $L_{n_{s_i}} L_{n_{s_j}} = L_{n_{s_j}} L_{n_{s_i}}$. Define $h_2 = L_v h_i L_v^{-1}$ when $v(i) = 2$. Consequently, the W_θ-cosets give a basis indexed by (I, v). In both algebras the generators are h_i, L_v, taking $L_v = U_v$ on Y_G. Moreover, F_r fixes both sets, and $\mathfrak{S}_3$ permutes their indices identically.

Given $a \in T_{\mathrm{ad},t}$, choose a spin lift a' and put $z = F_t(a')(a')^{-1} \in \mathrm{Z}(X)$. On the operator for $w \in W_\theta$, the scalar from a is the quadratic value on $(l - wl)(a)$. Thus, for $\lambda_w = (l - wl)/2 \in P_{\mathrm{wt}}$, the scalar is $\lambda_w(z)$. Indeed, the exponent on $\lambda_w(a')$ is $q - 1$ for $t = 1$, and $-(q + 1)$ for $t = -1$. Hence, inversion has no effect on $\lambda_w(z) \in \{\pm 1\}$. Given s_i, we have $\lambda_{s_i} \in Q_{\mathrm{rt}}$. The classes for k, g are e_2 and $(e_1 + e_2 - e_3 - e_4)/2 \bmod Q_{\mathrm{rt}}$. Thus, they generate $P_{\mathrm{wt}}/Q_{\mathrm{rt}} \cong C_2^2$. Consequently, the quotient D_{ad} fixes h_i and multiplies L_v by the characters of V, with boundary z on Y_G, Y_M. □

Lemma 6.113. *If $t = -1$, the module Y_M of* Lemma 6.112 *admits a field operator B with $\mathcal{H}_p$-fixed coefficients and $[B, \mathfrak{S}_3] = 1$. Moreover, on $\widetilde{Y}_M$, it has $B^f = R(n_0^{-1})$.*

Proof. First, let $\mathbf{H}$ be generated by SL_2 at $e_2 - e_3$ and the diagonal SL_2 on the outer roots l_1, l_3, l_4. The subgroup $\mathbf{H}$ is connected, defined over $\mathbb{F}_r$ and fixed pointwise by $\mathfrak{S}_3$. It contains n_0. Indeed, for the central simple reflection s_0, we have that the element $c = s_0 s_1 s_3 s_4$ has $c^3 = -I$. The word for c^3 has length $12 = |\Phi^+|$, so its Tits lift is $n_0 \in \mathbf{H}$. By Lang's theorem [58, Theorem 4.2.9], $g \in \mathbf{H}$ with $g^{-1} F_q(g) = n_0$. After conjugation by g, we obtain the split field lift $g F_r(g)^{-1} F_r$, whose factor $g F_r(g)^{-1}$ is F_q-fixed since $F_r(n_0) = n_0$.

Write $R(h)u(x) = u(xh)$ and $B_0 u(x) = u(F_r^{-1} x)$. On the F_t-fixed group, $F_q = \mathrm{Int}(n_0^{-1})$, so $B_0^f = L_{n_0} R(n_0^{-1})$, where $L_{n_0} u(x) = u(n_0 x)$. Since n_0 fixes $\widetilde{\theta}_t$, the adjoint module $\widetilde{Y}_M$ has the operator L_{n_0}. Moreover, $L_{n_0}^2 = 1$ on $\widetilde{Y}_M$, and L_{n_0} satisfies $L_{n_0} D = D L_{n_0}$ for $D \in \{R(h), h_i, L_v, B_0\} \cup \mathfrak{S}_3$. With $e_\pm = (1 \pm L_{n_0})/2$, we put $D = \mathrm{ord}_p(r)$. Since $q \equiv -1 \bmod p$, $D/\gcd(D, f) = 2$. Thus, a primitive $\alpha \in \mu_D$ has $\alpha^f = -1$, and $\mathcal{H}_p$ fixes α because $D \mid p - 1$. Consequently, $B = (e_+ + \alpha e_-) B_0$ has $B^f = (e_+ - e_-) B_0^f = R(n_0^{-1})$. Since $e_\pm$ are central in $\mathrm{End}_{M_N}(Y_M)$, B and B_0 give equal actions on $\mathrm{End}_{M_N}(Y_M)$. □

Proposition 6.114. *Let p and $\mathcal{E}_1$ satisfy* Proposition 6.111. *Then, $\mathcal{E}_1$ corresponds to the characters of $\mathrm{N}_X(P)$ over the Weyl orbit of θ_t with trivial central character on $\mathrm{Z}(X)$. Moreover, on pairs (χ, ψ),* Definition 2.1 *holds for $D_{\mathrm{ad}}, F_r, \mathfrak{S}_3$.*

Proof. In $\mathbb{Q}[W_c \rtimes V]$, we index the weights of W_c by $I \subseteq \{1, 2, 3, 4\}$. Its trivial extension to $V_I = \mathrm{Stab}_V(I)$ gives the source for each $\xi \in \mathrm{Irr}(V_I)$. By Clifford theory [43], we obtain $L_{I,\xi}$, with signed permutations in the V/V_I-basis. For $|I| = 0, 4$, there are eight modules of dimension 1. For $|I| = 1, 3$, we find two of dimension 4. Moreover, $|I| = 2$ gives three V-orbits with $|V_I| = 2$. Hence, we obtain 6 modules of dimension 2.

On N, the Borel induction Y_G is torus Lusztig induction at $l/2$. Thus, every constituent is in $\mathcal{E}_1$. By [208, proof of Proposition 7.41, Step 2], the V-orbit of a label I contributes $|V_I|$ constituents. Thus, $\mathcal{E}_1$ has $8 + 2 + 6 = 16$ characters, all in Y_G. On M_N, the constituents of Y_M lie over $W(D_4)\theta_t$ and exhaust that set. Since $[W(D_4) : W_\theta] = 3$, their degrees are $3, 6, 12$, all prime to p.

By Lemma 6.112, a primitive central idempotent e of $\mathbb{Q}[W_c \rtimes V]$ gives the paired summands eY_G, eY_M. Thus, eY_G is χ-isotypic and eY_M is ψ-isotypic. On $\widetilde{Y}_G, \widetilde{Y}_M$, the adjoint actions are over $\mathbb{Q}$, as are $F_r, \mathfrak{S}_3$ in the split models. At $t = -1$, Lemma 6.113 gives B with $\mathcal{H}_p$-fixed entries. Thus, on $\widetilde{Y}_G, \widetilde{Y}_M$, the parent maps act identically on e and are fixed by $\mathcal{H}_p$. Since e is over $\mathbb{Q}$, the pairs χ, ψ have equal stabilizers for $D_{\mathrm{ad}}, F_r, \mathfrak{S}_3, \mathcal{H}_p$.

Fix an idempotent e and a common module $L = L_{I,\xi}$. Given any y in its outer stabilizer, let A_y implement the action on $\mathrm{End}(L)$, with $A_1 = 1$. Next, pair lines by $I \mapsto y(I)$, and extend by V. Since y fixes (I, ξ) up to V, the stabilizer action ξ has the indicated correction from D_{ad}. Thus, A_y has entries in $\{0, \pm 1\}$ as a signed permutation matrix. Hence, $A_y A_{y'} A_{yy'}^{-1} \in \{\pm I_L\}$.

On eY_G, eY_M, the action at y factors as $P_y \otimes A_y$ and $Q_y \otimes A_y$, respectively. Thus, P_y, Q_y give the projective actions on χ, ψ with the inverse factor of A_y. Given (y, σ) in the mixed stabilizer, the parent matrices are fixed by σ, with common factor A_y on L. Consequently, $\mu_{y,\sigma} = \mu'_{y,\sigma}$ on the common inertia group. Since $\mathrm{C}_{\mathrm{Aut}(N)}(N) = 1$, the inflated actions of $\mathrm{C}_{X \rtimes D}(X)$ have scalar 1. Finally, every outer coset has y with $P^y = P$ by Sylow conjugacy. If $t = -1$, the field power is $B^f = R(n_0^{-1})$. Thus, (χ, ψ) satisfies $\succeq_c$. □

Lemma 6.115. *In* Section 6.5, *we fix $X, N, q = r^f$. By* Zhang [208, Proposition 7.40], *we set $\alpha = -e_1 - e_4$, $\beta_1 = e_1 - e_2$, $\beta_2 = e_1 + e_2$ and $\beta_3 = e_3 + e_4$. Next, we put $l = \sum_i \beta_i$, $x = l/4 \bmod P_{\mathrm{wt}}$ and $\mu = 2\alpha + l$. Thus, the roots integral on x are $\{\pm\mu\}$. The cocharacter μ defines a Levi $\mathbf{L}$ with roots $\{\pm\beta_i\}_{i=1}^3$. Moreover, we put $w_i = s_{\beta_i}$,*

$J = w_1w_2w_3$ and $V = \langle w_1w_2, w_1w_3\rangle$. Let $p \neq r$ be an odd divisor of $|X|$, and put $t = \eta_q(-1)$. Then, the split $\mathrm{Q}(1,2)$ *series has characters of p'-degree exactly when $p \geq 5$, $p \mid q - t$ and the rational class has component $v = 1$ in V. Consequently, both labels* $1, \mathrm{St}$ *contribute at μ.*

Proof. Put $a = (q-1)/2 \bmod 2$. By [208, (7.44) and Proposition 7.40], the rationalizing Weyl element is vJ^a, with $v \in V$, and μ remains split. Thus, for the signs t_i of vJ^a on β_i, the centralizer order away from $2, r$ is $(q^2-1)\prod_{i=1}^3(q-t_i)$. By Jordan degrees [58, Proposition 11.5.6], we must have $p \mid q - t'$ for some $t' = \pm 1$. Next, we put $A = v_p(q - t')$. The valuations of $|X|$ and the centralizer order are $4A + v_p(4!)$ and $A(1 + |\{i : t_i = t'\}|)$, respectively. Hence, p'-degree requires $p \geq 5$ and $t_i = t'$ for each i. Since $v \in \langle w_1w_2, w_1w_3\rangle$, $v = 1$ and $t' = (-1)^a = t$. Conversely, the two valuations equal $4A$, and $1, q$ are prime to p. □

Lemma 6.116. *For each label $\nu \in \{1, \mathrm{St}\}$ at μ, the four constituents are eigenspaces for J_I, for even $I \subseteq \{1,2,3\}$. We have $J_IJ_{I'} = J_{I'}J_I$ for even $I, I' \subseteq \{1,2,3\}$. Moreover, we have that F_r, σ fix J_I, and $\gamma \in \mathfrak{S}_3$ acts by $J_I \mapsto J_{\gamma I}$. A diagonal with boundary z multiplies J_I by $\lambda_I(z)$, with $\lambda_I = \frac{1}{2}\sum_{i\in I}\beta_i$.*

Proof. Let $\mathbf{L}_{\mathrm{ad}}$ be the adjoint Levi for μ. Its map to PGL_2^3 has connected torus kernel since $\alpha, \beta_1, \beta_2, \beta_3$ form a character lattice basis for $\mathbf{L}_{\mathrm{ad}}$. By Lang's theorem [58, Lemma 4.2.13(i)], we obtain $L_{\mathrm{ad}} \twoheadrightarrow \mathrm{PGL}_2(q)^3$. On each factor, we choose π from Lemma 6.44 with central scalar 1. Thus, for $q \equiv 1 \bmod 4$, the pair on $\mathrm{SL}_2(q)$ is quadratic principal. Given $q \equiv 3 \bmod 4$, it is quadratic cuspidal. In dual SL_2, the eigenvalues are $i, -i$. Consequently, the quartic weight on $\mathrm{PGL}_2(q)$ gives the quadratic weight on $\mathrm{SL}_2(q)$. With central label 1, the three factors give $x = l/4$.

Next, on L_{ad}, take the pullback of $\pi^{\boxtimes 3}$. Write d_i for the quadratic determinant of factor i. Then, the image of L consists of $d_1 = d_2 = d_3$, since the coroots $\beta_i^\vee$ have even pairings with β_j and $\alpha^\vee$ has pairings $(-1,-1,-1)$. Moreover, the image contains $\mathrm{PSL}_2(q)^3$. On factor i, we put J_i as in Lemma 6.44. Thus, $J_i^2 = F_r^f = 1$ and $[F_r, J_i] = 1$ on π. The coefficient factor on J_i is $\eta_q(b_\sigma)$, so $J_I = \prod_{i\in I} J_i$ is fixed for even I. Thus, the four quotient characters $\delta_I = \prod_{i\in I} d_i$ fix $\pi^{\boxtimes 3}$. By reciprocity, the four J_I span the commutant on L. Moreover, the permutation $\gamma \in \mathfrak{S}_3$ acts by $J_I \mapsto J_{\gamma I}$ and satisfies $\gamma F_r = F_r\gamma$, $\gamma\sigma = \sigma\gamma$.

By [208, proof of Proposition 7.40], induction from L_{ad} has two constituents $\widetilde{\rho}_1, \widetilde{\rho}_{\mathrm{St}}$, each with multiplicity 1. By Jordan degrees, $\widetilde{\rho}_{\mathrm{St}}(1) = q\widetilde{\rho}_1(1)$. Thus, both summands $\widetilde{\rho}_\nu$ are invariant. Extend J_I by $f(x) \mapsto \delta_I(x)J_If(x)$, with δ_I on D_{ad}. For the diagonal with boundary z, $\delta_I = \lambda_I(z)$ by Lemma 6.112, with $\lambda_I = \frac{1}{2}\sum_{i\in I}\beta_i$. The four classes $\lambda_I \bmod Q_{\mathrm{rt}}$ give all characters of D_{ad}. Thus, D_{ad} is transitive on the lifted J_I-eigencharacters. By [208, proof of Proposition 7.40], $\widetilde{\rho}_\nu|_N$ has four distinct constituents of multiplicity 1, so the J_I-eigenspaces give the four χ. □

Proposition 6.117. *Under* Lemma 6.115*, the eight characters at $x = l/4$ correspond to* $\mathrm{Irr}(M_N \mid W(D_4)\theta)$*, with θ on $T_{N,t}$. Moreover,* Definition 2.1 *holds on the pairs (χ, ψ) for $\mathfrak{S}_3$ and the coefficient action, including $\sigma(i) = -i$.*

Proof. Let $\widetilde{\theta} = \xi_t \circ l$ on $T_{\mathrm{ad},t}$, with ξ_t of order 4 on μ_{q-t}, and put $\theta = \widetilde{\theta}|_{T_{N,t}}$. By [208, Proposition 7.40 and (7.44)], J exchanges the two choices of ξ_t. If $t = -1$, the torus of type $-I = Js_\mu$ differs from J by the connected reflection s_μ, so the torus has $v = 1$. By [208, Proposition 7.40], the stabilizer of x is $W_\theta = \langle s_\mu\rangle \times V$. Since $\lambda_I \notin Q_{\mathrm{rt}}$ for nonempty even I, the stabilizer of $\widetilde{\theta}$ is $\langle s_\mu\rangle$.

Take Chevalley lifts m_i, m_μ for β_i, μ with $m_im_\mu = m_\mu m_i$. Given m_μ, take $e_\mu = [e_\alpha, (\prod_i \mathrm{ad}\, e_{\beta_i})e_\alpha]$ and its normalized negative root vector $e_{-\mu}$. Since $D_iD_j = D_jD_i$ for $D_i = \mathrm{ad}\, e_{\beta_i}$ and string constants are ± 1, the vectors $e_{\pm\mu}$ are fixed by $\mathfrak{S}_3$. Thus, m_μ is diagram fixed. On $T_{\mathrm{ad},t}$, all lifts are rational since $[n_0, m_i] = [n_0, m_\mu] = 1$. The pairings $\langle l, \beta_i^\vee\rangle = 2$ and $\langle l, \mu^\vee\rangle = 0$ give $\widetilde{\theta}(m_i^2) = \widetilde{\theta}(m_\mu^2) = 1$. Hence, on $Y(\theta) = \mathrm{Ind}_{T_{N,t}}^{M_N}\theta$, the products $L_{m_\mu}^\varepsilon L_{m_I}$ for $\varepsilon = 0, 1$ give the basis indexed by W_θ. On the adjoint model, the stabilizer $\langle s_\mu\rangle$ gives the two irreducible parents $\widetilde{\tau}_\pm$, with $L_{m_\mu} = \pm 1$. Moreover, each $\widetilde{\tau}_\pm$ has four eigenspaces for L_{m_I} on M_N. A diagonal with boundary z fixes L_{m_μ}. Since $l - w_Il = 2\sum_{i\in I}\beta_i$ for $w_I = \prod_{i\in I} w_i$, its scalar on L_{m_I} is $\lambda_I(z)$ by Lemma 6.112, for $t = 1, -1$.

On the two induced spaces $Y(\widetilde{\theta}^{\pm 1})$, we put $m_J = m_1m_2m_3$. Then, L_{m_J} exchanges the two spaces, with $L_{m_J}^2 = 1$ and $F_r, \mathfrak{S}_3$ fixing L_{m_J}. Write C_σ for coefficient action and B_0 for inverse field pullback. With $a_r = [r \equiv 3 \bmod 4]$, put $u_\sigma = [\sigma(i) = -i]$. Thus, $B_1 = L_{m_J}^{a_r}B_0$ and $T_\sigma = L_{m_J}^{u_\sigma}C_\sigma$ act on $Y(\widetilde{\theta})$. Moreover, T_σ satisfies $T_\sigma D = DT_\sigma$ for $D \in \{R(g), B_1, L_{m_I}, L_{m_\mu}\} \cup \mathfrak{S}_3$. If $t = 1$, then $2 \mid fa_r$ and we set $B = B_1$, with $B^f = 1$. If $t = -1$, then $a_r = 1$ and $2 \nmid f$. Thus, $B_1^f = CR(n_0^{-1})$ by Lemma 6.113, with $C = L_{m_J}L_{n_0}$. Both factors invert $\widetilde{\theta}$, and $L_{m_J}L_{n_0} = L_{n_0}L_{m_J}$. Moreover, $L_{m_J}^2 = L_{n_0}^2 = 1$. Consequently, $C^2 = 1$ on $Y(\widetilde{\theta})$, and C satisfies $CD = DC$ for $D \in \{B_1, T_\sigma, L_{m_I}, L_{m_\mu}\} \cup \mathfrak{S}_3$. Thus, $B = CB_1$ has $B^f = R(n_0^{-1})$. By Lemma 6.113, conjugation by g gives the marked field on M_N.

Pair $(\widetilde{\rho}_1, \widetilde{\tau}_+)$ and $(\widetilde{\rho}_{\mathrm{St}}, \widetilde{\tau}_-)$ by matching the J_I- and L_{m_I}-eigencharacters. Thus, the diagonal scalars $\lambda_I(z)$, the permutations of I and the actions of B, T_σ give equal mixed stabilizers of (χ, ψ). The multiplicities in $\widetilde{\rho}_\nu|_N, \widetilde{\tau}_\pm|_{M_N}$ are 1. Thus, on I_χ, I_ψ, restriction gives ordinary extensions of χ, ψ. On I_ψ, the extensions are equivariant for T_σ, and on I_χ for the global coefficient maps. Hence, $\mu_{y,\sigma} = \mu'_{y,\sigma} = 1$. On $\mathrm{Z}(X)$ both models have scalar 1. Moreover, by Proposition 6.114, the scalar on $\mathrm{C}_{X \rtimes D}(X)$ is 1. The local degree is $\psi(1) = [W(D_4) : W_\theta] = 24$ for each of the eight ψ, and $p \nmid 24$. Finally, $\langle l, \alpha^\vee \rangle = -3$ gives $\mathrm{ord}(\theta) = 4$. Thus, the orbit $W(D_4)\theta$ is disjoint from the quadratic orbit $W(D_4)\theta_t$. □

Lemma 6.118. *Let $X = \mathrm{Spin}_8^+(q)$. By* Zhang [208, Lemma 7.38(ii)]*,* $\mathrm{I}(1,3)$ *and the two* $\mathrm{U}(4)$ *classes form one $\mathfrak{S}_3$-orbit. First, compare the vector class* $\mathrm{I}(1,3)$*, then its two translates of type* $\mathrm{U}(4)$*. Hence, for the characteristic toral base Q of P in* Section 6.2*, we compare on $M = \mathrm{N}_X(Q)$ even when $Q < P$. In $X = \mathrm{Spin}_8^+(q)$, we let V_-, V_+ be the eigenspaces of dimensions $2, 6$ for the vector class s. Next, we write $t \in \{\pm 1\}$ for their common sign. Then, for $d = \mathrm{ord}_p(q)$, the eligible pairs (t, d) are $(+,1), (+,3), (-,2), (-,6)$. Set $e = 1$ at $d = 1, 2$ and $e = 3$ at $d = 3, 6$. If $e = 1$ and $p \geq 5$, every partition $\lambda \vdash 4$ gives p'-degree. If $e = 3$ or $p = 3$, the partitions are $(4), (2,2), (1^4)$. Moreover, each λ gives constituents χ_1, χ_2.*

Proof. By [208, Proposition 7.40], the signs of V_-, V_+ multiply to $+1$. Thus, both signs are t. Jordan degrees [58, Proposition 11.5.6] give the index $(q^4-1)(q^3+t)/(q-t)$ away from $2, r$, since the centralizer order contributes $(q-t)(q^3-t)(q^2-1)(q^4-1)$. If $p \mid q-t$, then $(q^4-1)/(q-t) \equiv 4t \bmod p$ and $q^3 + t \equiv 2t \bmod p$. Thus, p does not divide the index. If $d = 3, 6$ and $q^3 \equiv t \bmod p$, $p \nmid (q^4-1)(q^3+t)$. With $-t$ in place of t, the index has positive p-valuation in either case. At $d = 4$, the factor $q^4 - 1$ gives positive valuation for both signs t. Finally, d divides one of $2, 4, 6$, so the list is exhaustive.

With $u = tq$, we let χ_λ be a constituent at λ. If $e = 1$, the hook degree formula gives $v_p(\chi_\lambda(1)) = v_p(f^\lambda)$, with $f^\lambda = 4!/\prod_{h \in \mathrm{Hook}(\lambda)} h$. Indeed, for each hook h, $v_p(u^h - 1) = v_p(u-1) + v_p(h)$, giving $4v_p(u-1)$ in each product. If $p = 3$, $f^{(3,1)} = f^{(2,1,1)} = 3$, so these diagrams are excluded. If $e = 3$, the only numerator factor divisible by p is $u^3 - 1$. Thus, the hook set contains 3 exactly for $(4), (2,2), (1^4)$, and $p \geq 7$ since $d \mid p-1$. The order-two component fixes λ, so [208, Proposition 7.40] gives χ_1, χ_2. □

Lemma 6.119. *Given any allowed λ, a similitude h of nonsquare multiplier exchanges χ_1, χ_2. Moreover, isometries, field maps and $\mathcal{H}_p$ fix χ_1, χ_2 individually. Moreover, the scalar at z_0 is 1, and the scalar at $-I$ on Ω is $t\eta_q(-1)$. Consequently, we have that χ_i has an extension $\widetilde{\chi}_i$ to its isometry and field inertia I_i, fixed by $\mathcal{H}_p$.*

Proof. First, for $\eta_1 = \eta_2$, take the parent Φ and field operator B on $O_4^+(q)$ from Lemma 6.57. Restore the hyperbolic planes $E_j \oplus E_j^\vee$ for $j = 1, 2$ in V_+, in that order. Thus, on each $\mathrm{GL}(E_j)$, the extracted character is 1. At E_1 the rank-two label is 1 or St. If $t = +$, both rank-one flags give coefficient 1. Similarly, for $t = -$, the flag over $\mathbb{F}_{q^2}$ gives coefficient 1. Thus, the unequal ranks at E_1 distinguish the factor of L_1.

By [208, proof of Proposition 7.40], take the labels at E_2. Thus, for $t = +$, the $W(D_2)$-label is $1 \boxtimes 1$ for $\lambda = (4), (3,1), (2,2)$, and $\mathrm{sgn} \boxtimes \mathrm{sgn}$ for $\lambda = (2,1,1), (1^4)$. Similarly, for $t = -$, restrict $W(B_2)$ to $W(B_1)$ and choose one sign in dimension 2. Thus, the branches at ranks $2, 3$ pair with multiplicity 1. Both centralizers have component k. Moreover, the opposite flags at E_2 form one orbit under $C^\circ(s)^{F_q}$.

On the line Levi L_j, the character ν restricts to η_q on $\mathrm{GL}(E_j)$, whereas the extracted character is 1. Hence, the twists $\nu, \det \nu$ move the parent Φ_j. Also, $\Phi_j \det \neq \Phi_j$, by Lemma 6.57 at E_1 and irreducible restriction at E_2. Thus, on L_j, Proposition 6.74 gives multiplicity 1 for the marked branch χ_i. Thus, $R(g), F_r, T_\sigma$ act on V_{χ_i}. Over $\Omega_4^+(q)$, the parent twist groups are $\langle \nu \rangle, \langle \det \nu \rangle$. Consequently, at E_1, the restrictions differ. At E_2, the pairing gives $\chi_1 \neq \chi_2$. The nonsquare similitude h gives $\widehat{\chi}_1^h = \widehat{\chi}_2$ on the chosen extensions. Finally, on $\mathrm{GL}(E_j)$ the scalar at $-I$ is 1, so $-I$ acts on each final branch by $t\eta_q(-1)$. □

Proposition 6.120. *On $M = \mathrm{N}_X(Q)$, each vector form (t, d) gives one orbit of weights on $T \cap \Omega$. Given any allowed λ, we find two p'-degree characters ψ_1, ψ_2 over that orbit. Moreover, the pairing $\chi_i \leftrightarrow \psi_i$ from* Lemma 6.119 *extends to the $\mathfrak{S}_3$-orbit and satisfies $\succeq_c$ on all three classes.*

Proof. At $e = 1$, we put $H = T \rtimes (C_2 \wr \mathfrak{S}_4)$, with $T = (C_{q-t})^4$, and $H_0 = H \cap \Omega = \ker \det \cap \ker \nu$. The torus planes V_i are the reciprocal weight pairs of Q. Thus, we have that M is the spin preimage of H_0, since Q is the p-part of T.

Put $\theta = \eta_1 \otimes 1 \otimes 1 \otimes 1$ on T, with η_1 quadratic on V_1. Extend θ trivially on $C_2 \times W(B_3)$. On the first C_2 take sign $+1$. On $W(B_3) = W(D_3) \times \langle -I_3 \rangle$, we choose S^λ on $W(D_3) \cong \mathfrak{S}_4$ and sign $+1$ on $-I_3$. Then, the induced ρ_λ has degree $4f^\lambda$ by [43].

On T, the twist ν takes θ to $1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4$. The twist det acts by $(\epsilon_1, \epsilon_3) \mapsto (-\epsilon_1, -\epsilon_3)$. Hence, $\rho_\lambda|_{H_0}$ is irreducible by reciprocity. Let h be a similitude of nonsquare multiplier m. On a split plane V_i, take $\mathrm{diag}(m, 1)$, whose torus correction at inversion is m. Next, on a nonsplit V_i, take multiplication by a with $a^{q+1} = m$. Its correction a/a^q has quadratic value $\eta_q(m) = -1$. Thus, on V_1, the sign ϵ_1 changes. On V_2, V_3, V_4, the weights 1 give trivial corrections and the maps $h|_{V_i}$ are $\mathfrak{S}_{\{2,3,4\}}$-equivariant. Thus, the restrictions have signs $(+1, +1)$ and $(-1, +1)$ modulo simultaneous sign change. Conversely, $\theta|_{T \cap H_0}$ has lifts $\theta, \theta\nu|_T$. Since det acts by $(\epsilon_1, \epsilon_3) \mapsto (-\epsilon_1, -\epsilon_3)$, the lift θ gives two restrictions at each λ, of degree $4f^\lambda$.

At $e = 3$, the active space V_A has three reciprocal pairs of nontrivial Q-weights, and $V_R = V^Q$ is the residual plane. Thus, $\mathrm{C}_{\mathbf{X}}(Q)^\circ$ is a torus and $H = (C_{q^3-t} \rtimes C_6) \times (C_{q-t} \rtimes C_2)$. The weight θ is trivial on C_{q^3-t} and quadratic on C_{q-t}. Write j mod 6 for the C_6-label and ϵ for the residual C_2-sign. The twist det takes (j, ϵ) to $(j + 3, -\epsilon)$, since the C_6-generator inverts three coordinates. On C_{q^3-t}, the twist ν is nontrivial, by the norm onto $\mathbb{F}_q^\times$ for $t = +$ and the quadratic spinor norm for $t = -$. Hence, each parent restricts irreducibly to H_0.

Take a common nonsquare multiplier m. For split $V_A = U \oplus U^\vee$, we put $h|_U = mI$ and $h|_{U^\vee} = I$. Similarly, on the nonsplit V_A, take multiplication by $a \in \mathbb{F}_{q^6}$ with $\mathrm{N}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^3}}(a) = m$. Thus, both maps centralize C_{q^3-t}, and the weight 1 gives trivial corrections on C_6. With $h|_{V_R}$ as above, the similitude changes only ϵ. Pair $j = 0, 1, 2$ with $\lambda = (4), (2, 2), (1^4)$, respectively, and take $\epsilon = \pm 1$. Since the two lifts differ by $\nu|_T$, the relation $(j, \epsilon) \sim (j + 3, -\epsilon)$ gives precisely these six restrictions. Moreover, $6 \mid p - 1$, so the roots ζ_6^j are fixed by $\mathcal{H}_p$.

For the torus T, we set $F_v = \mathrm{Int}(v)F_q$ on the split model. With $c = (234)$, let ϵ_{all} invert the four planes. If $e = 1$, we set $v = 1$ at $t = +$ and $v = \epsilon_{\mathrm{all}}$ at $t = -$. If $e = 3$, we set $v = c$ at $t = +$ and $v = c\epsilon_{\mathrm{all}}$ at $t = -$. Thus, v lifts to spin over $\mathbb{F}_r$: the inversions lift by products of $e_i + f_i$, with even length and norm 1. Also, c has determinant 1 on the isotropic half U. The signed complements are over $\mathbb{F}_r$. By Lang's theorem [58, Theorem 4.2.9], we choose $g^{-1}F_q(g) = v$. Consequently, the field lift is $gF_r(g)^{-1}F_r$, with f-th power $gv^{-1}g^{-1}$.

On the local model ρ_λ, the field F_r fixes θ and the matrices on the complement C in $H = T \rtimes C$. Thus, its matrix is 1. Since v^{-1} is central in the complement at $e = 1, 3$ and fixes θ, its scalar is $\zeta \in \mu_d$. The natural v has order dividing d. Next, put $D = \mathrm{ord}_p(r)$. Since $D/\gcd(D, f) = d$, we choose $\alpha \in \mu_D$ with $\alpha^f = \zeta$. Thus, the field matrix αI has the marked f-th power, and $\mathcal{H}_p$ fixes it since $D \mid p - 1$. The matrices of S^λ and the labels ζ_6^j are also $\mathcal{H}_p$-fixed. Finally, the scalar at $-I$ is $(-1)^{(q-t)/2} = t\eta_q(-1)$, and the scalar at z_0 is 1.

Thus, the graph γ and field F_r give extensions of χ_i, ψ_i, fixed by $\mathcal{H}_p$. A nonsquare h acts by $(\chi_1, \psi_1) \mapsto (\chi_2, \psi_2)$. In the natural overgroup, an element gF_r^j centralizing Ω has $j = 0$ mod f, since F_r^j is outer for $0 < j < f$. By Schur's lemma on V, $g \in \{\pm I\}$. Hence, the scalars at $-I, z_0$ coincide on χ_i, ψ_i. Pullback to X gives ordinary extensions, with $\mu_{y,\sigma} = \mu'_{y,\sigma} = 1$.

The quadratic weight on $T \cap \Omega$ has point $x_{\mathrm{v}} = e_1/2$ mod P_{wt}, since $(qw - 1)x_{\mathrm{v}} = (qt - 1)e_1/2$ for the Weyl element w on the singleton. If $e = 1$, the four points $e_i/2$ are rationally conjugate. If $e = 3$, w cycles the active e_i, so x_{v} is the only rational point in $W(D_4)x_{\mathrm{v}}$. By [208, Lemma 7.38], its translates are $x_- = (1, 1, 1, -1)/4$ and $x_+ = (1, 1, 1, 1)/4$. Even signed permutations and the shift $\frac{1}{2}(1, 1, 1, 1)$ fix sign parity modulo Z^4. Thus, $W(D_4)x_-, W(D_4)x_+, W(D_4)x_{\mathrm{v}}$ are pairwise disjoint. Since $2x_-, 2x_+, 2x_{\mathrm{v}} \in P_{\mathrm{wt}}$, $\mathcal{H}_p$ fixes each of x_-, x_+, x_{v}. By [208, Lemma 7.38(iii)], the $\mathfrak{S}_3$-stabilizer of x_{v} is the natural graph group C_2. By Sylow conjugacy, we choose lifts a with $P^a = P$. Hence, $Q^a = Q$. Any two lifts with equal x_{v}^a differ by the compared stabilizer of x_{v}. Thus, each pair (χ_i^a, ψ_i^a) has equal mixed factors. □

Corollary 6.121. *With $X = \mathrm{Spin}_8^+(q)$ and q odd, we let $p \neq r$ be odd. Then, the p'-parts of the series $\mathcal{E}(X, s)$ for nonidentity quasi-isolated s correspond to disjoint subsets of $\mathrm{Irr}(\mathrm{N}_X(Q))$. Moreover, the pairs satisfy $\succeq_c$ equivariantly for $\mathfrak{S}_3, \mathcal{H}_p$.*

Proof. By [208, Lemma 7.38], the types are $\mathrm{I}(2, 2), \mathrm{Q}(1, 2), \mathrm{I}(1, 3), \mathrm{U}(4)$. Thus, at $d = 1, 2$, the comparison for $\mathrm{I}(2, 2)$ is in Proposition 6.114. At $d = 4$, Lemma 6.33 gives ten branches at k, with central character λ. Since $\ker \lambda = \langle z_0 \rangle$, its stabilizer in $\mathfrak{S}_3$ is the natural graph group C_2. The comparison at k covers C_2, D_{ad} and F_r, including mixed stabilizer actions. Moreover, $\mathcal{H}_p$ fixes λ, so each mixed stabilizer action fixes $\ker \lambda$. Given a second kernel Z_1, transport the models by a lift a with $\langle z_0 \rangle^a = Z_1$ and $P^a = P$. Two such lifts differ by a λ-stabilizer. Hence, on $\mathrm{Z}(X)$, distinct λ give disjoint images.

By Proposition 6.117, the $\mathrm{Q}(1,2)$ orbit at $d=1,2$ has weight of order 4. Thus, at $d=3,6$, only the orbits of x_{v}, x_-, x_+ occur, by Proposition 6.120. At $d=1,2$, the quadratic point for $\mathrm{I}(2,2)$ is $x_2=(e_1+e_2)/2$. The point x_2 has two half-integral coordinates modulo P_{wt}, so $W(D_4)x_2$ is disjoint from the orbits of x_{v}, x_-, x_+. Moreover, the $\mathrm{I}(2,2)$ identity-component and $\mathrm{Q}(1,2)$ comparisons have $Q=P$ for the eligible p. Thus, the disjoint images lie in $\mathrm{Irr}(M)$, with $M=\mathrm{N}_X(Q)$. □

Lemma 6.122. *With $X=\mathrm{Spin}_8^+(q)$ and $q=r^f$ odd, we let $p\neq r$ be an odd divisor of $|X|$. Thus, we put $d=\mathrm{ord}_p(q)$. Write $X=\mathbf{X}^F$ and $X^*=(\mathbf{X}^*)^{F^*}$. Next, we fix $P\in\mathrm{Syl}_p(X)$ and a Sylow d-torus $\mathbf{S}$ normalized by P. There is a characteristic abelian $Q\leq P$ such that $\mathbf{T}=\mathrm{C}_{\mathbf{X}}(Q)=\mathrm{C}_{\mathbf{X}}(\mathbf{S})$ is a maximal torus and $M=\mathrm{N}_X(Q)=\mathrm{N}_{\mathbf{X}}(\mathbf{T})^F=\mathrm{N}_{\mathbf{X}}(\mathbf{S})^F$. Moreover, we have that $Q\in\mathrm{Syl}_p(T)$ for $T=\mathbf{T}^F$. On X^*, the relations hold with $Q^*,\mathbf{T}^*,\mathbf{S}^*,M^*$.*

Proof. Thus, the degrees $2,4,4,6$ give $d\in\{1,2,3,4,6\}$. With $c=(234)$, we let ε_i change the sign of e_i. Then, choose $w=I,-I,c,-c,(12)(34)\varepsilon_2\varepsilon_4$ for $d=1,2,3,6,4$, respectively. The torus $\mathbf{T}$ of type w contains a Sylow d-torus $\mathbf{S}$. Next, for $d=1,2$, we have that the space $\ker\Phi_d(w)$ is all of $\mathbb{C}^4$. If $d\in\{3,6\}$, take $(0,1,\omega,\omega^2)$ with ω a primitive cube root. Similarly, for $d=4$, take (a,ia,b,ib) with independent a,b. Thus, each $\ker\Phi_d(w)$ contains a vector on which all $e_i\pm e_j$ are nonzero, so $\mathrm{C}_{\mathbf{X}}(\mathbf{S})=\mathbf{T}$.

By [63, Theorems 3.2 and 3.3], $Q=(\mathbf{S}^F)_p=T_p$ is normal in P, with $P/Q\in\mathrm{Syl}_p(W_d)$ for $W_d=M/T$. At $d=3,6$, only degree 6 contributes to $|X|_p$, and at $d=4$ the two contributions have degree 4. Thus, $Q=P$ for these d. At $d=1,2$, $|Q|=(q-t)_p^4$ for $t=1,-1$, respectively, and $|P/Q|=(4!)_p$. If $p\neq 3$, $Q=P$. If $p=3$, put $a=v_3(q-t)$, so $Q=(C_{3^a})^4$. The cycle $c=(234)$ acts on Q with a lift of order 3, giving $P=Q\rtimes C_3$. Thus, Q is characteristic in P by Lemma 4.3.

On cocharacters modulo p, $Q[p]=\ker(qw-1)$. At $d=1,2$, this kernel is $\mathbb{F}_p^4$. At $d=3,6$, it has the line $\mathbb{F}_p(0,1,\omega,\omega^2)$ from the indicated cube root in $\mathbb{F}_p$. At $d=4$, the tuples are (a,ia,b,ib) over $\mathbb{F}_p$. Thus, $(e_i\pm e_j)|_{Q[p]}\neq 0$. The nonzero restrictions $e_i|_{Q[p]}$ are distinct modulo sign, and $\#\{i:e_i|_{Q[p]}=0\}\leq 1$. Hence, even parity gives $C_{W(D_4)}(Q[p])=1$. By [133, Theorem 14.2], $C_{\mathbf{X}}(Q[p])^\circ=\mathbf{T}$. The component group embeds in $C_{W(D_4)}(Q[p])$, so $C_{\mathbf{X}}(Q[p])=C_{\mathbf{X}}(Q)=\mathbf{T}$. The dual argument gives $C_{\mathbf{X}^*}(Q^*)=\mathbf{T}^*$ for $Q^*=((\mathbf{S}^*)^{F^*})_p$. From $\mathrm{C}_{\mathbf{X}}(Q)=\mathrm{C}_{\mathbf{X}}(\mathbf{S})=\mathbf{T}$, we obtain $\mathrm{N}_X(Q),\mathrm{N}_X(\mathbf{S})\leq\mathrm{N}_X(\mathbf{T})$. Conversely, $\mathrm{N}_X(\mathbf{T})$ normalizes $Q=T_p$ and the unique Φ_d-subtorus $\mathbf{S}$ of maximal dimension in $\mathbf{T}$. Thus, all three normalizers equal M. By [63, Theorem 3.3], $\mathrm{N}_X(P)\leq M$. Moreover, Q is characteristic in P, so $\mathrm{Aut}(X)_P$ stabilizes M. Since $Q\not\leq\mathrm{Z}(X)$, $M<X$. □

Proposition 6.123. *Let $X=\mathrm{Spin}_8^\pm(q)$ with q odd, and let p be an odd prime dividing $|X|$ with $p\nmid q$. For $P\in\mathrm{Syl}_p(X)$, take the Sylow d-torus $\mathbf{S}$ normalized by P, with $d=\mathrm{ord}_p(q)$. Suppose $\mathbf{T}=\mathrm{C}_{\mathbf{X}}(\mathbf{S})$ is a maximal torus, and put $T=\mathbf{T}^F$, $M=\mathrm{N}_X(\mathbf{S})$. Put $W_d=M/T$. Then, we have that the W_d-orbits of p'-length in $\mathrm{Irr}(T)$ correspond to the rational classes $[s]$ of p'-index in X^*. For θ_s in the orbit, let $W_{d,s}=\mathrm{Stab}_{W_d}(\theta_s)$. Consequently, we find $\big|\mathcal{E}(X,s)\cap\mathrm{Irr}_{p'}(X)\big|=\big|\mathrm{Irr}_{p'}(W_{d,s})\big|=\big|\mathrm{Irr}_{p'}(M\mid W_d\theta_s)\big|$. The partition into these subsets is $\mathrm{Aut}(X)_P\times\mathcal{H}_p$-equivariant.*

Proof. By [63, Theorem 3.3], $\mathbf{S}$ is unique, so M is $\mathrm{Aut}(X)_P$-stable. Moreover, $\mathrm{C}_{\mathbf{X}}(\mathbf{S})=\mathbf{T}$ gives $M=\mathrm{N}_X(\mathbf{T})$. Take the dual $\mathbf{S}^*,\mathbf{T}^*$ and $M^*=\mathrm{N}_{X^*}(\mathbf{S}^*)$. Apply [132, Theorem 7.5 and proof of Theorem 7.8]. Since $\mathrm{C}_{\mathbf{X}^*}(\mathbf{S}^*)=\mathbf{T}^*$, the unipotent label is 1_{T^*}. The semisimple labels are the classes s^{M^*} of p'-size with $s\in T^*$. By [132, Theorem 5.9 and Proposition 5.11], these correspond to the X^*-classes of p'-size. By torus duality, s^{M^*} corresponds to $W_d\theta_s$, with stabilizer $W_{d,s}$. By [132, proof of Theorem 7.8], both p'-degree sets have labels in $\mathrm{Irr}_{p'}(W_{d,s})$. Given a field or diagram map a, torus duality gives $\theta_s^a=\theta_{s^a}$, with s^a the dual image. Moreover, diagonals fix $[s]$. If $\sigma\in\mathcal{H}_p$ acts with exponent u, $\theta_s^\sigma=\theta_s^u$ and $\mathcal{E}(X,s)^\sigma=\mathcal{E}(X,s^u)$, by [180, Lemma 3.2]. □

By Ruhstorfer et al. [156, Theorem D and its proof], the comparison at $s=1$ is over 1_T. Thus, inflation from W_d gives its image $\mathrm{Irr}_{p'}(M\mid 1_T)$. At $d=4$, the quadratic character of C_{q^2+1} on one cycle has lift $(-1,-1,1,1)$. Thus, we obtain that $\dim V_-=\dim V_+=4$, as in class k of Lemma 6.33. Moreover, we have that $\mathfrak{S}_3\cdot k$ gives the other two classes. If $d\in\{1,2,3,6\}$, by Corollary 6.121, the remaining rational forms $[s]$ have local subsets over $W_d\theta_s$.

Corollary 6.124. *Let $d=3,6$. Then, we find an $\mathrm{Aut}(X)_P\times\mathcal{H}_p$-equivariant bijection from $\mathrm{Irr}_{p'}(X)$ to $\mathrm{Irr}_{p'}(\mathrm{N}_X(P))$ satisfying $\succeq_c$ at every pair.*

Proof. At $d=3,6$, exactly one of the degrees $2,4,4,6$ is divisible by d and $p\geq 7$. Lemma 2.11 gives cyclic P and $\mathrm{N}_X(P[p])=\mathrm{N}_X(P)$. Thus, Proposition 2.9 gives the comparison on $\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(\mathrm{N}_X(P))$. □

Lemma 6.125. *Let $p \mid q-1$ and $X = \mathrm{Spin}_8^+(q)$. Thus, for the split torus $T \le B$, take $M = \mathrm{N}_X(T)$ as in* Lemma 6.122*, with $M/T = W = W(D_4)$. Put $\Psi_\theta = \{\alpha \mid \theta \circ \alpha^\vee = 1\}$ and $W_\theta = \mathrm{Stab}_W(\theta)$ for $\theta = \theta_s$. Thus, Ψ_θ is the root system of $\mathrm{C}_{\mathbf{X}^*}(s)^\circ$ by torus duality and* Digne–Michel [58, Proposition 3.5.1]*. Suppose that s is non-quasi-isolated. Then, Ψ_θ is empty, a union of orthogonal A_1-systems, or of type A_2, A_3. For $\Psi_\theta = A_1^k$ with $k \ge 0$, $|W_\theta|$ is a power of 2. In the path case $W_\theta = W(\Psi_\theta)$, and some $w \in W$ takes Ψ_θ to a standard subpath $\Psi_{w\theta}$.*

Proof. Since s is non-quasi-isolated, $\mathrm{C}_{\mathbf{X}^*}(s) \le \mathbf{L}^*$ for a proper Levi $\mathbf{L}^*$. The torus $\mathbf{T}^*$ is in $\mathbf{L}^*$, so some $w \in W$ makes $\mathbf{L}^*$ standard. Since T is split, the lift n_w is rational. In D_4, the standard $\mathbf{L}^*$ lies in an A_3-path or the outer factors A_1^3. In A_3, a diagonal lift of s has coordinates $x_1, \dots, x_4$, with roots $e_i - e_j$ in Ψ_θ exactly when $x_i = x_j$. Thus, a class of size $m = 3, 4$ gives A_{m-1} and an ordering gives a standard A_{m-1}. Classes of size ≤ 2 give orthogonal A_1-factors. Since $W(\Psi_\theta) \le W_\theta \le W(\mathbf{L}^*)$, in type A_3 we have $W_\theta = W(\Psi_\theta)$. In type A_2, the normalizer of $W(\Psi_\theta) \cong \mathfrak{S}_3$ in $\mathfrak{S}_4$ fixes the singleton, so $W_\theta = W(\Psi_\theta)$.

Next, take the simply connected cover $\pi : \widetilde{\mathbf{X}}^* \to \mathbf{X}^*$ and a lift $\widetilde{s}$ of s. The map $g \mapsto [\widetilde{g}, \widetilde{s}]$ on $\mathrm{C}_{\mathbf{X}^*}(s)$ is a homomorphism to $\ker \pi$, independent of $\widetilde{g}$. By Steinberg's theorem [40, §3.5], the kernel is $\pi(\mathrm{C}_{\widetilde{\mathbf{X}}^*}(\widetilde{s})) = \mathrm{C}_{\mathbf{X}^*}(s)^\circ$. Thus, $W_\theta / W(\Psi_\theta)$ embeds in $\ker \pi$ of order 4. In the orthogonal case $W(\Psi_\theta)$ is elementary abelian of 2-power order, so W_θ is a two-group. □

Proposition 6.126. *Given $\nu \in \mathrm{Irr}(T)$, we put $Y_G(\nu) = \mathrm{Ind}_B^X \nu$ and $Y_M(\nu) = \mathrm{Ind}_T^M \nu$, with ν inflated to B. Next, we put $Y_G(\mathcal{D}) = \bigoplus_{\nu \in \mathcal{D}} Y_G(\nu)$ and $Y_M(\mathcal{D}) = \bigoplus_{\nu \in \mathcal{D}} Y_M(\nu)$ over $\mathbb{Q}^{\mathrm{ab}}$. Write $E_G(\mathcal{D}), E_M(\mathcal{D})$ for their commutants. The functions f satisfy $f(tx) = \nu(t) f(x)$ for $t \in T$. Suppose that Ψ_θ is an orthogonal union. Let $\mathcal{D}$ consist of $W\theta$ and its translates under $F_r, \mathfrak{S}_3, \mathcal{H}_p$. Then, we find an isomorphism $E_G(\mathcal{D}) \cong E_M(\mathcal{D})$ with matched projections at $\nu \in \mathcal{D}$. Moreover, the actions of $D_{\mathrm{ad}}, F_r, \mathfrak{S}_3, \mathcal{H}_p$ correspond, including their products.*

Proof. Let n_i be the positive Tits lift of $s_i = s_{\alpha_i}$, with $n_i^2 = \alpha_i^\vee(-1)$. On $Y_M(\nu)$, the operator $L_i(\nu)$ acts by transport with n_i, so $L_i(s_i\nu) L_i(\nu) = \nu(n_i^2) I$. On $Y_G(\nu)$, write $T_i(\nu)$ for the Bruhat sum with lift n_i. If $\nu \circ \alpha_i^\vee \ne 1$, put $U_i(\nu) = q^{-1/2} T_i(\nu)$. By Lemma 6.112, $T_i(s_i\nu) T_i(\nu) = q\nu(n_i^2) I$ since $\sum_{u \in \mathbb{F}_q^\times} \nu(\alpha_i^\vee(u)) = 0$. If $\nu \circ \alpha_i^\vee = 1$, we put $U_i(\nu) = (2T_i(\nu) - (q-1)I)/(q+1)$. Thus, the eigenvalues $q, -1$ of $T_i(\nu)$ become $1, -1$ for $U_i(\nu)$. Consequently, $U_i(s_i\nu) U_i(\nu)$ and $L_i(s_i\nu) L_i(\nu)$ have equal scalars $\nu(n_i^2)$.

For orthogonal α, β, we have $s_\alpha \beta = \beta$ and $s_\beta \alpha = \alpha$, so $U_\alpha(s_\beta\nu) U_\beta(\nu) = U_\beta(s_\alpha\nu) U_\alpha(\nu)$. For adjacent α, β, the two galleries are $(\alpha, \alpha+\beta, \beta)$ and $(\beta, \alpha+\beta, \alpha)$. Put $S = \Psi_\nu \cap \{\alpha, \alpha+\beta, \beta\}$, with $|S| \le 1$. If $S = \varnothing$, both ordinary Bruhat products have coefficient $q^{-3/2}$. If $S = \{\gamma\}$, both full products have coefficient $2/(q(q+1))$. If $\gamma = \alpha$, both shortened products contain $T_\alpha T_\beta$, and for $\gamma = \beta$ they both have $T_\beta T_\alpha$. If $\gamma = \alpha + \beta$, the shortened products are $q^{-1} T_\alpha^2, q^{-1} T_\beta^2$. Since their signs have product $\nu((\alpha+\beta)^\vee(-1)) = 1$, the two products satisfy $q^{-1} T_\alpha^2 = q^{-1} T_\beta^2$. Thus, the braid holds for U_i.

The braid relations and squares of L_i, U_i coincide. Thus, each word reduces to cL_w or cU_w, with c the evaluated torus scalar. If $w\nu = \mu$, the local operators have distinct cosets Tn_w, and each U_w has nonzero leading coefficient at Bn_wB. Indeed, the identity in U_i contributes only to smaller lengths $\ell(w') < \ell(w)$. The dimension $|\{w \in W \mid w\nu = \mu\}|$ follows from [95, Lemma 5.2 and Problem 5.6]. Thus, $U_w \mapsto L_w$ gives $E_G(\mathcal{D}) \cong E_M(\mathcal{D})$ with matching projections at ν.

The pinned diagrams permute n_i, x_{α_i}, and F_r fixes n_i with $F_r(x_{\alpha_i}(u)) = x_{\alpha_i}(u^r)$. For t in the adjoint torus, $tn_it^{-1} = \alpha_i^\vee(\alpha_i(t)) n_i$. In the Bruhat sum T_i, t also changes u to $\alpha_i(t)u$. Thus, the factor $\alpha_i^\vee(\alpha_i(t))$ has equal evaluations on U_i, L_i, and value 1 if $\nu \circ \alpha_i^\vee = 1$. By [156, Lemma 2.2], $\mathcal{H}_p$ fixes $\sqrt{q}$ since $q \equiv 1 \bmod p$. Consequently, σ acts by $U_i(\nu) \mapsto U_i(\nu^\sigma)$ and $L_i(\nu)$ to $L_i(\nu^\sigma)$. Therefore, the correspondence on U_i, L_i is equivariant for all products of $D_{\mathrm{ad}}, F_r, \mathfrak{S}_3, \mathcal{H}_p$. □

Proposition 6.127. *Suppose that Ψ_θ has type A_{m-1}, with $m = 3, 4$. In each preceding W-orbit take the characters ν with standard Ψ_ν in $\mathcal{D}$. Then, we obtain that $E_G(\mathcal{D}) \cong E_M(\mathcal{D})$ with matching projections at ν. The actions of $D_{\mathrm{ad}}, F_r, \mathfrak{S}_3, \mathcal{H}_p$ match as in* Proposition 6.126*. Moreover, for $\lambda \vdash m$, it pairs S_q^λ with S_1^λ.*

Proof. By Lemma 6.125, each $W\theta$ contains a chosen ν. Moreover, $\mathcal{D}$ is stable under $D_{\mathrm{ad}}, F_r, \mathfrak{S}_3, \mathcal{H}_p$. For ν, μ in one W-orbit, the elements with $w\nu = \mu$ comprise a coset of $W_\nu = W(\Psi_\nu)$. Thus, exactly one w maps Ψ_ν^+ onto Ψ_μ^+, and its inversions avoid Ψ_ν. Match $J_w = q^{-\ell(w)/2} T_w$ with the Tits operator L_w. In a composition from ν through μ, the first gallery avoids Ψ_ν, and the second avoids $\Psi_\mu = w\Psi_\nu$. Thus, each root α in source coordinates has $\nu \circ \alpha^\vee \ne 1$. Each

deletion then contributes $\nu(\alpha^\vee(-1))$ to both normalized products. Hence, for J_uJ_w, L_uL_w, the torus factors agree, including when $uw\nu = \nu$.

If s_i is simple in Ψ_ν, write $s_{i'}$ for its image under w. Then, $ws_i = s_{i'}w$, with both products length-additive because $w\alpha_i = \alpha_{i'}$ is ambient simple and positive. Thus, $J_wT_i = T_{i'}J_w$ and $L_wL_i = L_{i'}L_w$. At ν, $W_\nu = W(\Psi_\nu)$ and $\nu \circ \alpha_i^\vee = 1$ for simple roots of Ψ_ν. Thus, the algebras are $\mathcal{H}_q(\mathfrak{S}_m)$ and $\mathbb{Q}^{\mathrm{ab}}[\mathfrak{S}_m]$. Since J_w, L_w are invertible, the Hom spaces to μ are $J_w \operatorname{End}_X(Y_G(\nu))$ and $L_w \operatorname{End}_M(Y_M(\nu))$.

Next, we put $K = \mathbb{Q}(\sqrt{q})$, $h = \binom{m}{2}$ and $c(\lambda) = \sum_{(i,j)\in\lambda}(j-i)$. For the longest w_0, put $A_\lambda(q) = q^{-(h+c(\lambda))/2}T_{w_0}$ on S_q^λ. By [208, Lemma 4.4, (4.1)], $A_\lambda(q)^2 = 1$, and the braid gives $A_\lambda(q)T_iA_\lambda(q)^{-1} = T_{m-i}$. Moreover, at $q = 1$, $A_\lambda(1) = w_0$ on S_1^λ. In the seminormal model of [90, §3], $A_\lambda(u)$ varies continuously for $u > 0$, since $[a]_u \neq 0$ for content differences $a \neq 0$, including at $u = 1$. Since $A_\lambda(u)^2 = 1$, the trace takes values in Z and is constant on $[1, q]$. Since $\dim S_u^\lambda = f^\lambda$, both eigenspace dimensions agree at $u = 1, q$. Taking K-bases for the eigenspaces yields $C_\lambda : S_q^\lambda \to S_1^\lambda$ with $C_\lambda A_\lambda(q) = A_\lambda(1)C_\lambda$. After conjugation by C_λ, we obtain the block isomorphism over K.

On each Ψ_ν, we choose C_λ in path coordinates. The relation $C_\lambda A_\lambda(q) = A_\lambda(1)C_\lambda$ intertwines the two orientations on Ψ_ν. Thus, J_w matches L_w on each block. The maps for $\mathfrak{S}_3$ also agree. Diagonals fix T_i, L_i on Ψ_ν and have equal torus scalars on J_w, L_w. Moreover, we have that F_r fixes the generators and $\mathcal{H}_p$ fixes K. Consequently, extending the block maps by $J_w \mapsto L_w$ intertwines $D_{\mathrm{ad}}, F_r, \mathfrak{S}_3, \mathcal{H}_p$ on $E_G(\mathcal{D}), E_M(\mathcal{D})$. □

Lemma 6.128. *Since* Section 6.5 *gives $Y_G(\mathcal{D}), Y_M(\mathcal{D})$, we compare their constituents (χ, ϕ) and degrees $\chi(1), \phi(1)$. Put $\Gamma = T_{\mathrm{ad}}^F \rtimes (\langle F_r\rangle \times \mathfrak{S}_3)$. Moreover, the multiplicity space U compares the stabilizers Γ_χ, Γ_ϕ. For $\succeq_c$, we compare the factors on $I' \times I'$ and the twists at (g, σ). By* Propositions 6.126 and 6.127*, $E_G(\mathcal{D}) \cong E_M(\mathcal{D})$ pairs the irreducibles χ, ϕ of $Y_G(\mathcal{D}), Y_M(\mathcal{D})$. Moreover, the associated models $\mathcal{P}, \mathcal{P}'$ have equal factors on $I' \times I'$, equal twists $\mu_{g,\sigma} = \mu'_{g,\sigma}$, and equal centralizing scalars at $z \in \mathrm{C}_A(X)$.*

Proof. The group Γ normalizes B, T, M and acts on $\mathcal{D}$. Given $a \in \Gamma$, we put $(a \cdot f)(x) = f(a^{-1}(x))$ on $Y_G(\mathcal{D})$ and $Y_M(\mathcal{D})$. Thus, we have actions of $A = X \rtimes \Gamma$ and $B_0 = M \rtimes \Gamma$, respectively. Moreover, $f^\sigma(x) = \sigma(f(x))$ gives the summand at ν^σ, with $(a \cdot f)^\sigma = a \cdot f^\sigma$. Over $\mathbb{Q}^{\mathrm{ab}}$, corresponding primitive central idempotents e of $E_G(\mathcal{D}), E_M(\mathcal{D})$ give χ, ϕ. The correspondence at e is $\Gamma \times \mathcal{H}_p$-equivariant. Thus, the summands are $V_\chi \otimes U, V_\phi \otimes U$ with common U, and $\Gamma_\chi = \Gamma_\phi =: J$. Moreover, $(\Gamma \times \mathcal{H}_p)_\chi = (\Gamma \times \mathcal{H}_p)_\phi$. By [141, Corollary 1.2], choose $\mathcal{P}'$ associated with ϕ, with factors $c'(y, z)$ of finite order in $\mathbb{Q}^{\mathrm{ab}}$. Then, the local action of $y \in J$ is $\mathcal{P}'(y) \otimes L_y$, with $L_1 = 1$. Indeed, division by $\mathcal{P}'(y)$ leaves an M-linear operator. On $\operatorname{End}(U)$, we write the action of y as conjugation by L_y. Thus, the global parent operator is $\mathcal{P}(y) \otimes L_y$ for a matrix $\mathcal{P}(y)$ on V_χ. Together with X, these give the associated $\mathcal{P}$ on $X \rtimes J$. Since both parent actions are genuine, $L_yL_z = c(y, z)L_{yz}$ gives factors $c(y, z)^{-1}$ for $\mathcal{P}, \mathcal{P}'$. Consequently, both factors on $M \rtimes J$ are the inflation of c^{-1}.

Given (g, σ) in the joint stabilizer, the two actions on $\operatorname{End}(U)$ coincide. Write $D = S \circ \sigma_U$, where σ_U acts on each coordinate of U and S comes from Skolem–Noether. Thus, D implements the common semilinear action. Given $y \in J$, we write $DL_yD^{-1} = b(y)L_{gyg^{-1}}$. For either $\mathcal{R} \in \{\mathcal{P}, \mathcal{P}'\}$, the parent map has form $S_\mathcal{R} \otimes D$. Hence, $S_\mathcal{R}\mathcal{R}(y)S_\mathcal{R}^{-1} = b(y)^{-1}\mathcal{R}(gyg^{-1})$. Since $\mu_{g,\sigma}|_M = \mu'_{g,\sigma}|_M = 1$, we have $\mu_{g,\sigma} = \mu'_{g,\sigma}$ on $M \rtimes J$.

If $(x, a) \in \mathrm{C}_A(X)$, $a = \mathrm{Int}(x)^{-1}$. Hence, $a \in T_{\mathrm{ad}}^F$, since its projections to $\langle F_r\rangle, \mathfrak{S}_3$ are trivial. The preimage of T_{ad}^F in X is T, so $\mathrm{C}_A(X) = \{(t^{-1}, \mathrm{Int}(t)) \mid t \in T\} \leq B_0$. On $Y_G(\nu)$ and $Y_M(\nu)$, this pair acts by $f(x) \mapsto f(t^{-1}x) = \nu(t)^{-1}f(x)$. Thus, the operator $\sum_{\nu\in\mathcal{D}} \nu(t)^{-1}e_\nu$ has the corresponding projections e_ν onto $Y_G(\nu), Y_M(\nu)$. Consequently, the action D_t on U agrees, and $D_tL_{\mathrm{Int}(t)}^{-1}$ gives both scalars at $(t^{-1}, \mathrm{Int}(t))$, including $t \in \mathrm{Z}(X)$.

Next, we let $\Delta = \operatorname{Aut}(X)_P$ and let $\epsilon : A \to \operatorname{Aut}(X)$ be conjugation on X. By Lemma 6.122, Δ normalizes $Q, \mathbf{T}$. Given $a \in \Delta$, we write $a = a_0F_r^j\gamma$, with $F_r^j\gamma \in \Gamma$. Since a_0 normalizes $\mathbf{T}_{\mathrm{ad}}$, we have $a_0 = \mathrm{Int}(n_w)t$, with $n_w \in M$ a rational Tits lift and $t \in T_{\mathrm{ad}}^F$. Thus, $\Delta \leq \epsilon(B_0)$. Put $B_1 = B_0 \cap \epsilon^{-1}(\mathrm{Int}_X(M)\Delta)$ and $A_1 = XB_1$, where $\mathrm{Int}_X(M)$ acts on X. Then, $A_1 \cap B_0 = B_1$ and $\epsilon(A_1) = \mathrm{Inn}(X)\Delta$. In particular, $\epsilon(B_1) = \mathrm{Int}_X(M)\Delta$. On restriction to $(A_1)_{\chi_p^\mathcal{H}}, (B_1)_{\phi_p^\mathcal{H}}$, the images are those of $\widehat{A} = X \rtimes \Delta_{\chi_p^\mathcal{H}}$ and $\widehat{B} = M \rtimes \Delta_{\chi_p^\mathcal{H}}$. Moreover, $(x, a) \in \mathrm{C}_{\widehat{A}}(X)$ has $a = \mathrm{Int}(x)^{-1} \in \Delta$, so $x \in \mathrm{N}_X(P) \leq M$. Thus, $\mathrm{C}_{\widehat{A}}(X) \leq \widehat{B}$, and [141, Theorem 2.9] gives $\succeq_c$ on $\widehat{A}, \widehat{B}$, including at $p = 3$. □

Proposition 6.129. *Given an admissible orbit $W\theta_s$ with s non-quasi-isolated and $p \mid q - 1$,* Lemma 6.128 *matches the sets of constituents χ, ϕ of p'-degree. On a path A_{m-1}, in particular, the labels λ are those with $p \nmid [W : \mathfrak{S}_m]f^\lambda$.*

Proof. If $\Psi_\theta = A_1^k$, the order $|W_\theta|$ is a power of 2 by Lemma 6.125. Since $|W| = 192$, we have $3 \mid [W : W_\theta]$, so an eligible prime p satisfies $p \geq 5$. Thus, the local degrees are p'-numbers by Proposition 6.123. Choose $\widetilde{s} \in \widetilde{T}^*$ for a regular inclusion $X \leq \widetilde{X}$. Since $p \geq 5$ and $p \mid q-1$, the split torus $\widetilde{T}^*$ contains a Sylow p-subgroup of $\widetilde{X}^*$. Thus, $|\widetilde{X}^* : \mathrm{C}_{\widetilde{X}^*}(\widetilde{s})|_p = 1$. The connected centralizer has split $A_1(q)$-factors with unipotent degrees $1, q$. By Jordan decomposition [58, Proposition 11.5.6], each $\widetilde{\chi} \in \mathcal{E}(\widetilde{X}, \widetilde{s})$ has $\widetilde{\chi}(1)_p = 1$. By Lemma 6.125, $C_{\mathbf{X}^*}(s)/C_{\mathbf{X}^*}(s)^\circ$ is a 2-group, and each restriction index divides $|D_{\mathrm{ad}}|$. Hence, the constituents of $Y_G(\theta_s)$ have p'-degree and lie in $\mathcal{E}(X, s)$. Since the local induction covers $\mathrm{Irr}(M \mid W\theta_s)$, the constituents of $Y_G(\theta_s)$ exhaust $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ by the count in Proposition 6.123.

If $\Psi_\theta = A_{m-1}$, let L be the standard Levi of P_L. The simply connected $\mathbf{L}_{\mathrm{der}}$ has split torus $T \cap \mathbf{L}_{\mathrm{der}}^F = \langle \alpha^\vee(\mathbb{F}_q^\times) \mid \alpha \in \Delta_L \rangle$. Since $\theta \circ \alpha^\vee = 1$ for $\alpha \in \Delta_L$, the product $L = T\mathbf{L}_{\mathrm{der}}^F$ gives $\widetilde{\theta}_L(tl) = \theta(t)$ for $t \in T$ and $l \in \mathbf{L}_{\mathrm{der}}^F$. Thus, the constituents on L are $\widetilde{\theta}_L u_\lambda$. Induction from P_L to X embeds $\mathrm{End}_L(\mathrm{Ind}_{B\cap L}^L \theta)$ in $\mathrm{End}_X(Y_G(\theta))$. Since $W_\theta = \mathfrak{S}_m$, both endomorphism algebras have dimension $m!$. Consequently, the characters $\widetilde{\theta}_L u_\lambda$ induce to distinct irreducibles χ_λ, of degrees $[X : P_L]u_\lambda(1)$.

The Bruhat formula [40, §2.8] gives $[X : P_L] = P_W(q)/P_{\mathfrak{S}_m}(q)$, where $P_W(q) = [2]_q[4]_q^2[6]_q$, $P_{\mathfrak{S}_m}(q) = \prod_{j=2}^m [j]_q$ and $[k]_q = (q^k - 1)/(q-1)$. Since p is odd and $p \mid q-1$, we have $v_p([k]_q) = v_p(k)$. Thus, $[X : P_L]_p = [W : \mathfrak{S}_m]_p$. In Green's formula [84, Theorem 14], the factors $[k]_q$ and $[h]_q$ give $u_\lambda(1)_p = (f^\lambda)_p$. Since $\theta \circ \alpha^\vee = 1$ on A_{m-1}, the local module S^λ gives $\phi_\lambda(1) = [W : \mathfrak{S}_m]f^\lambda$ by Proposition 6.123. Consequently, $\chi_\lambda(1)_p = \phi_\lambda(1)_p$, including at $p = 3$. Thus, the characters χ_λ of p'-degree exhaust $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ by the count in Proposition 6.123. □

Theorem 6.130. *Let q be odd and let $p \mid q-1$ be odd. Then, for $X = \mathrm{Spin}_8^+(q)$ and $M = \mathrm{N}_X(Q)$, there is an $\mathrm{Aut}(X)_P \times \mathcal{H}_p$-equivariant bijection $\mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(M)$ satisfying $\succeq_c$ at every pair (χ, ϕ).*

Proof. The partition of $\mathrm{Irr}_{p'}(X)$ over $[s]$ corresponds to the partition of $\mathrm{Irr}_{p'}(M)$ over $W\theta_s$ in Proposition 6.123. If $s = 1$, the result of [156, Theorem D] gives the bijection on $\mathcal{E}(X, 1) \cap \mathrm{Irr}_{p'}(X)$. Moreover, Corollary 6.121 gives the map on $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ for quasi-isolated $s \neq 1$. For the other s, Lemma 6.125 gives $\Psi_{\theta_s} = A_1^k$ or A_{m-1}. Thus, Proposition 6.129 pairs $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ with $\mathrm{Irr}_{p'}(M \mid W\theta_s)$ and Lemma 6.128 gives $\succeq_c$. Since the map on $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ is equivariant under the stabilizer of $[s]$, it extends to each class $[s]^a$. Moreover, $P^\gamma = P^x$ for some $x \in X$ by Sylow conjugacy, so $\gamma\,\mathrm{Int}(x)^{-1} \in \mathrm{Aut}(X)_P$ for each diagram element $\gamma \in \mathfrak{S}_3$. Thus, the union over $[s]$ is an $\mathrm{Aut}(X)_P \times \mathcal{H}_p$-equivariant bijection from $\mathrm{Irr}_{p'}(X)$ to $\mathrm{Irr}_{p'}(M)$. □

Lemma 6.131. *Let $X = \mathrm{Spin}_8^+(q)$ for q odd and $d = \mathrm{ord}_p(q) = 4$. Then, by* Proposition 6.123, *$p \geq 5$, $P = Q \leq T$ and $M = \mathrm{N}_X(P)$. The torus $\mathbf{T}$ has two negative cycles of length 2, so $p \mid |T^*|$ and we allow $p \mid |s|$. Write $s = ju$ with j of 2-power order and u of odd order. If $u \neq 1$, we have that $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(u)$ is a connected proper Levi containing $\mathrm{C}_{\mathbf{X}^*}(s)$. Either $\mathbf{L}^* = \mathbf{T}^*$ or its roots are $\{\pm\alpha, \pm\beta\}$ with $\alpha \perp \beta$. Moreover, F^* acts by $\{\pm\alpha\} \mapsto \{\pm\beta\}$. For the dual Levi $\mathbf{L}$, we have $I = \mathrm{Stab}_M(\theta_s) = \mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$. Thus, we obtain that $|I/T| = 1$ if $\mathbf{L} = \mathbf{T}$, and $|I/T| = 2$ otherwise.*

Proof. Put $h = (q^2+1)/2$, so $\gcd(h, 2(q^2-1)) = 1$. The isogenies $\mathbf{T} \to \mathbf{T}_{\mathrm{SO}} \to \mathbf{T}^*$ have 2-power kernels. Thus, u has a unique lift u_0 of odd order in $\mathbf{T}_{\mathrm{SO}}^F$. A coordinate $a \neq 1$ of u_0 has eigenvalues $\Omega(a) = \{a, a^q, a^{-1}, a^{-q}\}$, where $|\Omega(a)| = 4$ since $q^2 \equiv -1 \bmod |a|$. By [58, Proposition 11.2.1(iii) and Remark 11.2.2(i)], the group $\mathrm{C}_{\mathbf{X}^*}(u)/\mathrm{C}_{\mathbf{X}^*}(u)^\circ$ embeds in $\mathrm{Z}(\mathbf{X})$ and has exponent dividing $|u|$. Hence, $\mathbf{L}^*$ is connected.

Write a_1, a_2 for the two coordinates of u_0. The roots of $\mathbf{L}^*$ are $\gamma \in D_4$ with $\gamma(u) = 1$ by [58, Proposition 3.5.1]. If $a_1, a_2 \neq 1$ and $\Omega(a_1) \neq \Omega(a_2)$, all eigenvalues of u_0 are distinct, so $\mathbf{L}^* = \mathbf{T}^*$. If $a_i = 1$, the 1-eigenspace of u_0 gives SO_4 in the natural group, with roots $\pm(e_k - e_l), \pm(e_k + e_l)$ on the i-th cycle. The signed cycle $(e_k, e_l) \mapsto (e_l, -e_k)$ acts by $e_k - e_l \mapsto e_k + e_l$ and $e_k + e_l$ to $-(e_k - e_l)$. If $\Omega(a_1) = \Omega(a_2)$ with $a_1, a_2 \neq 1$, their reciprocal eigenvalues give two GL_2 factors in SO_8. Frobenius exchanges $\{a_1, a_1^{-1}\}$ and $\{a_1^q, a_1^{-q}\}$. Thus, both cases give A_1^2, and the signed action $q^{-2}(F^*)^2$ is -1 on the roots $\pm\alpha, \pm\beta$. Their real span meets D_4 in exactly $\{\pm\alpha, \pm\beta\}$. Consequently, $\mathrm{C}_{\mathbf{X}^*}(\mathrm{Z}(\mathbf{L}^*)^\circ) = \mathbf{L}^*$, a Levi. Since $u \in \langle s \rangle$, we have $\mathrm{C}_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$. Moreover, an automorphism with $s \mapsto s'$ acts by $u \mapsto s'_{2'}$. If $\langle s' \rangle = \langle s \rangle$, we have $\langle s'_{2'} \rangle = \langle u \rangle$, so $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(s'_{2'})$.

The subgroup of rotations in $W^F = M/T$ consists of $(i_1, i_2) \in (\mathbb{Z}/4)^2$ with $i_1 + i_2$ even. The block exchange $(1\ 2)$ doubles its order, so $|W^F| = 16$. The i_k-th rotation acts by $a_k \mapsto a_k^{q^{i_k}}$. Thus, different orbits $\Omega(a_1), \Omega(a_2)$ for $a_1, a_2 \neq 1$ give $W_u^F = 1$. If $a_i = 1$ and $a_{3-i} \neq 1$, $i_{3-i} = 0$ and $i_i \in \{0, 2\}$. If $a_2 = a_1^{q^j} \neq 1$, exactly one exchange fixes u, with opposite rotation exponents $j, -j$. Moreover, $T_2 = \mathrm{Z}(X)$ since $|T| = (q^2+1)^2$. Thus, W^F fixes every 2-power

order character of T, so $W_s^F = W_u^F$. By Lang's theorem for $\mathbf{T}$ [58, Theorem 4.2.9], $\mathrm{N}_{\mathbf{L}}(\mathbf{T})^F/T = W(\mathbf{L},\mathbf{T})^F$. This quotient is contained in W_s^F and has order 1 for $\mathbf{L} = \mathbf{T}$, or 2 for A_1^2. Hence, $\mathrm{N}_{\mathbf{L}}(\mathbf{T})^F = I$. □

Proposition 6.132. *Suppose that $u \neq 1$ and put $L = \mathbf{L}^F$. There is a bijection from $\mathcal{E}(L,s) \cap \mathrm{Irr}_{p'}(L)$ to $\mathrm{Irr}(I \mid \theta_s)$ with the $\succeq_c$ relation for automorphisms of L, T. Thus, $\varepsilon_X \varepsilon_L R_{\mathbf{L}}^{\mathbf{X}}$ and Ind_I^M give a bijection from $\mathcal{E}(X,s)$ to $\mathrm{Irr}(M \mid W^F\theta_s)$, jointly equivariant on the orbit of $[s]$ under $\mathrm{Aut}(X)_P \times \mathcal{H}_p$.*

Proof. If $L = I = T$, we put $\beta = \rho = \theta_s$. Otherwise, we put $R = [\mathbf{L},\mathbf{L}]^F$ and $T_R = T \cap R$. The simply connected factors of $[\mathbf{L},\mathbf{L}]$ are exchanged by F. Thus, $R \cong \mathrm{SL}_2(q^2)$ and $q^{-2}F^2 = -1$ on either root torus. In particular, T_R is nonsplit and $|T_R| = q^2 + 1 = 2h$. By Lang's theorem on $[\mathbf{L},\mathbf{L}]$ and its torus $\mathbf{T} \cap [\mathbf{L},\mathbf{L}]$ [58, Lemma 4.2.13], $L = TR$. Since $T_2 = \mathrm{Z}(X)$, the image of T lies in the odd subgroup of $T_{R,\mathrm{ad}}$, where $|T_{R,\mathrm{ad}}| = 2h$. The image of T_R has order h, so $C = \mathrm{C}_T(R)$ gives $T = CT_R$ and $L = CR$. Also, $C \cap R = \mathrm{Z}(R) = \langle z_R \rangle$ and $I = C\mathrm{N}_R(T_R)$. Since $z_R \in T_2$, we have $z_R \in \mathrm{Z}(X)$.

With $\kappa = \theta_s|_C$, we write $\lambda = \theta_s|_{T_R}$. As I fixes θ_s, $\lambda = \lambda^{-1}$, so λ is 1 or the quadratic character ϵ of T_R. Moreover, $\epsilon(z_R) = -1$ since h is odd. Let $U_R = \mathrm{N}_R(T_R)$ and $P_R \in \mathrm{Syl}_p(T_R)$. Since $\mathrm{C}_R(P_R) = T_R$, we have $U_R = \mathrm{N}_R(P_R)$. If $\lambda = 1$, the global rank-one constituents are $1_R, \mathrm{St}_R$ and the local pair over 1_{T_R} is $1_{U_R}, \mathrm{sgn}_{U_R/T_R}$. If $\lambda = \epsilon$, the pair η_+, η_- of odd Weil characters has degree $(q^2-1)/2$. Its local characters ρ_+, ρ_- extend ϵ, with $\rho_\pm(n) = \pm i$ for $n \in U_R \setminus T_R$, since $n^2 = z_R$.

By [186, Theorem 4.2], the bijection at $p \mid q^2+1$ pairs $1_R, \mathrm{St}_R$ with $1_{U_R}, \mathrm{sgn}_{U_R/T_R}$, respectively, and pairs $\{\eta_+, \eta_-\}$ with $\{\rho_+, \rho_-\}$. Thus, each pair satisfies $\succeq_c$ at $\mathrm{Aut}(R)_{P_R}$. At $q = 3$, [186, Theorem 4.4] gives these pairs for $R = 6.A_6/K \cong \mathrm{SL}_2(9)$, with $K = \mathrm{Z}(6.A_6)_3$. Since K is characteristic, $\mathrm{Aut}(6.A_6) \cong \mathrm{Aut}(R)$. The associated matrices $\mathcal{P}, \mathcal{P}'$ are constant on K-cosets, since $\mathcal{P}|_K = \mathcal{P}'|_K = 1$. Thus, the factors α, α', mixed functions $\mu_{a,\sigma}, \mu'_{a,\sigma}$ and centralizing scalars factor through $6.A_6 \to R$, giving $\succeq_c$ on R, U_R.

Given a pair (ξ, ξ') of characters of R, U_R just described, we let D be the common stabilizer of (κ, ξ) and (κ, ξ') in $\mathrm{Aut}(L)_T$. Since D fixes T_R, its image on R lies in $\mathrm{Aut}(R)_{P_R}$. Both rank-one representations have scalar $\kappa(z)$ at $z \in C \cap R$. Thus, from the rank-one models $\mathcal{P}_R, \mathcal{P}'_R$, define $\mathcal{P}_L(cx,a) = \kappa(c)\mathcal{P}_R(x, a|_R)$ and $\mathcal{P}_I(cy,a) = \kappa(c)\mathcal{P}'_R(y, a|_R)$, for $c \in C$, $x \in R$, $y \in U_R$ and $a \in D$ respectively. Since $\kappa^a = \kappa$, their factor sets agree on $D \times D$. Moreover, for (b,σ) jointly stabilizing (κ,ξ) and (κ,ξ'), we have $\kappa^{(b,\sigma)} = \kappa$, so $\mu_{b,\sigma} = \mu'_{b,\sigma}$. If $(cx,a) \in \mathrm{C}_{L \rtimes D}(L)$, $a = \mathrm{Int}(x)^{-1}$, so $x \in U_R$ since $a(T_R) = T_R$. Thus, both scalars are $\kappa(c)\omega_R(x, a|_R)$ with ω_R from the comparison on R, U_R. Consequently, the tensor $\mathcal{T}(l,i;a) = \mathcal{P}_L(l,a) \otimes \mathcal{P}_I(i,a)^\vee$ is a representation on $(L \times I) \rtimes D$ with $\mathcal{T}^{(b,\sigma)} \cong \mathcal{T}$. At $((i^{-1}, i^{-1}), \mathrm{Int}(i))$ the two scalars cancel, giving $\mathcal{T}(i^{-1}, i^{-1}; \mathrm{Int}(i)) = 1$. If $L = I = T$, we set $\mathcal{T}(l,i;a) = \theta_s(l)\theta_s(i)^{-1}$ for $a \in \mathrm{Aut}(T)_{\theta_s}$.

If $L \neq T$, the constituents of $R_{\mathbf{T}}^{\mathbf{L}}(\theta_s)$ lie over κ. By [58, Proposition 11.3.10], $R_{\mathbf{T}}^{\mathbf{L}}(\theta_s)|_R = R_{\mathbf{T}\cap[\mathbf{L},\mathbf{L}]}^{[\mathbf{L},\mathbf{L}]}(\lambda)$. Thus, ξ, ξ' with κ give (β, ρ) on L, I. In either case, $\beta(1) \in \{1, q^2, (q^2-1)/2\}$, so $p \nmid \beta(1)$. Since $\mathrm{C}_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$, we obtain the bijection $\chi = \varepsilon_X \varepsilon_L R_{\mathbf{L}}^{\mathbf{X}}(\beta)$ by [58, Theorem 11.4.3], with $\chi(1) = |X : L|_{r'}\beta(1)$ by [58, Proposition 10.2.9], where $r = \mathrm{char}\,\mathbb{F}_q$. As $P \leq T \leq L$, we have $p \nmid |X : L|$. By Proposition 6.123, $|\mathcal{E}(X,s) \cap \mathrm{Irr}_{p'}(X)| = 1$ for $\mathbf{L} = \mathbf{T}$ and 2 otherwise. By Clifford correspondence [95, Theorem 6.11], $\phi = \mathrm{Ind}_I^M \rho$ with $\phi(1) = [M : I]\rho(1)$ and $\phi \in \mathrm{Irr}(M \mid W^F\theta_s)$. Since (a,σ) fixes $[s]$, we choose $m \in M$ with $(\theta_s^{(a,\sigma)})^m = \theta_s$. Two choices differ by $i \in I$, whose inner action fixes $\mathrm{Irr}(L), \mathrm{Irr}(I)$. Moreover, σ replaces s by s^k with $\langle s^k \rangle = \langle s \rangle$, so $\mathrm{C}_{\mathbf{X}^*}(u^k) = \mathbf{L}^*$. By equivariance of $R_{\mathbf{L}}^{\mathbf{X}}$ on the series and of Ind_I^M, the pair associated with $(\beta^{(a,\sigma)}, \rho^{(a,\sigma)})$ is $(\chi^{(a,\sigma)}, \phi^{(a,\sigma)})$. Thus, the map is equivariant on $[s]^{\mathrm{Aut}(X)_P \times \mathcal{H}_p}$. □

Lemma 6.133. *Let $u \in \mathbf{T}^*$ in type D_4, with $\gcd(|u|, 6) = 1$, and suppose that $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(u)$ is a Levi. Then, its dual $\mathbf{L}$ is a Levi of a parabolic $\mathbf{P}_{\mathbf{L}} = \mathbf{L}\mathbf{U}$. Every element of $\mathbf{L}$ normalizes $\mathbf{U}$. Moreover, each automorphism of $\mathbf{X}$ with field degree 0 that fixes $\mathbf{T}$ and u normalizes $\mathbf{U}$.*

Proof. With $K = (W(D_4) \rtimes \mathfrak{S}_3)_u$, we put $\mathbf{D} = ((\mathbf{T}^*)^K)^\circ$. The morphism $N_K(t) = \prod_{g \in K} g(t)$ from $\mathbf{T}^*$ has connected image in $\mathbf{D}$. Since $|K| \mid 192 \cdot 6$ and $\gcd(|u|, 6) = 1$, we choose $c|K| \equiv 1 \bmod |u|$. Thus, $N_K(u^c) = u$, so $u \in \mathbf{D}$. If $\alpha(u) = 1$, $s_\alpha \in K$ and $\alpha|_{\mathbf{D}} = 1$. Conversely, if $\alpha(u) \neq 1$, then $\alpha|_{\mathbf{D}} \neq 1$. Thus, we obtain a vector in $Y(\mathbf{D}) \otimes \mathbb{Q}$ whose zero roots are precisely $\Phi(\mathbf{L}^*, \mathbf{T}^*)$. Since D_4 is simply laced, the invariant pairing $(\cdot,\cdot)$ gives $Y(\mathbf{T}^*) \otimes \mathbb{Q} \cong Y(\mathbf{T}) \otimes \mathbb{Q}$. Thus, choose $\lambda \in Y(\mathbf{T})$ with $K \leq \mathrm{Stab}(\lambda)$ and $\langle \alpha, \lambda \rangle = 0$ exactly at the roots of $\mathbf{L}$. By [137, Theorems 13.33 and 25.1], $P(\lambda)$ has Levi $\mathbf{L} = C_{\mathbf{X}}(\lambda(\mathbb{G}_m))$ and radical $\mathbf{U} = U(\lambda)$. The toral components act trivially on $Y(\mathbf{T})$ and K fixes λ. Thus, the indicated automorphisms fix $U(\lambda)$, and $\mathbf{L}$ normalizes it. □

From the pairs (β,ρ) in Proposition 6.132, we now obtain $\succeq_c$ on X, M. The odd part u of s has centralizer $\mathbf{L}^*$. By Lemma 6.133, choose the radical $\mathbf{U}$ of $\mathbf{P}_\mathbf{L}$. Thus, a field-free automorphism a fixing $\mathbf{T}, u$ satisfies $a(\mathbf{U}) = \mathbf{U}$ and $a(V^\mathbf{U}) = V^\mathbf{U}$ on a module V with this action. Let $I = M_{\theta_s} \leq L$ and $(\beta,\rho) \in \mathcal{E}(L,s) \times \mathrm{Irr}(I \mid \theta_s)$. Write $\chi = \varepsilon_X\varepsilon_L R^\mathbf{X}_\mathbf{L}(\beta)$ and $\phi = \mathrm{Ind}_I^M \rho$. Assume that $\mathrm{C}_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$ and $\mathbf{L}$ is the Levi for every generator s' of $\langle s\rangle$. Thus, by Digne–Michel [58, Theorem 11.4.3], we obtain $\mathcal{E}(L,s) \cong \mathcal{E}(X,s)$ via signed induction. Assume that $\mathcal{E}(X,s)$ and $\mathrm{Irr}(M \mid M\theta_s)$ have the equivariance of Proposition 6.132. Put $\Delta = \mathrm{Aut}(X)_P$, $D_0 = \Delta_\chi = \Delta_\phi$, $A = X \rtimes D_0$ and $B = M \rtimes D_0$. Let $B' = B_{\theta_s}$ and $H = LB'$. Since D_0 fixes the M-orbit of θ_s, we have $B = MB'$ and $B' \cap M = I$. Moreover, B' fixes β since $\chi = \varepsilon_X\varepsilon_L R^\mathbf{X}_\mathbf{L}(\beta)$ determines β. Hence, B' fixes ρ as well by Clifford correspondence (Isaacs [95, Theorem 6.11]). Hence, H is a subgroup. We have $A = XH$, $H \cap X = L$, and the projections give $H/L \cong B'/I \cong B/M \cong A/X \cong D_0$. If $(x^{-1}, \mathrm{Int}(x)) \in \mathrm{C}_A(X)$, then $\mathrm{Int}(x) \in D_0$, so $x \in M$. Thus, $\mathrm{C}_A(X) \leq B'$, since it acts trivially on θ_s. Let $H_1 = \ker(d_F : H \to \mathbb{Z}/f)$, where $q = r^f$.

Proposition 6.134. *Suppose that H_1 normalizes the radical $\mathbf{U}$ of a parabolic with Levi $\mathbf{L}$. Suppose that $\beta \boxtimes \rho^\vee$ has an extension V to $H \times_{D_0} B'$, with $V^{(b,\sigma)} \cong V$ for each (b,σ) stabilizing (β,ρ,θ_s). Assume that $\{(z,z) \mid z \in \mathrm{C}_A(X)\} \leq \ker(V)$. Then, the models on A, B give $\succeq_c$ at (χ,ϕ).*

Proof. Given $y \in H_1$, the field degree 0 lift K_y lies in $\mathbf{X}^F_{\mathrm{ad}} \rtimes \Gamma_{\mathrm{graph}}$, with $\Gamma_{\mathrm{graph}} = \mathfrak{S}_3$ in type D_4. The field degree 0 algebraic group action is faithful on X. Thus, $K_z = 1$ for $z \in \mathrm{C}_A(X)$ and $K_l = \mathrm{Int}(l)$ for $l \in L$. Also, $K_yF = FK_y$. If $|H/H_1| = m > 1$, we choose $a \in B'$ of degree f/m and let $\psi = \mathrm{Int}(t)\gamma F_r^{f/m}$ be its lift, with $t \in \mathbf{X}^F_{\mathrm{ad}}$ and γ pinned. Thus, $\psi(\mathbf{L}) = \mathbf{L}$, since a fixes θ_s. Put $v = a^m$. If $m = 1$, put $a = v = 1$ and $\psi = F$. Since the degree 0 lifts are unique and $F = F_r^f$, we have $\psi^m = K_vF$ and $\psi K_y = K_{aya^{-1}}\psi$ for $y \in H_1$.

The lifts K_y, ψ satisfy [208, Lemma 4.19(i)–(iv)], with $N = X$ and the subgroup for β equal to H. By Steps 1–2, we put $\tau(x_0,\dots,x_{m-1}) = (x_1,\dots,x_{m-1},K_v(x_0))$ and $F' = \tau^{-1}\,\mathrm{diag}(\psi)$. Thus, $x \mapsto \widehat{x} = (x,\psi x,\dots,\psi^{m-1}x)$ gives isomorphisms from X, L to the F'-fixed groups of $\mathbf{X}^m, \mathbf{L}^m$. The action $D(y)_j = K_{a^jya^{-j}}$ for $y \in H_1$ and $D(a) = \tau$ gives the action on $Y^{F'}_{\mathbf{U}^m}$. In particular, $A \times_{D_0} H$ acts by left X and inverse right L translations on $X \times L$, and the diagonal $\mathrm{C}_A(X)$ acts trivially. Let $\Xi_\mathbf{U}$ denote the alternating cohomology character of $Y^{F'}_{\mathbf{U}^m}$. Moreover, by [53, Proposition 3.3], $\Xi_\mathbf{U}(g,h) \in \mathbb{Z}$. Since $\mathrm{C}_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$, [208, proof of Lemma 4.19(b)] gives $\langle \Xi_\mathbf{U}|_{X\times L}, \chi \boxtimes \beta^\vee\rangle = \pm 1$ on the rational series of s. We choose the cohomology prime ℓ coprime to $r|X||L||A||H||s|$, so the series of s is available also when $p \mid |s|$.

For a mixed operation stabilizing (χ,ϕ), we choose the correction b by M so that $\theta_s^{(b,\sigma)} = \theta_s$. Thus, $\theta_s^b = \theta_s^{\sigma^{-1}}$, and b normalizes B', H, H_1 and $\mathbf{L}$. Lift b to $Z = \mathrm{Int}(t_b)\gamma_b F_r^j$, where $0 \leq j < f$. If $0 \leq i < m$, we let $h_i = a^iba^{-i}b^{-1} \in H_1$ and $Z_i = K_{h_i}Z$. Since Z_i and $\psi^iZ\psi^{-i}$ have equal degrees and actions on X, we have $Z_i = \psi^iZ\psi^{-i}$ on geometric points. In particular, $Z_i = K_{h_i}Z$ is a morphism. Thus, whenever $i > 0$, $Z_i\psi = \psi Z_{i-1}$. Moreover, $\psi^m = K_vF$ and $ZF = FZ$ give $ZK_v^{-1}\psi = K_v^{-1}\psi Z_{m-1}$. Hence, $\mathcal{Z} = \prod_i Z_i$ satisfies $\mathcal{Z}\,F' = F'\,\mathcal{Z}$.

Given $y \in H_1$, the coordinate identities give $\mathcal{Z}D(y) = D(byb^{-1})\mathcal{Z}$. Given a, the corresponding identity is $\mathcal{Z}\tau = D(bab^{-1}a^{-1})\tau\mathcal{Z}$. Since b normalizes H_1, each K_{h_i} normalizes $Z\mathbf{U}$. Thus, $\mathcal{Z}$ maps $Y^{F'}_{\mathbf{U}^m}$ to $Y^{F'}_{(Z\mathbf{U})^m}$, with action conjugated by b. Since $\mathcal{Z}$ is a universal homeomorphism, $\mathcal{Z}^*$ is an isomorphism on $H^j_c(-,\overline{\mathbb{Q}}_\ell)$. Consequently, $\Xi_{Z\mathbf{U}}(bgb^{-1},bhb^{-1}) = \Xi_\mathbf{U}(g,h)$.

Since $D(H)$ stabilizes $\mathbf{U}^m, (Z\mathbf{U})^m$ and fixes the series of β, [160, Lemma 3.1] compares the $D(H)$-actions by Lemma 7.168. Indeed, $\mathrm{C}_{\mathbf{X}^*}(s_i) \leq \mathbf{L}^*$ for each i gives the hypotheses of [19, Corollary 6.5 and Remark 6.7] on $\mathcal{E}(L,s)$. Moreover, $\dim Y^{F'}_{\mathbf{U}^m} = \dim Y^{F'}_{(Z\mathbf{U})^m}$ and $\overline{\mathbb{Q}}_\ell(1)$ is trivial under $A \times_{D_0} H$. Thus, at $\vartheta = \chi \boxtimes \beta^\vee$, we have $\Xi_\mathbf{U}[\vartheta] = \Xi_{Z\mathbf{U}}[\vartheta]$, where $\Xi[\vartheta]$ denotes the isotypic part. Since $\mathrm{C}_{\mathbf{X}^*}(s^k) = \mathrm{C}_{\mathbf{X}^*}(s)$ whenever $\langle s^k\rangle = \langle s\rangle$, the comparison also holds on $\mathcal{E}(L,s^k)$. Since $\Xi_\mathbf{U}^\sigma = \Xi_\mathbf{U}$ and $\vartheta^{(b,\sigma)} = \vartheta$, the projection at ϑ is fixed by (b,σ). By Proposition 2.4, we obtain an extension V_1 of ϑ with $V_1^{(b,\sigma)} \cong V_1$ and $\{(z,z) \mid z \in \mathrm{C}_A(X)\} \leq \ker(V_1)$.

By [141, Corollary 1.2], we choose an associated $\mathcal{P}_\rho$ on B', with factors α of finite order inflated from D_0. On $(h,b') \in H \times_{D_0} B'$, the ordinary extension of $\beta \boxtimes \rho^\vee$ has matrix $\mathcal{P}_\beta(h) \otimes \mathcal{P}_\rho(b')^\vee$. Indeed, division by $\mathcal{P}_\rho(b')^\vee$ leaves a matrix in $\mathrm{End}(V_\beta) \otimes 1$ by Schur's lemma for I. Replacing b' by ib' multiplies the dual factor by $\rho(i)^\vee$, so $\mathcal{P}_\beta(h)$ is independent of b'. Since the tensor is ordinary, $\mathcal{P}_\beta$ has factor set α. Similarly, the extension on $A \times_{D_0} H$ has matrices $\mathcal{P}_\chi(g) \otimes \mathcal{P}_\beta(h)^\vee$ with factor set α for $\mathcal{P}_\chi$. At (b,σ), we have $V^{(b,\sigma)} \cong V$ and $V_1^{(b,\sigma)} \cong V_1$. Thus, $\mu_\chi = \mu_\beta = \mu_\rho$ on D_0. At $z \in \mathrm{C}_A(X)$, the two tensors act trivially on (z,z), so the three scalar values at z agree.

Induce $\mathcal{P}_\rho$ from B' to B on the finite central extension of D_0 defined by α. Thus, the projective representation $\mathcal{P}_\phi$ has factor set α, and $\mathcal{P}_\phi|_M$ affords $\mathrm{Ind}_I^M \rho = \phi$ since $B = MB'$. The twist μ_ρ is inflated from D_0. Hence, $\mu_\rho(d)$ is constant on the summands over M/I for $d \in D_0$, and induction on the cover gives $\mu_\phi = \mu_\rho$ at (b,σ). A second

correction differs by $m \in M$, with intertwiner $\mathcal{P}_\phi(m)$. Moreover, $z \in \mathrm{C}_A(X)$ has $zm = mz$ for each representative $m \in M$, so $\mathcal{P}_\phi(z)$ has scalar $\mathcal{P}_\rho(z)$ on each summand. From $\mathcal{P}_\chi, \mathcal{P}_\phi$, we obtain $\succeq_c$ for $X \rtimes \Delta_{\chi_p^{\mathcal{H}}}, M \rtimes \Delta_{\chi_p^{\mathcal{H}}}$. □

Theorem 6.135. *Let $X = \mathrm{Spin}_8^+(q)$ for q odd, $P \in \mathrm{Syl}_p(X)$, and $\mathrm{ord}_p(q) = 4$. Then, we find an $\mathrm{Aut}(X)_P \times \mathcal{H}_p$-equivariant bijection from $\mathrm{Irr}_{p'}(X)$ to $\mathrm{Irr}_{p'}(\mathrm{N}_X(P))$ with the $\succeq_c$ relation at each pair (χ, ϕ).*

Proof. Given $u \neq 1$, we have $|u| \mid (q^2+1)/2$ and $\gcd(|u|, 6) = 1$. By Lemmas 6.131 and 6.133, $\mathbf{L}$ has a parabolic radical $\mathbf{U}$ stable under $H_1 = L(B' \cap H_1)$, since $B' \cap H_1$ fixes u. With D as in Proposition 6.132, each $(lb, b) \in H \times_{D_0} B'$ has $l \in L$ uniquely determined. Moreover, $f(lb, b) = ((l, 1), \mathrm{Int}(b)|_L)$ defines a homomorphism to $(L \times I) \rtimes D$. At $b = i \in I$, the tensor $\mathcal{T}$ in Proposition 6.132 has $((i^{-1}, i^{-1}), \mathrm{Int}(i))$ in its kernel. Thus, $V = \mathcal{T} \circ f$ extends $\beta \boxtimes \rho^\vee$ from $L \times I$. Moreover, $V^{(b,\sigma)} \cong V$ for each corrected mixed stabilizer operation (b, σ), and (z, z) lies in $\ker(V)$ since $\mathrm{Int}(z)|_L = 1$ for $z \in \mathrm{C}_A(X)$. By Proposition 6.134, we obtain $\succeq_c$ on A, B. If $u = 1$, $s \in T_2^*$. If $s = 1$, the result of [156, Theorem D] gives $\succeq_c$ on $\mathcal{E}(X, 1)$. By Corollary 6.121, we obtain $\succeq_c$ for the three nonidentity elements $s \in T_2^*$. Consequently, by Proposition 6.123, the union over $[s]$ gives the equivariant map $\mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(M)$ with $\succeq_c$ at each pair (χ, ϕ). □

Lemma 6.136. *With $q = r^f$ odd, we let p be an odd divisor of $q+1$. On the torus $\mathbf{T}$ of Weyl type -1, the action of F on $Y(\mathbf{T})$ is $-q$. Consequently, each rational subspace of $Y(\mathbf{T}) \otimes \mathbb{Q}$ gives an F-stable subtorus $\mathbf{T}_E$. We determine the Levi at $s \in T^*$ from E below. Let $K = (W(D_4) \rtimes \mathfrak{S}_3)_s$ for $s \in T^*$, and let $E = (\mathbb{Q}^4)^K$. If $E \neq 0$, then the subtorus $\mathbf{T}_E^*$ with rational cocharacters E gives a proper F^*-stable Levi $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(\mathbf{T}_E^*)$ containing $\mathrm{C}_{\mathbf{X}^*}(s)$. Its dual $\mathbf{L}$ has a parabolic radical $\mathbf{U}$ normalized by the field degree 0 stabilizer of θ_s. Moreover, each mixed operation stabilizing θ_s normalizes $\mathbf{L}$.*

Proof. Write $s = x + \Lambda$, where $\Lambda = \mathbb{Z}^4 + \mathbb{Z}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the adjoint lattice $Y(\mathbf{T}^*)$. Thus, fix x with denominator coprime to r, via the isomorphism of prime-to-r roots of unity with $(\mathbb{Q}/\mathbb{Z})_{r'}$. Let Ξ be the root system F_4 with short roots $\pm e_i \pm e_j$ and long roots $\pm 2e_i$ and $(\pm 1, \pm 1, \pm 1, \pm 1)$. The long roots have coroots $\pm e_i$ and $(\pm 1, \pm 1, \pm 1, \pm 1)/2$. Moreover, the short coroots $\pm e_i \pm e_j$ lie in $\mathbb{Z}^4$. Thus, the coroot lattice is Λ. Since $W(\Xi)$ permutes $\{\pm e_i \pm e_j \mid i < j\}$ and has order 1152, it is $W(D_4) \rtimes \mathfrak{S}_3$. Given $w \in K$, the translation $x - wx \in \Lambda$ gives the unique lift of w in the point stabilizer $(\Lambda \rtimes W(\Xi))_x$. Thus, $(\Lambda \rtimes W(\Xi))_x \cong K$. By Humphreys, §4.8, the walls $\langle \alpha, x \rangle \in \mathbb{Z}$ give the generators, so K is generated by s_α for $\alpha \in \Xi$ with $\langle \alpha, x \rangle \in \mathbb{Z}$.

For α a root of $\mathrm{C}_{\mathbf{X}^*}(s)^\circ$, we have $s_\alpha \in K$, so $\alpha|_E = 0$. In $E \setminus \{0\}$, choose y with $\alpha(y) \neq 0$ whenever $\alpha|_E \neq 0$. Consequently, $\Phi(\mathbf{L}^*, \mathbf{T}^*) = \{\alpha \in D_4 \mid \alpha(y) = 0\}$, so $\mathbf{L}^*$ is proper. Moreover, $K \cap W(D_4)$ fixes y. The chamber stabilizer theorem (Humphreys, §1.12) gives $W(D_4)_y = W(\mathbf{L}^*, \mathbf{T}^*)$. By [58, Proposition 3.5.1], the full $\mathrm{C}_{\mathbf{X}^*}(s)$ is generated by $\mathrm{C}_{\mathbf{X}^*}(s)^\circ$ and representatives of $K \cap W(D_4)$ in $\mathrm{N}_{\mathbf{X}^*}(\mathbf{T}^*)$. Hence, $\mathrm{C}_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$. Since F^* acts by $-q$ on $Y(\mathbf{T}^*)$, the torus $\mathbf{T}_E^*$ and $\mathbf{L}^*$ are F^*-stable.

Let $y_0 \in Y(\mathbf{T}) \otimes \mathbb{Q}$ correspond to y under the invariant form. Choose $\lambda = ny_0 \in Y(\mathbf{T})$ with $n > 0$. Thus, the roots with $\langle \alpha, \lambda \rangle > 0$ give $\mathbf{U}$, as in Lemma 6.133. The field degree 0 stabilizer acts on the lattice through K, so it normalizes $\mathbf{U}$. A mixed operation stabilizing θ_s acts on s by $g(s)^a = s$, where $g \in W(\Xi)$ and $\gcd(a, |s|) = 1$. Thus, $W(\Xi)_{g(s)} = W(\Xi)_s = K$, so $gKg^{-1} = K$ and $g(E) = E$. Consequently, the mixed operation normalizes $\mathbf{L}$. □

Lemma 6.137. *If $E = 0$, then up to $\mathfrak{S}_3$ the class of s is $[1]$, one of the quasi-isolated $[s] \neq [1]$ in* Corollary 6.121, *or $[x_3']$ for $x_3' = (0, -1/3, -1/3, -1/3)$. The last orbit has length 32 and occurs if and only if $3 \mid q+1$. Moreover, its centralizer in $\mathbf{X}^*$ is connected of semisimple type A_2.*

Proof. By the affine stabilizer description for Ξ, $E = 0$ exactly at a vertex x_i of the closed alcove $\overline{\mathcal{A}}$. Take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (e_2 - e_3, e_3 - e_4, 2e_4, e_1 - e_2 - e_3 - e_4)$. Then, we have that $2e_1$ is highest with coefficients $(2, 4, 3, 2)$. The vertex x_0 is 0. The vertices $x_1, \ldots, x_4$ on $\langle 2e_1, x \rangle = 1$ are $(\frac{1}{2}, \frac{1}{2}, 0, 0)$, $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$, $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, $(\frac{1}{2}, 0, 0, 0)$. Thus, x_1, x_2, x_4 give the eigenvalue types $I(2,2), Q(1,2), I(1,3)$, respectively [208, Lemma 7.38(i)]. Moreover, the triality orbit of $I(1,3)$ contains both classes of type $U(4)$ [208, Lemma 7.38(ii)]. Their rational classes $[s]$ are listed in Corollary 6.121.

If $x_3' = x_3 - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, we have that the short roots α with $\alpha(x_3') \in \mathbb{Z}$ are $e_i - e_j$ for distinct $i, j \in \{2, 3, 4\}$. Moreover, the long roots α with $\alpha(x_3') \in \mathbb{Z}$ are $\pm 2e_1$ and $(\pm 1, t, t, t)$ for $t = \pm 1$. Thus, the subsystem is $A_2 \times A_2$, so $|K| = 36$. Given $w \in K \cap W(D_4)$, the shift $x_3' - wx_3'$ has denominator dividing 3. Since it lies in Λ, it is integral.

Thus, w fixes coordinate 1 and permutes coordinates $2,3,4$ without changing signs, including at 1 by even parity. Hence, $K \cap W(D_4) = \mathfrak{S}_{\{2,3,4\}}$ is the Weyl group of $\mathrm{C}_{\mathbf{X}^*}(s)^\circ$. By [58, Proposition 3.5.2], $\mathrm{C}_{\mathbf{X}^*}(s) = \mathrm{C}_{\mathbf{X}^*}(s)^\circ$. The image of K in $\mathfrak{S}_3$ has order $36/6 = 6$, so the full orbit is one $W(D_4)$-orbit of length $192/6 = 32$. Since s has order 3, $F^*(s) = s^{-q} = s$ if and only if $3 \mid q+1$. In particular, the length 32 is prime to every odd $p \mid q+1$. □

Proposition 6.138. *Suppose that $v = s_{6'} \neq 1$ and $\mathrm{C}_{\mathbf{X}^*}(v)$ is a proper Levi $\mathbf{L}^*$, or that $E \neq 0$ and $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(\mathbf{T}_E^*)$ as in* Lemma 6.136. *Assume that $s \in \mathrm{Z}(\mathbf{L}^*)$ and the roots of $\mathbf{L}^*$ have type A_1^k, with $k \geq 0$. Then, $W(D_4)\theta_s$ contributes to $\mathrm{Irr}_{p'}(M)$ exactly when $p \geq 5$. In that case all 2^k pairs satisfy $\succeq_c$ by* Proposition 6.134.

Proof. By Lemma 6.133 for v, or Lemma 6.136 for s, the field degree 0 stabilizer of θ_s fixes $\mathbf{U}$. Thus, $H_1 = L(B' \cap H_1)$ normalizes $\mathbf{U}$. In either case $\mathrm{C}_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$, since $v \in \langle s \rangle$ in the first. From $s \in \mathrm{Z}(\mathbf{L}^*)$, $\mathrm{C}_{\mathbf{X}^*}(s) = \mathbf{L}^*$. Since the torus twist -1 is central in $W(D_4)$, we have $W(D_4)^F = W(D_4)$. Thus, $I = M_{\theta_s} = \mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$ and $I/T \cong (C_2)^k$. Since $[M : I] = 192/2^k$, Proposition 6.123 gives no p'-degree characters at $p = 3$. Given $p \geq 5$, we have $p \nmid [M : I]$ and $P = Q \leq T$.

Each root pair in A_1^k is F-stable, so $[\mathbf{L}, \mathbf{L}]^F = \prod_j R_j$ with $R_j \cong \mathrm{SL}_2(q)$ and nonsplit $T_j = R_j \cap T$. By Lang's theorem [58, Lemma 4.2.13], $L = T \prod_j R_j$. Since $s \in \mathrm{Z}(\mathbf{L}^*)$, by [58, Proposition 11.4.8], we obtain the extension λ of θ_s with $\prod_j R_j \leq \ker(\lambda)$, uniquely determined by $L = T \prod_j R_j$. Thus, the labels on L are $\beta_J = \lambda \, \mathrm{St}_J$ for $J \subseteq \{1, \ldots, k\}$. S_J has augmentation factor $\mathbb{Q}[\mathbb{P}^1(\mathbb{F}_q)]_0$ for $j \in J$ and constant factor $\mathbb{Q}$ for $j \notin J$. Thus, S_J affords St_J by [58, §7.4]. Locally, put $\rho_J = \lambda|_I \, \epsilon_J$, where $\epsilon_J = \prod_{j \in J} \epsilon_j$ and ϵ_j is the Weyl sign at R_j [186, Theorems 4.2 and 4.4]. Moreover, restriction to $\prod_j R_j$ and to I/T distinguish the respective 2^k labels after the twist by λ^{-1}. The signed induction of β_J has degree $|X : L|_{r'} q^{|J|}$, which is prime to p since $P \leq L$. Consequently, $2^k = |\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)|$ by Proposition 6.123.

For the ordinary stabilizers H, B' of Proposition 6.134, both λ and J are fixed. Thus, H acts on S_J by its actions on $\mathbb{P}^1(\mathbb{F}_q)$ and on the factors R_j. Let $V_J = \bigoplus_{j \in J} \Omega_1(P \cap R_j)$, where $\Omega_1(P \cap R_j) \cong \mathbb{F}_p$ consists of elements of order dividing p. Put $\eta_J(b) = \iota(\det(b|_{V_J}))$ for $b \in B'$, where $\iota : \mathbb{F}_p^\times \to \mu_{p-1}$ is an isomorphism. Since s_{α_j} inverts just the line $\Omega_1(P \cap R_j)$, $\eta_J|_I = \epsilon_J$. Moreover, $\mathcal{H}_p$ fixes μ_{p-1} pointwise. Consequently, $V(lb, b) = \lambda(l) S_J(lb) \eta_J(b)^{-1}$ is a representation of $H \times_{D_0} B'$, since B' fixes λ. At $(x, i) \in L \times I$ we have $l = xi^{-1}$, giving the character $\beta_J \boxtimes \rho_J^\vee$. Moreover, $z \in \mathrm{C}_A(X)$ acts trivially on S_J, V_J, so (z, z) acts as 1. A mixed operation (a, σ) fixing θ_s and the selected pair fixes J, acts by $\lambda \mapsto \lambda^{(a,\sigma)} = \lambda$, and gives conjugate actions on S_J, V_J. The action on S_J is over $\mathbb{Q}$ and the values of η_J are in μ_{p-1}. Thus, the character $\mathrm{tr}\, V$ of this representation is fixed by (a, σ), including permutations of J. By Proposition 6.134, we obtain $\succeq_c$. □

If $q = 67^{2j+1}$ and $p = 17$, we choose a coordinate a of order 17. The coordinates (a, a, a^2, a^3) give one root pair, while (a, a^2, a^3, a^4) give a regular s. Thus, we find $v = s$ in both cases. If $q = 89^{2j+1}$, $p = 5$ and $x = (1, 1, 2, 3)/9$, we have $v = 1$ and $s \in T^*$ since $9 \mid q+1$. The short roots α with $\alpha(x) \in \mathbb{Z}$ are $\pm(e_1 - e_2)$. Moreover, we have that $2e_i(x) \notin \mathbb{Z}$ and a sign vector has value $k/9$ with k odd and $|k| \leq 7$. Hence, $K = \langle s_{e_1 - e_2} \rangle$, $\mathbf{L}^*$ has type A_1, and $s \in \mathrm{Z}(\mathbf{L}^*)$ gives $[M : I] = 96$.

Proposition 6.139. *Suppose that $\mathrm{C}_{\mathbf{X}^*}(s) = \mathbf{L}^* < \mathbf{X}^*$ is a Levi of type A_{m-1} with $m \in \{3, 4\}$ on the torus of type -1. The partition $\mu \vdash m$ gives the pair (β_μ, ρ_μ) on L, I, where $\beta_\mu = \lambda_s U_\mu$ and $\rho_\mu = \lambda_s|_I \, \mathrm{Inf}_{I/T}^I S^\mu$. The induced pair (χ, ϕ) has p'-degree exactly when $p \nmid (192/m!) \dim S^\mu$. Moreover, if an inertia subgroup admits $\mathbf{U}$ as in* Proposition 6.134, *we have $\succeq_c$ at (χ, ϕ).*

Proof. With $\mathbf{R} = [\mathbf{L}, \mathbf{L}]$, we put $T_R = (\mathbf{T} \cap \mathbf{R})^F$. The Levi $\mathbf{L}$ has simply connected $\mathbf{R}$ of type A_{m-1}. Since $F|_{Y(\mathbf{T} \cap \mathbf{R})} = -q$, we have $F = \mathrm{Int}(t)\gamma F_q$ on $\mathbf{R}$, with t in the adjoint torus and $\gamma(g) = (g^{-1})^t$. By Lang's theorem [58, Theorem 4.2.9], we choose x in the adjoint torus with $t = x(\gamma F_q)(x)^{-1}$. Thus, $F \, \mathrm{Int}(x) = \mathrm{Int}(x)\gamma F_q$, and $R = \mathbf{R}^F \cong \mathrm{SU}_m(q)$. By Lang's theorem, $L = TR$. Since $s \in \mathrm{Z}(\mathbf{L}^*)$, by [58, Proposition 11.4.8], we obtain the unique extension λ_s of θ_s trivial on R, and $\mathcal{E}(L, s) = \lambda_s \mathcal{E}(L, 1)$. Given $\mu \vdash m$, let $U_{\mu,\mathrm{ad}} \in \mathcal{E}(\mathrm{PGU}_m(q), 1)$ correspond to μ. Then, $U_\mu = U_{\mu,\mathrm{ad}} \circ c_R$ for the adjoint map $c_R : L \to \mathrm{PGU}_m(q)$. The partition on restriction to R is μ by [156, proof of Theorem D]. Since -1 is central in $W(D_4)$, the relative Weyl group is $W(D_4)$. Thus, $\mathrm{C}_{\mathbf{X}^*}(s) = \mathbf{L}^*$ gives $I = \mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$ and $I/T \cong \mathfrak{S}_m$, so we choose the Specht module S^μ on I/T.

Let $a = v_p(q+1)$. Since p is odd, $v_p(q^i - (-1)^i) = a + v_p(i)$ for $i \geq 1$. The order formulas and $L = TR$ give $v_p(|X|) = 4a + v_p(192)$ and $v_p(|L|) = 4a + v_p(m!)$. By the hook formula for $\mathrm{SU}_m(q)$ [40, §13.8], the factors $q^i - (-1)^i$ and $q^{h(\square)} - (-1)^{h(\square)}$ give $v_p(U_\mu(1)) = v_p(\dim S^\mu)$. Hence, the factors $|X : L|_{r'}$ and $[M : I] = 192/m!$ in the two

inductions give $v_p(\chi(1)) = v_p(\phi(1))$. By [58, Theorem 11.4.3], we obtain $\mathcal{E}(L,s) \to \mathcal{E}(X,s)$ by signed induction, and [95, Theorem 6.11] gives $\mathrm{Irr}(I \mid \theta_s) \to \mathrm{Irr}(M \mid W(D_4)\theta_s)$ by Ind_I^M.

Let $\widehat{U}_\mu$ be the matrix representation on $\mathrm{PGU}_m(q) \rtimes \langle F_r \rangle$ with $(\mathrm{tr}\,\widehat{U}_\mu)|_{\mathrm{PGU}_m(q)} = U_{\mu,\mathrm{ad}}$. In the model with Hermitian matrix I_m, the graph map γ equals F_q. Since $p \mid q+1$, we have $\mathrm{ord}_p(-q) = 1$. The proof of [156, Proposition 8.5] gives $\widehat{U}_\mu$ with values in $\mathbb{Q}_p$, by the choice at 1_{T_R}. Thus, the character of $\widehat{U}_\mu$ is $\mathcal{H}_p$-fixed. The group $\mathrm{PGU}_m(q) \rtimes \langle F_r \rangle$ acts by projective semilinear maps on $\mathbb{F}_{q^2}^m$. Its stabilizer of T_R permutes the weight lines $\langle e_i \rangle$. Whenever $i \neq j$, a diagonal element with $t_i = a_0$, $t_j = 1$ and $t_k = a_0^{-1}$ for $k \notin \{i,j\}$ separates the two coordinate characters, where $a_0 \neq 1$ and $a_0^{q+1} = 1$. Thus, the action π on these lines restricts to $I \to I/T$ on I.

For the groups H, B' of Proposition 6.134, take the representation

$$V(lb,b) = \lambda_s(l)\,\widehat{U}_\mu(\mathrm{Int}(lb)|_R) \otimes S^\mu(\pi(\mathrm{Int}(b)|_R))^\vee \ ((lb,b) \in H \times_{D_0} B').$$

Since B' fixes λ_s, V is multiplicative. At $(x,i) \in L \times I$ we have $l = xi^{-1}$, so the restriction is $\beta_\mu \boxtimes \rho_\mu^\vee$. At (z,z) with $z \in \mathrm{C}_A(X)$, we have $l = 1$ and $\mathrm{Int}(z)|_R = 1$, so $V(z,z) = 1$. Moreover, (a,σ) stabilizing (β_μ, ρ_μ) gives $\lambda_s^{(a,\sigma)} = \lambda_s$ by uniqueness. On $\mathrm{PGU}_m(q) \rtimes \langle F_r \rangle$, conjugation by $a|_R$ intertwines the matrices via $\widehat{U}_\mu(a|_R)$. $a|_R$ acts on the lines by $\pi(a|_R)$. Since S^μ is rational and $\widehat{U}_\mu$ is $\mathcal{H}_p$-fixed, we have $V^{(a,\sigma)} \cong V$. By Proposition 6.134, we obtain $\succeq_c$ on this inertia group. □

Proposition 6.140. *If $3 \mid q+1$, then all three pairs (χ,ϕ) over s of* Lemma 6.137 *satisfy $\succeq_c$ for every odd $p \mid q+1$, on the full inertia in $\mathrm{Aut}(X)_P \times \mathcal{H}_p$.*

Proof. Take $s = (0,1/3,1/3,1/3) + \Lambda$ and $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(s)$, with roots $e_i - e_j$ for distinct $i,j \in \{2,3,4\}$. The three degrees on X are $|X : L|_{r'}$ times $1, q(q-1), q^3$ by [40, §13.8]. Moreover, their partners on M have degrees $32, 64, 32$ by Proposition 6.139. Thus, each pair satisfies $\gcd(3p, \chi(1)\phi(1)) = 1$. Since the orbit of s is stable under triality and -1 acts by $s \mapsto s^{-1}$, the series $\mathcal{E}(X,s)$ is stable under $\Delta = \mathrm{Aut}(X)_P$ and under $\mathcal{H}_p$. Moreover, the degree $\chi(1)$ determines χ in $\mathcal{E}(X,s)$. Since $\theta_s^{(a,\sigma)}$ lies in $W(D_4)\theta_s$, choose $b \in M$ with $(\theta_s^{(a,\sigma)})^b = \theta_s$. Thus, $(\lambda_s^{(a,\sigma)})^b = \lambda_s$, and the automorphism of $I/T \cong \mathfrak{S}_3$ is inner. Hence, both Δ and $\mathcal{H}_p$ fix ϕ. Since $\lambda_s^3 = 1$ and $\mathrm{Z}(X)$ is a 2-group, the characters on L, I over λ_s have scalar 1 at $\mathrm{Z}(X)$. Thus, their inductions also have scalar 1 there, and $\vartheta = \chi \boxtimes \phi^\vee$ is a character of $K_0 = (X \times M)/\operatorname{diag}\mathrm{Z}(X)$.

As in Section 2, let Π act on X by left X, inverse right M and Δ. Thus, the map $\pi_0 : (X \times M) \rtimes \Delta \to \Pi$ is given by $\pi_0((x,m),a)(t) = xa(t)m^{-1}$. Its kernel consists of $((x^{-1},x^{-1}), \mathrm{Int}(x))$ for $\mathrm{Int}(x) \in \Delta$ and $x \in \mathrm{N}_X(P)$. Thus, $\Pi/K_0 \cong \mathrm{Out}(X)$ because Δ surjects onto $\mathrm{Out}(X)$ by Sylow conjugacy. Write Π_2 for the preimage of $\langle \overline{s}_{2e_1} \rangle$ under $\Pi \to \mathfrak{S}_3$, and Δ_2 for the corresponding subgroup of Δ. Since s_{2e_1} has graph image $\overline{s}_{2e_1}$ of order 2, we have $[\Pi : \Pi_2] = 3$. Over Π_2, the stabilizer of s in the extended Weyl group is $\langle s_{2e_1} \rangle \times \mathfrak{S}_{\{2,3,4\}}$ by Lemma 6.137. It fixes $y = (0,1,1,1)$, whose zero D_4-roots are precisely $\Phi(\mathbf{L}^*, \mathbf{T}^*)$. Thus, the cocharacter defined by y gives $\mathbf{U}$ for the field degree 0 group over Π_2.

By Propositions 6.139 and 6.134, the pair over Δ_2 satisfies $\succeq_c$. Thus, $\mathcal{P}_\chi \otimes \mathcal{P}_\phi^\vee$ is ordinary on $(X \times M) \rtimes \Delta_2$, since its factors cancel. Given $z \in \mathrm{C}_{X \rtimes \Delta_2}(X)$, the scalars at z agree, so the tensor descends through π_0 and affords an extension ϑ_2 on Π_2. Moreover, $\mu_{\chi,\sigma} = \mu_{\phi,\sigma}$ gives $\vartheta_2^\sigma = \vartheta_2$ for $\sigma \in \mathcal{H}_p$. If σ inverts s, we choose a lift $n \in M$ of -1, so $(\theta_s^\sigma)^n = \theta_s$. Since the graph image of n is 1, the correction is $\mathrm{Int}(n)$ on Π_2.

Let $\Pi_3/K_0 \in \mathrm{Syl}_3(\Pi/K_0)$. Since a lift of -1 inverts T, the image of T in $M/[M,M]$ has exponent dividing 2. Moreover, $M/T = W(D_4)$ is generated by involutions, so $3 \nmid |M/[M,M]|$. As $X = [X,X]$, we have $3 \nmid |K_0/[K_0,K_0]|$. Hence, $\gcd(\vartheta(1)|\det\vartheta|, 3) = 1$. By [95, Corollary 8.16], we find a unique extension ϑ_3 of ϑ to Π_3 with $3 \nmid |\det\vartheta_3|$. Thus, $\vartheta_3^\sigma = \vartheta_3$ for $\sigma \in \mathcal{H}_p$, since $\vartheta^\sigma = \vartheta$ and $|(\det\vartheta_3)^\sigma| = |\det\vartheta_3|$.

Since $d = [\Pi : \Pi_3]$ is prime to 3, we choose $a, b \in \mathbb{Z}$ with $3a + db = 1$. Then, the virtual character $\Xi = a\,\mathrm{Ind}_{\Pi_2}^\Pi \vartheta_2 + b\,\mathrm{Ind}_{\Pi_3}^\Pi \vartheta_3$ is $\mathcal{H}_p$-fixed, with $\Xi|_{K_0} = \vartheta$ by invariance of ϑ. By Proposition 2.4, we obtain an $\mathcal{H}_p$-fixed extension of ϑ to Π. Consequently, the inflation along π_0 is trivial on $\ker(\pi_0)$. The associated factors $\mathcal{P}_\chi, \mathcal{P}_\phi$ give $\succeq_c$ over Δ, by the extraction in Proposition 6.134. □

Lemma 6.141. *Given $E \neq 0$, let $\mathbf{L}^*$ be the Levi in* Lemma 6.136. *Then, $s^2 \in \mathrm{Z}(\mathbf{L}^*)$. If $s \notin \mathrm{Z}(\mathbf{L}^*)$, the table lists $W(D_4)_s$ and its types. The counts (a,b) record $+1, -1$ in the natural torus T_Ψ. Moreover, the last column gives $\phi(1)$*

on M.

type in F_4	(a,b)	$\mathbf{L}^*_{\mathrm{ss}}$	$\Phi(\mathrm{C}(s)^\circ)$	$W(D_4)_s$	*degrees on M*
B_2	$(0,2)$	A_1^2	$\varnothing$	C_2	$96, 96$
B_3	$(1,2)$	A_1^3	A_1	C_2^2	$48, 48, 48, 48$
B_3	$(0,3)$	A_1^3	$\varnothing$	C_2^2	$48, 48, 48, 48$
C_3	$\{1,2\}$	A_3	A_1^2	D_8	$24, 24, 24, 24, 48$

$|D_8| = 8$, and $3 \mid [W(D_4) : W(D_4)_s]$ in each case.

Proof. Let $\Psi = \{\alpha \in \Xi \mid \alpha|_E = 0\}$ in the root system $\Xi = F_4$ of Lemma 6.136. The roots α with $\alpha(x) \in \mathbb{Z}$ span $E^\perp$, since their reflections generate K. Thus, they span Ψ over $\mathbb{Q}$, and the short roots of Ψ are $\Phi(\mathbf{L}^*, \mathbf{T}^*)$. In F_4, a proper Ψ has type $\prod_i A_{j_i}$ or B_2, B_3, C_3. Given a factor A_j, the roots $e_i - e_l$ are integral when $a_i = a_l$, so all a_i coincide at rank j. Given B_j, C_j, the reciprocal eigenvalue pair (a_0, a_0^{-1}) of multiplicity t, with $a_0 \neq \pm 1$, contributes rank $t-1$ on t coordinates [58, §3.5]. Hence, rank j requires $a_i \in \{1, -1\}$ for all i. In all cases the roots of Ψ take value 1 on s^2. Consequently, $s^2 \in \mathrm{Z}(\mathbf{L}^*)$, and $s \in \mathrm{Z}(\mathbf{L}^*)$ if $|s|$ is odd.

If $s \notin \mathrm{Z}(\mathbf{L}^*)$, then $W_s = W(D_4)_s \leq W(\mathbf{L}^*)$ because W_s fixes E pointwise. In B_k, the short roots of Ψ have type A_1^k. At signs (a,b), the integral subsystem is $B_a D_b$ since e_i is integral at $a_i = 1$ and $e_i \pm e_j$ at $a_i = a_j$. Thus, $b \geq 2$, since $b = 0$ gives central s and $b = 1$ has rank $k-1$. The intersection $W(B_a D_b) \cap W(A_1^k)$ has the sign group $C_2^a \times C_2^{b-1}$, with even parity on the b negative coordinates. Hence, $W_s \cong C_2^{k-1}$. In C_3, the short roots of Ψ have type $D_3 = A_3$, and nonconstant signs have counts $\{1, 2\}$. Thus, the integral roots have type $C_1 C_2$ and their short roots have type $D_1 D_2 = A_1^2$. The intersection with $W(D_3)$ has the even sign changes C_2^2 and the swap $(i\ j)$ with $a_i = a_j$. Consequently, $W_s \cong C_2^2 \rtimes C_2 = D_8$, with $(i\ j)$ exchanging two involutions in C_2^2. The degrees of W_s are all 1 for the B_k cases, and $1, 1, 1, 1, 2$ for D_8. Thus, multiplication by $192/|W_s|$ gives the table by Proposition 6.123. □

At $p = 3$, no row contributes to $\mathrm{Irr}_{p'}(M)$. If $s \in \mathrm{Z}(\mathbf{L}^*)$ and $E \neq 0$, then the Levi has type A_1^k, A_2 or A_3, covered by Propositions 6.138 and 6.139. Moreover, for $E = 0$, Lemma 6.137 lists $[s]$, and Proposition 6.140 treats $|s| = 3$. Thus, each admissible s at $p = 3$ is covered, and for odd $p \mid q+1$ this includes all s of odd order. Suppose now that $p \geq 5$ and $s \notin \mathrm{Z}(\mathbf{L}^*)$. The simple roots of $\mathbf{L}^*$ extend to a basis Δ of $X(\mathbf{T}^*)$, since $\mathbf{X}^*$ is adjoint. Let $t \in \mathbf{T}^*$ have $\alpha(t) = \alpha(s)$ on the roots of $\mathbf{L}^*$ in Δ, and $\alpha(t) = 1$ on the others. Thus, we find $t^2 = 1$ by $s^2 \in \mathrm{Z}(\mathbf{L}^*)$, so $t^{F^*} = t^{-q} = t$. Moreover, $st^{-1} \in \mathrm{Z}(\mathbf{L}^*)^{F^*}$ gives a linear twist $\lambda_{st^{-1}}$ by Digne–Michel [58, Proposition 11.4.8]. Since a stabilizer operation may change $\Delta \setminus \Phi(\mathbf{L}^*)$, we keep $\theta_s|_{\mathrm{Z}(\mathbf{L})^F}$ for the mixed comparison.

Lemma 6.142. *For the B_k cases let $C = \mathrm{Z}(\mathbf{L})^F$, we have that $R = \prod_{j=1}^k R_j$ with $R_j = \mathrm{SL}_2(q)$, and $T_R = T \cap R$. With $N = \mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$, we set $N_R = N \cap R$. Then, we find $L/(RC) \cong N/(N_R C) \cong T/(T_R C) \cong C_2$. Thus, $L/(RC) \setminus \{1\}$ induces a nonsquare diagonal on every R_j.*

Proof. Choose a D_4 base with outer nodes α_j for the Levi $\mathbf{L}$. Since $Y(\mathbf{T}) = \bigoplus_{i=0}^3 \mathbb{Z}\alpha_i^\vee$, write $T \cong C_{q+1}^4$ with coordinates t_i. At the middle node take t_0. Then, $\alpha_j(t) = t_j^2 t_0^{-1}$, so $[\alpha_j(t)] = [t_0]$ in C_{q+1}/C_{q+1}^2 for every selected j. For square $\alpha_j(t)$, we choose an element of T_R with those values. Since $C = \bigcap_j \ker \alpha_j$ in T, we have $T_R C = \ker(t \mapsto [t_0])$. Moreover, $L = RT$ by Lang's theorem [58, Lemma 4.2.13], and $N = N_R T$ since $\mathrm{N}_{R_j}(T \cap R_j)$ gives the j-th reflection in $N/T \cong C_2^k$. Consequently, both quotients are isomorphic to $T/(T_R C)$, and $[t_0] \neq 1$ gives the nonsquare diagonal on each R_j. □

Proposition 6.143. *Suppose that $p \geq 5$ and s lies in a B_k case of* Lemma 6.141. *Then, all 2^{k-1} pairs over s satisfy $\succeq_c$ on X, M.*

Proof. With $\xi = \theta_s|_C$, put $D = RC$ and $D_N = N_R C$. Moreover, choose $t_d \in T \setminus T_R C$. With $T_j = T \cap R_j$, we put $U_j = \mathrm{N}_{R_j}(T_j)$. Thus, torus duality gives $\theta_s|_{T_j} = 1$ at $+1$ and $\theta_s|_{T_j} = \epsilon_j$ at -1, with ϵ_j quadratic. At $+1$, pair $1, \mathrm{St}$ with $1, \mathrm{sgn}$ of U_j/T_j. At -1, pair the Weil characters $\eta_{j,+}, \eta_{j,-}$ of degree $(q-1)/2$ with the extensions $\epsilon_{j,+}, \epsilon_{j,-}$ of ϵ_j. By [186, Theorem 4.2 and proof of Theorem 4.4 at $q = 9$], we obtain $\succeq_c$ for each R_j, U_j. Write $\alpha = \boxtimes_j \alpha_j$ and $\delta = \boxtimes_j \delta_j$ for the products. Their scalars at $C \cap R$ are $\xi|_{C \cap R}$, so their products with ξ define $\alpha\xi, \delta\xi$ on D, D_N. By Lemma 6.142, the nonidentity coset exchanges $\eta_{j,+}, \eta_{j,-}$ and $\epsilon_{j,+}, \epsilon_{j,-}$ at -1. It fixes $1, \mathrm{St}$ and $1, \mathrm{sgn}$ at $+1$. Since $b \geq 2$, the inertia groups in L, N are D, D_N. Hence, $\beta = \mathrm{Ind}_D^L(\alpha\xi)$ and $\nu = \mathrm{Ind}_{D_N}^N(\delta\xi)$ are irreducible by [95, Theorem 6.11]. Their labels are $T/(T_R C)$-orbits. Moreover, the restriction of $R_{\mathbf{T}}^{\mathbf{L}}(\theta_s)$ to R is $R_{\mathbf{T} \cap [\mathbf{L},\mathbf{L}]}^{[\mathbf{L},\mathbf{L}]}(\theta_s|_{T_R})$ by [58, Proposition 11.3.10],

with scalar ξ on C. By [58, §12.5], the factor at T_j is $1-\mathrm{St}$ or $-\eta_{j,+}-\eta_{j,-}$. Since the diagonal coset fixes $\theta_s|_{T_R}$, the coefficients at α and α^{t_d} agree. Thus, β occurs in $R^{\mathbf{L}}_{\mathbf{T}}(\theta_s)$, so $\beta\in\mathcal{E}(L,s)$.

Since $N=TD_N$ and $T\cap D_N=CT_R$, Mackey restriction gives $\nu|_T=\mathrm{Ind}_{CT_R}^{T}(\theta_s|_{CT_R})$. The two extensions of $\theta_s|_{CT_R}$ are the $W(\mathbf{L})$-orbit of θ_s, since $[W(\mathbf{L}):W_s]=2$. Therefore, $\nu=\mathrm{Ind}_I^N\rho$ for the Clifford correspondent ρ at $I=M_{\theta_s}$. Consequently, $\phi=\mathrm{Ind}_N^M\nu=\mathrm{Ind}_I^M\rho$ is irreducible. The degrees of ϕ from Lemma 6.141 are prime to $p\geq 5$. Also, $\chi(1)=|X:L|_{r'}\,2\alpha(1)$ is prime to p, since $P\leq L$ and the factors of $\alpha(1)$ are $1,q,(q-1)/2$. Thus, the 2^{k-1} pairs exhaust $\mathcal{E}(X,s)\cap\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M\mid W(D_4)\theta_s)$ by Proposition 6.123. Moreover, (a,σ) acts by $\xi\mapsto\xi^{(a,\sigma)}$ and (α,δ) to the paired orbits of $(\alpha^{(a,\sigma)},\delta^{(a,\sigma)})$. In the notation of Proposition 6.134, put $V=NB'$, $H=LV=LB'$ and $O=B/M$. Then, $V/N\cong H/L\cong O$. Let $V_0=V_\alpha$, so $V_0=V_\delta$ by the pairing at R,N_R. The coset t_dT_RC exchanges α,α^{t_d} and δ,δ^{t_d} simultaneously. Thus, $V=NV_0$, $V_0\cap N=D_N$, and $H_0=DV_0$ gives $H=LH_0$ and $H_0\cap L=D$. In particular, $H_0/D\cong V_0/D_N\cong O$.

By [141, Theorem 3.2 and Lemmas 2.5–2.7], we have matched models $\mathcal{P}_\alpha,\mathcal{P}_\delta$ on the stabilizers of α,δ. Thus, for equal labels, a permutation π of the factors R_j acts on both tensor spaces by π. For distinct labels in $\{\eta_{j,+},\eta_{j,-}\}$, we choose d_j with $\eta_{j,-}^{d_j}=\eta_{j,+}$. Then, transport the model by d_j^{-1}. Thus, the ordinary tensor $W=\mathcal{P}_\alpha\otimes\mathcal{P}_\delta^\vee$ is trivial on $((n^{-1},n^{-1}),\mathrm{Int}(n))$ for $n\in N_R$. On $H_0\times_O V_0$ put

$$V_\xi(dv,v)=\xi(c)\,W\big((r_0,1),\mathrm{Int}(v)|_R\big),\ d=cr_0,\ c\in C,\ r_0\in R.$$

Since α,δ have scalar ξ on $C\cap R$, V_ξ is independent of the expression $d=cr_0$. Moreover, V_0 fixes ξ, so V_ξ is a representation. At $v=c'n\in D_N$, the diagonal inner pair at $n\in N_R$ acts trivially. Hence, V_ξ restricts to $(\alpha\xi)\boxtimes(\delta\xi)^\vee$. At $z\in\mathrm{C}_A(X)$ we have $d=1$ and $\mathrm{Int}(z)|_R=1$, so $V_\xi(z,z)=1$. Moreover, $W^{(a,\sigma)}\cong W$ and $\xi^{(a,\sigma)}=\xi$ for (a,σ) stabilizing $(\alpha\xi,\delta\xi)$, so $V_\xi^{(a,\sigma)}\cong V_\xi$.

By Proposition 6.134, the tensor V_ξ gives associated models on H_0,V_0 with common factor set γ on O. Induce both models on the cover $\widetilde{O}$ defined by γ. Thus, restriction to L,N affords β,ν by $H=LH_0$ and $V=NV_0$. For an operation stabilizing (β,ν), one element $t\in T$ corrects both product labels. Consequently, $\mu_\beta=\mu_\nu$ on O by induction. Moreover, $z\in\mathrm{C}_A(X)$ has $zt=tz$ for each representative $t\in T$, so both induced scalars at z equal the scalar on V_0. Since $H=LB'$, by Lemma 6.136, we obtain $\mathbf{U}$ for H_1. Thus, the extension on $A\times_O H$ in Proposition 6.134 pairs χ with β. Thus, composition with (β,ν) gives the models of (χ,ν). Finally, $V\cap M=N$ and $B=MV$, so induction from V to B has base character $\phi=\mathrm{Ind}_N^M\nu$. By Proposition 6.134, we obtain $\succeq_c$ at (χ,ϕ). □

Lemma 6.144. *For the C_3 case of* Lemma 6.141*, we have $p\geq 5$ with $p\mid q+1$. Thus, we put $C_0=\mu_{q+1}$ and $G=\mathrm{GU}_4(q)$, with Hermitian matrix I_4. Let T_G be its diagonal torus and $N_G=\mathrm{N}_G(T_G)$. Moreover,* Lemma 3.48 *gives matched operators on G,N_G. In the C_3 case, we have that L and N are the kernels of $u(g,z)=\det(g)z^{-2}$ on $G\times C_0$ and $N_G\times C_0$, respectively. The labels (λ,μ) with $\lambda,\mu\vdash 2$ restrict to five irreducibles on L and on N. Moreover, the five degrees on N are $3,3,3,3,6$ and the induced degrees on M are $24,24,24,24,48$.*

Proof. Choose the Levi roots e_i-e_j up to triality, with $Y(\mathbf{T})=\{y\in\mathbb{Z}^4\mid\sum_i y_i\in 2\mathbb{Z}\}$. Thus, $y\mapsto(y,\sum_i y_i/2)$ gives the diagonal cocharacters of $\{(g,z)\in\mathrm{GL}_4\times\mathbb{G}_m\mid\det(g)=z^2\}$. The difference coroots have last coordinate 0. Consequently, $(g,z)\mapsto((F_q(g)^{-1})^t,z^{-q})$ acts by $-q$ on $Y(\mathbf{T})$. Lang's theorem on $\mathbf{T}_{\mathrm{ad}}$ [58, Theorem 4.2.9] conjugates this Frobenius to F. Thus, fixed points give $L=\ker(u|_{G\times C_0})$ and $N=\ker(u|_{N_G\times C_0})$. Both maps u are onto C_0 by diagonal matrices. In particular, $\det(g)=d^2$ gives $z=d,-d$.

Take a lift of θ_s to $T_G\times C_0$. Thus, the blocks $\{1,2\},\{3,4\}$ in Lemma 6.141 give the lift $(a,a,a\epsilon,a\epsilon,\kappa)$, with quadratic $\epsilon\in\mathrm{Irr}(C_0)$. Its other lifts are $(a\delta,a\delta,a\epsilon\delta,a\epsilon\delta,\kappa\delta^{-2})$, where $\delta\in\mathrm{Irr}(C_0)$. Moreover, the dual centralizer in G^* has factors $\mathrm{GU}_2(q)\times\mathrm{GU}_2(q)$. By Section 3.4, the labels (λ,μ) with $\lambda,\mu\vdash 2$ give paired characters $\widetilde{\chi}_{\lambda,\mu},\widetilde{\phi}_{\lambda,\mu}$. On N_G the degree is $\binom{4}{2}=6$, and on G it is prime to $p\geq 5$ by [40, §13.8]. Thus, on $G\times C_0,N_G\times C_0$, the labels are $\widetilde{\chi}_{\lambda,\mu}\boxtimes\kappa,\widetilde{\phi}_{\lambda,\mu}\boxtimes\kappa$.

A self-twist by $\delta\circ u$ has $\delta^2=1$ since it fixes κ. If $\delta=\epsilon$, the action on labels is $(\lambda,\mu)\mapsto(\mu,\lambda)$. Thus, for $\lambda\neq\mu$ the twist orbit has size 2 and stabilizer 1. If $\lambda=\mu$, the stabilizer is $\langle\epsilon\rangle$. By [95, Corollaries 6.17 and 11.22], restrictions to L,N have multiplicity one. Thus, (λ,μ) and (μ,λ) for $\lambda\neq\mu$ give one common constituent on L and one on N. Each (λ,λ) gives two constituents by [58, Lemma 11.3.9].

Since $u(T_G\times C_0)=C_0$ and $\mathrm{SL}_4\times 1\leq\mathbf{L}$, [58, Proposition 11.3.10] gives $R^{\mathbf{L}}_{\mathbf{T}}(\theta_s)$ by restriction. Each $\mathrm{GU}_2(q)$ factor has coefficients $1,-1$ by [58, §12.5]. Thus, (λ,μ) and (μ,λ) have equal coefficients $\varepsilon_\lambda\varepsilon_\mu=\varepsilon_\mu\varepsilon_\lambda$, so their common

restriction β occurs in $R_{\mathbf{T}}^{\mathbf{L}}(\theta_s)$. The orbit of (λ,λ) meets the four labels in $\{(\lambda,\lambda)\}$, so both constituents occur. On N, the torus weights lie in $W(\mathbf{L})\theta_s$ since $W(\mathbf{L}) = \mathfrak{S}_4$ acts on the coordinates. Moreover, $I = M_{\theta_s} \leq N$, so each constituent ν has $\mathrm{Ind}_N^M \nu = \mathrm{Ind}_I^M \rho \in \mathrm{Irr}(M)$ by Clifford correspondence. We have $[M:N] = 192/24 = 8$. Thus, the degrees $3,3,3,3,6$ on N give $24,24,24,24,48$ on M. Since $P \leq L$ and $p \geq 5$, we have $\chi(1)_p = \phi(1)_p = 1$. Thus, the five pairs exhaust $\mathcal{E}(X,s) \cap \mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M \mid W(D_4)\theta_s)$ by Proposition 6.123. □

Lemma 6.145. *Let b act on $\mathbf{L},\mathbf{T}$ with field degree j. Write $e \in \{1,-1\}$ for its diagram sign on A_3 and $e_c \in \{1,-1\}$ for its sign on $\mathrm{Z}(\mathbf{L})^\circ$ after composition with F_r^{-j}. Thus, we put $k = r^j(e_c - e)/2$, $v = e_c r^j$ and $h = j$ for $e = 1$ or $h = j + f$ for $e = -1$, where $q = r^f$. Then, with $n \in N_G$ and F_r entrywise Frobenius, the parent action is $(g,z) \mapsto (z^k n F_r^h(g) n^{-1}, z^v)$. Its restriction $b|_L$ determines b on the products, with $u(b(g,z)) = u(g,z)^{r^h}$.*

Proof. Since b stabilizes $\mathbf{T}$, the adjoint part b_0 normalizes the diagonal torus of SL_4. By Lang's theorem on $\mathrm{Z}(\mathrm{GL}_4)$ [58, Theorem 4.2.9], we obtain a rational lift $n \in N_G$. On $\mathrm{SU}_4(q)$, inverse transpose equals F_q, so the exponent on matrices is h. The connected centre is parametrized by (tI, t^2). The field and graph maps give $tI \mapsto t^{er^j} I$, and b acts on t by $v = e_c r^j$. Thus, $2k + er^j = v$. Since the maps agree on $[\mathbf{L},\mathbf{L}]$ and $\mathrm{Z}(\mathbf{L})^\circ$, they agree on $\mathbf{L}$. Since $r^h \equiv er^j \bmod (q+1)$, the contribution from z to u has exponent $4k - 2v = -2er^j$. Consequently, $u(b(g,z)) = u(g,z)^{r^h}$.

The parent group has $k, v \bmod (q+1)$, $h \bmod 2f$, and n modulo $\mathrm{Z}(G)$. Indeed, for $b_1 b_2$, the new exponents are $k_1 v_2 + r^{h_1} k_2,\ v_1 v_2,\ h_1 + h_2$, with monomial matrix $n_1 F_r^{h_1}(n_2)$. Since v is a unit and F_r^h is invertible, so is b. If $b|_L = 1$, $\mathrm{Int}(n) F_r^h$ fixes $\mathrm{SU}_4(q)$ pointwise. By the unitary automorphism description [79, Theorem 2.5.1], F_r has order $2f$ modulo the inner-diagonal group. Thus, $h = 0 \bmod 2f$, and $n \in \mathrm{Z}(G)$ since the module $\mathbb{F}_{q^2}^4$ is irreducible. Since every $z \in C_0$ occurs in L, the formula on L now gives $v = 1$ and $k = 0 \bmod (q+1)$. Hence, the restriction to L determines the parent action. In particular, $(g_0, z_0) \in N$ acts with $n = g_0,\ h = k = 0,\ v = 1$. □

Proposition 6.146. *The five pairs in* Lemma 6.144 *satisfy $\succeq_c$ on the C_3 case.*

Proof. Since $\mathrm{ord}_p(-q) = 1$, we choose the paired models P_1, P_2 for the four diagonal coordinates of G, N_G from Lemma 3.48. Write ω for their common scalar character on $\mathrm{Z}(G)$. Thus, the product models are $P_i \otimes \kappa$. Write a_u for the generator in Lemma 3.48, with $a_u^{2f} = a_0$. Moreover, a_u has field degree 1 on $\mathbb{F}_{q^2}^4$. Thus, $F_r^h a_u^{-h} = m_h \in N_G$, and the action of nF_r^h is represented by $nm_h a_u^h$.

For (b,σ) and the compensating character $\delta \circ u$, we choose $A_{i,(\delta,h,\sigma)}$ from Lemma 3.48 and put $D_{i,b} = P_i(nm_h) A_{i,(\delta,h,\sigma)}$. Thus, the covariance on $(g,1)$ is the determinant identity at h from that lemma. On $(1,z)$, the parent action gives $\sigma(\kappa(z)) = \delta(z^{-2r^h})^{-1}\omega(z^k)\kappa(z^v)$. Thus, the stabilizers modulo $\mathrm{Irr}(C_0) \circ u$ coincide for $P_i \otimes \kappa$, $i = 1,2$. Given $D_{i,b} D_{i,b'}$, moving $P_i(n' m_{h'})$ through $A_{i,(\delta,h,\sigma)}$ gives a scalar $\delta(\det(a_u^h n' m_{h'} a_u^{-h}))^{-1}$. This scalar is common to $i = 1,2$. Moreover, the monomial lifts for b, b' multiply to the lift at bb' times cI, with scalar $\omega(c)$. Moreover, we have that $m_{h+2f} = m_h a_0^{-1}$, so reduction of h chooses the matrices $P_i(a_0)$ through Lemma 3.47. Thus, $\alpha_1(b,b') = \alpha_2(b,b')$ for the semilinear factors, including $k \neq 0$. The factor κ enters through the condition on $(1,z)$.

Let $D_{\mathrm{tw}} \leq \mathrm{Irr}(C_0)$ be the self-twist group. Since $\alpha_1 = \alpha_2$, the factors cancel on $V_1 \otimes V_2^\vee$. If $D_{\mathrm{tw}} = \langle \epsilon \rangle$, let J_i be the operator at ϵ on the space V_i. Since $J_1^2 = J_2^2$ as scalars, normalize both to $J_i^2 = 1$. Put $V_i^\pm = \ker(J_i \mp 1)$. By Lemma 6.144, V_i^+, V_i^- afford the two constituents, since (g,z) with $(\epsilon \circ u)(g,z) = -1$ exchanges V_i^+, V_i^-. Thus, $\ker(J_1 \otimes J_2^\vee - 1) = \bigoplus_\pm V_1^\pm \otimes (V_2^\pm)^\vee$. At (b,σ), the common factors give $D_{i,b} J_i D_{i,b}^{-1} = c_b J_i$ with $c_b \in \{1,-1\}$ independent of i. Thus, $D_{1,b}, D_{2,b}$ transport the two signs together. If $D_{\mathrm{tw}} = 1$, we find only the tensor $V_1 \otimes V_2^\vee$. Moreover, $\delta \circ u = 1$ on L, N, so each orbit under $\mathrm{Irr}(C_0)$ has constant restrictions to L, N.

Fix one pair (β,ν) on L, N with tensor space $V_{\beta,\nu}$. Thus, put $V = NB'$, $H = LV$ and $O = B/M$ as in Proposition 6.143. Given $b \in V$, we choose δ restoring $P_1 \otimes \kappa$ by cyclic Clifford theory. Thus, δ also restores $P_2 \otimes \kappa$ by the scalar condition at $(1,z)$. Any ambiguity lies in D_{tw}, which acts trivially on $V_{\beta,\nu}$. Moreover, $n \mapsto cn$ has common scalar $\omega(c)$, which cancels. Thus, the operators D_b on the chosen tensor satisfy $D_b D_c = D_{bc}$. At $b = (g_b, z_b) \in N$, set $n = g_b$ and $\delta = 1$ by Lemma 6.145. The two scalars $\kappa(z_b)$ cancel, and $D_b = \beta(b) \otimes \nu(b)^\vee$. Consequently, $(lb, b) \mapsto (\beta(l) \otimes 1) D_b$ extends $\beta \boxtimes \nu^\vee$ to $H \times_O V$. Given $z \in \mathrm{C}_A(X)$, we have $z|_L = 1$, so the parent action is 1 and $D_z = 1$. Moreover, at (b,σ), the common factors $\alpha_1 = \alpha_2$ give invariance of $\beta \boxtimes \nu^\vee$ and its extension. This includes $(\lambda,\mu) \mapsto (\mu,\lambda)$ via $\delta = \epsilon$ for unequal labels. Since $H = LB'$, Lemma 6.136 gives $\mathbf{U}$ for H_1. Thus, the extension on $A \times_O H$ from Proposition 6.134 gives the comparison between χ, β. Compose with (β,ν) and induce from V to B as in Proposition 6.143. Consequently, $\phi = \mathrm{Ind}_N^M \nu$ is paired with χ and satisfies $\succeq_c$. □

Theorem 6.147. *Let $S = \mathrm{P}\Omega_8^+(q)$ for q odd. Then,* $\mathrm{iMN}(S,p)$ *holds, with universal cover X, for each $p \mid |S|$. For odd $p \nmid q$, we set $M = \mathrm{N}_X(Q)$ from* Lemma 6.122.

Proof. Let $d = \mathrm{ord}_p(q)$ for odd $p \nmid q$. Thus, at $d = 2$, Proposition 6.123 partitions both sets over $[s]$. If $E = 0$, apply [156, Theorem D] at $s = 1$, Corollary 6.121 at the other quasi-isolated s, and Proposition 6.140 at $|s| = 3$. For $E \neq 0$ and $s \in \mathrm{Z}(\mathbf{L}^*)$, take Propositions 6.138 and 6.139 at $\mathbf{L}^*$. Otherwise, by Lemma 6.141, we obtain B_k or C_3. Thus, Propositions 6.143 and 6.146 give $\succeq_c$. Consequently, the maps on $[s]$ extend over $[s]^{\mathrm{Aut}(X)_P \times \mathcal{H}_p}$ and give the equivariant bijection at $d = 2$. If $d \in \{1, 4\}$, take Theorems 6.130 and 6.135. Moreover, Corollary 6.124 covers $d = 3, 6$. By Lemma 6.122, $d \in \{1, 2, 3, 4, 6\}$. At $p \mid q$, apply [162, Theorem 1.3]. At $p = 2$, apply [159, Theorem A(3)]. If S has an exceptional multiplier, apply [157, Lemma 4.1(2)]. □

6.6 Rank-four minus forms

Proposition 6.148. *Put $X = \mathrm{Spin}_8^-(q)$, $q = r^f$ odd, and $S = \mathrm{P}\Omega_8^-(q)$. Thus, $\mathrm{Z}(X) = \langle z_0 \rangle$ is the kernel of $X \to \mathrm{SO}_8^-(q)$ and has order 2. For odd $p \mid |X|$ with $p \neq r$, put $d = \mathrm{ord}_p(q)$. The rational Frobenius is $F = \gamma F_r^f$, where γ is the natural graph involution. Thus, $F_r^f|_X = \gamma|_X$. If $d \in \{3, 4, 6, 8\}$ and $P \in \mathrm{Syl}_p(X)$, then $X, \mathrm{N}_X(P)$ satisfy* Proposition 2.9. *Thus, the comparison covers $\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(\mathrm{N}_X(P))$.*

Proof. The order formula gives multiplicity 1 for $\Phi_d(q)$ at $d = 3, 4, 6, 8$. As $d \mid p-1$, $p \geq 5$ and $p \nmid |W(D_4)|$. By [133, Theorem 25.14], $P = (\mathbf{S}^F)_p$ is cyclic for a Sylow d-torus $\mathbf{S}$. By [130, Proposition 2.2 and its proof], $\mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{S})$ and $C_{\mathbf{X}}(P[p]) = C_{\mathbf{X}}(\mathbf{S})$. Thus, P is the unique Sylow p-subgroup of $C_X(P[p])$, giving $\mathrm{N}_X(P[p]) = \mathrm{N}_X(P)$. By Proposition 2.9, the pairs for $X, \mathrm{N}_X(P)$ satisfy $\succeq_c$. □

Lemma 6.149. *Given $d \in \{1, 2\}$, we put $t = 1$ or $t = -1$, respectively. Let $\mathbf{T}$ have three rational planes of sign t and one of sign $-t$. A plane of sign e has torus order $q - e$, so $T = \mathbf{T}^F$ has order $(q-t)^3(q+t)$. In $\mathrm{Syl}_p(T)$, we fix Q with $Q \leq P \in \mathrm{Syl}_p(X)$. Then, Q is characteristic in P, and $M = \mathrm{N}_X(Q) = \mathrm{N}_{\mathbf{X}}(\mathbf{T})^F$ contains $\mathrm{N}_X(P)$. Moreover, we have that $W = M/T \cong W(B_3)$. The W-orbits on $\mathrm{Irr}(T)$ of p'-length correspond to the rational classes $[s]$ of p'-index in X^*. Moreover, the quasi-isolated orbits $[s] \neq [1]$ on T^* are as follows.*

natural lift type	*planes*	*length*
$I(1,3)$	*distinguished singleton*	1
$I(1,3)$	*active singleton*	3
$I(2,2)$	*two active sign planes*	3
$Q(1,2)$	*two active quartic planes,* $tq \equiv 1 \pmod 4$	12

Proof. The Sylow d-torus $\mathbf{S}$ has cocharacter space spanned by e_1, e_2, e_3. Thus, no D_4 root is zero on $Y(\mathbf{S})$, so $\mathrm{C}_{\mathbf{X}}(\mathbf{S}) = \mathbf{T}$. The rational Weyl elements act by $(\pi, \epsilon_1, \epsilon_2, \epsilon_3)$, with $\pi \in \mathfrak{S}_3$ and $\epsilon_i = \pm 1$. The fourth coordinate has sign $\epsilon_1\epsilon_2\epsilon_3$ by even parity. Hence, on $\mathbf{T}$, Lang's theorem [58, Theorem 4.2.9] gives $W \cong W(B_3)$, of order 48. By [63, Theorem 3.2], $P/Q \in \mathrm{Syl}_p(W)$ and $Q = T_p \cong (C_{p^a})^3$ for $a = v_p(q-t)$. Thus, $P = Q$ for $p \geq 5$, whereas $P/Q \cong C_3$ for $p = 3$. At $p = 3$, let c cycle the three planes. Its order-3 lift, through the kernel of order 2, gives $P = Q \rtimes C_3$. By Lemma 4.3, Q is characteristic in P. In the natural torus coordinates, $Q[p]$ consists of tuples $(a_1, a_2, a_3, 0)$ over $\mathbb{F}_p$. Thus, $(e_i \pm e_j)|_{Q[p]} \neq 0$ for every root, and $C_{W(D_4)}(Q[p]) = 1$. As in Lemma 6.122, this gives $C_{\mathbf{X}}(Q[p]) = C_{\mathbf{X}}(Q) = \mathbf{T}$. Since $Q = T_p$, we have $\mathrm{N}_X(Q) = \mathrm{N}_{\mathbf{X}}(\mathbf{T})^F$. From $\mathrm{C}_{\mathbf{X}}(\mathbf{S}) = \mathbf{T}$, we obtain $M = \mathrm{N}_X(\mathbf{S})$. By [63, Theorem 3.3], M contains $\mathrm{N}_X(P)$, and $M < X$ since $Q \not\leq \mathrm{Z}(X)$.

By Proposition 6.123, the p'-length orbits $W\theta_s$ correspond to the rational p'-index classes $[s]$. Over $W\theta_s$, both sets have size $\left|\mathrm{Irr}_{p'}(W_s)\right|$ for $W_s = \mathrm{Stab}_W(\theta_s)$. Write $s = [(z_1, z_2, z_3, y)]$ modulo negation. Thus, F^* has exponents $(tq, tq, tq, -tq)$. For sign lifts in [208, Lemma 7.38], choose $y = 1$. The first three cases have $3, 1, 2$ negative active entries, giving lengths $\binom{3}{3}, \binom{3}{1}, \binom{3}{2}$. In type $Q(1,2)$, the sign entries give $F^*(\widetilde{s}) = \widetilde{s}$ for the lift $\widetilde{s}$. Thus, the two entries in $\{i, -i\}$ must both be active with $tq \equiv 1 \bmod 4$. The active sign position has 3 choices, and each quartic entry has 2 since inversion at $y = \pm 1$ corrects parity. Hence, $|W\theta_s| = 12$ and $|W_s| = 4$. In type $U(4)$, all entries lie in $\{i, -i\}$. Since the exponents $tq, -tq$ give opposite signs, $F^*(\widetilde{s}) \notin \{\widetilde{s}, -\widetilde{s}\}$. Thus, this torus has no $U(4)$ point. At $p = 3$, only the distinguished singleton has p'-length. □

Proposition 6.150. *Let $s \neq 1$ be admissible on T^*, and suppose its lift $\widetilde{s}$ has at most one of $1, -1$ as an eigenvalue. Then, $\mathrm{C}_{\mathbf{X}^*}(s)$ is a connected proper Levi $\mathbf{L}^*$. The comparison on $\mathcal{E}(X,s) \cap \mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M \mid W\theta_s)$ satisfies $\succeq_c$ by the unipotent comparison on L. Moreover, at $p = 3$ the sole exception among admissible $s \neq 1$ is $[(-1,-1,-1,1)]$.*

Proof. Put $k = tq$. Thus, a rational lift $\widetilde{s} = (z_1, z_2, z_3, y)$ satisfies $z_i^{k-1} = c$ and $y^{k+1} = c$, with $c \in \{1, -1\}$. Both solution sets are invariant under $u \mapsto u^{-1}$ and $u \mapsto -u$. If u lies in both, $u^2 = 1$. Thus, for $y \neq \pm 1$, the pair $\{y, y^{-1}\}$ cannot equal $\{-z_i, -z_i^{-1}\}$. If $\widetilde{s}$ is conjugate to $-\widetilde{s}$, $\{y, y^{-1}\} = \{-y, -y^{-1}\}$, so $y = \pm i$. If $y = \pm i$, every active pair differs from $\{i, -i\}$. Consequently, each active class $\{u, u^{-1}\}$ would be paired with $\{-u, -u^{-1}\}$ of multiplicity $m_u = m_{-u}$, contrary to $\sum_{\{u,u^{-1}\}} m_u = 3$ on the active coordinates. If $y = \pm 1$, conjugacy to $-\widetilde{s}$ requires both eigenvalues $1, -1$. Thus, $\widetilde{s}$ is not conjugate to $-\widetilde{s}$ under the hypothesis. In particular, $c = -1$ excludes $1, -1$.

By [58, Proposition 3.5.1], the root factors on $V_u, V_{u^{-1}}$ are $\mathrm{GL}(V_u)$, with $\mathrm{SO}(V_\epsilon)$ for the sign $\epsilon \in \{1, -1\}$ if present. Since the determinants on $V_u, V_{u^{-1}}$ cancel, the centralizer in SO_8 is connected. Any extra component in $\mathbf{X}^*$ requires $g\widetilde{s}g^{-1} = -\widetilde{s}$. Thus, $\mathrm{C}_{\mathbf{X}^*}(s)$ is connected. Choose a cocharacter ν^* of weights $m_u, -m_u$ on $V_u, V_{u^{-1}}$, with distinct $m_u > 0$ over reciprocal pairs, and weight 0 on $V_{\pm 1}$. Then, $\mathrm{C}_{\mathbf{X}^*}(s) = \mathrm{C}_{\mathbf{X}^*}(\nu^*(\mathbb{G}_m))$ by the zero roots of ν^*. Thus, $\mathbf{L}^* = \mathrm{C}_{\mathbf{X}^*}(s)$ is a proper F^*-stable Levi. At field degree 0, a restorer of s fixes $\widetilde{s}$, so it gives $V_u \mapsto V_u$ and fixes ν^*. Hence, the primal cocharacter ν gives a stable radical $\mathbf{U}$. Moreover, replacing s by a generator s^j of $\langle s \rangle$ gives $\mathrm{C}(s^j) = \mathrm{C}(s) = \mathbf{L}^*$.

Since $s \in \mathrm{Z}(\mathbf{L}^*)$, we choose $\lambda_s \in \mathrm{Irr}(L)$ with $\mathcal{E}(L,s) = \lambda_s \mathcal{E}(L,1)$ by [58, Proposition 11.4.8]. From $\mathrm{C}_{\mathbf{X}^*}(s) = \mathbf{L}^*$, $I = M_{\theta_s} = \mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$. Since $\mathbf{S} \leq \mathbf{L}$ and $\mathrm{C}_{\mathbf{L}}(\mathbf{S}) = \mathbf{T}$, it is a regular Sylow d-torus of $\mathbf{L}$. Moreover, $p \nmid [X : L]$ because a representative of $[s]$ centralizes P^*. The factors are of type A on $V_u, V_{u^{-1}}$, and type D_m with $m \leq 3$ on $V_{\pm 1}$. Thus, on each derived factor R_i, choose the p'-degree pairs of [156, Theorem D and its proof]. Then, on L, I, Proposition 6.97 gives the unipotent pairs (β_0, ρ_0) via the adjoint models of R_i and permutation of the tensor factors R_i. Consequently, $\beta = \lambda_s \beta_0$ and $\rho = \lambda_s|_I \rho_0$ give the pairs over s. For a marked mixed operation (b, σ) fixing θ_s, uniqueness gives $\lambda_s^{(b,\sigma)} = \lambda_s$. Thus, $\mathbf{U}$ gives the hypothesis in Proposition 6.134. If $|H/H_1| = m$, the identity $\varphi^m = K_{a^m} F$ gives $\varphi^m = K_{a^m} \gamma F_r^f$. Thus, the last map is $K_{a^m}^{-1}\varphi$, giving $(F')^m = \mathrm{diag}(\gamma F_r^f)$ as in the transfer. Moreover, [19, Corollary 6.5 and Remark 6.7] compares $\mathbf{U}$ with its translate on $\mathcal{E}(L,s)$. Consequently, $\succeq_c$ holds for the triples over X and M. Finally, we let $p = 3$. Since $[W : W_s]$ is prime to 3, conjugate a Sylow 3-subgroup of W_s to the unsigned cycle (123). Since $(123)^3 = 1$, it fixes $\widetilde{s}$ and gives $\widetilde{s} = (x, x, x, y)$. Both signs occur only for $x = -y = \pm 1$, the distinguished singleton. If $x = y = \pm 1$ then $s = 1$, which is treated by [156, Theorem D]. □

Let C_n be a cyclic torus with $2 \mid n$, and let η_n be its quadratic character. Also, on $\mathbb{F}_q^\times$, we write η_q^{fld} for its quadratic character. Let $B = \mathbf{B}^F$ be the pinned Borel of X, with torus T_B. For either t, each singleton class $[s]$ has a representative on T_B^*. Indeed, the (-1)-eigenspace V_- has type $+$ or $-$. Isometries on V_-, V_+ conjugate the decompositions, and on V_+ one can correct the determinant to 1. Thus, the rational type of V_- gives the class $[s]$ on both tori.

Lemma 6.151. *Each rational series $\mathcal{E}(X,s)$ of type $I(1,3)$ consists of five pairs $\{\chi, \chi^d\}$ for the nonidentity diagonal class d of S. Moreover, each χ has a rational extension to $S \rtimes \langle F_r \rangle$, with $\langle F_r \rangle$ the field and graph subgroup on S.*

Proof. Let $\theta \in \mathrm{Irr}(T_B)$ be quadratic at the marked plane V_i. Thus, torus duality for $\pi : X \to \mathrm{SO}_8^-(q)$ gives $\theta(z_0) = 1$. Inflate θ to B and write $Y_\theta = \mathrm{Ind}_B^X \theta$ over $\mathbb{Q}$. In $W(B_3)$, put $a = (12)$, $b = (23)$, and let u invert planes $3, 4$. Moreover, the absolute nodes α_3, α_4 for u are orthogonal and F exchanges them. Thus, the Hecke values q_a, q_b, q_u are q, q, q^2 on trivial coroot characters. Given u, the pinned representative n_u is $n_{\alpha_3} n_{\alpha_4}$.

In the presentation of [58, §§6.1–6.2], let T_h be the operator over $\mathbb{Q}$. Thus, for $\theta \circ h^\vee \neq 1$, the composition $T_h(h\theta)T_h(\theta)$ is $q_h \theta(n_h^2)$ times the identity. With $i = 3, 4$, the short coroot on $\mathbb{F}_{q^2}^\times$ has coordinates z^{q+1} and z^{1-q}, onto $\mathbb{F}_q^\times$ and C_{q+1}, respectively. Hence, $\theta \circ u^\vee$ is quadratic and has value 1 at -1. Thus, $U = q^{-1}T_u$ squares to 1 for $i \in \{3, 4\}$.

At $i = 4$, the generators T_b, T_a, UT_bU satisfy the type A_3 relations. Indeed, the braid on b, u gives $T_b(UT_bU) = (UT_bU)T_b$, and U exchanges T_b, UT_bU. Thus, $\mathcal{H}_q(A_3) \rtimes \langle U \rangle$ maps onto $\mathrm{End}_{\mathbb{Q}X}(Y_\theta)$, and both have dimension 48 by the double cosets Bn_wB. Given $\lambda \vdash 4$, we put $c(\lambda) = \sum_{(i,j)\in\lambda}(j - i)$. By [208, Lemma 4.4, (4.1)], on S_q^λ the operator $A_\lambda = q^{-(6+c(\lambda))/2} T_{w_0}$ has square 1. Moreover, we have that $A_\lambda T_b A_\lambda^{-1} = UT_bU$ by the longest word w_0, as in Proposition 6.127. Since $c(\lambda) \in \{6, 2, 0, -2, -6\}$, each $U \mapsto \pm A_\lambda$ gives a rational module above S_q^λ.

At $i = 1$, the stabilizer is $W(B_2) \times \langle w \rangle$, where $w = abuba$ inverts planes $1, 4$. Since $\theta \circ b^\vee = \theta \circ u^\vee = 1$, the generators T_b, T_u have values q, q^2. The maps T_a, T_b from $i = 1$ to $i = 3$ are invertible by the nonzero scalars $q_h \theta'(n_h^2)$ on the respective torus characters θ'. Thus, conjugation of $q^{-1}T_u$ by the composition from $i = 1$ to $i = 3$ gives an involution V proportional to T_w over $\mathbb{Q}$, since $\ell(w) = 5$. Moreover, w fixes the positive simple roots of $W(B_2)$. Thus, $\ell(wh) = \ell(hw) = \ell(w) + 1$ and $wh = hw$ for $h = b, u$, so $VT_h = T_h V$. The double cosets Bn_xB give dimension 16, and the generators give $\mathrm{End}_{\mathbb{Q}X}(Y_\theta) \cong \mathcal{H}_{q,q^2}(B_2) \otimes \mathbb{Q}[C_2]$.

The roots $q, -1$ and $q^2, -1$ of the relations for T_b, T_u give four rational linear modules η. By Tits deformation [58, §6.2], the remaining block $A_{\mathbb{C}}$ has degree 2, so $A_{\mathbb{Q}}$ is central simple of dimension 4. At least one of T_b, T_u acts nonscalarly on $A_{\mathbb{Q}}$. Since the roots x_1, x_2 for T_h are distinct in $\mathbb{Q}$, $(T_h - x_1)/(x_2 - x_1)$ is an idempotent on $A_{\mathbb{Q}}$ different from $0, 1$. Thus, $A_{\mathbb{Q}} \cong \mathrm{M}_2(\mathbb{Q})$, and the factor $\mathbb{Q}[C_2]$ doubles the number of blocks to 10.

By [208, Lemma 7.38 and the proof of Proposition 7.39], each label $\lambda \vdash 4$ gives a pair $\{\chi, \chi^d\}$ on S via the regular overgroup $\widetilde{X}$. Thus, the ten constituents of Y_θ exhaust $\mathcal{E}(X, s)$. On Y_θ, the field map F_r permutes the summands of T_h and fixes the pinned representatives n_h. Therefore, the image eY_θ of each rational primitive idempotent e is F_r-stable. Since $F_r^{2f} = 1$ on S, the image eY_θ is a rational module for $S \rtimes \langle F_r \rangle$. Moreover, d exchanges the constituents over each λ, whereas F_r fixes each χ. Thus, the outer stabilizer of χ is $\langle \overline{F}_r \rangle$, including the graph action F_r^f. □

Lemma 6.152. *For either t, the local torus on S admits a coordinate field automorphism f_t of order $2f$ fixing the permutations $\mathfrak{S}_3$ and the plane swaps s_{v_i} in the orthogonal normalizer. Moreover, $\overline{f}_t = \overline{F}_r$ in $\mathrm{Out}(S)$.*

Proof. On the split algebraic group, set $F = \gamma F_r^f$. If $t = 1$, F_r acts on rotation entries by $z \mapsto z^r$ and fixes the orthogonal complement representatives n_w over $\mathbb{F}_r$. Instead, for $t = -1$, let $n \in \mathbf{X}$ lift $-I_4 \in W(D_4)$. More precisely, let $\pi(n) = s_{v_1} s_{v_2} s_{v_3} s_{v_4}$, where each v_i is in hyperbolic plane i and has norm $\kappa \in \mathbb{F}_r^\times$. Their product has determinant 1 and spinor norm κ^4 modulo squares, so n can be chosen over $\mathbb{F}_r$. The four reflections s_{v_i} lie on distinct planes and $\pi(n)$ satisfies $\pi(n)\gamma = \gamma\pi(n)$. Thus, for $F' = \mathrm{Int}(n)F$, the signs are $(-, -, -, +)$. The permutations $\mathfrak{S}_3$ on $1, 2, 3$ and the swaps s_{v_i} are fixed by F' and F_r. Moreover, $\pi(nF(n)) = 1$, so $F_r^{2f} = 1$ on the natural F'-fixed group. By Lang–Steinberg [58, Theorem 4.2.9], take g with $g^{-1}F(g) = n$. Then, $\mathrm{Int}(g) : \mathbf{X}^{F'} \xrightarrow{\sim} \mathbf{X}^F$, and F_r becomes $f_t = \mathrm{Int}(gF_r(g)^{-1})F_r$. As $F_r(n) = n$, $F(gF_r(g)^{-1}) = gF_r(g)^{-1}$. Hence, $\overline{f}_t = \overline{F}_r$ in $\mathrm{Out}(S)$, so $\overline{f}_t$ has order $2f$. Since $f_t^{2f} = 1$, $|\langle f_t \rangle| = 2f$. □

Proposition 6.153. *The distinguished mark $i = 4$ gives six p'-characters at $p = 3$ and ten at $p \geq 5$. In contrast, the active mark $i = 1$ gives zero at $p = 3$ and ten at $p \geq 5$. Moreover, the corresponding triples on X, M satisfy $\succeq_c$.*

Proof. In $\mathrm{O}_8^-(q)$, we choose the rotation torus T_O. Then, its normalizer $N_O = T_O \rtimes (W(B_3) \times C_2)$ has the final C_2 inverting plane 4. Let $N_0 = N_O \cap \Omega_8^-(q) = \pi(M)$. Then, the quotient characters are $\delta = \det$ and the quadratic spinor norm ν. On one plane inversion $\delta = -1$, and on T_O $\nu|_{T_O} = \prod_{i=1}^4 \eta_{q-t_i}$, where t_i is the plane sign. Thus, $N_O/N_0 \cong C_2^2$. Also, $\pi(T) = T_O \cap \Omega_8^-(q)$ because $\pi^{-1}(\mathbf{T}_O) = \mathbf{T}$, where $\mathbf{T}_O = \pi(\mathbf{T})$.

Write θ_i for the quadratic character on plane i, inflated to T_O. Moreover, the two lifts of $\theta_i|_{\pi(T)}$ are θ_i and $\theta_i\nu|_{T_O}$. They have $1, 3$ marked planes, respectively. Thus, the two N_O-orbits on $\mathrm{Irr}(T_O)$ are distinct. Let W_i be the complement stabilizer of θ_i, and write $\widetilde{\theta}_i$ for its extension trivial on W_i. By [95, Theorem 6.11], above θ_i the characters are $\alpha_\eta = \mathrm{Ind}_{T_O \rtimes W_i}^{N_O}(\widetilde{\theta}_i \otimes \eta)$ for $\eta \in \mathrm{Irr}(W_i)$. At $i = 4$, $W_i \cong \mathfrak{S}_4 \times C_2 \times C_2$ since $W(B_3) = W(D_3) \times \langle -I_3 \rangle$. At $i = 1$, $W_i \cong C_2 \times W(B_2) \times C_2$. Hence, each W_i has twenty rational characters η.

The determinant δ is nontrivial on the final C_2 of both W_i, so δ fixes no α_η. Moreover, $\nu, \delta\nu$ change the T_O-orbit to that of $\theta_i\nu|_{T_O}$. Thus, each $\alpha_\eta|_{N_0}$ is irreducible by [95, Corollary 6.17]. $\alpha_\eta|_{N_0} = \alpha_{\eta'}|_{N_0}$ exactly for $\alpha_{\eta'} \in \{\alpha_\eta, \alpha_\eta\delta\}$ by Clifford theory on N_O/N_0. Conversely, a character of N_0 above $\theta_i|_{\pi(T)}$ lies below a character of N_O above θ_i or $\theta_i\nu|_{T_O}$. Thus, the character or its ν-twist is above θ_i. Hence, the count on N_0 is $20/2 = 10$ for each i.

Choose nonsquare $m \in \mathbb{F}_q^\times$ and let g_m have multiplier m on V_i for each i, with equal matrices on $1, 2, 3$. On a split plane, $\mathrm{diag}(m, 1)$ changes the swap by a rotation m, of quadratic value $\eta_q^{\mathrm{fld}}(m)$. Similarly, on a nonsplit plane V_i, multiplication by z of norm m changes the swap by z/z^q, with $(z/z^q)^{(q+1)/2} = m^{-(q-1)/2}$. Thus, on plane i the rotation has value -1 under θ_i, and the other rotations have value 1. The operator g_m gives a factor -1 at $w_i \in W_i$, the inversion of plane i. In contrast, for $i = 1$, the character δ has $\delta(w_4) = -1$. If $i = 4$, $\delta(-I_3) = -1$. Therefore, g_m gives five diagonal pairs on N_0. By Lemma 6.152, f_t fixes θ_i, W_i and the chosen coset representatives in $W(B_3) \times C_2$. Thus, on the rational model Y_{α_η}, the operator for f_t is 1 and has $2f$-th power 1. Consequently, each $\alpha_\eta|_{N_0}$ extends rationally to $N_0 \rtimes \langle f_t \rangle$.

At $i = 4$, the six-space V_+ has sign t, so set $x = tq \equiv 1 \bmod p$. By [40, §§11.3, 13.8], the p-part of the degree for λ is that of $\prod_{j=1}^{4}(x^j - 1)/\prod_{h\in\mathcal{H}(\lambda)}(x^h - 1)$, with hook multiset $\mathcal{H}(\lambda)$. Indeed, the centralizer index $[X^* : \mathrm{C}_{X^*}(s)]$ and the component factor 2 are prime to p. Thus, v_p is that of $4!/\prod_{h\in\mathcal{H}(\lambda)} h = \dim S^\lambda$, as on the local $\mathfrak{S}_4$. At $p = 3$, the partitions $(4), (2, 2), (1^4)$ remain, each with both signs. At $p \geq 5$, every $\lambda \vdash 4$ remains. Moreover, the orbit of $i = 1$ has length 3. Thus, for $p = 3$, the orbit contributes no p'-character by Lemma 6.149. At $p \geq 5$, V_+ has sign $-t$, so $x = -tq \equiv -1 \bmod p$. Each $\lambda \vdash 4$ has two even hooks in $\mathcal{H}(\lambda)$, matching the numerator exponents $2, 4$. Thus, v_p of the degree is zero, as $p \geq 5$ and all exponents are at most 4. Also, on N_0, the degrees from $W(B_2)$ are $1, 2$ and the orbit length is 3.

Pair the orbits $\{\chi, \chi^d\}$ and $\{\rho, \rho^d\}$ bijectively, with $\chi \mapsto \rho$ and $\chi^d \mapsto \rho^d$ for the class d of g_m. By the two lemmas, the extensions $\widehat{\chi}, \widehat{\rho}$ have values in $\mathbb{Q}$ and $\overline{F}_r = \overline{f}_t$ in $\mathrm{Out}(S)$. Moreover, the spinor norm $\nu(-I) = -1$ in $\mathrm{SO}_8^-(q)$ gives $S \cong \Omega_8^-(q)$. In $\mathrm{Aut}(S)_{\pi(Q)}$, the inner part is $N_0 = \mathrm{N}_S(\pi(Q))$. Since d exchanges ρ, ρ^d, the stabilizer of ρ in $\mathrm{Aut}(S)_{\pi(Q)}$ is $N_0 \rtimes \langle f_t\rangle$. Consequently, we consider the extension representations $\mathcal{P}, \mathcal{P}'$ via the respective maps to $\mathrm{Aut}(S)$. Their ordinary factor sets are 1, and the $\mathcal{H}_p$ twist functions are 1 since $\mathcal{P}, \mathcal{P}'$ are rational. Moreover, $\mathcal{P}, \mathcal{P}'$ are trivial on the kernel containing z_0. Thus, the triples satisfy $\succeq_c$. □

Together, Propositions 6.150 and 6.153 cover $p = 3$. Thus, only $p \geq 5$ dividing $q^2 - 1$ remains. Suppose that $p \geq 5$ and $p \mid q - t$. In Lemma 6.149, the $I(2, 2)$ orbit has a lift $\widetilde{s} = (-1, -1, 1, 1)$. The eigenspaces V_-, V_+ have types $+, -$, since $t \cdot t = 1$ on V_- and $t \cdot (-t) = -1$ on V_+. Thus, for both choices of t, the root permutation on A_1^4 is a transposition τ, which determines $[s]$ by Zhang [208, proof of Proposition 7.41]. In contrast, the second $[s']$ has a 4-cycle on A_1^4 and is absent from T^*.

Lemma 6.154. *In $\mathcal{E}(S, s)$, the nonidentity diagonal class d fixes two χ and gives four pairs $\{\chi, \chi^d\}$. Moreover, if $\chi^d = \chi$, the character χ extends rationally to $D \rtimes \langle F_r\rangle$, where $D = \mathrm{InnDiag}(S)$. Otherwise, we have that χ extends rationally to $S \rtimes \langle F_r\rangle$.*

Proof. Since τ exchanges the factors on V_+, its labels are $1, \mathrm{St}$ over $\mathbb{F}_{q^2}$. By [208, proof of Proposition 7.41, Step 2], the equal pairs $(1, 1), (\mathrm{St}, \mathrm{St})$ on V_- each have two constituents χ, χ^d, whereas $\{1, \mathrm{St}\}$ has one. Thus, we count $2(2 + 2 + 1) = 10$ characters. Their degrees are prime to p by [58, Proposition 11.5.6], since $[X^* : \mathrm{C}_{X^*}(s)]$ is prime to p, the connected labels have degrees $1, q, q^2$, and the rational component has order 2.

Set $S = \mathrm{SO}_8^-(q)/\{\pm I\}$ by Proposition 6.153. Then, $D = \mathrm{GO}_8^-(q)^\circ/Z$, with scalar subgroup Z, and D/S has order 2 with character $\eta_q^{\mathrm{fld}} \circ \mu$. Let B_D be its pinned Borel, and write $z_i, \mu/z_i$ on each split plane i. The character $\theta_D = \eta_q^{\mathrm{fld}}(z_1/z_2)$ on B_D is trivial on Z and the unipotent radical. Moreover, $\theta_D^{F_r} = \theta_D$. Since B_D maps onto D/S, the rational induced module $Y_D = \mathrm{Ind}_{B_D}^{D}\theta_D$ restricts to $Y_\theta = \mathrm{Ind}_{B_D\cap S}^{S}\theta$, where $\theta = \theta_D|_{B_D\cap S}$. Thus, Y_θ has rational operators for D and F_r.

Take a, b, u as in Lemma 6.151. On θ with marks $1, 2$, we put $A = T_a$ and $C = T_u$. Since $\theta \circ a^\vee = \theta \circ u^\vee = 1$, their quadratic relations have values q, q^2. The operator T_b to the character θ^b with marks $1, 3$ is invertible by the scalar $q\theta(n_b^2)$ in Lemma 6.151. Thus, $V = T_b^{-1}(q^{-1}T_u)T_b$ is an involution proportional over $\mathbb{Q}$ to T_w, where $w = bub$. Put $A' = VAV$. The words $awaw, wawa$ have length 8 and invert planes $1, 2$. Thus, the braid gives $AA' = A'A$. Moreover, $wu = uw$ has length 4 and $au = ua$. Hence, A, A', C give the joint eigenvalues (q or $-1, q$ or $-1, q^2$ or -1), and V exchanges A, A'.

The elements $1, a, w, aw, wa, awa, waw, awaw$ have lengths $0, 1, 3, 4, 4, 5, 7, 8$, and multiplication by u adds 1 to each length since they fix e_3. Thus, the words x, xu in this list give all 16 double cosets for the stabilizer W_θ. By the Bruhat presentation [58, §§6.1–6.2], $\mathrm{End}_{\mathbb{Q}S}(Y_\theta) \cong \mathbb{Q}^8 \rtimes \langle V\rangle$. The eight factors $\mathbb{Q}$ correspond to the eigenvalues of A, A', C, and V swaps the first two coordinates. Moreover, F_r fixes A, A', C, V since $F_r(n_x) = n_x$. Conjugation by D fixes A, C and acts on V by the character $\eta_q^{\mathrm{fld}}(\mu/z_2^2) = \eta_q^{\mathrm{fld}}(\mu)$ from the action of w on θ_D. Hence, D also fixes A'.

Write $e_{\mathbf{a}}$ for the joint idempotent at $\mathbf{a} = (a_1, a_2, a_3)$ for A, A', C. If $a_1 \neq a_2$, $e_{\mathbf{a}}$ is primitive in a block $\mathrm{M}_2(\mathbb{Q})$ and its image is stable under D, F_r. The two choices of a_3 give two blocks. If $a_1 = a_2$, $e_{\mathbf{a}}(1 \pm V)/2$ are rational primitive idempotents for $\mathbf{a}$. They are fixed by F_r and exchanged by d. Thus, the ten constituents χ of Y_θ have rational extensions. By [58, §11.3], each χ lies in $\mathcal{E}(S, s)$, so these exhaust $\mathcal{E}(S, s)$. □

Proposition 6.155. *Every χ of* Lemma 6.154 *has a corresponding p'-character on M with $\succeq_c$ for either $t = 1$ or $t = -1$.*

Proof. Let $T_S = T_{\mathrm{SO}}/\{\pm I\}$ and M_S be the image of M on S. Then, the signed swaps $w \in W(B_3)$ with parity correction on V_4 give $M_S = T_S \rtimes W(B_3)$. Indeed, on the algebraic torus $\mathbf{T}_{\mathrm{SO}}/\{\pm I\}$, each $w \neq 1$ still acts nontrivially. In $N_D(T_S) = T_D \rtimes W(B_3)$, put $\theta_{D,t} = \eta_{q-t}(z_1/z_2)$ on the similitude torus T_D. At $t = -1$, both z_i^{q+1} equal μ, so z_1/z_2 has norm 1. Thus, $\theta_{D,t}$ is defined for both t, is trivial at each scalar zI and restricts to $\theta_t = \eta_{q-t}(z_1)\eta_{q-t}(z_2)$ on T_S. Moreover, $\theta_{D,t}^{f_t} = \theta_{D,t}$. The image of T_D in D/S has order 2. Hence, on M_S, we obtain $\mathrm{Ind}_{T_S}^{M_S}\theta_t$ as the restriction of $\mathrm{Ind}_{T_D}^{N_D(T_S)}\theta_{D,t}$.

The endomorphism algebra of $\mathrm{Ind}_{T_S}^{M_S}\theta_t$ is $\mathbb{Q}[W_{\theta_t}]$, since θ_t extends trivially to $T_S \rtimes W_{\theta_t}$. Thus, $|W_{\theta_t}| = 16$ as above. With $A = a$, we put $V = w$ and $C = u$ and $A' = waw$. On V, the diagonal acts by the character $\eta_{q-t}(\mu/z_2^2) = \eta_q^{\mathrm{fld}}(\mu)$. Indeed, for $t = -1$, $\mu/z_2^2 = z_2^{q-1}$ and $(z_2^{q-1})^{(q+1)/2} = \mu^{(q-1)/2}$. Moreover, D acts trivially on A, A', C through the normalizer $N_D(T_S)$. Match $q, -1$ and $q^2, -1$ with $1, -1$ on A, A', C. If $a_1 = a_2$, keep the sign of V. Otherwise, the block is indexed by a_3. Thus, the rational idempotents match under d, F_r on Y_θ and d, f_t on $\mathrm{Ind}_{T_S}^{M_S}\theta_t$. By [95, Theorem 6.11], the degrees $\rho(1)$ are $3 \cdot 1$ or $3 \cdot 2$, so $p \geq 5$ divides none.

By Lemma 6.152, f_t and F_r have equal images in $\mathrm{Out}(S)$. The coordinate map f_t acts on $T_D \rtimes W(B_3)$ with $2f$-th power 1 by the description of F' on similitudes, since $\pi(nF(n)) = 1$. Thus, if $\rho^d = \rho$, then the local idempotent $e_{\mathbf{a}}$ gives a rational extension of ρ to $N_D(T_S) \rtimes \langle f_t \rangle$. Otherwise, it gives one to $M_S \rtimes \langle f_t \rangle$. Indeed, the inner part of $\mathrm{Aut}(S)_{\pi(Q)}$ is M_S. By Proposition 6.153, compose the rational representations $\mathcal{P}, \mathcal{P}'$ with the maps to $\mathrm{Aut}(S)$. Thus, both ordinary factors and $\mathcal{H}_p$-twist functions are 1, and $\mathcal{P}, \mathcal{P}'$ are trivial on the centralizer kernel $K \geq \langle z_0 \rangle$. Consequently, the triples satisfy $\succeq_c$. □

Lemma 6.156. *Suppose that* $4 \mid q - t$ *and* $p \geq 5$ *divides* $q - t$. *For the orbit of length* 12 *in* Lemma 6.149, *the connected centralizer* $\mathrm{C}^\circ_{\mathbf{X}^*}(s)$ *has semisimple type* A_1. *Thus, each label* $\nu \in \{1, \mathrm{St}\}$ *occurs with two orientations* β_+, β_- *of the source. The standard Levi* $\mathbf{L}$ *has simple roots* $\Delta \setminus \{\alpha\}$, *with* α *the central node of* D_4. *Thus, on* A_1^3 *the adjoint group has rational image* $\mathrm{PGL}_2(q^2) \times \mathrm{PGL}_2(q)$. *The image of* $L = \mathbf{L}^F$ *is* $\mathrm{PSL}_2(q^2) \times \mathrm{PGL}_2(q)$. *Moreover, the class has* $v = 1$ *in* Zhang [208, (7.44)].

Proof. Place the distinguished plane V_2 at coordinate 2 with eigenvalue 1. Then, by [208, Proposition 7.40], take $\beta_1 = e_1 - e_2,\ \beta_2 = e_1 + e_2,\ \beta_3 = e_3 + e_4$, and $\mu_0 = e_3 - e_4$. The point $x = (\beta_1 + \beta_2 + \beta_3)/4$ gives $\widetilde{s} = (-1, 1, i, i)$. Moreover, the graph γ changes the sign of e_2, so it exchanges β_1, β_2 and fixes μ_0, β_3. Thus, $\mathbf{L}$ is the Levi for μ_0 in the cited construction.

Write $J = -s_{\mu_0}$ for $s_{\beta_1}s_{\beta_2}s_{\beta_3}$ and $e = (q-1)/2 \bmod 2$. The torus twists for $t = 1, -1$ are $\gamma, s_{\mu_0}J\gamma$, respectively. Since $4 \mid q - t$, their ratios with $J^e\gamma$ are $1, s_{\mu_0}$, respectively. The reflection s_{μ_0} and the torus $\mathbf{T}^*$ have representatives in $\mathrm{C}^\circ_{\mathbf{X}^*}(s)$. Thus, $v = 1$ in [208, (7.44)] by Lang–Steinberg [58, §4.2] in $\mathrm{C}^\circ_{\mathbf{X}^*}(s)$. Since $n_J^e \in \mathbf{L}$, by Lang–Steinberg in $\mathbf{L}$ we choose the model $F = \gamma F_q$ on $\mathbf{L}$. The map $\mathbf{L}_{\mathrm{ad}} \to \mathrm{PGL}_2^3$ has connected torus kernel, since $\alpha, \beta_1, \beta_2, \beta_3$ form the basis of $X^*(\mathbf{T}_{\mathrm{ad}})$. Thus, L_{ad} maps onto $\mathrm{PGL}_2(q^2) \times \mathrm{PGL}_2(q)$, since γ exchanges the factors at β_1, β_2 and fixes β_3. The group $[\mathbf{L}, \mathbf{L}]^F$ gives $\mathrm{PSL}_2(q^2) \times \mathrm{PSL}_2(q)$. In simple coroot coordinates let $c_0 \in \mathbb{F}_q^\times$ be the central coordinate, and c_i the coordinates at $\beta_i^\vee$. Then, the root values are c_i^2/c_0, with $c_2 = c_1^q$ and $c_3 \in \mathbb{F}_q^\times$. The denominator c_0 is a square in $\mathbb{F}_{q^2}$ and takes both values of $\eta_q^{\mathrm{fld}}(c_0)$. Consequently, the image of L is $\mathrm{PSL}_2(q^2) \times \mathrm{PGL}_2(q)$. In the pinned model, F_r has order $2f$ on $\mathrm{PGL}_2(q^2)$ and f on $\mathrm{PGL}_2(q)$. □

Proposition 6.157. *Given any* t, *the quartic orbit gives four* p'*-characters on* X *and four on* M. *Moreover, on either set,* d *has two orbits* $\{\chi, \chi^d\}$, *and the paired triples satisfy* $\succeq_c$.

Proof. On $A = \mathrm{PGL}_2(q^2)$ and $B = \mathrm{PGL}_2(q)$, take the parents Θ_A, Θ_B of Lemma 6.44 with central sign $+1$. The trace-zero dual elements s_A, s_B have eigenvalues $i, -i$. The class s_A is split since $q^2 \equiv 1 \bmod 4$, and the type of s_B is split exactly when $q \equiv 1 \bmod 4$. Moreover, the tuple of $\mathrm{diag}(i, -i)$ at $\beta_1, \beta_2, \beta_3$ maps to x by $\beta_1^\vee, \beta_2^\vee, \beta_3^\vee$ and is fixed by $\mathrm{Int}(n_J^e)\gamma F_q$. In the pinned model, $\mathrm{C}_{\mathrm{SL}_2}(s_A)$ and $\mathrm{C}_{\mathrm{SL}_2}(s_B)$ are connected tori $\mathbf{T}_A, \mathbf{T}_B$. Thus, by [58, Proposition 4.2.14(ii)], each of s_A, s_B gives one rational class. From $\Theta_A \boxtimes \Theta_B$, we obtain the source at $v = 1$ via the quotient $L_{\mathrm{ad}} \to A \times B$. In particular, the kernel, including z_0 on L, acts trivially.

On L, Θ_A has two constituents on $\mathrm{PSL}_2(q^2)$, while Θ_B remains irreducible on B. In Lemma 6.44, the involution J_A satisfies $J_A\, B_A = B_A\, J_A$. Under a Galois automorphism σ acting on the r-th roots by b_σ, its scalar ratio is $\eta_{q^2-1}(b_\sigma) = 1$ since $b_\sigma \in \mathbb{F}_r^\times$. Thus, the two J_A-eigenspaces give rational-valued extensions $\widehat{\beta}_\pm$. The field operators satisfy $B_B^f = 1$ and $B_A^{2f} = 1$, so $B_A \otimes B_B$ gives the extension on $\langle F_r \rangle$ of order $2f$. A nonsquare diagonal on A exchanges the J_A-eigenspaces.

For $\nu \in \{1, \mathrm{St}\}$, [208, Proposition 7.40] pairs each $\beta_\pm$ with one $\chi_{\nu,\pm}$ of Harish–Chandra multiplicity 1. The labels $1, \mathrm{St}$ have distinct degrees $1, q$, so they are fixed by F_r, σ and d. By [208, Lemma 3.13], induction from $\widehat{\beta}_\pm$ gives the σ-fixed field extension $\widehat{\chi}_{\nu,\pm}$. Moreover, d exchanges $\chi_{\nu,+}, \chi_{\nu,-}$. Consequently, $[X^* : \mathrm{C}_{X^*}(s)]$, q and 2 are prime to p, and the Jordan formula [58, Proposition 11.5.6] gives $p \nmid \chi_{\nu,\pm}(1)$.

On T_S, set $\widetilde{s} = (-1, i, i, 1)$ with the distinguished plane at coordinate 4. Then, we set $\theta(z) = \eta_{q-t}(z_1)\xi(z_2 z_3)$, where $\mathrm{ord}(\xi) = 4$ and $\xi^2 = \eta_{q-t}$. Since $4 \mid q - t$, $\theta(-I) = 1$. In the complement $W(B_3)$, the stabilizer of θ is $\langle a, v\rangle \cong C_2^2$, where $a = (23)$ and v inverts planes $1, 4$. Thus, we obtain four extensions $\theta_{\varepsilon,\delta}$ with values ε, δ on a, v, and their inductions have degree 12 by [95, Theorem 6.11]. The element j inverting planes $2, 3$ centralizes $\langle a, v\rangle$ and takes θ to θ^{-1}.

Let $P_0 = f_t$, as in Lemma 6.152. Put $h = 0$ if $r \equiv 1 \bmod 4$ and $h = 1$ otherwise. Thus, $\theta^{P_0} = \theta^{j^h}$, so the inertia group in $M_S \rtimes \langle P_0\rangle$ is generated by $T_S \rtimes \langle a, v\rangle$ and $j^h P_0$. The latter fixes a, v, has $2f$-th power 1 and has order $2f$ modulo M_S. Then, give $j^h P_0$ value 1 in the extension of $\theta_{\varepsilon,\delta}$. For Galois inversion of i, the element j acts by $\theta^\sigma \mapsto \theta$ and fixes all three generators $a, v, j^h P_0$. Thus, $\widehat{\rho}^\sigma_{\varepsilon,\delta} = \widehat{\rho}_{\varepsilon,\delta}$ for either image of i. A common nonsquare multiplier m acts on v by the sign $\eta_q^{\mathrm{fld}}(m) = -1$ on the marked plane 1, as in Proposition 6.153. Its matrices on V_2, V_3 are identical, so it centralizes a. Consequently, ε matches $1, \mathrm{St}$, and d matches $\delta = \pm 1$ with $\chi_{\nu,+}, \chi_{\nu,-}$. By Proposition 6.153, the extensions $\widehat{\chi}_{\nu,\pm}, \widehat{\rho}_{\varepsilon,\delta}$ give $\succeq_c$ via the maps to $\mathrm{Aut}(S)$. □

By Proposition 6.150, the remaining classes $[s]$ have both eigenvalues $1, -1$ and $p \geq 5$. Moreover, let m count the planes with eigenvalues in $\{1, -1\}$. The case $m = 4$ has only the sign eigenvalues ± 1. Exclude those classes and $[s_{\mathrm{q}}]$ of Proposition 6.157. Thus, we find $m \in \{2, 3\}$. Since only plane 4 has type $-t$, some sign plane has type t. On that plane, we have that $x = \pm 1$ satisfies $F(x) = x$, so the common sign is $+1$.

Lemma 6.158. *With $G_0 = \mathrm{SO}_8^-(q)$, we put $T_0 = \mathbf{T}_0^F$ and $M_0 = \mathrm{N}_{\mathbf{G}_0}(\mathbf{T}_0)^F$. The series $\mathcal{E}(X, s)$ and the local orbit $M\theta_s$ factor through S. Moreover, their lifts to G_0, M_0 are unique with scalar 1 at $-I$. The proper F-stable Levi $\mathbf{L}$ has $L = \mathrm{SO}(Y)^F \times \prod_i L_i$, with $\dim Y = 2m$ and L_i linear or unitary for a pair $\{x_i, x_i^{-1}\}$. Moreover, we have that $\mathrm{C}_{\mathbf{G}_0^*}(s_0) \leq \mathbf{L}^*$ for the lift s_0, and $(M_0)_{\theta_0} \leq N = \mathrm{N}_{\mathbf{L}}(\mathbf{T}_0)^F$ for the lifted weight θ_0.*

Proof. A rational lift of s to T_0^* corresponds to a character ϑ of T_0 with $\theta_s = \vartheta \circ \pi$. Thus, $\theta_s(z_0) = 1$, and z_0 acts trivially on $\mathcal{E}(X, s)$ by [58, §11.3]. Since $\nu(-I) = -1$ for the spinor norm, $G_0 = \Omega_8^-(q) \times \langle -I\rangle$. Moreover, $\langle -I\rangle$ is a direct factor of T_0, M_0. The two lifts of θ_s differ by $\nu|_{T_0}$. Hence, only θ_0 has $\theta_0(-I) = 1$, selecting s_0 from the lifts $\{\widetilde{s}, -\widetilde{s}\}$. Field, graph and similitude maps fix $-I$, and Galois fixes 1. Thus, a map b carries θ_0 to the lift of θ_s^b with scalar 1. By [58, Proposition 11.3.10], $R_T^X(\theta_s) = (R_{T_0}^{G_0}(\theta_0)) \circ \pi$. Thus, $\mathcal{E}(G_0, s_0)$ gives the unique lifts of $\mathcal{E}(X, s)$ trivial on $-I$. Also, the weight orbit $M_0\theta_0$ gives the lifts on M_0 by [95, Theorem 6.11].

Given any nonsign pair $\{x, x^{-1}\}$, let $E_x, E_{x^{-1}}$ be the dual eigenspaces. On E_x, we choose distinct positive cocharacter weights a_x. Then, the weights on $E_{x^{-1}}$ and $Y = E_1 \oplus E_{-1}$ are $-a_x, 0$, respectively. Write λ for this cocharacter. If x occurs on planes of both types $t, -t$, then $x^{q-t} = x^{q+t} = 1$ implies $x^2 = 1$. Thus, for each nonsign pair $\{x, x^{-1}\}$, Frobenius fixes or swaps $E_x, E_{x^{-1}}$. Consequently, $\mathbf{L}^* = \mathrm{C}_{\mathbf{G}_0^*}(\lambda(\mathbb{G}_m))$ is F-stable, and L_i is linear or unitary as F fixes or exchanges $E_x, E_{x^{-1}}$. Given an element of $\mathrm{C}_{\mathbf{G}_0^*}(s_0)$, the determinants on $E_x, E_{x^{-1}}$ are reciprocal, so the determinant on Y is 1. Hence, $\mathrm{C}_{\mathbf{G}_0^*}(s_0) \leq \mathbf{L}^*$, also for disconnected $\mathrm{C}_{\mathbf{G}_0^*}(s_0)$. The Weyl image of $(M_0)_{\theta_0}$ fixes s_0. Thus, $(M_0)_{\theta_0}$ fixes each E_x setwise and lies in N.

An ordinary restorer b of field degree 0 fixes s_0, also when b includes the graph γ. Thus, b gives $b(E_x) = E_x$ for every x, fixes λ and normalizes the radical $\mathbf{U}_\lambda$. For a mixed operation stabilizing θ_0, the coefficient exponent is prime to $|s_0|$. Consequently, the nonsign pairs $\{x, x^{-1}\}$ may be permuted, and Y remains stable. Given $s_i \in \mathrm{Z}(L_i^*)$, the linear character λ_{s_i} of [58, Proposition 11.4.8] twists the unipotent comparison on $L_i, \mathrm{N}_{L_i}(T_0 \cap L_i)$. Finally, if the m planes in Y have signs $t_1, \ldots, t_m$, then the sign of Y is $\prod_j t_j$. □

Proposition 6.159. *With $m = 2$, we put $H = \mathrm{SO}(Y)^F$ and $N_H = \mathrm{N}_H(T_0 \cap H)$. The two characters χ over the character label on H and their partners ρ on N_H satisfy $\succeq_c$ for the induced actions on Y. Moreover, the degrees $(\chi(1), \rho(1))$ are $((q^2+1)/2, 1)$ for Y^- and $((q+t)^2/2, 2)$ for Y^+. The scalar at $-I_Y$ on both models is $\eta_{q-t_i}(-1)$ for the marked plane of type t_i.*

Proof. If Y has type $-$, its planes have signs $t, -t$. The spinor norm gives $H = \Omega_4^-(q) \times \langle -I_Y\rangle$. In type D_2 [58, §4.3], F swaps the absolute factors A_1, A_1, and F^2 is trivial on the Weyl group $W(D_2)$ for this torus. Thus, $\mathrm{Spin}_4^-(q) = \mathrm{SL}_2(q^2)$, with inverse-image torus of order $q^2 - 1$. The maps from $\mathbb{F}_{q^2}^\times$ to the planes are $z \mapsto z^{q+1}$ and $z \mapsto z^{1-q}$, up to inversion

in the targets C_{q-1}, C_{q+1}. Thus, the lifted weight is η_{q^2-1}, of value 1 at -1. By [186, Theorem 4.1 and its proof], the two Weil characters $\chi_\pm$ on $\mathrm{SL}_2(q^2)$ pair with the extensions $\rho_\pm$ of η_{q^2-1} to $\mathrm{N}_{\mathrm{SL}_2(q^2)}(C_{q^2-1})$. Both pairs have scalar 1 on the kernel $\langle -I\rangle$ of the map $\pi_Y : \mathrm{SL}_2(q^2) \to \Omega_4^-(q)$. Moreover, C_{q^2-1} is the full inverse image under π_Y of the torus in $\Omega_4^-(q)$, so π_Y maps $\mathrm{N}_{\mathrm{SL}_2(q^2)}(C_{q^2-1})$ onto $N_H \cap \Omega_4^-(q)$. On $\langle -I_Y\rangle$, both models have scalar $\eta_{q-t_i}(-1)$. An induced automorphism b fixes $-I_Y$ and gives $b([H,H]) = [H,H] = \Omega_4^-(q)$. Hence, b fixes each direct factor of H setwise. Tensor the comparison for $\Omega_4^-(q)$ with the common linear character ω of $\langle -I_Y\rangle$. Since b fixes ω and $\omega^\sigma = \omega$, the tensor has the original functions $\mu_{(b,\sigma)}$ and scalar $\omega(-I_Y)$.

If Y has type $+$, each plane has sign t. With $R = \mathrm{SL}_2(q)^2$, we write $K = \Omega_4^+(q) = R/\langle(-I,-I)\rangle$. The roots $e_1 - e_2, e_1 + e_2$ give rank-one tori T_1, T_2 of order $q-t$. Thus, for $m(z_1,z_2) = z_1z_2$ or z_1/z_2, we have $\theta_H \circ m = \eta_{q-t} \boxtimes \eta_{q-t}$. On each $\mathrm{SL}_2(q)$, the Weil characters $\chi_\pm$ of degree $(q+t)/2$ match the extensions $\rho_\pm$ by [186, Theorems 4.1, 4.2 and proof of Theorem 4.4], including $q = 9, p = 5$. Moreover, the scalars at $-I$ match for both signs $\pm$. All four pairs are trivial on $\langle(-I,-I)\rangle$ and give pairs on $K, N_K = K \cap N_H$.

On $Y = \mathrm{M}_2(\mathbb{F}_q)$, the simultaneous conjugation $y \mapsto d_0 y d_0^{-1}$ for nonsquare $\det(d_0)$ represents the other coset of H/K. Choose the projective actions of d_0 in the adjoint tori of T_1, T_2 in R, so the coset has a representative $d \in N_H$. Thus, d exchanges the signs of $\chi_\pm$ and $\rho_\pm$ in both positions $1, 2$. Given a product pair (α, δ) on K, N_K, the d-orbits have order 2. Hence, by [95, Theorem 6.11], the inductions $\chi = \mathrm{Ind}_K^H \alpha$ and $\rho = \mathrm{Ind}_{N_K}^{N_H} \delta$ are irreducible. Each has scalar $t\eta_q^{\mathrm{fld}}(-1) = \eta_{q-t}(-1)$ at $-I_Y$. By [58, Proposition 11.3.10 and §12.5], the torus induction $R_{\mathbf{T}_H}^{\mathbf{H}}(\theta_H)$ on restriction to R is, up to sign, $\sum_{\varepsilon,\varepsilon'} \chi_\varepsilon \boxtimes \chi_{\varepsilon'}$. Thus, the two induced χ give the series for θ_H. On N_H, the weight θ_H has orbit length 2 and stabilizer quotient of order 2. Consequently, the two induced ρ exhaust the orbit of θ_H.

The models for (α, δ) are tensors by [141, Theorem 3.2 and Lemmas 2.5–2.7], as for R in Proposition 6.143. Then, in each R_j, transport the $+$ models by the nonsquare diagonal d_j to the $-$ models. On tensor spaces $V_1 \otimes V_2, W_1 \otimes W_2$, a swap acts by $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ and by $w_1 \otimes w_2 \mapsto w_2 \otimes w_1$, with the rank-one operators $P_j(b), P_j'(b)$ at the signs $\varepsilon, \varepsilon'$. Thus, the swap can have $\varepsilon \neq \varepsilon'$. For (b, σ) in the mixed stabilizer, the tensor isomorphism $C_1 \otimes C_2$ intertwines the two actions of (12). Moreover, the kernel $\langle(-I,-I)\rangle$ acts trivially on α, δ.

Let B be the local ordinary stabilizer of ρ over N_H, and $B_0 = B_\delta$. Then, we have that B permutes δ, δ^d, and correction by d restores δ and α together. Thus, $B = N_H B_0$ and $B_0 \cap N_H = N_K$, giving $O = B/N_H \cong B_0/N_K$. In the global overgroup put $A_0 = KB_0$, so $A_0/K \cong O$. On $A_0 \times_O B_0$, the operator at (kb, b) with $k \in K$ is $(\mathcal{P}_\alpha(k) \otimes 1)W(b)$, with $\mathcal{P}_\alpha(k)$ the ordinary matrix and $W(b)$ the matched tensor operator. Since $(-I,-I)$ acts trivially on α, the matrix $\mathcal{P}_\alpha(k)$ is independent of the lift to R. Moreover, at $b \in N_K$ the inner relation at b gives the ordinary matrices on α, δ. By Proposition 6.143, the models on A_0, B_0 induce models of χ, ρ on the common cover $\widetilde{O}$. Thus, μ is also the mixed function on O. At a centralizer element z, on each summand indexed by N_H/N_K the two scalars at z coincide. Consequently, χ, ρ satisfy $\succeq_c$ for the induced actions on Y. □

Lemma 6.160. *For $m = 3$ in* Lemma 6.158*, we write $H = \mathrm{SO}_6^\epsilon(q)$ for the form of sign ϵ and $N_H = \mathrm{N}_H(T_0 \cap H)$. Moreover, the dual class for H determines the scalar at $-I_Y$. Take $(G, C) = (\mathrm{GL}_4(q), \mathbb{F}_q^\times)$ for $\epsilon = +$ and $(\mathrm{GU}_4(q), \mu_{q+1})$ for $\epsilon = -$. On $E = G \times C$, we put $u(g,z) = \det(g)/z^2$, $K = \ker u$ and $Z_1 = \{(cI, c^2) : c \in C\}$. Then, $H = K/Z_1$, with $-I_Y$ represented by $(I, -1)$. For the parent torus $\mathbf{T}_G$, we put $N_G = \mathrm{N}_{\mathbf{G}}(\mathbf{T}_G)^F$. Thus, we find $N_H = (K \cap (N_G \times C))/Z_1$. The plane signs t,t,t and $t,t,-t$ give parent cycle types $1^4, 2^2$, respectively.*

Proof. Put $\varphi(g,z) = z^{-1} \bigwedge^2 g$ [Taylor, *Geometry of the Classical Groups*, (12.19)] on the algebraic kernel $\mathbf{K}$. Thus, φ maps onto $\mathrm{SO}(\bigwedge^2 \overline{\mathbb{F}}_r^4)$, with connected kernel $\mathbf{Z}_1 = \{(cI, c^2)\}$. If $\epsilon = -$, the Frobenius on $\mathbf{K}$ is $(g,z) \mapsto ((g^{(q)})^{-T}, z^{-q})$. By Taylor, Lemma 12.27, the map corresponds under φ to the diagram twist of F_q on D_3. Therefore, Lang–Steinberg [58, §4.2] for $\mathbf{Z}_1$ gives $H = K/Z_1$, and $\varphi(I,-1) = -I_Y$. Moreover, $k \in K$ normalizes $\mathbf{T}_K = \varphi^{-1}(\mathbf{T}_0 \cap \mathbf{H})$ exactly when $\varphi(k)$ normalizes $\mathbf{T}_0 \cap \mathbf{H}$. Thus, $\mathrm{N}_K(\mathbf{T}_K) = K \cap (N_G \times C)$ gives N_H after quotienting by Z_1. On T_G, $\det : T_G \to C$ is onto since each cycle torus T_j maps by its norm to C. Thus, u induces $E/K \cong C$ and $(N_G \times C)/(K \cap (N_G \times C)) \cong C$.

For plane signs t,t,t, we have $\epsilon = t$, $d' = \mathrm{ord}_p(\epsilon q) = 1$ and normalized Frobenius eigenvalues $1,1,1$. In contrast, for signs $t,t,-t$, we have $\epsilon = -t$, $d' = 2$ and eigenvalues $-1,-1,1$ after dividing Frobenius by ϵq. Thus, in $\mathfrak{S}_4$, an involution w with this spectrum corresponds to (12)(34). Thus, the cycle type in A_3 is 1^4 or 2^2, on all 4 positions of T_G. □

Proposition 6.161. *The dual class for G has eigenvalues $a, a, -a, -a$. If $a, -a$ are rational, the series on H, N_H have five characters each. When F^* swaps $a, -a$, the counts on H, N_H are both 4. Moreover, every degree is prime to*

p, and the pairs have scalar $\kappa(-1)$ at $-I_Y$ for the torus extension $\theta_G \boxtimes \kappa$.

Proof. Lift the weight of H to $\mathbf{T}_K^F$ and extend it to $\theta_G \boxtimes \kappa$ on $T_G \times C$. Moreover, two extensions differ by $\delta \circ u$ for $\delta \in \mathrm{Irr}(C)$. At complementary pairs $\{i,j\},\{k,l\}$, the exterior-square weights are $z_i z_j/z$ and $z_k z_l/z$. They are inverse since $\det(g) = z^2$. For the eigenvalues $\varepsilon_i \in \{1,-1\}$ on the three planes Y_i, two ε_i agree and ε_k differs. Thus, the roots of value 1 form A_1^2. Thus, in A_3, the four entries are a, a, b, b. The projective point has order 2, so $b/a = -1$. Moreover, $\kappa(-1)$ is the scalar at $-I_Y$.

At $d' = 1$, both entries $a, -a$ are rational. At $d' = 2$, if F^* fixes the entries $a, -a$, each 2-cycle is constant, one with a and one with $-a$. Otherwise, each cycle contains $a, -a$. Thus, $C_2 \wr \mathfrak{S}_2$ gives one rational W_G-orbit in T_G^* in each case. Its labels are (λ,μ), with $\lambda,\mu \vdash 2$ for the two rational entries, or $\lambda \vdash 2$ for the degree 2 orbit. At $d' = 1$, the hook formula [40, §13.8] gives $p \nmid \chi_G(1)$ since $p > 4$. At $d' = 2$, put $x = \epsilon q$. Then, $x^2 - 1, x^4 - 1$ are the numerator factors with positive p-valuation. With rational entries each partition $\lambda,\mu \vdash 2$ has one even hook h, while the degree 2 orbit gives two even exponents $2h$. Thus, each even exponent k contributes $v_p(x^2-1)$, since $k \in \{2,4\}$ and $p \geq 5$. Hence, the numerator and denominator contribute $2v_p(x^2-1)$ each. Also, the local Weyl groups $\mathfrak{S}_4$ and $C_2 \wr \mathfrak{S}_2$ have p'-order.

The twist $\delta \circ u$ acts by $\delta \det$ on G and δ^{-2} on C. Thus, any self-twist has $\delta^2 = 1$. For rational entries $a, -a$, the quadratic twist acts by $(\lambda,\mu) \mapsto (\mu,\lambda)$. By [95, Corollaries 6.17 and 11.22], a parent (λ,λ) gives two constituents, and $\{(\lambda,\mu),(\mu,\lambda)\}$ with $\lambda \neq \mu$ gives one. For the degree 2 orbit, the quadratic twist fixes both $\lambda \vdash 2$, so each gives two constituents on K. The local determinant twist also gives $(\lambda,\mu) \mapsto (\mu,\lambda)$ or fixes λ, respectively. Hence, the counts on H, N_H are $5, 4$, with all degrees prime to p.

The dual map of $\mathbf{K} \hookrightarrow \mathbf{E}$ has connected central kernel $\mathbf{C}^*$. By [58, Lemma 4.2.13(i)], the lifts of $[s_H]$ form one $(\mathbf{C}^*)^F$-orbit of rational classes, with s_H the dual element for K. By [58, Proposition 11.3.10], the restrictions from $\mathcal{E}(E, \widetilde{s}_H)$ exhaust $\mathcal{E}(K, s_H)$, up to the twists $\delta \circ u$. At (cI, c^2) each parent has scalar $\omega(c)\kappa(c^2) = 1$, since the weight comes from H and ω is the scalar character on G. Thus, each restriction descends through Z_1. Locally, $\mathbf{T}_K$ and its image in $\mathbf{H}$ have equal Weyl groups by φ. Given $N_H\theta_H$, twist the parent by $\delta \circ u$ to give weight $\theta_G \boxtimes \kappa$ on $T_G \times C$. Consequently, the scalar at $-I_Y$ is $\kappa(-1)$, fixed by δ^{-2}. □

Proposition 6.162. *For the induced actions on H, N_H, the pairs (β,ν) of* Proposition 6.161 *satisfy $\succeq_c$ with scalar $\kappa(-1)$ at $-I_Y$.*

Proof. By [79, Theorem 2.5.1], write an action in type A_3 as $\mathrm{Int}(\bar{n})\gamma^e F_r^h$, with $\bar{n}$ in the adjoint group. Thus, on G, we choose $\gamma(g) = g^{-T}$ in standard coordinates. Moreover, on C the power is $k = (-1)^e r^h$. Thus, u changes by the k-th power and $\mathbf{Z}_1$ is stable. The duality identity for φ gives the induced action on $H = K/Z_1$. Moreover, we choose $\gamma^e F_r^h$ in the coordinates of T_G. Then, the adjoint correction $\bar{n}$ lifts to $n \in N_G$ by Lang–Steinberg in $\mathrm{Z}(\mathbf{G})$. Changing n to cn contributes $\omega(c)$ on both parent models.

In $\mathrm{Out}(H')$ for the simple quotient H' of $[H,H]$, the class of $\gamma^e F_r^h$ determines e, h up to the unitary relation $F_q = \gamma$. Moreover, that relation holds on G and C. Thus, a lift acting as 1 on H has $\gamma^e F_r^h = 1$ in the parent action and n scalar. The lift then acts as 1 on E. The coordinate change to T_G conjugates each lift u_0 and its powers u_0^j. In particular, we have that $(g,z) \in K \cap (N_G \times C)$ induces $\mathrm{Int}(g)$ on E.

If $G = \mathrm{GL}_4(q)$, take Proposition 3.14 at $d' = 1$, and the cochain of Proposition 3.20 at $d' = 2$. If $G = \mathrm{GU}_4(q)$, we choose the models P_i from Lemma 3.48, with normalization at a_u^{2f}. Thus, the factors for P_1, P_2 satisfy $\alpha_1 = \alpha_2$ when the parent labels return with a determinant twist δ. If u_0 is a field lift and $g \in N_G$, $(gu_0)^m = (\prod_{j=0}^{m-1} u_0^j g u_0^{-j}) u_0^m$. Both determinant corrections evaluate at $(gu_0)^m u_0^{-m}$ as in Section 6.7.

Tensor the models P_i with κ. Then, for a mixed operation (b,σ) returning the parent labels up to a determinant twist, choose the correction $\delta \circ u$. On $G \times 1$, the covariance of P_i has the determinant correction δ on the inner matrix g. On $1 \times C$, the covariance of κ is the comparison with $\kappa\delta^{-2}$, common to $i = 1, 2$. Thus, at $u(e_1) = u(e_2)$ the two twists cancel, and $\alpha_1/\alpha_2 = 1$. By Proposition 6.146, we choose $V_1 \otimes V_2^\vee$ when the self-twist group is 1. Otherwise, for the nontrivial quadratic δ take J_i normalized by a common scalar to $J_i^2 = 1$. Each $V_i^\pm = \ker(J_i \mp 1)$ affords one constituent, and an element g_0 with $\delta(u(g_0)) = -1$ swaps V_i^+, V_i^-. Thus, for each sign take $V_1^\pm \otimes (V_2^\pm)^\vee$ in $\ker(J_1 \otimes J_2^\vee - 1)$. For such (b,σ), the operators J_1, J_2 change by a common scalar $c_{b,\sigma}$. Thus, the summands have signs $(+,+)$ or $(-,-)$.

Given a selected pair (β,ν), we write D_b for the paired operator at $\delta \circ u$. Another correction δ' differs by a self-twist δ'/δ, trivial on $V_\beta \otimes V_\nu^\vee$. Hence, $D_b D_c = D_{bc}$ on this space. At (xb, b), with $x \in H$, the fibre product matrix

is $(\mathcal{P}_\beta(x)\otimes 1)D_b$, where $\mathcal{P}_\beta(x)$ is the ordinary matrix for x. At an inner endpoint (g,z), the two factors $\kappa(z)$ cancel in D_b, and Z_1 acts trivially. Therefore, the restriction is $\beta\boxtimes\nu^\vee$ on $H\times N_H$. For b acting as 1 on H, the lift and correction are both 1. For (b,σ) in the mixed stabilizer, the paired operator $D_{b,\sigma}$ intertwines the extension with its (b,σ)-translate. Consequently, the fibre product gives $\succeq_c$ with scalar $\kappa(-1)$ on each side at $-I_Y$. □

Theorem 6.163. *Let q be an odd prime power, $S=\mathrm{P}\Omega_8^-(q)$, and $p\mid|S|$. Then,* iMN(S,p) *holds, with universal cover X. At odd p with $d=1,2$, take $M=\mathrm{N}_X(Q)$ from* Lemma 6.149. *Instead, at $d=3,4,6,8$, take $M=\mathrm{N}_X(P)$.*

Proof. For the non-quasi-isolated $[s]$ with both eigenvalues $1,-1$, set $L=\mathrm{SO}(Y)^F\times\prod_i L_i$ from Lemma 6.158. Then, on $\mathrm{SO}(Y)^F$, the comparison is Proposition 6.159 for $\dim Y=4$ and Proposition 6.162 for $\dim Y=6$. Moreover, L_i is a torus, $\mathrm{GL}_j(q)$ or $\mathrm{GU}_j(q)$ with $j\le 2$. On L_i, apply [156, Theorem D] on $\mathcal{E}(L_i,1)$ with the twist λ_{s_i} of Lemma 6.158. At (b,σ), the permutation τ of the L_i acts on V by [141, proofs of Theorems 2.7 and 3.2], with the comparison above on Y. Moreover, $V^{(b,\sigma)}\cong V$. On each side the scalar at $-I$ is $\theta_0(-I)=1$.

With $I=(M_0)_{\theta_0}$, write $N=\mathrm{N}_L(T_0)$. Then, $I=N_{\theta_0}$ by Lemma 6.158. Let $\beta\in\mathcal{E}(L,s_0)$ be paired with $\nu\in\mathrm{Irr}(N)$. By [95, Theorem 6.11], the θ_0-space of ν affords $\rho\in\mathrm{Irr}(I\mid\theta_0)$. Thus, $\nu=\mathrm{Ind}_I^N\rho$ and $\phi=\mathrm{Ind}_I^{M_0}\rho$. By [58, Theorem 11.4.3], signed induction $\chi=\pm R_{\mathbf{L}}^{\mathbf{G}_0}(\beta)$ gives the bijection with $\mathcal{E}(G_0,s_0)$, since $\mathrm{C}_{\mathbf{G}_0^*}(s_0)\le\mathbf{L}^*$. The index $[G_0:L]$ is prime to p since T_0 contains a Sylow p-subgroup for $p\ge 5$. Moreover, $[M_0:I]$ divides 48. Hence, $p\nmid\chi(1)\phi(1)$.

For a mixed operation (b,σ) stabilizing (χ,ϕ), the weights of ϕ lie in $M_0\theta_0$, with scalar 1 at $-I$. Choose $m_0\in M_0$ so that $\theta_0^{(m_0b,\sigma)}=\theta_0$. By Lemma 6.158, the corrected action m_0b gives $(m_0b)(\mathbf{L})=\mathbf{L}$, since the coefficient exponent k satisfies $\gcd(k,|s_0|)=1$. Thus, the reciprocal pairs $\{x,x^{-1}\}$ are permuted. By signed induction on $\mathcal{E}(L,s_0)$ and Clifford theory at θ_0 $\beta^{(m_0b,\sigma)}=\beta$ and $\rho^{(m_0b,\sigma)}=\rho$. Moreover, any other correction m_0' differs by $m_0'm_0^{-1}\in I$, with ordinary matrices on β,ρ.

In Proposition 6.134, we let B' be the local ordinary stabilizer of θ_0 and put $H'=LB'$. Then, $B'\cap M_0=I$, and $H'/L\cong B'/I\cong A/G_0$ for the ambient stabilizer A. Let $e_{\theta_0^{-1}}$ project the $\nu^\vee$-factor of V onto the weight θ_0^{-1}. Then, $e_{\theta_0^{-1}}$ satisfies $e_{\theta_0^{-1}}B'=B'e_{\theta_0^{-1}}$ and is fixed by (m_0b,σ). Thus, the module $e_{\theta_0^{-1}}V$ gives the ordinary extension of $\beta\boxtimes\rho^\vee$ with the paired diagonal scalars on $\mathrm{C}_A(G_0)$.

By Lemma 6.158, each $b\in H_1'$ of field degree 0 normalizes $\mathbf{U}_\lambda$, also if its diagram part is γ for $F=\gamma F_r^f$. If a generates H'/H_1' of order $h>1$, its lift φ has $\varphi^h=K_{a^h}F$ by the field-lift relations. At $h=1$, set $\varphi=F$. The diagram group is $\langle\gamma\rangle$, with $F_r^f=\gamma$ on G_0, so K_b is unique at degree h. Thus, for $z\in\mathrm{C}_A(G_0)$, we have $K_z=1$. By Proposition 6.134, the radical comparison on $\mathcal{E}(L,s_0)$ and induction from I give $\succeq_c$ at χ,ϕ. Moreover, their scalar 1 at $-I$ gives pairs on S, so inflation gives $\succeq_c$ on X,M.

At $d=1,2$, Lemma 6.149 gives the admissible classes $[s]\ne[1]$. The argument for s_0 gives the remaining pairs (χ,ϕ) after Propositions 6.150, 6.153, 6.155 and 6.157. Moreover, at $s=1$ the comparison on M is [156, Theorem D]. At $p=3$, the sole admissible quasi-isolated $[s]\ne[1]$ is $I(1,3)$ on plane 4. Thus, on each orbit of $\Delta\times\mathcal{H}_p$, transport the corresponding pairs. At $d=3,4,6,8$, we apply Proposition 6.148. At $p=r$, apply [162, Theorem 1.3]. At $p=2$, apply [159, Theorem A(3)]. □

6.7 Rank-five tori

Proposition 6.164. *Let $X=\mathrm{Spin}_{2n}^\epsilon(q)$, with q odd and $n\ge 5$. For odd $p\nmid q$, we put $d=\mathrm{ord}_p(q)$, $e=d/\gcd(d,2)$ and $\tau=(-1)^{d+1}$, as in* Section 6.2. *If P has one active block of rank e, then* iMN(S,p) *holds on X with $\mathrm{N}_X(P)$.*

Proof. By the block model of Section 6.2, $P\simeq C_{(q^e-\tau)_p}$. The single active block gives $\Phi_d(q)$ with multiplicity 1 in the generic order. By [130, Proposition 2.2 and its proof], $\mathrm{N}_X(P)=\mathrm{N}_X(\mathbf{S})$ and $C_{\mathbf{X}}(P[p])=C_{\mathbf{X}}(\mathbf{S})$ for a Sylow d-torus $\mathbf{S}$ containing P. Thus, P is central and Sylow in $C_X(P[p])$, so $\mathrm{N}_X(P[p])=\mathrm{N}_X(P)$. Hence, Proposition 2.9 gives iMN(S,p) on X. □

Lemma 6.165. *With $n=5$, we put $d=4$. The torus has two negative 2-cycles and one singleton of type ϵ. Moreover, $W=\mathrm{N}_X(T)/T\cong C_4\wr\mathfrak{S}_2$. The nonidentity quasi-isolated orbits on T^* are $I(2,3)$ of length 2, $I(1,4)$ of length 1, and $Q(2,1)$. The last length is 2 if $\epsilon q\equiv 1 \bmod 4$ and 4 otherwise.*

Proof. At $n=5$, the order factors away from r are $q^5-\epsilon$ and $q^{2i}-1$ for $1\le i\le 4$. Thus, the block count m is at least 2 exactly at $d=1,2,4$. At $d=3,6,8$, and at $d=5$ for $\epsilon=1$ or $d=10$ for $\epsilon=-1$, we have $m=1$. Then, at

$d = 4$, each negative 2-cycle has a signed rotation r_i of order 4. The total parity of $r_1^a r_2^b$ determines the inversion on plane 5. Moreover, the exchange π permutes the two blocks. Hence, $W = \langle r_1, r_2, \pi\rangle \cong C_4 \wr \mathfrak{S}_2$ has order 32. The order formula gives $|P| = (q^2+1)_p^2$, so $P = T_p$. $\mathrm{C}_X(P) = T$, since the active weights on P are distinct and the residual group $\mathrm{SO}_2^\epsilon(q)$ is toral. By [132, Theorem 5.9 and Proposition 5.11], the W-orbits on T^* give the p'-index classes $[s]$.

Given a lift x in the natural dual torus, put $F(x) = cx$, with $c = \pm 1$. Thus, each active block has coordinates (a, ca^q) with $a^{q^2+1} = 1$. The singleton coordinate y satisfies $y^{\epsilon q - 1} = c$. Since $q^2 + 1 \equiv 2 \bmod 8$, the active entries cannot be $\pm i$. By [208, Lemma 7.15], the possible types are $I(2,3), I(1,4), Q(2,1)$. For a sign involution, $c = 1$ and the lift modulo $\{\pm I\}$ can have $y = 1$. Hence, $I(2,3)$ has one negative block, and $I(1,4)$ has two.

Given $Q(2,1)$, the quartic slot is $y = \pm i$, so $c = 1$ exactly when $\epsilon q \equiv 1 \bmod 4$. If $c = 1$, the sign blocks are $(1,1)$ and $(-1,-1)$. Modulo common negation, π, r_1 act transitively on $y \in \{i, -i\}$, giving length 2. If $c = -1$, each sign block is $(a, -a)$. Encode a_1, a_2 and y/i by three bits in $\mathbb{F}_2^3$. The rotations add $(1,0,1), (0,1,1)$, and common negation adds $(1,1,1)$. Thus, the three generators span $\mathbb{F}_2^3$, giving one orbit of length 4 on the quotient. □

Lemma 6.166. *For $I(2,3)$ and $I(1,4)$, the counts of p'-characters on X, M are 16, 14, respectively, on the torus T of* Lemma 6.165. *Given $Q(2,1)$, the counts are 10 if $c = 1$ and 5 if $c = -1$.*

Proof. Given $I(2,3)$, the orthogonal factors are $\mathrm{SO}_4^-(q)$ and $\mathrm{SO}_6^{-\epsilon}(q)$. The labels on $\mathrm{SO}_4^-(q)$ are $1, \mathrm{St}$, and those on $\mathrm{SO}_6^{-\epsilon}(q)$ are the four hooks $\lambda \vdash 4$ by the hook formula [40, §13.8]. Indeed, at q or $-q$ the only numerator index divisible by $d = 4$ is 4, so $p \nmid \chi_\lambda(1)$ exactly when λ has a hook of length 4. By [208, §7, following Lemma 7.15 and (7.19)], the component C_2 fixes the eight pairs (ν, λ) on $\mathrm{C}_{\mathbf{X}^*}(s)^\circ$. Thus, restriction from $\widetilde{X}$ gives two constituents per pair. Locally $W_{\theta_s} = \langle r_1, r_2\rangle \cong C_4^2$, with 16 linear characters.

In contrast, for $I(1,4)$, the connected nontoral factor is $\mathrm{SO}_8^+(q)$. By Lemma 6.59, p'-degree requires quotient weight 2 and cohook core of rank 0, since $u < 2$ leaves positive v_p. The relative degrees at weight 2 are prime to p. Write $k \in \mathbb{Z}/4\mathbb{Z}$ for a colour, with row exchange $k \mapsto k+2$. Thus, there are $4 \cdot 2 = 8$ labels supported in one colour and $\binom{4}{2} = 6$ with two. The fixed supports are $\{0,2\}, \{1,3\}$. Hence, the other 12 labels give six nondegenerate symbols Λ, each with two extensions under C_2. By [208, §7, following Lemma 7.15], C_2 exchanges the characters u_Λ^+, u_Λ^- for each degenerate Λ. Thus, the global count is $6 \cdot 2 + 2 = 14$, and the local count on W is $8 + 6 = 14$.

Given $Q(2,1)$ with $c = 1$, the orthogonal factors are both $\mathrm{SO}_4^-(q)$. By [208, (7.16)–(7.17)], C_4 exchanges the two factors $\mathrm{SO}_4^-(q)$, with its subgroup C_2 fixing the labels $1, \mathrm{St}$. Thus, each label (ν, ν) has four extensions. The orbit $\{(1,\mathrm{St}), (\mathrm{St}, 1)\}$ has stabilizer C_2 and gives two. If $c = -1$, F exchanges the sign spaces V_+, V_-. The bits $(e_\mathrm{F}, y_\mathrm{F}) = (1,1)$ of [208, (7.17)] and quartic multiplicity 1 give $\mathrm{SO}_4^+(q^2)$. The rational component C_2 swaps its two A_1's. Thus, the labels give $2 \cdot 2 + 1 = 5$ characters. Their connected degrees are r-powers. Moreover, the component orders $2, 4$ and the indices $[X^* : \mathrm{C}_{X^*}(s)]$ are prime to p.

At $c = 1$, a word $r_1^a r_2^b \pi^j$ stabilizes the weight if $a + b \equiv j \bmod 2$, since π swaps $(1,1), (-1,-1)$, requiring inversion on y. The stabilizer has order 16 and generators $h = r_1\pi, x = r_1^2$, with $h^8 = x^2 = 1$ and $xhx = h^5$. Thus, the four characters η of $\langle h\rangle$ with $\eta^x = \eta$ give 8 extensions, and the other 4 give two inductions of degree 2. If $c = -1$, we choose first coordinates $a_1 = a_2 = 1$. The action on $\mathbb{F}_2^3$ gives a, b even for both values of j. Thus, $W_{\theta_s} = \langle r_1^2, r_2^2, \pi\rangle$ is dihedral of order 8 and has character degrees $1, 1, 1, 1, 2$. By [176, Theorem 1.1], θ_s extends to M_{θ_s}, so induction from M_{θ_s} gives the characters over $W\theta_s$. □

Put $A = C_{q^2+1}$, $B = C_{q-\epsilon}$, and write $N_O = ((A \rtimes C_4) \wr \mathfrak{S}_2) \times (B \rtimes C_2)$ for the normalizer in $\mathrm{O}_{10}^\epsilon(q)$. Thus, we have that r_i acts by q-power on $\mathbb{F}_{q^4}$ in block i. Write η for the quadratic character of A. Then, the weights on $A^2 \times B$ are $\theta_1 = (\eta, 1, 1)$ and $\theta_2 = (\eta, \eta, 1)$. Extend θ_j trivially on the singleton inversion w_B and by ζ on its active stabilizer W_j. Thus, induction gives $\alpha_{j,\zeta}$ on N_O. Let $N_0 = \ker(\det, \nu)$ for the spinor norm ν on N_O. By Isaacs [95, Corollary 6.17], each $\alpha_{j,\zeta}|_{N_0}$ is irreducible. Indeed, we have that $\det(w_B) = -1$, and the twists involving ν change the weight 1_B to $\nu|_B \neq 1$. Thus, distinct $\alpha_{j,\zeta}$ of a given j have distinct restrictions to N_0 by Clifford theory. The pullbacks along π_0 are trivial at z_0.

Choose nonsquare $m \in \mathbb{F}_q^\times$ and $v \in \mathbb{F}_{q^4}^\times$ with $v^{q^2+1} = m$. Thus, multiplication by v has multiplier m on each active block. On plane 5, take a similitude g_5 of multiplier m. Moreover, conjugation changes r_i by v^{1-q}, with $\eta(v^{1-q}) = m^{(1-q)/2} = -1$. The multiplier satisfies $\mu \circ \pi = \mu$, and $\theta_j|_B = 1$. Therefore, on $I(2,3)$ the label (k,l) goes to $(k+2, l) \bmod 4$, giving eight pairs. On $I(1,4)$ each occupied colour k goes to $k+2$, with the same partition λ_k at $k+2$. Hence, the fixed supports are $\{0,2\}, \{1,3\}$. The parent values lie in $\mathbb{Q}(i)$, fixed by $\mathcal{H}_p$ since $p \equiv 1 \bmod 4$. Their scalar at the natural $-I$ is -1 on $I(2,3)$ and 1 on $I(1,4)$, since $q^2 + 1 \equiv 2 \bmod 4$.

Proposition 6.167. *Fix r_1, r_2, π and the labels $k \in \mathbb{Z}/4\mathbb{Z}$ of* Section 6.7. *Also, put $\delta = \eta_q^{\mathrm{fld}}(-1)$. Moreover, the mark on a flag E is the eigenvalue 1 or -1 on E. For either sign ϵ, the sixteen characters of $I(2,3)$ and the twelve moved characters of $I(1,4)$ satisfy $\succeq_c$ with their partners on M.*

Proof. First, for $I(2,3)$, the eigenspaces V_-, V_+ have dimensions $4, 6$ and signs $-1, -\epsilon$. Choose one marked line E_- in V_- and two in V_+ along nondegenerate symbols. Thus, on V_-, the labels $1, \mathrm{St}$ have coefficient 1 at the flag E_-. On V_+, take the steps of [208, Proposition 7.40] from rank 3, as in Lemma 6.119. Thus, Y_- is anisotropic and Y_+ has sign $-\epsilon$. Thus, if $\epsilon = 1$, take the odd Weil parent Φ of Lemma 6.57. If $\epsilon = -1$, choose the parent on $Y_- \perp Y_+$ from Lemma 6.88. In either case, $-I_{Y_- \perp Y_+}$ has scalar $-\delta$. The three line characters are $1, 1, \eta_q^{\mathrm{fld}}$, so the restored scalar at $-I$ is $(-\delta)\delta = -1$.

On $L_j = \mathrm{GL}_1(q) \times \mathrm{O}(Y_j)$, the determinant and spinor norm are $\det_{Y_j}$ and $\eta_q^{\mathrm{fld}} \boxtimes \nu_{Y_j}$. Thus, a twist by ν or $\det \nu$ changes the line character ξ_j, even if the parent on Y_j has a norm self-twist. Moreover, det changes the selected character on $\mathrm{O}(Y_j)$. By [95, Corollary 6.17], each source on L_j restricts irreducibly to $L_j \cap \Omega$. A reflection u_j and an isometry v_j with nonsquare norm on Y_j give both generators of O/Ω in the parabolic P_j. Hence, by [95], $\mathrm{Ind}_{P_j}^{\mathrm{O}}(\beta_j)|_\Omega = \mathrm{Ind}_{P_j \cap \Omega}^{\Omega}(\beta_j|_{P_j \cap \Omega})$ for the inflated source β_j.

By [208, Proposition 7.18, Steps 2–4], the matrix $\mathcal{B}_j$ at (Λ_-, Λ_+) is a permutation matrix. At (Λ_-, Λ_+) the connected coefficient is 1, and γ fixes Λ_-, Λ_+. Moreover, ξ_j distinguishes V_-, V_+ even at $j_- = j_+$. At the aligned rank 1 endpoint, $\langle \nu \rangle, \langle \det \nu \rangle$ distinguish the restrictions to L_j. At opposite forms, choose χ_1, χ_2 from Lemma 6.88. Moreover, $j_- \neq j_+$ for $j \geq 2$. Thus, the column of $\mathcal{B}_j$ gives multiplicity 1 in V_j. The comparison at (Λ_-, Λ_+) shows that $\mathcal{H}_p$ fixes the extended source and χ. By [208, Lemma 3.13], $\widehat{\chi}$ is an ordinary $\mathcal{H}_p$-fixed extension. The multiplier m transports $\widehat{\chi}$ to the other orientation.

Let $v = r_1 r_2$ for $\epsilon = 1$, and $v = r_1 r_2 w_B$ for $\epsilon = -1$, with singleton reflection w_B. Thus, in the field group for vF_q, the generator a satisfies $a^f = v^{-1}$. Put $R_j = r_j^{-1}$. The element a acts by r-power on T, fixes r_1, r_2, π, w_B and fixes the cosets represented by $1, \pi$. Moreover, the weight θ_s on T is a-fixed. Since $R_1 R_2 \in \mathrm{Z}(W)$ and the scalar on w_B is 1, the power a^f acts by i^{k+l} on both cosets at label (k, l). Put $D = \mathrm{ord}_p(r)$. Since $D/\gcd(D, f) = 4$, $(\mu_D)^f = \mu_4$. In μ_D, choose α with $\alpha^f = i^{k+l}$. As $D \mid p - 1$, $\mathcal{H}_p$ fixes α. Thus, $a \mapsto \alpha I$ gives the ordinary local extension with $a^f = v^{-1}$. Also, at any W-label the same choice applies if $R_1 R_2$ has scalar in μ_4.

On the global model write w for its twist over $\mathbb{F}_r$. Since $\det w = \det v$, Lang–Steinberg [58, §4.2] in SO gives g with $F_q(g) = w^{-1} g v$. Then, a becomes $g F_r(g)^{-1} a_w$, and $g F_r(g)^{-1}$ is rational for wF_q. Its f-th power is $g F_q(g)^{-1} w^{-1} = g v^{-1} g^{-1}$. Moreover, the ordinary extension $\widehat{\chi}$ on $\mathrm{O}(V)$ includes a correction u with $\nu(u) = -1$ when needed. At $I(2,3)$, we let $k = 0, 1$ correspond to $1, \mathrm{St}$ on V_-, and take $l = \ell(\lambda) - 1$ for the hook λ on V_+. The multiplier m changes the orientation by $k \mapsto k+2$. Moreover, the local normalizer N_O maps onto O/Ω via $(\det, \nu)$. Both extensions have scalar -1 at $-I$ and are trivial on z_0. Thus, the ordinary $\mathcal{H}_p$-fixed extensions give $\succeq_c$. On changing the field representative a to ua, we choose the matrices $P_\chi(u) P_\chi(a)$ and $P_\rho(u) P_\rho(a)$ on the respective models.

Given $I(1,4)$, the nondegenerate split symbols Λ of rank 4 have defect 0 or absolute defect 4. Thus, at defect 4, rank 4 is minimal, giving the cuspidal label Λ_C of 2-cohook weight 2. The other five Λ have defect 0. Consequently, for these Λ, restore the negative lines E_1, E_2, E_3 by [208, Proposition 7.18] from the rank-one symbol Λ_1. On split Y_+, take the even Weil parent Φ of Lemma 6.57. If Y_+ is anisotropic, we apply Lemma 6.88 with Y_- split. The initial scalar is δ, and the three lines contribute δ^3, so the final scalar at $-I$ is 1. In each matrix $\mathcal{B}_j$, the mark on Y_+ remains 1.

Given Λ_C, take [208, Proposition 7.25(ii)] with cuspidal rank 2^2 and positive rank 1. Thus, on anisotropic Y_+, we extend ϑ to the isometries fixing (V_-, V_+). Write C_ϑ for its source centralizer. On split Y_+, the factors of C_ϑ are O_1 and a dihedral $\mathrm{O}_2^{\pm}(q)$, with a linear character β_C on L_C valued in $\{1, -1\}$. Thus, on the multiplicity plane U_2, take a reflection u and a rotation v_0 of nonsquare norm. Their images under $(\det, \nu)$ generate O/Ω. Two extensions of a character of Ω differ by $\lambda \in \mathrm{Irr}(\mathrm{O}/\Omega)$. Hence, the values at u, v_0 on the source line L_C determine λ. Both F_r and the corrected coefficient action (d_σ, σ) fix β_C.

In $\mathrm{O}(V)(\mathbb{F}_r)$, we choose w with line signs ± 1 and reflections on U_2. Then, w centralizes the source ϑ and its grading. In the wF_q-model, we have that $a_w^f = w^{-1}$. The source ϑ is from $\mathrm{Tr}_{\mathbb{F}_{q^f}/\mathbb{F}_r}$, so $\vartheta^{a_w} = \vartheta$. Thus, on its multiplicity-one line L_C choose the field eigenvalue $\alpha \in \mu_D$ with $\alpha^f = \beta_C(w^{-1}) \in \{1, -1\}$. Since $(\mu_D)^f = \mu_4$, we may choose α as above. Schur's lemma determines a_w^f on the module from L_C. The half-grading isometry d_σ centralizes w and C_ϑ. Thus, on the line $d_\sigma L_C$, the ratio is $\sigma(\alpha)/\alpha = 1$ for $\sigma \in \mathcal{H}_p$. Hence, the field extension is $\mathcal{H}_p$-fixed. At $-I$, the two negative blocks give $\eta_{q^2+1}(-1)^2 = 1$. Locally, on $I(1,4)$ the scalar of $R_1 R_2$ lies in μ_4. Thus, the normalization of a

extends each local parent ρ. By Lemma 6.166, $k \mapsto k+2$ has 6 moved pairs. Then, for each Λ, match one pair (χ, ρ) and transport by m. Both extensions have scalar 1 at $-I$ and z_0. Hence, all twelve pairs (χ, ρ) give $\succeq_c$. □

Proposition 6.168. *For the remaining two pairs (χ, ρ) in $I(1,4)$, there are extensions $\widehat{\chi}, \widehat{\rho}$ on the similitude-field groups $\mathcal{A}, \mathcal{B}$. Whenever $\sigma \in \mathcal{H}_p$, we have that $\widehat{\chi}^\sigma = \widehat{\chi}$, $\widehat{\rho}^\sigma = \widehat{\rho}$, and both have scalar 1 at zI.*

Proof. On Y_+ of sign ϵ, we choose the trivial extension 1 on the similitude-field group $\Gamma(Y_+)$. On the marked isotropic plane E_1, we choose $\eta_q^{\mathrm{fld}} \circ \det_{E_1}$ on the flag stabilizer. Thus, on zI its value is $\eta_q^{\mathrm{fld}}(z^2) = 1$. In the split space V_- of dimension 4, the connected stabilizer in $A_1 \times A_1$ is $A_1 \times B_1$ or $B_1 \times A_1$, with B_1 a Borel. By [72, Comparison Theorem 3.2.7], the source 1 has coefficient 1 on one of $(1, \mathrm{St}), (\mathrm{St}, 1)$ and 0 on its mate. The component γ_1 acts by $(1, \mathrm{St}) \leftrightarrow (\mathrm{St}, 1)$, so the regular characters $\widetilde{\chi}_1^{\pm}$ restrict irreducibly to χ_1. By Lemma 6.79, in the flag module $V(E_1)$ the fraction for χ_1 is $q/(q+1)$, since the factor on Y_+ is a torus. On Y_+, the isometries give both values of $\det, \nu$, and the proper similitudes give all $m \in \mathbb{F}_q^\times$. The field generator a_1 also has a representative fixing E_1. Hence, $V(E_1)|_\Omega$ has fraction $q/(q+1) > 1/2$ at χ_1. By Lemma 6.78, the χ_1-space of $V(E_1)$ gives $\widehat{\chi}_1$.

Next, add the marked plane E_2 with character $\eta_q^{\mathrm{fld}} \circ \det_{E_2}$. The labels $\xi = (2), (1,1)$ arise from one-box additions to (1). By [208, Proposition 7.20(1), Step 1], the factor on Y_+ may have rank 1. Moreover, in $\widetilde{X}$, the coefficients at ξ^+, ξ^- are $1, 0$, with both restrictions irreducible on Ω. On the flag Levi L_2, the source is $\chi_1 \boxtimes (\eta_q^{\mathrm{fld}} \circ \det_{E_2})$ since the added plane has eigenvalue -1. Thus, each spin character χ_ξ occurs with multiplicity 1 in the induction on Ω. The field and similitude actions fix ξ, and the two orientations of ξ give the same χ_ξ. The coefficient action fixes ξ by [208, Proposition 7.20(1)]. Thus, the χ_ξ-space in $V(E_2)$ affords the ordinary extension fixed by $\mathcal{H}_p$ for either Y_+. The factors $\eta_q^{\mathrm{fld}} \circ \det_{E_j}$ have value 1 at zI.

Let $\widetilde{T}$ be the similitude torus with coordinates z_1, z_2 satisfying $z_j^{q^2+1} = m$. Moreover, on the singleton Y_0, the multiplier is m as well. Thus, all $m \in \mathbb{F}_q^\times$ occur, and the kernel of m on the similitude normalizer $\widetilde{N}$ is the isometry normalizer N_O of Section 6.7. $\widetilde{N} = \widetilde{T} \rtimes (W \times \langle r_0 \rangle)$, where r_0 inverts the singleton. The weight $\widehat{\theta} = \eta_{q^2+1}(z_1/z_2)$ has value 1 on zI and restricts to $\eta_{q^2+1} \boxtimes \eta_{q^2+1}$ on the two active tori. Moreover, π, r_0 fix $\widehat{\theta}$. Each rotation r_j changes it by $\eta_q^{\mathrm{fld}}(m)$, since $(z_j^{q-1})^{(q^2+1)/2} = m^{(q-1)/2}$. Thus, its complement inertia is $W_0 \times \langle r_0 \rangle$, where $W_0 = \{r_1^a r_2^b \pi^j : a + b \equiv 0 \bmod 2\}$.

If $k \in \{0, 1\}$, define a linear character ξ_k on W_0 by $\xi_k(r_1^a r_2^b) = i^{ka+(k+2)b}$ and $\xi_k(\pi) = 1$. The exchange π fixes the rotation values since $a + b$ is even. Conjugation by r_1 changes the value on π to -1. Thus, $\mathrm{Ind}_{W_0}^{W} \xi_k$ is the two-colour character on $\{k, k+2\}$, with π exchanging its two basis positions. Extend $\widehat{\theta}$ by ξ_k and value 1 at r_0, then induce to $\widetilde{N}$. By Clifford theory [95, Theorem 6.11], the restriction to N_O is the parent α_k on $\{k, k+2\}$. Moreover, we have that $\alpha_k|_{N_0}$ is irreducible by Section 6.7. Then, a fixes the cosets $1, r_1$ and their matrices. Thus, a^f acts by i^{2k+2} on both. In μ_D, we choose α with $\alpha^f = i^{2k+2}$ by Proposition 6.167. Hence, $a \mapsto \alpha I$ gives the extension fixed by $\mathcal{H}_p$ with scalar 1 at zI.

The two-cohook quotient matches $\xi = (2), (1,1)$ with $\{k, k+2\}$ for $k = 0, 1$. Moreover, the multipliers on $\widetilde{T}$ run through $\mathbb{F}_q^\times$, the complement gives $\det = \pm 1$, and the singleton torus gives $\nu = \pm 1$. Thus, $\widetilde{N}$ meets all such cosets modulo Ω. Moreover, transport a through SO by Proposition 6.167. By [79, Theorem 2.5.1], an action centralizing Ω has field degree 0 mod f. Its linear part lies in $\mathrm{C}_{\mathrm{GL}(V)}(\Omega) = \{zI\}$ since Ω acts absolutely irreducibly on V. Thus, both extensions have scalar 1 on the centralizer, giving $\succeq_c$. □

Corollary 6.169. *Let $X = \mathrm{Spin}_{10}^{\epsilon}(q)$ for odd $q = r^f$, and let $\mathrm{ord}_p(q) = 4$. Then, the fourteen characters of $I(1,4)$ and the sixteen of $I(2,3)$ satisfy $\succeq_c$ with their partners on M.*

Proof. Given $I(2,3)$, we have that the flag on $\mathrm{SO}_4^-(q)$ is over $\mathbb{F}_{q^2}$, and both $1, \mathrm{St}$ have coefficient 1. On the split positive six-space V_+, restriction from $\mathfrak{S}_4$ to $\mathfrak{S}_2 \times \mathfrak{S}_2$ has coefficient 1 at $1 \boxtimes 1$ on $(4), (3,1)$, and at $\mathrm{sgn} \boxtimes \mathrm{sgn}$ on $(2,1,1), (1^4)$. Instead, on nonsplit V_+, all four hooks λ have empty 2-core. Thus, restriction from $W(B_2)$ to $W(B_1)$ gives a sign of multiplicity 1. Then, the rank-two label 1 or St has coefficient 1 at a rational line E. By [208, Propositions 7.18(i) and 7.39], these give the marked coefficients 1. Hence, the endpoint planes have signs $-1, -\epsilon$ as in Proposition 6.167.

In $I(1,4)$, the negative space V_- is split of dimension 8. For each noncuspidal Λ of defect 0, Proposition 6.58 gives three steps $\Lambda_j \mapsto \Lambda_{j-1}$ through unequal rows to split Y_-. Thus, the weights are η_q^{fld} on the restored lines in V_-, also at equal rank. For $\xi = (2), (1,1)$ take the plane columns $\mathcal{B}_{E_j}$ of Proposition 6.168, with ratio $q/(q+1)$ at E_1 and coefficients $1, 0$.

Let Λ_C have absolute defect 4, and first take Y_+ anisotropic. The torus $\mathrm{SO}(Y_+)$ is anisotropic, and Λ_C is cuspidal on V_-. Thus, by [72, §4.7], the character χ_C on SO is cuspidal. Moreover, the component C_2 inverts $\mathrm{SO}(Y_+)$, so the involution s is quasi-isolated. By [165, Theorem 10.8], χ_C extends to $\widehat{\chi}_C$ on O, fixed by F_r and rational-valued. [208, Proposition 7.25(ii), Steps 2–4] gives sources $(U_\pm, \vartheta_\pm)$ of multiplicity 1. On their lines $L_\pm$, take the normalization of Lemma 6.238 with $(\mu_D)^f = \mu_4$.

If Y_+ is split, we let P_1 stabilize a line $E \leq Y_+$. Its source is $\beta = 1_{\mathrm{GL}_1} \boxtimes (\nu_- \psi_C)$, where ψ_C is the unipotent cusp on $\mathrm{SO}_8^+(q)$ and ν_- is the spinor norm. Moreover, γ fixes ψ_C. For the extension of ψ_C, apply [165, Theorem 10.8] at $s = 1$, and Lemma 6.238 for the power of a_w. The rational ν_--twist gives the extension of β. The relative Weyl group is C_2, since the exchange of the lines E, E' fixes 1_{GL_1} and ψ_C. By [72, Comparison Theorem 3.2.7], the squared norm of $R_{P_1}(\beta)$ is 2. Thus, it has two constituents $\chi_{C,1}, \chi_{C,2}$ of multiplicity 1. By [72, §4.7], both have connected label Λ_C on V_-. The component C_2 fixes Λ_C, so restriction from $\widetilde{X}$ gives a diagonal orbit of size 2. Thus, by [165, Theorem A], γ, F_r and $\mathcal{H}_p$ fix both $\chi_{C,i}$ on SO.

Induce the extended β from the isometry and field stabilizer P_E. The restriction to SO is $R_{P_1}(\beta)$, since P_E contains a reflection on V_-. By [208, Lemma 3.13], each $\chi_{C,i}$ has an $\mathcal{H}_p$-fixed extension $\widehat{\chi}_{C,i}$. For either Y_+, the spinor twist on SO changes s to $-s$, so the dimensions at $+1, -1$ change from $2, 8$ to $8, 2$. Hence, the restrictions to Ω are irreducible by [95, Corollary 6.17]. The operators for a^f and scalars at $-I$ match by Propositions 6.167 and 6.168. Thus, at (χ, ρ) we have $\succeq_c$ for the counts $14, 16$ of Lemma 6.166 at $I(1,4), I(2,3)$. □

Corollary 6.170. *Let $X = \mathrm{Spin}_{10}^\epsilon(q)$, with q odd and $d = \mathrm{ord}_p(q) = 4$. Moreover, for $[s] \neq [1]$ quasi-isolated on T^*, the pairs (χ, ϕ) between $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M \mid W\theta_s)$ satisfy $\succeq_c$.*

Proof. Propositions 6.167 and 6.168 give the pairs for $I(2,3)$ and $I(1,4)$. Next, for $Q(2,1)$ with $\epsilon q \equiv 1 \bmod 4$, we have $c = 1$ and two rational factors $\mathrm{SO}_4^-(q)$. By Lemma 6.27, $z_0 \in \ker \chi$ when $c = 1$. Each label $1, \mathrm{St}$ has weight 1 and core rank 0 at $e = 2$. The singleton label is $\lambda = (1)$, of weight 0 at $e = 2$, so $R = 1 < e$. Moreover, $p \geq 5$, and the relative degrees at weight 1 are prime to p. Thus, Theorems 6.70 and 6.90 give $\succeq_c$ for (ν, ν) and $(1, \mathrm{St})$, respectively.

By Lemma 6.68, on $C^2_{q^2+1}$ the weights are $1, \eta$, with $\eta^2 = 1 \neq \eta$, and the singleton weight is ξ of order 4. Moreover, the norm character is 1 on X, and η gives entries $(-1, -1)$. Thus, the two block orders and $y = \pm i$ give $Q(2,1)$ by Lemma 6.165. The squares r_1^2, r_2^2 have scalars $(-1)^k, (-1)^l$ on the weight spaces, with $k, l \in \mathbb{Z}/4\mathbb{Z}$. Moreover, at weight 1, the labels $1, \mathrm{St}$ give opposite values of $(-1)^k$, and row exchange acts by $k \mapsto k + 2$. On r_i^2 the determinant and norm characters equal 1. The block exchange acts by $(k, l) \mapsto (l, k)$, and the nonsquare multiplier adds 2 to a colour. Thus, $\{(-1)^k, (-1)^l\}$ distinguishes the orbits of $(1,1)$, $(\mathrm{St}, \mathrm{St})$ and $(1, \mathrm{St})$. The cited comparisons distinguish $4, 4, 2$ images for $(1,1), (\mathrm{St}, \mathrm{St}), (1, \mathrm{St})$, respectively. Hence, all 10 characters in $\mathrm{Irr}_{p'}(M \mid W\theta_s)$ occur by Lemma 6.166.

If $\epsilon q \equiv -1 \bmod 4$, we have $c = -1$ and $e = 2$. Proposition 6.32 gives the pairs on $\mathrm{SO}_4^+(q^2)$ at $j = e/2 = 1$, with $w = 2$, $a_0 = 0$, and $R = 1$. Each local model ρ has $\rho(z_0) = -I_\rho$. Moreover, the block weights θ_1, θ_2 and the spin plane weight θ_0 have order dividing 2. Thus, the torus isogenies of 2-power index give trivial $T_{2'}$-action on ρ. By Lemma 6.27, the 2-power order lift t has $F(t) = -t$. Since $v_2(q^2 + 1) = v_2(\epsilon q - 1) = 1$, the blocks are $(a, -a)$ with $a = \pm 1$, and $y^{\epsilon q - 1} = -1$ gives $y = \pm i$. Hence, each image lies over $Q(2,1)$, of length 4 by Lemma 6.165. The inverse on $\mathrm{Irr}(M)$ in Proposition 6.29 distinguishes the 5 images of the two nondegenerate labels $(1,1), (\mathrm{St}, \mathrm{St})$ and the orbit $\{(1, \mathrm{St}), (\mathrm{St}, 1)\}$. Thus, all 5 characters in $\mathrm{Irr}_{p'}(M \mid W\theta_s)$ occur by Lemma 6.166. For these pairs (χ, ϕ), the cited models give $\succeq_c$. Moreover, for $s = 1$, by [156, Theorem D], we obtain the relation $\succeq_c$. □

Lemma 6.171. *Keep the hypotheses of* Lemma 6.165. *Thus, $X = \mathrm{Spin}_{10}^\epsilon(q)$, $q = r^f$ is odd. Moreover, $d = \mathrm{ord}_p(q) = 4$, and $M = \mathrm{N}_X(P) = \mathrm{N}_X(T)$. For a lift t of $s \in T^*$, we write $F(t) = ct$ with $c \in \{1, -1\}$ and put $[v] = \{v, v^{-1}\}$. Moreover,* Lemma 6.165 *gives the entries (a, ca^q) with $a^{q^2+1} = 1$, and the singleton has entry y with $y^{\epsilon q - 1} = c$. Outside the quasi-isolated classes, t and $-t$ are conjugate in $\mathrm{O}_{10}(\overline{\mathbb{F}}_q)$ exactly when $y^2 = -1$, both active blocks have $a^2 \neq 1$, and one block has reciprocal pairs $[-a], [-ca^q]$ for the entries a, ca^q of the other block. Every conjugator g then has $\det(g) = -1$. Moreover, both 1 and -1 occur in t precisely for t as in (1)–(3). In (1), we have that $y = \pm 1$ and $c = 1$. Moreover, one block has $a = -y$ and the other has $a^2 \neq 1$. In (2), $y^4 \neq 1$ and $a^2 = 1$ on both blocks. Moreover, $1, -1$ each occur twice. In (3), $y^2 \neq 1$ and $c = -1$. Moreover, one block has entries $(a, -a)$ for $a^2 = 1$ and the other has $a^2 \neq 1$.*

Proof. Put $m = q^2 + 1$. Since q is odd, $\gcd(m, 2(q \pm 1)) = 2$ and $m \equiv 2 \bmod 8$. Thus, $a^q = \pm a^{\pm 1}$ implies $a^2 = 1$, and the pairs $[a], [ca^q]$ on a block with $a^2 \neq 1$ are distinct. Also, $[a] \neq [-a]$ and $[ca^q] \neq [-ca^q]$ since $a^2 = -1$ is impossible.

Moreover, $y^{q^2-1}=1$. If $[y]$ meets either $[a]$ or $[-a]$, $y^{\gcd(q^2-1,m)}=1$ and $y^2=1$. If $y^2=1$, then $c=1$ and each block with $a^2=1$ has constant sign (a,a). Hence, the counts for $1,-1$ have odd sum $1+2k$, where k counts blocks with $a^2=1$. If $y^2\neq 1$, conjugacy of $t,-t$ requires $[y]=[-y]$. Hence, $y^2=-1$. On a lone block with $a^2\neq 1$, neither $[-a]$ nor $[-ca^q]$ appears in t. If both blocks have $a^2=1$, equal multiplicities of $1,-1$ give $Q(2,1)$, by Lemma 6.165. Thus, outside $Q(2,1)$, the two blocks are paired by $[u]\mapsto[-u]$.

Conversely, assume $y^2=-1$ with block pairs $[u]\mapsto[-u]$. Put $V_u=\ker(t-u)$. On four active lines we choose A with $A(V_u)=V_{-u}$. Moreover, the map $A\oplus A^{-T}$ on the eight lines has determinant $\det(A)\det(A^{-T})=1$. On the singleton we swap the y,y^{-1} lines, with determinant -1. Since the ten eigenvalues of t are distinct, $\mathrm{C}_{\mathrm{O}_{10}}(t)$ consists of reciprocal scalings (z_u,z_u^{-1}) and has determinant 1. Consequently, every conjugator g from t to $-t$ has determinant -1.

We now suppose that $1,-1$ both occur. If $y^2=1$, then two blocks with $a^2=1$ give $I(1,4)$ or $I(2,3)$. Thus, the block with $a=-y$ is the only one with $a^2=1$, giving (1). If $y^2\neq 1$ and both blocks have $a^2=1$, then $1,-1$ each occur twice. At $c=1$, the blocks are $(1,1),(-1,-1)$, and at $c=-1$ both are $(a,-a)$. If $y^2=-1$, the class is $Q(2,1)$, so $y^4\neq 1$ gives (2). Finally, if $a^2=1$ on exactly one block and $y^2\neq 1$, the signs (a,ca^q) are opposite exactly when $c=-1$, giving (3). □

Proposition 6.172. *Suppose that $s\neq 1$, that at most one of $1,-1$ occurs in t, and that $t,-t$ are in distinct $\mathrm{O}_{10}(\overline{\mathbb{F}}_q)$ classes. Then, we have that $\mathrm{C}_{\mathbf{X}^*}(s)$ is a proper connected Levi $\mathbf{L}^*$. Moreover, the correspondence on $\mathcal{E}(L,s)$ gives pairs (χ,ϕ) between $\mathcal{E}(X,s)\cap\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M\mid W\theta_s)$ that satisfy $\succeq_c$.*

Proof. Write $V_u=\ker(t-u)$. By [58, Proposition 3.5.1], the factors on $V_u\oplus V_{u^{-1}}$ for $u^2\neq 1$ are $\mathrm{GL}(V_u)$, with inverse transpose on $V_{u^{-1}}$. Thus, the sign space V_δ, if present for $\delta=\pm1$, has factor $\mathrm{SO}(V_\delta)$. Since $t,-t$ have distinct O_{10}-classes, $\mathrm{C}_{\mathbf{X}^*}(s)$ is the image of $\mathrm{C}_{\mathrm{SO}_{10}}(t)$. Give $V_u,V_{u^{-1}}$ weights $m_u,-m_u$, with distinct $m_u>0$ over the pairs $[u]$, and give V_δ weight 0. Then, the cocharacter ν^* has zero roots for $\mathbf{L}^*=\mathrm{C}_{\mathbf{X}^*}(\nu^*(\mathbb{G}_m))=\mathrm{C}_{\mathbf{X}^*}(s)$. Thus, $\mathbf{L}^*$ is an F-stable Levi, proper since $s\neq 1$ in the adjoint group $\mathbf{X}^*$.

An ordinary restorer b of field degree 0 acts on t by orthogonal conjugation. Since b fixes s and no isometry conjugates t to $-t$, it fixes each V_u and ν^*. Thus, the dual cocharacter ν gives a radical $\mathbf{U}$ with $b(\mathbf{U})=\mathbf{U}$ for ordinary b of field degree 0. For a mixed stabilizer operation (b,σ) corrected to fix θ_s, conjugation by b acts by $s\mapsto s^j$ for $\gcd(j,|s|)=1$. Hence, b stabilizes $\mathbf{L}$ since $\mathrm{C}_{\mathbf{X}^*}(s^j)=\mathrm{C}_{\mathbf{X}^*}(s)$.

Let $\mathbf{S}$ be the Sylow 4-torus. On block j, its weights are $v_j,\zeta_4 v_j$, with independent v_1,v_2 and $\zeta_4^2=-1$. Moreover, only the singleton has weight 0. Thus, no root $e_i\pm e_j$ is zero on $\mathbf{S}$, so $\mathrm{C}_{\mathbf{L}}(\mathbf{S})=\mathbf{T}$. Moreover, torus duality gives $I=M_{\theta_s}=\mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$ because $\mathrm{C}_{\mathbf{X}^*}(s)=\mathbf{L}^*$. Since $P\leq T\leq L$ and $[M:T]=32$, both $[X:L]$ and $[M:I]$ are prime to p.

Given any adjoint factor R_j of L, let $\mathbf{S}_j$ be the image of $\mathbf{S}$. By [156, Theorem D and its proof], we have unipotent pairs (β_j,ρ_j) on R_j and $\mathrm{N}_{R_j}(\mathbf{S}_j)$. Moreover, an F-orbit of h simple factors gives a factor R_j over $\mathbb{F}_{q^h}$. On $\bigotimes_j V_j$, the permutation of factors R_j permutes the spaces V_j, where $V_j=V_{\beta_j}\boxtimes V_{\rho_j}^\vee$, by [141, proofs of Theorems 2.7 and 3.2]. Thus, the adjoint models give (β_0,ρ_0) on L,I as in Proposition 6.150. From $s\in\mathrm{Z}(\mathbf{L}^*)$, we obtain the linear character λ_s of L by [58, Proposition 11.4.8]. Its restriction to T is θ_s, so uniqueness gives $\lambda_s^{(b,\sigma)}=\lambda_s$ for every corrected (b,σ) fixing θ_s. Thus, $\beta=\lambda_s\beta_0$ and $\rho=\lambda_s|_I\rho_0$ give the pairs on $\mathcal{E}(L,s)$ and $\mathrm{Irr}(I\mid\theta_s)$. The matching factor sets on H,B' give an extension V of $\beta\boxtimes\rho^\vee$ to $H\times_{D_0}B'$. Moreover, the matching operators for the corrected (b,σ)-actions give $V^{(b,\sigma)}\cong V$, and the common central scalars give $(z,z)\in\ker V$ for $z\in\mathrm{C}_A(X)$. The radical $\mathbf{U}$ is fixed by H_1 in Proposition 6.134. If $F=\gamma F_r^f$, the field lift φ satisfies $\varphi^m=K_{a^m}F$ by the relation at φ^m. Moreover, on $\mathcal{E}(L,s)$, the comparison of $\mathbf{U}$ and $b(\mathbf{U})$ is [19, Corollary 6.5 and Remark 6.7]. By Proposition 6.134, we obtain $\succeq_c$ for $\chi=\pm R_{\mathbf{L}}^{\mathbf{X}}(\beta)$ and $\phi=\mathrm{Ind}_I^M(\rho)$. □

Lemma 6.173. *In* Lemma 6.171*, we choose t with complementary blocks and $y^2=-1$, or take t as in* (3)*. Then, a proper F-stable Levi $\mathbf{L}^*$ contains $\mathrm{C}_{\mathbf{X}^*}(s)$. Moreover, $b(\nu^*)=\nu^*$ for its cocharacter ν^* and every ordinary restorer b of field degree 0. Put $L=\mathbf{L}^F$, $R=[\mathbf{L},\mathbf{L}]^F$ and $Q=q^2$. Moreover, we put $T_R=T\cap R$, $C=\mathrm{C}_T(R)$ and $J=CR$. Then, $R\cong\mathrm{SL}_2(Q)$, $|T_R|=Q+1$, and $\theta_s|_{T_R}$ is quadratic. For the complementary blocks, $[L:J]=2$. In case* (3)*, $L=J$.*

Proof. First, for complementary blocks put $\mathbf{L}^*=\mathrm{C}_{\mathbf{X}^*}(s_{2'})$. The odd part $s_{2'}$ has a unique odd lift u. Thus, a degree 0 restorer of s fixes u, even when it acts by $t\mapsto -t$. On the singleton, u has eigenvalue 1. On the two active blocks, u has reciprocal pairs $[b],[b^q]$ for $b=a_{2'}$. Since $a^2\neq 1$, then $b\neq 1$ since $v_2(q^2+1)=1$. Moreover, $[b]$ and $[b^q]$ are

distinct since $b^{q^2+1}=1$. Each pair occurs twice, and F acts by $[b]\mapsto[b^q]$. By [58, Proposition 3.5.1], the factors on u are $\mathrm{GL}(V_b)$ and $\mathrm{GL}(V_{b^q})$, where $V_v=\ker(u-v)$. There is also the torus $\mathrm{SO}(V_1)$ on the singleton. Since $|u|$ is odd, no isometry conjugates u to $-u$. Thus, the weights on $V_b, V_{b^{-1}}$ are $1,-1$, on $V_{b^q}, V_{b^{-q}}$ are $2,-2$, and on V_1 are 0 for a cocharacter ν^* with $\mathbf{L}^*=\mathrm{C}_{\mathbf{X}^*}(\nu^*(\mathbb{G}_m))$.

Next, in case (3), put weight 0 on $V_+\oplus V_-$ and weights $m_v,-m_v$ on the other pairs $[v]$, with distinct $m_v>0$. On $V_+\oplus V_-$, the roots are $\pm(e_1-e_2)$ and $\pm(e_1+e_2)$. Moreover, F exchanges the two A_1 factors. By Lemma 6.171, t and $-t$ are in distinct O_{10}-classes. Thus, $b(\nu^*)=\nu^*$ for ordinary b of degree 0, and $\mathrm{C}_{\mathbf{X}^*}(s)\le\mathbf{L}^*$. For either Levi, $s\mapsto s^k$ with $\langle s^k\rangle=\langle s\rangle$ gives the same root set $\Phi(\mathbf{L}^*,\mathbf{T}^*)$. Hence, the primal radical $\mathbf{U}$ is fixed by the field degree 0 subgroup H_1 of the transfer.

The factors of $[\mathbf{L},\mathbf{L}]$ are simply connected of type A_1 and are exchanged by F. Thus, by [58, §4.3], $R\cong\mathrm{SL}_2(Q)$. On the first root torus, F^2 acts by $x\mapsto x^{-Q}$, so $|T_R|=Q+1$. For the complementary blocks, $v,-v$ on aligned coordinates have ratio -1. In case (3), $a(-a)=a/(-a)=-1$. Thus, the relative coroot evaluates to -1 on s, so $\theta_s|_{T_R}$ is quadratic. By [58, Lemma 4.2.13], Lang's theorem for $\mathbf{T}\cap[\mathbf{L},\mathbf{L}]$ gives $L=TR$. Hence, $J\cap T=CT_R$ and $[L:J]=[T:CT_R]$.

For complementary blocks put $d=x_1/x_2$ on aligned entries. Given any $d\in\mu_{Q+1}$, we set $x_1=d, x_2=1$. Moreover, sn maps the singleton torus T_0 onto $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ by Section 6.3. Thus, a coordinate of T_0 with norm $\mathrm{sn}(d)^{-1}$ gives a lift to T by [137, §24.h]. Hence, T maps onto the adjoint torus T_{ad} of order $Q+1$, with image of T_R of order $(Q+1)/2$. The kernel on T is C, so $[L:J]=2$.

In case (3), T maps on the sign space into $\mathrm{SO}_4^-(q)$, with $\mathrm{sn}(-I_4)$ the nonsquare discriminant of $V_+\oplus V_-$. Thus, $\mathrm{SO}_4^-(q)=\Omega_4^-(q)\times\langle -I_4\rangle$ as in Proposition 6.159. Consequently, T acts on $R/\mathrm{Z}(R)=\Omega_4^-(q)$ by inner automorphisms. Since R is perfect, automorphisms of R agreeing modulo $\mathrm{Z}(R)$ agree on R. Given $x\in T$, an inner representative $r\in R$ for its action centralizes T_R. Hence, $r\in\mathrm{C}_R(T_R)=T_R$, so $xr^{-1}\in C$ and $L=J$. □

Proposition 6.174. *In* Lemma 6.173*, the characters $\beta\in\mathcal{E}(L,s)\cap\mathrm{Irr}_{p'}(L)$ have degrees $(Q-1)/2,(Q-1)/2$ in case* (3)*. For $[L:J]=2$ there is one β of degree $Q-1$. Moreover, the pairs (χ,ϕ) on X,M satisfy $\succeq_c$.*

Proof. With $D=\mathrm{N}_R(T_R)$, we put $N=\mathrm{N}_L(T)=TD$ and $N_0=\mathrm{N}_J(T)=CD$ and $\kappa=\theta_s|_C$. The restriction formula [58, Proposition 11.3.10] gives $R_{\mathbf{T}}^{\mathbf{L}}(\theta_s)|_R=R_{\mathbf{T}\cap[\mathbf{L},\mathbf{L}]}^{[\mathbf{L},\mathbf{L}]}(\theta_s|_{T_R})$. Thus, the signed restriction to R is $\eta_++\eta_-$ with odd Weil characters $\eta_\pm$ of degree $(Q-1)/2$. On D the corresponding characters $\rho_\pm$ extend $\theta_s|_{T_R}$. By [186, Theorem 4.2 and Claim 1 in its proof], the pairs satisfy $\succeq_c$ at $\mathrm{Aut}(R)_{T_R}$ for $Q\ne9$. At $Q=9,p=5$, take the $\mathrm{SL}_2(9)$ pairs in [186, proof of Theorem 4.4] on the quotient of $6.A_6$ by its central C_3. The scalar at $z_R=-I_R$ is -1, since $(Q+1)/2$ is odd. Hence, $\eta_\pm$ and $\rho_\pm$ satisfy $\rho_\pm=\kappa$ on $C\cap R=\langle z_R\rangle$. By [141, Theorem 2.8], the products $\beta_\pm=\eta_\pm\cdot\kappa$ and $\xi_\pm=\rho_\pm\cdot\kappa$ on J,N_0 give the relation $\succeq_c$ at the stabilizers of $(\beta_\pm,\xi_\pm)$. Moreover, for (b,σ) in their joint stabilizer, $\kappa^{(b,\sigma)}=\kappa$. In case (3), $J=L$, $I=M_{\theta_s}=N$ and $(\beta,\rho)=(\beta_\pm,\xi_\pm)$.

Next, for complementary blocks choose $v\in T\setminus CT_R$. On R its action represents the coset outside $\mathrm{PSL}_2(Q)$ in $\mathrm{PGL}_2(Q)$. Thus, by equivariance, v exchanges $(\eta_+,\rho_+),(\eta_-,\rho_-)$. Since v centralizes C, it exchanges β_+,β_- and ξ_+,ξ_-. Thus, Clifford correspondence [95, Theorem 6.11] gives irreducible characters $\beta=\mathrm{Ind}_J^L\beta_+$ and $\widetilde{\rho}=\mathrm{Ind}_{N_0}^N\xi_+$. Moreover, the restriction of β to R is $\eta_++\eta_-$, so β is the sole character of $\mathcal{E}(L,s)$. The centralizer of s is toral since t has ten distinct eigenvalues and $gtg^{-1}=-t$ in O_{10} implies $\det(g)=-1$. Hence, $I=M_{\theta_s}=T$. Since $N=TN_0$, [95, Problem 5.2] gives $\widetilde{\rho}|_T=\mathrm{Ind}_{CT_R}^T(\theta_s|_{CT_R})$. By [95, Lemma 5.2], θ_s occurs with multiplicity 1. Since its inertia is T, [95, Theorem 6.11] gives $\widetilde{\rho}=\mathrm{Ind}_T^N\theta_s$.

For complementary blocks put $\widetilde{B}=NB'$ and $H=LB'$ as in the transfer. With $H_0=H_{\beta_+}$, put $B_0=\widetilde{B}_{\xi_+}$. Since v exchanges β_+,β_- and ξ_+,ξ_-, the quotient maps give $H_0/J\cong B_0/N_0\cong D_0$. Moreover, $H_0=JB_0$ by $L=JN$ and the pairing on J,N_0. The models on H_0,B_0 have common factor set α on D_0. Thus, on the central cover for α, we induce from H_0,B_0 with representatives $1,v$. For (b,σ) in the mixed stabilizer, we choose $w\in\{b,bv\}$ with $(\beta_+,\xi_+)^{(w,\sigma)}=(\beta_+,\xi_+)$. The shared comparison function $\mu_{w,\sigma}$ lies on D_0. Hence, each induced block for h has the comparison scalar $\mu_{w,\sigma}(\bar{h})$, with $\bar{h}\in D_0$, because v maps to 1 in D_0. The inner correction by $v\in N$ is conjugation by the matrix at v. Moreover, for $z\in\mathrm{C}_A(X)$, we have $zv=vz$, so each summand for $1,v$ has the original scalar at z. The θ_s-line in $\widetilde{\rho}$ is invariant under $B'=B_{\theta_s}$ and each corrected (b,σ) fixing θ_s. Thus, the models for β and $\rho=\theta_s$ share α and $\mu_{w,\sigma}$ on D_0, and share scalars at all $z\in\mathrm{C}_A(X)$.

In both cases $\mathrm{C}_{\mathbf{X}^*}(s)\le\mathbf{L}^*$, so $\chi=\pm R_{\mathbf{L}}^{\mathbf{X}}(\beta)$ gives $\mathcal{E}(X,s)$ by [58, Theorem 11.4.3]. By Clifford correspondence, $\phi=\mathrm{Ind}_I^M\rho$ over $W\theta_s$. Since $P\le T\le L$, we have $p\nmid[X:L]$, and $p\nmid(Q-1)$ since $p\mid Q+1$. Also, $[M:I]$ divides

32. Thus, both bijections restrict to p'-degree. Finally, the radical $\mathbf{U}$ of Lemma 6.173 is fixed by H_1, and the field lift satisfies $\varphi^m = K_{a^m} F$ by the relation for φ. By Proposition 6.134, we obtain $\succeq_c$ for (χ, ϕ). □

Take t in case (1) of Lemma 6.171 with $y = 1$. Thus, we obtain that $\dim V_+ = 2$ and $\dim V_- = 4$ for $V_\pm = \ker(t \mp 1)$. The two dimensions distinguish t from $-t$. Moreover, each odd power of t leaves the signs on $V_\pm$ fixed. With $Y = V_- \perp V_+$, we put $U_A = Y^\perp$. The torus A on U_A has order q^2+1. Its character λ has $\lambda_{2'} \neq 1$, since $v_2(q^2+1) = 1$ and the coordinate on U_A has $a^2 \neq 1$. The spinor norms are ν_Y, ν_A with values in $\{1, -1\}$.

Proposition 6.175. *For every λ above, the spin Levi L gives four pairs (β, ρ) on L, M_{θ_s}. Moreover, the induced pairs (χ, ϕ) on X, M satisfy $\succeq_c$ for either ϵ.*

Proof. The distinct reciprocal pairs on U_A are $[a]$ and $[a^q]$ by Lemma 6.171. Thus, on $Y \oplus U_A$ the natural Levi is $G_Y \times A$, where $G_Y = \mathrm{SO}(Y)$ and Y has sign $-\epsilon$. By [137, §24.h], the spin map on L has kernel $\langle z_0 \rangle$ and image $H = \ker(\nu_Y \boxtimes \nu_A)$ in $G_Y \times A$. Moreover, the image of $\mathrm{N}_L(T)$ is $H_N = H \cap (\mathrm{N}_{G_Y}(T_Y) \times A)$ for the torus T_Y of Y. Since $F(t) = t$, Lemma 6.27 gives trivial scalar at z_0.

On V_- of type $-$, the two labels are $1, \mathrm{St}$. In V_-, choose a marked isotropic line E with character η_q^{fld} on $\mathrm{GL}(E)$. Moreover, the plane $E^\perp/E$ in V_- is anisotropic, and the plane V_+ has type ϵ. Thus, for $\epsilon = -1$, choose the odd Weil source Φ of Lemma 6.57. If $\epsilon = 1$, we choose χ_1, χ_2 on $\Omega_4^-(q)$ from Lemma 6.88. By [208, Proposition 7.18, Steps 2–4], on E the coefficients at $1, \mathrm{St}$ are both 1. Moreover, the line weight η_q^{fld} selects $E \leq V_-$, fixed by the paired reflections on $(E^\perp/E) \perp V_+$. Consequently, the constituents of multiplicity 1 have $\mathcal{H}_p$-fixed extensions on $\mathrm{O}(Y)$ with field lift a_Y as in Proposition 6.167. Put $\delta = \eta_q^{\mathrm{fld}}(-1)$. The source at the residual $-I_4$ has scalar $-\delta$, and the line contributes δ. Thus, at $-I_Y$, we obtain scalar -1. Moreover, a nonsquare multiplier on Y gives two orientations of 1 and two of St. Write χ_k^Y for the four characters on G_Y, with $0 \leq k < 4$.

Next, put $B = C_{q^2+1}$ on V_- and $C_0 = C_{q-\epsilon}$ on V_+. By Section 6.7, the group $\mathrm{N}_{\mathrm{O}(Y)}(T_Y)$ is $(B \rtimes \langle R_B \rangle) \times (C_0 \rtimes \langle w_0 \rangle)$, with $|R_B| = 4$ and $|w_0| = 2$. Take ξ_k^Y with the quadratic weight η_B on B, value i^k on R_B, and value 1 on $C_0 \rtimes \langle w_0 \rangle$. Moreover, $\nu_Y|_{C_0} \neq 1$. Thus, twists by ν_Y or $\det_Y \nu_Y$ change the restriction to C_0, and the twist by $\det_Y$ changes the value at w_0. Consequently, on $\mathrm{N}_{\Omega(Y)}(T_Y)$ the restrictions of ξ_k^Y are irreducible and distinct by [95, Lemma 5.2]. Also, $\eta_B(-1) = -1$, so $\xi_k^Y(-I_Y) = -1$. For a block similitude u of multiplier m, the change of rotation has $\eta_B(u^{1-q}) = \eta_q^{\mathrm{fld}}(m)$, as in Section 6.7. Hence, a nonsquare m acts on the index by $k \mapsto k+2$.

On Y we choose the field lift a_Y of Proposition 6.167, with a_Y^f the rotation R_B and a possible w_0, on which $\xi_k^Y(w_0) = 1$. Put $D = \mathrm{ord}_p(r)$. Since $D/\gcd(D, f) = 4$, choose $\alpha_k \in \mu_D$ with $\alpha_k^f = i^k$. Moreover, $D \mid p-1$, so $\mathcal{H}_p$ fixes α_k. We choose $\xi_k^Y(a_Y) = \alpha_k$. Match $k = 0, 1$ with $1, \mathrm{St}$, respectively, and transport by a multiplier m with $\eta_q^{\mathrm{fld}}(m) = -1$ to $k = 2, 3$. Thus, the paired models on Y are $\mathcal{H}_p$-fixed, with scalar -1 at $-I_Y$. For an operation stabilizing χ_k^Y, a Galois automorphism cannot compensate for $k \mapsto k+2$, since χ_k^Y is $\mathcal{H}_p$-fixed.

Restrict $\chi_k^Y \boxtimes \lambda$ to H and $\xi_k^Y \boxtimes \lambda$ to H_N. For either kernel the quotient character $\nu_Y \boxtimes \nu_A$ changes λ to $\lambda\nu_A \neq \lambda$. Thus, by [95, Lemma 5.2], each kernel H, H_N gives 4 distinct irreducibles. After inflation to $L, \mathrm{N}_L(T)$, we obtain β_k, ρ_k. On Y the dual centralizer has connected factors $\mathrm{SO}(V_-)$ and $\mathrm{SO}(V_+)$, with component C_2 from simultaneous reflection. By [208, §7, following Lemma 7.15 and (7.19)], it fixes $1, \mathrm{St}$ on $\mathrm{SO}(V_-)$. Thus, the orientations of $1, \mathrm{St}$ exhaust $\mathcal{E}(G_Y, t|_Y)$. The other lift $-t$ gives the twist $\nu_Y \boxtimes \nu_A$, with the same restriction to H. By [58, Proposition 11.3.10], the characters β_k exhaust $\mathcal{E}(L, s)$ at $z_0 = 1$. Locally $\lambda_{2'}$ has an orbit of length 4 under q-power, since its order $h > 1$ is odd and $q^2 \equiv -1 \bmod h$. In particular, $\lambda^{q^j} \in \{\lambda, \lambda\nu_A\}$ implies $j \equiv 0 \bmod 4$. Moreover, the block swap places $\lambda_{2'} \neq 1$ on V_-. Thus, $I = M_{\theta_s} = \mathrm{N}_L(T)$ has $I/T \cong C_4$ from the rotations on B, with determinant compensated by w_0. The four restrictions ρ_k exhaust $\mathrm{Irr}(I \mid \theta_s)$.

On $U_A = \mathbb{F}_{q^4}$ take the form $x \mapsto \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x^{q^2+1})$. Moreover, the normalizer of A in the semilinear isometries is $A \rtimes \langle \varphi \rangle$ by Section 6.7, with $\varphi : x \mapsto x^r$ and $\varphi^f : x \mapsto x^q$. On $A \rtimes \langle \varphi \rangle_\lambda$ take $\widehat{\lambda}(z\varphi^b) = \lambda(z)$. Thus, on A the mixed stabilizer operation (g, σ) has $\lambda^{(g,\sigma)} = \lambda$. Given $u \in A$ and $\varphi^b \in \langle \varphi \rangle_\lambda$, the factor under conjugation by u is $\lambda(u/\varphi^b(u)) = 1$. Consequently, $\widehat{\lambda}^{(g,\sigma)} = \widehat{\lambda}$ on the inertia group, with $g \in A \rtimes \langle \varphi \rangle$.

For an operation stabilizing (β_k, ρ_k), the multiplier on Y is $m = d^2$, since a nonsquare acts by $k \mapsto k+2$. Multiplication by $d^{-1}I$ makes the multiplier 1 on Y and U_A. The choices $d, -d$ differ by $-I$, with scalar $-\lambda(-1)$ on both models. Moreover, on $Y \oplus U_A$ we couple the models on Y with $\widehat{\lambda}$ on U_A at equal field degrees modulo f. Their f-th powers are a_Y^f and φ^f, respectively. The ordinary quotients agree because $\mathrm{N}_{\mathrm{O}(Y)}(T_Y)$ maps onto $\mathrm{O}(Y)/\Omega(Y)$ via $(\det_Y, \nu_Y)$, by w_0 and C_0. Thus, a chosen action on U_A permits a correction in $\mathrm{N}_{\mathrm{O}(Y)}(T_Y)$ with any chosen $(\det_Y, \nu_Y)$. With $A_{\mathrm{tr}} = X \rtimes D_0$, we put $H_{\mathrm{tr}} = LB'$. Given $z \in \mathrm{C}_{A_{\mathrm{tr}}}(X)$, the scalar on Y agrees by Proposition 6.167, and on

U_A both scalars are $\widehat{\lambda}(g_A)$ at the component g_A. Consequently, the tensor V of the two models extends $\beta_k \boxtimes \rho_k^\vee$ on $H_{\mathrm{tr}} \times_{D_0} B'$. For each (b,σ) stabilizing (β_k, ρ_k), $V^{(b,\sigma)} \cong V$, and $(z,z) \in \ker V$ for $z \in \mathrm{C}_{A_{\mathrm{tr}}}(X)$.

Finally, the cocharacter ν^* gives weight 0 on Y and weights $1, -1$ on $V_a, V_{a^{-1}}$ and $2, -2$ on $V_{a^q}, V_{a^{-q}}$, with $V_v = \ker(t - v)$. Since $\dim V_+ \neq \dim V_-$, a degree 0 restorer of s fixes t and ν^*. The Levi $\mathbf{L}^*$ contains $\mathrm{C}_{\mathbf{X}^*}(s)$. Moreover, a power s^j generating $\langle s \rangle$ has the Levi $\mathbf{L}^*$ since $V_v(t) = V_{v^j}(t^j)$. Thus, the primal radical $\mathbf{U}$ is fixed by H_1. On Y, the dual centralizer contains the Sylow p-subgroup of G_Y^* in T_Y^*. Thus, $1, \mathrm{St}$ give p'-degrees by [58, Proposition 11.5.6]. Also, $P \leq T \leq L$ and $[M : I] \mid 32$. Hence, $\chi_k = \pm R_{\mathbf{L}}^{\mathbf{X}}(\beta_k)$ by [58, Theorem 11.4.3] and $\phi_k = \mathrm{Ind}_I^M(\rho_k)$ by [95, Theorem 6.11] give the p'-pairs on X, M. Moreover, the field lift for $F = \gamma F_q$ satisfies $\varphi_{\mathrm{tr}}^m = K_{a^m} F$ by the relation for φ_{tr}. By Proposition 6.134, we obtain $\succeq_c$ for (χ_k, ϕ_k). □

Lemma 6.176. *In* Lemma 6.171*, take t as in* (2)*. On the active blocks, Y is split with two coordinates* 1 *and two* -1*. Moreover, the singleton coordinate y has $y^4 \neq 1$. Let $\mathbf{L}$ be the Levi for $2e_5$, with $L = \mathbf{L}^F$, $R = [\mathbf{L}, \mathbf{L}]^F = \mathrm{Spin}_8^+(q)$. On $\mathbf{U}_0 = \mathrm{Z}^\circ(\mathbf{L})$, we put $U = \mathbf{U}_0^F = \mathrm{Spin}_2^\epsilon(q)$. Moreover, we put $T_R = T \cap R$, $C = \mathrm{C}_T(R)$, $J = CR$, $I = \mathrm{N}_L(T)$, $I_R = \mathrm{N}_R(T_R)$, and $I_J = CI_R$. We have that $L = TR$ and $J \cap I = I_J$, and the map $I/I_J \to L/J$ is an isomorphism of order* 2*. Moreover, the union in L for case* (2) *consists of the p'-characters β whose restriction to R meets the three eligible* $\mathrm{I}(2,2)$ *series at $d = 4$ and whose scalar character ν on U has $\nu^2 \neq 1$. In I the corresponding set consists of p'-characters ρ whose restriction to I_R meets the paired sets for* $\mathrm{I}(2,2)$ *and whose character ν on U has $\nu^2 \neq 1$.*

Proof. By Lang's theorem on $\mathbf{T} \cap [\mathbf{L}, \mathbf{L}]$ [58, Lemma 4.2.13], $L = TR$. Moreover, the singleton torus T_0 has spinor norm image $\mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$ by Section 6.3. Thus, the projection of T to $\mathrm{SO}(Y)$ is the full natural torus T_Y. Given $t_1 \in T$ with projection $\bar{t}_1$, its action on R is inner exactly when $\bar{t}_1 \in \Omega(Y)$. Indeed, an inner action on R projects to an action of $\Omega(Y)$, and the possible scalar discrepancy $\pm I_Y$ lies in $\Omega(Y)$ since Y is split of rank 4. Conversely, lift a point in $\Omega(Y)$ to R by [137, §24.h]. Two such actions on R agreeing modulo $\mathrm{Z}(R)$ agree because R is perfect. An inner representative centralizing T_R lies in T_R. Hence, the map $T \to \mathrm{Out}(R)$ has kernel CT_R. As $\nu_Y(T_Y) = \{1, -1\}$ for the spinor norm ν_Y, $[T : CT_R] = 2$. Next, we put $\mathbf{T}_R = \mathbf{T} \cap [\mathbf{L}, \mathbf{L}]$. On $\mathbf{T}_R \times \mathbf{U}_0 \to \mathbf{T}$, the kernel is μ_2 and the singleton U has order $q - \epsilon$ prime to p. Thus, $P \leq T_R$. Moreover, $\mathrm{C}_{[\mathbf{L},\mathbf{L}]}(P) = \mathbf{T}_R$ by Lemma 6.122 at $d = 4$. Thus, I_R normalizes $\mathbf{T} = (\mathbf{T} \cap [\mathbf{L}, \mathbf{L}])\mathrm{Z}^\circ(\mathbf{L})$ and so T. Conversely, $R \cap I$ normalizes T_R. Therefore, $I = TI_R$ by $L = TR$, and $J \cap I = CI_R$. Thus, $I/I_J \cong L/J \cong T/CT_R$.

In the dual Levi $(\mathrm{SO}_8 \times \mathrm{GL}_1)/\langle(-I, -1)\rangle$, the projection to $\mathbf{R}^* = \mathrm{PSO}_8$ acts by $(t|_Y, y) \mapsto [t|_Y]$. The cocharacter of U is $2e_5$ in the coroot lattice of D_5. Thus, the point for ν is y^2, independent of replacing $(t|_Y, y)$ by $(-t|_Y, -y)$. In particular, $\nu^2 \neq 1$ means $y^4 \neq 1$. For the projected point s_R, Lemma 6.33 and Corollary 6.121 give the three $\mathrm{I}(2,2)$ classes of R on the balanced patterns, with $F(t) = t$ or $F(t) = -t$. By [58, Proposition 11.3.10], restriction from L to R gives constituents in $\mathcal{E}(R, s_R)$. Moreover, on I_R the paired sets are $\mathrm{Irr}(I_R \mid I_R\theta_R)$, with $\theta_R = \theta_s|_{T_R}$ for each of the three s_R. Conversely, the projected point fixes $t|_Y$ up to negation, and $y^4 \neq 1$ selects case (2). Finally, extending from R, I_R to J, I_J by C leaves degrees equal, and the indices $[L : J] = [I : I_J] = 2$ allow only a factor 2 by [95, Theorem 6.11 and Corollary 6.17]. Thus, the degrees on R, I_R are prime to p exactly when the degrees on L, I are prime to p. □

Proposition 6.177. *The pairs (β, ρ) of* Lemma 6.176 *on L, I satisfy $\succeq_c$ and have character ν on U. Moreover, above each invariant pair on J, I_J, both extensions occur.*

Proof. Take (ξ, ζ) on R, I_R from Corollary 6.121. By [141, Theorem 2.9], the comparison on (ξ, ζ) holds with the image $\Gamma_R \leq \mathrm{Aut}(R)$ of the Levi normalizer. Next, choose $\kappa \in \mathrm{Irr}(C)$ with $\kappa|_U = \nu$ and $\kappa|_{C \cap R} = \xi|_{C \cap R}/\xi(1)$. Thus, the products $\xi_0 = \xi \cdot \kappa$ on J and $\zeta_0 = \zeta \cdot \kappa$ on I_J satisfy $\succeq_c$ by [141, Theorem 2.8]. On $C \times C$ the tensor has scalar $\kappa(c_1 c_2^{-1})$. Moreover, an operation stabilizing (ξ_0, ζ_0) fixes κ, since both restrictions to C have scalar character κ. We consider all such κ with their translates.

Choose $v \in T \setminus CT_R$, representing the nonidentity element of $I/I_J \cong L/J$. If $\xi_0^v \neq \xi_0$, $\zeta_0^v \neq \zeta_0$. Thus, [95, Theorem 6.11] gives irreducible characters $\beta = \mathrm{Ind}_J^L(\xi_0)$ and $\rho = \mathrm{Ind}_{I_J}^I(\zeta_0)$. Moreover, an operation stabilizing (β, ρ) is corrected by 1 or v to an operation stabilizing (ξ_0, ζ_0). Consequently, on the common cover for (ξ_0, ζ_0), induction with representatives $1, v$ as in Proposition 6.174 gives $\succeq_c$ for (β, ρ). The scalar of $u \in U$ on either induced summand is $\nu(u)$, since v centralizes U.

Now suppose $\xi_0^v = \xi_0$. Then, $\zeta_0^v = \zeta_0$. On the stabilizers of ξ_0, ζ_0, choose $\mathcal{P}, \mathcal{P}'$ with common factor α on the quotient by J, I_J. Its subgroup $D = L/J \cong I/I_J$ has order 2, with generator $d = vJ$. Moreover, choose a

cochain τ with $\tau(1)=1$ and $\tau(d)^2=\alpha(d,d)^{-1}$. Then, the factor set of $(\tau\mathcal{P},\tau\mathcal{P}')$ is trivial on $D\times D$. Thus, the restrictions to L,I are ordinary models of β,ρ. By [95, Corollary 6.17], the other pair is $(\beta\varepsilon_D,\rho\varepsilon_D)$, where $\varepsilon_D(d)=-1$. Moreover, [141, Lemma 1.9(a)] pairs the transports along (b,σ) with $\tau|_D$. For an operation stabilizing (ξ_0,ζ_0), the comparison function restricts to $\mu|_D\in\mathrm{Irr}(D)$. Thus, the extensions β,ρ change by the sign $\mu(d)$.

Write $\widehat{L},\widehat{I}$ for the overgroups, with $\widehat{L}=L\widehat{I}$ and $L\cap\widehat{I}=I$. Next, on the stabilizers of (β,ρ), put $S=\tau\mathcal{P}$, $S'=\tau\mathcal{P}'$ and choose $b\in\widehat{I}_\rho$ for each coset of I. After conjugation by $S(b)$, we obtain $S(x^b)$ up to a character of D on $x\in L$. Since b fixes β and $\beta\varepsilon_D\neq\beta$, this character is 1_D. $S'(b)$ gives exact conjugation on I because b fixes ρ. Thus, with the common representatives b, we choose $Q(xb)=S(x)S(b)$ and $Q'(xb)=S'(x)S'(b)$. Put $c(xb)=\alpha_\tau(\bar{x},\bar{b})$ for the factor $\alpha_\tau=\alpha\,\partial\tau$ on the quotient by J,I_J. Thus, $Q=cS$ and $Q'=cS'$, with factors inflated from $\widehat{L}_\beta/L\cong\widehat{I}_\rho/I$. For (g,σ) in the joint stabilizer of (β,ρ), both comparison functions acquire $c^{(g,\sigma)}/c$. Moreover, for centralizing z, both scalars are $c(z)\omega(z)$ with the original common scalar $\omega(z)$. Consequently, the models give $\succeq_c$ for (β,ρ) and $(\beta\varepsilon_D,\rho\varepsilon_D)$. Finally, $U\leq J$ gives $\varepsilon_D|_U=1$, so both pairs have scalar character ν on U. □

Proposition 6.178. *Take (β,ρ) in* Proposition 6.177. *The fibres of $\beta\mapsto\pm R^{\mathbf{X}}_{\mathbf{L}}(\beta)$ and $\rho\mapsto\mathrm{Ind}_I^M(\rho)$ are the w-orbits, where w inverts the singleton. Moreover, the w-orbits have length 2, and the induced pairs (χ,ϕ) satisfy $\succeq_c$.*

Proof. By Lemma 6.165, M/T fixes the singleton plane setwise. Take $w=r_1$, which inverts e_5 and rotates block 1 on Y. Both actions have determinant -1, so w has determinant 1. Thus, w lifts to M by [58, §4.2]. The cocharacter $2e_5$ has stabilizer L in X and gives the radical $\mathbf{U}$. Moreover, $\mathrm{N}_X(\mathbf{L})$ acts on $\mathbf{U}_0$ by 1 or inversion, with kernel L. Hence, $\mathrm{N}_X(\mathbf{L})/L\cong M/I\cong C_2$. On U the lift of w acts by $\nu\mapsto\nu^{-1}$. Since $\nu^2\neq1$, its orbits on β,ρ have length 2. By Clifford correspondence [95, Theorem 6.11], $\phi=\mathrm{Ind}_I^M(\rho)$ and its two constituents ρ,ρ^w on I.

On the dual torus, $[y]$ is the sole pair outside $\{[1],[-1]\}$, with $[y]\neq[-y]$ since $y^4\neq1$. Consequently, no isometry conjugates t to $-t$. Every element of $\mathrm{C}_{\mathbf{X}^*}(s)$ fixes the lines $V_y,V_{y^{-1}}$. Thus, $\mathrm{C}_{\mathbf{X}^*}(s)\leq\mathbf{L}^*$, and signed induction gives a bijection $\mathcal{E}(L,s)\to\mathcal{E}(X,s)$ by [58, Theorem 11.4.3]. If two L-series with points s,s' give the same X-series, a rational dual conjugator from s to s' stabilizes $V_y\oplus V_{y^{-1}}$. Thus, it lies in $\mathrm{N}_{\mathbf{X}^*}(\mathbf{L}^*)^F$, acting by 1 or w on the singleton. Moreover, w exchanges the radicals $\mathbf{U},\mathbf{U}^w$. The signed inductions from $\mathbf{U},\mathbf{U}^w$ agree on $\mathcal{E}(L,s)$ by [19, Corollary 6.5 and Remark 6.7] because $\mathrm{C}_{\mathbf{X}^*}(s)\leq\mathbf{L}^*$. Hence, the fibres of $\beta\mapsto\pm R^{\mathbf{X}}_{\mathbf{L}}(\beta)$ are $\{\beta,\beta^w\}$. Since $P\leq L$ and $[M:I]=2$, the degrees of χ,ϕ are prime to p exactly when those of β,ρ are prime to p.

Next, we let β lie in $\mathcal{E}(L,s)$. By Lemma 6.176, both labels of (β,ρ) project to s_R and to ν on U. Modulo I_R/T_R, the lifts of (s_R,ν) differ by $(t|_Y,y)\mapsto(t|_Y,-y)$. Moreover, $(t|_Y,-y)=(-t|_Y,y)$ in $\mathbf{L}^*$. If $F(t)=t$, the block swap π acts by $t|_Y\mapsto-t|_Y$, since the blocks have signs $1,-1$. If $F(t)=-t$, we have that the product r_1r_2 acts by $t|_Y\mapsto-t|_Y$ since each block is $(a,-a)$ with $a=\pm1$. Both elements fix e_5 and lie in I/T. Thus, (s_R,ν) determines one I-orbit of θ_s. In particular, ρ lies over θ_s.

For the transfer take $I_s=M_{\theta_s}\leq I$, since inversion on U gives $\nu^{-1}\neq\nu=\theta_s|_U$. Let ρ_s be the Clifford correspondent of ρ over θ_s. Thus, $\rho=\mathrm{Ind}_{I_s}^I(\rho_s)$ and $\phi=\mathrm{Ind}_{I_s}^M(\rho_s)$. An ambient operation (b,σ) stabilizing (χ,ϕ) can be corrected by M to fix θ_s. Then, it fixes ν, so the fibres $\{\beta,\beta^w\},\{\rho,\rho^w\}$ allow a correction that stabilizes (β,ρ). With $B'=B_{\theta_s}$, put $H=LB'$. Next, on H,B' restrict the models of Proposition 6.177. On B' take the weight space V_s for θ_s, which affords ρ_s on I_s. Moreover, V_s is invariant under (b,σ) fixing θ_s. Thus, on D_0 the common factor is α and the comparison function is $\mu_{b,\sigma}$. On V_s, the scalar at $z\in\mathrm{C}_A(X)$ remains $\omega(z)$ from the paired models. The tensor V extends $\beta\boxtimes\rho_s^\vee$ on $H\times_{D_0}B'$. Moreover, $V^{(b,\sigma)}\cong V$ and $(z,z)\in\ker V$ for $z\in\mathrm{C}_A(X)$.

Given b of field degree 0, an operation stabilizing θ_s fixes $2e_5$, since inversion on $\mathbf{U}_0$ cannot fix $\nu\neq\nu^{-1}$. Thus, the radical $\mathbf{U}$ for $2e_5$ is fixed by H_1. Moreover, if $\langle s^j\rangle=\langle s\rangle$, the lines $V_y,V_{y^{-1}}$ still give $\mathbf{L}^*$ at s^j. On $\mathcal{E}(L,s)$, [19, Corollary 6.5 and Remark 6.7] compares $\mathbf{U}$ and its translates at (b,σ). For the minus form $F=\gamma F_q$, the lift satisfies $\varphi^m=K_{a^m}F$ by the relation for φ. Consequently, Proposition 6.134, with I_s,ρ_s in the middle, gives $\succeq_c$ at (χ,ϕ). □

Theorem 6.179. *Let $X=\mathrm{Spin}^\epsilon_{10}(q)$ with q odd and $\epsilon=\pm1$. Let p be an odd prime with $d=\mathrm{ord}_p(q)=4$, and take $P\in\mathrm{Syl}_p(X)$. Then, we find an $\mathrm{Aut}(X)_P\times\mathcal{H}_p$-equivariant bijection from $\mathrm{Irr}_{p'}(X)$ to $\mathrm{Irr}_{p'}(\mathrm{N}_X(P))$ with relation $\succeq_c$ at each pair (χ,ϕ).*

Proof. At $s=1$, by [156, Theorem D], we obtain $\succeq_c$. By Corollary 6.170, we obtain $\succeq_c$ at quasi-isolated $[s]\neq[1]$. For the remaining $[s]$, Lemma 6.171 describes the lift t. With at most one of $1,-1$, the pairs for $t,-t$ in distinct O_{10}-classes are in Proposition 6.172. Moreover, for $t,-t$ conjugate in O_{10}, we apply Proposition 6.174. With both $1,-1$, the pairs (χ,ϕ) in (1) are in Proposition 6.175. In (2), $\succeq_c$ follows from Proposition 6.178. Finally, in (3),

$\succeq_c$ is Proposition 6.174. By Lemma 6.165, the classes $[s]$ of p'-index correspond to $W\theta_s$ on $\mathrm{Irr}(T)$. Thus, the sets $\mathrm{Irr}_{p'}(M \mid W\theta_s)$ are disjoint as $[s]$ varies. In case (2), the fibres $\{\beta, \beta^w\}$ and $\{\rho, \rho^w\}$ correspond by Proposition 6.178. Consequently, $\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M)$ are in the stated correspondence, with $M = \mathrm{N}_X(P)$. □

Lemma 6.180. *With $q = r^f$ odd, we let p be an odd prime dividing $q-\epsilon$, where $\epsilon = \pm 1$. Moreover, we set $n = p^a \geq 5$ with $a \geq 1$. In $X = \mathrm{Spin}^{\epsilon}_{2n}(q)$ let $\mathbf{T}$ have n singleton planes of sign ϵ. Thus, their total sign is $\epsilon^n = \epsilon$. With $T = \mathbf{T}^F$, we put $M = \mathrm{N}_X(T)$ and $b = v_p(q-\epsilon)$. There is $P \in \mathrm{Syl}_p(X)$ with $P \leq M$ and T_p characteristic in P. Moreover, the semisimple classes $[s]$ of p'-index in X^* have representatives on T^* with natural lifts $t = (x, \dots, x)$, where $x^{\epsilon q - 1} = \pm 1$. Conversely, we have that every such t gives a class $[s]$ of p'-index. The classes $[s]$ correspond to orbits of p'-size on $\mathrm{Irr}(T)$ under M/T.*

Proof. The action of F on $W(D_n)$ at $\mathbf{T}$ is conjugation by the signed scalar ϵI. Thus, $M/T = W(D_n)$, with even sign subgroup of order 2^{n-1}. Choose $S_p \in \mathrm{Syl}_p(\mathfrak{S}_n)$ containing an n-cycle g, possible since $n = p^a$. On the natural planes S_p permutes the n factors of $(C_{p^b})^n$. The determinant and spinor norm of S_p are trivial since $|S_p| = p^{v_p(n!)}$. Thus, $P \cong (C_{p^b})^n \rtimes S_p$ by Schur–Zassenhaus, with base $B = T_p$. The valuation of $|X|$ is $nb + v_p(n!)$ by the factors $q^n - \epsilon$ and $q^{2j} - 1$ for $1 \leq j < n$. Hence, the lift gives $P \in \mathrm{Syl}_p(X)$.

By [Brewster–Passman–Wilcox, Proposition 2.2], the base B is characteristic in P, since C_{p^b} is generated by its squares. Moreover, $\mathrm{C}_{\mathbf{X}}(B) = \mathbf{T}$ by the $2n$ distinct weights ξ_i, ξ_i^{-1} on B. Thus, $\mathrm{N}_X(P) \leq M$, and M is $\mathrm{Aut}(X)_P$-stable. Moreover, through the 2-power isogenies $P^* \in \mathrm{Syl}_p(X^*)$ with base B^* and $\mathrm{C}_{\mathbf{X}^*}(B^*) = \mathbf{T}^*$. By [132, Theorem 5.9 and Proposition 5.11], the M/T-orbits on T^* distinguish the classes $[s]$ whose centralizers contain a Sylow p-subgroup. Choose s centralizing P^*. Then, $s \in T^*$, and g fixes s. On a lift t, $g(t) = ct$ with $c = \pm 1$. Since $g^n(t) = t$ and n is odd, $c = 1$. Thus, $t = (x, \dots, x)$, and $F(t) = \pm t$ is $x^{\epsilon q - 1} = \pm 1$. Conversely, such a point centralizes B^* and S_p. Moreover, by duality a $W(D_n)$-orbit on $\mathrm{Irr}(T)$ has p'-size exactly when S_p fixes some θ. Thus, $[s]$ gives the orbit of the corresponding θ. □

Lemma 6.181. *Given $x \neq \pm 1$, the full centralizer of s is a connected Levi $\mathbf{L}^*$, and $I = M_{\theta_x} = \mathrm{N}_L(T)$ on the dual weight θ_x. Moreover, we have that $I/T \cong \mathfrak{S}_n$ and $[M : I] = 2^{n-1}$. The p'-characters in $\mathcal{E}(X, s)$ and $\mathrm{Irr}(M \mid M\theta_x)$ are indexed by the hooks $(n-j, 1^j)$ for $0 \leq j < n$.*

Proof. Let $V_x, V_{x^{-1}}$ be the isotropic eigenspaces of t. Then, the centralizer in SO_{2n} acts on them as $A \oplus A^{-T}$ with $A \in \mathrm{GL}(V_x)$, by [58, Proposition 3.5.1]. Moreover, an isometry that conjugates t to $-t$ requires $x^2 = -1$, and then exchanges $V_x, V_{x^{-1}}$ with determinant $(-1)^n = -1$. Thus, the full centralizer of s is the image of $\mathrm{GL}(V_x)$, the Levi $\mathbf{L}^*$ for the two opposite weights on $V_x, V_{x^{-1}}$. The natural factor has rational type $\mathrm{GL}^{\epsilon}_n(q)$. A signed permutation fixing s either inverts 0 coordinates or inverts n coordinates when $x^2 = -1$. Since n is odd, $I/T = \mathfrak{S}_n$ and $[M : I] = 2^{n-1}$. The linear character β_x of L corresponding to s comes from [58, Proposition 11.4.8], and $\beta_x|_T = \theta_x$. Given $\lambda \vdash n$, let u_λ be the unipotent character of L with label λ. Then, $\pm R^{\mathbf{X}}_{\mathbf{L}}(\beta_x u_\lambda)$ gives $\mathcal{E}(X, s)$ by [58, Theorem 11.4.3]. On I, we put $\rho_{x,\lambda} = \beta_x|_I \cdot \mathrm{Inf}^I_{I/T} \mathrm{Sp}_\lambda$, with $\mathrm{Sp}_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$. Moreover, $\mathrm{Ind}^M_I \rho_{x,\lambda}$ gives the characters over $M\theta_x$ by [95, Theorem 6.11].

Since $p \mid q-\epsilon$, the index $|X : L|_{r'} = \prod_{j=1}^{n-1}(q^j + \epsilon^j)$ is prime to p. Moreover, $[M : I] = 2^{n-1}$ is prime to p. By the hook degree formula [40, §13.8] and $v_p(q^j - \epsilon^j) = b + v_p(j)$, we have that $v_p(u_\lambda(1)) = v_p(n!) - v_p\big(\prod_{\square \in \lambda} h(\square)\big) = v_p(\mathrm{Sp}_\lambda(1))$. By [Macdonald, 1971, 189–192], for $n = p^a$ we have $p \nmid \mathrm{Sp}_\lambda(1)$ exactly when λ has a hook of length p^a. Thus, $\lambda = (n-j, 1^j)$ for $0 \leq j < n$. □

Theorem 6.182. *With q odd, we put $n = p^a \geq 5$ with p an odd prime. If $p \mid q - \epsilon$, then $\mathrm{iMN}(\mathrm{P}\Omega^{\epsilon}_{2n}(q), p)$ holds on X with $M = \mathrm{N}_X(T)$ for the singleton torus T. In particular, at $n = p = 5$ the congruences are $q \equiv \epsilon \bmod 5$.*

Proof. At $x = \pm 1$, $s = 1$, and [156, Theorem D] gives $\succeq_c$ on the unipotent series. Moreover, at $x^2 = -1$, Proposition 6.16 gives $\succeq_c$ with $\mathrm{ord}_p(\epsilon q) = 1$ and residue rank 0. Its partition labels are $\lambda = (n-j, 1^j)$ by Lemma 6.181. Suppose that $x^2 \notin \{1, -1\}$. An ordinary restorer b of field degree 0 fixes $V_x, V_{x^{-1}}$: exchanging them replaces x by $x^{-1} \notin \{x, -x\}$. Thus, b fixes the cocharacter ν^* with weights $1, -1$ on $V_x, V_{x^{-1}}$. The primal cocharacter $2(e_1 + \cdots + e_n)$ defines a radical $\mathbf{U}$ fixed by H_1. Moreover, for each generator s^j of $\langle s \rangle$ we have $\mathrm{C}_{\mathbf{X}^*}(s^j) = \mathrm{C}_{\mathbf{X}^*}(s) = \mathbf{L}^*$. For (b, σ) stabilizing θ_x, uniqueness gives $\beta_x^{(b,\sigma)} = \beta_x$.

As in Lemma 6.19, $\mathrm{Z}(\mathbf{L})$ has equation $a^n = z^2$ and is connected for odd n. By Lang's theorem, we obtain $L \twoheadrightarrow L_{\mathrm{ad}}$. By [156, Theorem D and its proof], the models on L_{ad} give unipotent pairs on L, I. On $I/T \cong \mathfrak{S}_n$ take the Specht

module V_λ over $\mathbb{Q}$ for Sp_λ. Moreover, F_r and inverse transpose act trivially on I/T. Thus, on $H \times_{D_0} B'$ the twist by $\beta_x(hb'^{-1})$ gives the relative tensor V for $\beta_x u_\lambda, \rho_{x,\lambda}$ as in Lemma 6.19. Since $\beta_x^{(b,\sigma)} = \beta_x$, $V^{(b,\sigma)} \cong V$, and the diagonal $\mathrm{C}_A(X)$ acts trivially. If $F = \gamma F_q$, apply the identity $\varphi^m = K_{a^m} F$. By Proposition 6.134, we obtain $\succeq_c$ for (χ, ϕ). Finally, by Lemma 6.180, M contains $\mathrm{N}_X(P)$ and is $\mathrm{Aut}(X)_P$-stable. Since $1 < T/\mathrm{Z}(X) < X/\mathrm{Z}(X)$ and $X/\mathrm{Z}(X)$ is simple, $M = \mathrm{N}_X(T) < X$. Moreover, the p'-index classes $[s]$ correspond to the orbits $M\theta_x$ of p'-size. Thus, the comparisons on $\mathcal{E}(X, s)$ give the equivariant bijection with $\mathrm{Irr}_{p'}(M)$ and the relation $\succeq_c$. □

With $X = \mathrm{Spin}_{10}^\epsilon(q)$ and $q = r^f$ odd, we choose $t = \pm 1$ with $p \mid q - t$ for odd p. The torus $\mathbf{T}$ has five planes of type t if $\epsilon = t$. Otherwise, it has four of type t and a distinguished plane of type $-t$. Put $T = \mathbf{T}^F$, $M = \mathrm{N}_X(T)$, and $m = 5$ or 4, respectively. As in Lemma 6.180, take $P = T_p \rtimes S_p$ with $T_p \cong (C_{p^b})^m$, $b = v_p(q - t)$, and $S_p \in \mathrm{Syl}_p(\mathfrak{S}_m)$. The order of P is $p^{mb + v_p(m!)} = |X|_p$, so $P \in \mathrm{Syl}_p(X)$. Moreover, for $m = 4$ the distinguished torus has order $q + t$, prime to p. Thus, we find $\mathrm{C}_{\mathbf{X}}(T_p) = \mathbf{T}$ by the distinct weights ξ_j, ξ_j^{-1} and the factor SO_2^{-t} if $m = 4$. By [Brewster–Passman–Wilcox, Proposition 2.2], T_p is characteristic in P. Hence, we obtain that $\mathrm{N}_X(P) \leq M$ and M is $\mathrm{Aut}(X)_P$-stable. Write $A = C_{q-t}$, $B = C_{q+t}$, with quadratic characters η_A, η_B.

Proposition 6.183. *Suppose that $\epsilon = -t$. There is a bijection between the p'-characters of X in the series with distinguished coordinate $y^4 \neq 1$ and the p'-characters of M over the corresponding orbits $M\theta_s$. Moreover, the pairs (χ, ϕ) satisfy $\succeq_c$.*

Proof. With $k = tq$, write $u = (x_1, \ldots, x_4, y)$ for the natural lift of s. Thus, $F(u) = cu$ gives $x_j^{k-1} = c$ and $y^{k+1} = c$ with $c = \pm 1$. Both solution sets are stable under $z \mapsto -z$ and $z \mapsto z^{-1}$. Moreover, their intersection has $z^2 = 1$. Thus, for $[z] = \{z, z^{-1}\}$ we have $[y] \neq [\pm x_j]$, and $[y] \neq [-y]$ since $y^4 \neq 1$. A rational projective conjugator g lifts to $\widetilde{g}$ with $F(\widetilde{g}) = \pm\widetilde{g}$. Thus, $\widetilde{g}u\widetilde{g}^{-1}$ has Frobenius sign c, since $F(\widetilde{g}u\widetilde{g}^{-1}) = c\widetilde{g}u\widetilde{g}^{-1}$. The equations distinguish $[y]$ within each rational class, even when $x_j^2 \neq 1$ for every j. Also, no isometry acts by $u \mapsto -u$, by $[y] \neq [-y]$ and $[y] \neq [\pm x_j]$.

The active planes give Y of type $t^4 = 1$. Moreover, for the Levi $\mathbf{L}$ at $2e_5$ with radical $\mathbf{U}$, put $L = \mathbf{L}^F$, $R = [\mathbf{L}, \mathbf{L}]^F = \mathrm{Spin}_8^+(q)$ and $U = \mathrm{Spin}_2^{-t}(q)$ on the singleton. The singleton has $|U| = q + t$, prime to p. Thus, $P = T_p \rtimes S_p$ lies in R, with S_p permuting the four planes of Y. Moreover, sn is onto $\{1, -1\}$ on both natural tori. With $T_R = T \cap R$ and $C = \mathrm{C}_T(R)$, the proof of Lemma 6.176 gives $L = TR$, $J = CR$, and $I/I_J \cong L/J \cong C_2$ for $I = \mathrm{N}_L(T)$ and $I_J = C\mathrm{N}_R(T_R)$. On $2e_5$, the weight ν is dual to y^2. Hence, $y^4 \neq 1$ is equivalent to $\nu^2 \neq 1$.

By Theorem 6.147, the p'-characters of R correspond to those of $I_R = \mathrm{N}_R(T_R)$. Next, adjoin C by [141, Theorem 2.8], with each compatible $\kappa \in \mathrm{Irr}(C)$ and $\nu = \kappa|_U$. On $I/I_J \cong L/J$, apply Proposition 6.177 with the starting pair (ξ, ζ) on R, I_R. Thus, we have that (β, ρ) satisfy $\succeq_c$ for every $\nu^2 \neq 1$, with scalar $\nu(u)$ at $u \in U$. Moreover, the degrees of β, ρ differ from those of ξ, ζ by factors in $\{1, 2\}$. Thus, the pairs exhaust the indicated p'-sets of L, I.

The rational Weyl element w inverts coordinates $1, 5$. Its lift generates $M/I \cong C_2$. Moreover, w acts by $\nu \mapsto \nu^{-1}$, so the orbits of β, ρ have length 2. By Clifford correspondence [95, Theorem 6.11], $\phi = \mathrm{Ind}_I^M \rho$ with fibres $\{\rho, \rho^w\}$. On the dual side, $\mathrm{C}_{\mathbf{X}^*}(s)$ fixes the lines of y, y^{-1}, so $\mathrm{C}_{\mathbf{X}^*}(s) \leq \mathbf{L}^*$. If the two points s, s' fuse in X^*, the rational conjugator fixes $[y]$ by the equations for c. Thus, it lies in $\mathrm{N}_{\mathbf{X}^*}(\mathbf{L}^*)^F$ with action 1 or w on that plane. By [58, Theorem 11.4.3], signed induction $\chi = \pm R_{\mathbf{L}}^{\mathbf{X}}\beta$ is bijective on $\mathcal{E}(L, s)$. Moreover, [19, Corollary 6.5 and Remark 6.7] compares $\mathbf{U}, \mathbf{U}^w$, giving fibres $\{\beta, \beta^w\}$. Since $P \leq L$ and $[M : I] = 2$, the ambient degrees are prime to p for the p'-pairs on L, I.

Given ρ, choose a weight θ on T under ρ. Then, $\theta|_U = \nu$, so $M_\theta \leq I$, and its Clifford correspondent ρ_θ induces to ϕ on M. Correct an operation stabilizing (χ, ϕ) by M to fix θ. Thus, the corrected operation fixes ν and so β, ρ in their two-element fibres. On $B' = B_\theta$ the θ-space of ρ gives the model for ρ_θ, as in Proposition 6.178. Consequently, the tensor V for $\beta \boxtimes \rho_\theta^\vee$ on $LB' \times_{D_0} B'$ satisfies $V^{(b,\sigma)} \cong V$ and $(z, z) \in \ker V$ for $z \in \mathrm{C}_{X \rtimes D_0}(X)$. Given b of field degree 0, fixing $\nu \neq \nu^{-1}$ gives $b(2e_5) = 2e_5$. Thus, $\mathbf{U}$ is H_1-stable. Moreover, if $\langle s^j \rangle = \langle s \rangle$, the lines $V_y, V_{y^{-1}}$ still give $\mathbf{L}^*$. If $F = \gamma F_q$ and $h = |LB' : H_1|$, the relation $\varphi^h = K_{a^h} F$ is the relation for φ. By Proposition 6.134, we obtain $\succeq_c$ with $\mathcal{E}(L, s)$ in the cohomology comparison and M_θ for local induction. Consequently, (χ, ϕ) satisfy $\succeq_c$. □

Lemma 6.184. *Take u in $\mathrm{I}(1, 4)$ with its 1-eigenspace of dimension 2 and the -1-eigenspace of dimension 8. If $\epsilon = t$, we have that the orbit on T^* has size 5. If $\epsilon = -t$, we have that the 1-plane is distinguished at orbit size 1 and active at size 4. Moreover, $[s]$ is eligible exactly when $p \nmid |M\theta_s|$. The -1-eigenspace is split at sizes $5, 1$ and of minus type at size 4.*

Proof. Since the dimensions are $2, 8$, the normalized lift u has $F(u) = u$. The inversions fix the entries ± 1 of u, so M permutes the planes of type t and those of type $-t$. Thus, the 1-plane has 5 choices for $\epsilon = t$, and 1 or 4 for

$\epsilon = -t$. Moreover, the type of $V_- = \ker(u+1)$ is $t^4 = 1$ or $t^3(-t) = -1$, respectively. By [132, Theorem 5.9 and Proposition 5.11], the orbits of p'-size on T^* give the classes $[s]$ of p'-index.

With $W_j = C_2 \wr \mathfrak{S}_j$, we write w for the weight of Λ. Moreover, at split V_- the active count is $m_- = 4$. By Lemma 6.59, a label of p'-degree has $w = 4$ and core rank 0, since $(4-w)v_p(q-t) = 0$. Take the 1-hook core at $t = 1$ and the 1-cohook core at $t = -1$. Hence, the two runners give (α, β) of size 4, with p'-degree on W_4. Row exchange acts by $(\alpha, \beta) \mapsto (\beta, \alpha)$. If $\alpha \neq \beta$, the symbol Λ has two orientations. If $\alpha = \beta$, the component group C_2 exchanges the degenerate labels $u_\Lambda^\pm$, giving one character of X by [208, §7, following Lemma 7.15]. For the equal bipartitions $\alpha = \beta \vdash 2$, the degree on W_4 is $\binom{4}{2} = 6$, so $p = 3$ excludes both. Moreover, by [40, §13.8], absolute defect 4 has least rank 4, with the unique cuspidal symbol $(\{0,1,2,3\}, \varnothing)$ up to shifts of Λ and exchange of its rows.

Given V_- of minus type, we set $m_- = 3$. Thus, the formula gives $w = 3$, core rank 1, and p'-degree on W_3. The 1-hook core at $t = 1$ is the rank 1 symbol of type -1 and absolute defect 2. Moreover, for $t = -1$ the 1-cocore has type $+1$ after 3 cohooks. Order its normalized rows as $(\{1\}, \{0\})$. Thus, with (α, β) of size 3 gives a nondegenerate Λ, of absolute defect 2 and two orientations of u_Λ. □

Proposition 6.185. *Let $[s]$ be an eligible* I$(1,4)$ *class of* Lemma 6.184. *Then, the pairs (χ, ϕ) between $\mathcal{E}(X,s) \cap \mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M \mid M\theta_s)$ satisfy $\succeq_c$.*

Proof. Let Y_+ be the positive plane and Λ the symbol on V_- of sign ϵ_-. For nondegenerate Λ of absolute defect 0 or 2, [208, Lemma 7.17(i)] gives ordinary hooks $\Lambda_4, \dots, \Lambda_1$, with $\Lambda_4 = \Lambda$ and all Λ_i nondegenerate. Moreover, for $\epsilon(Y_-) = \epsilon(Y_+)$, choose Φ from Lemma 6.57. Given $\epsilon(Y_-) \neq \epsilon(Y_+)$, we choose χ_1, χ_2 from Lemma 6.88. On E_i, with $1 \le i \le 3$, the weight on $\mathrm{GL}(E_i)$ is η_q^{fld}. By [208, Proposition 7.18, Steps 2–4], the marked coefficient at Λ_i is 1. Consequently, the isotypic space for χ gives the field extension $\widehat{\chi}$ as in Proposition 6.167. Write $\delta = \eta_q^{\mathrm{fld}}(-1)$. On $Y_- \perp Y_+$ the scalar at $-I$ is $\epsilon_-\delta$. Hence, the restored scalar is $(\epsilon_-\delta)\delta^3 = \epsilon_-$. A nonsquare multiplier on V gives the other orientation.

At $\Lambda_C = (\{0,1,2,3\}, \varnothing)$, the 1-hook core has rank 4, so $t = -1$. For anisotropic Y_+, by [165, Theorem 10.8], we obtain $\widehat{\chi}_C$ on $\mathrm{O}(V)$. Moreover, for split Y_+, the source is $1 \boxtimes (\nu_-\psi_C)$ on $\mathrm{GL}_1(q) \times \mathrm{SO}_8^+(q)$, with cusp ψ_C and spinor twist ν_- as in Corollary 6.169. At $d = 2$, Lemma 6.238 gives the field operator B_C, with scalar 1 at $-I$. Moreover, for equal labels (α, α) with $\alpha = (2), (1,1)$, choose the global flag E_1, E_2 of Proposition 6.168. At E_1, the fraction for χ_1 is $q/(q+1)$. At E_2, the addition to (1) has coefficient 1 by [208, Proposition 7.20(1)]. The flag multiplicity is 1. For either Y_+, [208, Lemma 3.13] gives the similitude and field extension $\widehat{\chi}_\alpha$, fixed by $\mathcal{H}_p$. On zI their scalar is 1, since the weight $\eta_q^{\mathrm{fld}} \circ \det_{E_i}$ takes $\eta_q^{\mathrm{fld}}(z^2) = 1$.

Let T_O be the natural torus and put $N_O = \mathrm{N}_{\mathrm{O}(V)}(T_O)$, $N_0 = N_O \cap \Omega(V)$. On T_O, set $\theta_O = \eta_1 \boxtimes \cdots \boxtimes \eta_4 \boxtimes 1$, where η_i is quadratic on negative plane i. Moreover, the stabilizer of Y_+ acts with value 1 at its reflection w_+ and label (α, β) on W_4 if $\epsilon_- = 1$. If $\epsilon_- = -1$, the label is on W_3, with sign ε_B at reflection w_B on the distinguished plane. Induction to N_O over the position of Y_+ gives the parent ξ_O. Moreover, $(\det, \nu)$ gives $N_O/N_0 \cong C_2^2$, with w_+ and $u \in T_O$ with $\nu(u) = -1$. The twist by ν changes the quadratic support size from 4 to 1. The twist by det changes the value at w_+ from 1 to -1 on the θ_O-space. Thus, by [95, Lemma 5.2], the restrictions $\xi_O|_{N_0}$ are distinct irreducibles. After inflation to M, $z_0 \in \ker\phi$. If $I = M_{\theta_s}$, the quotient I/T is W_4 if $\epsilon_- = 1$ and $W_3 \times \langle w_B \rangle$ if $\epsilon_- = -1$. Hence, by [95, Theorem 6.11], we obtain all ϕ over $M\theta_s$ with degree factor $[M : I]$, from Lemma 6.184.

For a split plane, $\mathrm{diag}(m, 1)$ changes the inversion by coordinate m, with value $\eta_q^{\mathrm{fld}}(m)$. On a nonsplit plane, $z^{q+1} = m$ gives correction z^{1-q} with value $\eta_q^{\mathrm{fld}}(m)$. Thus, for $\eta_q^{\mathrm{fld}}(m) = -1$ the W_j-label changes by $(\alpha, \beta) \mapsto (\beta, \alpha)$, and $\varepsilon_B \mapsto -\varepsilon_B$ for $\epsilon_- = -1$. On W_3, the ordered core corresponds to $\varepsilon_B = 1$. Transport by m gives $\varepsilon_B = -1$. Moreover, the scalar at $-I$ is $\eta_A(-1)^4 = 1$ for $\epsilon_- = 1$, and $\eta_A(-1)^3\eta_B(-1) = -1$ for $\epsilon_- = -1$.

Take the prime-field lift a and put $w_{\mathrm{ns}} = \prod_{i \in J_{\mathrm{ns}}} w_i$, where w_i inverts plane i and J_{ns} indexes the nonsplit planes. Thus, $a^f = w_{\mathrm{ns}}$ and w_{ns} is central modulo T_O. Its scalar on ξ_O is $\zeta = \xi_O(w_{\mathrm{ns}})/\xi_O(1)$. If $t = 1$, the base model has $\zeta = 1$, since w_{ns} is $1, w_+$ or w_B. If $t = -1$, $\zeta \in \{1, -1\}$. Moreover, $D = \mathrm{ord}_p(r)$ has $D/\gcd(D, f) = 2$ in the latter case, so $\alpha_0^f = \zeta$ for some $\alpha_0 \in \mu_D$. Since $D \mid p-1$, this α_0 is fixed by $\mathcal{H}_p$. Thus, we choose $a \mapsto \alpha_0 I$ for $t = -1$ and $a \mapsto I$ for $t = 1$. By [58, §4.2], Lang transport by g as in Proposition 6.167 gives the power $gw_{\mathrm{ns}}g^{-1}$ on $\widehat{\chi}$. The nonsquare similitude g_m gives the other orientation of (χ, ϕ), with lift $g_m a g_m^{-1}$ on the local side and power $g_m w_{\mathrm{ns}} g_m^{-1}$. Consequently, for $\eta_q^{\mathrm{fld}}(m) = -1$ and moved labels, the extensions $\widehat{\chi}, \widehat{\phi}$ are $\mathcal{H}_p$-fixed with common scalar at $-I$. Moreover, an operation stabilizing (χ, ϕ) has $\eta_q^{\mathrm{fld}}(m) = 1$ since $\chi^\sigma = \chi$ and $\phi^\sigma = \phi$ for $\sigma \in \mathcal{H}_p$.

With the fixed labels (α, α), we choose the multiplier torus $\widetilde{T}$ and put $\widehat{\theta} = \eta_A(z_1 z_2/(z_3 z_4))$. The z_i are the first

coordinates when $t=1$ and have $z_i^{q+1}=m$ when $t=-1$. Thus, the quotient in $\widehat{\theta}$ belongs to A, with $\widehat{\theta}|_{T_O}=\theta_O$ and $\widehat{\theta}(zI)=1$. A permutation changes $z_1z_2/(z_3z_4)$ by a square in A. Moreover, inversion at coordinate i changes $\widehat{\theta}$ by $\eta_q^{\mathrm{fd}}(m)$. Thus, its inertia in W_4 is $W(D_4)$.

If $\alpha=(2)$, we choose a basis v_S for $S\subseteq\{1,2,3,4\}$ with $|S|=2$, with an inversion set J_0 acting by $(-1)^{|J_0\cap S|}$ on v_S. If $\alpha=(1,1)$, we multiply the permutation action by sgn. Moreover, $Kv_S=v_{S^c}$ has $K^2=1$. For the inversion set J_0, its matrix $P(J_0)$ satisfies $KP(J_0)=(-1)^{|J_0|}P(J_0)K$. Thus, $\ker(K\mp 1)$ are the rational $W(D_4)$-constituents by [95, Theorem 6.11]. Extend $\widehat{\theta}$ by either eigenspace and $w_+\mapsto 1$, then induce to $\mathrm{N}_{\mathrm{GO}(V)}(\widetilde{T})$. Thus, the restriction to N_O is the parent ξ_O of (α,α), induced over Y_+. For the field action F_r, the matrices of W_4 are fixed and the coordinates become z_i^r. Moreover, $\widehat{\theta}(z^r)=\widehat{\theta}(z)$ and w_{ns} acts by 1. Indeed, at $t=-1$ the negative inversions act by $(-1)^{|S|}=1$. At $t=1$, w_{ns} fixes every negative coordinate. Thus, $a\mapsto I$ gives the extension $\widehat{\phi}_\alpha$, fixed by $\mathcal{H}_p$ and with scalar 1 at zI.

In the local overgroup, N_O meets the $(\det,\nu)$ classes, $\widetilde{T}$ has every multiplier m, and a has field degree 1. For moved (χ,ϕ), choose the isometry-field groups $\mathcal{A},\mathcal{B}$. Moreover, for fixed (χ,ϕ), choose the similitude-field groups $\mathcal{A},\mathcal{B}$. Thus, $\mathcal{A}_\chi=\Omega(V)\mathcal{B}_\phi$ and $\Omega(V)\cap\mathcal{B}_\phi=N_0$. By [79, Theorem 2.5.1], centralizing $\Omega(V)$ implies degree 0 mod f, and $\mathrm{C}_{\mathrm{GL}(V)}(\Omega(V))=\mathbb{F}_q^\times I$. Consequently, at $-I$ both scalars are ϵ_-. At zI the fixed labels give scalar 1. Since the lift u of s has $F(u)=u$, by Lemma 6.27, $z_0\in\ker\chi$. Thus, by [141, Definition 1.5 and Theorem 2.9], the extensions on $\Omega(V),N_0$ give $\succeq_c$ on X,M. □

Lemma 6.186. *Given* $\mathrm{I}(2,3)$*, we write* u *for the natural lift with* $Y_-=\ker(u+1)$ *of dimension* 4 *and* $Y_+=\ker(u-1)$ *of dimension* 6*. Moreover, keep* T,t,p *from* Section 6.7*. The orbit of* s *on* T^* *has size* 10 *for* $\epsilon=t$*. For* $\epsilon=-t$ *it has size* 4 *or* 6 *as the distinguished plane lies in* Y_- *or* Y_+*, respectively. Moreover, we have that* $[s]$ *is eligible exactly when* $p\nmid|M\theta_s|$*. The labels* $1,\mathrm{St}$ *on* Y_- *each give two orientations. If* Y_- *is split, the orbit* $\{(1,\mathrm{St}),(\mathrm{St},1)\}$ *on its two* A_1 *factors gives the label* χ_{mix}*.*

Proof. The unequal dimensions of Y_-,Y_+ give $F(u)=u$. Thus, the signed Weyl group permutes u within each type. The counts for Y_- are $\binom{5}{2}=10$ at $\epsilon=t$, 4 with the distinguished plane in Y_-, and $\binom{4}{2}=6$ with it in Y_+. By [132, Theorem 5.9 and Proposition 5.11], the orbits of p'-size give exactly the classes $[s]$ of p'-index. Moreover, we have that Y_- is split for plane types (t,t). The simultaneous reflection on Y_- exchanges $(1,\mathrm{St}),(\mathrm{St},1)$ and fixes $(1,1),(\mathrm{St},\mathrm{St})$, by [208, §7, following Lemma 7.15]. For minus Y_-, the labels $1,\mathrm{St}$ come from $\mathrm{SL}_2(q^2)$. Thus, the connected degrees on Y_- are powers of r.

The A_3 factor on Y_+ has labels $\lambda\vdash 4$, fixed by the graph on A_3. Moreover, for homogeneous Y_+ the relative group is $W(D_3)\cong\mathfrak{S}_4$. By [40, §13.8] and $p\mid q-t$, its degrees satisfy $v_p(u_\lambda(1))=v_p(\mathrm{Sp}_\lambda(1))$. For opposite singleton type on Y_+, the relative group is $W(B_2)$, and $\mathrm{ord}_p(-tq)=2$. Since the staircase sizes up to 4 are $0,1,3$, the 2-core of $\lambda\vdash 4$ is $\varnothing$. Thus, the five 2-quotients $(\lambda^{(0)},\lambda^{(1)})$ have size 2. By [40, §13.8], only even indices contribute to $v_p(u_\lambda(1))$. Each even length $2h$ gives the hook length h in $\lambda^{(0)},\lambda^{(1)}$. Hence, the degrees on $W(B_2)$ have the same p-valuation, which is 0 since they are $1,1,1,1,2$ and p is odd. Also, $p=3$ is excluded by the orbit size 6.

Let T_O be the natural torus and $N_O=\mathrm{N}_{\mathrm{O}(V)}(T_O)$. Put $\theta_O=\eta_1\boxtimes\eta_2\boxtimes 1$ on T_O, with quadratic weights η_1,η_2 on Y_-. Moreover, $W_\pm$ consists of signed permutations on $Y_\pm$ with each plane type invariant. Let $j_+\in\mathrm{Z}(W_+)$ invert all three planes for homogeneous Y_+, or the distinguished plane for mixed Y_+. Thus, $\det(j_+)=-1$ and $W_+=W_+^{\mathrm{SO}}\times\langle j_+\rangle$. Extend $\zeta_+\in\mathrm{Irr}(W_+^{\mathrm{SO}})$ to $\widehat{\zeta_+}$ by the scalar 1 at j_+. Moreover, we choose $\zeta_-\in\mathrm{Irr}(W_-)$. With $\widehat{\theta_O}$ trivial on $W_-\times W_+$, induce $\widehat{\theta_O}(\zeta_-\boxtimes\widehat{\zeta_+})$ to the parent ξ_O on N_O. By Proposition 6.185, the twist by ν changes the support size of θ_O from 2 to 3, and det changes the scalar 1 at j_+ to -1. Thus, on $N_0=N_O\cap\Omega(V)$ the restrictions of ξ_O are distinct irreducibles by [95, Lemma 5.2]. Moreover, $I=M_{\theta_s}$ has $I/T\cong W_-\times W_+^{\mathrm{SO}}$, with the determinant correction j_+. By [95, Theorem 6.11], the inductions to M exhaust $\mathrm{Irr}(M\mid M\theta_s)$.

For different types on Y_-, $W_-=C_2\times C_2$. Moreover, a multiplier m acts on its reflection values by $(a,b)\mapsto(-a,-b)$ when $\eta_q^{\mathrm{fd}}(m)=-1$, as in Proposition 6.185. In each orbit $\{(a,b),(-a,-b)\}$, set $b=1$. Moreover, for homogeneous Y_-, $W_-=W(B_2)$. Write $\varepsilon_{\mathrm{inv}}$ for the total inversion character. Thus, $\varepsilon_{\mathrm{inv}}$ gives 2 linear label orbits of size 2, with $\varepsilon_{\mathrm{inv}}\zeta_-=\zeta_-$ on the label of degree 2. Hence, the free pairs correspond to $1,\mathrm{St}$ and the fixed label to $\{(1,\mathrm{St}),(\mathrm{St},1)\}$, with ζ_+ determined by λ. The local degree is $|M\theta_s|\zeta_-(1)\zeta_+(1)$, with $\zeta_-(1)\in\{1,2\}$. Thus, the p'-degree test is the test on λ. □

Proposition 6.187. *Take eligible* $[s]$ *in* Lemma 6.186. *Then, the pairs* (χ,ϕ) *between* $\mathcal{E}(X,s)\cap\mathrm{Irr}_{p'}(X)$ *and* $\mathrm{Irr}_{p'}(M\mid M\theta_s)$ *satisfy* $\succeq_c$.

Proof. For $1,\mathrm{St}$ on Y_-, we choose an isotropic line E_-. The trivial torus source has coefficient 1 at $1,\mathrm{St}$, on $A_1\times A_1$ in split type and on $\mathrm{SL}_2(q^2)$ in minus type. Moreover, on Y_+ take the lines $E_{+,1},E_{+,2}$ by Lemma 6.119 and [208, Proposition 7.40]. On R_-,R_+ at the endpoints take Φ of Lemma 6.57 if $\mathrm{sign}(R_-)=\mathrm{sign}(R_+)$. Otherwise, we choose χ_1,χ_2 of Lemma 6.88. The weights on $\mathrm{GL}(E_-),\mathrm{GL}(E_{+,i})$ are $\eta_q^{\mathrm{fld}},1$, respectively. By [208, Proposition 7.18], each of $E_-,E_{+,1},E_{+,2}$ has marked coefficient 1. Thus, on the isotypic χ-space the extension is $\widehat{\chi}$ of Proposition 6.167. Write $\epsilon_-=\mathrm{sign}(Y_-)$. Thus, the negative weights at $-I$ multiply to ϵ_-, the scalar of $\widehat{\chi}$. If $\eta_q^{\mathrm{fld}}(m)=-1$, the multiplier m gives the other orientation of χ.

Next, for χ_{mix}, Y_- is split and $H_\lambda\in\mathrm{Irr}_{p'}(\mathrm{SO}(Y_+))$ is unipotent. Put $d=\mathrm{ord}_p(q)$ and $e=\mathrm{ord}_p(\epsilon_+q)$ with $\epsilon_+=\mathrm{sign}(Y_+)$. Moreover, $\mathrm{core}_e(\lambda)=\varnothing$ for $e\in\{1,2\}$, so the d-cuspidal pair is $(\mathbf{T}_+,1)$. On the adjoint graph-field group Γ_{ad}, the extension $\widehat{H}_{\lambda,\mathrm{ad}}$ is fixed by $\mathcal{H}_p$ by [156, §8.1 and proof of Theorem D]. Thus, its pullback to $\Gamma(Y_+)$ gives $\widehat{H}_\lambda$ with scalar 1 on zI. Moreover, by [156, proof of Theorem D], $H_\lambda|_{\Omega(Y_+)}$ is irreducible.

For an isotropic plane $E\leq Y_-$ take the source $(\eta_q^{\mathrm{fld}}\circ\det_E)\boxtimes\widehat{H}_\lambda$ on the stabilizer of E. At zI, its scalar is $\eta_q^{\mathrm{fld}}(z^2)=1$. Moreover, on Y_+ the group $\Gamma(Y_+)$ meets all $(m,\det,\nu)$ classes. Thus, on $\Omega(V)$ Mackey's formula [95, Problem 5.6] gives induction from $\Omega(V)_E$. By [72, Comparison Theorem 3.2.7] and Proposition 6.168, the flag E gives $A_1\times B_1$ or $B_1\times A_1$. The coefficients at $(1,\mathrm{St}),(\mathrm{St},1)$ are $1,0$, and the component exchanges the two A_1's. Moreover, for χ_{mix} the fraction is $q/(q+1)$ by Lemma 6.79, since $H_\lambda(1)$ cancels. By Lemma 6.78, we obtain $\widehat{\chi}_{\mathrm{mix}}$, with $\widehat{\chi}_{\mathrm{mix}}^\sigma=\widehat{\chi}_{\mathrm{mix}}$ for $\sigma\in\mathcal{H}_p$ and scalar 1 at zI.

For moved (χ,ϕ), choose ξ_O from Lemma 6.186 and the field lift a of Proposition 6.185. Then, $a^f=w_{\mathrm{ns}}$ inverts the nonsplit planes. If $t=1$, its scalar on ξ_O is 1: on Y_+ the reflection is j_+, and on Y_- it has value 1 by the choice of orientation. If $t=-1$, its scalar is $\zeta=\pm1$, so the cited proposition gives $\alpha_0\in\mu_{\mathrm{ord}_p(r)}$ with $\alpha_0^f=\zeta$. Thus, the local extension with $a\mapsto\alpha_0 I$ is fixed by $\mathcal{H}_p$, with $\alpha_0=1$ for $t=1$. The scalar at $-I$ is $\eta_A(-1)^2=1$ in split type and $\eta_A(-1)\eta_B(-1)=-1$ for minus Y_-. Moreover, the nonsquare similitude g_m transports a to $g_mag_m^{-1}$ by Proposition 6.185. For $(\chi_{\mathrm{mix}},\phi)$, on $\widetilde{T}$ put $\widehat{\theta}=\eta_A(z_1/z_2)$ on the two negative coordinates. Thus, $\widehat{\theta}(zI)=1$ and the inertia in W_- is the even-inversion subgroup $W(D_2)$. On the wreath module of degree 2, let K exchange the vectors $v_{\{1\}},v_{\{2\}}$. Thus, $K^2=1$ and $\ker(K\mp1)$ are rational $W(D_2)$-lines. An inversion w_i acts by $\ker(K-1)\mapsto\ker(K+1)$ by Proposition 6.185. Extend $\widehat{\theta}$ by either line and $\widehat{\zeta}_+$ on W_+, then induce over the positions of Y_-. By [95, Theorem 6.11], restriction to N_O gives ξ_O, and inflation from N_0 gives ϕ. Moreover, we have that F_r fixes the Weyl action and gives $z_i\mapsto z_i^r$, so $\widehat{\theta}^{F_r}=\widehat{\theta}$. The scalar of w_{ns} is 1 at $t=1$ by j_+, and has an $\mathcal{H}_p$-fixed f-th root α_0 at $t=-1$. Consequently, $\widehat{\phi}^\sigma=\widehat{\phi}$ for $\sigma\in\mathcal{H}_p$, with scalar 1 at zI.

The local groups meet the $(\det,\nu)$ classes and contain a degree-1 field lift. Given χ_{mix}, their multipliers run through $\mathbb{F}_q^\times$. Thus, $\mathcal{A}_\chi=\Omega(V)\mathcal{B}_\phi$ as in Proposition 6.185. At $-I$ the common scalar is ϵ_-, and at zI on fixed pairs it is 1. Consequently, the $\mathcal{H}_p$-fixed extensions give $\succeq_c$ by [141, Definition 1.5 and Theorem 2.9]. Moreover, at $F(u)=u$, by Lemma 6.27, $z_0\in\ker\chi\cap\ker\phi$. □

Lemma 6.188. *At* $p\mid q-t$, *we put* $u=\eta_q^{\mathrm{fld}}(-1)$. *Thus, for* $i^2=-1$, *the pair* $\{i,i^{-1}\}$ *lies on a plane of type* u. *Moreover, in type* $Q(a,c_0)$, *we have that the natural lift* v *has* a *coordinates* 1, a *coordinates* -1 *and* c_0 *coordinates in* $\{i,-i\}$. *Thus, we find* $2a+c_0=5$. *Each entry* $|Ws|$ *counts one orbit of* $W=M/T$ *on* T^*. *A dash means that* Ws *is absent. Thus, we have that* $[s]$ *has* p'*-index precisely when* $p\nmid|Ws|$.

	$Q(1,3)$	$Q(2,1)$
$\epsilon=t,\ u=t$	*80*	*30*
$\epsilon=t,\ u=-t$	–	–
$\epsilon=-t,\ u=-t$	–	*6*
$\epsilon=-t,\ u=t$	*32*	*24*

Proof. By [208, Lemma 7.15], $Q(1,3)$ and $Q(2,1)$ are the two types with $2a+c_0=5$ and $a,c_0>0$. Moreover, on each plane F fixes the coordinates $1,-1$. Thus, $F(v)=v$ since $F(v)\in\{v,-v\}$ and $a>0$. $4\mid q-u$ and $4\nmid q+u$, so each quartic coordinate of v lies on a plane of type u. The group W permutes planes of type t and planes of type $-t$ separately. Moreover, for any subset J of the quartic coordinates, W inverts J and one sign coordinate if $|J|$ is odd.

When $\epsilon = t$, $u = t$ and Wv has $2^{c_0}5!/(a!^2c_0!)$ elements. Thus, the quotient $v \sim -v$ gives $|Ws| = 2^{c_0-1}5!/(a!^2c_0!)$. When $\epsilon = -t$ and $u = -t$, the distinguished plane has coordinate $v_5 = \pm i$, so $c_0 = 1$. Hence, $|Ws| = 2\binom{4}{2}/2 = 6$. When $\epsilon = -t$ and $u = t$, we choose the lift v with coordinate 1 on the distinguished plane. Thus, $|Ws| = 2^{c_0}4!/((a-1)!a!c_0!)$. By [132, Theorem 5.9 and Proposition 5.11], the orbits of p'-size give exactly the classes $[s]$ of p'-index. In particular, $Q(2,1)$ gives no such class at $p = 3$, and at $p = 5, \epsilon = t$ both orbit sizes are divisible by p. □

Proposition 6.189. *For $[s]$ of type $Q(1,3)$ and p'-index, the mixed comparison holds at M on $\mathcal{E}(X,s) \cap \mathrm{Irr}_{p'}(X)$. The set has size 12 for $\epsilon = t$ and 6 for $\epsilon = -t$.*

Proof. Take $v = (1,-1,i,i,i)$, with the first plane distinguished when $\epsilon = -t$. Then, $t = u$ and the two sign planes $V_\pm = \ker(v \mp 1)$ have unipotent labels $1, 1$. Moreover, $F(v) = v$, so Lemma 6.27 gives that z_0 lies in their kernels for $\chi \in \mathcal{E}(X,s)$. Moreover, we have that $e = 1$ since $p \mid q - t$, and the factor $\mathrm{GL}_3^t(q)$ has labels $\lambda \vdash 3$. By [40, §13.8], the degrees $u_\lambda(1)$ have $v_p(u_\lambda(1)) = v_p(\mathrm{Sp}_\lambda(1)) = 0$, since the Specht degrees are $1, 2, 1$. By [208, (7.16)–(7.19)], the rational component group is C_4 for $\epsilon = t$ and C_2 for $\epsilon = -t$. The action on the partition λ fixes λ by contragredience on $\mathrm{GL}_3^t(q)$. Thus, each λ gives four or two characters in $\mathcal{E}(X,s)$, respectively, all of p'-degree.

Write $W_s = \mathrm{Stab}_W(s)$. Let j swap V_+, V_- and invert coordinates $1, 3, 4, 5$. Thus, $jv = -v$ and j^2 inverts coordinates $1, 2$. If $\epsilon = t$, $j \in W_s$. If $\epsilon = -t$, $j^2 \in W_s$. Moreover, the unsigned $\mathfrak{S}_3$ on coordinates $3, 4, 5$ centralizes j. Thus, $\mathfrak{S}_3 \times \langle j \rangle$ and $\mathfrak{S}_3 \times \langle j^2 \rangle$ lie in the respective groups W_s. Their orders $24, 12$ equal $|W|/|Ws|$ by Lemma 6.188. Hence, $W_s = \mathfrak{S}_3 \times \langle j \rangle$ or $\mathfrak{S}_3 \times \langle j^2 \rangle$, respectively. Since the Sylow d-torus has centralizer $\mathbf{T}$, θ_s extends to $I_M(\theta_s)$ by [176, Theorem 1.1]. By [95, Theorem 6.11], we obtain 12 or 6 characters over $W\theta_s$, with degrees $|Ws|\,\mathrm{Sp}_\lambda(1)$ prime to p.

If $\epsilon = t$, the symbols $\Lambda_\pm$ have weights $u_+ = u_- = 1$ and empty cores, and λ has weight $w = 3$. Thus, the degree factor $5!/(1!^2 3!) = 20$ is prime to p, and Theorem 6.70 gives the comparison. If $\epsilon = -t$, the weights are $(u_+, u_-, w) = (0, 1, 3)$ with core ranks $(1, 0, 0)$ and residual sign $-t$. Then, $R = e = 1$ and $4!/(0!1!3!) = 4$. Thus, $\succeq_c$ follows from Theorem 6.90 with Proposition 6.89 for $u_+ = 0$. Moreover, at fixed λ, the local characters ψ have distinct component labels.

Let T_O be the natural torus and ν the spinor character. By Lemma 6.62 at $e = 1$ and $t = u$, $\mathfrak{S}_3 \le \Omega(V)$. Moreover, in the parent on $\mathrm{N}_{\mathrm{O}(V)}(T_O)$, the space of weight ϑ associated to v affords Sp_λ on $\mathfrak{S}_3$. Moreover, $[T_O : T_O \cap \Omega(V)] = 2$, so the only other weight above $\vartheta|_{T_O \cap \Omega(V)}$ is $\vartheta\nu|_{T_O}$, associated to $-v$. If both $\vartheta, \vartheta\nu|_{T_O}$ occur, then $\epsilon = t$ and the swap of V_+, V_- with inversions on coordinates $3, 4, 5$ gives $\vartheta^j = \vartheta\nu|_{T_O}$. Since j acts on the weights and centralizes $\mathfrak{S}_3$, both spaces have type Sp_λ. Thus, the θ_s-weight space of each local image ψ is isotypic of type Sp_λ. Consequently, ψ determines λ, and the 12 or 6 images exhaust $\mathrm{Irr}_{p'}(M \mid W\theta_s)$. □

Proposition 6.190. *Let $\mathbf{G}$ be simply connected of type D_n with $n \ge 5$. Keep $F, q, p, s, \mathbf{S}$ from* Section 6.1. *In particular, $\mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathbf{Z}_s)$ for the torus $\mathbf{Z}_s$ of* Zhang [208, Proposition 7.3, Step 1]. *We have that $s^4 \in \mathbf{Z}_s$. Moreover, the lift of $\mathbf{L}^*$ in SO_{2n} is $\prod_i \mathrm{GL}(U_i) \times \mathrm{SO}(Y)$, with $\dim Y$ even. For the lift t of s, we have that $t|_{U_i} = xI$ or $t|_{U_i} = x\,\mathrm{diag}(I_b, -I_b)$. Also, $\mathrm{Spec}(t|_Y) \subseteq \{1, -1, i, -i\}$, with $i^2 = -1$. If $\mathrm{Spec}(t|_Y)$ lies in $\{1, -1\}$ or in $\{i, -i\}$, we find $s^2 \in \mathrm{Z}(\mathbf{L}^*)$.*

Proof. Let t lift s to SO_{2n} and put $V_\xi = \ker(t - \xi I)$. By [208, Proposition 7.3, Step 1], the lift of $\mathbf{Z}_s$ acts trivially on $V_1 \oplus V_{-1}$. Moreover, put $\mathbf{A}_0 = \prod_{[\xi]} \mathbb{G}_m$, with $[\xi] = \{\xi, \xi^{-1}\}$ for $\xi \ne \pm 1$. The $[\xi]$-coordinate of $\mathbf{A}_0$ acts by a on V_ξ and a^{-1} on $V_{\xi^{-1}}$. By [208, Proposition 7.3, Step 1], we obtain the lift of $\mathbf{Z}_s$ as $\mathbf{A}_0$ or $(\mathbf{A}_0^j)^\circ$. In the fixed torus case, $jtj^{-1} = -t$, with j above a component of $\mathrm{C}_{\mathbf{G}^*}(s)$ or above the corrected graph action γ_s.

Since $t^4 = 1$ on $V_1 \oplus V_{-1}$, $t^4 \in \mathbf{A}_0$. In the fixed torus case, j acts by $[\xi] \mapsto [-\xi]$. For the orbit $\{[\xi], [-\xi]\}$ of size 2, the fixed torus is $\{(a, a) : a \in \mathbb{G}_m\}$ on $V_\xi, V_{-\xi}$. Thus, we have that $(\xi^4, (-\xi)^4)$ is fixed. Only $[i]$ has orbit size 1, and j inverts the $\mathbb{G}_m$ on $[i]$. Since $i^4 = 1$, $t^4 \in (\mathbf{A}_0^j)^\circ$. Consequently, $s^4 \in \mathbf{Z}_s$ also when $F^*(t) = -t$. Let U_i have weight $\omega_i \ne 1$ for the lift of $\mathbf{Z}_s$. Then, $U_i^\vee$ has weight ω_i^{-1}. Thus, $\mathrm{GL}(U_i)$ acts on U_i and dually on $U_i^\vee$. If $U_i = V_\xi$, $t|_{U_i} = \xi I$. Otherwise, $U_i = V_\xi \oplus V_{-\xi}$, with $\dim V_\xi = \dim V_{-\xi} = b$ by j. Hence, $t|_{U_i} = \xi\,\mathrm{diag}(I_b, -I_b)$. The zero-weight space is $Y = V_1 \oplus V_{-1}$, with $V_i \oplus V_{-i}$ adjoined when j inverts $[i]$. If V_1, V_{-1}, V_i are all nonzero in Y, then $jV_1 = V_{-1}$, giving type $Q(a, c)$ with $2a = \dim V_1 = \dim V_{-1}$ and $c = \dim V_i$. If $Y = V_1 \oplus V_{-1}$ and $V_1, V_{-1} \ne 0$, $t|_Y$ has type I. If $Y = V_i \oplus V_{-i}$, its rank $\dim V_i$ may be odd. Thus, t^2 is scalar on each U_i, and $t^2|_Y$ is scalar unless Y contains both a sign eigenspace and V_i. □

Write $s = s_2 s_{\mathrm{odd}}$ for the 2-part and odd part. Since $s_{\mathrm{odd}} \in \langle s^4 \rangle$, we have that $s_{\mathrm{odd}} \in \mathbf{Z}_s^{F^*}$. By Digne–Michel [58, Proposition 11.4.8], we have that s_{odd} gives a linear character λ of L. Moreover, its character on $J = \mathrm{C}_{\mathbf{L}}(\mathbf{S})^F$ is $\lambda|_J$. Thus, for λ the torus formula of Digne–Michel [58, Proposition 11.4.8] gives $\lambda^{-1}\mathcal{E}(L,s) = \mathcal{E}(L,s_2)$ and $(\lambda|_J)^{-1}\mathcal{E}(J,s) = \mathcal{E}(J,s_2)$. The normalizer $I = \mathrm{N}_{\mathbf{L}}(\mathbf{S})^F$ has the corresponding twist $(\lambda|_I)^{-1}$. On the fibre product for L, I, the paired twists give $\lambda^{-1}(x)$ at the inner difference $x \in L$. Moreover, a joint stabilizer element (a,σ) fixing the corrected s fixes s_{odd} and so the linear character λ. Thus, the twists by λ^{-1} are equivariant for (a,σ). Consequently, the comparison on L, I may be made at s_2, whose image in $\mathbf{L}^*/\mathrm{Z}(\mathbf{L}^*)$ has order dividing 4. The signed inductions $\mathcal{R}_{G,L}, \mathcal{R}_{K,J}$ are over s.

Lemma 6.191. *Suppose that* $\dim Y \in \{0,2\}$*, and for* $\dim Y = 2$ *treat* $\mathrm{SO}(Y)$ *as the block* GL_1*. For blocks* U_i*, we have that* $1 \le i \le k$*, put* $m_i = \dim U_i$*,* $D_0 = \ker(\prod_i : \mu_2^k \to \mu_2)$*,* $\mathbf{Z}' = \mathbb{G}_m^k/D_0$ *and* $\mathbf{A} = \mathbb{G}_m^k$*. Then,* $\mathbf{L}$ *has a regular inclusion in* $\mathbf{E} = (\prod_i \mathrm{GL}(U_i)) \times \mathbf{Z}'$ *with* $\mathbf{L} = \ker u$*, for the map* $u(g,[z]) = (\det(g_i)/z_i^2)_i \in \mathbf{A}$*. Moreover, we have that* u *is equivariant for* F*, permutations of* U_i *and duality* $U_i \leftrightarrow U_i^\vee$*. For* $I = \mathrm{N}_{\mathbf{L}}(\mathbf{S})^F$ *and* $I_E = \mathrm{N}_{\mathbf{E}}(\mathbf{S})^F$*, it gives* $E/L \cong \mathbf{A}^F \cong I_E/I$*.*

Proof. For the natural diagonal torus of SO_{2n}, the spin cocharacter lattice is $\{a \in \mathbb{Z}^n : \sum_j a_j \in 2\mathbb{Z}\}$. Thus, on each block U_i, put $\mathbf{B}_i = \{(g_i, z_i) : \det(g_i) = z_i^2\}$. A block cocharacter a_i has exponent $\frac{1}{2}\sum_j a_{ij}$ on z_i. Moreover, $z_i^2, \prod_i z_i$ generate $X(\mathbf{Z}')$. Thus, the quotient by D_0 admits a exactly when $\sum_{i,j} a_{ij}$ is even. Also, the block coroots $e_j - e_l$ coincide. By [72, §1.3], $\mathbf{L} \cong (\prod_i \mathbf{B}_i)/D_0 = \ker u$. Moreover, $\mathbf{L}, \mathbf{E}$ both have derived group $\prod_i \mathrm{SL}(U_i)$, and $\mathrm{Z}(\mathbf{E})$ is connected. Thus, $\mathbf{L} \hookrightarrow \mathbf{E}$ is regular.

Permuting U_i permutes z_i. On a dual block, both z_i and $\det(g_i)$ are inverted. Thus, u is equivariant for signed permutations on $\mathbf{A}$, and D_0 is invariant. A field map F_r^a raises $z_i, \det(g_i)$ to r^a. Moreover, inner conjugation in $\mathrm{GL}(U_i)$ fixes z_i and $\det(g_i)$. Two lifts on $\mathbf{L}$ of a natural action on SO_{2n} differ by a map $\mathbf{L} \to \mu_2$. Since $\mathbf{L}$ is connected, its image is $\{1\}$. Consequently, an F-orbit of h blocks U_i with orientation sign v gives $\mathrm{GL}_{m_i}^v(q^h)$. If $v = -1$, this group is $\mathrm{GU}_{m_i}(q^h)$, even if $F(U_i) \ne U_i$. By Lang's theorem [58, §4.2] on $\mathbf{L}$, $u(E) = \mathbf{A}^F$. Take an F-stable maximal torus $\mathbf{T}_L$ of $\mathrm{C}_{\mathbf{L}}(\mathbf{S})$. Then, $\mathbf{T}_E = \mathbf{T}_L \mathrm{Z}(\mathbf{E})^\circ$ maps onto $\mathbf{A}$ with kernel $\mathbf{T}_L$. Thus, $u(\mathbf{T}_E^F) = \mathbf{A}^F$ by Lang's theorem on $\mathbf{T}_L$. Since $\mathbf{T}_E^F \le I_E$, $I_E/I \cong \mathbf{A}^F$.

Write $\mathbf{S}_i$ for the projection of $\mathbf{S}$ to the i-th natural block. Every element of I_E normalizes each $\mathbf{S}_i$. Conversely, $g_i \in \mathrm{N}_{\mathrm{GL}(U_i)}(\mathbf{S}_i)$ normalizes the block centralizer of $\mathbf{S}_i$. Thus, the product of the g_i normalizes $\mathrm{C}_{\mathbf{L}}(\mathbf{S})$ by its block root groups. Moreover, $\mathbf{S}$ is the Sylow d-torus of $\mathrm{Z}(\mathrm{C}_{\mathbf{L}}(\mathbf{S}))^\circ$. Thus, I_E is the product of $\mathbf{Z}'^F$ and the block normalizers of $\mathbf{S}_i$. If $\mathbf{C}_i$ is a central d-torus on U_i, the normalizer and centralizer of $\mathbf{S}_i\mathbf{C}_i$ are those of $\mathbf{S}_i$. Moreover, on an F-orbit of length h, the period is $d_h = \mathrm{ord}_p(q^h)$ and the signed block length is $e_h = \mathrm{ord}_p(vq^h)$. □

Lemma 6.192. *A character* $\Theta \in \mathrm{Irr}(E)$ *or* $\Theta \in \mathrm{Irr}(I_E)$ *has quotient self-twists only in* $\{1,\eta\}$*, where* $\eta = \boxtimes_j \eta_j$ *and* η_j *is quadratic on the* j*-th factor of* $\mathbf{A}^F$*. Thus,* $\Theta|_L$ *or* $\Theta|_I$ *has* $h_0 \in \{1,2\}$ *constituents of degree* $\Theta(1)/h_0$ *and multiplicity* 1*. In particular, we have that* Θ *has* p'*-degree if any constituent has* p'*-degree.*

Proof. The isogeny $\beta : \mathbf{Z}' \to \mathbf{A}$, $[z] \mapsto (z_i^2)_i$, has kernel $\mu_2^k/D_0 \cong \mu_2$ via $(\delta_i) \mapsto \prod_i \delta_i$. By Lang's theorem [58, Propositions 4.2.10 and 4.2.11], $\mathbf{A}^F/\beta(\mathbf{Z}'^F) \cong H^1(F,\mu_2) = C_2$. For an orbit of length h and sign v, its rational determinant coordinate is $a \in \mu_{q^h - v}$. A lift of a has first coordinate z with $z^2 = a$. Transport by F^h gives the ratio $z^{vq^h - 1}$. Thus, the connecting map on a is $a^{(q^h - v)/2}$, because z^{vq^h-1} has order dividing 2. The product of the signs $a^{(q^h-v)/2}$ gives η. Since $\mathbf{Z}'^F$ acts by scalars in Θ, a self-twist is trivial on $\beta(\mathbf{Z}'^F)$. Write $(H,N) = (E,L)$ or (I_E, I). By [95, Lemma 5.2], the squared norm $\langle \Theta|_N, \Theta|_N \rangle$ is $|\{\alpha \in \mathrm{Irr}(H/N) : \Theta\alpha = \Theta\}|$. Moreover, [43] gives that this norm is $e^2 h_0$ for h_0 constituents of multiplicity e. Thus, $e = 1$, $h_0 \in \{1,2\}$, and each constituent has degree $\Theta(1)/h_0$. Since $p \nmid h_0$, its p-part is $\Theta(1)_p$.

By [58, Proposition 11.4.8], the twist by η_j acts on block eigenvalues by $x \mapsto -x$. Thus, a scalar block has $\Theta_j\eta_j \ne \Theta_j$. On a balanced block, we let F_v be the signed Frobenius on $\{x,-x\}$. If F_v fixes each eigenvalue, then η swaps the partition labels λ_+, λ_-. Thus, $\Theta_j\eta_j = \Theta_j$ exactly when $\lambda_+ = \lambda_-$. If $F_v(x) = -x$, the single partition on the orbit $\{x,-x\}$ is fixed by η. Consequently, $\Theta\eta = \Theta$ requires $\Theta_j\eta_j = \Theta_j$ on every rational block j. In particular, $\mathrm{SO}(Y)$ treated as GL_1 gives $\Theta\eta \ne \Theta$ when $\dim Y = 2$. □

Lemma 6.193. *Take* $\mathbf{L} \le \mathbf{E}$ *from* Lemma 6.191 *and put* $\mathbf{J} = \mathrm{C}_{\mathbf{L}}(\mathbf{S})$*,* $\mathbf{J}_E = \mathrm{C}_{\mathbf{E}}(\mathbf{S})$*. On a rational block, we write* $A = \mathrm{GL}_m^v(Q)$ *with* $Q = q^h$*,* $v = 1$ *for the linear group and* $v = -1$ *for the unitary group* $\mathrm{GU}_m(Q)$*. Moreover, we let*

$\mathbf{S}_A$ *be the projection of* $\mathbf{S}$ *to* A *and put* $N_A = \mathrm{N}_A(\mathbf{S}_A)$. *Given* $s \in J^*$, *one rational lift* $\widetilde{s}$ *in* $\mathbf{J}_E^*$ *gives the series on* E *and the eligible local orbit on* I_E. *Moreover, a change of* $\widetilde{s}$ *twists both series by* $\delta \in \mathrm{Irr}(E/L) \cong \mathrm{Irr}(I_E/I)$. *On a block* $A = \mathrm{GL}_m^v(Q)$ *with* $\mathrm{ord}_p(vQ) = 1$, *the two balanced eigenvalues* $x, -x$ *of an eligible block lie in* μ_{Q-v}.

Proof. Let $\pi : \mathbf{E}^* \to \mathbf{L}^*$ have central torus kernel Z. Then, $\mathbf{J}_E^* = \pi^{-1}(\mathbf{J}^*)$, and Lang's theorem [58, §4.2] gives surjections $E^* \to L^*$ and $J_E^* \to J^*$. If $\pi(\widetilde{s}') = s^g$ with $g \in J^*$, choose a rational lift $\widetilde{g} \in J_E^*$ of g. Thus, $\widetilde{s}'^{\widetilde{g}^{-1}} = \widetilde{s}z$ for $z \in Z^{F^*}$, dual to $\delta \in \mathrm{Irr}(E/L)$. By [58, Proposition 11.3.10], the constituents of $\widetilde{\chi}|_L$ for $\widetilde{\chi} \in \mathcal{E}(E, \widetilde{s})$ lie in $\mathcal{E}(L, s)$. Similarly, $\mathcal{E}(J_E, \widetilde{s})$ restricts to $\mathcal{E}(J, s)$. By [58, Proposition 11.4.8], multiplication by z gives the twists by δ on E and by $\delta|_{J_E}$ on J_E.

For a parent $\widetilde{\nu} \in \mathrm{Irr}(I_E)$ above the marked orbit on I, we choose a constituent $\widetilde{\gamma}$ on J_E above $\mathcal{E}(J, s)$. Thus, the twist by δ puts $\widetilde{\gamma}$ in $\mathcal{E}(J_E, \widetilde{s})$. On A, N_A, $\widetilde{s}$ determines the active labels x and residual cores ρ_x. Moreover, its eligible J_E^*-classes form one I_E^*-orbit by Lemma 6.7. In each A, the residual factor A_R has p'-order, and Q_A is the toral base. Thus, $\mathbf{Z}'^F$ adds $(\mathbf{Z}'^F)_p$ to the base and has classes $\{z\}$.

Now let $m = 2b$ and $d_A = \mathrm{ord}_p(vQ) = 1$. Put $a = v_p(Q - v) > 0$. If the signed Frobenius F_v exchanges $x, -x$, their centralizer in A^* is $C = \mathrm{GL}_b(Q^2)$. By the order formulas and $v_p(Q^2 - 1) = a$, we have that $v_p(|A|) = 2ba + v_p((2b)!)$ and $v_p(|C|) = ba + v_p(b!)$. Hence, $p \mid [A^* : C]$, so $\widetilde{\chi}(1)$ is divisible by p by [58, §11.5]. Moreover, a parent $\widetilde{\chi}$ of a p'-constituent has p'-degree by Lemma 6.192. Consequently, F_v fixes $x, -x$ separately. □

Write $Q = r^a$ with field prime r. On A, N_A, Section 3 gives the comparison at $d_A = \mathrm{ord}_p(vQ)$ before restriction to L, I. Moreover, we have that $\mathbf{S}_A$ has active blocks of length d_A by Lemma 6.191. If $p \nmid |A|$ or $m = 1$, we have that $N_A = A$, so take $\tau = \rho$. Otherwise, take a p'-degree pair (ρ, τ). If $d_A > 1$, the core and ordered quotients of λ_x determine λ_x, also when the eigenvalue orbit $[x]$ has size 2. If $d_A = 1$, we have that the dual centralizer $C_{A^*}(\widetilde{s}_A)$ contains a Sylow p-subgroup. Hence, it contains $(C_p)^m$ after conjugation. Thus, the weights of $(C_p)^m$ give diagonal $\widetilde{s}_A$ with entries in μ_{Q-v}. The group N_A is monomial, and λ_x is the quotient label with sign det from Section 3. Moreover, at $m = 2$, we have that $d_A = 1$ gives two slots of length 1 and $d_A = 2$ gives one slot of length 2. If $v = -1$, the degrees on A, N_A follow from Proposition 3.46.

Let ω_x be the scalar character for $[x]$ of degree e. Thus, an active torus of length d_A has scalar character $\omega_x^{d_A/e}$. Thus, the exponent in τ is $|\mathrm{core}(\lambda_x)| + (d_A/e)\sum_j |\lambda_x^{(j)}| = |\lambda_x|$, as in ρ. By Section 3, the inverse from $(\rho_x, (\lambda_x^{(j)})_j)$ to λ_x is equivariant. Thus, on (ρ, τ), field, diagram and coefficient actions follow (x, λ_x). Moreover, the twist δ multiplies x by the dual scalar z_δ on both labels. Consequently, ρ, τ have the same correcting twist δ. For $v = 1$ and odd d_A, $\alpha_\rho = \alpha_\tau$ by Proposition 3.14. If d_A is even, Proposition 3.20 gives $\alpha_\rho = \alpha_\tau$ with cochain c, where $c(\delta) = 1$ on pure determinant self-twists δ. Moreover, the field lift u has $u^a \in N_A$ with matrices $\rho(u^a), \tau(u^a)$, by the computation for Proposition 3.22 on A, N_A.

If $v = -1$, by Lemma 3.48, we obtain $\alpha_\rho = \alpha_\tau$ and the matrices at u^{2a}. Moreover, we let H be a Hermitian matrix with $g^{\mathsf{T}} H F_Q(g) = H$ for $g \in A$. Then, we find $g^{-\mathsf{T}} = H F_Q(g) H^{-1}$. The cochain c has $c(u^{2a}) = 1$ and $c(ju^{2a}) = c(j)$. If $c(\delta) \neq 1$ at a self-twist δ, pair the eigenspaces of the normalized δ-operators. For $g \in N_A$ and $k \geq 1$, we have that the lift gu has power $(gu)^k = g\, {}^u g \cdots {}^{u^{k-1}} g\, u^k$. Thus, a correcting twist δ at g_0 gives $\delta(\det g_0)^{\pm 1}$ in both ρ, τ, with the same exponent. Consequently, for gu, the power relation agrees on A, N_A.

Proposition 6.194. *For parents above* $[s]_{J^*}$, *we have that the comparisons on* A, N_A *give pairs* (β, ν) *of* p'*-constituents on* L, I. *Moreover, the tensor extension for* (β, ν) *in* Proposition 6.9 *exists even when* $V|_L$ *has* 2 *constituents.*

Proof. Let V, V' be product parents on E, I_E with common character κ on $\mathbf{Z}'^F$. Given $\delta \in \mathrm{Irr}(\mathbf{A}^F)$, we have that the twist $\delta \circ u$ has factors $\delta_i \circ \det$ on A and $[z] \mapsto \delta((z_i^{-2})_i)$ on $\mathbf{Z}'^F$. Thus, on $\mathbf{Z}'^F$ the twists agree, and $\rho(zI), \tau(zI)$ have the same scalar for $zI \in \mathrm{Z}(A)$. By Lemma 6.193, we obtain the pairs (V, V') over $[s]_{J^*}$. By Lemma 6.191, the actions on U_i are permutations, $U_i \leftrightarrow U_i^\vee$, and field actions. Thus, z_i goes to $z_j^{\pm r^a}$. Write $A = \mathbf{A}_v^{F_v}$ with $\mathbf{A}_v = \mathrm{GL}_m$. For $\bar{g} \in (\mathbf{A}_v/\mathrm{Z}(\mathbf{A}_v))^{F_v}$, Lang's theorem [58, §4.2] gives $g \in A$ above $\bar{g}$. If $\bar{g}$ fixes $\mathbf{S}_A$, $g \in N_A$. Two such lifts differ by zI, acting by the same scalar on ρ, τ. Moreover, on $\mathrm{GU}_m(Q)$ we have $g^{-\mathsf{T}} = H F_Q(g) H^{-1}$. Thus, the inner powers on A, N_A give the powers on V, V'. Also, F is the identity on E and L.

Choose (V_0, V_0') at a base label 0 in each orbit. Moreover, let H_i implement the transport $h_i : 0 \to i$. For $b : i \to j$, the loop $h_j^{-1} b h_i$ has operator $D_*(h_j^{-1} b h_i)$ on $V_0 \otimes (V_0')^\vee$. Since $\alpha_\rho = \alpha_\tau$, the operators D_* multiply with coefficient maps σ. Thus, the transport for b is $H_j D_*(h_j^{-1} b h_i) H_i^{-1}$. In a composition $i \to j \to k$, we have $H_j^{-1} H_j = 1$ with

coefficient action 1. Consequently, on $V \otimes (V')^\vee$ take the tensor of $H_j D_* H_i^{-1}$ followed by the factor permutation π. When the paired labels return up to δ, the κ-scalars and the evaluations $\delta(\det g)$ at inner matrices g are common to V, V'.

If V, V' have only the self-twist 1, the restrictions $V|_L, V'|_I$ are irreducible by Lemma 6.192. Otherwise, let J_g, J_l be the intertwiners for η. The matched factors give $J_g^2 = J_l^2 = cI$. Thus, we divide both by $c^{1/2}$. An element x with $\eta(u(x)) = -1$ exchanges the $+1, -1$ eigenspaces. Consequently, $V_\pm, (V')_\pm$ are the constituents of $V|_L, (V')|_I$ by Lemma 6.192. On $V \otimes (V')^\vee$, the fixed space of $J_g \otimes (J_l^{-1})^\vee$ is $V_+ \otimes (V'_+)^\vee \oplus V_- \otimes (V'_-)^\vee$. A joint stabilizer element j fixes $\eta = \boxtimes_i \eta_i$, since field and inversion fix each η_i and j permutes the factors η_i. Thus, the common factors of P_j, Q_j give $P_j J_g P_j^{-1} = \varepsilon_j J_g$ and $Q_j J_l Q_j^{-1} = \varepsilon_j J_l$. The choice $c^{1/2}$ contributes the same coefficient ratio on J_g, J_l. Hence, $\varepsilon_j^2 = 1$ and j acts on both signs by ε_j. Given an inner matrix g, $\varepsilon_g = \eta(u(g))$. Changing $c^{1/2}$ swaps the signs on both V, V'. Thus, choose one pair (V, V') in each orbit under δ. The other pairings are the $\delta \circ u$-twists.

Let (β, ν) also denote the paired representations on V_β, V_ν. To return both parents at y, take a correcting quotient character δ_y. Two corrections differ by 1 or η. Thus, both diagonal operators are 1 on $V_\beta \otimes V_\nu^\vee$. Consequently, the ordinary $D(y)$ on that summand multiply, with $D(i) = \beta(i) \otimes \nu(i)^\vee$ for $i \in I$. On the fibre product, with $xy^{-1} \in L$, take $R(x, y) = (\beta(xy^{-1}) \otimes 1) D(y)$. Covariance under $D(y)$ gives $R(x, y)R(x', y') = R(xx', yy')$. Thus, $R(l, i) = \beta(l) \otimes \nu(i)^\vee$ for $l \in L$, $i \in I$. Moreover, a redundant centralizing lift z acts on β, ν by common scalars, so $R(z, z) = 1$. The semilinear maps for (a, σ) give the corrected mixed action on R. Consequently, (β, ν) satisfies Proposition 6.9. □

Corollary 6.195. *For $\epsilon \in \{1, -1\}$ and $n \geq 5$, a proper-Levi series with $\dim Y \leq 2$ has the central mixed comparison at $\mathrm{N}_{\mathbf{G}}(\mathbf{S})^F$ for each eligible odd nondefining prime p. In particular, this includes lifts t with no eigenvalue in $\{1, -1, i, -i\}$.*

Proof. For $[s]_{J^*}$, Proposition 6.194 gives the comparison on L, I. Moreover, both eigenspaces at $\widetilde{s}$ lie over $[s]_{L^*}$ and the local orbit of $[s]_{J^*}$ by Lemma 6.193. The eligible K-series form one D-orbit by Lemma 6.7. Thus, corrections in D, K return $\mathbf{L}$ and both induction marks, with stabilizer $KV^\#$ from Section 6.1. Then, the correction in I fixes the chosen constituent γ on J. A corrected field-free stabilizer operation a fixes $\mathbf{Z}_s$ pointwise by [208, Proposition 7.3, Steps 1–2]. Thus, choose a cocharacter μ of $\mathrm{Z}(\mathbf{L})^\circ$ with $\langle \alpha, \mu \rangle \neq 0$ for roots α outside $\mathbf{L}$. The radicals $\mathbf{U}, \mathbf{U} \cap \mathbf{K}$ defined by μ are stable under a. Moreover, Section 6.1 compares $\mathbf{U}, \mathbf{U} \cap \mathbf{K}$ under field stabilizer operations. Consequently, on G, D the comparison follows from Proposition 6.9. If t has no eigenvalue in $\{1, -1, i, -i\}$, then $Y = 0$ by Proposition 6.190. If $V_1 = V_{-1} = 0$ and $\dim V_i = 1$, then $\dim Y \leq 2$. Thus, for odd n and $\dim V_i = \dim V_{-i} = n$, Proposition 6.16 covers $V = V_i \oplus V_{-i}$. □

Lemma 6.196. *Let Y have weight 0 in* Proposition 6.190*, with $\dim Y = 2m \geq 4$ and rational sign ϵ_Y. Write $m_i = \dim U_i$ for the linear blocks. Since $\mathbf{L}$ is proper, some $U_i \neq 0$. Moreover, we let $\langle z_0 \rangle$ be the kernel of $\mathrm{Spin}(Y) \to \mathrm{SO}(Y)$, also the kernel of $\mathrm{Spin}_{2n} \to \mathrm{SO}_{2n}$. If some m_i is odd, we find $\pi_0\mathrm{Z}(\mathbf{L}) \cong C_2$. If all m_i are even, $\pi_0\mathrm{Z}(\mathbf{L})$ is C_4 for odd m and $C_2 \times C_2$ for even m. Given a regular injection $\mathbf{L} \hookrightarrow \mathbf{E}$, we put $\mathbf{Z}_E = \mathrm{Z}(\mathbf{E})$. Then, we have that $E/(L\mathbf{Z}_E^F) \cong H^1(F, \mathrm{Z}(\mathbf{L}))$ has order at most 4. Its order is 2 if some m_i is odd or if $\epsilon_Y \eta_q(-1)^m = -1$, with η_q the quadratic character of $\mathbb{F}_q^\times$. Moreover, we have that $I_E/(I\mathbf{Z}_E^F) \cong E/(L\mathbf{Z}_E^F)$.*

Proof. For the determinant covers $\mathbf{B}_i$ of Lemma 6.191, put $D_1 = \ker(\prod_{i=0}^k : \mu_2^{k+1} \to \mu_2)$. Then, $\mathbf{L} = (\mathrm{Spin}(Y) \times \prod_i \mathbf{B}_i)/D_1$, with $\delta_0 = -1$ denoting z_0. Indeed, for orthogonal summands X_i, X_j, the even Clifford algebras we have $xy = yx$ for $x \in \mathrm{Cl}^0(X_i)$, $y \in \mathrm{Cl}^0(X_j)$ by [137, Theorem 24.56]. The signs $(\delta_0, \ldots, \delta_k)$ multiply to $\prod_{i=0}^k \delta_i$ in Spin_{2n}. Thus, multiplication has kernel D_1. By Lemma 6.191, the quotient is $\mathbf{L}$. Moreover, F acts on $\mathrm{Spin}(Y)$ and permutes or dualizes the factors $\mathbf{B}_i$.

The preimage of $\{\pm I_Y\} \times \prod_i \mathbb{G}_m$ in $\mathbf{L}$ is $\mathrm{Z}(\mathbf{L})$ by central isogeny. Above $+I_Y$, put $a_i I$ on U_i. Then, the equation is $z^2 = \prod_i a_i^{m_i}$. Thus, if m_i is odd for some i, the group over $+I_Y$ is connected. Otherwise, z_0 represents the other component over $+I_Y$. Choose a rational orthogonal basis $e_1, \ldots, e_{2m}$ of (Y, Q_Y), put $d_Y = \prod_j Q_Y(e_j)$ and $v = e_1 \cdots e_{2m}/\sqrt{d_Y}$ in the Clifford algebra. By [137, Theorem 24.56 and §24.i], $v^2 = z_0^m$ and $F(v) = \eta_q(d_Y)v$. Since $\eta_q(d_Y) = \epsilon_Y \eta_q(-1)^m$, the component relations are $z_0^2 = 1$, $v^2 = z_0^m$, and $F(v)v^{-1} = z_0^c$ with $(-1)^c = \epsilon_Y \eta_q(-1)^m$. Consequently, $\pi_0\mathrm{Z}(\mathbf{L})$ has order 2 or 4, with relations as stated, and its F-coinvariants have order 2 or 4 by the value of $F(v)v^{-1}$.

The central product $\mathbf{E} = \mathbf{L}\mathbf{Z}_E$ has intersection $\mathbf{L} \cap \mathbf{Z}_E = \mathrm{Z}(\mathbf{L})$. Thus, the rational quotient is $H^1(F, \mathrm{Z}(\mathbf{L}))$ by Lang's theorem [58, §4.2] on $\mathbf{L}, \mathbf{Z}_E$. By Lang's theorem on $\mathrm{Z}(\mathbf{L})^\circ$, we obtain the F-coinvariants of $\pi_0\mathrm{Z}(\mathbf{L})$. Given I_E,

the torus $\mathbf{T}_L\mathbf{Z}_E \le \mathrm{C}_{\mathbf{E}}(\mathbf{S})$ gives all classes in E/L by Lemma 6.191 and Lang's theorem on $\mathbf{T}_L$. Since $\mathbf{Z}_E^F \le I_E$, the quotient is $I_E/(I\mathbf{Z}_E^F) \cong E/(L\mathbf{Z}_E^F)$. □

Lemma 6.197. *Suppose that* $\mathbf{R} = [\mathbf{L}, \mathbf{L}]$ *has only type A components. Then, we have that* $\mathbf{L}$ *has a regular inclusion in* $\mathbf{E}$ *with the ambient actions on* $\mathbf{L}$*. The parents on* E *are tensors of characters of* $\mathrm{GL}_{d_j}^v(q^h)$ *with a compatible character* κ *on* $\mathrm{Z}(\mathbf{E})^F$*. Moreover, we have that* I_E *has parents given by the normalizers* N_A *and compatible* κ*. In particular, the construction includes* $\dim Y = 4, 6$ *for either sign* ϵ_Y*.*

Proof. With $\mathbf{R} = \prod_j \mathrm{SL}_{d_j}$, write $\mathbf{C} = \mathrm{Z}(\mathbf{L})^\circ$. Put $K_0 = \{(z, z^{-1}) : z \in \mathbf{R} \cap \mathbf{C}\}$ and $\mathbf{P}_0 = (\prod_j \mathrm{GL}_{d_j}) \times \mathbf{C}$. Then, $\mathbf{L} = (\mathbf{R} \times \mathbf{C})/K_0$ embeds in $\mathbf{E} = \mathbf{P}_0/K_0$. Its derived group is $\mathbf{R}$, the image of $\mathrm{Z}(\mathbf{P}_0)$ is the connected centre $\mathbf{Z}_E$, and $\mathbf{E}/\mathbf{L} \cong \prod_j \mathbb{G}_m$ via the determinants. Thus, by [72, §1.7], $\mathbf{L} \hookrightarrow \mathbf{E}$ is regular. On SL_{d_j}, fix a pinning. Thus, conjugation by g on SL_{d_j} extends by g on GL_{d_j}. On $\mathrm{GL}(U_i)$, take inverse transpose when $U_i \mapsto U_i^\vee$, by Lemma 6.191. Moreover, on $\mathrm{Spin}(Y)$, a diagram involution has the map $g \mapsto Jg^{-\mathsf{T}}J^{-1}$ on GL_{d_j}, with the signed antidiagonal J of the pinning. On $\mathbf{R}$, permutations and field actions extend to $\prod_j \mathrm{GL}_{d_j}$. On $\mathbf{C}$ take the action from $\mathbf{L}$. Since the actions agree on $\mathbf{R} \cap \mathbf{C}$, they fix K_0 and descend to $\mathbf{E}$. In particular, the extension of F to $\mathbf{E}$ acts as the identity on $E = \mathbf{E}^F$. If $\dim Y = 4$, the two A_1 factors may be exchanged by F, yielding one rational factor over q^2. An inner action on A_1 is represented in PGL_2. If $\dim Y = 6$, $\mathrm{Spin}(Y)$ has type $D_3 = A_3$, so the graph gives $\mathrm{GU}_4(q)$.

Let $\pi : \mathbf{P}_0 \to \mathbf{E}$ be the quotient. Moreover, for $e \in E$, a lift $g \in \mathbf{P}_0$ has $g^{-1}F(g) \in K_0$. By Lang's theorem [58, §4.2] on $\mathrm{Z}(\mathbf{P}_0)$, there is central z with $gz \in \mathbf{P}_0^F$ and $\pi(z) \in \mathbf{Z}_E^F$. Thus, $E = \pi(\mathbf{P}_0^F)\mathbf{Z}_E^F$, and $\pi(\mathbf{P}_0^F) \cap \mathbf{Z}_E^F = \pi(\mathrm{Z}(\mathbf{P}_0)^F)$. The intersection formula follows from $\pi^{-1}(\mathbf{Z}_E) = \mathrm{Z}(\mathbf{P}_0)$. Consequently, a tensor character Θ on $\mathbf{P}_0^F$ trivial on K_0^F extends with $\kappa \in \mathrm{Irr}(\mathbf{Z}_E^F)$ when their scalars agree on $\pi(\mathrm{Z}(\mathbf{P}_0)^F)$. The character of E has degree $\Theta(1)$. Moreover, for $\mathbf{S}_0 = (\pi^{-1}\mathbf{S})^\circ$, the central correction gives $I_E = \pi(\mathrm{N}_{\mathbf{P}_0}(\mathbf{S}_0)^F)\mathbf{Z}_E^F$. By Lemma 6.191, the tori $\mathbf{S}_A$ give the factors $N_A, \mathrm{C}_A(\mathbf{S}_A)$. A central d-torus $\mathbf{C}_A$ in A gives $\mathrm{N}_A(\mathbf{S}_A\mathbf{C}_A) = N_A$ and $\mathrm{C}_A(\mathbf{S}_A\mathbf{C}_A) = \mathrm{C}_A(\mathbf{S}_A)$. Thus, on I_E the scalars for Θ, κ must again agree on $\pi(\mathrm{Z}(\mathbf{P}_0)^F)$. Moreover, $I_E/I \cong (\mathbf{E}/\mathbf{L})^F$ by Lemma 6.196. □

Proposition 6.198. *For* $\mathbf{L} \hookrightarrow \mathbf{E}$ *as in* Lemma 6.197*, the operators on* A, N_A *give the middle mixed comparison on* L, I *above* $[s]_{J^*}$*. Moreover, they pair individual constituents of* $V|_L, V'|_I$ *also when the group* D *of self-twists has order* 4*.*

Proof. A quotient self-twist δ is trivial on $\mathbf{Z}_E^F$, so the self-twist group D has order $1, 2$ or 4 by Lemma 6.196 for $\dim Y = 4, 6$. For $\dim Y \le 2$, $|D| \le 2$ by Lemma 6.192, which also gives $e^2h = |D|$ for h constituents of multiplicity e. Thus, e, h are powers of 2. Since $p \nmid eh$, a constituent of p'-degree gives $p \nmid \Theta(1)$ for the parent Θ. Consequently, each factor Θ_j has p'-degree. On a factor $A = \mathrm{GL}_m^v(Q)$ with $p \mid Q - v$, the argument with $(C_p)^m$ in Lemma 6.193 gives that s_A is diagonal with eigenvalues in μ_{Q-v}. Thus, at $d_A = 1$, the A_1, A_3 factors also have eigenvalues in μ_{Q-v}. If $d_A > 1$, take Section 3 on A, N_A. If $p \nmid |A|$, choose the identical character on $A = N_A$.

First, take $\pi^* : \mathbf{E}^* \to \mathbf{P}_0^*$. If $g \in P_0^*$ conjugates $\pi^*(s_1), \pi^*(s_2)$, choose a geometric lift $\widetilde{g}$ of g. Since $\widetilde{g}^{-1}F^*(\widetilde{g})$ is central, $s_1^{\widetilde{g}}$ is rational and differs from s_2 by $z \in (\ker \pi^*)^{F^*}$. By Steinberg's theorem [72, §2.2], $\mathrm{C}_{\mathbf{E}^*}(s_1)$ is connected since $\mathrm{Z}(\mathbf{E})$ is connected. The derived group of $\mathbf{J}_E^*$ is simply connected as for $\mathbf{E}^*$, so $\mathrm{C}_{\mathbf{J}_E^*}(s_1)$ is connected. Thus, Lang's theorem gives $s_1 \sim s_2 z$ in E^* or J_E^*. The translations z are dual to linear characters δ of $E/\pi(\mathbf{P}_0^F)$. By Lemma 6.197, δ is determined by $\delta|_{\mathbf{Z}_E^F}$. Thus, κ determines the lift on both E, I_E. For the connected torus kernel of $\mathbf{E}^* \to \mathbf{L}^*$, the lifts of s form one orbit of $\mathrm{Irr}(E/L)$ by Lemma 6.193.

Take the type A operators on A, N_A and a common character on $\mathbf{C}^F$. Thus, the scalar tests for V, V' agree on K_0^F and $\pi(\mathrm{Z}(\mathbf{P}_0)^F)$. By Proposition 6.194, on the simultaneous stabilizer H we have $\alpha_V = \alpha_{V'}$, including for factor permutations. Consequently, D is common to V, V'. By Lemma 6.197, we write $y = y_0 z \in I_E$, with y_0 from the natural normalizers and $z \in \mathbf{Z}_E^F$. Two choices of y_0, z differ on $\pi(\mathrm{Z}(\mathbf{P}_0)^F)$. Thus, the scalars there cancel on $V \otimes (V')^\vee$.

Given $\delta \in D$, take intertwiners J_δ, J'_δ with common factors α. Conjugation acts by δ on $\mathbb{C}J_\delta, \mathbb{C}J'_\delta$. Thus, the $|D|$ lines are independent, and $|D| = \dim \mathrm{End}_L(V) = \dim \mathrm{End}_I(V')$. Consequently, both algebras are $\mathcal{A} = \mathbb{C}^\alpha[D]$. By Lusztig's theorem [72, Theorem 1.7.15], $V|_L$ is multiplicity-free. Hence, $\mathcal{A}$ is commutative, and $V'|_I$ is also multiplicity-free. Write V_λ, V'_λ for the constituent spaces where $\mathcal{A}$ acts by λ. Then, on $V_\lambda \otimes (V'_\mu)^\vee$, the diagonal twist δ acts by $\lambda(J_\delta)/\mu(J_\delta)$. Thus, the common fixed space for D is $\bigoplus_\lambda V_\lambda \otimes (V'_\lambda)^\vee$.

Moreover, an ordinary or mixed stabilizer operation j acts identically on both factors $\mathcal{A}$, including on coefficients. Thus, j acts by $(V_\lambda, V'_\lambda) \mapsto (V_{\lambda^j}, V'_{\lambda^j})$. Two corrections differ by $\delta \in D$, acting by 1 on $V_\lambda \otimes (V'_\lambda)^\vee$. Consequently, on

its stabilizer, $R(x,y)$ of Proposition 6.194 extends the tensor for (V_λ, V'_λ) with the common central scalars at z and the mixed action at (a,σ). □

Theorem 6.199. *Let $n \geq 5$, with q,p odd and $p \nmid q$. If $\epsilon = \pm 1$, we choose an eligible $\mathcal{E}(G,s)$ with $\mathbf{L}$ from* Zhang [208, Proposition 7.3]. *If $s \in \mathrm{Z}(\mathbf{L}^*)$ or $\dim Y \leq 6$, we have that $\mathcal{E}(G,s)$ has the central mixed comparison at $\mathrm{N}_{\mathbf{G}}(\mathbf{S})^F$.*

Proof. Given $s \in \mathrm{Z}(\mathbf{L}^*)$, the scalar twist by λ_s has $B = L$. Thus, $\lambda_s|_J$ is dual to s and selects $\mathcal{E}(J,s)$. For $\dim Y \leq 2$, the comparison is Corollary 6.195. For $\dim Y = 4, 6$, Proposition 6.198 pairs constituents with common $\mathcal{A}$-label λ. Thus, with radicals $\mathbf{U}, \mathbf{U}\cap\mathbf{K}$ of Corollary 6.195, $\succeq_c$ holds on G, D by Proposition 6.9. Consequently, for the remaining $s \notin \mathrm{Z}(\mathbf{L}^*)$, the space Y has rank $4 \leq m < n$ and the blocks U_i have the forms of Proposition 6.190 at s. □

Lemma 6.200. *Take $n \geq 5$ and $p \mid q - t$, with $t = \pm 1$ and p,q odd. The centralizer of the Sylow d-torus is $\mathbf{T}$. If the ambient sign is t^n, we have that $\mathbf{T}$ has n planes of type t. Otherwise, its plane types are t with multiplicity $n-1$ and $-t$ with multiplicity 1. With $T = \mathbf{T}^F$, we put $M = \mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$. Suppose that the proper Levi $\mathbf{L}$ has residue Y of rank $n-1$ and $s \notin \mathrm{Z}(\mathbf{L}^*)$. On Y, the natural lift has sign ranks $a, b > 0$ with $a + b = n - 1$. Moreover, the marked plane $Y_0 = Y^\perp$ has coordinate $x \neq \pm 1$, with $|x^2| > 2$ or $|x^2| = 2$ and $a \neq b$. Let k count planes of type ϵ_{Y_0}. With $M_1 = \mathrm{Stab}_M(Y_0)$, we write $I = \mathrm{N}_{\mathbf{L}}(\mathbf{T})^F$. Then, $[M : M_1] = k$, $[M_1 : I] = 2$, and $[G : L]_p = [M : I]_p = k_p$. Consequently, we obtain that $p \nmid k$ for an eligible s.*

Proof. The blocks U_i of nonzero weight have total rank 1. Let v be the natural lift of s. If $\mathbf{Z}_s$ is a fixed torus for $gvg^{-1} = -v$, then Proposition 6.190 gives rank at least 2 from each orbit $\{[\xi], [-\xi]\}$ of size 2 and rank 0 from the fixed pair $[i]$. Thus, those choices cannot give rank 1. Consequently, Y_0 has the sole nonsign pair $\{x, x^{-1}\}$. Both $1, -1$ occur on Y since $s \notin \mathrm{Z}(\mathbf{L}^*)$. Moreover, Frobenius fixes $1, -1$, so $F^*(v) = v$. If $x^2 = -1$ and $a = b$, then [208, Lemma 7.15] gives the quasi-isolated $Q(a,1)$. Thus, either $|x^2| > 2$ or $x^2 = -1$ with $a \neq b$.

The Weyl group M/T is transitive on the k planes of type ϵ_{Y_0}, so $[M : M_1] = k$. The subgroup I fixes both orientations of Y_0. Inversion on Y_0 and a plane in Y gives an even signed permutation w. Thus, w lifts to $j \in M_1$ by Lang's theorem [58, Lemma 4.2.13(i)]. Thus, $M_1/I = \langle jI \rangle \cong C_2$. If $\mathbf{T}$ has m planes of type t, then $|G|_p = |M|_p = (q-t)_p^m (m!)_p$ by the order formulas. The group L contains T, with active permutation factor $((m-1)!)_p$ if Y_0 has type t, and $(m!)_p$ if Y_0 has type $-t$. Hence, $[G : L]_p = m_p$ or 1, respectively, which is k_p. Moreover, $[M : I]_p = (2k)_p = k_p$. The eligible series have $p \nmid [G : L]$, since $\mathrm{C}_{G^*}(s) \leq L^*$. Thus, $P \in \mathrm{Syl}_p(G)$ may be chosen in I, with $P \cap R \in \mathrm{Syl}_p(R)$ for $R = \mathrm{Spin}(Y)^F$. □

Proposition 6.201. *Let $R = \mathrm{Spin}(Y)^F$. Suppose that $\succeq_c$ holds for the eligible $\mathrm{I}(a,b)$ pairs on R and its torus normalizer I_R, with weights $\theta_s|_{T_R}$ and natural actions on R. Then, marked induction gives $\succeq_c$ at M on the series of* Lemma 6.200.

Proof. With $\mathbf{T}_R = \mathbf{T} \cap \mathrm{Spin}(Y)$, we put $T_R = \mathbf{T}_R^F$ and $I_R = \mathrm{N}_{\mathrm{Spin}(Y)}(\mathbf{T}_R)^F$. Let $C = \mathrm{C}_T(R)$ and $U \leq C$ be the rational spin torus on Y_0. Since $\mathbf{L} = \mathbf{T}\,\mathrm{Spin}(Y)$, Lang's theorem on $\mathbf{T}_R$ gives $L = TR$. Moreover, $I = TI_R$ since T normalizes $\mathbf{T}$. If $y \in T$ has square spinor norm on Y, its coordinate on Y lifts to T_R. Hence, $y \in CT_R$ and $L/CR \cong I/(CI_R)$ has order at most 2, with representatives in T. Moreover, a torus normalizer on R returns $P \cap R$ by Sylow conjugacy in I_R. Thus, the correction is inner on R, I_R.

Let (ξ, ζ) be a lower pair on R, I_R. The character ν of U is dual to x^2, since $2e_0$ is the cocharacter of Y_0 in the spin lattice. Take each $\kappa \in \mathrm{Irr}(C)$ with $\kappa|_U = \nu$ and $\kappa|_{C\cap R} = \xi|_{C\cap R}/\xi(1)$. By [141, Theorems 2.8–2.9], we obtain $\succeq_c$ for the pairs $(\xi\kappa, \zeta\kappa)$ on CR, CI_R. If $L/CR = C_2$, take both extensions of each invariant $(\xi\kappa, \zeta\kappa)$. On an orbit of size 2, induce the pair $(\xi\kappa, \zeta\kappa)$. Both steps are in Proposition 6.177. If $L/CR = 1$, take $(\xi\kappa, \zeta\kappa)$ directly. Thus, on L, I the mark ν and any extension signs ± 1 are matched. Under $L^* \to R^*$, we have that the lift $v = (v_Y, x)$ maps to v_Y modulo $\{\pm I_Y\}$. Similarly, restriction of a weight θ_s to T_R has label v_Y modulo $\{\pm I_Y\}$. Consequently, the union has marks $\nu, \{a, b\}$. The degree ratios are 1 or 2, so p'-degrees match.

By [58, Theorem 11.4.3], signed induction $\mathcal{R}_{G,L}$ is injective on each $\mathcal{E}(L,s)$. A dual conjugator between $[s]_{L^*}$ and $[s']_{L^*}$ fixes the unique nonsign plane Y_0. Thus, modulo L^* it gives 1 or the sign change j. If $|x^2| > 2$, we have that j gives $\nu^j = \nu^{-1} \neq \nu$. If $x^2 = -1$, j acts by $x \mapsto -x$, so returning s would exchange sign spaces of ranks $a \neq b$. Thus, j acts freely on the classes $[s]_{L^*}$ and the I-orbits of θ_s. Moreover, $\mathcal{R}_{G,L}$ is injective on $\mathcal{E}(L,s)$ and equivariant under j, so its fibres are $\{\beta, \beta^j\}$. Similarly, [95, Theorem 6.11] gives that $\mathrm{Ind}_I^{M_1} \rho$ is irreducible with fibre $\{\rho, \rho^j\}$. For weights

$\theta_s, \theta_{s'}$ at Y_0, any transporter lies in M_1, so $M_{\theta_s} \le M_1$. By [95, Theorem 6.11(a),(b)], we induce from M_{θ_s} via M_1 to obtain distinct irreducibles of M above the weight orbits $M_1\theta_s$ at Y_0. The two types $t, -t$ give separate M-orbits.

For an ambient mixed stabilizer element (g, σ), correct by M to fix Y_0. If (β, ρ) moves to its j-translate, also correct by j. An ordinary field-free operation stabilizing (β, ρ, θ_s) cannot induce the sign change on Y_0 by the test for j. Thus, it fixes $2e_0$ and the radical $\mathbf{U}$ defined by $2e_0$. For mixed operations stabilizing θ_s, we compare $\mathbf{U}$ by Section 6.1. Since $\mathbf{J} = \mathbf{K} = \mathbf{T}$, the local map is Ind_I^M by Clifford correspondence. Consequently, Proposition 6.9 gives $\succeq_c$ at the images under $\mathcal{R}_{G,L}$ and Ind_I^M. By Lemma 6.200, $p \nmid [G : L][M : I]$, so both maps restrict to the p'-degree sets. □

Corollary 6.202. *If $n = 5$ and $2 \nmid q$, take either ambient sign. For $p \mid q - t$ with $t = \pm 1$, the non-quasi-isolated series $\mathcal{E}(G, s)$ have the candidate comparison at M.*

Proof. If $n = 5$, Theorem 6.199 leaves only $\dim Y = 8$ among ordinary Levi cases. Thus, on R take $\mathrm{I}(1, 3), \mathrm{I}(2, 2)$. If $\epsilon_Y = 1$, Corollary 6.121 gives the pairs on R, I_R at the natural labels s_R, including when s_R has triality translates. Moreover, for $\epsilon_Y = -1$, the weights θ_s are in Lemma 6.149. For these θ_s, $\succeq_c$ on R, I_R follows from Propositions 6.153 and 6.155. Consequently, for $\epsilon_Y = \pm 1$, $\succeq_c$ on G, M follows by Proposition 6.201.

Given v with eigenvalues $i, -i$ only, consider $\mathbf{T}$. On $\mathbf{T}$ of mixed type, $i^{tq-1} = -i^{-tq-1}$, and replacing i by i^{-1} leaves both values in $\{\pm 1\}$ fixed. Thus, $F^*(v) \ne \pm v$ there. By [132, Theorem 5.9], an eligible s is conjugate into T^*. Hence, v with only $i, -i$ has $\epsilon = t^5$, where Proposition 6.16 gives the comparison at G, M. Thus, each case for s in [208, Corollary 7.14, proof] gives $\succeq_c$ on G, M. □

Lemma 6.203. *Take $s \in T^*$ of type $Q(2, 1)$ from* Lemma 6.188. *Its lift v has $F(v) = v$, so by* Lemma 6.27, *z_0 acts trivially. The spaces $V_\pm = \ker(v \mp 1)$ have rank 2. Moreover, the quartic plane $V_i \oplus V_{-i}$ has unipotent label 1. Let s have type $Q(2, 1)$ and p'-index. Then, we have that $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ has size 13 if V_+, V_- have equal type, and 10 otherwise. Moreover, each pair of symbols (A, B) has the central comparison at M.*

Proof. On a split $V_\pm$, put $T_0 = (1, 1)$, $S_0 = (\mathrm{St}, \mathrm{St})$, $X_1 = (\mathrm{St}, 1)$ and $X_2 = (1, \mathrm{St})$ on the two A_1 factors. Moreover, the graph g exchanges X_1, X_2 with common symbol X_0. On a minus $V_\pm$, the labels are $1, \mathrm{St}$. The rational flags of the A_1 factors give noncuspidal labels T_0, S_0 or $1, \mathrm{St}$, respectively. Thus, for a pair containing $T_0, S_0, 1$ or St, the line source is [208, Proposition 7.18]. For (X_0, X_0), the source is [208, Proposition 7.20(5)]. Moreover, Corollary 6.95 gives $\succeq_c$ for all eligible (A, B) at M.

If all sign planes have type t, both $V_\pm$ are split. By [208, (7.16)–(7.17)], the rational component C_4 acts by $(A, B) \mapsto (B, gA)$ after relabeling. Thus, the orbit of (A, B) contributes $|\mathrm{Stab}_{C_4}(A, B)|$, by [208, (7.18)–(7.19)] and [95, Theorem 6.11]. The counts are

$\{A, B\}$	T_0, T_0	S_0, S_0	T_0, S_0	T_0, X_0	S_0, X_0	X_0, X_0
number	4	4	2	1	1	1.

Indeed, the four ordered pairs (X_i, X_j), $i, j \in \{1, 2\}$, form one orbit. The total is 13.

Otherwise, we choose v with the distinguished sign plane in V_+. Then, V_+ is minus and V_- is split. By [208, (7.17)], the rational component C_2 fixes $1, \mathrm{St}$ on V_+ and acts by g on V_-. Thus, each minus label ξ gives $2 + 2 + 1 = 5$ characters. The connected degrees are r-powers for field prime r. Moreover, component factors have the form 2^a. Hence, for $p \nmid |Ws|$, every counted character has p'-degree. By Lemma 6.188, $p = 3$ gives no eligible $Q(2, 1)$. □

Proposition 6.204. *For distinct symbol pairs (A, B) in* Lemma 6.203, *the images at M are disjoint. Moreover, for s of type $Q(2, 1)$, they exhaust $\mathrm{Irr}_{p'}(M \mid M\theta_s)$.*

Proof. Let $B = \mathrm{N}_{\mathrm{O}(V)}(\mathbf{T}_{\mathrm{nat}})$ for the natural torus $\mathbf{T}_{\mathrm{nat}}$, and put $B_0 = B \cap \Omega(V)$. Given $\rho, \rho' \in \mathrm{Irr}(B)$, [95, Lemma 5.2 and Problem 5.3] gives $\langle \rho|_{B_0}, \rho'|_{B_0} \rangle = \sum_{\lambda \in \mathrm{Irr}(B/B_0)} \langle \rho, \lambda\rho' \rangle$. Thus, a common constituent requires $\rho = \lambda\rho'$ for $\lambda \in \{1, \det, \nu, \det \nu\}$, with ν the spinor character. Moreover, a similitude b with nonsquare $\mu(b)$ gives the other orientation ρ^b.

If the sign planes have type t, $V_\pm$ have complement $W(B_2)$ and core rank 0. If $t = 1$, the hook runners have lengths $|X| = |Y|$. If $t = -1$, take the split symbols $(\{2\}, \{0\})$, $(\{1, 2\}, \{0, 1\})$ and $(\{1\}, \{1\})$. Thus, the cohook runner sizes

satisfy $|B_{00}| = |B_{01}|$. By Lemma 6.59, the $W(B_2)$-label recovers the unordered pair $\{X, Y\}$. Moreover, the linear $W(B_2)$-labels give row orbits T_0, S_0, and the label of degree 2 gives X_0.

Write η_+, η_- for the complements at $1, \nu|_{T_{\mathrm{nat}}}$, with the factor $\nu|_{W(B_2)}$ absorbed in η_- as in Section 6.3. Thus, on $W(B_2)$, we write δ_2 for the inversion sign. Then, ν exchanges η_+, η_-, and the twist by det acts by $\eta_\pm \mapsto \eta_\pm \delta_2$. Choose b with one matrix b_i on all planes of type t. Thus, b fixes plane permutations. On inversions in $W(B_2)$, the scalars are $1, -1$ at $1, \nu|_{T_{\mathrm{nat}}}$. It acts by $(\eta_+, \eta_-) \mapsto (\eta_+, \eta_- \delta_2)$. Hence, $\nu, \det, b$ fix $\{A, B\}$, also when the quartic plane $V_i \oplus V_{-i}$ is passive. Consequently, distinct $\{A, B\}$ have disjoint constituents on B_0. With fixed $\{A, B\}$, the marked comparisons distinguish the constituents ψ. At $A = B$, we apply Theorem 6.70.

Suppose now that V_+ contains the passive plane V_0. Fix its torus weight $1_{T_{V_0}}$. The twists $\nu, \det \nu$ change $1_{T_{V_0}}$ to $\nu|_{T_{V_0}}$, while b fixes $1_{T_{V_0}}$. Thus, only det and b compare the chosen parents. On V_-, the $W(B_2)$-label gives the unordered split rows $\{X, Y\}$. Moreover, on V_+ take $1_{\mathrm{O}(V_0)}$ for both labels $1, \mathrm{St}$. The active $W(B_1)$ has reflection r_+. Given $t = 1$, the symbols $(\{0, 1, 3\}, \{0\})$, $(\{0, 1, 2\}, \{1\})$ pack to $(\{0, 1, 2\}, \{0\})$ on runners $0, 1$, respectively. If $t = -1$, take $(\{0\}, \{0, 1, 3\})$, $(\{0, 1, 2\}, \{1\})$. Thus, runner 0 has parity 0 on top and 1 on the bottom. Runner 1 has parity $1, 0$. Both symbols have runner sizes $(3, 1)$. Moreover, we have that the move $3 \mapsto 2$ on runner 0 in the first symbol and $1 \mapsto 0$ on runner 1 in the second give core $(\{0, 2\}, \{0, 1\})$. Thus, r_+ has opposite signs for $1, \mathrm{St}$ at each t. With passive reflection r_0, the product of scalars on r_+, r_0 is fixed by det and b. Indeed, det changes both signs, while b fixes the signs on r_+, r_0 at weight 1. Consequently, the product separates $1, \mathrm{St}$ on V_+, and the split symbol on V_- separates the other choices.

Let $W_s = \mathrm{Stab}_{M/T}(s)$. For homogeneous signs, we set $v = (1, 1, -1, -1, i)$. Thus, $W(D_2)^2 \leq W_s$ has order 16. Let j exchange V_+, V_- and invert coordinates $1, 5$. Then, j^2 inverts coordinates $1, 3$, and $j^4 = 1$. Thus, $W(D_2)^2 \rtimes \langle j \rangle$ has order $64 = |W_s|$ by Lemma 6.188. On $W(D_2)^2$-labels, j acts by $(A, B) \mapsto (B, gA)$. With a distinguished plane in V_+, we choose the base $W(D_2) \times C_2$ with C_2 the double inversion on V_+. Moreover, inverting one coordinate on each V_+, V_- gives a second coset in W_s. Thus, $|W_s| = 16$, and the quotient C_2 acts by g on $W(D_2)$ and fixes the base C_2.

Given an orbit of (A, B), the quotient stabilizer $C_{A,B}$ is cyclic. By [95, Theorem 6.11], we obtain $|C_{A,B}|$ characters over the orbit. Thus, $|\mathrm{Irr}(W_s)| = 13$ or 10, with 2-power degrees. Moreover, $\mathbf{T} = \mathrm{C}_{\mathbf{X}}(\mathbf{S})$ for the Sylow d-torus $\mathbf{S}$, so [176, Theorem 1.1] gives an extension of θ_s to M_{θ_s}. Thus, the characters over $M\theta_s$ have degrees $[M : M_{\theta_s}]\varphi(1)$ for $\varphi \in \mathrm{Irr}(W_s)$, prime to p. Consequently, the disjoint images exhaust $\mathrm{Irr}_{p'}(M \mid M\theta_s)$. Natural actions permute V_+, V_-, and coefficient actions fix $\{A, B\}$. Thus, the marked comparisons combine over $Q(2, 1)$. □

Theorem 6.205. *Let q be a prime power and $S = \mathrm{P}\Omega^\epsilon_{10}(q)$, with $\epsilon = \pm 1$. Assume the parent comparisons on $\mathrm{GL}^\pm_m(Q)$, the marked orthogonal comparisons at M, and the transfer by $\mathcal{R}_{G,L}$. Then,* $\mathrm{iMN}(S, p)$ *holds for every $p \mid |S|$.*

Proof. If $2 \mid q$, apply Theorem 6.2. For odd q, [159, Theorem A(3)] gives $\mathrm{iMN}(S, 2)$. At $p \mid q$, [162, Theorem 1.3] gives $\mathrm{iMN}(S, p)$. Thus, assume q is odd and $p \nmid 2q$. By Lemma 6.165, the number m of active blocks exceeds 1 only for $d = 1, 2, 4$. If $m = 1$, the comparison is Proposition 6.164. Moreover, for $d = 4$, it is Theorem 6.179. If $d \in \{1, 2\}$, the result of [156, Theorem D and its proof] gives the unipotent comparison over 1_T. Moreover, for s non-quasi-isolated, Corollary 6.202 gives $\succeq_c$. For quasi-isolated $s \neq 1$, the types are $I(1, 4), I(2, 3), Q(1, 3), Q(2, 1)$ by [208, Lemma 7.15]. On M, apply Propositions 6.185, 6.187, 6.189 and 6.204 for those four s. By [132, Theorem 5.9 and Proposition 5.11], each rational class $[s]$ of p'-index corresponds to one torus orbit Ws of p'-size. Moreover, $\mathcal{R}_{G,L}, \mathrm{Ind}_I^M$ have fibres $\{\beta, \beta^j\}, \{\rho, \rho^j\}$ by Proposition 6.201. Thus, the comparisons on $\mathrm{Irr}_{p'}(X)$ and $\mathrm{Irr}_{p'}(M)$ combine over the classes $[s]$. □

6.8 Scalar periods

Lemma 6.206. *Let Y be the space of weight 0 in* Proposition 6.190*, of rank $m \geq 4$. Moreover, we have that $V = Y \perp V_A$ with $V_A = \bigoplus_i (U_i \oplus U_i^\vee) \neq 0$. Put $\mathbf{R} = \mathrm{Spin}(Y)$. Let $\mathbf{H}$ be the spin preimage of $\prod_i \mathrm{GL}(U_i)$ on V_A. By* Lemma 6.191*, we have that $\mathbf{H}$ has spin kernel $\langle z_0 \rangle$ for every number of F-orbits of U_i. Write R, H, L for rational points and $I = \mathrm{N}_{\mathbf{L}}(\mathbf{S})^F$. The multiplication $\pi : \mathbf{R} \times \mathbf{H} \to \mathbf{L}$ is an F-equivariant central isogeny with kernel $\langle (z_0, z_0) \rangle$. Moreover, we let $\mathbf{S}_i$ be the projections of $(\pi^{-1}\mathbf{S})^\circ$, for $i = \mathbf{R}, \mathbf{H}$, and put $I_i = \mathrm{N}_i(\mathbf{S}_i)^F$. Then, for $B = \pi(R \times H)$, $B_I = \pi(I_{\mathbf{R}} \times I_{\mathbf{H}})$, we have that $B_I = I \cap B$ and $I/B_I \cong L/B \cong C_2$. Moreover, for $J = \mathrm{C}_{\mathbf{L}}(\mathbf{S})^F$, we have that the quotient $J/(J \cap B)$ has the same representatives as I/B_I.*

Proof. On $Y \perp V_A$, by Lemma 6.14, we obtain π, $\mathbf{S}_i$ and the three quotients of order 2. The marked actions on Y, V_A act on π. Moreover, the representatives lie in $\mathbf{T}_L^F$ for a rational maximal torus $\mathbf{T}_L$ containing $\mathbf{S}$. Since $|\ker \pi| = |L/B| = 2$, for $P_R \in \mathrm{Syl}_p(R)$, $P_H \in \mathrm{Syl}_p(H)$ $\pi(P_R \times P_H) \in \mathrm{Syl}_p(L)$ when p is odd. Specifically, $\pi(P_R \times P_H)$ has order $|R|_p|H|_p = |L|_p$. □

By Lemma 6.10, the dual map π^* has kernel $\langle k \rangle$ for $k = [(1_Y, -1_A)]$. Moreover, the torus correction for π^* holds for $\mathbf{J}^* = \mathrm{C}_{\mathbf{L}^*}(\mathbf{S}^*)$. Indeed, for a conjugator $\bar{g}$ of $\pi^*(s)$, take a rational maximal torus $\overline{\mathbf{T}}$ containing $\pi^*(s)$. Lang on $(\pi^*)^{-1}(\overline{\mathbf{T}})$ gives all boundary values on $\overline{\mathbf{T}}^{F^*}$. Thus, the fibres in L^*, J^* have classes $[s], [ks]$. Moreover, by torus duality k corresponds to $\lambda \neq 1$ on L/B. By Digne–Michel [58, Proposition 11.4.8], translation $s \mapsto ks$ corresponds to the tensor product with λ. By Lemma 6.206, we have that λ gives the nontrivial characters on I/B_I and $J/(J \cap B)$. If all U_i are balanced, then $[s] = [ks]$ by Lemma 6.10. For scalar U_i, suppose $Y = V_+ \perp V_-$ and $F^*(y) = y$ for the lift y on Y. If the ranks and types of V_+, V_- agree, a rational isometry g exchanges them. Compose g with a reflection in V_+ if $\det g = -1$. Thus, $g \in \mathrm{SO}(Y)^F$ exchanges the eigenspaces of y, and $g \perp 1_{V_A}$ conjugates s to ks.

Proposition 6.207. *Let $A_R \leq \mathrm{Aut}(R)$ be the natural group induced on R. Put $s_R = \mathrm{pr}_{R^*}(\pi^*(s))$. Suppose that $\succeq_c$ holds on $R, I_{\mathbf{R}}$ for $A_R \times \mathcal{H}_p$ over $[s_R]$ and its marked $I_{\mathbf{R}}$-orbit. On $H, I_{\mathbf{H}}$, take* Proposition 6.194. *Then, over $\{[s], [ks]\}$, we have that the comparison $\succeq_c$ on L, I is equivariant for λ. If $[s] = [ks]$ or $\lambda|_{\mathrm{Z}(\mathbf{L})^F} \neq 1$, it pairs each series $\mathcal{E}(L, s)$ separately.*

Proof. On $\mathbf{H}$, the orbits and permutations of U_i are covered by Proposition 6.194. Pair R, H characters with equal scalar at z_0. Thus, we have that (z_0, z_0) acts by 1 and the tensors give characters on B, B_I. Moreover, at z_0 the scalars match. On A, the simultaneous stabilizer, apply $\succeq_c$ through the maps to $\mathrm{Aut}(R), \mathrm{Aut}(H)$ by [141, Theorem 2.9]. Thus, the common factors are $\alpha_R \alpha_H$ and the common functions are $\mu_R \mu_H$. Given a conjugation by $u \in I_i$, the matrix at u is multiplied by a common scalar c_u. Thus, the tensor at $\pi(r, h) \in B_I$ has the inner action of (r, h). Moreover, the ambiguity (z_0, z_0) acts by 1.

Take $w \in T_L$ representing the nontrivial coset of I/B_I. Let (r, h) lift w to $\pi^{-1}(\mathbf{T}_L)$. Then, $F(r, h) = (z_0 r, z_0 h)$, so $r^2 \in I_{\mathbf{R}}$, $h^2 \in I_{\mathbf{H}}$ and $w^2 = \pi(r^2, h^2)$. Thus, the square relation for w matches on B, B_I. If a pair has orbit size 2, induce from B, B_I. Given a fixed (β_0, ν_0), both extensions have one cochain $c : C_2 \to \mathbb{C}^\times$, by Proposition 6.177. Thus, on wB_I each mixed operation stabilizing (β_0, ν_0) has sign ε in both models. The twist by λ exchanges extensions with signs ± 1 and fixes the pairs induced from B, B_I. Moreover, the degree ratios are 1 or 2.

By [169, Proposition 4.6], for $\chi \in \mathcal{E}(L, s)$, every constituent of $\chi \circ \pi$ lies in $\mathcal{E}(R \times H, \pi^*(s))$. Conversely, a character χ over $\mathcal{E}(R \times H, \pi^*(s))$ has label $[s]$ or $[ks]$, since the series on $R \times H$ are disjoint. Moreover, for $\psi \in \mathrm{Irr}(I)$, the formula on J projects the labels of $\psi|_J$. The corrections in $I_{\mathbf{R}} \times I_{\mathbf{H}}$ map to B_I. For an odd Sylow subgroup $\overline{P}^*$ centralizing $\pi^*(s)$, we choose a lift P^*. Then, $[s, x] \in \mu_2$ for $x \in P^*$ defines a homomorphism $P^* \to \mu_2$. Hence, $[s, P^*] = 1$. Consequently, the eligible orbits in J^* correspond to $[s], [ks]$ by Lemma 6.7, including for nonabelian J.

If $[s] = [ks]$, there is one class. Otherwise, suppose $z \in \mathrm{Z}(\mathbf{L})^F$ has $\lambda(z) = -1$. Then, we may take $w = z$. The scalar c of z^2 matches on B, B_I. Thus, at z take $c^{1/2}$ in both extensions of (β_0, ν_0). The series $\mathcal{E}(L, s)$ and $\mathcal{E}(L, ks)$ have opposite scalars at z. Moreover, the scalars at z agree on the corresponding series in L, J by torus duality. Since $z \in \mathrm{Z}(I)$, induction to I gives the same two values on z. Thus, they distinguish $[s], [ks]$.

Suppose $-I_Y$ has nonsquare spinor norm and m_i is odd for some U_i. On its F-orbit, the scalar torus $\mathbf{C}_A$ has spin parity $\sum_i m_i a_i \bmod 2$ on a cocharacter (a_i). Thus, the inverse image of $\mathbf{C}_A$ in $\mathbf{H}$ is connected. By Lang's theorem, choose $a \in \mathbf{C}_A^F$ with boundary $\partial(a) \neq 1$ in $H^1(F, \mu_2)$. Moreover, on Y we have $\partial(-I_Y) \neq 1$. Thus, for lifts $\tilde{y}, \tilde{a}$ of $-I_Y, a$ we have $F(\tilde{y}, \tilde{a}) = (z_0 \tilde{y}, z_0 \tilde{a})$. Then, $z = \pi(\tilde{y}, \tilde{a})$ is rational and central in $\mathbf{L}$, since $(-I_Y, a)$ is central downstairs. Moreover, $z \notin B$ by the nontrivial common defect. Consequently, $\lambda(z) = -1$. □

Theorem 6.208. *Let q, p be odd with $p \mid q - t$, where $t = \pm 1$. Thus, we obtain that $\mathrm{C}_{\mathbf{G}}(\mathbf{S}) = \mathbf{T}$ and $K = T$. In the* maps $\mathcal{R}_{G,L}, \mathcal{R}_{K,J}$, *we have that the series of K are single weights θ_s. Let $n \geq 5$ be odd, with ambient sign $\epsilon = \pm 1$. Assume the comparisons for $\mathrm{GL}_m^{\pm}(Q)$ and the marked transfers through L, I. Then, for eligible s with lift v satisfying $1, -1 \notin \mathrm{Spec}(v)$, the comparison $\succeq_c$ holds at $\mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$.*

Proof. For odd n and $1, -1 \notin \mathrm{Spec}(v)$, [208, Lemma 7.15] gives that s is non-quasi-isolated. Thus, by [208, Proposition 7.3 and Corollary 7.14], s has a proper Levi $\mathbf{L}$ or the exceptional spectrum $\{i, -i\}$ on V. On $\mathbf{L}$, Corollary 6.195 covers $\dim Y \leq 2$. If v has no $i, -i$, $Y = 0$. Otherwise, nonzero Y comes from $\{i, -i\}$, fixed under $[\xi] \mapsto [-\xi]$. By Proposition 6.190, the paired orbits give U_i balanced of dimension $2b_i$. Thus, $m = \frac{1}{2} \dim Y$ is odd since $n = m + 2 \sum_i b_i$.

Given v with only $i, -i$ on V, we apply Proposition 6.16 at rank n. An eligible label has odd $\mathrm{ord}_p(\epsilon q)$. Since $p \mid q - t$, this gives $\epsilon = t = t^n$ and so a homogeneous $\mathbf{T}$. Given $m < n$ with $m \leq 3$, we apply Theorem 6.199. If $5 \leq m < n$, the lift of $s_R \in R^*$ has only $i, -i$. Since s is eligible, we choose $P^* \in \mathrm{Syl}_p(L^*)$ in $\mathrm{C}_{L^*}(s)$. By Lang's theorem [58, Propositions 4.2.10 and 4.2.11], the dual map π^* has kernel and cokernel C_2 on rational points. Thus, $\pi^*(P^*)$ is Sylow in $R^* \times H^*$, with projection to R^* in $\mathrm{C}_{R^*}(s_R)$. By Proposition 6.16, at rank m the comparison on $R, I_{\mathbf{R}}$ is over the weight $\beta_0|_{T_R}$ of its quadratic character. Moreover, all U_i are balanced, so $[s] = [ks]$ by Lemma 6.10. Thus, the comparison at $[s]$ on L, I is Proposition 6.207. The graph-fixed $\mathbf{Z}_s$ of [208, Proposition 7.3, Step 1] gives the stable radical $\mathbf{U}$ for ordinary degree-zero actions. Since $K = T$, the local radical is 1. By Proposition 6.9, we obtain $\succeq_c$ on $G, \mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$. □

Proposition 6.209. *Suppose* $\mathrm{C}_{\mathbf{G}}(\mathbf{S}) = \mathbf{T}$, *with independent scalar blocks* U_i *and* $Y = V_+ \perp V_-$ *of rank* $m \geq 4$, *where* $V_\pm \neq 0$ *have eigenvalues* ± 1. *Take* $P \leq I$ *and its toral base* Q *from* Section 6.1. *Assume the comparison on* $R, I_{\mathbf{R}}$ *in* Proposition 6.207. *Then,* $\succeq_c$ *holds on* G, M *over* $[\pi^*(s)]$, *also if* $[s] \neq [ks]$ *share a central character on* L.

Proof. On $Y \perp V_A$, the lifts of s, ks are $(y, a), (y, -a)$. Thus, $s^\sharp = ks$ in Lemma 6.11, since the U_i are independent. By that lemma, $\mathrm{C}_{\mathbf{G}^*}(s) = \mathrm{C}_{\mathbf{G}^*}(ks)$, with common Levi $\mathbf{L}^*$. Natural and coefficient operations carry s, ks together. Consequently, $p \nmid |[s]|$ exactly when $p \nmid |[ks]|$. With $M = \mathrm{N}_{\mathbf{G}}(\mathbf{T})^F$, put $I = M \cap L$. Over $\{[s], [ks]\}$, Proposition 6.207 gives $\succeq_c$ on L, I for the joint stabilizer of $[\pi^*(s)]$ and for λ. Thus, (β, ν) meets the hypotheses of Proposition 6.13. $J = K = T$, so the local map is Ind_I^M, paired with $\mathcal{R}_{G,L}$. By that proposition, the fibres are W-orbits for $W = \mathrm{N}_G(\mathbf{L})/L$, and the p'-cuts agree. Thus, $\succeq_c$ holds on the ambient union of $[s], [ks]$. □

Corollary 6.210. *Let* $X = \mathrm{Spin}^\epsilon_{12}(q)$, *with* q, p *odd,* $\epsilon = \pm 1$, *and* $p \mid q - t$ *for* $t = \pm 1$. *Assume the comparisons on* L, I *and the transfers* $\mathcal{R}_{X,L}, \mathcal{C}_{M,I}$. *Then, the characters in* $\mathrm{Irr}_{p'}(X)$ *over non-quasi-isolated* s *have* $\succeq_c$ *with the corresponding union at* M.

Proof. By Theorem 6.199, we may assume $s \notin \mathrm{Z}(\mathbf{L}^*)$ and $m \geq 4$. If $m = 5$, the lower series on R are $I(1, 4)$ and $I(2, 3)$. Thus, on $R, I_{\mathbf{R}}$ take the marked weights θ_{s_R} from Propositions 6.185 and 6.187. Then, Proposition 6.201 gives $\succeq_c$ on X, M. If $m = 4$, the nonzero blocks have total rank 2. If a block U_i is balanced, then $\dim U_i = 2$, so $[s] = [ks]$ by Lemma 6.10. On Y, the spectrum is contained in $\{1, -1, i, -i\}$ by Proposition 6.190. Thus, for split Y and noncentral s_R, we apply Corollary 6.121 on $R, I_{\mathbf{R}}$. For minus Y, take Section 6.6 and Proposition 6.153 at s_R. Moreover, for $s_R \in \mathrm{Z}(R^*)$, the twist α_{s_R} takes the comparison at 1 to that at s_R. By Proposition 6.207, we obtain $\succeq_c$ on L, I, and Proposition 6.9 gives $\succeq_c$ on X, M.

For independent U_i, the dimensions are 2 or $1, 1$, with F permuting or dualizing the U_i. Thus, Y has spectrum in $\{1, -1\}$. If only one sign occurs on Y, $s \in \mathrm{Z}(\mathbf{L}^*)$. Otherwise, s_R has type $I(1, 3)$ or $I(2, 2)$. Take the pairs on $R, I_{\mathbf{R}}$ from Corollary 6.202. Thus, on X, M, Proposition 6.209 gives $\succeq_c$ even for lifts $[s], [ks]$ with equal scalars on $\mathrm{Z}(L)$. Since $n = 6$, [208, Proposition 7.3] assigns every non-quasi-isolated s to a proper $\mathbf{L}$. □

Lemma 6.211. *Let* $n \geq 5$, *with* q, p *odd and* $p \mid q - t$ *for* $t = \pm 1$. *Thus, in* $\mathbf{T} = \mathrm{C}_{\mathbf{G}}(\mathbf{S})$, *we find that the case* $I(1, b)$ *has a positive plane* P_+ *and a negative space* Y *of rank* $b = n - 1$. *Let* $1 \leq a \leq b$, *with* $a + b = n$. *If all* n *planes have type* t, $I(a, b)$ *has one orbit of size* $\binom{n}{a}$ *for* $a < b$ *and* $\frac{1}{2}\binom{n}{a}$ *for* $a = b$. *If the distinguished plane* P_0 *has type* $-t$, *we set* $v|_{P_0} = 1$. *Then, each distinct* $k \in \{a, b\}$ *gives an orbit of size* $\binom{n-1}{k}$ *with negative type* t^k *and positive type* $-t^{n-k}$. *Moreover, we have that* $[s]$ *is eligible exactly when* $p \nmid |\mathrm{Orb}(v)|$ *for the torus orbit* $\mathrm{Orb}(v)$.

Proof. The natural lift v has signs ± 1, so $F^*(v) = v$. On the homogeneous torus, $\mathfrak{S}_n$ acts transitively on the negative a-subsets. Thus, $v \sim -v$ divides $\binom{n}{a}$ by 2 exactly when $a = b$, and the two types are t^a, t^b. If $v|_{P_0} = 1$, the k negative coordinates lie among the $n - 1$ planes of type t. Hence, the orbit has size $\binom{n-1}{k}$ and types $t^k, -t^{n-k}$. By [132, Theorem 5.9 and Proposition 5.11], the p'-orbits in $\mathbf{T}^*$ give rational classes $[s]$ of p'-index. The trivial connected label 1 has p'-degree in the series of $[s]$ because the component group is a 2-group [58, §11.5]. Thus, Kummer's theorem gives the p'-test by adding $a + b$ or $k + (n - 1 - k)$ in base p without a carry. □

If $n = 6$, the homogeneous sizes for $I(1, 5), I(2, 4), I(3, 3)$ are $6, 15, 10$, respectively. Moreover, $(|\mathcal{O}_1|, \ldots, |\mathcal{O}_5|) = (5, 10, 10, 5, 1)$ for the mixed torus orbits. Thus, $p = 3$ allows $I(3, 3)$ and all five values of k, whereas $p = 5$ allows $I(1, 5)$ and $k = 5$. Thus, for orbit $\mathcal{O}$ and stabilizer degree d_χ, we find that the test is $p \nmid |\mathcal{O}| d_\chi$.

Lemma 6.212. *Let $b = n - 1 \geq 4$. Suppose that the lift v is negative on Y of rank b and positive on P_+, with an orbit of p'-size. Write m for the number of planes in Y of type t. If $m = b$, the eligible symbols correspond to $\{\alpha, \beta\}$ with $|\alpha| + |\beta| = b$ and $p \nmid \mathrm{Sp}_{\alpha,\beta}(1)$ on $W(B_b)$. Thus, we have that $\alpha \neq \beta$ gives two spin characters and $\alpha = \beta$ gives one. If $m = b - 1$, order the rows of the core Λ_0. Then, we have that each ordered (α, β) with $|\alpha| + |\beta| = m$ and $p \nmid \mathrm{Sp}_{\alpha,\beta}(1)$ on $W(B_m)$ gives two spin characters χ^+, χ^-.*

Proof. The orbit of P_+ has size $n, n-1$ or 1 by Lemma 6.211, and P_+ has unipotent label 1. Since $b > 1$, the rational component C_2 acts by improper isometries on P_+ and Y [208, §7, following Lemma 7.15]. On Y, the graph automorphism γ_Y fixes each nondegenerate symbol Λ and exchanges the two labels Λ^+, Λ^- for degenerate Λ. Thus, Λ gives the two orientations $\chi^\pm$ and $\{\Lambda^+, \Lambda^-\}$ gives one character. By Lemma 6.27, $\chi(z_0) = \chi(1)$ for each character χ so obtained.

Since C_2 has order prime to p, the degree test is on Y. Take the 1-hook runners if $t = 1$ and the 1-cohook runners if $t = -1$. If the quotient (α, β) has size w, then Lemma 6.59 gives

$$(m - w)v_p(q - t) + v_p(m!/w!) + v_p(\mathrm{Sp}_{\alpha,\beta}(1)), \quad \mathrm{Sp}_{\alpha,\beta} \in \mathrm{Irr}(W(B_w)).$$

Thus, the valuation is 0 exactly when $w = m$ and $p \nmid \mathrm{Sp}_{\alpha,\beta}(1)$. If $m = b$, the core has rank 0, so row exchange acts by $(\alpha, \beta) \mapsto (\beta, \alpha)$ and Λ is degenerate exactly when $\alpha = \beta$. If $m = b - 1$, the core of rank 1 has type $-t$. Its rows are $(\{0, 1\}, \varnothing)$ for $t = 1$ or $(\{1\}, \{0\})$ for $t = -1$, with symbol Λ_0 given by these rows. Thus, (α, β) determines Λ for the fixed order of Λ_0. Equal rows of Λ would give equal rows of Λ_0, which has rank 1. □

Proposition 6.213. *Suppose that the marked sources on $\mathrm{GL}(E_j)$ satisfy the comparisons of multiplicity 1 and that the cuspidal source for Λ_0 satisfies its comparison at M. Thus, for nondegenerate Λ in* Lemma 6.212*, both pairs $(\chi^\pm, \phi^\pm)$ at M satisfy the mixed comparison for Λ. The ordered volume scalar is $\epsilon_Y \eta_q(-1)^b$ on both pairs.*

Proof. Write $2h$ for the absolute defect of Λ, and put $q = r^f$ and $D = \mathrm{ord}_p(r)$. If $t = -1$, $D/\gcd(D, f) = 2$ and $(\mu_D)^f = \mu_2$. Moreover, $\mathcal{H}_p$ fixes μ_D because $D \mid p - 1$. If $h = 0, 1$, the marked 1-hooks of [208, Proposition 7.18] give a negative plane P_- of type $1, -1$, respectively, with P_+ fixed. Thus, if $\epsilon(P_-) = \epsilon(P_+)$, the parent is Φ from Lemma 6.57. If $\epsilon(P_-) \neq \epsilon(P_+)$, the orientations are χ_1, χ_2 from Lemma 6.88. If $h \geq 2$, the ordinary hook core Λ_0 is cuspidal of rank h^2 by [40, §13.8], so there are $b - h^2$ hooks. In fact, $t = -1$ because $t = 1$ gives core rank 0 or 1 by Lemma 6.212. By [208, Proposition 7.25(ii)], the source for Λ_0 has a toral factor on P_+. On the source line for wF_q, the field operator A_w has $A_w^f = w^{-1}$, as in Proposition 6.167. Thus, the scalar ± 1 of w^{-1} has an f-th root in μ_D, also when $b = h^2$.

At each E_j, the character on $\mathrm{GL}(E_j)$ is η_q. The spinor twist acts by $\eta_q \mapsto 1$, whereas the residual parent Φ_j differs from $\Phi_j \det$. If both ranks are 1, the columns of $\mathcal{B}_j$ are given by the restrictions of Φ_j in Proposition 6.167. Otherwise, the rank $\mathrm{rk}(Y_j)$ of the residual negative space exceeds 1. By [208, Proposition 7.18, Steps 2–4], at E_j the connected multiplicity is 1 and $\mathcal{B}_j$ belongs to $\{I_2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$. Thus, the summand V_j of multiplicity 1 has field operator $A|_{V_j}$. If T_σ fixes the source on E_j and the row of $\mathcal{B}_j$, it fixes V_j. If $\mu(g)$ is nonsquare, we have that gV_j has the opposite orientation. Thus, only Λ is subject to the p'-degree test. On a plane P_j of type v, the quadratic character has value $(-1)^{(q-v)/2} = v\eta_q(-1)$ at $-I$. Consequently, the product over Y has value $\epsilon_Y \eta_q(-1)^b$ at $-I$, also on the isometry parent when $-I \notin \Omega$.

On the local torus, we let θ be quadratic on the b negative planes and trivial on P_+. The reflection r_+ has scalar 1. If $m = b$, the complement has a label (α, β) of $W(B_b)$. If $m = b - 1$, it has a label (α, β) of $W(B_m)$ and a scalar at r_0 on the distinguished negative plane P_0. With $B = \mathrm{N}_{\mathrm{O}(V)}(\mathbf{T}_{\mathrm{nat}})$ and $B_0 = B \cap \Omega(V)$, we let B_θ be the inertia group of θ in B. Extend θ with value 1 on the complement in B_θ. Then, the parent is $\rho = \mathrm{Ind}_{B_\theta}^B(\theta\psi)$, where ψ has label $\mathrm{Sp}_{\alpha,\beta}$ and the signs above. The twist $\det$ gives a factor -1 at r_+, while $\nu, \det\nu$ change $(b, 1)$ to $(1, b)$ on θ. Thus, $\rho|_{B_0}$ is irreducible because the twist group $\langle \det, \nu \rangle$ acts freely on ρ [95, Lemma 5.2], as in Proposition 6.204. Distinct normalized ρ have distinct restrictions to B_0. Conversely, every orbit under $\langle \nu, \det \rangle$ has a parent with $r_+ \mapsto 1$ and b quadratic factors. Its degree is $[B : B_\theta]\,\mathrm{Sp}_{\alpha,\beta}(1)$ up to a factor prime to p of 2-power order. Thus, the p'-test is as in Lemma 6.212.

Take g with nonsquare multiplier $\mu(g)$ and identical matrices on the planes P_j of type t. By Proposition 6.185, conjugation by g gives a factor -1 at each negative reflection r_j and fixes the scalar at r_+. Thus, if $m = b$, g acts by $(\alpha, \beta) \mapsto (\beta, \alpha)$ and acts freely if $\alpha \neq \beta$. If $m = b - 1$, g also changes the sign at r_0, so it acts freely even if $\alpha = \beta$. If

$m = b$, we choose one representative of $\{(\alpha,\beta),(\beta,\alpha)\}$. If $m = b-1$, order the core and take scalar 1 at r_0. Thus, the reconstructed symbols Λ are distinct.

The prime field map F_r fixes the signed complement. Its f-th power is $\mathrm{Int}(w_{\mathrm{ns}})$, where $w_{\mathrm{ns}} = \prod_{\epsilon(P)=-1} r_P$ and r_P is the inversion on the plane P. Thus, w_{ns} is central in the complement of B_θ and has one scalar under ρ. If $t = 1$, then either $w_{\mathrm{ns}} = 1$ or w_{ns} is a reflection with scalar 1 on the chosen branch. If $t = -1$, we choose an f-th root in μ_D of the sign of w_{ns}. Since θ is quadratic and ψ is rational, $\widehat{\rho}$ is fixed by $\mathcal{H}_p$ and the field operator A satisfies $A^f = \rho(w_{\mathrm{ns}})$. At $-I$, its scalar is $\epsilon_Y \eta_q(-1)^b$. By Lang's theorem [58, §4.2], the pair (χ,ϕ) has the field correction from A_w, as in Proposition 6.167. Since $\chi^g \neq \chi$, the isometries and F_r give the stabilizer of χ modulo central scalars. Thus, (χ,ϕ) has equal scalars at $-I$ and matching mixed functions on M. The comparison also holds for (χ^g,ϕ^g). □

Proposition 6.214. *By* Lemma 6.212*, a degenerate symbol Λ occurs only when Y is homogeneous and $b = 2j$. Then, we have that Y is split since $t^{2j} = 1$. Write $\xi \vdash j$ for either row partition and (α,α) for the quotient of Λ. If $\Lambda = (A,A)$ is eligible, then the planes E_i give the mixed comparison on $\mathcal{A},\mathcal{B}$. Both stabilizer extensions at Λ act trivially on zI. Moreover, if $t = -1$, the local field operator A has $A^f = (-1)^j I$.*

Proof. On the positive plane P_+, choose the trivial extension 1. With character $\eta_q \circ \det_{E_1}$ on $\mathrm{GL}(E_1)$, the flag module $V(E_1)$ has a constituent $\chi_{(1)}$ of multiplicity 1 and dimension fraction $q/(q+1) > 1/2$ by Proposition 6.168. Thus, $V(E_1)_{\chi_{(1)}}$ affords the full extension $\widehat{\chi}_{(1)}$. Let $\xi^{(1)} = (1),\dots,\xi^{(j)} = \xi$ record the box additions to ξ. Then, for $2 \le i \le j$ the character on $\mathrm{GL}(E_i)$ is $\eta_q \circ \det_{E_i}$. By [208, Lemma 7.19 and Proposition 7.20(1), Step 1], the connected orientations of $\xi^{(i)}$ restrict to one spin character $\chi_{\xi^{(i)}}$ of multiplicity 1 in the flag module $V(E_i)$, including when P_+ has rank 1. The label $\xi^{(i)}$ on Y and 1 on P_+ are fixed under the stabilizer and $\mathcal{H}_p$. Hence, $V(E_i)_{\chi_{\xi^{(i)}}}$ affords $\widehat{\chi}_{\xi^{(i)}}$. At step i, zI has value $\eta_q(z^2) = 1$ on E_i. Thus, $\widehat{\chi}_\xi$ is $\mathcal{H}_p$-invariant and trivial on zI. The path to ξ has no degree test at p until the final label $\xi^{(j)}$.

Let $\widetilde{T}$ be the natural torus of multiplier μ. If $t = 1$, we choose the first coordinate z_i on each plane. If $t = -1$, choose $z_i^{q+1} = \mu$. Thus, for $A_0 = C_{q-t}$, put $\widehat{\theta}(z) = \eta_{A_0}(\prod_{i=1}^{j} z_i / \prod_{i=j+1}^{2j} z_i)$. The argument belongs to A_0, since the j norm factors cancel for $t = -1$. The character $\widehat{\theta}$ is trivial on zI and restricts to $\eta_{A_0}^{\boxtimes 2j}$ on $\widetilde{T} \cap \mathrm{O}(V)$ at the marked planes. A permutation π changes the argument by a^2 for $a \in A_0$. Moreover, $\widehat{\theta}^{r_i}/\widehat{\theta} = \eta_q \circ \mu$ by Proposition 6.185. Thus, the complement for $\widehat{\theta}$ is $W(D_{2j}) \times \langle r_+ \rangle$.

Write U_α for the $W(B_{2j})$-module of (α,α). In particular, $U_\alpha = \bigoplus_{|D|=j} S^\alpha(D) \otimes S^\alpha(D^{\mathrm{c}})$, where $D \subseteq \{1,\dots,2j\}$ and $S^\alpha(D)$ is the Specht module on D. The rational operator C acts by $u \otimes v \mapsto v \otimes u$ from D to D^{c}. Thus, $C^2 = I$, and C intertwines the twist by inversion parity ε. The eigenspaces $U_\alpha^{\pm} = \ker(C \mp I)$ afford the two irreducibles of $W(D_{2j})$ over (α,α) by the type D restriction rule.[1] Indeed, $Cr_i = -r_i C$ for a marked inversion r_i.

Put $\widetilde{B} = \mathrm{N}_{\mathrm{GO}(V)}(\mathbf{T}_{\mathrm{nat}})$ and $B = \widetilde{B} \cap \mathrm{O}(V)$. On the inertia subgroup $\widetilde{B}_{\widehat{\theta}}$, we choose $\widehat{\theta} \otimes U_\alpha^+$ with $r_+ \mapsto 1$, and induce to $\widetilde{B}$. Since $\widetilde{B} = B\widetilde{T}$, Mackey restriction [95, Chapter 5] gives the induction to B from $B \cap \widetilde{B}_{\widehat{\theta}}$. Induction from $W(D_{2j})$ to $W(B_{2j})$ acts by $U_\alpha^+ \mapsto U_\alpha$, by the restriction rule. Thus, at B the parent ρ has quotient (α,α) and position induction on P_+. By Proposition 6.213, no nontrivial twist in $\langle \det, \nu \rangle$ fixes ρ, so $\rho|_{B\cap\Omega(V)}$ is irreducible. Consequently, the $\widetilde{B}$-module extends that character on $B \cap \Omega(V)$ and affords an extension at M by inflation.

Put $w_{\mathrm{ns}} = \prod_{\epsilon(P)=-1} r_P$, with r_P the inversion on P. If $t = 1$, only r_+ can occur in w_{ns}, and its scalar is 1. If $t = -1$, the product $r_1 \cdots r_{2j}$ has scalar $(-1)^j$ on every D-summand and on $U_\alpha^{\pm}$. Again, r_+ has scalar 1. Thus, w_{ns} has scalar $(-1)^j$ on the induced model for $t = -1$, because w_{ns} is central in the complement of $\widetilde{T}$ on each plane type. The map F_r fixes $\widehat{\theta}$ and the rational model $U_\alpha^{\pm}$. If $t = -1$ and j is odd, take $\beta \in \mu_{\mathrm{ord}_p(r)}$ with $\beta^f = -1$, as in Proposition 6.213. Otherwise, $\beta = 1$. Thus, the field operator $A = \beta I$ is fixed by $\mathcal{H}_p$ and $A^f = \beta^f I$ on the complement, where β^f is the scalar of w_{ns}. By Proposition 6.168, Lang conjugation on $\widetilde{B}$ gives the field correction $\mathrm{Int}(w_{\mathrm{ns}})$. Both extensions have scalar 1 on zI. Hence, $(\widehat{\chi}_\xi,\widehat{\rho})$ gives the mixed comparison on $\mathcal{A},\mathcal{B}$. □

Theorem 6.215. *Let $b \ge 4$ and let p,q be odd, with p prime and $p \mid q - t$ for $t \in \{1,-1\}$. Suppose that the sources on E_i and the cuspidal core Λ_0 have their comparisons for $I(1,b)$. Then, in either form $X = \mathrm{Spin}^{\epsilon}_{2b+2}(q)$, each eligible rational series $\mathcal{E}(X,s)$ of type $I(1,b)$ has the central mixed comparison at M with the given orbit of θ_s.*

Proof. By Lemma 6.211, the rational orbits of v are eligible at p exactly when $p \nmid |\mathrm{Orb}(v)|$. On each $\mathrm{Orb}(v)$, the symbols Λ of p'-degree are given by Lemma 6.212. If $\Lambda = (A,B)$ with $A \neq B$, Proposition 6.213 gives both pairs

[1] M. Geck, *On Kottwitz's conjecture for twisted involutions*, J. Lie Theory **25** (2015), 395–429, Example 3.4.

$(\chi^{\pm},\phi^{\pm})$, including when Y has one plane of type $-t$. If $\Lambda=(A,A)$, the single pair (χ,ϕ) is given by Proposition 6.214. Thus, (α,β) and the core recover Λ, and the normalization by $\langle\det,\nu\rangle$ exhausts the parents ρ. Thus, the extensions $(\widehat{\chi},\widehat{\rho})$ with equal central scalars give $\succeq_c$ for each (χ,ϕ). In particular, we have that $b=5$ gives the $I(1,5)$ series on $\mathrm{Spin}^{\epsilon}_{12}(q)$. The odd rank 5 excludes equal rows, and $5\neq h^2$ excludes a cuspidal Λ. However, at $t=-1$, a symbol of absolute defect 4 has a cuspidal core of rank 4, with one step on E_1 to rank 5. □

Lemma 6.216. *Let $q=r^f$ be odd and let $p\nmid q$ be an odd prime. Assume that $d=\mathrm{ord}_p(q)$ is even. Suppose that ρ is unipotent on G_{ad} of split type D and rank at least 4, and that $\rho^{\tau}=\rho$ for the natural graph involution τ. Then, we have that ρ has an ordinary $\mathcal{H}_p$-invariant extension to $G_{\mathrm{ad}}\rtimes\langle\tau,F_r\rangle$.*

Proof. By [208, proof of Proposition 7.2, Step 2 and (7.1)], an extension $\widehat{\rho}_{\alpha}$ has rational values on $G_{\mathrm{ad}}\rtimes\langle\tau\rangle$. If $\sigma(i)=i$, its coefficient ratio at F_r is $\sigma(\alpha)/\alpha$, where $\alpha^f=\omega\in\{1,-1\}$. Put $k=\mathrm{ord}_p(r)$, so $k/\gcd(k,f)=d$. If f is even, $4\mid k\mid p-1$. Thus, $\sigma(i)=i$ for $\sigma\in\mathcal{H}_p$. Moreover, $(\mu_k)^f=\mu_d$ contains ω, so choose $\beta\in\mu_k$ with $\beta^f=\omega$. Thus, the twist by λ, with $\lambda(F_r)=\beta/\alpha$ and $\lambda(\tau)=1$, has coefficient ratio $\sigma(\beta)/\beta=1$ for all $\sigma\in\mathcal{H}_p$. If f is odd, fix the rational restriction ρ_g on $G_{\mathrm{ad}}\rtimes\langle\tau\rangle$. The quotient C_f has odd order. By [208, Lemma 3.2], the extension $\widehat{\rho}$ is rational. On $\mathrm{GO}(V)$ with F_r, the extension via $G_{\mathrm{ad}}\rtimes\langle\tau,F_r\rangle$ has scalar 1 at zI. □

Lemma 6.217. *Let (Λ_+,Λ_-) be an eligible pair for $I(2,4)$. If $d\in\{0,1,2\}$ of the symbols $\Lambda_{\pm}$ are degenerate, the counts at X and M are both $2,1,2$, respectively.*

Proof. Order the sign spaces Y_+,Y_-. Given a homogeneous space $Y_{\pm}$ of rank j, we choose the orbit $\{P,P\otimes\varepsilon\}$ of its $W(B_j)$-label, where ε is inversion parity. If $Y_{\pm}$ has a plane of type $-t$, the choices are $(P,1),(P\otimes\varepsilon,-1)$, with $P\in\mathrm{Irr}(W(B_{j-1}))$ for the ordered core $\Lambda_{0,\pm}$. Thus, the choice set $\mathcal{P}_{\pm}$ has size 2 for unequal rows and 1 for equal rows of $\Lambda_{\pm}$, by Lemma 6.212. Since p is odd, the p'-test is on the orbit size $|\mathcal{O}|$ and the degrees $P_{\pm}(1)$. The remaining factors have 2-power order.

Let θ be trivial on T_+ and quadratic on T_- in the natural torus. Then, $P_+\boxtimes P_-$ with the preceding signs gives ρ by induction to $B=\mathrm{N}_{\mathrm{O}(V)}(\mathbf{T}_{\mathrm{nat}})$. The twist by det changes $P_{\pm}$ to $P_{\pm}\otimes\varepsilon$ and gives a factor -1 at r_0 on the plane of type $-t$. The spinor twist ν exchanges $(2,4)$ with $(4,2)$ on θ. Thus, by Proposition 6.204, restriction gives $4/2=2$ irreducibles of $B\cap\Omega(V)$ from $d=0$ and $2/2=1$ from $d=1$. If $d=2$, we have that only det is a nontrivial self-twist of ρ. Consequently, $\rho|_{B\cap\Omega(V)}$ has two distinct irreducibles of multiplicity 1 by Clifford theory [95, Theorem 6.11]. In fact, Y_+ and Y_- are both homogeneous and V is split. Moreover, a parent over $\theta|_{\mathbf{T}_{\mathrm{nat}}\cap\Omega(V)}$ has the chosen order up to ν.

The global component C_2 acts by the two graph involutions γ_+,γ_-. By Lemma 6.212, $\gamma_{\pm}$ fixes nondegenerate $\Lambda_{\pm}$ and exchanges the orientations $\Lambda^+_{\pm},\Lambda^-_{\pm}$ otherwise. Thus, the stabilizer orders in C_2 are $2,1,1$ and the orbit counts are $1,1,2$ for $d=0,1,2$, respectively. Consequently, there are $2,1,2$ spin characters. If $d=1$, we choose Y_- for the degenerate symbol Λ_-. The twist by ν exchanges the weights on $T_{\pm}$ and multiplies $P_{\pm}$ by linear characters. Thus, the test $P_{\pm}\otimes\varepsilon=P_{\pm}$ is independent of the sign of $Y_{\pm}$. The quotients $(\alpha_{\pm},\beta_{\pm})$ and the cores recover $\Lambda_{\pm}$. □

Proposition 6.218. *Suppose that the pure involutions s and marked flags E_i have their source comparisons for $I(2,4)$. Then, for $t=1,-1$ and each $d\in\{0,1,2\}$ in* Lemma 6.217*, the eligible orbit $\mathcal{O}$ has the central mixed comparison on X for either sign ϵ.*

Proof. Write $\Lambda_{\pm}=(A_{\pm},B_{\pm})$. Suppose first that $A_+\neq B_+$ and $A_-\neq B_-$, with the negative space Y_- of rank 4. For 1 or St on Y_+, the hook on E gives the $I(1,4)$ source. By Proposition 6.213, its extensions $\widehat{\chi}^{\pm}$ are on the isometries with F_r. Thus, only $\Lambda_{\pm}$ at rank 6 are subject to the p'-degree test. On $\mathrm{GL}(E)$ the weight is 1, whose spinor twist is η_q. Thus, the marked columns of multiplicity 1 in [208, Proposition 7.18] select the two ordered branches $\chi^{\pm}$ after adjoining E. Their volume scalar is $\epsilon_-\eta_q(-1)^4$, and if $\mu(g)$ is nonsquare, g exchanges $\chi^{\pm}$. On the local side, g changes P_- to $P_-\otimes\varepsilon$. If $t=1$ and r_0 belongs to a plane of type $-t$, a twist by det sets its scalar to 1. If $t=-1$, choose the f-th root in $\mu_{\mathrm{ord}_p(r)}$ of the scalar of w_{ns} by Proposition 6.213. Thus, $(\chi^{\pm},\phi^{\pm})$ have equal powers of F_r and equal scalars at $-I$, giving $\succeq_c$.

Now suppose that exactly one Λ_i is degenerate. Choose its space Y_- of rank $2j=2$ or 4, and let H_+ be the nondegenerate unipotent character on Y_+. The trivial module affords $H_+=1$. Similarly, the building $\mathcal{B}(Y_+)$ gives St on the top reduced homology of $\mathcal{B}(Y_+)$. For other Λ_+ of defect 0 or 2, the hooks of [208, Proposition 7.18] give H_+ from a toral source of rank 1, with weight 1 on each $\mathrm{GL}(E_i)$. Moreover, the stabilizer generators F_r,γ_+ and the

similitudes fix Λ_+. The connected multiplicity at E_i is 1. The stabilizer of E_i meets the relevant cosets of $\mathrm{SO}(Y_+)$ for H_+. Thus, the H_+-isotypic subspace affords $\widehat{H}_+$ with its induced field operator. If H_+ is cuspidal of rank 4 and defect 4, $t=-1$, and $\widehat{H}_+$ is given by Lemma 6.216.

On $E \subseteq Y_-$, we choose the character $\eta_q \circ \det_E$ and take $\widehat{H}_+$ on Y_+. Since $\eta_q \circ \det_E$ is linear and $H_+|_{\mathrm{SO}(Y_+)}$ is irreducible, the source on $P \cap \Omega(V)$ is irreducible for the stabilizer P of E. The residual group on Y_+ and the stabilizer of E give representatives for the $(\det, \nu, \mu)$-cosets. Thus, the fraction of $V(E)$ for the mixed label on $A_1 \times A_1$ is $q/(q+1)$, after cancellation of $H_+(1)$. If the ranks of $Y_\pm$ differ, the component stabilizer is 1. If $Y_\pm$ are split of rank 2, the fraction is $q/(q+1)$ by Lemma 6.79. If $\epsilon(Y_+) \neq \epsilon(Y_-)$, we have that the component orbits $\mathcal{O}_+, \mathcal{O}_-$ are distinct. Thus, the fraction $q/(q+1) > 1/2$ gives the extension $\widehat{\chi}$ by Lemma 6.78. If $j=2$, the branching in Proposition 6.214 has weight $\eta_q \circ \det_{E_2}$ on E_2. In particular, the degenerate Λ_- of rank 2 with cuspidal Λ_+ of rank 4 is treated by [208, Proposition 7.20(4)].

Locally, take $\widehat{\theta}$ and U_α^+ of $W(D_{2j})$ from Proposition 6.214 on Y_-. On Y_+, take the full rational complement label P_+ with the chosen scalar at r_0 when present. After induction to $\widetilde{B}$, we obtain $\widehat{\rho}$, with one irreducible ϕ at M by Lemma 6.217. The $\det|_{\mathrm{O}(Y_+)}$-twist on Y_+ gives ϕ as well, since $P_- \otimes \varepsilon = P_-$ and $\det|_{\Omega(V)} = 1$. The field power scalar is c_+c_-, where $c_\pm$ is the scalar of $w_{\mathrm{ns},\pm}$ on the respective block. If $t=1$, we normalize $c_+c_-=1$. If $t=-1$, the f-th root lies in $\mu_{\mathrm{ord}_p(r)}$. Thus, the extensions of χ, ϕ have scalar 1 at zI.

If Λ_+, Λ_- are both degenerate, ξ_- is (2) or $(1,1)$, and $\xi_+ = (1)$. The box from (1) to ξ_- has source $\eta_q \circ \det_{E_2}$ on the new plane E_2, by Proposition 6.73 with the quartic rank $c=0$. Thus, the orientations of ξ_- give the constituents χ_1, χ_2 of $\Phi|_{\Omega(V)}$. If $\xi_+ \neq \xi_-$, the only nontrivial self-twist is det. Since $c=0$, the exponents for μ, σ, F_r in the construction are 0. Consequently, the eigenspaces $\ker(D_g \mp I)$ afford χ_1, χ_2 with multiplicity 1 under $\Omega(V)$. Both spaces are stable under $\mathrm{GSO}(V)$ and F_r. Moreover, g_- of improper sign exchanges them and zI acts trivially on $\ker(D_g \mp I)$.

For $\widehat{\rho}$, the positive $W(B_2)$-label is (α, α). Thus, the rational operator C_+ intertwines the positive inversion parity ε_+ as in Proposition 6.214. Let $\Delta(x) = \det(x)/\mu(x)^6$ be the sign on $\widetilde{B}$. On the inertia subgroup J, the negative parity is 1, and both permutations and $\widetilde{T}$ lie in $\ker\Delta$. Therefore, $\Delta|_J = \varepsilon_+$. On induced functions $f(ux) = \rho(u)f(x)$ for $u \in J$, we set $(D_l f)(x) = \Delta(x) C_+ f(x)$. Thus, $C_+\rho(u) = \Delta(u)\rho(u)C_+$ implies the covariance of $D_l f$ and $D_l^2 = I$. The operator D_l intertwines the twist by Δ, so g_- exchanges $\ker(D_l \mp I)$. By Lemma 6.217, these spaces afford irreducibles ϕ_i of $B \cap \Omega(V)$. Both $\ker\Delta$ and F_r act on each eigenspace.

If $t=-1$, $w_{\mathrm{ns}} = r_1 \cdots r_6$ has scalar $(-1)^{1+2} = -1$ and $\Delta(w_{\mathrm{ns}}) = 1$. Thus, $\beta^f = -1$ gives the power on $\ker(D_l \mp I)$, with β fixed by $\mathcal{H}_p$. If $t=1$, $w_{\mathrm{ns}} = 1$. Lang's conjugation within $\mathrm{SO}(V)$ gives the field correction by w_{ns}. Consequently, the eigenspaces of D_g, D_l give pairs (χ_i, ϕ_i) with equal central scalars at zI and matching mixed functions. Conjugation by g_- acts by $(\chi_1, \phi_1) \mapsto (\chi_2, \phi_2)$. □

Proposition 6.219. *By* Lemma 6.211*, we have that the* $I(3,3)$ *orbit has size* 10 *in either form, so* $p \neq 5$*. With* $\delta = \eta_q(-1)$*, we put* $\kappa = t\delta$*. At rank* 3*, each symbol* Λ *is nondegenerate of absolute defect* 0 *or* 2*. Suppose that the opposite-plane sources* M_i *have their comparisons at* M*. Then, an eligible pair* (Λ_+, Λ_-) *for* $\mathrm{Spin}_{12}^-(q)$ *gives two pairs* (χ_i, ϕ_i) *with mixed comparisons on* $\mathcal{O}$*.*

Proof. Take the distinguished plane P_0 in the positive space Y_+. Thus, Y_- consists of three planes of type t, and Y_+ has types $t, t, -t$. Since $\epsilon(Y_-) \neq \epsilon(Y_+)$, no rational isometry exchanges $Y_\pm$. The component C_2 fixes $\Lambda_\pm$, so it gives two branches χ_1, χ_2 [208, §7, following Lemma 7.15]. On Y_-, the quotient gives two labels of $W(B_3)$ related by inversion parity ε. In contrast, on Y_+, the label on $W(B_2)$ has ordered core of rank 1 and a scalar at r_0. Thus, $p \nmid P_\pm(1)$ by Lemma 6.212.

The source M_i has field power $A_i^f = \gamma$ by Lemma 6.88. Moreover, the weights on Y_- are η_q and on Y_+ are 1. By [208, Proposition 7.18], $\mathcal{B}_{E_i}$ is a permutation matrix on χ_1, χ_2. Thus, the summand of multiplicity 1 has the field operator from M_i and is fixed by $\mathcal{H}_p$, as in Section 6.3. At $-I$, its scalar is $(t\eta_q(-1))^3 = \kappa^3$. Moreover, g with nonsquare $\mu(g)$ exchanges χ_1, χ_2.

On the local parent ρ, the twist by det changes both orientations, whereas g changes only the one on Y_-. The ν-twist changes the weight 1 on P_0 to η_{q+t}. Since P_0 has type $-t$, no permutation restores the weight 1. By Lemma 6.217, $B \cap \Omega$ has two distinct irreducibles ϕ_i over (Λ_+, Λ_-). Normalize the scalar at r_0 to 1. By Proposition 6.213, the field power, volume κ^3 and action of g give $\succeq_c$ on the two pairs (χ_i, ϕ_i). □

Proposition 6.220. *In the split form X, the torus $\mathbf{T}$ gives two characters for $\{\Lambda_+, \Lambda_-\}$ eligible of type $I(3,3)$ if $\Lambda_+ \neq \Lambda_-$, and four if $\Lambda_+ = \Lambda_-$. Suppose that the involution parents Φ and marked sources on E_i have their comparisons. Then, each branch χ and its local character ϕ satisfy $\succeq_c$.*

Proof. Both sign spaces Y_+, Y_- have type t. Their exchange w has determinant 1 and satisfies $wr_+r_- = r_+r_-w$ for the matched reflections $r_\pm$. Thus, the rational component is $\langle w, r_+r_-\rangle \cong C_2 \times C_2$. The simultaneous graph r_+r_- fixes both nondegenerate labels $\Lambda_\pm$. Thus, the component gives 2 branches when w exchanges distinct labels and 4 when they are equal.

On $W(B_3)$, a fixed label (α_0, α_0) would need $2|\alpha_0| = 3$. Write P, M_0 for two complement labels and let α be the spinor character on $W(B_3)$. Thus, in the convention $(P, M_0 \otimes \alpha)$ for the torus weights $1, \eta_q$, the twists on the pair (P, M_0) are $\det : (P', M_0')$ and $\nu : (M_0, P)$, where $P' = P \otimes \varepsilon$ for inversion parity ε. Moreover, g of nonsquare multiplier gives (P, M_0'). For the rule for ν in Section 6.3, the scalar factor satisfies $\alpha^2 = 1$.

For distinct original labels Λ_+, Λ_-, no nontrivial $\lambda \in \langle \det, \nu\rangle$ fixes the parent ρ. By Proposition 6.204, $B \cap \Omega$ has two irreducibles ϕ_1, ϕ_2 for the unordered pair, and g exchanges them. By Proposition 6.74 with $c = 0$, the first marked column beyond $\Lambda_+ = \Lambda_-$ gives χ_1, χ_2 with field operators A_1, A_2. The local power A^f is the scalar of w_{ns} in Proposition 6.213. At $-I$ both parents have scalar κ^3. Thus, g carries the comparison to (χ_2, ϕ_2). For equal labels $\Lambda_+ = \Lambda_-$, the parent (P, P) has self-twist group $\langle\nu\rangle$. Thus, $\rho|_{B\cap\Omega}$ has two constituents of multiplicity 1. The parent ρ^g has self-twist group $\langle \det \nu\rangle$. Since $\mathrm{Stab}(\rho) \neq \mathrm{Stab}(\rho^g)$ under quotient twists, their restrictions at $B \cap \Omega$ are disjoint. Consequently, both counts are 4 by Proposition 6.58 for $c = 0$.

Write J for the inertia subgroup of θ. Thus, w exchanges the matched planes for P. Let C be the tensor flip on $U_P \otimes U_P$. For the source representation π of J, the twist rule gives $C\pi(whw^{-1})C^{-1} = \nu(h)\pi(h)$ for $h \in J$. Thus, on induced functions $f(hx) = \pi(h)f(x)$ the operator $(H_0 f)(x) = \nu(x)Cf(wx)$ has the source covariance, intertwines the ν-twist and satisfies $H_0^2 = \nu(w)I$. Given a matched plane transposition w_i, the negative plane $\ker(w_i + I)$ has form $2Q$, where Q is the plane form. Thus, $\nu(w_i) = t\eta_q(-1) = \kappa$, since the determinant changes by 2^2. Thus, $\nu(w) = \kappa^3$ and $H = \zeta H_0$ is an involution for $\zeta \in \mu_4$ with $\zeta^2 = \kappa^3$. Since H_0 is rational, its normalized operator H has coefficient sign $\sigma(\zeta)/\zeta$.

The global sign is g_σ^3 by Proposition 6.58, where $g_\sigma = \eta_q(b_\sigma)$ and $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$. If $\sigma \in \mathcal{H}_p$ acts by p^k on roots of p'-order, $g_\sigma = \eta_q(p)^k$. If f is odd, $\delta = (-1/r)$ and $(r/p) = (q/p) = (t/p)$. Thus, quadratic reciprocity gives $(p/r) = (-1/p)^{[\delta=-1]+[t=-1]}$. Hence, $g_\sigma = 1$ if $\kappa = 1$ and $g_\sigma = \sigma(i)/i$ if $\kappa = -1$. If f is even, $\delta = 1$ and $g_\sigma = 1$. Moreover, if $t = -1$, then $4 \mid p - 1$ by Lemma 6.216, so $\sigma(i) = i$. In either case $g_\sigma^3 = \sigma(\zeta)/\zeta$.

The scalar for w_{ns} gives the local field operator A as in Proposition 6.213. Thus, $AH = HA$. If $t = -1$, choose the torus twist v as $v = s_1 \cdots s_6$, with identical reflections s_i over $\mathbb{F}_r$ in the split model. Thus, $\det(v) = \nu(v) = 1$ over $\mathbb{F}_r$, so v has a spin lift $\widetilde{v}$ fixed by F_r. By Lang's theorem [58, Theorem 4.2.9], we choose $\widetilde{T}$ with $F_q(\widetilde{T}) = \widetilde{T}\widetilde{v}$. Then, $\widetilde{T}F_r(\widetilde{T})^{-1}$ is F_q-fixed since $F_r(\widetilde{v}) = \widetilde{v}$. Its image $u \in \Omega$ gives the inner correction $\mathrm{Int}(u)$ for the split field action. Thus, u acts within $\ker(H \mp I)$, and the corrected operator A_u has the given f-th power. If $t = 1$, we set $u = 1$.

On both norm involutions, x acts with sign $\nu(x)$, and (x, σ) acts with sign $\nu(x)g_\sigma^3$. Thus, the corresponding field extensions on $\ker(H \mp I)$ have the stabilizing pairs (x, σ) with $\nu(x)g_\sigma^3 = 1$. Both have volume scalar κ^3. Moreover, g gives the comparison on (χ^g, ϕ^g), including its field operator A^g. Thus, distinct pairs (Λ_+, Λ_-) give disjoint constituents of $\rho|_{B\cap\Omega}$ by the quotient twist test. □

Corollary 6.221. *With q odd, we let p be an odd prime with $p \mid q - t$. Suppose that the source comparisons for $I(a,b)$ hold for $a + b = 6$. Then, each eligible series $\mathcal{E}(X, s)$ for a sign involution s of $X = \mathrm{Spin}_{12}^\epsilon(q)$ has the comparison on the orbit $\mathcal{O}$ at M.*

Proof. The $I(1,5)$ series are covered by Theorem 6.215, and $I(2,4)$ by Proposition 6.218. Similarly, for $I(3,3)$ apply Propositions 6.219 and 6.220 to both forms X. Thus, each eligible $\mathcal{O}$ in Lemma 6.211 is covered. On each $\mathcal{O}$, the bijection Ω pairs $\chi \in \mathrm{Irr}_{p'}(X)$ with $\psi \in \mathrm{Irr}_{p'}(M)$ and gives $\succeq_c$. In particular, we have that $\Omega(\chi^\sigma) = \psi^\sigma$ for each $\sigma \in \mathcal{H}_p$, with $\Omega(\chi) = \psi$. □

Lemma 6.222. *Let $a, c \geq 1$, $n = 2a + c \geq 5$, and $p \mid q - t$, with p, q odd and $t = \pm 1$. Thus, we put $u = \eta_q(-1)$. Given $Q(a,c)$ with labels $(\Lambda_+, \Lambda_-, \lambda)$,* Theorem 6.70 *and* Corollary 6.95 *give pairs (χ, ϕ) at θ_s. Thus, we determine their union on $\mathbf{T}$. Each eligible $Q(a,c)$ series has a rational natural lift v in $\mathrm{SO}(V)$. Each row gives one torus orbit*

$\mathcal{O}$ *when its conditions hold. Thus,* $\mathcal{O}$ *is eligible exactly when* $p \nmid m_{\mathcal{O}}$.

plane types	*condition*	$m_{\mathcal{O}}$
all homogeneous	$u = t$	$n!/(a!^2 c!)$
distinguished quartic plane	$u = -t,\ c = 1$	$(2a)!/a!^2$
distinguished sign plane	$u = t$	$(n-1)!/((a-1)!a!c!)$

In rows $1, 2$*, the quotient weights are* a, a *with cores of rank* 0*. In the third row, we put* $P_0 \subseteq Y_+$*. Then, we have that* Y_+, Y_- *have quotient weights* $a-1, a$ *and core ranks* $1, 0$*, respectively. The labels are the* p'*-degree wreath characters* $P_\pm$ *and the* p'*-degree Specht character* Sp_λ *on the active quartic positions* $1, \ldots, c$.

Proof. On a sign coordinate, v has value ± 1, so $F^*(v) = v$. A quartic coordinate has plane type u, since $4 \mid q - u$. Thus, if all planes have type t, then $u = t$ and the ambient sign is t^n. With a distinguished plane of type $-t$, the sign is $-t^n$. If that plane is quartic, then $u = -t$ and $c = 1$. Otherwise, $u = t$ and the distinguished plane is in Y_+. Permutations within each type give $m_{\mathcal{O}}$ from the positions of $Y_\pm$ and the quartic planes P_i. Given a quartic inversion r_i, the inversion r_j of a sign plane gives $\det(r_i r_j) = 1$. Thus, the choice between $v, -v$ and the inversions change only the 2-part of the index. By [132, Theorem 5.9 and Proposition 5.11], each p'-orbit gives one eligible rational class $[s]$. By Lemma 6.59, the labels on $Y_\pm$ have p'-degree for the full weights a, a or $a-1, a$ as in the table and $p \nmid P_\pm(1)$. Moreover, the active quartic factor has $\mathrm{ord}_p(uq) = 1$. Thus, the hook formula [40, §13.8] gives $v_p(\chi_\lambda(1)) = v_p(\mathrm{Sp}_\lambda(1))$ by applying $v_p((uq)^j - 1) = v_p(uq-1) + v_p(j)$ to numerator indices j and hook lengths h. If $c = 1$ is inactive, the label is 1. Consequently, the p'-tests give the domains of $(\Lambda_+, \Lambda_-, \lambda)$, including cuspidal and staircase symbols $\Lambda_\pm$. □

Proposition 6.223. *At the spin torus normalizer* M*, distinct triples* $(\Lambda_+, \Lambda_-, \lambda)$ *of* Lemma 6.222 *have disjoint images modulo* (Λ_+, Λ_-) *and* (Λ_-, Λ_+) *when* $Y_\pm$ *admit a rational swap* w.

Proof. By Proposition 6.204, if two parents ρ, ρ' share a constituent on $B \cap \Omega$, then $\rho' = \rho\tau$ for $\tau \in \langle \det, \nu \rangle$. The nonsquare multiplier g acts on both orientations of ρ. On homogeneous sign planes, normalize by α on the negative $W(B_a)$, where α is its spinor character. Thus, on (P_+, P_-) the rules are $\det : (P'_+, P'_-)$, $\nu : (P_-, P_+)$ and $g : (P_+, P'_-)$ by Section 6.3, where $P'_\pm = P_\pm \otimes \varepsilon$ for inversion parity ε. Since both cores have rank 0, the orbit of $P_\pm$ under ε determines $\Lambda_\pm$. Thus, we have that $\{\Lambda_+, \Lambda_-\}$ is determined even when P_0 is a passive quartic plane.

If P_0 is a distinguished sign plane in Y_+, choose the trivial module on $\mathrm{O}(P_0)$, the core of rank 1. The ν-twist changes the label 1 on P_0 to η_{q+t}. Since P_0 is the only plane of type $-t$, such a twist cannot restore 1 on P_0. The det-twist gives a factor -1 at r_0 and acts by $P_+ \mapsto P_+ \otimes \varepsilon$. Thus, normalization at r_0 to 1 fixes the ordered quotient and recovers Λ_+ from $W(B_{a-1})$. The multiplier g fixes P_+, and the orbit of P_- under ε determines Λ_-. If $a = 1$, we have that $W(B_{a-1}) = 1$, so the core of Y_+ determines Λ_+.

On the quartic planes, $\mathfrak{S}_c$ consists of plane permutations in Ω. Indeed, a transposition w_i has $\nu(w_i) = u\eta_q(-1) = 1$ by Proposition 6.220. If L_c is the active quartic factor, then $\mathrm{ord}_p(uq) = 1$. Thus, the weight space for θ is isotypic of type Sp_λ under $\mathfrak{S}_c$. If a second weight θ' exists, $\theta' = \theta\nu$. Moreover, a conjugator g_0 from $\theta\nu$ to θ exchanges $Y_\pm$ and inverts the quartic coordinates P_i up to $\pi \in \mathfrak{S}_c$. Given $\pi \in \mathfrak{S}_c$, the quartic inversions satisfy $\pi(r_1 \cdots r_c)\pi^{-1} = r_1 \cdots r_c$, and $\nu|_{\mathfrak{S}_c} = 1$. Thus, each constituent over $\theta|_{\mathbf{T} \cap \Omega}$ has Sp_λ-isotypic action there, including when $\mathrm{Sp}_\lambda(1) > 1$. On a passive quartic plane, $\lambda = (1)$. Consequently, we recover $(\Lambda_+, \Lambda_-, \lambda)$. Given a fixed triple $(\Lambda_+, \Lambda_-, \lambda)$, the component labels at M follow from Theorem 6.70 and Corollary 6.95. □

Theorem 6.224. *Let* $n = 2a+c \geq 5$*,* $a, c \geq 1$*, with* p, q *odd and* $p \mid q-t$*. Suppose that the sources for* $Q(a, c)$ *have their comparisons on* E_i*. Then, in either form* $X = \mathrm{Spin}^\epsilon_{2n}(q)$ *the central mixed correspondence holds on* $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ *and its corresponding local orbit at* M.

Proof. It suffices to count $\mathrm{Irr}_{p'}(M \mid \theta_s)$ by Proposition 6.223. The Sylow torus $\mathbf{S}$ has centralizer $\mathbf{T}$, a maximal torus. Thus, θ_s has an extension $\widehat{\theta}_s$ to M_{θ_s} by [176, Theorem 1.1]. Consequently, the characters are $\mathrm{Ind}_{M_{\theta_s}}^M(\widehat{\theta}_s\psi)$ for $\psi \in \mathrm{Irr}_{p'}(M_{\theta_s}/T)$, since the index is prime to p by Lemma 6.222. If $Y_\pm$ are homogeneous, the Weyl inertia is $\mathfrak{S}_c$ times a factor with normal subgroup $K = W(D_a) \times W(D_a)$, where $W(D_1) = 1$. Modulo K, let z invert matched planes $P_\pm$ in $Y_\pm$. Let g exchange Y_+, Y_- and invert the c quartic planes. If c is odd, we multiply by r_{P_+} to obtain $\det(g) = 1$. Then, $g^2 = z$ and the quotient is C_4. If c is even, $g^2 = z^2 = 1$ and $gz = zg$. Thus, we obtain the quotient $C_2 \times C_2$. The cosets in $\langle g, z \rangle K/K$ exhaust the signed permutations modulo K, also for a passive quartic plane P_0.

By Proposition 6.214, the restriction from $W(B_a)$ to $W(D_a)$ gives one graph-fixed character for $\alpha \neq \beta$, and two exchanged by graph for $\alpha = \beta$. Thus, these give the choices for $\Lambda_\pm$ with cores of rank 0. Given a cyclic stabilizer over K, its invariant character of K extends by [95, Corollary 11.22]. Conversely, for a full $C_2 \times C_2$ stabilizer, the two labels ψ_0, ψ_0 are graph-fixed. Choose one model U for the extension to $W(B_a)$ of the common label. Then, g acts on $U \otimes U$ by the tensor flip and z acts by $R \otimes R$ for the matched inversion R on U. The operators satisfy $g^2 = z^2 = 1$ and $gz = zg$, so the operators extend $\psi_0 \boxtimes \psi_0$ to its stabilizer. Thus, a label ψ of K gives $|I_\psi/K|$ characters for its stabilizer I_ψ in the factor containing K. The Specht label λ is independent of ψ. Moreover, the remaining indices are 2-powers.

If P_0 is a distinguished positive sign plane, $K = W(B_{a-1}) \times W(D_a)$. On Y_+ even parity determines the P_0 inversion from the active signs. Since $\epsilon(Y_+) \neq \epsilon(Y_-)$, no rational element exchanges $Y_\pm$. The other coset of K inverts P_0 and $P_- \subset Y_-$. Thus, it acts trivially on $W(B_{a-1})$ and by graph on $W(D_a)$. Consequently, the cyclic count for Y_+ depends on the core of rank 1 and is independent of λ.

Thus, the rational action on $\Lambda_\pm$ matches the Weyl action on $P_\pm$. Moreover, duality fixes λ. By [208, §7, preceding Lemma 7.16], the self-twists of $\widetilde{\chi}$ correspond to $\mathrm{Stab}(\Lambda_+, \Lambda_-, \lambda)$. Indeed, after central multiplication of $\widetilde{s}$, the rational restoring conjugator acts on $\Lambda_\pm$ by the corresponding component Since $\widetilde{\chi}|_X$ is multiplicity-free, its length is $|\mathrm{Stab}(\widetilde{\chi})|$ under quotient twists. Thus, for $(\Lambda_+, \Lambda_-, \lambda)$ the count at X equals the Weyl count at M, with the p'-tests of Lemma 6.222. Consequently, $\mathrm{Irr}_{p'}(M \mid \theta_s)$ is covered by the comparisons for $(\Lambda_+, \Lambda_-, \lambda)$. □

Corollary 6.225. *Let p, q be odd and $p \mid q - t$. Then, under the preceding hypotheses, $\mathrm{Irr}_{p'}(\mathrm{Spin}^-_{12}(q))$ has the comparison at M.*

Proof. By [156, Theorem D] and Corollaries 6.210 and 6.221, the series $\mathcal{E}(X, s)$ for proper Levi subgroups and $I(a, b)$ have the comparison. Moreover, $Q(1, 4)$ and $Q(2, 2)$ are the quartic series, both with $u = t$. By Lemma 6.222, the homogeneous multinomials are $30, 90$, and the mixed ones are $5, 30$. Hence, the eligible orbits of $Q(1, 4)$ and $Q(2, 2)$ have the comparisons by Theorem 6.224. By [208, Lemma 7.15], $U(6)$ occurs only for $\mathrm{Spin}^+_{12}(q)$. Thus, the cited cases partition $\mathrm{Irr}_{p'}(\mathrm{Spin}^-_{12}(q))$ and have disjoint images at M. □

6.9 Orthogonal character spaces

By Zhang [208, Proposition 7.18, Steps 1 and 3], we have that a parabolic LV_u for a line or plane E gives restriction multiplicity 1. Thus, we determine the corresponding group M_E. Let $q = r^f$ be odd, V a nondegenerate quadratic space over $\mathbb{F}_q$ of even dimension and $P \neq 1$ a Sylow p-subgroup of $G = \Omega(V)$, where p is odd and $p \neq r$. Thus, we write $\delta = (\det, \nu)$, with ν the spinor square class and $\Delta = \{\pm 1\} \times \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$.

Lemma 6.226. *With $U = V^P$, we put $W = U^\perp$ and $A = \mathrm{O}(U)$ and $B = \mathrm{N}_{\mathrm{O}(W)}(P)$. Then, $V = U \perp W$ and $p \nmid |A|$. Moreover, we have that $M = \mathrm{N}_G(P) = \{(x, b) \in A \times B \mid \delta(x)\delta(b) = 1\}$. The projection $M \to A$ is surjective. Suppose that $p \nmid |\mathrm{GL}_k(q)|$. Then, the P-stable totally singular k-spaces form one M-orbit if U has Witt index at least k, and none otherwise. For such E, we have that M_E is the kernel of $\delta(x)\delta(b)$ on $A_E \times B$, where $A_E = \mathrm{Stab}_A(E)$.*

Proof. The averaging projection $|P|^{-1} \sum_{g \in P} g$ is self-adjoint with image U, so its kernel is W. Thus, $V = U \perp W$ and P acts faithfully on W. Since $[\mathrm{O}(V) : G]$ is a 2-power, P is Sylow in $\mathrm{O}(V)$. A subgroup $C_p \leq A$ would give $P \times C_p \leq \mathrm{O}(V)$, so $p \nmid |A|$. As $P \neq 1$, its faithful action on W implies $\dim W \geq 2$. A nondegenerate binary subspace $W_0 \leq W$ has form xy or $aN_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ for some $a \neq 0$. Thus, the reflection formula [137, §24.i] on W_0 gives both spinor classes with determinant -1, so $\delta(\mathrm{O}(W)) = \Delta$. Since P is Sylow in $\ker(\delta|_{\mathrm{O}(W)})$, [97, Lemma 1.13] gives $\delta(B) = \Delta$. Since $U = V^P$, the normalizer of P in $\mathrm{O}(V)$ is $A \times B$, with $\delta(x, b) = \delta(x)\delta(b)$. Consequently, M is the stated kernel and $M \to A$ is onto.

If $p \nmid |\mathrm{GL}_k(q)|$, a stable k-space E has $P|_E = 1$, so $E \subseteq U$. By Witt's theorem the singular k-spaces in U form one A-orbit when U has Witt index at least k. Moreover, for $k > 0$ a choice of opposite $E' \subseteq U$ gives an isometry exchanging E, E', and $M \to A$ is onto. Let LV_u be the parabolic at E in G, with L the common stabilizer of E, E'. Thus, $P \leq L$, and $M_E = \mathrm{N}_L(P)\mathrm{C}_{V_u}(P)$. Indeed, the image of M_E in L lies in $\mathrm{N}_L(P)$. The other factor $v \in V_u$ normalizes P, so $[v, P] \leq V_u \cap P = 1$. □

Proposition 6.227. *Fix E as in* Lemma 6.226. *Given $\rho \in \mathrm{Irr}(A)$, we have that $\beta \in \mathrm{Irr}(A_E)$ and $\tau \in \mathrm{Irr}(B)$, put $\Pi = \rho \boxtimes \tau$ and $\Theta = \beta \boxtimes \tau$. Moreover, we put $T = \{\lambda \in \mathrm{Irr}(\Delta) : \tau(\lambda \circ \delta) = \tau\}$. Then, $\langle \Pi|_{M_E}, \Theta|_{M_E}\rangle = \sum_{\lambda \in T} \langle \rho|_{A_E}, \beta(\lambda \circ \delta)\rangle$. Similarly, for $z = \pm 1$, the Pin formula holds on $\widetilde{M}, \widetilde{M_E}$.*

Proof. The quotient $(A_E \times B)/M_E$ is Δ. By Frobenius reciprocity [95, Lemma 5.2], the pairing on M_E has contributions for $\lambda \in \mathrm{Irr}(\Delta)$, with $\Theta_\lambda = \Theta \cdot (\lambda \circ \delta)$ on the product group. The B-factor is $\langle \tau, \tau(\lambda \circ \delta)\rangle$, which is 1 if $\lambda \in T$ and 0 otherwise. Thus, $\|\Pi|_M\|^2 = \sum_{\lambda \in T} \langle \rho, \rho(\lambda \circ \delta)\rangle$, with the analogous formula for $\Theta|_{M_E}$. If $T = 1$, both restrictions are irreducible and at M_E the inner product is $\langle \rho|_{A_E}, \beta\rangle$. In particular, linear τ has $T = 1$.

Let $\widehat{A}, \widehat{A_E}, \widehat{B}$ be the geometric Pin preimages of A, A_E, B over $\overline{\mathbb{F}}_q$. Let K, K_E be the product kernels at Δ on $\widehat{A} \times \widehat{B}, \widehat{A}_E \times \widehat{B}$. Thus, by [137, Theorem 24.56 and §24.i], each spin lift over M is ub with $(u, b) \in K$ and $\deg u = \deg b \bmod 2$. The only other pair for ub is $(-u, -b)$. Since $M \le \Omega(V)$, both lifts $ub, -ub$ are rational. The spin preimage $\widetilde{P}$ has order $2|P|$. Thus, P_0 is the unique Sylow subgroup of $\widetilde{P}$, and $\widetilde{M} = \mathrm{N}_{\mathrm{Spin}(V)}(P_0)$.

For a K-representation π with common sign $z = \pm 1$ at $(-1, 1)$ and $(1, -1)$, we choose c with $c^2 = z$. Put $\pi^{\mathrm{spin}}(ub) = c^i \pi(u, b)$, where $i = \deg u = \deg b$ in $\{0, 1\}$. Since $z^2 = 1$, the pairs (u, b) and $(-u, -b)$ give equal matrices. Given $u'b'$ of parity j, the Clifford product has scalar $(-1)^{ij}$ from exchanging b, u' [137, Theorem 24.56]. Thus, the sign $z^{ij} = c^{i+j-(i+j \bmod 2)}$ gives multiplicativity of π^{spin}. Multiplication by c^{-i} gives the inverse construction. Since c^i occurs on both K, K_E, the Hom spaces on K_E and on the spin flag subgroup $\widetilde{M_E}$ coincide. Thus, the formula for $\widehat{A}_E \times \widehat{B}$ gives the spin formula at sign z. Given a coefficient automorphism σ, the twist $(c^\sigma/c)^i$ occurs on both K, K_E. If $\tau(1)$ is prime to p, the parent degrees are prime to p since A, A_E have p'-order and the covers have kernel C_2. Thus, each constituent on $\widetilde{M}$ or $\widetilde{M_E}$ has p'-degree by Clifford's theorem [95, Theorem 6.2]. □

Lemma 6.228. *Let S_ρ, S_β be the stabilizers under $\mathrm{Irr}(\Delta)$, with a character λ acting via $\lambda \circ \delta$. Suppose that $\langle \rho|_{A_E}, \beta\rangle = 1$ and $S_\rho \cap T = S_\beta \cap T = S$. Moreover, assume $\langle \rho|_{A_E}, \beta\lambda\rangle = 0$ for $\lambda \in T \setminus S$. Then, the restriction matrix $\mathcal{B}$ on M, M_E is a permutation matrix. Similarly, on $\widetilde{M}, \widetilde{M_E}$, the matrix $\widetilde{\mathcal{B}}$ for common z is a permutation matrix.*

Proof. Let $i : \Theta \to \Pi$ be a nonzero parent map. Thus, i is injective and spans $\mathrm{Hom}_{A_E \times B}(\Theta, \Pi)$ or the Pin analogue on $\widehat{A}_E \times \widehat{B}$. Given K, K_E, Proposition 6.227 gives that $\mathrm{End}_K(\Pi)$, $\mathrm{End}_{K_E}(\Theta)$ and $\mathrm{Hom}_{K_E}(\Theta, \Pi)$ all have dimension $|S|$. The map $v \mapsto iv$ from $\mathrm{End}_{K_E}(\Theta)$ is injective. Moreover, for $u \mapsto ui$, a basis of $\mathrm{End}_K(\Pi)$ consists of invertible u_λ for the twists $\lambda \in S$. Since the flagged parent maps onto Δ, the nonzero maps $u_\lambda i$ lie in distinct spaces for λ. Thus, $u \mapsto ui$ and $v \mapsto iv$ are onto, and $ui = iv$ gives their isomorphism. For (π_j, θ_j), both blocks have size m_j. The map i gives a summand of type θ_j in $\pi_j|_{K_E}$. Thus, the matched coefficients contribute $\sum_j m_j^2 = |S|$ to $\dim \mathrm{Hom}_{K_E}(\Theta, \Pi)$. Therefore, each matched coefficient is 1, and all others are 0. Moreover, $i, u_\lambda i$ give the transported permutation for (E, Π, Θ) under conjugation or a coefficient automorphism σ. By Proposition 6.227, the spin subgroups $\widetilde{M}, \widetilde{M_E}$ give that permutation as well. □

With $q = r^f$ odd, we let p be an odd prime with $d = \mathrm{ord}_p(q) > 2$. Thus, we put $e = \mathrm{ord}_p(q^2)$ and $c = d/e$. Choose $0 < u < v$ so that $cu + cv$ has no carry in base p. Thus, with $a = du$ and $b = dv$, we set $n = a + b$, $w_+ = a/e$, $w_- = b/e$ and $m = w_+ + w_-$. Take t_0 in the natural dual $\mathrm{SO}^+_{2n}(q)$ with scalars $1, -1$ on split spaces of ranks a, b, respectively. Thus, for $X = \mathrm{Spin}^+_{2n}(q)$, we choose s to be the image of t_0 in X^*. By Zhang [208, Lemma 7.15], we have that s has type $I(a, b)$. The symbols $\Lambda_\pm$ for the labels $1, 1$ are nondegenerate. Moreover, we have that 1 is principal series on each $Y_\pm$, so $\Lambda_\pm$ are noncuspidal.

Proposition 6.229. *The constituents over $1, 1$ for $I(a, b)$ have p'-degree. Moreover, we have that $p \mid [X : LV_u]$ for each line or plane parabolic LV_u in* Zhang [208, Proposition 7.18]. *Both properties hold for unbounded n.*

Proof. Given $d \mid h$, the order formula for $\mathrm{SO}^+_{2h}(q)$ [40] gives its p-part as $(q^{2e} - 1)_p^{h/e}((h/e)!)_p$. Indeed, we have $\gcd(q^h + 1, p) = 1$, so $q^{2h} - 1$ can replace $q^h - 1$ in the p-part. The factor $q^{2i} - 1$ contributes when $e \mid i$, with valuation $v_p(q^{2e} - 1) + v_p(i/e)$. Thus, the connected centralizer index has p-part $\binom{m}{w_-}_p = 1$ by Kummer's theorem. The isogenies to X^* have 2-power kernels, so the same p-part holds in X^*.

Let $\widetilde{\mathbf{X}}$ be a regular overgroup of $\mathbf{X}$ and let $Z = \ker \pi$ for the dual map π. By Lang's theorem [58, Lemma 4.2.13(i)], s has a rational lift $\widetilde{s}$, with connected centralizer $\widetilde{\mathbf{C}} = \mathrm{C}_{\widetilde{\mathbf{X}}^*}(\widetilde{s})$. Next, we put $\mathbf{C}^\circ = \mathrm{C}_{\mathbf{X}^*}(s)^\circ$. Given $y \in \pi^{-1}(\mathbf{C}^\circ)$, $[\widetilde{s}, y] \in Z[2]$, since $\widetilde{s}^2 \in Z$. The inverse image $\pi^{-1}(\mathbf{C}^\circ)$ is connected. Thus, $[\widetilde{s}, \pi^{-1}(\mathbf{C}^\circ)] = 1$, and $\pi(\widetilde{\mathbf{C}}) = \mathbf{C}^\circ$. The two orders on F-fixed points gain the factor $|Z^F|$. Thus, by [58, Proposition 11.5.6], the parent ρ has p'-degree for the trivial label of $\widetilde{\mathbf{C}}$. By [95, Theorem 6.2], we find equal degrees in $\rho|_X$, each dividing $\rho(1)$.

Next, we set $V = D \oplus D^*$ for the hyperbolic space on $D = \mathbb{F}_{q^d}^{u+v}$ over $\mathbb{F}_q$. Let x act on D by an element of order p and on D^* by the contragredient. Thus, $V^{\langle x \rangle} = 0$. Since $[\mathrm{O}(V) : G]$ is a 2-power, $x \in G$. For a Sylow p-subgroup P containing x, we get that $V^P = 0$. Hence, this holds for every Sylow P. Since $d > 2$, both $\mathrm{GL}_1(q)$ and $\mathrm{GL}_2(q)$ have p'-order. Thus, by Lemma 6.226, no P-stable flags of dimension $1, 2$ occur even on the cover X. Consequently, $p \mid [X : LV_u]$ for both parabolic types. Finally, choose $u = p^j$ and $v = p^{j+1}$. Since $c \in \{1, 2\}$ and $c < p$, the digits of cu, cv are disjoint for all $j \geq 0$. □

Proposition 6.230. *The two constituents χ_i of X over $1, 1$ factor through G. Moreover, each is the restriction via $X \to G$ of a representation over $\mathbb{Q}$ of $E = \mathrm{O}(V) \rtimes \langle F_r \rangle$, with F_r the split field generator of order f. The scalar at $-I$ on each extension is $\eta_q(-1)^b$, where η_q is the quadratic character on $\mathbb{F}_q^\times$. Moreover, a proper similitude g with nonsquare $\mu(g)$ exchanges χ_1, χ_2.*

Proof. Keep t_0 in the dual of $S = \mathrm{SO}(V)$. Thus, on the split torus T_S, its character θ has a trivial coordinates and b quadratic coordinates. In the regular inclusion $S \leq H = \mathrm{GSO}(V)$, choose a rational lift $\widetilde{t}_0$ of t_0 in the split dual torus T_H^*, with character $\widetilde{\theta}$ and parent ρ. By the comparison theorem [72, Theorem 3.2.7], $\langle \mathrm{Ind}_{B_H}^H \widetilde{\theta}, \rho \rangle = 1$, where B_H is a split Borel in H. Indeed, in the centralizer $\mathrm{C}_{H^*}(\widetilde{t}_0)$, the split Borel induction contains 1 with multiplicity 1.

The dual centralizer of t_0 has component C_2. Moreover, $r_+ r_-$ represents its nonidentity component, with reflections $r_\pm$ on $Y_\pm$. By Proposition 6.229, the map $C_2 \to Z[2]$ is injective by $\pi(\mathrm{C}_{H^*}(\widetilde{t}_0)) = \mathrm{C}_{S^*}(t_0)^\circ$. Since C_2 fixes 1, exactly two multiplier twists fix $\mathcal{E}(H, \widetilde{t}_0)$ and ρ. By the cyclic quotient formulas for H/S, $\rho|_S$ has two constituents ξ_1, ξ_2. Both have multiplicity 1 and are exchanged by the nonsquare g. Any other parent $\rho\lambda$ over ξ_i has a different series from ρ, so its multiplicity is 0 in $\mathrm{Ind}_{B_H}^H \widetilde{\theta}$. Since $H = SB_H$, Mackey restriction [95, Chapter 5] gives multiplicity 1 for each ξ_i in $\mathrm{Ind}_{B_S}^S \theta$.

We have $\mathcal{E}(S, t_0)\nu = \mathcal{E}(S, -t_0)$, disjoint because $a < b$. Thus, on G the Clifford constituents $\xi_1|_G, \xi_2|_G$ are irreducible and distinct. Moreover, $\xi_i\nu$ has multiplicity 0 in $\mathrm{Ind}_{B_S}^S \theta$, which lies in $\mathcal{E}(S, t_0)$. Since ν is onto on T_S, $S = GB_S$. Thus, Mackey restriction gives multiplicity 1 for $\xi_i|_G$ in the flag induction on G. Under $X \to G$, the dual image of t_0 is s. Thus, by [208, Proposition 7.18, Step 3], the lifts are χ_1, χ_2 over the labels $1, 1$. By Witt's theorem, $Y_\pm$ determine the rational class of t_0, with determinant adjusted by r_+ or r_-.

By [165, Theorem A(i),(ii)], the characters ξ_i are rational-valued and fixed by F_r. Put $\mathbf{S} = \mathrm{SO}(V_{\overline{\mathbb{F}}_q})$ and let $H = \mathbf{H}^F$. By [165, §2.4], $\mathbf{S}_{\mathrm{ad}}^F$ induces the inner-diagonal group on S. Since $\mathbf{H}/\mathrm{Z}(\mathbf{H}) = \mathbf{S}_{\mathrm{ad}}$ and $\mathrm{Z}(\mathbf{H})$ is connected, [114] gives $H \twoheadrightarrow \mathbf{S}_{\mathrm{ad}}^F$. Moreover, $|H : S\mathrm{Z}(H)| = 2$, as $\mu(cI) = c^2$. As g exchanges ξ_1, ξ_2, their inner-diagonal stabilizers are $\mathrm{Inn}(S)$. Thus, [165, Theorem A(iii)] gives $\xi_i^\gamma = \xi_i$. Let γ exchange the last hyperbolic pair in V, with $\det(\gamma) = -1$. Thus, we have that γ normalizes B_S and inverts the final coordinate of T_S. Give θ the value 1 at γ and F_r. Indeed, $\theta^\gamma = \theta = \theta^{F_r}$, since $\theta^2 = 1$, and $\gamma F_r = F_r \gamma$. After induction to E, we obtain a rational module V_θ. Since $\det(\gamma) = -1$ and $\nu(T_S) = \{\pm 1\}$, Mackey restriction gives $V_\theta|_G = \mathrm{Ind}_{B_S \cap G}^G \theta$. The rational idempotents $e_{\xi_i|_G}$ are fixed by E. Thus, each $e_{\xi_i|_G} V_\theta$ affords a rational extension by multiplicity 1 on G. Finally, $-I$ is central in E and belongs to T_S. Thus, its scalar is $\theta(-I) = \eta_q(-1)^b$, including when $-I \notin G$. □

Take $e, m, w_\pm$ as in Section 6.9. Put $t = 1$ if $q^e \equiv 1 \bmod p$, and $t = -1$ otherwise. Thus, we have that V has m blocks V_i of rank e and type t, with $t^{w_+} = t^{w_-} = 1$. The torus on each V_i is $T = C_{q^e - t}$. If $t = 1$, the block is hyperbolic on $\mathbb{F}_{q^e}$ and its dual. The simultaneous q-Frobenius F_q on both halves and the exchange ι generate $C \cong C_{2e}$ since e is odd. If $t = -1$, we have that the space is $\mathbb{F}_{q^{2e}}$ with quadratic form $\mathrm{Tr}_{\mathbb{F}_{q^e}/\mathbb{F}_q}(x^{q^e+1})$. Then, we have that T is the norm-one group and $C = \langle F_q \rangle$. In both cases $\det|_C$ is quadratic and nontrivial. Indeed, for $t = -1$, this follows from the $2e$-cycle of F_q on a normal basis.

Lemma 6.231. *Put $Q = (T_p)^m$ and $P = Q \rtimes Y$, where Y is a Sylow p-subgroup of $\mathfrak{S}_m$. Let J be the set of Y-orbits, $T' = T_{p'}$, and $D = T' \rtimes C$. Then, we have that P is Sylow in $\mathrm{O}(V)$, and $B/Q = D^J \rtimes \mathrm{N}_{\mathfrak{S}_m}(Y)$ for $B = \mathrm{N}_{\mathrm{O}(V)}(P)$. Moreover, we have that B acts transitively on the P-stable markings of w_- negative coordinates, with orbit size $\prod_j \binom{m_j}{(w_-)_j}$ where $m_j, (w_-)_j$ are base-p digits.*

Proof. By Proposition 6.229, P is Sylow. Given an abelian subgroup $A \leq P$, we write $\bar{A} \leq Y$ for its image. Thus, $A \cap Q \leq Q^{\bar{A}}$ by Lemma 6.122. Thus, on an orbit J_A of size $p^h > 1$, the diagonal base has size $\leq p^\ell = |T_p|$. On the orbit, $\bar{A}$ has a regular image of order p^h. Since $\ell + h < \ell p^h$, any A with $\bar{A} \neq 1$ has $|A| < |Q|$. Consequently, Q is characteristic in P.

The Q-weights on distinct V_i have distinct kernels. On V_i, the weights are $\mathcal{W}_i \cup \mathcal{W}_i^{-1}$ for a Frobenius orbit $\mathcal{W}_i$, with $\mathcal{W}_i = \mathcal{W}_i^{-1}$ if $t = -1$. Thus, $\mathrm{N}_{\mathrm{O}(V)}(Q)$ permutes the V_i. On V_i the centralizer is T, and C gives the actions of F_q and inversion on the weights. Consequently, $\mathrm{N}_{\mathrm{O}(V)}(Q) = (T \rtimes C) \wr \mathfrak{S}_m$. Modulo Q, the subgroup P/Q is Y on the blocks. A tuple $(d_i) \in D^m$ can normalize Y only if $d_{y(i)} = d_i$ for $y \in Y$, and its permutation must belong to $\mathrm{N}_{\mathfrak{S}_m}(Y)$. Thus, B/Q has the stated form. Choose Y with m_j transitive orbits of size p^j by the Sylow structure of $\mathfrak{S}_m$. A fixed marking unites Y-orbits. Since $m_j < p$, it marks exactly $(w_-)_j$ orbits of size p^j. The no-carry condition ensures $(w_-)_j \le m_j$. Thus, the permutations of the m_j orbits give one B-orbit of the stated p'-length. □

Fix a marking, and let I be its stabilizer in B/Q. Let θ_- be quadratic on the marked T'-coordinates and trivial on T' at positive coordinates. Since C fixes the quadratic character, we extend θ_- with value 1 on C^J and the permutations in I. Let α_- be the product of $\det|_C$ on the marked factors. Given $i = 0, 1$, we put $\Pi_i = \mathrm{Ind}_I^{B/Q}(\theta_- \alpha_-^i)$. By Clifford theory (Isaacs [95, Theorem 6.11]), Π_i is irreducible because I is the inertia group of $\theta_-|_{(T')^J}$. Thus, $\Pi_i(1) = [B/Q : I]$ is given by Lemma 6.231. Moreover, Π_i has a signed permutation model over $\mathbb{Q}$. P acts trivially.

On $\mathrm{N}_{\mathrm{O}(V)}(Q)$, induce from the stabilizer of w_- negative blocks with their weights on $T \rtimes C$, with marking basis $\mathcal{X}$. Then, we have that P permutes $\mathcal{X}$ without signs, and $\mathcal{X}^P$ spans a B-submodule. A diagonal $d \in D$ on an orbit of length p^j contributes its sign to the power p^j, so the submodule affords Π_i. By Broué's formula,[2] at P the Brauer quotient in characteristic p has basis $\mathcal{X}^P$.

Proposition 6.232. *Put $M_0 = \mathrm{N}_{\Omega(V)}(P)$. Then, $\psi_i = \Pi_i|_{M_0}$ are distinct members of $\mathrm{Irr}(M_0)$ and are rational. Moreover, h is a proper similitude with satisfies $[h, P] = 1$ and $\psi_0^h = \psi_1$. The scalar at $-I$ on Π_i is $\eta_q(-1)^b$. Consequently, the inflations to $\mathrm{N}_X(\widetilde{P})$ are distinct, where $\widetilde{P}$ is the Sylow lift of P.*

Proof. By Lemma 6.226, $B/M_0 = \Delta$. The spin preimage of T is connected on V_i. Thus, by [58, Propositions 4.2.10 and 4.2.11], the character $\nu|_T$ is quadratic and nontrivial. On T', the character $\nu|_{T'}$ is nontrivial and $(\nu|_{T'})^{p^j} = \nu|_{T'}$ for a diagonal orbit. Thus, $\nu, \det\nu$ exchange the unequal counts (w_+, w_-). The twist det gives a factor -1 at $c_0 \in C$ generating C on each Y-orbit. Consequently, the twist stabilizer of Π_i is 1. By Proposition 6.227, $\Pi_i|_{M_0}$ is irreducible. The weights $\theta_-, \theta_-\alpha_-$ differ only on C for marked factors of I. Since $w_+, w_- > 0$, neither 1 nor det exchanges Π_0, Π_1. Thus, ψ_0, ψ_1 are distinct. The unordered counts $\{w_+, w_-\}$ give the sign ranks $\{ew_+, ew_-\}$, so distinct rank pairs at M_0 have disjoint images.

Fix a nonsquare $\mu \in \mathbb{F}_q^\times$ and take h with identical matrices on the blocks V_i. If $t = 1$, let $h(x, y) = (\mu x, y)$. If $t = -1$, let h multiply by z with $z^{q^e+1} = \mu$. Then, $[h, P] = 1$, and each block has determinant μ^e. If $t = 1$, the element $h\iota h^{-1}\iota^{-1}$ has quadratic value $\eta_{q^e}(\mu) = -1$ since e is odd, and $hF_q = F_q h$. If $t = -1$, the quadratic value of $hF_q h^{-1}F_q^{-1}$ is $(z^{1-q})^{(q^e+1)/2} = \mu^{(1-q)/2} = -1$. Thus, $\Pi_0^h = \Pi_1$. At $-I$, the scalar on either parent is $(-1)^{(q^e-t)w_-/2} = t^{w_-}\eta_q(-1)^{ew_-} = \eta_q(-1)^b$. Thus, $\mathrm{N}_X(\widetilde{P}) \to M_0$ is onto by Proposition 6.227. □

Theorem 6.233. *Given s of* Section 6.9*, the constituents χ_i over $1, 1$ and the local ψ_i have an $\mathrm{Aut}(X)_P \times \mathcal{H}_p$-equivariant pairing. Moreover, (χ_i, ψ_i) satisfies $\succeq_c$ of* Definition 2.1 *on $M = \mathrm{N}_X(\widetilde{P})$.*

Proof. Let F_0 be the r-power map on V_i. Thus, the trace forms on V_i make F_0 a semilinear isometry. Transport F_0 to the standard split coordinates of V. Then, $\mathrm{N}_E(P) = B\langle F_0\rangle$ for $E = \mathrm{O}(V) \rtimes \langle F_r\rangle$. The element $v_0 = F_0^f$ acts by F_q on every block. The map F_0 acts on T by r-th powers, fixes C and fixes the block permutations in Y. Thus, on the coset basis of Π_i, we have that F_0 acts by I. If $t = 1$, each F_q in v_0 has determinant 1. If $t = -1$, it has determinant -1 on each block, but $w_- = b/e$ is even. Thus, the scalar of v_0 is 1 on both Π_i, giving ordinary extensions to $\mathrm{N}_E(P)$.

Take $\omega(z) = \eta_q(z)^b$ on scalars zI. Since $\omega^{F_r} = \omega$ and the isometry intersection is $\{I, -I\}$, the models for Π_i, χ_i have scalar $\omega(-1)$ at $-I$ by Proposition 6.230. Thus, over $\mathbb{Q}$ the global model extends to $\mathbb{F}_q^\times E$ and the local model to $\mathbb{F}_q^\times \mathrm{N}_E(P)$. Given $n \ge 9$, the semilinear quotient by scalars zI induces $\mathrm{Aut}(X)$ by [79, Theorem 2.5.1]. Thus, $\mathrm{GSO}(V)$ induces $\mathrm{InnDiag}(X)$, and an isometry g with $\det(g) = -1$ represents the graph coset. Moreover, F_r generates the field group of order f. The actions lift uniquely to X. For square μ, each (χ_i, ψ_i) is fixed. Thus, by Proposition 6.232, $(\chi_1, \psi_1) = (\chi_0^h, \psi_0^h)$ for nonsquare $\mu(h)$.

Now write P for the Sylow lift and $M = \mathrm{N}_X(P)$. Then, $\Gamma = \mathrm{Aut}(X)_{P,\chi_i} = \mathrm{Aut}(X)_{P,\psi_i}$. Choose representatives s_γ of square multiplier that normalize $\bar{P}$, with $s_1 = 1$ and $\gamma \in \Gamma$. $\bar{x}$ denotes the natural image of $x \in X$. Thus, evaluating the ordinary models at $\bar{x}s_\gamma$ and $\bar{m}s_\gamma$ for $m \in M$ gives projective models $\mathcal{P}, \mathcal{P}'$. Their common factor is

[2] M. Broué, *On Scott modules and p-permutation modules*, Proc. Amer. Math. Soc. **93** (1985), 401–408, p. 402.

$\alpha(\gamma,\gamma') = \omega(s_\gamma s_{\gamma'} s_{\gamma\gamma'}^{-1})$ on the scalar coordinate. If (x,γ) centralizes X, $x \in M$ and $\bar{x}s_\gamma = zI$. Thus, the common scalar is $\omega(z)$. The rational matrices give coefficient function 1 for $\sigma \in \mathcal{H}_p$. Thus, each (a,σ) in the joint stabilizer of (χ_i,ψ_i) has the inner factor determined by α in both models. Hence, $\mu_{a,\sigma} = \mu'_{a,\sigma}$, and both pairs satisfy $\succeq_c$ with scalar 1 on the spin kernel. □

Lemma 6.234. *Let $\Lambda_\pm$ be p'-degree labels on $Y_\pm$ of* Section 6.9. *With $k = \mathrm{ord}_p(r)$, we put $K = \mathbb{Q}(\zeta_{p-1})$. Then, we find $k \mid p-1$. If d is odd, $e = d$ is an odd divisor of $p-1$, so $2e \mid p-1$. Otherwise, we have that $2e = d \mid p-1$. Thus, we have that $\mathcal{H}_p$ fixes K pointwise. Let Y_j be a transitive Sylow p-subgroup on p^j letters and set $K_j = \mathrm{N}_{\mathfrak{S}_{p^j}}(Y_j)$, with $K_0 = 1$. Then, $B/Q = \prod_{j\geq 0}\big((D \times K_j) \wr \mathfrak{S}_{m_j}\big)$, $K_j/[Y_j,Y_j] \cong (\mathbb{F}_p \rtimes \mathbb{F}_p^\times)^j$. The p'-degree irreducibles of K_j are precisely those of $K_j/[Y_j,Y_j]$. Moreover, the models over K are indexed by $x \in \mathbb{F}_p^j$, with the trivial model at $x = 0$.*

Proof. By Lemma 6.231, permutations of the m_j equal-length orbits give $\mathfrak{S}_{m_j}$. On an orbit of size p^j, the internal normalizer K_j fixes the constant coordinate D. Thus, the full factor at j is $(D \times K_j) \wr \mathfrak{S}_{m_j}$. The quotient $K_j/[Y_j,Y_j]$ is a product of j affine groups by Giannelli.[3] By [95, Theorem 6.2], a p'-degree irreducible of K_j lies above linear characters of Y_j, so it is trivial on $[Y_j,Y_j]$. For $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$, the linear characters of $\mathbb{F}_p^\times$ give $p-1$ models over K. The augmentation module U of $\mathbb{Q}[\mathbb{F}_p]$ has the nontrivial weights of $\mathbb{F}_p$ in one free $\mathbb{F}_p^\times$-orbit. Thus, each affine factor has p models of p'-degree. Choose the index $0 \in \mathbb{F}_p$ for 1. Then, external products give κ_x for $x \in \mathbb{F}_p^j$. □

Proposition 6.235. *For fixed runner conventions, $\Lambda_\pm$ with ordered rows of p'-degree correspond bijectively to parents of B/Q of p'-degree with colours $1, \eta_{T'}$ and digit totals $(w_\pm)_j$. Moreover, row exchange on both $\Lambda_\pm$ gives the twist by* det. *In particular, for $1,1$ with positive label P_+ fixed, the parents are Π_i of* Section 6.9.

Proof. Let $w = w_\pm$ on the chosen sign space $Y_\pm$. By Lemma 6.59, p'-degree gives weight w and p'-degree for the wreath label ρ_w. Thus, the packed core has rank 0 and equal rows. If $t = 1$, equal row lengths for $\Lambda_\pm$ give defect 0. If $t = -1$, the original defect can differ from 0. Take row length $L \equiv 1 \bmod 2e$ in the packed symbol. Then, $2e$-shifts fix each quotient λ_i. Take indices $i + eh$ with $0 \leq i < e$, $h = 0,1$ and the alternating assignment for $t = -1$ in Lemma 6.59. Then, row exchange gives $i + eh \mapsto i + e(h+1) \bmod 2e$. The lengths and quotients λ_i determine $\Lambda_\pm$ up to shift.

Given any λ_i, we choose the p-core tower of charge 0 with descendants indexed by $\mathbb{F}_p$. Let s_j be the total size at depth j across the runners, so $w = \sum_j p^j s_j$. Let H be the hook product for λ_i. Thus, the wreath degree is $d_w = w!/H$. By Macdonald's formula[4] and Legendre's formula, $v_p(d_w) = \frac{\sum_j s_j - \sum_j w_j}{p-1}$, where w_j are the digits of w. Indeed, for runner sizes n_i, the multinomial $w!/\prod_i n_i!$ contributes $(\sum_i s_p(n_i) - s_p(w))/(p-1)$, with s_p the digit sum. Thus, $p \nmid d_w$ exactly when $s_j = w_j$ for each j. Conversely, a vertex partition $\lambda_{j,i,x}$ of size at most $w_j < p$ is a p-core. The tower reconstructs λ_i from its cores and quotients.

Let $\lambda^\pm_{j,i,x}$ denote the partition at depth j, with whole runner index $i \in \mathbb{Z}/(2e)$ and vertex $x \in \mathbb{F}_p^j$. Moreover, let $\widetilde{\theta}_\pm$ be the extensions of $\theta_+ = 1, \theta_- = \eta_{T'}$ with value 1 on C. Write ξ_i for the cyclic character of C of index i. At depth j, attach $\lambda^\pm_{j,i,x}$ to $(\widetilde{\theta}_\pm \xi_i) \boxtimes \kappa_x$ of $D \times K_j$, with κ_x from Lemma 6.234. Since $w_+ + w_-$ has no carry, at depth j the sizes total m_j. Write U_j for the base tensor. Thus, U_j extends by the equal-slot permutations π. By [95, Theorem 6.11 and Corollary 6.17], the multiplicity modules are $S^{\lambda^\pm_{j,i,x}}$. Thus, the induced ρ_j recovers the partitions $\lambda^\pm_{j,i,x}$ up to slot permutation. Conversely, over $\theta_\pm$ the D-labels are $\widetilde{\theta}_\pm \xi_i$ and the K_j-labels have p'-degree. Since $m_j < p$, the degrees $S^{\lambda^\pm_{j,i,x}}(1)$ and the permutation indices divide $m_j!$. Thus, $\boxtimes_j \rho_j$ gives all the parents with totals $(w_\pm)_j$, with models over K.

Permutations of the blocks V_i have determinant 1. For a constant rotation $c_0 \in C$ on p^j blocks, $\det(c_0)^{p^j} = \det(c_0)$. Thus, the determinant twist acts by $i \mapsto i + e \bmod 2e$. The trivial symbol $(\{ew\},\{0\})$ has row (w) on runner 0. If $t = -1$, the parity $2 \mid w$ gives phase 0 as well. For the row (w), its beta set replaces the bead -1 by $w-1$. Thus, its quotient has one row $(\lfloor w/p \rfloor)$ if nonempty. At depth j, the only row is (w_j) if $w_j > 0$. Translate its vertex x to 0, uniformly over the cyclic indices i. Thus, the trivial orientation has $\kappa_0 = 1$ at each vertex and $S^{(w_j)} = 1$. Negative row exchange shifts the indices i by e. Consequently, the two orientations give Π_0, Π_1. □

[3] E. Giannelli, *McKay bijections for symmetric and alternating groups*, Algebra Number Theory **15** (2021), 1809–1835, p. 1815, formula $(*)$.

[4] I. G. Macdonald, *On the degrees of the irreducible representations of symmetric groups*, Bull. London Math. Soc. **3** (1971), 189–192.

Theorem 6.236. *In* Section 6.9*, an eligible pair* (Λ_+, Λ_-) *with unequal rows gives* χ_0, χ_1*. Thus, the orientations give an* $\mathrm{Aut}(X)_P \times \mathcal{H}_p$*-equivariant pairing with* ψ_i *on* $M = \mathrm{N}_X(P)$*. Given* $z \in \mathrm{Z}(X)$*, we have that* $\chi_i(z)/\chi_i(1) = \psi_i(z)/\psi_i(1)$*. Distinct* (Λ_+, Λ_-) *give disjoint images at* M*. Moreover, we have that* ψ_i *has an ordinary extension fixed by* $\mathcal{H}_p$ *on* $N = \mathbb{F}_q^\times \mathrm{N}_E(P)$*, via* $M \to M_0$*.*

Proof. On each nondegenerate $\Lambda_\pm$, translation by e exchanges its two arrays. Thus, det translates both arrays by e, by Proposition 6.235. The twists $\nu, \det\nu$ exchanges the unequal counts w_+, w_-. Thus, a parent ρ has trivial stabilizer in $\mathrm{Irr}(\Delta)$. By Proposition 6.232, the restriction $\rho|_{M_0}$ is irreducible, and ρ, ρ' have equal restrictions exactly when $\rho' = \rho$ or $\rho' = \rho\det$. Consequently, the four orientations give ψ_0, ψ_1. By Proposition 6.235, either ψ_i recovers the two original symbols $\Lambda_\pm$. Moreover, h adds e to the marked indices i and fixes κ_x and S^λ, by Proposition 6.232. Thus, $\psi_0^h = \psi_1$. At the toral element $-I$, both parents have scalar $\eta_q(-1)^b$.

The map F_0 fixes the operators for $C, K_j, \mathfrak{S}_{m_j}$ and its r-power on T' fixes $1, \eta_{T'}$. Thus, the identity implements F_0. The element $v_0 = F_0^f$ is a rotation of order dividing d on every block, so its scalar z belongs to μ_d. Indeed, the product for v_0 of all cyclic values $\xi_i(F_q)$ is constant across all cosets. Since $d = k/\gcd(k, f)$, the map $A \mapsto A^f$ has image $\{A^f : A \in \mu_k\} = \mu_d$. In μ_k, we choose A with $A^f = z$ as the scalar of F_0. Thus, over K, the extension is $\mathcal{H}_p$-fixed. By Theorem 6.233, we extend at $z'I$ by $\omega(z') = \eta_q(z')^b$.

The simultaneous graph fixes the nondegenerate $\Lambda_\pm$. Thus, for $S \le H$, by Proposition 6.230, we obtain two constituents ξ_i exchanged by h. The character $\xi_i\nu$ lies in the series of $-t_0$. Thus, $\xi_i|_{\Omega(V)}$ are distinct and irreducible. The inflations χ_i exhaust these labels on X by [208, Proposition 7.18, Step 3]. By [165, Theorem A], ξ_i are rational with $\xi_i^\gamma = \xi_i^{F_r} = \xi_i$. Moreover, their scalar at $-I$ depends on t_0 [72, Theorem 3.3.12] and is $\eta_q(-1)^b$ by Proposition 6.230. Thus, pair (χ_0, ψ_0) and transport by h. Consequently, the two pairs (χ_i, ψ_i) are equivariant and the extensions at N apply for both defects of $\Lambda_\pm$. □

Lemma 6.237. *The ordinary extensions of* χ_i *in* Theorem 6.236 *need an operator* F_r*. We apply Harish–Chandra induction at* t_0*, including at equal ranks of* Y_+, Y_-*. Thus, only the final* χ_i *is subject to the* p'*-degree test. Suppose that* $\Lambda_\pm$ *are nondegenerate and have rank* > 0*. An ordinary* 1*-hook with nondegenerate result* Λ' *gives a permutation of multiplicity* 1 *on the two constituents of* $\mathrm{SO}(V)$*. Thus, a source extension on* $E = \mathrm{O}(V) \rtimes \langle F_r\rangle$ *gives an extension at* (Λ_+, Λ_-)*. In particular, if both defects are* 0*, the extensions are over* $\mathbb{Q}$ *for every* f*.*

Proof. First, take $H = \mathrm{GSO}(V)$, with $S = \mathrm{SO}(V)$. The split line Levi is $\mathrm{GL}_1(q) \times H'$. Its element (x, h') acts by $\mu(h')/x$ on E'. Thus, on the Levi of S put $\gamma = 1$ or η_q according to the sign of E. By [208, Lemma 7.17(i)], the coefficient of the 1-hook in the connected centralizer is 1. Thus, the regular parents have coefficient 1 by [72, §4.7]. Only $1, \lambda$ from r_+r_- fix the parent labels $\Lambda_\pm$ in the series of t_0. The marked space has residual rank > 0, so the reflections $r_\pm$ lie off $E \oplus E'$. For the dual representative u of r_+r_-, $[\widetilde{t}_0, u] = [\widetilde{t}_0', u]$ before and after the hook. If $Y_\pm$ have equal ranks, exchange acts by $t_0 \mapsto -t_0$, so it adds no component to $\mathrm{C}_{S^*}(t_0)$. By [208, Lemma 7.16(a)], the two twists $1, \lambda$ contribute 1 each and the other contributions are 0. The nonsquare multiplier h exchanges both pairs ξ_i, ξ_i'. Thus, $\mathcal{B}$ has entry sum 2 and $\mathcal{B}_{ij} = \mathcal{B}_{3-i,3-j}$ for $i, j \in \{1, 2\}$, so $\mathcal{B}$ is a permutation matrix.

Given $\Lambda = (A, B)$ with $A \ne B$ and $|A| = |B|$, by [208, Lemma 7.17(i)], we obtain a 1-hook when $\mathrm{rk}(\Lambda) \ge 2$. By [72, Theorem 4.4.28], defect 0 has only the cusp of rank 0 with equal rows. Thus, at rank 1 the endpoint is $(\{1\}, \{0\})$. At the toral endpoints, V_θ of Proposition 6.230 has both S-constituents with multiplicity 1, also when both ranks are 1. Consequently, V_θ gives rational extensions $\widehat{\xi_i}$.

For a hook addition, tensor with γ on E and inflate across V_u. The opposite lines E, E' are over $\mathbb{F}_r$. Since the parabolic contains F_r and an odd residual isometry r_0, the restriction to S has one double coset by [95, Problem 5.6]. By [165, Theorem A], ξ is rational and fixed by F_r and the graph, since h exchanges ξ_1, ξ_2. Thus, e_ξ projects onto a summand of type ξ, giving the ordinary extension, over $\mathbb{Q}$ if the source is over $\mathbb{Q}$. If the source is $\mathcal{H}_p$-fixed, the induced character Ξ is fixed and its constituent over ξ is unique. Thus, the scalar at $-I$ gains $\gamma(-1)$ from E. □

Lemma 6.238. *At positive defect* $2h$*, we have that the cuspidal endpoints* Λ_0 *in* Lemma 6.237 *have ordinary* $\mathcal{H}_p$*-fixed extensions to* E *for eligible* d, f*.*

Proof. For absolute defect $2h > 0$, the ordinary hook core is Λ_0 of rank h^2 by [72, Theorem 4.4.28]. Since $Y_\pm$ are split, $2 \mid h$. At defect 0, keep the toral endpoint of rank 1. If both endpoints are cuspidal, $\mathbf{C} = \mathrm{C}_{\mathbf{H}^*}(\widetilde{t}_0)$ has two semisimple factors modulo $\mathrm{Z}(\mathbf{H}^*)$. For a proper split Levi $\mathbf{L}^*$ containing $\widetilde{t}_0$, its torus lies in $\mathbf{C}$ and is noncentral in $\mathbf{H}^*$. Thus,

$\mathbf{L}^* \cap \mathbf{C}$ is proper in $\mathbf{C}$ since $\mathrm{Z}(\mathbf{C})^\circ = \mathrm{Z}(\mathbf{H}^*)^\circ$. By [72, §4.7], cuspidality on $\mathbf{C}$ give zero Harish–Chandra restriction. Since V_u is the unipotent radical in H, S, the constituents ξ_i of S are cuspidal. By [165, Theorem A], $\xi_i^\gamma = \xi_i$ for the graph γ, so ξ_i extends across $\mathrm{O}(V)/S \cong C_2$. By [165, Theorem 10.8], the extensions Θ_i are rational-valued and fixed by F_r.

Next, at a toral endpoint take γ on E. For the residual cusp ξ_0, we put $\beta_0 = \xi_0$ or $\nu\xi_0$ at eigenvalue 1 or -1, respectively. By [165, Theorem 10.8], ξ_0 has a rational-valued extension Θ_0 to $\mathrm{O}(V')$ fixed by F_r. Moreover, ν extends to $\mathrm{O}(V')$. The regular cover coefficient is 1. Since the residual lift s' is central, the S-Levi source is just $\beta = \gamma \boxtimes \beta_0$. Thus, at β, the two coefficients sum to 2 by [208, Lemma 7.16(a)]. Since h exchanges ξ_1, ξ_2 and fixes β, the coefficients are $1, 1$. Consequently, the field extension of β_0 gives extensions of ξ_1, ξ_2 by parabolic induction. Given a toral $Y_\pm$, set $\Theta = \Theta_0$ on $\mathrm{O}(V')$. Otherwise, set $\Theta = \Theta_i$ on $\mathrm{O}(V)$. If $2 \nmid f$, [208, Lemma 3.2] gives an extension of Θ over C_f with values in $\mathbb{Q}$.

Suppose now that $2 \mid f$. Then, positive defect gives $2 \mid d$ by Proposition 6.235. Thus, $S_\Theta = \mathrm{SO}(V')$ or $\mathrm{SO}(V)$, respectively. By [165, proof of Theorem 10.8], a representative N of the wave-front class gives Γ_N of multiplicity 1 at $\Theta|_{S_\Theta}$. By [165, Proposition 10.4], the partition of N has only odd parts. Thus, N has even grading. Consequently, $\Gamma_N = \mathrm{Ind}_U^{S_\Theta} \theta_N$, with θ_N linear by [208, Lemma 7.23]. By reciprocity on S_Θ, $\dim \mathrm{Hom}_U(\theta_N, \Theta) = 1$.

On a block of length $2h+1$, we set $Nx_i = x_{i+1}$, $Nx_{2h} = 0$, with antidiagonal pairings of x_i, x_{2h-i}. Thus, tensor with M_h, with a reference diagonal form over $\mathbb{F}_r$. To change the discriminant on M_h, choose a reference line $\mathbb{F}_r x$ with Frobenius $-F_q$. If $z^q = -z$, its fixed basis is zx. Thus, the discriminant changes by $z^2 \in \mathbb{F}_q^\times$, a nonsquare. By [208, Lemma 7.23], these involutions on M_h give the forms on M_h. Thus, their product w acts trivially on x_i. On the wF_q-fixed space, $F_0 = F_r$ stabilizes (U, θ_N) and $F_0^f = w^{-1}$. Thus, isometric transport of F_0 gives the field action for Θ.

By [95, Corollary 11.22], Θ extends across the quotient C_f. On the θ_N-line, the eigenvalue A_0 of F_0 has $A_0^f = \pm 1$, the scalar of w^{-1}. Since $2 \mid d$ and $(\mu_k)^f = \mu_d$, we choose $A \in \mu_k$ with $A^f = A_0^f$. Thus, twist by λ with $\lambda(F_0) = A/A_0$. Given $\sigma \in \mathcal{H}_p$, we write $\zeta_r^\sigma = \zeta_r^l$. The isometry $d_l : x_i \mapsto l^{i-h} x_i$ satisfies $d_l w = w d_l$, $d_l F_0 = F_0 d_l$ and $d_l N d_l^{-1} = lN$. Thus, d_l or d_l^{-1} carries the θ_N-line to the θ_N^σ-line by the trace formula. There F_0 still has scalar A, and the conjugate model has scalar $A^\sigma = A$. By [95, Corollary 6.17], the twist $\kappa \in \mathrm{Irr}(C_f)$ compares the extensions. Its value at F_0 is 1, so the extension is fixed by $\mathcal{H}_p$. In either case, ν extends with $F_r \mapsto 1$. Thus, choose γ on the toral line E. At each hook E_i, we extend by Lemma 6.237. □

Theorem 6.239. *Each pair (χ_i, ψ_i) of* Theorem 6.236 *satisfies $\succeq_c$ for every f. If each of w_+, w_- has an odd base-p digit, the pairs exhaust the p'-part of the rational series $I(a, b)$.*

Proof. Restrict the ordinary extension of χ_i on E via $X \to \Omega(V)$. Its scalar at $-I \in S$ is determined by t_0. Thus, it extends on zI by $\omega(z) = \eta_q(z)^b$. The local model of ψ_i has scalar $\omega(z)$ there. Thus, the section s_γ in Theorem 6.233 gives the common factor $\alpha(\gamma, \gamma') = \omega(s_\gamma s_{\gamma'} s_{\gamma\gamma'}^{-1})$ and common centralizer scalars $\omega(z)$. Since $\widehat{\chi}_i$ is fixed by $\mathcal{H}_p$, one intertwiner compares the models on $\mathbb{F}_q^\times E$. Thus, on s_γ the pure coefficient function is 1, and α gives $\mu_{a,\sigma} = \mu'_{a,\sigma}$ for every (a, σ) in the joint stabilizer of (χ_i, ψ_i).

Given a space of weight w, degeneracy means the quotients on runners $i, i+e$ agree, since the core has rank 0. Thus, each size at depth j occurs in pairs. By Proposition 6.235, depth j has total w_j, so $2 \mid w_j$. Conversely, if $2 \mid w_j$ for all j, put the row $(w/2)$ on runners $i, i+e$ and $\varnothing$ elsewhere. Its wreath degree is $\binom{w}{w/2}$. By Kummer's theorem, the even digits w_j give p'-degree. Thus, packing back gives degenerate Λ with $t^w = 1$, also at $t = -1$. Consequently, an odd digit of each $w_\pm$ excludes a degenerate $\Lambda_\pm$. By Proposition 6.235, the parents on B/Q are all obtained with colours $1, \eta_{T'}$ and digit totals $(w_\pm)_j$. For unbounded n, take disjoint digit supports for w_+, w_- with unit digits, choosing $2 \mid w_\pm$ when $d = 2e$. Thus, $p^j w_\pm$ gives examples for all $j \geq 0$. □

Theorem 6.240. *Suppose that a symbol Λ in* Section 6.9 *is degenerate. Choose $\Lambda_- = \Lambda$ on Y_-, replacing t_0 by $-t_0$ if needed. Thus, the character on S changes by ν, with $\nu|_{\Omega(V)} = 1$. Let b' be the rank of Y_- and $w = b'/e$. By* Theorem 6.239*, all digits w_j are even, so b' is even. The ranks of Y_+, Y_- are distinct. For $d > 2$ in* Section 6.9*, we have that the p'-part of $I(a, b)$ has an $\mathcal{H}_p$-equivariant bijection onto the sector at $M = \mathrm{N}_X(P)$ with colours $1, \eta_{T'}$ and equal central scalars at $z \in \mathrm{Z}(X)$. If one of $\Lambda_\pm$ is degenerate, the count is 1. If both are degenerate, it is 2.*

Proof. On the split torus of $H = \mathrm{GSO}(V)$, choose the character $\widetilde{\theta} = \eta_q(\prod_{i \in Y_-} z_i)$ with multiplier value 1, with hyperbolic coordinates z_i on Y_-. Thus, $\widetilde{\theta}(zI) = \eta_q(z)^{b'} = 1$, and the dual lift $\widetilde{t}_0$ has order 2. The connected labels

$\Lambda_\pm^\pm$ are rational-valued. By Jordan equivariance, $\rho^\sigma = \rho$ for every coefficient automorphism σ. On $S = \mathrm{SO}(V)$, [165, Theorem A(i)] gives rationality of $\rho|_S$.

The nonidentity component $\gamma_+\gamma_-$ exchanges the negative signs Λ_-^+, Λ_-^-. Thus, the multiplier twist of ρ is distinct, and $\rho|_S$ is irreducible. Its orbit under $\gamma_+\gamma_-$ determines the restriction. Since $\dim Y_+ \neq \dim Y_-$, ν pairs the disjoint series at $t_0, -t_0$. Thus, $\rho|_{\Omega(V)}$ is irreducible, and distinct restrictions to S stay distinct on $\Omega(V)$. By [208, Lemma 7.15 and Propositions 7.18 and 7.20], the spin component acts by $\gamma_+\gamma_-$. Moreover, $t_0, -t_0$ give twists by ν, with $\nu|_{\Omega(V)} = 1$. After inflation to X, we obtain one χ for one degenerate symbol and two χ_i for two degenerate symbols.

Let $V_\pm$ be the models without $1, \eta_{T'}$ in Proposition 6.235. Write $\epsilon_\pm$ for their determinant characters. If $\Lambda_\pm$ is degenerate, $V_\pm \otimes \epsilon_\pm \cong V_\pm$. On $V_\pm$, act by $(i,x) \mapsto (i+e,x)$. The affine labels κ_x and partitions $\lambda_{j,i,x}$ agree at the two indices. Thus, $(i,x) \mapsto (i+e,x)$ acts on the tensors and on $S^{\lambda_{j,i,x}}$. Thus, the operator $J_\pm$ has $J_\pm^2 = I$ and $J_\pm\rho(g) = \epsilon_\pm(g)\rho(g)J_\pm$, with matrices over K. The row action from [208, Lemma 7.19] translates i by e.

Since $w_+ \neq w_-$, no twist containing ν is a self-twist. If Λ_- alone is degenerate, the twist det switches the rows of Λ_+. Thus, $\psi = \rho|_{M_0}$ is irreducible for both orders of Λ_+ by Proposition 6.232. If both $\Lambda_\pm$ are degenerate, the twist stabilizer $\langle \det \rangle$ gives two constituents of multiplicity 1. On the source module take $J_+ \otimes J_-$. Thus, on functions $f(g)$ the determinant operator is $Jf(g) = \det(g)(J_+ \otimes J_-)f(g)$. Its two eigenspaces give the two ψ_i over K. An odd rotation r_o with $\det(r_o) = -1$ exchanges the spaces. Thus, ψ_i is fixed by $\mathcal{H}_p$ and has the central scalar of χ_i. Its inflation is trivial on z_0. Moreover, the tower inverse recovers $\Lambda_\pm$ modulo $\langle \det, \nu \rangle$. With Theorem 6.236, this gives the p'-sector at M and the series in X. □

Proposition 6.241. *Let $N^\sharp$ be the semilinear similitude normalizer of P. If one $\Lambda_\pm$ is degenerate, we have that ψ extends to $N^\sharp$. If both $\Lambda_\pm$ are degenerate, each ψ_i extends to the proper subgroup of $N^\sharp$. Moreover, the extensions are trivial at zI and fixed by $\mathcal{H}_p$.*

Proof. Let $B^\sharp = \mathrm{N}_{\mathrm{GO}(V)}(P)$. By Proposition 6.232, common block similitudes centralize P and have every multiplier μ. Thus, modulo Q, the torus $T^\sharp$ has p'-coordinates $(\mu, (z_o)_{o \in J})$. If $t = 1$, the block coordinates are $z_o, \mu/z_o$. If $t = -1$, $z_o^{q^e+1} = \mu$. Since $d > 2$, the multiplier group $\mathbb{F}_q^\times$ has p'-order. Given $x \in B^\sharp$, choose $g \in T^\sharp$ with multiplier $\mu(g) = \mu(x)$. Then, $g^{-1}x$ lies in B modulo Q. Thus, $B^\sharp/Q = T^\sharp \rtimes W$, with complement W from Lemma 6.231.

Write $J_- \subseteq J$ for the marked orbits. At size p^j, we choose $\delta_o = \pm 1$ in equal numbers on J_-, possible since $2 \mid w_j$. Then, $\widehat{\theta}(\mu, (z_o)) = \eta_T(\prod_{o \in J_-} z_o^{\delta_o |o|})$. The exponent sum $\sum_{o \in J_-} \delta_o |o|$ is 0. Thus, the argument has norm 1 if $t = -1$, and is in T also when $t = 1$. In particular, $\widehat{\theta}(zI) = 1$. On T', the odd exponent $\delta_o |o|$ gives η_T at every $o \in J_-$. For another choice δ'_o, the half-differences satisfy $\sum_o (\delta'_o - \delta_o)|o|/2 = 0$. Thus, their ratio belongs to T^2. Consequently, all permutations of J_- in the complement fix $\widehat{\theta}$. For a rotation c_o on $o \in J_-$, the calculations in Proposition 6.232 give $\widehat{\theta}^{c_o}/\widehat{\theta} = \eta_q(\mu)$ if $\det(c_o) = -1$, and 1 otherwise.

Let $W_\pm$ be the complements for $1, \eta_T$. Since the torus weights determine J_+, J_-, the inertia of $\widehat{\theta}$ is $T^\sharp \rtimes (W_+ \times \ker \epsilon_-)$. Take the tensor of $\widehat{\theta}, V_+$ and one eigenspace V_-^+ of J_-. By index-two Clifford theory [95, Theorem 6.11], V_-^+ is irreducible on $\ker \epsilon_-$, and its induction to W_- gives V_-. Consequently, induce the tensor to $B^\sharp/Q$. Since the inertia meets each class μ, [95, Problem 5.2] gives one double coset for B/Q. Induction from $\ker \epsilon_-$ gives V_-. Hence, it gives the parent ρ of Theorem 6.240. Thus, with one degenerate Λ_-, the extension over K has scalar 1 at zI. If both are degenerate, set $\partial(g) = \det(g)/\mu(g)^n$. On the source subgroup $\partial = \epsilon_+$. Thus, $J^\sharp f(g) = \partial(g) J_+ f(g)$ defines an involution intertwining ∂. Its eigenspaces extend ψ_0, ψ_1 on $\ker \partial$. An isometry g with $\det(g) = -1$ exchanges the eigenspaces.

Moreover, we let v represent block Frobenius on $\mathbf{T}$, with v over $\mathbb{F}_r$. Then, $\det(v) = 1$, since for $t = -1$ there are m negative cycles and m is even. Thus, on the vF_q-fixed space, $F_r(v) = v$ and $F_r^f = v^{-1}$. By Lang's theorem [58, Theorem 4.2.9], we choose $g \in \mathbf{SO}(V)$ with $g^{-1}F_q(g) = v$. After transport by g, the field map is $gF_r(g)^{-1}F_r$. The element $gF_r(g)^{-1}$ is rational since $F_r(v) = v$ and belongs to $\mathrm{SO}(V)$. The field map on $T^\sharp$ raises z_o to z_o^r and fixes the complement representatives. Thus, $\widehat{\theta}$ is invariant. The uniform rotation v^{-1} has even marked parity and scalar $z \in \mu_d$ on the parent ρ by Theorem 6.236. By restriction to B, the extension has scalar z at v^{-1}. Thus, choose $A \in \mu_k$ with $A^f = z$. Thus, AI is the field operator over K. If there are two branches, AI fixes both eigenspaces of $J^\sharp$. Consequently, the extensions over K are fixed by $\mathcal{H}_p$. □

Proposition 6.242. *If both $\Lambda_\pm$ in* Theorem 6.240 *are degenerate, then (χ_i, ψ_i) satisfies $\succeq_c$ on the proper semilinear stabilizer* $\ker \partial$. *An isometry g with* $\det(g) = -1$ *exchanges the pairs $(\chi_0, \psi_0), (\chi_1, \psi_1)$.*

Proof. First, choose the lift $\widetilde{t}_0$ of order 2 in the dual of $H = \mathrm{GSO}(V)$. By Proposition 6.214, the Weyl labels on $W(D_{n_\pm})$ come from $(\alpha_\pm, \alpha_\pm)$ on $W(B_{n_\pm})$. Thus, the regular parent ρ occurs in $Y_{\widetilde{\theta}} = \mathrm{Ind}_{B_H}^H \widetilde{\theta}$ by [72, Theorem 3.2.7]. By Theorem 6.240, $\rho|_S$ and $\rho|_{\Omega(V)}$ are irreducible. On one hyperbolic plane $\langle u_i, v_i\rangle \subset Y_+$, let γ_+ exchange u_i, v_i. Thus, on the torus $x_i \mapsto \mu/x_i$. The dual map acts by $(s_i, z) \mapsto (s_i^{-1}, zs_i)$, where z is the central coordinate. Since $s_i = 1$, the lift $\widetilde{t}_0$ is fixed. By [169, Theorem 6.2], the graph on Y_+ exchanges Λ_+^+, Λ_+^- and fixes Λ_-. The two pinnings of $\mathbf{H}$ are H-conjugate because $\mathrm{Z}(\mathbf{H})$ is connected. Thus, γ_+ exchanges χ_0, χ_1, since matching values on $\Omega(V)$ requires both signs on Λ_+, Λ_- to change.

The field F_r acts on $Y_{\widetilde{\theta}}$ over $\mathbb{Q}$. On $B_H w B_H$, a kernel K_w is determined by $K_w(w)$. Moreover, $K_w(w) = 0$ unless $\widetilde{\theta}^w = \widetilde{\theta}$. Each representative w is over $\mathbb{F}_r$, so the normalization with value 1 gives an F_r-fixed basis of $\mathrm{End}_H(Y_{\widetilde{\theta}})$. Thus, F_r fixes the block e_ρ and its endomorphism algebra. Since ρ is rational-valued, its central idempotent e_ρ gives a $\mathbb{Q}$-module. Over a splitting field, $e_\rho Y_{\widetilde{\theta}} = V_\rho \otimes L$ with endomorphism algebra $\mathrm{End}(L)$. Consequently, F_r acts by $A \otimes I_L$ with $A^f = I_{V_\rho}$. The character on $H \rtimes \langle F_r\rangle$ is $\dim(L)\widehat{\rho}$ for one ordinary extension $\widehat{\rho}$. Thus, $\widehat{\rho}$ is rational-valued and trivial on zI even if $\dim L > 1$. The local extensions $\widehat{\psi}_i$ of Proposition 6.241 are fixed by $\mathcal{H}_p$ and trivial on zI. Thus, on the section s_γ of Theorem 6.233, both extensions give $\omega, \alpha, \mu_{a,\sigma}$ with identical values. □

Proposition 6.243. *Suppose Λ_- alone is degenerate for (χ_B, ψ) in* Theorem 6.240. *Thus, we have that (χ_B, ψ) satisfies $\succeq_c$ on its full semilinear stabilizer.*

Proof. Take Y_- degenerate of rank $b' = 2|\lambda|$, and let t_0 have sign -1 on Y_-. Let H_+ denote the unipotent label on Y_+. Thus, the flag space $E \subset Y_-$ has dimension $b' < n$. On $\mathrm{GL}_{b'}(q)$, take $(\eta_q \circ \det)U_\tau$ for $\tau \vdash b'$. Thus, at torus weight 1, the proof of Proposition 6.242 extends U_τ rationally as a character across F_r. Moreover, $\eta_q \circ \det$ is field-fixed and has value $\eta_q(z^{b'}) = 1$ at zI.

Given H_+ of defect 0, take the trivial source at rank 1 and line weight 1. By [208, Lemma 7.17(i)], each hook coefficient is 1. Moreover, the unipotent lift $\widetilde{H}_+$ is irreducible on $\mathrm{SO}(Y_+)$, and its adjoint label H_+ is graph-fixed. Thus, the $\widetilde{H}_+$-spaces of multiplicity 1 give an extension to $\mathrm{GO}(Y_+)$ with F_r and scalar 1 at zI. If H_+ has positive defect, $2 \mid d$. By Lemma 6.216, the adjoint label H_+ has an $\mathcal{H}_p$-fixed extension $\widehat{H}_+$ across the natural graph and F_r. Its inflation has scalar 1 at zI. Induce the source with $\tau = 2\lambda$ to give Y. Thus, on $\mathrm{O}(V)$ its character is $F_{2\lambda}(h)$, where $h \in \mathrm{Irr}(\mathrm{O}(Y_+))$ is as in Section 6.1. The parent ρ in Theorem 6.240 restricts to χ_B on $\Omega(V)$ by Section 6.1. Consequently, the coefficient of χ_B is $\langle Y|_{\Omega(V)}, \chi_B\rangle = [s_{2\lambda}]s_\lambda^2 = 1$. By Section 6.1, χ_B is rational and fixed by $\mathrm{GO}(V) \rtimes \langle F_r\rangle$. Thus, Y_{χ_B} gives an $\mathcal{H}_p$-fixed extension trivial on zI. The extension $\widehat{\psi}$ of Proposition 6.241 is also fixed by $\mathcal{H}_p$ and trivial on zI. Thus, the section s_γ from Theorem 6.233 gives $\succeq_c$. □

Let $X = \mathrm{Spin}_{2n}^\epsilon(q)$, where $q = r^f$ is odd and $n \geq 5$. Let $p \neq r$ be an odd prime dividing $|X|$. Then, we put $d = \mathrm{ord}_p(q)$, $e = \mathrm{ord}_p(q^2)$ and $q^e \equiv t \bmod p$ with $t = \pm 1$. In a rational class s of type $I(a, b)$, $1 \leq a < b$ and $a + b = n$. Thus, we find $F^*(t_s) = t_s$ for a natural lift t_s, since $-t_s$ has ranks b, a at $1, -1$. We fix Y_+ of rank $n_+ = a$ and Y_- of rank $n_- = b$, with types ϵ_+, ϵ_-. By Witt's theorem, $Y_+ \perp Y_-$ is determined up to isometry by n_i, ϵ_i for $i = +, -$. Moreover, a reflection in either Y_i corrects the determinant to 1. Moreover, scaling the form on Y_i by μ changes its determinant by μ^{2n_i}, so the types ϵ_i are invariant under projective conjugation of t_s.

Lemma 6.244. *For $i = +, -$, we put $w_i = m(n_i, \epsilon_i)$ count the rank-e blocks of type t in* Section 6.3. *Accordingly, let U_i have rank $r_i = n_i - ew_i$ and type $\delta_i = \epsilon_i t^{w_i}$, and put $U = U_+ \perp U_-$. Then, $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X) \neq \varnothing$ precisely when $w_+ + w_-$ has no base-p carry and either $r_+ + r_- < e$ or $r_+ + r_- = e$ with $\delta_+\delta_- = -t$. Moreover, when s meets both tests, the possible ξ_i on Y_i are exactly the unipotent characters with $p \nmid \xi_i(1)$. Each ordered symbol for ξ_i has an ordered core on U_i and a quotient tower of size w_i and p'-degree.*

Proof. $0 \leq r_i < e$ except when $e \mid n_i$ and $\epsilon_i \neq t^{n_i/e}$. Then, $r_i = e$ and $\delta_i = -t$. If $r_i = 0$, $\delta_i = 1$. Thus, $p \nmid |\mathrm{O}(U_i)|$. By the orthogonal order formula [40], $p \nmid |\mathrm{O}(U)|$ precisely under the stated test on r_i, δ_i. Then, $m = w_+ + w_-$ counts the rank-e blocks of type t in the ambient space. With $\ell = v_p(q^e - t)$, the ambient valuation is $m\ell + v_p(m!)$, so the centralizer index has valuation $v_p\binom{m}{w_+}$. Thus, Kummer's theorem gives $p \nmid \binom{m}{w_+}$ precisely when $w_+ + w_-$ has no carry. If U has $j > 0$ further blocks of rank e and type t, then the ambient valuation increases by $j\ell + v_p((m + j)!/m!) > 0$. The component group at s and the isogeny kernel both have 2-power order. Consequently, for $\chi \in \mathcal{E}(X, s)$, the Jordan degree formula [58, Proposition 11.5.6] gives $p \nmid \chi(1)$ exactly when the index has p'-order and $p \nmid \xi_i(1)$ for both i. In particular, $(\xi_+, \xi_-) = (1, 1)$ suffices for nonemptiness.

The symbol for ξ_i has weight w_i and a quotient of p'-degree by Lemma 6.59. The core Λ_i^0 on U_i has rank r_i and type δ_i. Conversely, each symbol on U_i has weight 0 by Lemma 6.59. Choose reduced rows A_i, B_i and a common shift h with fixed residue modulo $2e$. A further shift by $2e$ adds initial beads to A_i, B_i and leaves the partitions on the $2e$ runners fixed. Thus, any tuple $(\pi_j)_j$ of weight w_i fits for large h. The runners are ordinary for $t = 1$ and alternate rows for $t = -1$ as in Section 6.3, also at $r_i = 0$. Row exchange transposes (A_i, B_i) and acts by $j \mapsto j + e \bmod 2e$. Thus, the symbol for ξ_i is degenerate exactly when (A_i, B_i) and the quotient tuple $(\pi_j)_j$ are fixed by row exchange. Since $2e \mid p - 1$, the inverse at $e = 1$ follows from Proposition 6.235 as well. □

Lemma 6.245. *Let $O_0 = \mathrm{O}(U)$, $S_0 = \mathrm{SO}(U)$, and let t_U have eigenvalues $1, -1$ on U_+, U_-, respectively. Then, we have that the O_0-parents of the rational t_U-series correspond to the ordered cores $(\Lambda_+^0, \Lambda_-^0)$ on U_+, U_-, with* det *transposing both. Each parent ρ satisfies $\rho(g) \in \mathbb{Q}$ for $g \in O_0$.*

Proof. By [200, Propositions 6.1–6.2], the Jordan map for O_0 accounts for the index 2 of S_0. Given t_U, the full centralizer is $\mathrm{O}(U_+) \times \mathrm{O}(U_-)$ and its character $\det_+ \det_-$ corresponds to det. Thus, a graph orbit $\{\chi, \chi^\gamma\}$ of size 2 gives one induced parent fixed by det. If $\chi^\gamma = \chi$, the two extensions differ by det, and the Jordan map pairs $\widehat{\chi}, \det \widehat{\chi}$ in either order. Consequently, a nondegenerate Λ_i^0 gives extensions $\widehat{\xi_i}, \det_i \widehat{\xi_i}$, and the two degenerate signs $\Lambda_i^{0,\pm}$ give one det-fixed parent. If $U_- = 0$, the label on U_- is 1 and $t_U = I_U$. If $U_+ = 0$, then $t_U = -I_U$ and the series consists of $\nu\xi$ for unipotent ξ, where ν is the spinor character of O_0. If t_U is noncentral, the S_0-characters are rational by [165, Theorem A(i)]. For central $t_U = \pm I_U$, each is ξ or $\nu|_{S_0}\xi$ with ξ unipotent. Thus, if $\chi^\gamma \neq \chi$, $\mathrm{Ind}_{S_0}^{O_0} \chi$ has values in $\mathbb{Q}$. If $\chi^\gamma = \chi$, it suffices that one extension $\widehat{\chi}$ has values in $\mathbb{Q}$, because the other is $\det \widehat{\chi}$.

By Lemma 6.237, a nondegenerate ξ_i has ordinary hooks from a cusp ξ_c or from $\mathrm{SO}_2^+(q)$. The toral extension is 1 and $\widehat{\xi_c}$ has values in $\mathbb{Q}$ by [165, Theorem 10.8]. If a hook target W has minus type, W has rank at least 2, so W contains a singular line D. The parabolic P_D contains a reflection on $D^\perp/D$. Thus, on $\mathrm{SO}(W)$, [95, Problem 5.2] gives one line induction. At ξ_i, the coefficient is 1. Consequently, the extension $\widehat{\xi_i}$ in the induction has values in $\mathbb{Q}$. If $t_U = -I_U$, fix $\nu\widehat{\xi_i}$ on O_0.

Given two nondegenerate labels and $t_U \neq \pm I_U$, the component with determinant -1 on each of U_+, U_- fixes both labels ξ_+, ξ_-. Moreover, a proper similitude h with nonsquare multiplier exchanges the two S_0-constituents. By [149, Corollary 8.13], $\rho \det$ transposes the symbols $\Lambda_\pm^0$. Since they are nondegenerate, $\rho \det \neq \rho$. By the cyclic-quotient formulas, $\rho|_{S_0} = \chi$ is irreducible, so $\chi^\gamma = \chi$. Two twists of ρ give the hook matrix on S_0 by Lemma 6.237 for $r'_+, r'_- > 0$, including $r'_+ = r'_-$. If neither endpoint U'_i is split toral, both ξ'_i are cuspidal, including 1 on $\mathrm{SO}_2^-(q)$. Moreover, the connected centralizer at $t_{U'}$ in $\mathrm{SO}(U')$ has split central rank 0. Thus, the characters χ' are cuspidal by Lemma 6.238, with extensions $\widehat{\chi}'$ of values in $\mathbb{Q}$ by [165, Theorem 10.8]. If an endpoint is $\mathrm{SO}_2^+(q)$, the last line leaves $t_{U'} = \pm I_{U'}$. Thus, the one column for $t_{U'}$ in Lemma 6.238 has coefficient 1 for each target χ. Consequently, the hook coefficients 1 give rational-valued extensions $\widehat{\chi}$.

If only one Λ_i^0 is degenerate, take $i = -$ after the twist by ν if necessary. On a maximal singular $E \subset U_-$, take $(\eta_q \circ \det)U_{2\lambda}$. On U_+, choose the extension $\widehat{\xi_+}$ just constructed. Thus, the induced character Y has values in $\mathbb{Q}$. By the flag formula, $\langle Y|_{\Omega(U)}, \chi_B \rangle = [s_{2\lambda}]s_\lambda^2 = 1$. Consequently, the unique extension of the S_0-character in Y has values in $\mathbb{Q}$, and the other differs by det. If both symbols are degenerate, then U_+, U_- are split. By Proposition 6.242, the graph γ_+ on U_+ gives $\chi^\gamma \neq \chi$, also when $r_+ = r_-$ with the fixed lift t_U. Thus, the induced character on O_0 has values in $\mathbb{Q}$. Consequently, every parent ρ has $\rho(g) \in \mathbb{Q}$ for $g \in O_0$. □

Theorem 6.246. *Let $X = \mathrm{Spin}_{2n}^\epsilon(q)$, with q odd and $n \geq 5$. With $p \mid |X|$ odd and $p \nmid q$, we let $\widehat{P}$ be Sylow in X. If s has rational type $I(a, b)$ with $1 \leq a < b$, $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$ has an $\mathcal{H}_p$-equivariant bijection onto the sector $\mathcal{S}_s$ of $M = \mathrm{N}_X(\widehat{P})$. For each pair (χ, ψ) and $z \in \mathrm{Z}(X)$, we have that $\chi(z)/\chi(1) = \psi(z)/\psi(1)$. Distinct rational classes s give disjoint sectors in $\mathrm{Irr}_{p'}(M)$. Both sets are empty if s fails the tests in* Lemma 6.244.

Proof. Set $\mathcal{S}_s = \varnothing$ when either test on s in Lemma 6.244 fails. Otherwise, we put $m = w_+ + w_-$ and take the sum W of the m blocks of rank e and type t. Then, $V = W \perp U$ and $p \nmid |\mathrm{O}(U)|$. We choose $P = (T_p)^m \rtimes Y$ on W, with Y Sylow in $\mathfrak{S}_m$. Thus, $|P| = |\mathrm{O}(V)|_p$ by Lemma 6.244. Moreover, $Q = (T_p)^m$ is characteristic in P and $V^Q = U$ by Lemma 6.231. Hence, $B = \mathrm{N}_{\mathrm{O}(V)}(P) = B_A \times O_0$, where $B_A = \mathrm{N}_{\mathrm{O}(W)}(P)$ and $O_0 = \mathrm{O}(U)$. By Lemma 6.234, B_A/Q is the product over the sizes p^j of the Y-orbits. If $e = 1$, T_p has distinct weights α, α^{-1} on its plane. Thus, C acts by inversion on T and has order 2. Therefore, the formulas for Q and B_A/Q hold for $d = 1, 2$ as well. By Lemma 6.226, $(\det, \nu)$ on B has image Δ and kernel $M_0 = \mathrm{N}_{\Omega(V)}(P)$. Thus, the spin preimage is $M = \mathrm{N}_X(\widehat{P})$, where $\widehat{P}$ lifts P.

For ordered labels Λ_+, Λ_-, denote the core pair by $(\Lambda_+^0, \Lambda_-^0)$ and the two arrays by $\boldsymbol{\lambda}_\pm$ as in Lemma 6.244. Let Π_A be the parent of $\boldsymbol{\lambda}_\pm$ furnished by Proposition 6.235 with colours $1, \eta_{T'}$. Thus, the totals at depth j are $(w_+)_j, (w_-)_j$. If $w_i = 0$, the tensor for the i-colour is 1 on zero slots. Let Θ_0 be the parent of $(\Lambda_+^0, \Lambda_-^0)$ on O_0 in Lemma 6.245. Thus, $\Pi = \Pi_A \boxtimes \Theta_0$ lies in $\mathrm{Irr}_{p'}(B/Q)$ and $\Pi^\sigma = \Pi$ for $\sigma \in \mathcal{H}_p$, since Π_A has a model over $K = \mathbb{Q}(\zeta_{p-1})$ and Θ_0 has values in $\mathbb{Q}$. Define $\mathcal{S}_s$ by inflation of the constituents of $\Pi|_{M_0}$ for all such Π. For another Sylow subgroup $\widehat{P}'$, conjugate $\mathcal{S}_s$ by $g \in X$ with $\widehat{P}^g = \widehat{P}'$. Equivalently, the parents in $\mathrm{Irr}_{p'}(B/Q)$ are indexed by the two colours $1, \eta_{T'}$, the totals $(w_\pm)_j$ and the residual series $\mathcal{E}(S_0, t_U)$. Thus, $\mathcal{S}_s$ depends only on s, with no choice of labels $\Lambda_\pm$.

By Proposition 6.235, $\Pi_A \det|_{B_A}$ shifts the runner index by $e \bmod 2e$. Moreover, $\Theta_0 \det|_{O_0}$ transposes the two cores $\Lambda_\pm^0$ by Lemma 6.245. Thus, $\Pi \det$ transposes the two complete symbols $\Lambda_\pm$. By Proposition 6.232, $\nu|_{T'} = \eta_{T'}$ on each active orbit, whose length p^j is odd. Moreover, on $S_0 = \mathrm{SO}(U)$, the dual central twist by $-I_U$ acts on series by $t_U \mapsto -t_U$. In rank 1, it is the quadratic twist on $\mathrm{SO}_2^\delta(q)$. Consequently, $\nu, \det\nu$ exchange the two totals $ew_+ + r_+ = a$ and $ew_- + r_- = b$. Since $a < b$, neither $\Pi\nu$ nor $\Pi \det\nu$ has positive rank a, even if $w_+ = w_-$ or $r_+ = r_-$.

By [95, Lemma 5.2], for parents Π, Π' $\langle \Pi|_{M_0}, \Pi'|_{M_0}\rangle = \sum_{\lambda \in \mathrm{Irr}(\Delta)} \langle \Pi, \Pi'\lambda\rangle$, where λ is inflated to B. When at least one Λ_i is nondegenerate, the stabilizer of Π in $\mathrm{Irr}(\Delta)$ is trivial, so $\Pi|_{M_0} \in \mathrm{Irr}(M_0)$. Thus, $\Pi'|_{M_0} = \Pi|_{M_0}$ exactly when $\Pi' \in \{\Pi, \Pi\det\}$. With two nondegenerate Λ_i, there are 2 relative row orders. With one degenerate Λ_i, there is 1 row order up to det. If both Λ_i are degenerate, the stabilizer is $\langle\det\rangle$ and $\|\Pi|_{M_0}\|^2 = 2$. Consequently, $\Pi|_{M_0} = \psi_0 + \psi_1$ for distinct $\psi_i \in \mathrm{Irr}(M_0)$.

In the doubly degenerate case, J_A is the determinant operator on Π_A over K from Theorem 6.240. Indeed, $i \mapsto i + e \bmod 2e$ fixes both arrays $\boldsymbol{\lambda}_\pm$, with operator I on any empty factor. If $U \neq 0$, Θ_0 is fixed by $\det|_{O_0}$ and $\Theta_0|_{S_0} = \theta_0 + \theta_1$ with $\theta_0 \neq \theta_1$. If $t_U = \pm I_U$, each θ_i equals ξ_i or $\nu|_{S_0}\xi_i$ for a degenerate unipotent ξ_i. If t_U is noncentral, the values of θ_i lie in $\mathbb{Q}$ by [165, Theorem A(i)]. Thus, the central idempotents e_{θ_i} lie in $\mathbb{Q}[S_0]$ and $J_0 = e_{\theta_0} - e_{\theta_1}$. Set $J_0 = I$ when $U = 0$. Consequently, $J_A \otimes J_0$ is a determinant operator on Π and its two eigenspaces give ψ_0, ψ_1. An element $g \in B$ with $\det(g) = -1$ exchanges these spaces. Given $\sigma \in \mathcal{H}_p$, any equivalence $\Theta_0^\sigma \cong \Theta_0$ takes e_{θ_i} to itself. Thus, the operator $J_A \otimes J_0$ gives $\psi_i^\sigma = \psi_i$ from the rational values of Θ_0.

On each rational space Y_i, we choose a reflection γ_i. By [208, Lemma 7.15], the component $\gamma_+\gamma_-$ induces the graph involution on each $\mathrm{SO}(Y_i)$. Thus, over the pair (Λ_+, Λ_-), restriction from $\mathrm{GSO}(V)$ to $S = \mathrm{SO}(V)$ gives $2, 1, 2$ characters as the number of degenerate Λ_i is $0, 1, 2$, as in Theorems 6.236 and 6.240. Each character χ_S has values in $\mathbb{Q}$ by [165, Theorem A(i)]. Moreover, ν takes the t_s-series to the disjoint $-t_s$-series. Thus, the characters in the t_s-series restrict irreducibly and distinctly to $\Omega(V) = \ker(\nu|_S)$ by [95, Lemma 5.2]. By [208, Lemma 7.15 and Proposition 7.18] and Lemma 6.244, inflation to X gives $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X)$. Consequently, for each pair (Λ_+, Λ_-) with unordered rows, we match χ_i with ψ_i in either order if there are 2 characters. Both χ_i and ψ_i are fixed by $\mathcal{H}_p$. The inverse on B_A/Q recovers $\boldsymbol{\lambda}_\pm$ and that on O_0 recovers $\Lambda_\pm^0$. Thus, the formula on M_0 gives bijectivity. The local labels recover $(ew_i + r_i, t^{w_i}\delta_i)$ up to the exchange $i \mapsto -i$. Consequently, $a < b$ gives disjoint sectors $\mathcal{S}_s$.

For the scalar at $-I$ on S, choose a dual torus T^* from nonsingular planes in Y_+, Y_-. On a plane of type τ in Y_-, the character θ_τ of order 2 on $C_{q-\tau}$ has value $\tau\eta_q(-1)$ at -1. Moreover, each plane in Y_+ contributes 1. Thus, the torus gives scalar $\epsilon_-\eta_q(-1)^b$ in the t_s-series. The scalar on Π is $\left(t\eta_q(-1)^e\right)^{w_-}\delta_-\eta_q(-1)^{r_-} = \epsilon_-\eta_q(-1)^b$. Indeed, each of the p^j slots contributes the scalar at $-I$ on its block. The planes of U_- give the scalar on Θ_0. The kernel of $X \to \Omega(V)$ has scalar 1 on both inflations. Thus, if $z \in \mathrm{Z}(X)$ maps to $-I$, the two scalars are $\epsilon_-\eta_q(-1)^b$. If $-I \notin \Omega(V)$, $\mathrm{Z}(X) = \ker(X \to \Omega(V))$. □

Corollary 6.247. *The correspondence $\chi \mapsto \psi$ in* Theorem 6.246 *holds for s of type $I(a, a)$ when the rational spaces $Y_\pm$ have opposite signs $\epsilon_+\epsilon_- = -1$, for every eligible e, with ambient sign $\epsilon = -1$.*

Proof. First, take $\epsilon_+ = 1$ and $\epsilon_- = -1$. Thus, $\mathrm{disc}(Y_+) \neq \mathrm{disc}(Y_-)$ modulo squares, also after scaling by μ, since μ^{2a} is a square. Consequently, t_s and $-t_s$ are in distinct rational S-classes. By [208, Lemma 7.15 and (7.16)–(7.17)], the rational components are $1, k$, where $k = \gamma_+\gamma_-$. Indeed, for the component g exchanging Y_+, Y_-, the Frobenius has $F(g) = kg$ modulo the connected component. Locally, ν exchanges the two quadratic labels $(ew_i + r_i, t^{w_i}\delta_i)$ and gives $\epsilon_+ = -1$. Thus, the split-positive convention excludes $\Pi\nu, \Pi\det\nu$ from $\mathcal{S}_s$. The proof of Lemma 6.244 needs only w_i, r_i, δ_i if Y_+, Y_- are rational. Thus, the test on w_+, w_- and that on U both apply. Consequently, with $1, k$ as above, by Theorem 6.246, we obtain $\chi \mapsto \psi$ with equal scalars on $\mathrm{Z}(X)$. □

Proposition 6.248. *Let $n = 2a$, with individually rational sign spaces $Y_+ \cong Y_-$ of type ϵ_0. Thus, the split sum is $V = Y_+ \perp Y_-$. Isometry gives $w_+ = w_- = w$ and $r_+ = r_- = r_0$. Moreover, the tests at s are those of* Lemma 6.244. *In*

particular, $a = ew + r_0$. The twist by ν can return the t_s-series of $\mathrm{SO}(V)$ because Y_+, Y_- are isometric. For rational isometric Y_+, Y_-, the parents ρ on $\mathrm{O}(V)$ and Π on B admit a bijection with twist actions $\det \mapsto D$ and $\nu \mapsto T$, where $D(\Lambda_+, \Lambda_-) = (\Lambda_+^{\mathrm{t}}, \Lambda_-^{\mathrm{t}})$ and $T(\Lambda_+, \Lambda_-) = (\Lambda_-, \Lambda_+)$, respectively. If $\Lambda_+ \notin \{\Lambda_-, \Lambda_-^{\mathrm{t}}\}$, the induced pairing is $\mathcal{H}_p$-equivariant on constituents at $\Omega(V)$ and M_0 for all eligible e, r_0. After inflation to X, M, each pair (χ, ψ) has $\chi(z)/\chi(1) = \psi(z)/\psi(1)$ for $z \in \mathrm{Z}(X)$.

Proof. First, let B_i be the active normalizer on Y_i for $i = +, -$. On $B_+ \times B_-$, we choose the models V_i of Proposition 6.235 with torus colour 1. On B_-, take $V_- \otimes \nu_-$, where ν_- is the spinor character. Thus, ν_- is trivial on $Q \cap B_-$ and equals $\eta_{T'}$ on each T'-coordinate of the negative torus T_-. Let $\widetilde{\theta}_-$ be the extension equal to 1 on the complement C_-. Then, $\delta = \nu_-/\widetilde{\theta}_-$ is linear and trivial on T_-. Consequently, the ν_--twist recovers V_-. On $B_+ \times B_-$, the total spinor character is $\nu_+ \boxtimes \nu_-$. At depth j, match the $(w)_j$ orbits of each colour. Then, let s_A swap the matched orbits isometrically. Since ν_+ corresponds to ν_- under s_A, the composition of the ν-twist with conjugation by s_A acts by $V_+ \boxtimes (V_-\nu_-) \mapsto V_- \boxtimes (V_+\nu_-)$. The product $\nu_+ \boxtimes \nu_-$ also gives the values on K_j. Moreover, det shifts the runner indices by e mod $2e$ as in Proposition 6.235. Thus, if $r_0 = 0$, the arrays determine Λ_i.

On $\mathrm{O}(V)$ and on $\mathrm{O}(U)$ if $U \neq 0$, we choose the Jordan maps $\mathcal{L}_{t_s}, \mathcal{L}_{t_U}$ of [149, Theorem 8.12 and Corollaries 8.13–8.14]. At t_s, t_U, the eigenvalues are $1, -1$. Thus, det acts by D, and ν acts by T with the dual element $t_s \mapsto -t_s$ or $t_U \mapsto -t_U$. Given a nonsquare rescaling μ, the spinor character is $\nu \det$ by [137, §24.i]. In the rescaled convention, transpose Λ_- by the operation $L(\Lambda_+, \Lambda_-) = (\Lambda_+, \Lambda_-^{\mathrm{t}})$. Then, $LDL = D$ and $LTL = DT$ give the twists by $\det, \nu$ in the given forms. On both Y_i, we choose matching phases in $\mathbb{Z}/2e\mathbb{Z}$, tower vertices in $\mathbb{F}_p^j$ and padding lengths modulo $2e$. Thus, we have that D, T act on the cores Λ_i^0 and arrays $\boldsymbol{\lambda}_i$. The inverse recovers every Π over $1, \eta_{T'}$ with digit totals $(w)_j, (w)_j$ and residual series at t_U.

Now, order the unordered symbols Λ_+, Λ_-. Since they are distinct, neither T nor DT fixes (Λ_+, Λ_-). Thus, the twist stabilizer in $\mathrm{Irr}(\Delta)$ is 1 or $\langle \det \rangle$. By Theorem 6.246, the restriction to $\Omega(V)$ or M_0 has 1 or 2 distinct constituents of multiplicity 1, respectively. Moreover, distinct $\mathrm{Irr}(\Delta)$-orbits give disjoint constituents. On $S = \mathrm{SO}(V)$, each constituent θ has values in $\mathbb{Q}$ by [165, Theorem A(i)]. If $\theta\nu = \theta$, the parent $\rho\nu$ must be ρ or $\rho \det$. Thus, $\theta\nu \neq \theta$ and $\theta|_{\Omega(V)}$ is irreducible with values in $\mathbb{Q}$. On M_0, a det-fixed parent has constituents ψ_0, ψ_1 on the eigenspaces of $J_A \otimes J_0$, by Theorem 6.246. Thus, $\psi_i^\sigma = \psi_i$ for $\sigma \in \mathcal{H}_p$. By Theorem 6.246, the scalars at $-I$ and on $\ker(X \to \Omega(V))$ agree for χ, ψ. Moreover, the component swapping Y_+, Y_- moves either ordered pair (Λ_+, Λ_-). Consequently, inflation gives the claimed pairs (χ, ψ).

If $r_0 = 0$, we have that $d > 2$, both Y_i are split and $d \mid a$, then (χ, ψ) satisfies $\succeq_c$ when both distinct Λ_i are nondegenerate or both degenerate. Indeed, ν cannot fix a constituent θ on $\mathrm{SO}(V)$ since $\Lambda_+ \notin \{\Lambda_-, \Lambda_-^{\mathrm{t}}\}$. At each hook on E, choose t_s on the residual space $E^\perp/E$. Moreover, δ on B_- is rational and field-fixed. If h has nonsquare multiplier $\mu(h)$, the reflection formula gives $\nu_-^h = \nu_- \det|_{B_-}$. Thus, on C_-, each $\delta(c)I$ satisfies $\delta(c)I\, J_- = J_-\, \delta(c)I$. Moreover, $\delta(v_0) \in \mu_d$ since the rotation v_0 satisfies $v_0^d = 1$. The field root in μ_k exists by Theorem 6.236, since $x \mapsto x^f$ maps μ_k onto μ_d. The scalar at $-I$ is the original scalar since $\delta(-I) = 1$. Thus, on zI, we choose $\eta_q(z)^a$ if both Λ_i are nondegenerate. If both Λ_i are degenerate, take scalar 1 and extend with $\delta|_{C_-}$ in Proposition 6.241 on the eigenspaces of J_-. Consequently, on s_γ, the extensions $\widehat{\chi}, \widehat{\psi}$ give $\succeq_c$ by Theorem 6.239 and Proposition 6.242. □

For ρ in $\mathcal{L}_{t_s}, \mathcal{L}_{t_U}$, the proof of Lemma 6.245 gives $\rho(g) \in \mathbb{Q}$ on $\mathrm{O}(V)$ or $\mathrm{O}(U)$. Thus, we find $\rho^\sigma = \rho$ for $\sigma \in \mathcal{H}_p$. Suppose that $\widehat{\rho}$ extends ρ to the field overgroup A_ρ, with ordinary isometry group H. By Isaacs [95, Corollary 6.17], $\widehat{\rho}^\sigma = \widehat{\rho}\lambda_\sigma,\ \lambda_\sigma \in \mathrm{Irr}(A_\rho/H),\ \lambda_\sigma(1) = 1$. Thus, the multiplier at $x \in A_\rho$ is $\lambda_\sigma(xH)$.

Lemma 6.249. *For equal unordered symbols Λ, the orbit of ρ matches that of Π by* Proposition 6.248. *Thus, the stabilizers of ρ, Π in $\mathrm{Irr}(\Delta)$ contain ν or $\nu \det$. We determine how $\sigma \in \mathcal{H}_p$ permutes the constituents of $\rho|_{\Omega(V)}$ and $\Pi|_{M_0}$. Suppose that both split spaces Y_i of rank a have principal symbol Λ with unordered bipartition (u, v). Put $\tau_i = u_i + v_i$, with missing parts equal to 0. If $u \neq v$, we have that each parent ρ on $\mathrm{O}(V)$ has two constituents on $\Omega(V)$, with permutation sign $\eta_q(b_\sigma)^a$ for $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$. If $u = v$, we have that each $\rho|_{\Omega(V)}$ has 4 distinct constituents with values in $\mathbb{Q}$.*

Proof. First, choose a maximal singular E of dimension $2a$. Its stabilizer $P_E = LR_u$ in $\mathrm{O}^+_{4a}(q)$ lies in $\mathrm{SO}(V)$, with $L = \mathrm{GL}_{2a}(q)$. Moreover, let $\xi \in \mathrm{Irr}(L)$ have dual eigenvalues $1, -1$, each with multiplicity a and partition τ. Thus, ξ has values in $\mathbb{Q}$ and we inflate it across R_u. To compute $\langle \rho|_{P_E}, \xi \rangle$, take $\widetilde{P}_E \leq \mathrm{GSO}(V)$ a Siegel parabolic. On its Levi $L \times \mathbb{F}_q^\times$, we extend ξ by 1 on the multiplier coordinate $\mathbb{F}_q^\times$. Since $\widetilde{P}_E \mathrm{SO}(V) = \mathrm{GSO}(V)$, [95, Problem 5.2] gives a single

induction from P_E. By [72, Theorem 3.2.7], each factor Y_i has the branching coefficient from $W(D_a)$ to $\mathfrak{S}_a$. For the $W(B_a)$-label (u,v), the branching polynomial is $s_u s_v$. Consequently, at $\tau = u+v$ $[s_\tau]s_u s_v = 1$. Thus, for $u \neq v$, the irreducible restriction to $W(D_a)$ has coefficient 1 for either inclusion of $\mathfrak{S}_a$. The regular parent restricts to $\chi_0 + \chi_1$ on $\mathrm{SO}(V)$. Consequently, $\langle \rho|_{P_E}, \xi\rangle = 1$ for each full parent ρ. If $u = v$, the $W(B_a)$-label restricts to $D_u^+ + D_u^-$ on $W(D_a)$ by [208, Lemma 7.19]. Thus, for either inclusion of $\mathfrak{S}_a$, the coefficients at $D_u^\pm$ sum to 1. The product over Y_+, Y_- also gives 1. In the component notation of [208, Lemma 7.15], only $1, k$ fix the source series at E for the regular lift $\widetilde{t}_s$. Moreover, two regular parents $\widetilde{\rho}, \widetilde{\rho}'$ have equal irreducible restrictions to $\mathrm{SO}(V)$ precisely under simultaneous sign change $(+,+) \leftrightarrow (-,-)$ or $(+,-) \leftrightarrow (-,+)$. Thus, the branches of $\rho|_{\mathrm{SO}(V)}$ account for the coefficient 1.

Since R_u has odd order, it lies in $\Omega(V)$. On L, the spinor character is $\eta_q \circ \det$. Thus, $P_E \cap \Omega(V) = L_0 R_u$ with $L_0 = \ker(\eta_q \circ \det)$. Among the determinant twists of ξ, the stabilizer consists of $1, \eta_q \circ \det$, since the dual eigenvalues are exactly $1, -1$ with equal labels τ. Consequently, the constituents of $\xi|_{L_0}$ are distinct β_1, β_2, and each induces to ξ. By [95, Lemma 5.2], each β_i occurs with multiplicity 1 on $L_0 R_u$ in $\rho|_{L_0 R_u}$. Moreover, the restrictions $\beta_i|_{\mathrm{SL}_{2a}(q)}$ are irreducible and distinct, since $\xi|_{\mathrm{SL}_{2a}(q)}$ also has two constituents of multiplicity 1. By [163, Theorem 6.1 and Remark 6.2], each $\beta_i|_{\mathrm{SL}_{2a}(q)}$ has field $\mathbb{Q}$ unless q is nonsquare and $(2a)_2 \mid q-1$ with ξ fixed by a twist of order $(2a)_2$. Since the stabilizing group has order 2, the exceptional case is $2 \nmid fa$ for $q = r^f$. Then, we obtain the field $\mathbb{Q}(\sqrt{(-1)^{(r-1)/2} r})$. Thus, the permutation of β_1, β_2 under σ has sign $(b_\sigma/r)^{fa} = \eta_q(b_\sigma)^a$.

If $u \neq v$, the stabilizer of ρ is generated by $\lambda = \nu$ or $\nu \det$ by Proposition 6.248. Thus, $\rho|_{\Omega(V)} = \chi_1 + \chi_2$ with inertia group $\ker \lambda$ in $\mathrm{O}(V)$, by [95, Theorem 6.11]. An element $g \in L$ of nonsquare determinant satisfies $\lambda(g) = -1$. Thus, $\chi_1^g = \chi_2$ and $\beta_1^g = \beta_2$. Consequently, the multiplicities 1 at β_1, β_2 give the permutation sign $\eta_q(b_\sigma)^a$ on $\rho|_{\Omega(V)}$. If $u = v$, all four twists fix ρ. Thus, $\|\rho|_{\Omega(V)}\|^2 = 4$. By Clifford homogeneity, the coefficient 1 at β_1 excludes multiplicity 2 on $\Omega(V)$. Thus, $\rho|_{\Omega(V)}$ has 4 distinct constituents. Exactly one constituent χ_1 contains β_1 on $L_0 R_u$. Since $a = 2|u|$ is even, β_1 has values in $\mathbb{Q}$. Moreover, ρ has values in $\mathbb{Q}$ by Lemma 6.245. Thus, χ_1 and its 3 conjugates under $\mathrm{O}(V)$ have values in $\mathbb{Q}$. □

Lemma 6.250. *Let both active spaces W_i have weight w and rank $h = ew$. Suppose that Λ_- is the i-th transpose of Λ_+, where $i \in \{0,1\}$. Then, for $\lambda = \nu \det^i$, the active model has a λ-intertwiner J_A over $K = \mathbb{Q}(\zeta_{p-1})$ with $J_A^2 = c_A = t^w \eta_q(-1)^h$. Given $\sigma \in \mathcal{H}_p$, we have that the sign on $J_A/\sqrt{c_A}$ is $\eta_q(b_\sigma)^h$.*

Proof. First, the colour stabilizer is $H \leq B_A/Q$. By Proposition 6.248, match the $(w)_j$ orbits of each colour. Thus, s_A exchanges the matched orbits and $s_A^2 = 1$. On H, the source model is $V_1 \boxtimes (V_1 \otimes \nu_- \det_-^i)$ over K by Theorem 6.240. Thus, the flip A has $A^2 = I$ and $A\rho_H(x) = \lambda(x)\rho_H(s_A^{-1} x s_A)A$ for $x \in H$, on the tensor ρ_H. Consequently, on $\mathrm{Ind}_H^{B_A/Q} \rho_H$, $J_A(g \otimes v) = \lambda(g) g s_A \otimes Av$ is well defined and implements λ. Moreover, $J_A^2 = \lambda(s_A)$. The -1-space of s_A is the antidiagonal subspace for W_+, of dimension $2h$ with its form multiplied by 2. Thus, $\det(s_A) = 1$, $J_A/\sqrt{c_A}$ has sign $\eta_q(b_\sigma)^h$. By the reflection formula [137, §24.i], $\nu(s_A) = t^w \eta_q(-1)^h$, also at $h = 0$.

On the prime-to-p roots involved, we write σ as the p^j-power. Then, $\sigma(\sqrt{c_A})/\sqrt{c_A} = c_A^{j(p-1)/2}$ since $c_A = \pm 1$. Moreover, $(q/p)^h = (q^e/p)^w = (t/p)^w$ in Legendre symbols. By quadratic reciprocity, $(p/r)^{fh} = (q/p)^h (-1)^{((p-1)/2)((r-1)/2)fh}$, which equals $(t^w \eta_q(-1)^h)^{(p-1)/2}$. Thus, $b_\sigma \equiv p^j \bmod r$ gives $c_A^{j(p-1)/2} = \eta_q(b_\sigma)^h$. Since J_A is over K, $J_A/\sqrt{c_A}$ has sign $\eta_q(b_\sigma)^h$. □

Proposition 6.251. *Suppose that the common symbol Λ is eligible and degenerate. Then, each $\rho|_{\Omega(V)}$ and $\Pi|_{M_0}$ has four distinct $\mathcal{H}_p$-fixed constituents. The paired inflations to X, M have equal central characters on $\mathrm{Z}(X)$, including when $U \neq 0$.*

Proof. Since Λ, Λ^0 have equal rows, Y_i, U_i are split with even ranks a, r_0. Thus, W_i is split of even rank h, and $c_A = 1$ with $i = 0$ in Lemma 6.250. On Π_A, we choose the det-intertwiner D_A from Theorem 6.240. The operators on V_1 in both factors coincide, so $D_A A = A D_A$. Moreover, $\det(s_A) = 1$ gives $D_A J_A = J_A D_A$ after induction. Both D_A, J_A are involutions over K.

If $U \neq 0$, by Lemma 6.249, $\Theta_0|_{\Omega(U)}$ has four constituents with values in $\mathbb{Q}$. Moreover, Δ acts regularly on the 4 characters. Choose a constituent θ and let e_{θ^c} be the central idempotent of $\mathbb{Q}[\Omega(U)]$ at θ^c for $c \in \Delta$. For $\mu = \det, \nu$, put $J_{\mu,0} = \sum_{c \in \Delta} \mu(c) e_{\theta^c}$. Thus, $J_{\det,0}, J_{\nu,0}$ are involutions on Θ_0 with $J_{\det,0} J_{\nu,0} = J_{\nu,0} J_{\det,0}$ for $\det, \nu$, respectively. If $U = 0$, we set $J_{\mu,0} = I$. On Π, the operators $D = D_A \otimes J_{\det,0}$ and $J = J_A \otimes J_{\nu,0}$ satisfy $DJ = JD$. Each $g \in B$ acts on eigenvalues by $(u,v) \mapsto (u \det(g), v\nu(g))$. Thus, the map $B \twoheadrightarrow \Delta$ gives 4 nonzero joint eigenspaces. By [95, Lemma 5.2], $\|\Pi|_{M_0}\|^2 = 4$. Consequently, the 4 eigenspaces give distinct $\psi_i \in \mathrm{Irr}(M_0)$. Since D_A, J_A are over

K and $J_{\mu,0}$ comes from $\mathbb{Q}[\Omega(U)]$, $\psi_i^\sigma = \psi_i$ for $\sigma \in \mathcal{H}_p$. Since equal rows give principal Λ, by Lemma 6.249, we obtain the 4 fixed constituents on $\Omega(V)$. The parents ρ, Π match by Proposition 6.248, and χ, ψ have equal scalars on $\mathrm{Z}(X)$ by Theorem 6.246. □

If $t = 1$, we have that each e-hook lies in one row of $\Lambda = (A, B)$. On a row $D \in \{A, B\}$, the hook replaces D by $D' = (D \setminus \{x\}) \cup \{x - e\}$, $x \in D$, $x - e \geq 0$, $x - e \notin D$. Thus, the core Λ^0 consists of the two e-cores of A, B. If w hooks give Λ^0, we find $\operatorname{rk}\Lambda - \operatorname{rk}\Lambda^0 = ew$. In particular, a principal Λ has principal core Λ^0.

Lemma 6.252. *Let $q = r^f$ be odd, with isometric rational Y_i. Suppose that $a = s^2$, $s \geq 2$, and both Y_i have cuspidal symbol Λ. Then, each ξ of $\mathrm{SO}(V)$ with symbol pair (Λ, Λ) has permutation sign on its two $\Omega(V)$-constituents given by $\eta_q(b_\sigma)^a$, where $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$.*

Proof. Take $m = y = 0$ in [208, Proposition 7.26]. Then, (7.23) gives type $(-1)^s$ on Y_i, and Step 1 gives $z_0 \in \ker\rho$. By (7.27) and Step 2, the nilpotent N has Jordan sizes $1, 3, \ldots, 4s-1$ with total multiplicity 1 for the regular parent ρ. Thus, at a contributing pair (U_N, θ_N), Step 3 gives exactly one constituent χ of $\rho|_X$ with θ_N in $\chi|_{U_N}$. Moreover, the central character on $\mathrm{Z}(X)$ is rational, so Step 5 gives that σ permutes the constituents of $\rho|_X$.

By [208, Lemma 7.23(i)–(ii)], γ has even natural weights and $\mathrm{Ad}(\gamma(z))N = z^{-2}N$. Thus, the half weights define γ_0 in $\mathrm{SO}(V)$ over $\mathbb{F}_q$ with $\mathrm{Ad}(\gamma_0(z))N = z^{-1}N$. Put $h_\sigma = \gamma_0(b_\sigma^{-1})$, with the convention $\theta_N^h(u) = \theta_N(h^{-1}uh)$. Since $\mathrm{Ad}(h_\sigma)N = b_\sigma N$, the trace formula gives $\theta_N^{h_\sigma} = \theta_N^\sigma$. By Lang's theorem, the image of h_σ in $\mathbf{X}_{\mathrm{ad}}^F$ lifts to $\widetilde{\mathbf{X}}^F$ through the connected centre. Thus, χ^{h_σ} and χ^σ both belong to $\rho|_X$ and contain θ_N^σ on U_N. By uniqueness at (U_N, θ_N^σ), $\chi^\sigma = \chi^{h_\sigma}$.

Moreover, on a block of size $2j+1$, the positive γ_0-weights are $1, \ldots, j$. By the reflection formula [137, §24.i], $\nu(h_\sigma) = \eta_q(b_\sigma)^L$ for $L = \sum_{j=0}^{2s-1} j(j+1)/2 = \binom{2s+1}{3}$. Since $3L = s(2s-1)(2s+1)$, $L \equiv s \equiv a \bmod 2$. The two constituents of $\xi|_{\Omega(V)}$ are exchanged by the coset with $\nu = -1$, since $\xi\nu = \xi$ and $[\mathrm{SO}(V) : \Omega(V)] = 2$. Thus, the permutation sign is $\nu(h_\sigma) = \eta_q(b_\sigma)^a$. Moreover, for $a = 1$, by Lemma 6.57, we obtain $\eta_q(b_\sigma)$ for $\mathrm{SO}_2^\pm(q)$. □

Proposition 6.253. *Let the rational isometric Y_i have rank a and common nondegenerate symbol Λ. Then, for either $\xi \in \mathrm{Irr}(\mathrm{SO}(V))$ over (Λ, Λ), the two constituents of $\xi|_{\Omega(V)}$ have permutation sign $\eta_q(b_\sigma)^a$.*

Proof. If Λ is cuspidal or $a = 1$, we apply Lemma 6.252. If Λ is principal, apply Lemma 6.249. Otherwise, take $h \in H_1(\Lambda)$ giving a nondegenerate Λ' of rank $a-1$ by [208, Lemma 7.17(i)]. Take h on both Y_i, with singular lines E_i and $E = E_+ \oplus E_-$. In $S = \mathrm{SO}(V)$, we let $P_E = LR_u$ with $L = \mathrm{GL}_2(q) \times \mathrm{SO}(V')$, where $V' = E^\perp/E$. Thus, on L, the source is $\beta = \sigma_2 \boxtimes \xi'$, with σ_2 of colours $1, -1$ and ξ' over (Λ', Λ').

The spinor character on L is $\nu_L(g, x) = \eta_q(\det g)\nu(x)$, and $R_u \leq \Omega(V)$. Thus, the spin Levi maps to $L_0 = \ker\nu_L$. On σ_2, we choose the involution J_2 for $\eta_q \circ \det$ from Lemma 6.56. On ξ', we set $J' = e_{\chi_1'} - e_{\chi_2'}$ for the two constituents on $\Omega(V')$. Thus, $J_2 \otimes J'$ implements ν_L and has square I. Since the parents σ_2, ξ' are rational-valued, the signs on J_2, J' are $\eta_q(b_\sigma), \eta_q(b_\sigma)^{a-1}$, by Lemma 6.56 and induction on a. Consequently, the two constituents on L_0 have permutation sign $\eta_q(b_\sigma)^a$.

In $H = \mathrm{GSO}(V)$, take the regular parents $\rho, \widetilde{\beta}$ for S, L, respectively. The extracted plane has labels $1, -1$ with unipotent labels $1, 1$. Thus, the connected coefficient is $1 \cdot 1 = 1$ by [208, Lemma 7.17 and Proposition 7.18, Step 3]. For an isometry x centralizing t_s, conjugation on the volume of Y_- is multiplication by $\det(x|_{Y_-}) = \pm 1$. Thus, the series stabilizer of ρ under multiplier twists is $\{1, \eta_q \circ \mu\}$. The paired reflection k gives $\eta_q \circ \mu$. On $\widetilde{L}$, the reflections on Y_+', Y_-' give k, also at $a = 2$ when each Y_i' has rank 1. The two twists fix (Λ, Λ) and (Λ', Λ'). Consequently, $\rho|_S, \widetilde{\beta}|_L$ each has 2 constituents by [95, Lemma 5.2]. Moreover, $\mu(\widetilde{L}) = \mathbb{F}_q^\times$, so $H = S\widetilde{L}$. Thus, the sum in [208, Lemma 7.16(a)] is 2. The matrix on $\{\xi_1, \xi_2\}$ and $\{\beta_1, \beta_2\}$ is a permutation matrix.

Choose (ξ, β) with coefficient 1. Since $\beta\nu_L = \beta$, each of the two constituents on L_0 induces to β. Thus, each occurs with multiplicity 1 in $\xi|_{P_E \cap \Omega(V)}$, after inflation across R_u. An element $g \in L$ with $\nu_L(g) = -1$ swaps the constituents on $L_0, \Omega(V)$. Consequently, multiplicity 1 pairs the constituents on L_0 and $\Omega(V)$. Since ξ, β have values in $\mathbb{Q}$, the pairing is equivariant under σ. Thus, the sign $\eta_q(b_\sigma)^a$ on the source also holds on $\xi|_{\Omega(V)}$. □

Theorem 6.254. *Let $X = \mathrm{Spin}_{2n}^\epsilon(q)$ with q odd and $n \geq 5$. Let $p \mid |X|$ be odd with $p \nmid q$. Then, we put $M = \mathrm{N}_X(\widehat{P})$ for $\widehat{P} \in \mathrm{Syl}_p(X)$. Let s have type $I(a,b)$ or $I(a,a)$ and rational spaces Y_+, Y_-. Then, we have that $\mathcal{E}(X,s) \cap \mathrm{Irr}_{p'}(X)$ has an $\mathcal{H}_p$-equivariant bijection onto the sector $\mathcal{S}_s \subseteq \mathrm{Irr}_{p'}(M)$ of* Theorem 6.246. *For isometric Y_+, Y_-, we choose the*

spinor character ν_- on the negative factor B_- as in Proposition 6.248. *Moreover, each pair (χ,ψ) has $\chi(z)/\chi(1) = \psi(z)/\psi(1)$ for $z \in \mathrm{Z}(X)$. Both sets are empty exactly when s fails a test on w_i, U in* Lemma 6.244.

Proof. If $Y_+ \ncong Y_-$, we apply Theorem 6.246 or Corollary 6.247. If $Y_+ \cong Y_-$, the component swapping Y_+, Y_- changes the index by a factor 2 [208, Lemma 7.15]. Thus, at (w_+, w_-, U), apply Lemma 6.244. If the unordered Λ_i differ, we apply Proposition 6.248. If $\Lambda_+ = \Lambda_-$ is degenerate, apply Proposition 6.251. Thus, let $\Lambda = \Lambda_+ = \Lambda_-$ be nondegenerate. Let Λ^0 be the core of Λ on each U_i. Put $\lambda = \nu \det^i$ in the orientation i of Lemma 6.250. Then, if Λ^0 is nondegenerate and $r_0 > 0$, the parent Θ_0 has irreducible restriction θ_0 to $\mathrm{SO}(U)$. Let $J_0 = e_{\theta_1} - e_{\theta_2}$ for its two constituents on $\Omega(U)$. Thus, J_0 implements λ on Θ_0. By Proposition 6.253, J_0 has sign $\eta_q(b_\sigma)^{r_0}$. If $r_0 > 0$ and Λ^0 is degenerate, we set $J_0 = J_{\nu,0} J^i_{\det,0}$ from Proposition 6.251. Then, its sign is $1 = \eta_q(b_\sigma)^{r_0}$, since r_0 is even. If $r_0 = 0$, we set $J_0 = I$.

On Π, we set $J = (J_A/\sqrt{c_A}) \otimes J_0$ with J_A, c_A from Lemma 6.250. Since Π has stabilizer $\langle\lambda\rangle$ of order 2, the eigenspaces of J give the constituents of $\Pi|_{M_0}$. Thus, their permutation sign under $\sigma \in \mathcal{H}_p$ is $\eta_q(b_\sigma)^{ew+r_0} = \eta_q(b_\sigma)^a$. By Proposition 6.253, χ_0, χ_1 have sign $\eta_q(b_\sigma)^a$. Thus, either pairing $\chi_i \mapsto \psi_i$ is $\mathcal{H}_p$-equivariant. The inverse on Π in Proposition 6.248 recovers $\mathcal{S}_s$. Moreover, if z maps to $-I$, both scalars are $\epsilon_- \eta_q(-1)^a$ by Theorem 6.246. The kernel $\langle z_0\rangle$ is trivial on χ_i, ψ_i. □

If F^* swaps the geometric sign spaces Y_+, Y_-, then by Zhang [208, (7.17)], we obtain one factor $\mathrm{SO}^\epsilon_{2a}(q^2)$ in the centralizer at s. By the order formulas (Carter [40]), $q^{2a} - \epsilon$ and $q^{4j} - 1$ for $1 \le j < a$ cancel. Thus, the index at s has valuation $v_p(\prod_{j=1}^a (q^{4j-2} - 1))$. If e is even, it divides no index $2j - 1$. If $2 \nmid e$, we have that the case $c = 0$ is excluded by Proposition 6.29. Thus, $\mathcal{E}(X, s) \cap \mathrm{Irr}_{p'}(X) \ne \varnothing$ exactly when $2 \mid e$, or $4 \mid d$. At $2 \mid e$, we have that the label 1 gives a character of p'-degree by Digne–Michel [58, Proposition 11.5.6]. Moreover, Lemma 6.27 gives $\chi(z_0) = -\chi(1)$. The inverse in Proposition 6.29 is on $\mathrm{N}_X(Q)$ for the characteristic toral base Q. Moreover, $\mathrm{N}_X(P)$ includes K_j on the Sylow orbits.

Let $X = \mathrm{Spin}^\epsilon_{2n}(q)$ with q odd. Fix an odd prime $p \nmid q$, and put $e = \mathrm{ord}_p(q^2)$. Then, take $w > 0$ blocks V_i of dimension $2e$ and type τ, with residual space V_R satisfying $p \nmid |\mathrm{O}(V_R)|$. Let T_b have order $q^e - \tau$, with $p \mid q^e - \tau$ and $(q^e - \tau)_2 = 2$. Thus, the toral base is $Q = (T_b)^w_p$ and $P = Q \rtimes Y$, where $Y \in \mathrm{Syl}_p(\mathfrak{S}_w)$. Moreover, the odd lift of P is Sylow in X. Moreover, we put $M_Q = \mathrm{N}_X(Q)$, $M_P = \mathrm{N}_X(P)$, and let z_0 generate $\ker(X \to \Omega(V))$. If $D \in \{Q, P\}$, we let T_D be the active torus in M_D and put $U_D = (T_D)_{2'}$. Thus, we find $Q \le U_D$.

Theorem 6.255. *Let $\mathcal{F}_D$ consist of $\chi \in \mathrm{Irr}_{p'}(M_D)$ with $\chi(z_0) = -\chi(1)$ and $U_D \le \ker\chi$, for $D = Q, P$. Then, there is an $\mathcal{H}_p$-equivariant bijection $\mathcal{F}_Q \to \mathcal{F}_P$. Each pair (χ_Q, χ_P) has equal central characters on $\mathrm{Z}(X)$. Moreover, the residual $\langle a, d\rangle$-orbit of Γ and restriction multiplicities agree.*

Proof. By Lemma 6.231, Q is characteristic in P, and the distinct supports of the Q-weights give the blocks V_i. Thus, $V^Q = V_R$ and the normalizers stabilize V_A, V_R, where $V_A = \perp_i V_i$. On V_A, the natural Q-normalizer is $(T_b \rtimes C_{2e}) \wr \mathfrak{S}_w$. Moreover, modulo Q, its P-normalizer is $((T_{b,p'} \rtimes C_{2e})^w)^Y \rtimes \mathrm{N}_{\mathfrak{S}_w}(Y)$ by the lemma, including $e = 1$ as in Theorem 6.246. On V_R both groups have factor $\mathrm{O}(V_R)$. Given odd $L \le \Omega(V)$, we obtain a unique odd lift $\widehat{L}$ by [97, Theorems 3.8 and 3.12], since z_0 is central. Thus, M_Q, M_P are the even rational lifts. If $D = Q$, let $\mathcal{O}$ consist of singletons and put $K_* = \mathfrak{S}_w$. If $D = P$, choose the Y-orbits as $\mathcal{O}$ and $K_* = \mathrm{N}_{\mathfrak{S}_w}(Y)$. Thus, the count is $m = |\mathcal{O}|$. Let $\widetilde{A}$ be the active cover on $V_A \perp \langle u_0\rangle$. The formulas give $U_D \lhd \widetilde{A}$. Thus, put $A = \widetilde{A}/U_D$.

On $V_A \perp V_R \perp \langle u_0\rangle$, we choose the covers $\widetilde{A}, R$ with determinant bits b_A, b_R. The homogeneous lifts of $x \in \widetilde{A}$ and $y \in R$ are $x_0 u_0^{b_A(x)}$ and $y_0 u_0^{b_R(y)}$. Thus, the Clifford product gives $yx = z_0^{b_A(x) b_R(y)} xy$ by [137, Theorem 24.56]. Choose ω_A over $-I_{V_A}$ with a common sign at Q, P. It has square z_0^{ew}, and conjugation by ω_A acts by $x \mapsto z_0^{b_A(x)} x$ and fixes R. Consequently, we put $R^\flat = \{\omega_A^{b_R(y)} y : y \in R\}$, with $[\widetilde{A}, R^\flat] = 1$. Given $y, y' \in R$, the extra product factor is $(\omega_A^2)^{b_R(y) b_R(y')}$. Thus, $R^\flat$ is a subgroup with $\widetilde{A} \cap R^\flat = \langle z_0\rangle$ by projection to V_R. Since $\omega_A \in \widetilde{A}$, the full preimage is generated by $\widetilde{A}, R^\flat$. Thus, $R^\flat$ depends only on R, ω_A. The rational group is $\ker a \cap \ker d$, for parity a and Frobenius boundary d. On A, we put $a_A = (-1)^{b_A}$ and take the active boundary d_A. If the residual lift F_R has parity t_R, then $a|_A = a_A$ and $d|_A = d_A a_A^{t_R}$ by the Clifford sign formula. On $R^\flat$, we restrict a, d. If $V_R = 0$, take $\ker a_A \cap \ker d_A$ in A.

Given any block V_i, choose h_i over $-I_{V_i}$ and a rational k_i over its rotation of order $2e$, with common coordinates at Q, P. Thus, $h_i h_j = h_j h_i$ and $h_i^2 = z_0^\epsilon$. The lift k_i conjugates h_i to $z_0 h_i$ and fixes h_j for $j \ne i$. Moreover, $k_j k_i = z_0 k_i k_j$ for $j \ne i$ and $k_i^{2e} = z_0^{e(e+1)/2}$. The volume h_i is $(-\iota)^e \prod_{\nu=1}^e T_\nu(-1)$ in the exterior algebra, with $\iota^2 = -1$ and T_ν scaling the positive coordinate ν. For exterior multiplication ε_ν and contraction ι_ν, we put $R_\nu = (\varepsilon_\nu + \iota_\nu)(-1)^{\deg}$.

Thus, $R_\nu^2 = z_0$ and $R_\nu R_\mu = z_0 R_\mu R_\nu$ for $\mu \neq \nu$. Write $L = \bigwedge P((1,\dots,e))$. The norm-one negative lift is $v = T_1((-1)^{e-1})R_1 L$. Since L permutes the R_ν, $(R_1 L)^e = R_1 \cdots R_e$, whose square is $z_0^{e(e+1)/2}$. If $2 \mid e$, the extra torus element contributes each $T_\nu(-1)$ and $-T_\nu(-1)$ in exactly one of the $2e$ conjugates, with product $(-1)^e = 1$. On w negative blocks, the active Frobenius gives $v_i \mapsto z_0^{w-1} v_i$ and h_i to $z_0 h_i$. Thus, $k_i = h_i^{(w-1) \bmod 2} v_i$ has $d_A(k_i) = 1$. Since $(h_i v_i)^2 = z_0^{e+1} v_i^2$ and $2 \mid e(e+1)$, $k_i^{2e} = v_i^{2e}$. On a positive block V_i, e is odd and $k_i = (R_1 \cdots R_e)L$. Since $L^e = 1$ and L fixes $R_1 \cdots R_e$, the power is again $z_0^{e(e+1)/2}$. In both cases $d_A(h_i) = -1$ and $d_A(k_i) = 1$. Thus, an orbit product of the h_i has boundary -1 and parity 1, and an orbit product of the k_i has parity -1. Consequently, a_A, d_A are independent at P, Q.

Given $\pi \in K_*$, we let $b(\pi)$ be its parity bit on w letters. Its lift is $\widehat{\pi} = \iota^{(e \bmod 2) b(\pi)} \bigwedge P(\pi)$. Thus, conjugation acts by $\big(h_i, k_i\big) \mapsto \big(h_{\pi(i)}, k_{\pi(i)}\big)$. $\widehat{\pi}\widehat{\pi'} = z_0^{eb(\pi)b(\pi')} \widehat{\pi\pi'}$ and $d_A(\widehat{\pi}) = (-1)^{b(\pi)}$. On negative blocks V_i, reordering the cycles gives $d_A(\widehat{\pi}) = (-1)^{b(\pi)}$. In this case $q \equiv 1 \bmod 4$ if e is odd. On positive blocks, e is odd and $q \equiv 3 \bmod 4$, so $\iota^q/\iota = -1$.

Thus, for $O \in \mathcal{O}$, we put $H_O = \prod_{i \in O} h_i$ and $Z_O = \prod_{i \in O} k_i^2$. In $K_O = \prod_{i \in O} k_i$, order the factors by increasing i. Since $|O|$ is odd, $H_O^2 = z_0^e$, and K_O acts by $H_O \mapsto z_0 H_O$ and fixes $H_{O'}$ for $O' \neq O$. Moreover, $K_O^2 = z_0^{\binom{|O|}{2}} Z_O$. The torus in A is generated by z_0, H_O, with independent H_O modulo z_0. Take $\lambda(z_0) = -1$ and $\mathrm{i} = \sqrt{-1}$. If $e = 2j$, set $(\lambda(H_O), \beta, \widetilde{\lambda}(\widehat{\pi})) = (1, \mathrm{i}^{e+1}, 1)$. If e is odd, the triple is $(\mathrm{i}^{|O|}, (-1)^{(e+1)/2}, (\beta\mathrm{i})^{b(\pi)})$. In both cases $\widetilde{\lambda}(Z_O) = \beta^{|O|}$. Thus, every faithful torus weight is A-conjugate to λ. The inertia group I is generated by the torus, the Z_O, and the $\widehat{\pi}$. Indeed, a rotation on O fixes λ exactly at even exponent, and K_* takes each orbit O to one of length $|O|$. Hence, $[A : I] = 2^m$ and the quotient of I by the torus is $W_* = C_e^m \rtimes K_*$. Thus, these values define $\widetilde{\lambda}$, since $Z_O Z_{O'} = Z_{O'} Z_O$, $\beta^e = (-1)^{e(e+1)/2}$ and the permutation values account for $z_0^{eb(\pi)b(\pi')}$. If e is even, $2e \mid p-1$ and $|O| \equiv 1 \bmod 4$. If e is odd, the entry for Z_O includes $|O| \equiv 3 \bmod 4$.

If $\pi = (i, i+1)$ swaps singleton blocks, the lift $h_i \widehat{\pi}$ has value 1 under $\widetilde{\lambda}$ when e is even. Moreover, for odd e, the operator $\mathrm{i}^{e-1} h_i \widehat{\pi}$ has value 1. Put $k_* = \prod_i k_i$. If e is even, k_* fixes the torus weight on $\ker d_A$. Thus, extending $\widetilde{\lambda}|_{I \cap \ker d_A}$ across k_* gives values $\pm\sqrt{\widetilde{\lambda}(k_*^2)}$. Conjugation by H_O exchanges these two extensions in A. Thus, both occur under $\Phi_*(\eta)$. By [95, Theorem 6.11 and Corollary 6.17], the faithful parents are $\Phi_*(\eta) = \mathrm{Ind}_I^A(\widetilde{\lambda} \otimes \mathrm{Inf}\, \eta)$ for $\eta \in \mathrm{Irr}(W_*)$, with degree $2^m \eta(1)$.

At Q, the relative group is $W_* = C_e \wr \mathfrak{S}_w$. Thus, write $w = \sum_h m_h p^h$ with $0 \leq m_h < p$. At P, the m_h orbits of length p^h give $W_* = \prod_h ((C_e \times K_h) \wr \mathfrak{S}_{m_h})$, where K_h normalizes the Sylow group on p^h slots. Thus, in each colour $u \in \mathbb{Z}/e\mathbb{Z}$, the towers of Proposition 6.235 give $\eta \mapsto \eta^\sharp$. At depth h, the total is m_h, and vertices $x \in \mathbb{F}_p^h$ give labels κ_x of K_h by Lemma 6.234. Conversely, if $\lambda_{u,h,x}$ has size at most $m_h < p$, it is a p-core. Thus, the inverse towers recover η. Both models are over $K = \mathbb{Q}(\zeta_{p-1})$, fixed by $\mathcal{H}_p$. Moreover, if $e = 2j$, both $\eta, \eta^\sharp$ have the translation $u \mapsto u + j$. The total exponent of K_O is odd since $|O| = p^h$. On the source module, $c = \prod_O Z_O$ has scalar $\beta^w \gamma_\eta$. If $m_{u,h}$ counts boxes at colour u and depth h, the exponent of ζ_e in γ_η is $\sum_{u,h} u p^h m_{u,h}$ at Q and $\sum_{u,h} u m_{u,h}$ at P. Since $e \mid p-1$, the exponents are equal modulo e. Consequently, both scalars are $\beta^w \gamma_\eta$.

Put $\omega_A = \prod_i h_i$, $T = \Phi_*(\eta)(\omega_A)$, and $S(g \otimes v) = d_A(g)\, g k_* \otimes v$ on the representation for $\Phi_*(\eta)$. Moreover, $k_*, \omega_A \in A$ at Q, P. Conjugation by k_* changes H_O by z_0 and $\widehat{\pi}$ by $z_0^{b(\pi)}$, and fixes Z_O. Thus, conjugation by k_* fixes W_* pointwise and multiplies $\widetilde{\lambda}$ by d_A, including on K_h. Consequently, S is well defined and implements d_A, while T implements a_A. Since $d_A(k_*) = 1$, $T^2 = (-1)^{ew}$, $ST = (-1)^w TS$ and $S^2 = (-1)^{\binom{w}{2}} \beta^w \gamma_\eta$. Thus, the three constants agree for $\eta, \eta^\sharp$. Hence, compare the coefficient action on T, S with these squares.

If e is even, S, T have matrices over K. If e is odd, take $K(\mathrm{i})$. If $\sigma \in \mathcal{H}_p$ acts by $\mathrm{i} \mapsto -\mathrm{i}$, then these values on $H_O, \widehat{\pi}$ gain factors $-1, (-1)^{b(\pi)}$, respectively, while η is fixed. Thus, $g \otimes v \mapsto g k_* \otimes \sigma(v)$ gives $\Phi_*(\eta)^\sigma \simeq \Phi_*(\eta)$. The map matches T by equivariance and S since $d_A(k_*) = 1$. If $\sigma(\mathrm{i}) = \mathrm{i}$, conjugate the coefficients alone. Consequently, $\Phi_*(\eta)^\sigma = \Phi_*(\eta)$.

Let $\Gamma \in \mathrm{Irr}(R^\flat)$ be faithful on z_0. Since z_0 has scalar -1 on both factors, $\rho = \Phi_*(\eta) \boxtimes \Gamma$ is a character of $AR^\flat$. Let L_Γ consist of $a^u d^v$ with residual restriction fixing Γ. Given any $a^u d^v \in L_\Gamma$, choose a residual intertwiner R_{uv}. Thus, $\mathrm{End}_{M_D/U_D}(V_\rho)$ has basis $T^{u+t_R v} S^v \otimes R_{uv}$ for $a^u d^v \in L_\Gamma$. By Clifford theory [43], the summand at $\mu = a^u d^v$ is $\mathrm{Hom}_{AR^\flat}(V_\rho, V_{\rho\mu})$. Its dimension is 1 for $\mu \in L_\Gamma$ and 0 otherwise. If $V_R = 0$, we choose all 4 characters with trivial residual factor. Since R_{uv} agree at Q, P, the relations on S, T give an isomorphism f of commutants. Thus, f pairs their primitive central idempotents e_θ. Given any θ on M_D/U_D, the block at e_θ is $\mathrm{Mat}_{m_\theta}(\mathbb{C})$, where m_θ is its restriction multiplicity.

Moreover, for $\sigma \in \mathcal{H}_p$, the active map fixes S, T. On $R^\flat$, take the common map with Γ^σ. If Γ^σ differs from Γ by $a^u d^v$, the active operator $T^{u+t_R v} S^v$ acts by equal conjugation at Q, P. Moreover, $\mathrm{Ad}(cC) = \mathrm{Ad}(C)$ for an isomorphism C and $c \neq 0$. Thus, the correspondence of idempotents is $\mathcal{H}_p$-equivariant, also when Γ returns modulo $a^u d^v$. If the total volume ω is rational, then $\rho(\omega) = \pm T \otimes \Gamma(\omega_R)$ for residual volume ω_R and one choice of sign at Q, P. Consequently, the scalars on ω, z_0 agree.

By induction, each $\chi \in \mathcal{F}_D$ lies below a full parent ρ. Since U_D is normal, $U_D \leq \ker \rho$. Moreover, the quotient by M_D/U_D has order 4. By [95, Theorem 6.2], $p \nmid \rho(1)$ exactly when $p \nmid \chi(1)$. By [95, Lemma 5.2], parents ρ, ρ' have a common constituent precisely when $\rho' = \rho a^u d^v$ for some $u, v \in \{0, 1\}$. Consequently, the tower inverse on $\eta^\sharp$ and the idempotents e_θ give all pairs in $\mathcal{F}_Q, \mathcal{F}_P$. □

Proposition 6.256. *Let $s \geq 2$ and let Λ be the cuspidal symbol over $\mathbb{F}_{q^2}$ in the exchanged series $I(s^2, s^2)$. Suppose that $d = \mathrm{ord}_p(q) = 4j$, the j-cocore is $\varnothing$, and the relative label η has p'-degree. With $w = s^2/j$, the fibre of Λ admits an $\mathcal{H}_p$-equivariant bijection at $M_P = \mathrm{N}_X(P)$ with equal central characters. If s is odd, the fibre has two $\mathcal{H}_p$-fixed characters. If s is even, it has four $\mathcal{H}_p$-fixed characters at each value on ω.*

Proof. If $\Lambda = (\{0, \ldots, 2s-1\}, \varnothing)$, the runners of Proposition 6.29 give $\#\{v \in \mathbb{Z}_{\geq 0} : u + jv < 2s\} \equiv 0 \pmod 2$ for every $0 \leq u < j$. Thus, $j \mid s$ and $w \equiv s \bmod 2$. In [208, Proposition 7.26], set $m = 0, y = 1$. Then, (7.24) gives $\epsilon = (-1)^s$. By Step 1 and (7.24), even s gives two rational classes $[s_0], [s_1]$ with opposite scalars at ω and 4 constituents each. If $2 \nmid s$, the class has 2 constituents. Moreover, the scalar at z_0 is -1 in both cases.

Let $q = r^f$ and $\sigma(\zeta_r) = \zeta_r^{b_\sigma}$, with $b_\sigma \in \mathbb{F}_r^\times$. On the Jordan blocks for N in [208, Lemma 7.23 and (7.27)], choose γ_0 from Lemma 6.252. Thus, $h_\sigma = \gamma_0(b_\sigma^{-1})$ takes θ_N to θ_N^σ and has spinor character $\nu(h_\sigma) = \eta_q(b_\sigma)^s$. If s is even, h_σ lifts to X. If s is odd, the natural space V has type -1 and rank $2s^2$. Then, $\nu(-I_V) = -1$, so replace h_σ by $-h_\sigma$ when $\nu(h_\sigma) = -1$, with $\theta_N^{-h_\sigma} = \theta_N^{h_\sigma}$. Thus, we have a spin lift $\widehat{h}_\sigma \in X$.

On each Jordan block, h_σ before correction has determinant 1 and so fixes its Clifford volume ω_i. The correction $-I_V$ fixes every even product of the ω_i. Thus, $\eta_E^{\widehat{h}_\sigma} = \eta_E$ for the character η_E on E^F selected in Step 3 of [208, Proposition 7.26]. By Step 5, η_E has values in $\mathbb{Q}(\mathrm{i})$, with $\mathrm{i}^2 = -1$. Since $4 \mid p-1$, $\eta_E^\sigma = \eta_E$ for $\sigma \in \mathcal{H}_p$. Moreover, the scalars on $\mathrm{Z}(X)$ are rational, so Step 5 gives χ^σ in the packet of χ. Consequently, at (U_N, θ_N^σ), both $\chi^\sigma, \chi^{\widehat{h}_\sigma}$ have the same η_E. By Step 3, (U_N, θ_N^σ) and η_E determine the constituent. Thus, $\chi^\sigma = \chi^{\widehat{h}_\sigma} = \chi$.

Locally $e = 2j$. By Proposition 6.29, Λ with empty cocore gives labels $\eta, \eta\varepsilon_W$, where ε_W shifts colours by j. Since Λ has unequal rows, $\eta \neq \eta\varepsilon_W$. By Theorem 6.255, their images Φ_1, Φ_2 are fixed by all twists $a^u d^v$. Thus, Φ_1, Φ_2 have disjoint restrictions. Since $V_R = 0$, the commutant has basis $1, S, T, ST$. If w is odd, then $ST = -TS$ with $S^2, T^2 \neq 0$. Thus, the algebra is $\mathrm{Mat}_2(\mathbb{C})$, and each Φ_i contributes one $\mathcal{H}_p$-fixed constituent of multiplicity 2. If w is even, $ST = TS$ and $T^2 = 1$. Moreover, in $S^2 = (-1)^{\binom{w}{2}} \beta^w \gamma_\eta$, the sign has root in $\mu_4 \subset K = \mathbb{Q}(\zeta_{p-1})$, and $\gamma_\eta \in \mu_e$ has a root in $\mu_{2e} \subset K$. Thus, we have that S, T give 4 joint idempotents over K. Given any Φ_i, two idempotents have $T = 1$ and two have $T = -1$. Since T acts as the chosen rational volume ω, there are 4 local characters over each value $\kappa(\omega) = \pm 1$. Consequently, pair χ, ψ with equal central character κ. □

Corollary 6.257. *Under* Proposition 6.29*, the image ψ at M_P determines the class $[s]$ and labels (Λ, λ). The relative label is recovered by $\eta^\sharp \mapsto \eta$. If e is even, the orbit of Γ separates $c = 0$ from $c > 0$.*

Proof. Let θ in the residual orbit be on $X_0 = \mathrm{Spin}(V_R)^F$, and let t_0 be its natural dual lift. Put $c_0 = \frac{1}{2} \dim \ker(t_0^2 + I)$, with $c_0 = 0$ if $V_R = 0$. By Proposition 6.29, the orbit of θ gives t_0 up to ± 1 and orthogonal conjugacy. Thus, c_0 depends only on the orbit of θ. At even e, the active weights are quadratic, so U_Q acts trivially. At odd e, the weight on $\{(g, z) : \det(g) = z^2\}$ is quadratic in z. Thus, the weight is trivial on U_Q. Moreover, $y = 1$ gives scalar -1 at z_0 by [208, (7.25)]. For both periods we have $(q^e - \tau)_2 = 2$, by the order for even e and by Lemma 6.27 for odd e. Consequently, Theorem 6.255 gives the bijection at M_P, with the orbit of Γ and corresponding idempotents e_θ.

By Lemma 6.27, an active eigenvalue ξ satisfies $\xi^{q^e - \tau} = (-1)^e$ and $\xi_{2'} = 1$. Thus, at even e, the active eigenvalues are $1, -1$, and at odd e they are $\mathrm{i}, -\mathrm{i}$. Both have multiplicity e in each case. Consequently, $t_0^4 = I$ is equivalent to $t^4 = I$ on V. By Proposition 6.29, we recover $c = c_0$ if e is even and $c = c_0 + ew$ otherwise, then $a = (n - c)/2$. In particular, $c_0 = 0$ gives the involution case when e is even. The inverse $\eta^\sharp \mapsto \eta$ and the cores recover (Λ, λ) by Proposition 6.29. If there are 2 rational classes, the scalar at ω gives $[s]$ by [208, (7.25)]. Consequently, the residual orientation and e_θ give the individual constituent ψ. □

Proposition 6.258. *Let $n \geq 5$, with e odd and $\tau = -\eta_q(-1)$. Take the faithful pure type A series $\mathcal{E}(X, s)$ of sign τ, where $X = \mathrm{Spin}^{\epsilon}_{2n}(q)$ and $\epsilon = \tau^n$. Then, we write $n = ew + r_0$ with $0 \leq r_0 < e$. For each e-core ρ of size r_0 and p'-label η of $C_e \wr \mathfrak{S}_w$, we find four members at M_P if $2 \mid n$ and one if $2 \nmid n$. If n is odd, pair $\lambda \vdash n$ with $\psi \in \mathrm{Irr}_{p'}(M_P)$ $\mathcal{H}_p$-equivariantly with equal scalars on $\mathrm{Z}(X)$, also at $e = 1$. If n is even, the marked M_Q-branches in* Lemma 6.36 *have the corresponding $\mathcal{H}_p$-action at M_P in each class $[s]$.*

Proof. On a pure type A space of rank r_0, the type is τ^{r_0}. Thus, rank e would have type $\tau^e = \tau$, whereas the Sylow residue of rank e has type $-\tau$ by Lemma 6.96. Consequently, $r_0 < e$. If $r_0 > 0$, we let Γ_R be the parent on R at ρ, and put $J_R = \mathrm{Spin}(V_R)^F$. For even $r_0 > 0$, the proper parent has restriction $\theta_1 + \theta_2$ on J_R at $[s]$ by Lemma 6.21. At $r_0 = 2$, the comparison follows from the rank 2 restriction. Moreover, the graph γ exchanges the classes and acts by $\omega_R \mapsto z_0\omega_R$. Thus, $\Gamma_R|_{J_R}$ has 4 distinct constituents. If $2 \nmid r_0$, we let θ be the spin character at ρ. The eigenspace exchange γ has determinant -1 and acts by $t_0 \mapsto -t_0$. Since inverse transpose fixes ρ, $\theta^\gamma = \theta$, also at the torus for $r_0 = 1$. Moreover, $\nu(-I_{V_R}) = \tau^{r_0}\eta_q(-1)^{r_0} = -1$. Thus, $R^+ = \ker a_R = J_R\langle\omega_R\rangle$, with $\omega_R^2 = z_0$. The two extensions $\widehat{\theta}_+, \widehat{\theta}_-$ on R^+ have scalars $\mathrm{i}, -\mathrm{i}$ at ω_R. Since $\omega_R^\gamma = z_0\omega_R$, conjugation by γ swaps the extensions. By [95, Theorem 6.11], $\mathrm{Ind}_{R^+}^R \widehat{\theta}_+ = \Gamma_R$ and $\Gamma_R|_{J_R} = 2\theta$. Thus, the orbit at ρ gives one parent Γ_R, with $\|\Gamma_R|_{J_R}\|^2 = 4$. Thus, all twists by a_R, d_R fix Γ_R.

On $R^\flat$, the product formula of Theorem 6.255 contributes $(\omega_A^2)^{b_R(y)b_R(y')}$. Thus, on a model for Γ_R, we choose the matrix $\mathrm{i}^{ewb_R(y)}\Gamma_R(y)$ at $\omega_A^{b_R(y)}y$. On J_R, the factor is 1 since $b_R(y) = 0$. Thus, the residual commutant is $\mathbb{C}^4$ or $\mathrm{Mat}_2(\mathbb{C})$ according to r_0 mod 2. For the two signs a_R, d_R, this gives commutator $(-1)^{r_0}$. A basis change in $\langle a, d\rangle \cong \mathbb{F}_2^2$ has determinant 1, so the alternating pairing agrees. Moreover, $ST = (-1)^w TS$ on $\Phi_*(\eta^\sharp)$ by Theorem 6.255. Thus, on M_P, the commutator is $(-1)^{w+r_0} = (-1)^n$, with residual factor 1 at $r_0 = 0$. The commutant on M_P has dimension 4, so the restriction has 4 distinct constituents for $2 \mid n$ and one of multiplicity 2 for $2 \nmid n$. The inverse $\eta^\sharp \mapsto \eta$ then recovers the ordered e-quotient. Together with ρ, it determines $\lambda \vdash n$.

If n is odd, the eigenspace exchange g at t has $\det(g) = -1$. Thus, $C_{\mathbf{X}^*}(s)$ is connected. By Lang's theorem, $[s]$ is unique. Each λ gives one global χ with values in $\mathbb{Q}$ by [208, Propositions 7.6(i) and 7.13(i)]. Moreover, τq has odd order e modulo p, so each factor $q^k + \tau^k$ of the semisimple index is a p-unit for $1 \leq k < n$. Thus, the hook formulas at λ give the cut at (ρ, η) [84] and [190]. At $e = 1$, the core is $\varnothing$. Then, the identity $v_p((\tau q)^k - 1) = v_p(\tau q - 1) + v_p(k)$ for hook lengths and $1 \leq k \leq n$ gives $v_p(u_\lambda(1)) = v_p(f^\lambda)$. The local $\eta^\sharp$ is $\mathcal{H}_p$-fixed by Theorem 6.255. The orbit at ρ is $\mathcal{H}_p$-stable, and any change in $\mathrm{i}^{ewb_R(y)}$ is a twist by a_R. Consequently, its unique constituent ψ is $\mathcal{H}_p$-fixed. Since $\epsilon q \equiv -1 \bmod 4$, the centre is $\langle z_0\rangle$, where both χ, ψ are faithful. If n is even, by Lemma 6.36, we obtain 2 branches at M_Q in each class $[s]$. The torus weights are trivial on U_Q and the residual core is ρ. Thus, (ρ, η) has all 4 constituents at M_Q. Apply Theorem 6.255 at $[s_0], [s_1]$. The graph γ exchanges $[s_0], [s_1]$ and acts by $\omega \mapsto z_0\omega$. Thus, the opposite scalars at ω and the $\mathcal{H}_p$-action within each pair agree at M_Q, M_P. □

Theorem 6.259. *Let $X = \mathrm{Spin}^{\epsilon}_{2n}(q)$ with q odd. Fix an odd prime $p \nmid q$, with toral base Q of $P \in \mathrm{Syl}_p(X)$. Thus, for $D = Q, P$, we put $\mathcal{Y}_D = \{\chi \in \mathrm{Irr}_{p'}(\mathrm{N}_X(D)) : z_0 \in \ker\chi\}$. Then, we find an $\mathcal{H}_p$-equivariant bijection $\mathcal{Y}_Q \to \mathcal{Y}_P$ with equal scalars on $\mathrm{Z}(X)$. Every torus colour ξ is allowed, including weights nontrivial on Q, with the relative labels κ_x fixed throughout.*

Proof. First, take $B_D = \mathrm{N}_{\mathrm{O}(V)}(D)$ on the natural images of Q, P. For determinant d and spinor character b, we put $B_D^0 = \ker(d|_{B_D}) \cap \ker(b|_{B_D})$. Thus, the odd lifts of Q, P in Theorem 6.255 give $\mathrm{N}_X(D)/\langle z_0\rangle \cong B_D^0$. Put $H = T_b \rtimes C_{2e}$, with $T_b = C_{q^e-\tau}$. Then, $B_Q = (H \wr \mathfrak{S}_w) \times \mathrm{O}(V_R)$. Let ϑ be a weight of Q in $\chi \in \mathrm{Irr}_{p'}(B_P)$. Its inertia I has p'-index by [95, Theorem 6.2]. Thus, I contains the normal Sylow subgroup $P = Q \rtimes Y$, so $[Q, Y] \leq \ker\chi$. On an orbit O of Y, put $Q_O = \prod_{i\in O}(T_b)_{p,i}$. The product map $Q_O \to (T_b)_p$ has kernel $[Q_O, Y]$. Thus, $Q_O/[Q_O, Y] \cong (T_b)_p$. By Lemma 6.231, the $T_{b,p'} \rtimes C_{2e}$ coordinates are constant on O. Thus, the constants and $\mathrm{N}_{\mathfrak{S}_w}(Y)$ split the quotient by Q. A rotation acts on $(T_b)_p$ by its block automorphism, and K_h fixes the coordinate product on O. Consequently, with $w = \sum_h m_h p^h$ and $0 \leq m_h < p$, $B_P/[Q, Y] \cong \prod_h((H \times K_h) \wr \mathfrak{S}_{m_h}) \times \mathrm{O}(V_R)$. The construction holds for every 2-part of T_b.

Each weight θ on T_b extends with value 1 on $(C_{2e})_\theta$. By [95, Theorem 6.11], $p \nmid \xi(1)$ for all $\xi \in \mathrm{Irr}(H)$, since $p \nmid 2e$. At Q, a parent has partitions λ_ξ with $m_\xi = |\lambda_\xi|$ and residual character Θ. The active degree is $w! \prod_\xi \xi(1)^{m_\xi}\chi^{\lambda_\xi}(1)/\prod_\xi m_\xi!$. Thus, p'-degree requires p'-degree for $\Theta, \chi^{\lambda_\xi}$ and no carry in $\sum_\xi m_\xi = w$. With $m_\xi = \sum_h m_{\xi,h}p^h$, the towers of Proposition 6.235 have $m_{\xi,h}$ boxes at level h. Moreover, $\sum_\xi m_{\xi,h} = m_h$, and each

node at $x \in \mathbb{F}_p^h$ gives a label κ_x of K_h by Lemma 6.234. Thus, the partitions at x give the relative Specht labels at P. Conversely, we have that every node $\lambda_{\xi,h,x}$ has size less than p, so the inverse tower recovers λ_ξ. Thus, all p'-parents are paired, with Θ common and all weights on $Q/[Q,Y]$ allowed. Given $\sigma \in \mathcal{H}_p$, the colours become ξ^σ, and κ_x and $\chi^{\lambda_{\xi,h,x}}$ are fixed.

Moreover, for ξ, choose the runners of [30, Section 5, (62)]. Runner $(p-1)/2$ gives the augmentation character of $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$. Moreover, opposite runners $u, p-1-u$ pair under the quadratic twist. On $\mathbb{F}_p$, multiplication by v has sign (v/p), and translations have sign 1. At higher vertices x, each sign s has $s^{p^j} = s$. Moreover, a swap of two p^h-orbits has sign -1. On the abacus, $x \mapsto -1-x$ exchanges beads with gaps. Thus, $u \mapsto p-1-u$ and $\lambda_{\xi,h,x} \mapsto \lambda^{\mathsf{t}}_{\xi,h,x}$ give the sign twist at Q, P. A block twist acts on ξ, and each sign has odd exponent $|O|$. Moreover, d, b are trivial on $(T_b)_p$. On a block transposition π, $d(\pi) = 1$. Its -1-space is the antidiagonal V_i, with form scaled by 2. Thus, by the reflection formula [137, §24.i], $b(\pi) = \eta_q(-1)^e\tau$. Thus, the two twists d, b match at Q, P. Let $\omega_\xi = \xi(-I_{V_i})/\xi(1)$. At Q, its contribution is $\omega_\xi^{m_\xi}$, and at P it is $\omega_\xi^{m'_\xi}$ for $m'_\xi = \sum_h m_{\xi,h}$. Since p is odd, $m_\xi \equiv m'_\xi \bmod 2$, and $\omega_\xi^2 = 1$. Thus, with the residual factor Θ, the scalars on $-I_V$ agree. The image of $\mathrm{Z}(X)$ lies in $\{\pm I_V\}$.

Let $L \leq \langle d, b\rangle$ be the stabilizer of Π. Thus, the paired parent $\Pi^\sharp$ also has stabilizer L. Given a generator α, we choose $J_{\alpha,\xi}$ from ξ to $\alpha\xi$, with $J_{\alpha,\alpha\xi}J_{\alpha,\xi} = I$. Moreover, on Θ, choose a residual intertwiner of square I. Given two generators α, δ, we define c_ξ by $J_{\alpha,\delta\xi}J_{\delta,\xi} = c_\xi J_{\delta,\alpha\xi}J_{\alpha,\xi}$. The inverse choices give $c_{\alpha\xi} = c_{\delta\xi} = c_\xi^{-1}$. Thus, if $\alpha\xi \neq \xi$ or $\delta\xi \neq \xi$, the factors c_ξ, c_ξ^{-1} occur with equal multiplicities. If $\alpha\xi = \delta\xi = \xi$, $c_\xi = \pm 1$ and $c_\xi^{m_\xi} = c_\xi^{m'_\xi}$. The residual factor at Θ is common. If $b(\pi) = 1$, these factors give the commutator on Π.

For a p'-partition $\lambda \vdash l$, we choose the Specht module U_λ and the tower module V_λ on $K(l) = \mathrm{N}_{\mathfrak{S}_l}(Y_l)$ with $Y_l \in \mathrm{Syl}_p(\mathfrak{S}_l)$. Moreover, we let D_λ implement sign on l letters, from λ to λ^{t}. The affine models are over $K = \mathbb{Q}(\zeta_{p-1})$ by Lemma 6.234. The Specht matrices for λ are rational, so U_λ, V_λ are over K. Thus, for $\lambda \neq \lambda^{\mathsf{t}}$, we choose inverse maps $D_\lambda, D_{\lambda^{\mathsf{t}}}$ over K by the intertwining equations. If $\lambda = \lambda^{\mathsf{t}}$, normalize $D_\lambda^2 = I$ over $\overline{\mathbb{Q}}$. Thus, the ratio for D_λ under $\sigma \in \mathcal{H}_p$ is the sign on the two D_λ-eigenspaces. By [30, Section 5, (62)–(64)], the ratios on U_λ, V_λ agree. If $l < p$, we set $V_\lambda = U_\lambda$ and take $D_\lambda = I$ at $l = 0, 1$. Thus, $D_{\lambda^{\mathsf{t}}}D_\lambda = I$ with matching coefficient ratios.

Choose subsets I_ξ of the original letters with $m_{\xi,h}$ entire Y-orbits at height h, possible since $\sum_\xi m_{\xi,h} = m_h$. The permutation inertia is $\prod_\xi \mathfrak{S}(I_\xi)$ at Q and $\prod_\xi K(m_\xi)$ at P. Indeed, the orbits in I_ξ and their normalizers give $K(m_\xi)$. The block tensor contains ξ on m_ξ slots at Q and m'_ξ slots at P. Moreover, K_h fixes each block slot of ξ. Thus, the relative module at ξ affords sign on m_ξ letters.

Given any generator α, let n_α permute the subsets by $\xi \mapsto \alpha\xi$, with common ordered Y-sets for equal profiles. Thus, $n_\alpha \in \mathrm{N}_{\mathfrak{S}_w}(Y)$ at both ends and $n_\alpha^2 = 1$. Given α, δ, we have that the choices give $n_\alpha n_\delta = n_\delta n_\alpha$. On the source tensor let A_α combine the maps $J_{\alpha,\xi}$, the relative D_λ if the permutation exponent is $v_\alpha = 1$, and the residual map for Θ. Then, on Π or $\Pi^\sharp$, put $J_\alpha(g \otimes v) = \alpha(g)gn_\alpha \otimes A_\alpha v$. Each factor of A_α implements the corresponding restriction of α. Thus, J_α is well defined and $J_\alpha^2 = \alpha(n_\alpha)I$. Given α, δ, the coset factor is $\alpha(n_\delta)/\delta(n_\alpha)$. The relative factors agree by $D_{\lambda^{\mathsf{t}}}D_\lambda = I$, and the factors c_ξ match. Consequently, the monomials in J_α have matching products.

Under σ followed by μ, we choose the transport on I_ξ and a common map on Θ. On U_λ, V_λ, we apply σ and then D_λ if $v_\mu = 1$. Thus, the coset multipliers on n_α agree. On $\xi, \alpha\xi$ with $\alpha\xi \neq \xi$, the ratios cancel. On a fixed ξ, the signs occur to powers m_ξ, m'_ξ of equal parity. If $v_\alpha = 1$, the ratio on D_λ occurs with multiplicity 1 per ξ and agrees at Q, P. Thus, σ acts compatibly on the commutants modulo μ.

The monomials in J_α span the commutant on B_D^0 by Theorem 6.255. Thus, e_θ determines the constituent θ. They correspond equivariantly for $\mathcal{H}_p$ and for changes by μ. By [95, Lemma 5.2], different $\langle d, b\rangle$-orbits have disjoint restrictions. Moreover, since $[B_D : B_D^0]$ is a 2-power, each $\theta \in \mathrm{Irr}_{p'}(B_D^0)$ lies below a p'-parent by [95, Theorem 6.2]. Consequently, inflation gives $\mathcal{Y}_Q \to \mathcal{Y}_P$ with equal scalars at $\mathrm{Z}(X)$. □

6.10 Spin characters

Proposition 6.260. *Let $n \geq 5$. By* Theorem 6.259*, we choose B_Q, B_Q^0. Write $T_b = C_{q^e-\tau}$, with $\tau \in \{1, -1\}$. On $\Pi \in \mathrm{Irr}(B_Q)$, impose $\theta^4 = 1$ on each torus weight and $t_0^4 = I$ on the residual lift t_0. Then, each constituent on B_Q^0 determines both tests. The p'-parents Π satisfying it correspond bijectively to p'-parents ρ on $\mathrm{O}(V)$ with natural lift $t^4 = I$. Moreover, the bijection is equivariant for $\det, \nu$, has equal scalars at $-I$, and recovers the signed dimensions of t.*

Proof. Let w_+, w_-, v count the weights of orders $1, 2, 4$, respectively. The torus T_b has quartic weights precisely when e is odd and $\tau = \eta_q(-1)$. Then, inversion in C_{2e} exchanges θ, θ^{-1}, with inertia C_e. On V_R, let t_0 have sign ranks $r_\pm$, types $\delta_\pm$ and multiplicity c_0 at i. A space of rank 0 has type 1. Thus, the invariants of t are $a_\pm = r_\pm + ew_\pm$, $c = c_0 + ev$ and $\epsilon_\pm = \delta_\pm \tau^{w_\pm}$. Indeed, a sign block contributes rank e and type τ, whereas a quartic block contributes e reciprocal pairs. Moreover, the type A factor at $i, -i$ has sign $\eta_q(-1)$.

Restrict Π to $\prod_b T_b \times \mathrm{SO}(V_R)$ for its weights θ and series at t_0. Cyclic torus duality gives orders $1, 2, 4$. Thus, the lift t projects to s on the spin Levi L. The twist by $\det$ fixes t and the twist by ν gives $-t$. Consequently, the condition $t^4 = I$ descends to B_Q^0 by Theorem 6.259. If $a_+ + a_- > 0$, a reflection in V_+ or V_- centralizes t. Thus, $t^{\mathrm{O}(V)} = t^{\mathrm{SO}(V)}$. The dimensions and types $(a_\pm, \epsilon_\pm), c$ determine the rational class of t. Given s, we choose the orbits of $(a_\pm, \epsilon_\pm)$ under $t \mapsto -t$. Moreover, a similitude of multiplier μ scales each discriminant by $\mu^{2a_\pm}$, a square. If $a_+ = a_- = 0$, see Proposition 6.261 for the orientations at t. [132, Theorem 7.5 and Proposition 7.7] gives the ordinary labels at s.

Take the Jordan maps $\mathcal{L}_t, \mathcal{L}_{t_0}$ of [149, Theorem 8.12 and Corollaries 8.13–8.14]. Thus, the residual symbols C_+, C_- accompany the partition ρ_A at $i, -i$. For rank 0, set $C_\pm = \varnothing$. On sign colours $\pm$, take labels $\eta_\pm$ of $C_{2e} \wr \mathfrak{S}_{w_\pm}$. Moreover, on the negative group B_-, tensor the positive convention by ν_-, as in Proposition 6.248. Let I_A be the quartic inertia group. Thus, θ of order 4 extends with value 1 on C_e and $\mathfrak{S}_v$. Thus, the label η_A is an e-multipartition. The common odd rotation k acts by $\theta \mapsto \theta^{-1}$ and fixes $C_e, \mathfrak{S}_v$ by conjugation. Given a block swap π, $\nu(\pi) = 1$ by Theorem 6.259. Since $\nu(k_b^2) = 1$ for block rotations k_b, inversion fixes η_A.

Unpack $\eta_\pm$ over $C_\pm$ as in Lemma 6.244. At $\tau = 1$, take hooks, and at $\tau = -1$ take cohooks. Thus, pad modulo $2e$ with common origins. Then, m beads with partition π have heights $\pi_j + m - j$ for $1 \le j \le m$. Thus, row exchange maps index u to $u + e \bmod 2e$ and transposes $C_\pm$. The twist by $\det$ transposes both $\Lambda_\pm$. Thus, transposing only Λ_- can distinguish constituents on B_Q^0. If $4 \mid |T_b|$, unpack η_A over the e-core ρ_A. Otherwise, the quartic label is ρ_A.

To check the cores, put $R = r_+ + r_- + c_0$. By the order formula, $R \le e$, with type $-\tau$ if $R = e$. Thus, $r_\pm = e$ leaves no other space in V_R, and an e-move would give rank 0 of type -1, whether it is a hook for $\tau = 1$ or a cohook for $\tau = -1$. If $4 \mid |T_b|$, then $c_0 = e$ would give type $\tau^e = \tau$, contrary to type $-\tau$. Otherwise, the period at ρ_A is $2e > c_0$.

With $L = v_p(q^e - \tau)$, write $w = w_+ + w_- + v$. On B_Q, the valuation is that of $\frac{w!}{w_+!w_-!v!}\eta_+(1)\eta_-(1)\eta_A(1)$, since $p \nmid |\mathrm{O}(V_R)|$ and the inertia indices are 2-powers. On $\mathrm{O}(V)$, the formula in Proposition 6.71 assigns $bL + v_p(b!)$ to b blocks before division by $\eta(1)$. Losing a block adds at least L to $v_p(\rho(1))$, since the multinomial is integral. Consequently, p'-degree requires the full count w, no carry in $w_+ + w_- + v$ and p'-degree for η_+, η_-, η_A. Conversely, these tests suffice at $R \le e$, with type $-\tau$ at $R = e$. Thus, the inverse gives all ρ of p'-degree, since the remaining indices are 2-powers.

By the convention at ν_-, spinor twist exchanges $(\Lambda_+, \epsilon_+), (\Lambda_-, \epsilon_-)$ with η_A fixed. Thus, the correspondence matches the $\langle \det, \nu \rangle$-orbits. Put $\epsilon_- = (-1)^{x_-}$, $s_q = (q-1)/2 \bmod 2$ and $l_q = (q^2-1)/8 \bmod 2$. By [165, Proposition 2.7(i)], torus evaluation gives $(-1)^{a_- s_q + x_- + c l_q}$ at $-I$. Moreover, each negative block contributes $\tau\eta_q(-1)^e$, and each quartic block contributes $(-1)^{(q^e-\tau)/4} = (-1)^{e l_q}$. Thus, the scalar from t_0 times the block contributions gives $(-1)^{a_- s_q + x_- + c l_q}$. Since $\mathrm{Z}(X)$ maps into $\{I, -I\}$, this gives the central comparison. □

Proposition 6.261. *Let $X = \mathrm{Spin}_{2n}^\epsilon(q)$, with $n \ge 5$ and q odd. Let $p \mid |X|$ be odd with $p \nmid q$. Moreover, take the marked maps at M_Q cited below. Then, the series with $t^4 = I$ have an $\mathcal{H}_p$-equivariant bijection on the p'-characters to their sector at $M_P = \mathrm{N}_X(P)$. Each pair (χ, ψ) has equal scalars on $\mathrm{Z}(X)$. If $\chi(z_0) = \chi(1)$, we choose the sector at M_P from* Proposition 6.260 *and* Theorem 6.259. *Otherwise, take $t_0^4 = I$ by* Corollary 6.257*, with pure A as in* Proposition 6.258.

Proof. First, suppose that $\chi(z_0) = \chi(1)$ and put $W = V_+ \perp V_-$. If $c = 0$, then $\rho(g) \in \mathbb{Q}$ on $\mathrm{O}(V)$ by Lemma 6.245. If $c > 0$, the dual centralizer of t lies in the Levi dual to $L = \mathrm{GL}_c^{\eta_q(-1)}(q) \times \mathrm{SO}(W)$. Thus, take the signed induction of $\theta(\det)u_\lambda \boxtimes \tau_W$ from L by [72, Theorem 3.3.22], where $|\theta| = 4$ and τ_W belongs to an involution series. By [165, Theorem A(i)], τ_W has values in $\mathbb{Q}$. At $t|_W = \pm I$ or rank 1, see Lemma 6.245. Hence, ξ has values in $\mathbb{Q}(i)$.

If an improper isometry moves ξ, $\rho = \mathrm{Ind}_{\mathrm{SO}(V)}^{\mathrm{O}(V)} \xi$ has values in $\mathbb{Q}(i)$. For $W = 0$ this applies since $C_{\mathrm{O}(V)}(t) \le \mathrm{SO}(V)$. By [164, Lemma 4.7], every $g \in \mathrm{O}(V)$ is conjugate to g^{-1}. Thus, ρ has values in $\mathbb{Q}(i) \cap \mathbb{R} = \mathbb{Q}$. If ξ is graph-stable and $W \ne 0$, extend τ_W rationally to $\mathrm{O}(W)$ by Lemma 6.245. A reflection r_W on W normalizes L and the geometric radical $\mathbf{U}$. Thus, we choose the virtual character Z of $\mathrm{O}(V)$ from Lemma 6.82. The Lefschetz trace on $Y_{\mathbf{U}}$ and source values in $\mathbb{Q}(i)$ give $Z(g) \in \mathbb{Q}(i) \cap \mathbb{R} = \mathbb{Q}$. Moreover, $Z|_{\mathrm{SO}(V)} = \pm\xi$. Thus, the extension coefficients n_+, n_- in Z sum to ± 1. A Galois swap gives $n_+ = n_-$, contradicting $n_+ + n_- = \pm 1$. Consequently, both extensions have values in $\mathbb{Q}$.

On B_Q, the sign-colour modules are over $\mathbb{Q}(\zeta_{2e})$, fixed by $\mathcal{H}_p$. If $\sigma(i) = -i$, we apply the common rotation k to restore the weight θ. By Proposition 6.260, k fixes η_A. Moreover, the residual parent Θ has values in $\mathbb{Q}$. Thus, the full parents ρ, Π are fixed by $\mathcal{H}_p$. If the signed labels $(\Lambda_\pm, \epsilon_\pm)$ differ up to row order, neither ν nor $\det\nu$ fixes ρ, Π. If either Λ_i has unequal rows, det moves ρ, Π. Thus, by [95, Lemma 5.2], the restrictions to $\Omega(V), B_Q^0$ are irreducible and $\mathcal{H}_p$-fixed.

For the equal nondegenerate pair (Λ, Λ), take these row orders. Then, $\rho\nu = \rho$, and det transposes Λ_+, Λ_-. In $\mathrm{N}_{\mathrm{GO}(V)}(Q)$, we choose h with nonsquare multiplier μ as in Proposition 6.232. By [149, Corollary 8.3], h transposes Λ_-. If Λ^0 is nondegenerate, $\Theta\nu = \Theta$ gives the common core order $C_+ = C_-$. Thus, twist by $\det_{V_R}$ to select $C_\pm$ with $\eta_\pm$ fixed. Since $\det_{V_R}$ is rational and $\det_{V_R}(-I) = 1$, this keeps the action of $\mathcal{H}_p$ and the scalar. Conjugation by h gives the other relative order of $\Lambda_\pm$. Since $\nu^h/\nu \in \{1, \det\}$, the element h normalizes B_Q^0. On B_Q, the shift $u \mapsto u + e$ transposes η_-. By Proposition 6.260, η_A is fixed. Thus, the stabilizers are $\langle\nu\rangle, \langle\det\nu\rangle$. For degenerate or empty Λ^0, the shift on η_- distinguishes them. If Λ is degenerate, row transposition fixes $\Lambda^0, \eta_\pm$. Thus, the idempotents e_θ pair by Theorem 6.70. For nondegenerate Λ, we apply Corollary 6.67. Moreover, at $c = 0$, we apply Theorem 6.254. At M_Q, its inverse keeps $w_\pm$ and the spectrum of t_0.

If $a_+ = a_- = 0$, every active weight is quartic. Thus, e is odd, $\tau = \eta_q(-1)$ and $n = ew + r_0$, with $r_0 < e$ and type τ^n. The partition λ gives one full parent ρ and one Π. Both are fixed by $\det, \nu$, since $\det|_I = 1$ for the quartic inertia and ν acts by $\theta \mapsto \theta^{-1}$ with η_A fixed. The two constituents on $\mathrm{SO}(V)$ lie in distinct classes since $C_{\mathrm{O}(V)}(t) \leq \mathrm{SO}(V)$. An inverting isometry g acts by $t \mapsto -t = t^{-1}$. If $\tau = 1$, g exchanges the isotropic halves. If $\tau = -1$, g conjugates each Hermitian line. Thus, $\det(g) = (-1)^n$, and the ν-operator swaps the determinant eigenspaces if n is odd. On V_R, the commutator has sign $(-1)^{r_0}$. The active rotation k has determinant $(-1)^w$. Consequently, the coupled sign is $(-1)^{w+r_0} = (-1)^n$, also for $r_0 = 0$. The restriction of ρ or Π has multiplicity 2 at one constituent if $2 \nmid n$ and multiplicity 1 at each of 4 constituents if $2 \mid n$. If $2 \nmid n$, the constituent is $\mathcal{H}_p$-fixed, also at $e = 1$. If $2 \mid n$, apply Lemma 6.36 to each orientation of t. Since $\det(g) = 1$, the orientations remain distinct under ν. If $\chi(z_0) = \chi(1)$, the graph fixes the scalar at $-I$. Suppose that $\Lambda_\pm$ have equal rows with $(\Lambda_+, \epsilon_+) \neq (\Lambda_-, \epsilon_-)$. Thus, ρ, Π have stabilizer $\langle\det\rangle$. By [95, Lemma 5.2], the two constituents on $\mathrm{SO}(V)$ restrict irreducibly and distinctly to $\Omega(V)$. $\Pi|_{B_Q^0}$ has 2 distinct constituents.

Given $g \in \mathrm{SO}(V)$, by [164, Lemma 4.7], $h_g \in \mathrm{O}(V)$ with $h_g g h_g^{-1} = g^{-1}$ and $\det(h_g) = (-1)^n$. Given an arbitrary reflection r, we have $h_g r^{-n} \in \mathrm{SO}(V)$. Thus, g^{-1} and $r^n g r^{-n}$ are conjugate in $\mathrm{SO}(V)$, and $\overline{\xi} = \xi^{r^n}$. The reflection r exchanges the two constituents of $\rho|_{\mathrm{SO}(V)}$. Consequently, they are real if n is even and conjugate if n is odd. The values of ξ lie in $\mathbb{Q}(i)$, and the equal rows give $2 \mid a_\pm$. Thus, $n \equiv c \bmod 2$ gives permutation sign $(\sigma(i)/i)^c$ on $\rho|_{\mathrm{SO}(V)}$. The equal rows of $C_\pm$ also give $2 \mid r_\pm$. Consequently, the sign there is $(\sigma(i)/i)^{c_0}$.

On $\eta_\pm$, the determinant operator gives $u \mapsto u + e$ and swaps equal Specht factors over $\mathbb{Q}(\zeta_{2e})$. Moreover, on η_A it acts by $\det(g)$ on gI_A. Thus, for $\sigma(i) = -i$, the correcting rotation gives $(-1)^v$. If $\sigma(i) = i$, the factor is 1. Consequently, the local sign is $(\sigma(i)/i)^{v+c_0} = (\sigma(i)/i)^c$, since $c = c_0 + ev$ and e is odd if $v > 0$. If $t = \pm I$, we choose the representative with $\theta = 1$ and unipotent residue at $t_0 = I$. Thus, the local dual class for $\mathrm{C}_X(Q)$ is 1, with converse by torus duality. At $s = 1$, we apply [156, Theorem D]. Consequently, at M_Q, the parent inverse recovers all e_θ. By Theorem 6.259, transfer to M_P. If $\chi(z_0) = -\chi(1)$, take Corollary 6.257 or Proposition 6.258. Moreover, for Λ cuspidal, we apply Proposition 6.256. Thus, z_0 separates the sectors with scalars $1, -1$. □

Proposition 6.262. *Suppose that q and p are odd with $p \nmid q$. For the toral base Q of* Theorem 6.259*, we set $B_Q = N_{\mathrm{O}(V)}(Q)$. The p'-characters of $\mathrm{O}(V)$ and B_Q admit a bijection equivariant under $d = \det$ and the spinor twist ν. Moreover, the B_Q-labels determine the primary partitions λ_ξ, the ordered pair (Λ_+, Λ_-) and the form on V, including roots ξ of order divisible by p.*

Proof. With $e = \mathrm{ord}_p(q^2)$, let t_b be the sign of a block V_j of dimension $2e$. For $\xi \in \overline{\mathbb{F}}_q^\times \setminus \{1, -1\}$, we let h be least positive with $\xi^{q^h} = \xi^\delta$, where $\delta \in \{1, -1\}$ is unique. If $\delta = 1$, $x \mapsto x^q$ has two orbits of size h, paired by $x \mapsto x^{-1}$. Otherwise, $x \mapsto x^q$ has one orbit of size $2h$ and $\xi^{q^h} = \xi^{-1}$. Thus, at multiplicity m, the centralizer factor is $\mathrm{GL}_m(q^h)$ for $\delta = 1$ and $\mathrm{GU}_m(q^h)$ for $\delta = -1$, by [149, Theorem 8.12].

Set $k = \mathrm{ord}_p(\delta q^h)$. The torus T_b admits ξ exactly when $\xi^{q^e} = \xi^{t_b}$, or equivalently when $h \mid e$ and $\delta^{e/h} = t_b$. In that case $k = e/h$. Indeed, $(\delta q^h)^{e/h} \equiv 1 \bmod p$, since $q^e \equiv t_b \bmod p$. Conversely, $(\delta q^h)^j \equiv 1 \bmod p$ implies $q^{2hj} \equiv 1 \bmod p$ and so $e \mid hj$. Thus, the signed cycle C_{2e} has inertia C_k at ξ, since its orbit has size $2h$. Also, V_j contributes each of the $2h$ eigenvalues with multiplicity k.

Let v_ξ be the number of blocks at ξ. Thus, the weight on $C_k \wr \mathfrak{S}_{v_\xi}$ has a k-multipartition $\boldsymbol{\mu}_\xi$. On the residual space V_R, the Jordan label is a partition ρ_ξ of r_ξ by [149, Theorem 8.12]. Absent labels have $v_\xi = 0$ or $r_\xi = 0$ as appropriate. Since $p \nmid |\mathrm{O}(V_R)|$, its subgroup $\mathrm{GL}^\delta_{r_\xi}(q^h)$ has $r_\xi < k$, by the order factor $q^{hk} - \delta^k$. Consequently, ρ_ξ is a k-core, and $(\rho_\xi, \boldsymbol{\mu}_\xi)$ determines $\lambda_\xi \vdash m_\xi = r_\xi + kv_\xi$. If T_b has no weight at ξ, $v_\xi = 0$ and $\lambda_\xi = \rho_\xi$. At $1, -1$, by Proposition 6.260, we obtain Λ_+, Λ_- from the signed cores $\Lambda_{+,R}, \Lambda_{-,R}$, with the rank boundary on V_R. Thus, the primary spaces V_ξ give $V = \bigoplus_\xi V_\xi$ orthogonally. If $V_1 = V_{-1} = 0$, restriction from $\mathrm{O}(V)$ to $\mathrm{SO}(V)$ accounts for the proper orientations of t.

Conversely, the Jordan degree formula [58, Proposition 11.5.6] gives $p \nmid [\mathrm{SO}(V) : C_{\mathrm{SO}(V)}(t)]$ for the rational semisimple label t of a p'-character. Passage from $\mathrm{SO}(V)$ to $\mathrm{O}(V)$ changes degrees only by powers of 2. Therefore, we may assume that $Q \rtimes Y \le C_{\mathrm{SO}(V)}(t)$. By Lemma 6.231, Q has the blocks V_j with torus T_b on each block, and t has constant entries on the Y-orbits. Thus, the counts v_ξ, including the counts at $1, -1$, add without carries in base p.

Put $L = v_p(q^e - t_b) > 0$. Given an available ξ, a factor $\mathrm{GL}^\delta_{m_\xi}(q^h)$ with $b = \lfloor m_\xi/k \rfloor$ has order of valuation $bL + v_p(b!)$. Indeed, $hk = e$, $\delta^k = t_b$, and the contributing order factors are $q^{hkj} - \delta^{kj}$ for $1 \le j \le b$. By [84] and [190], a hook of λ_ξ of length a contributes to the denominator if $k \mid a$. For k-weight b', let $f_{\boldsymbol{\mu}_\xi}$ be the wreath character degree at $\boldsymbol{\mu}_\xi$. Thus, the unipotent degree has valuation $(b-b')L + v_p(b!/b'!) + v_p(f_{\boldsymbol{\mu}_\xi})$. Thus, its p'-degree condition is $b' = b$ and $p \nmid f_{\boldsymbol{\mu}_\xi}$. At $1, -1$, Lemma 6.244 gives the corresponding tests for Λ_+, Λ_-. Conversely, the multinomial has p'-order, so the tests on $\lambda_\xi, \Lambda_+, \Lambda_-$ give p'-degrees in $\mathrm{O}(V)$ and B_Q, by [58, Proposition 11.5.6] and Theorem 6.259. Thus, $(\rho_\xi, \boldsymbol{\mu}_\xi)$ reconstructs λ_ξ.

Given a nonsign weight ξ, the inertia $C_k = \langle c^{2h} \rangle$ lies in the signed cycle $\langle c \rangle = C_{2e}$. Thus, $d|_{C_k} = 1$, and $d(\pi) = 1$ for block swaps π because $\dim V_j = 2e$. Moreover, ν acts by $\xi \mapsto -\xi$. Suppose that $\nu(\pi) = 1$. We extend the weight ξ trivially on C_k and $\mathfrak{S}_{v_\xi}$. A simultaneous rotation g on the v_ξ coordinates has $gxg^{-1} = x$ for x in these two groups. Since $\nu|_{C_k \times \mathfrak{S}_{v_\xi}} = 1$, the twist changes the index ξ to $-\xi$ with multipartition $\boldsymbol{\mu}_\xi$. If $\nu(\pi) = -1$, the swap formula in Theorem 6.259 gives $(q^e - t_b)_2 = 2$. Consequently, the two orders are $a, 2a$ with a odd. Write ξ_o for the root of order a. We choose the extension at $-\xi_o$ to be the twist by ν of the extension at ξ_o, also on the permutations $\mathfrak{S}_{v_\xi}$. Because $\nu^2 = 1$, both directions fix $\boldsymbol{\mu}_\xi$. Given a coefficient automorphism σ, the orders of the weights $\xi_o^\sigma, (-\xi_o)^\sigma$ are $a, 2a$. Thus, the actions of d, ν are those of [149, Corollaries 8.13–8.14] on $\lambda_\xi, \Lambda_+, \Lambda_-$. □

Lemma 6.263. *Suppose that $\xi \mapsto \xi^{uq^j}$ changes a nonsign marking of type (h, δ), with $u \in \{1, -1\}$. Its determinant at multiplicity 1 is $\alpha = u^h \delta^j$. Thus, at $m = r + kv$, the primary determinant is $\alpha^m = \alpha^r (\alpha^k)^v$, with local determinants α^r on the residue and α^k on each block V_j. Similarly, the global scalar $c_V(-I)$ is $c_R(-I) \prod_{j=1}^v c_j(-I)$.*

Proof. The change $(u, j) \mapsto (u\delta, j + h)$ generates the ambiguity in uq^j on the marked orbit and fixes α. Indeed, h is least positive with $\xi^{q^h} \in \{\xi, \xi^{-1}\}$. If $\delta = 1$, we pair the dual $\mathbb{F}_{q^h}$-lines by $\mathrm{Tr}_{q^h/q}(xy)$. By the normal basis theorem, F_q on one line is an h-cycle. Thus, on the paired space $\det(F_q \oplus F_q) = (-1)^{2(h-1)} = 1$. The exchange $(x, y) \mapsto (y, x)$ has determinant $(-1)^h$. If $\delta = -1$, the multiplicity-one space is $\mathbb{F}_{q^{2h}}$ with form $\mathrm{Tr}_{q^h/q}(xx^{q^h})$. On a normal basis, $x \mapsto x^q$ is a $2h$-cycle of determinant -1, and $x \mapsto x^{q^h}$ inverts the torus C_{q^h+1}. Thus, the determinant is α for either δ. At multiplicity m, the rational centralizer $\mathrm{GL}^\delta_m(q^h)$ in $\mathrm{O}(V)$ has determinant 1 on V_ξ since the geometric factors act by A and $A^{-\mathrm{T}}$ by [149, Theorem 8.12]. Also, exchanging spaces V_ξ of dimension $2hm$ has determinant 1.

On each available block V_j, the determinant is $u^e t_b^j = \alpha^k$. Two representatives differ by $c \in C_k$, on which $d = 1$. Consequently, the determinant product on the v blocks and residue is $\alpha^{kv+r} = \alpha^m$. For the primary torus character θ of $\mathrm{GL}^\delta_m(q^h)$, the scalar at $-I$ is $\theta(-1)^m$ by [165, Proposition 2.7(i)]. Moreover, the signed norm on T_b has exponent $\frac{q^e - t_b}{q^h - \delta} = \sum_{\ell=0}^{k-1} \delta^\ell q^{h(k-1-\ell)} \equiv k \bmod 2$. Its sign has no effect at $-I$. Thus, the scalar product is $\theta(-1)^{kv} \theta(-1)^r = \theta(-1)^m$. For an unavailable orbit, $v = 0$ and $r = m$. □

Proposition 6.264. *Suppose that the eigenvalues of t lie in $\mathbb{F}_q^\times$ and that the two sign labels (Λ_+, ϵ_+), (Λ_-, ϵ_-) differ as unordered labels, including $\mathrm{rk}\,\Lambda_\pm$ and the rational types $\epsilon_\pm$. The bijection is $\mathcal{H}_p$-equivariant between the families in $\mathrm{Irr}_{p'}(\Omega(V))$ and $\mathrm{Irr}_{p'}(N_{\Omega(V)}(P))$, with equal scalars on $Z(\Omega(V))$. Suppose that $\Lambda_\pm = (A_\pm, A_\pm)$. If $\sigma \in \mathcal{H}_p$ stabilizes the marked label, then σ acts on χ^+, χ^- with sign $(-1)^{\sum_i m_i b_i}$. The multiplicity m_i is at $\{\xi_i, \xi_i^{-1}\}$, and $b_i = 1$ precisely for an exchange of $E_{\xi_i}, E_{\xi_i^{-1}}$, the two eigenspaces of t.*

Proof. Choose one ξ_i per inverse pair and set $W = V_1 \perp V_{-1}$. Moreover, for $U_j = \bigoplus_{i \le j} E_{\xi_i}$, let $\mathcal{F} = (U_1 \subset \cdots \subset U_s)$. Thus, its stabilizer is $P_\mathcal{F} = LU$, with $L = \mathrm{O}(W) \times \prod_i \mathrm{GL}_{m_i}(q)$ and unipotent radical U. Write τ_W for the involution parent on W and $\tau_i = (\theta_i \circ \det) u_{\lambda_i}$ at ξ_i, where u_{λ_i} is unipotent. Thus, $\chi = \mathrm{Ind}_{P_\mathcal{F}}^{\mathrm{O}(V)} \mathrm{Inf}_L^{P_\mathcal{F}} (\tau_W \otimes \bigotimes_i \tau_i)$. Since $W \ne 0$,

$P_{\mathcal{F}}$ contains a reflection r_W on W. Thus, Mackey restriction to $S = \mathrm{SO}(V)$ has one double coset $SP_{\mathcal{F}}$. The induction from $P_{\mathcal{F}} \cap S$ to S is Harish–Chandra induction $R^S_{L\cap S}$. Since $\mathbf{C}_{S^*}(t)$ lies in the dual Levi $\mathbf{L}^*_S$, [72, Theorem 3.3.22] takes distinct constituents of $\tau_W|_{\mathrm{SO}(W)}$, with the factors τ_i, to distinct irreducibles on S. If $\tau_W|_{\mathrm{SO}(W)} = \tau_W^+ + \tau_W^-$, r_W exchanges $\tau_W^{\pm}$ and their images $\chi^{\pm}$. By Clifford theory, $\chi|_S = \chi^+ + \chi^-$ with χ irreducible. Otherwise, $\chi|_S$ is irreducible, so replacing τ_W by $\tau_W \det_W$ gives $\chi d \neq \chi$ by [95, Lemma 5.2].

By Lemma 6.265, the character χ depends only on the Levi L and its character $\tau_W \otimes \bigotimes_i \tau_i$. If the halves $E_{\xi_i}, E_{\xi_i^{-1}}$ are exchanged, inverse transpose on $\mathrm{GL}_{m_i}(q)$ acts by $\theta_i \mapsto \theta_i^{-1}$ and fixes λ_i. Moreover, $\sigma \in \mathcal{H}_p$ acts by $\theta_i \mapsto \theta_i^{\sigma}$, powers the corresponding ξ_i, and fixes the λ_i at their transported roots. The character τ_W is rational by Lemma 6.245. Also, $\nu|_{\mathrm{GL}_{m_i}(q)} = \eta_q \circ \det$ by the reflection formula [137, §24.i]. On B_Q, the actions of d, ν follow from Proposition 6.262. Indeed, $h = \delta = 1$, so such a label requires $t_b = 1$. Consequently, $q^e \equiv 1 \bmod p$ implies $e = \mathrm{ord}_p(q)$ is odd and $k = e$. Thus, a simultaneous rotation g fixes $\boldsymbol{\mu}_{\xi_i}$ by Proposition 6.262, and values on C_e have order dividing $2e \mid p-1$.

Fix the order of $(\Lambda_+, \epsilon_+), (\Lambda_-, \epsilon_-)$. The $\mathcal{H}_p$-action fixes the positions $+, -$, while $\nu, d\nu$ exchange them. Thus, if $d\chi \neq \chi$, the restriction $\chi|_{\Omega(V)}$ is irreducible by [95, Lemma 5.2]. Otherwise, $\chi^+|_{\Omega(V)}$ and $\chi^-|_{\Omega(V)}$ are distinct irreducibles by [95, Lemma 5.2]. Suppose that $\sigma \in \mathcal{H}_p$ stabilizes the marking of t. The half exchange at ξ_i has determinant $(-1)^{m_i b_i}$ by Lemma 6.263. Also, permuting $E_{\xi_i} \oplus E_{\xi_i^{-1}}$ contributes determinant 1. Thus, the proper transporter includes r_W exactly when $\sum_i m_i b_i \equiv 1 \bmod 2$, and r_W exchanges χ^+, χ^-. If $V_{1,R} = V_{-1,R} = 0$, the determinant distinguishes the two $\mathrm{SO}(V_R)$-classes.

On the local weight coset gI with inertia I, the determinant operator is multiplication by $d(g)$. Inversion at ξ_i contributes $(-1)^{b_i}$ on each active block V_j, and the residual contribution is $(-1)^{r_i b_i}$. Since σ fixes the operators at $1, -1$, the sign on $\psi^{\pm}$ is $(-1)^{\sum_i (r_i + v_i) b_i}$. The exponents agree modulo 2 because $m_i = r_i + e v_i$ and $e \equiv 1 \bmod 2$. Thus, we match $\chi^{\pm}$ with $\psi^{\pm}$ over one $\mathcal{H}_p$-orbit representative λ. Agreement on $(\mathcal{H}_p)_\lambda$ extends the matching along $\mathcal{H}_p \cdot \lambda$. By Lemma 6.263 and the scalar of τ_W in Lemma 6.245, $\chi^{\pm}, \psi^{\pm}$ have equal scalars at $-I$. Finally, Theorem 6.259 gives the bijection to $\mathrm{Irr}_{p'}(N_{\Omega(V)}(P))$. □

Lemma 6.265. *Suppose that q is odd and $\mathbf{G} = \mathrm{O}(V)$ has Frobenius F. Let $\mathbf{L}^\circ$ be the F-stable Levi for the primary halves E_ξ and the sign space $W \neq 0$. Enlarge $\mathbf{L}^\circ$ by $\mathrm{O}(W)$ to $\mathbf{L}$. Then, we have that $R^{\mathbf{G}}_{\mathbf{L}}$ is independent of the parabolic $\mathbf{P}^\circ$ with Levi $\mathbf{L}^\circ$ from the marked halves. For $\sigma \in \mathcal{H}_p$ and $\alpha \in \{d, \nu\}$, we have $(R^{\mathbf{G}}_{\mathbf{L}} f)^\sigma = R^{\mathbf{G}}_{\mathbf{L}}(f^\sigma)$, $\quad (R^{\mathbf{G}}_{\mathbf{L}} f)\alpha = R^{\mathbf{G}}_{\mathbf{L}}(f\, \alpha|_{\mathbf{L}^F})$.*

Proof. A reflection r_W on W lies in each flag stabilizer. Thus, $\mathbf{L} = N_{\mathbf{G}}(\mathbf{L}^\circ, \mathbf{P}^\circ)$ for every such $\mathbf{P}^\circ$. Next, for semisimple $x \in \mathbf{G}^F$, we set $\mathbf{H} = C_{\mathbf{G}}(x)^\circ$. Let d_x take f to $u \mapsto f(xu)$ on the unipotent elements of $\mathbf{H}^F$ and to zero elsewhere. By [42, Proposition 3.3], $d_x R^{\mathbf{G}}_{\mathbf{L}} f$ is a rational linear combination of $R^{\mathbf{H}}_{\mathbf{J}_g} d_x({}^g f)$, where $\mathbf{J}_g = C_{{}^g\mathbf{L}}(x)^\circ$ for $g \in \mathbf{G}^F$ with $x \in {}^g\mathbf{L}$. Since $q > 2$, the operator $R^{\mathbf{H}}_{\mathbf{J}_g}$ depends only on $\mathbf{J}_g$ by [16, Theorem 3.9]. Also, a unipotent $u \in C_{\mathbf{G}}(x)^F$ lies in $\mathbf{H}^F$, since the component group of $C_{\mathbf{G}}(x)$ is a 2-group. Consequently, the sections d_x determine the whole class function $R^{\mathbf{G}}_{\mathbf{L}} f$ by Jordan decomposition. The matrix of $R^{\mathbf{H}}_{\mathbf{J}_g}$ is rational by [58, Lemma 9.1.5 and Proposition 9.1.6]. Since $d_x(f^\sigma) = (d_x f)^\sigma$, the identity for σ follows from the exchange formula. Moreover, $\alpha(u) = \alpha(v) = 1$ for unipotent u, v. Thus, $\alpha(g^{-1}xvg) = \alpha(xu) = \alpha(x)$ in every summand, so $d_x R(f\, \alpha|_{\mathbf{L}^F}) = d_x((Rf)\alpha)$ for $R = R^{\mathbf{G}}_{\mathbf{L}}$. □

Proposition 6.266. *We may choose the bijection Φ of* Proposition 6.262 *equivariant under $\mathcal{H}_p$ and the twists d, ν. If $\mathrm{Stab}_{\langle d,\nu\rangle}(\chi) = 1$, then $\chi|_K$ is irreducible for $K = \ker d \cap \ker \nu$. Moreover, we have that Φ induces an $\mathcal{H}_p$-equivariant matching of the restrictions $\chi|_K$.*

Proof. With the marked $\mathbf{L}$, we set $\tau = \tau_W \otimes \bigotimes_\xi \tau_\xi$, with involution parent τ_W on $\mathrm{O}(W)$ and $\tau_\xi = (\theta_\xi \circ \det) u_{\lambda_\xi}$ on $\mathrm{GL}^{\delta_\xi}_{m_\xi}(q^{h_\xi})$. With $S = (\mathbf{G}^\circ)^F$, we write $L = \mathbf{L}^F$ and $L_S = (\mathbf{L}^\circ)^F$. If $W \neq 0$, $[L : L_S] = [\mathbf{G}^F : S] = 2$. Given $x \in S$, [42, Proposition 3.1] groups the summation over $\mathbf{G}^F$ into two equal contributions via $g \mapsto g r_W$ with $r_W \in L \backslash L_S$. Thus, $(R^{\mathbf{G}}_{\mathbf{L}}\tau)|_S = R^{\mathbf{G}^\circ}_{\mathbf{L}^\circ}(\tau|_{L_S})$. The dual centralizer of t lies in $(\mathbf{L}^\circ)^*$. By [72, Theorem 3.3.22], we obtain a signed bijection $\mathcal{E}(L_S, t) \to \mathcal{E}(S, t)$.

If $\tau|_{L_S}$ is irreducible, let $\epsilon\chi_0 = R^{\mathbf{G}^\circ}_{\mathbf{L}^\circ}(\tau|_{L_S})$, with $\epsilon = \pm 1$. Since $R^{\mathbf{G}}_{\mathbf{L}}\tau$ restricts to $\epsilon\chi_0$, the character χ_0 is $\mathbf{G}^F$-invariant. Thus, the two extensions $\chi, \chi d$ [95, Theorem 6.2 and Corollary 11.22] have coefficients a, a' in $R^{\mathbf{G}}_{\mathbf{L}}\tau$ with $a + a' = \epsilon$. Thus, choose χ when a is odd, and χd otherwise. By Lemma 6.265, the twist $\tau d|_L$ swaps a, a'. Consequently, $\tau, \tau d|_L$ give distinct extensions $\chi, \chi d$. If $\tau|_{L_S} = \tau_1 + \tau_2$ with $\tau_2 = \tau_1^{r_W}$, the distinct images χ_1, χ_2 are conjugate by

r_W. Thus, by [95, Theorem 6.2], the full parent is $\mathrm{Ind}_S^{\mathbf{G}^F}\chi_1$. If $W=0$, the two orientations of t give one full parent χ by [200, Propositions 6.1–6.2]. Consequently, the signed bijection on S and the choice of χ give the inverse of Φ.

Under $\sigma\in\mathcal{H}_p$, the torus character θ_ξ becomes θ_ξ^σ. By [84] and [190], u_{λ_ξ} has rational values, and Frobenius F_q and inverse transpose $g\mapsto g^{-\mathrm{T}}$ fix its label λ_ξ. Thus, the primary permutation acts by $(\xi,\lambda_\xi)\mapsto(\xi^\sigma,\lambda_\xi)$. Moreover, $\tau_W^\sigma=\tau_W$ by Lemma 6.245. The coefficient a is fixed by flag transport and σ by Lemma 6.265. Also, by [149, Corollary 8.14], the twist ν acts as $\xi\mapsto-\xi$ and $(\Lambda_+,\Lambda_-)\mapsto(\Lambda_-,\Lambda_+)$. On $A_\xi=\mathrm{GL}_{m_\xi}^{\delta_\xi}(q^{h_\xi})$, the restriction $\nu|_{A_\xi}$ is $\eta_\xi\circ\det$, where η_ξ is quadratic on $C_{q^{h_\xi}-\delta_\xi}$. Indeed, this torus T_ξ lies on a $2h_\xi$-space, with spin preimage $\widehat{\mathbf{T}}_\xi$ and kernel μ_2. By Lang's theorem, the image of $\widehat{\mathbf{T}}_\xi^F$ has index 2 in T_ξ. The kernel of $\det|_{A_\xi}$ is generated by unipotent elements u with $\nu(u)=1$. Thus, $\nu|_{A_\xi}$ is determined on T_ξ, and Lemma 6.265 gives equal coefficients a under ν. If $W=0$, the full parent χ lies over either orientation on S.

On V_R, the extension χ_R again has odd coefficient a_R. On the active blocks V_j, a common rotation g restores the representative ξ^σ. Moreover, $gxg^{-1}=x$ for $x\in C_k\wr\mathfrak{S}_{v_\xi}$. Since the order of ξ is invariant under σ, the extensions at $\xi,-\xi$ follow Proposition 6.262. Also, $\mathcal{H}_p$ fixes the values on C_k, since $k\mid 2e\mid p-1$. Consequently, the active labels $(\xi,\boldsymbol{\mu}_\xi)$ map to $(\xi^\sigma,\boldsymbol{\mu}_\xi)$. The order of (Λ_+,Λ_-) is fixed under σ on V and V_R. Thus, Φ is equivariant under $\mathcal{H}_p,d,\nu$. If $B=\mathrm{O}(V)$ or $B=B_Q$, we write $K=\ker(d|_B)\cap\ker(\nu|_B)$. By [95, Lemma 5.2], $\|\chi|_K\|^2=|\mathrm{Stab}_{\langle d,\nu\rangle}(\chi)|$. If this stabilizer is trivial, then $\chi|_K$ is irreducible and $(\chi\alpha)|_K=\chi|_K$ for each twist α. Since $\chi^\sigma|_K=(\chi|_K)^\sigma$, the correspondence includes the case $\chi^\sigma=\chi\alpha$ for $\alpha\in\langle d,\nu\rangle$. □

Proposition 6.267. *Suppose that the stabilizer of χ under $\langle d,\nu\rangle$ is $\langle d\rangle$, at either end. If $\sigma\in\mathcal{H}_p$ stabilizes its twist orbit, then $\chi^\sigma\nu^b=\chi$ for a unique $b\in\{0,1\}$. Given any nonsign orbit at ξ, we let $u_\xi q^{j_\xi}$ restore the marking of $(-1)^b\xi^\sigma$ by* Lemma 6.263. *Thus, the permutation by σ of $\theta_1,\theta_2\in\mathrm{Irr}(K)$ has sign $\prod_\xi(u_\xi^{h_\xi}\delta_\xi^{j_\xi})^{m_\xi}$. Consequently, the pairs of constituents match $\mathcal{H}_p$-equivariantly at $N_X(P)$, with equal scalars on $Z(X)$.*

Proof. Since $d\chi=\chi$, the proper constituents χ^+,χ^- are distinct by [95, Theorem 6.2]. Thus, for $K=\ker d\cap\ker\nu$, [95, Lemma 5.2] gives $\|\chi|_K\|^2=2$. Hence, $\chi^+|_K,\chi^-|_K$ are distinct irreducibles. Given $W\neq0$, the involution source has $\Lambda_\pm=(A_\pm,A_\pm)$ and rational constituents $\tau_W^\pm$ by Lemma 6.245. An improper correction r_W transposes τ_W^+,τ_W^-. If $b=1$, $\Lambda_+=\Lambda_-$. By Proposition 6.251, $\tau_W|_{\Omega(W)}$ has four distinct constituents. Thus, $\tau_W^\pm\nu_W=\tau_W^\pm$. If $W=0$, the transporter determinant distinguishes the $\mathrm{SO}(V)$-orientations of t. Since $\nu|_K=1$, the actions of σ and $\sigma\nu^b$ on $\chi^\pm|_K$ coincide.

If $\alpha_\xi=u_\xi^{h_\xi}\delta_\xi^{j_\xi}$, Lemma 6.263 gives determinant $\alpha_\xi^{m_\xi}$. The operator $R_{\mathbf{L}}^{\mathbf{G}}$ is independent of $\mathbf{P}^\circ$ by Lemma 6.265. If σ permutes the primary indices ξ, the pairs (ξ,λ_ξ) move together, and permuting V_ξ contributes determinant 1. Consequently, the proper correction at $W\neq0$, or the orientation at $W=0$, has sign $\prod_\xi\alpha_\xi^{m_\xi}$. On V_R, the determinant factors are $\alpha_\xi^{r_\xi}$. At $1,-1$, the determinant operator D_A pairs cyclic labels $j,j+e$. Thus, $D_A^\sigma=D_A$ for $\sigma\in\mathcal{H}_p$. If $b=1$, then Proposition 6.251 gives $D_AJ_A=J_AD_A$. At a nonsign weight ξ, the summand at gI has $D_A=d(g)I_{gI}$, since $d|_I=1$. Thus, the restoring rotation g contributes $d(g)$ and block permutations contribute 1. By Lemma 6.263, the active blocks at ξ contribute $\alpha_\xi^{kv_\xi}$. Consequently, the local sign has factor $\alpha_\xi^{r_\xi+kv_\xi}=\alpha_\xi^{m_\xi}$. Over a representative λ, we pair the constituents $\chi^\pm|_K$ with the local constituents $\psi^\pm$. Since the signs agree on $(\mathcal{H}_p)_\lambda$, the pairing extends to $\mathcal{H}_p\cdot\lambda$. By Lemma 6.263, the scalars at $-I$ agree on each V_ξ. On W, the involution scalar $c_W(-I)$ agrees as well. Finally, Theorem 6.259 gives $\psi^\pm\mapsto\psi_P^\pm$ at $N_X(P)$. □

Lemma 6.268. *Suppose that d,ν fix the full parent χ. Let c be the multiplicity of i, with $i^2=-1$, in the natural spectrum. Thus, $c=0$ if i is absent. For the operators D,J_ν of the two twists, we have $DJ_\nu=(-1)^cJ_\nu D=(-1)^nJ_\nu D$ on $\mathrm{O}(V)$ and B_Q. If n is odd, then $\chi|_K=2\theta$ for one $\theta\in\mathrm{Irr}(K)$.*

Proof. The sign on $\chi^\pm$ before restriction to K is given by Proposition 6.267. Thus, with $\sigma=1$ and $b=1$, it computes the sign of J_ν on $\chi^\pm$. If V_1,V_{-1} are nonzero, $\Lambda_+=\Lambda_-=(A,A)$. Their sign is 1 by Proposition 6.251. If $V_1=V_{-1}=0$, the empty product is 1. If the orbits of $\xi,-\xi$ differ, we have that their transporters $g_\xi,g_{-\xi}$ are inverse and $m_\xi=m_{-\xi}$. Consequently, their factors multiply to 1.

Suppose that $-\mathcal{O}_\xi=\mathcal{O}_\xi$ on a nonsign orbit $\mathcal{O}_\xi$. The group $A_\xi=\langle F_q,\iota\rangle$ acts regularly on $\mathcal{O}_\xi$, with $F_q:x\mapsto x^q$ and $\iota:x\mapsto x^{-1}$. Since $x\mapsto-x$ centralizes the abelian A_ξ, it is translation by $a\in A_\xi$ with $|a|=2$. If $\delta=-1$, $A_\xi=C_{2h}$. If $\delta=1$ and h is odd, $A_\xi=C_h\times C_2$ is also cyclic. In both cases $a=\iota$, so $-\xi=\xi^{-1}$. Thus, $\xi=\pm i$ and $h=1$ give determinant $(-1)^c$. Otherwise, $\delta=1$ and h is even, so $u^h\delta^j=1$. By Proposition 6.262, such an orbit has $v_\xi=0$,

since a nontrivial label requires $t_b = 1$ and $h \mid e$ with e odd. The spaces V_1, V_{-1} have ranks $n_1 = n_{-1}$. Moreover, each pair $\mathcal{O}_\xi, \mathcal{O}_{-\xi}$ contributes $2hm_\xi$. A fixed orbit $\mathcal{O}_\xi \neq \mathcal{O}_i$ has even h. Consequently, $n \equiv c \bmod 2$.

Thus, the sign $(-1)^n$ on the D-eigenspaces gives $DJ_\nu = (-1)^n J_\nu D$. Normalize $D^2 = J_\nu^2 = I$ over $\mathbb{C}$. By [95, Lemma 5.2], $\dim \mathrm{End}_K(E_\chi) = 4$ for a module E_χ affording χ. Hence, the distinct twist lines $1, D, J_\nu, DJ_\nu$ give a basis. If $2 \nmid n$, $\mathrm{End}_K(E_\chi) \cong M_2(\mathbb{C})$, so $\chi|_K = 2\theta$. The action on θ follows from the orbit of χ. If n is even, $DJ_\nu = J_\nu D$, and the idempotents $(I + \epsilon D)(I + \eta J_\nu)/4$ give the four constituents $\theta_{\epsilon,\eta}$ for $\epsilon, \eta \in \{1, -1\}$. □

Proposition 6.269. *Suppose that $V_1, V_{-1} \neq 0$. Assume that each nonsign pair $\{x, x^{-1}\}$ is split over $\mathbb{F}_q$, with $x^2 \neq -1$ and $v_x = 0$. If $W = V_1 \perp V_{-1}$, the involution correspondence gives an $\mathcal{H}_p$-match on the constituents $\chi|_{\ker d \cap \ker \nu}$. In particular, the match includes parents χ fixed by ν or $d\nu$.*

Proof. Propositions 6.266 and 6.267 cover twist stabilizers $1, \langle d \rangle$. Suppose that $d^a\nu$ fixes the parent. Since $x^2 \neq -1$, the pairs $\{x, x^{-1}\}$ and $\{-x, -x^{-1}\}$ differ. The two partitions λ_x, λ_{-x} coincide and have size m_x. For one x per pair of primaries, we put $U_x = E_x \oplus E_{-x}$ and $U_x^\vee = E_{x^{-1}} \oplus E_{-x^{-1}}$. Thus, $\dim U_x = 2m_x$. The orthogonal sum $Z = \bigoplus_x (U_x \oplus U_x^\vee)$ lies in V_R since $v_x = 0$. Write $W = Z^\perp$ in V and $W_0 = W \cap V_R$. All active blocks V_j lie in W, so the involution correspondence at Q gives the p'-degree conditions on W.

Set $A = \prod_x \mathrm{GL}(U_x)$. The global flag has Levi $\mathrm{O}(W) \times A$, and the flag in B_Q has Levi $B_W \times A$ with $B_W = N_{\mathrm{O}(W)}(Q)$. Moreover, the residual factor of B_W is $\mathrm{O}(W_0)$. On both Levis take $F = \bigotimes_x F_x$ on A. On the flag with halves E_x, E_{-x} in U_x, the two characters are $(\theta_x \circ \det)u_{\lambda_x}$ and $(\theta_{-x} \circ \det)u_{\lambda_x}$. The dual centralizer $C^*(t|_{U_x})$ lies in $\mathrm{GL}(E_x) \times \mathrm{GL}(E_{-x})$, so F_x is irreducible by [72, Theorem 3.3.22]. The twist $\eta_q \circ \det$ acts by $\theta_x \mapsto \theta_{-x}$ and exchanges the two halves. Thus, F_x is fixed by $\eta_q \circ \det$ by parabolic independence [16, Theorem 3.9]. Moreover, on A, we have $d|_A = 1$ and $\nu|_A = \prod_x \eta_q \circ \det_x$ by Proposition 6.266.

For $B = \mathrm{O}(V)$ or B_Q, let H_B be the flag stabilizer. Inflate $\tau_W \otimes F$ to ρ_B on H_B. Thus, $\chi_B = \mathrm{Ind}_{H_B}^B \rho_B$. By induction in stages, χ_B is irreducible by Proposition 6.264. If $W_0 = 0$, the two orientations of $t|_{V_R}$ on $\mathrm{SO}(V_R)$ are conjugate under $\mathrm{O}(V_R)$. By [200, Proposition 6.2], the full residual parent is $\mathrm{Ind}_{\mathrm{SO}(V_R)}^{\mathrm{O}(V_R)} \chi_R^+$ for one proper orientation χ_R^+.

Put $K_B = \ker(d|_B) \cap \ker(\nu|_B)$. Since $W \neq 0$, on $\mathrm{O}(W)$, the map (d, ν) is onto $\{\pm 1\}^2$. On B_W, a block rotation r_Q has $d(r_Q) = -1$, while a torus generator t_Q has $d(t_Q) = 1$ and $\nu(t_Q) = -1$. If $Q = 1$, $B_W = \mathrm{O}(W)$. Thus, $B = K_B H_B$. By the primary inverse and $(\nu|_A)F = F$, the twist stabilizers correspond via $H_B/(H_B \cap K_B) \cong B/K_B$. By [95, Problem 5.6], $\chi_B|_{K_B} = \mathrm{Ind}_{H_B \cap K_B}^{K_B}(\rho_B|_{H_B \cap K_B})$. By [95, Lemma 5.2 and Problem 5.3], $\|\chi_B|_{K_B}\|^2 = \|\rho_B|_{H_B \cap K_B}\|^2 = |S_{\rho_B}|$. Write $\rho_B|_{H_B \cap K_B} = \sum_j e_j \beta_j$ and put $\gamma_j = \mathrm{Ind}_{H_B \cap K_B}^{K_B} \beta_j$. Since $e_j > 0$, the identity $\|\sum_j e_j \gamma_j\|^2 = \sum_j e_j^2$ gives distinct irreducibles γ_j.

Take $\alpha = d^a\nu^b$ fixing ρ_B. On τ_W, choose the operator $J_{\alpha,W}$ of Theorem 6.254. If both twists fix τ_W, take the involutions $D_W, J_{\nu,W}$ of Proposition 6.251. Thus, on ρ_B the operator is $J_{\alpha,W} \otimes J_A^b$, where J_A gives the quadratic determinant twist on F. At $b = 0$, the factor is I_F. By [95, Lemma 5.2], the dimension of $\mathrm{End}_{H_B \cap K_B}(V_{\rho_B})$ is $|S_{\rho_B}|$ for the self-twist group S_{ρ_B}. Thus, $\{J_{\alpha,W} \otimes J_A^b : \alpha \in S_{\rho_B}\}$ is a basis. A permutation of U_x or exchange with $U_x^\vee$ has the same rational isometry g_Z on Z at both ends. Thus, g_Z gives equal scalar ratios on J_A. A correcting isometry g scales the line for α by $\alpha(g)$. If the flag changes from $\mathcal{F}$ to $\mathcal{F}'$ with the marked Levi L_B fixed, the projections $\pi_\mathcal{F}, \pi_{\mathcal{F}'}$ agree on $H_B \cap H_B'$. Both unipotent radicals lie in K_B, so the restricted sources agree on $H_B \cap H_B' \cap K_B$. By [95, Problem 5.6], the coset 1 contributes a nonzero intertwiner between γ_j, γ_j'. Thus, $\gamma_j = \gamma_j'$ also for $\mathcal{F}'$ marked by ξ^σ. The involution scalars $c_W(z)$ agree. On Z, we have that the common F gives $c_Z(z)$ on either side. By Theorem 6.259, γ_j has a partner on $N_X(P)$. □

Lemma 6.270. *Let t have nonzero sign spaces V_1, V_{-1} and a nonquartic primary. Suppose that $gtg^{-1} = -t$ for a rational $g \in \mathrm{O}(V)$. In the Levi of* Zhang [208, Proposition 7.3, Step 1]*, we put $W = E_1 \oplus E_{-1} \oplus E_i \oplus E_{-i}$, with $i^2 = -1$. The spin Levi L and its toral normalizer $N_L(\widetilde{Q}_E)$ have images $\ker d \cap \ker \nu$ in $E = \mathrm{O}(W) \times \prod_i A_i$ and $D_E = B_W \times \prod_i I_i$, respectively, with $A_i = \mathrm{GL}_{2m_i}^{\delta_i}(q^{h_i})$. Moreover, we have that $I_i = N_{A_i}(Q_i)$ for the projected toral base Q_i, so $I_i = A_i$ if $Q_i = 1$. On E, D_E, we have $d = d_W$ and $\nu = \nu_W \prod_i \eta_i \circ \det_i$, with quadratic η_i on $C_{q^{h_i} - \delta_i}$. If $V_1 = V_{-1} = 0$ and $g \in \mathrm{SO}(V)$, the formulas hold with $W = E_i \oplus E_{-i}$.*

Proof. If $d(g) = -1$ and $V_1 \neq 0$, a reflection r_1 on V_1 gives a proper negator gr_1. The torus $\mathbf{S}^*$ of [208, Proposition 7.3, Step 1] acts trivially on W by Proposition 6.190. Its nonzero weights give $U_x = E_x \oplus E_{-x}$ and $U_x^\vee = E_{x^{-1}} \oplus E_{-x^{-1}}$. Since $gE_x = E_{-x}$, we have $\dim E_x = \dim E_{-x} = m_x$. Thus, Frobenius cycles the pairs $U_x, U_x^\vee$. After h_i steps,

Frobenius carries U_x to U_x or $U_x^\vee$. The corresponding rational factor is A_i, with $\delta_i = 1$ when U_x maps to U_x and $\delta_i = -1$ when U_x maps to $U_x^\vee$. Thus, h_i is the orbit length of $\{U_x, U_x^\vee\}$, whereas h in Proposition 6.262 counts the orbit of $\{x, x^{-1}\}$. Let $\pi : \mathbf{Spin}(V) \to \mathbf{SO}(V)$ be the spin covering and set $\widetilde{\mathbf{L}} = \pi^{-1}(\mathbf{L}^\circ)$ for the connected Levi $\mathbf{L}^\circ$. Since $\pi(\mathbf{Spin}(V)^F) = \Omega(V)$, on rational points $\pi(\widetilde{\mathbf{L}}^F) = \mathbf{L}^{\circ F} \cap \Omega(V)$. The natural determinant on A_i is 1, and by Proposition 6.266 we have $\nu|_{A_i} = \eta_i \circ \det_i$. Thus, the image of L is $E \cap \ker d \cap \ker \nu$ with kernel $\langle z_0 \rangle$.

Since p is odd, $Q_E = Q_W \times \prod_i Q_i$ lies in $E \cap \ker d \cap \ker \nu$. By Theorem 6.255, the odd lift $\widetilde{Q}_E$ is unique. Consequently, any lift $\widetilde{y}$ of $y \in N_E(Q_E) \cap \ker d \cap \ker \nu$ normalizes $\widetilde{Q}_E$. Since $N_E(Q_E) = D_E$, the second image is $D_E \cap \ker d \cap \ker \nu$. Given $W \neq 0$, both quotient signs occur independently on $\mathrm{O}(W)$. If $Q_W \neq 1$, a block rotation r_Q has $d(r_Q) = -1$, and a torus generator t_Q has $d(t_Q) = 1$ and $\nu(t_Q) = -1$. Otherwise, $B_W = \mathrm{O}(W)$. □

Proposition 6.271. *Take the comparison on A_i, I_i from* Proposition 6.194 *and that on W for $t^4|_W = I$. Suppose that n is odd and $V_1, V_{-1} \neq 0$. Let $K_E = E \cap \ker d \cap \ker \nu$ and $K_D = D_E \cap \ker d \cap \ker \nu$ as in* Lemma 6.270. *Then, $\chi|_{K_E}$ and $\psi|_{K_D}$ have constituents with an $\mathcal{H}_p$-equivariant match with equal scalars at $z \in Z(X)$. Moreover, the match pairs identical markings $(U_x, U_x^\vee)$ and is equivariant under changes of (χ, ψ) modulo $\langle d, \nu \rangle$.*

Proof. Let $r = \frac{1}{2}\dim W$. Since $n - r = \sum_i 2m_i h_i$, r is odd. The sign spaces V_1, V_{-1} are isometric of rank a, so $r = 2a + c$ with $c = \dim E_i$ odd at the root $i^2 = -1$. Given $r \leq 3$, the comparison on W is Theorem 6.199. If $r \geq 5$, the ordered labels (Λ_+, Λ_-) on W follow Proposition 6.261. If $p \nmid |\mathrm{O}(W)|$, then $B_W = \mathrm{O}(W)$ and $\tau_W \mapsto \tau_W$. Otherwise, the degree condition is at Q_W, since $p \nmid [X : L]$, and kernel restriction changes degrees by 2-powers.

Write S_0 for the twist stabilizer of the W-parent τ_W, and S_1 for that of χ, both in $\langle d, \nu \rangle$. Moreover, $\tau_W^\sigma = \tau_W$ for $\sigma \in \mathcal{H}_p$ by Proposition 6.261. If $\alpha = d^a \nu^b \in S_1$, we let J_i give the quadratic determinant twist on A_i or I_i when $b = 1$. Thus, the operator on χ is $J_{\alpha,W} \otimes \bigotimes_i J_i^b$. Indeed, $d|_{A_i} = 1$ by Lemma 6.270. If $S_1 = 1$, the restriction $\chi|_K$ is irreducible by [95, Lemma 5.2], with $K = \ker d \cap \ker \nu$ in the respective product.

Suppose that $S_1 = \langle \alpha \rangle$, with $\alpha = \nu$ or $d\nu$. Then, $S_0 = S_1$, since a further self-twist on W would include d and so also fix χ. On W, we set $J_{\alpha,W} = e_+ - e_-$ with $e_\pm$ the projections onto $\theta^\pm$. Consequently, the permutation of e_+, e_- gives the ratio ± 1 on $J_{\alpha,W}$. Any compensating quotient twist on W lies in $S_0 = \langle \alpha \rangle$, so its operator centralizes $J_{\alpha,W}$. Under a change of basis by g_W, conjugation on $J_{\alpha,W}$ has scalar $\alpha(g_W)$.

If $S_1 = \langle d \rangle$, Proposition 6.261 determines the sign on $J_{d,W}$, also for a pure quartic residue $V_R \cap W$. If the compensating twist $d^a\nu$ fixes τ_W, we have that ν fixes τ_W because $d \in S_0$. Thus, $J_{\nu,W} J_{d,W} J_{\nu,W}^{-1} = (-1)^c J_{d,W} = -J_{d,W}$ at both ends by Lemma 6.268. If $S_1 = \langle d, \nu \rangle$, the tensor operators satisfy $J_d J_\nu = -J_\nu J_d$, since J_d has identity on A_i. Consequently, the restriction algebra is $M_2(\mathbb{C})$ by Lemma 6.268 and $\chi|_K = 2\theta$ for $\theta \in \mathrm{Irr}(K)$.

Given a single twist line α, the ratios on J_i agree by Proposition 6.194. The operations are $\sigma \in \mathcal{H}_p$, field actions F_q, inverse transpose $x \mapsto x^{-\mathrm{T}}$, conjugations $\mathrm{Int}(g_i)$ and permutations of A_i. Thus, the transport permutes the J_i-eigenspaces with matching signs ± 1. On W, the rational transporter is $g_W \in \mathrm{O}(W)$. Since $\tau_W^\sigma = \tau_W$, the coefficient map on W has identity marking before correction by d, ν. If a marking stabilizer fixes Q_E, its projection on W lies in B_W. By Lemma 6.270, $(d, \nu)(B_W) = \{\pm 1\}^2$. Consequently, the ratios on $J_{\alpha,W} \otimes \bigotimes_i J_i^b$ agree on the stabilizer of $[\chi]$. For distinct twist orbits $[\chi], [\chi']$, their restrictions to K are disjoint by [95, Lemma 5.2]. Conversely, each $\theta \in \mathrm{Irr}(K)$ lies under a full parent χ by [95, Theorem 6.2]. Given a central z, the matched scalars are $c_W(z) \prod_i c_{A_i}(z)$. □

Theorem 6.272. *Let $S = \mathrm{P}\Omega_{2n}^\epsilon(q)$ with $n \geq 5$ and $P_S \in \mathrm{Syl}_p(S)$. If q, p are odd with $p \nmid q$, we choose the maps on A_i, I_i from* Proposition 6.194 *and those at $t^4 = I$ from* Proposition 6.261. *Assume the transport of $\mathcal{E}(L, s)$ in* Section 6.1. *Then, $\mathrm{Irr}_{p'}(S)$ and $\mathrm{Irr}_{p'}(N_S(P_S))$ admit an $\mathcal{H}_p$-equivariant bijection. For odd q, p with $p \nmid q$, we have that the comparison on $X = \mathrm{Spin}_{2n}^\epsilon(q)$ includes all $\chi \in \mathrm{Irr}_{p'}(X)$ with $z_0 \in \ker \chi$, with equal scalars at $z \in Z(X)$.*

Proof. At $p = 2$, [159, Corollary B] gives the bijection. Given $p \mid q$, it is from [162]. Suppose that q, p are odd with $p \nmid q$. By Lemma 6.27, the domain $z_0 \in \ker \chi$ corresponds to rational lifts t of $s \in X^*$. Moreover, the lifts $t, -t$ differ on parent labels by ν, by [149, Corollary 8.14]. For the Levi L in Lemma 6.270, put $K_E = \ker(d|_E) \cap \ker(\nu|_E)$. Since $K_E \cong L/\langle z_0 \rangle$, every character χ_L with $z_0 \in \ker \chi_L$ inflates a constituent of $\chi_E|_{K_E}$ by [95, Theorem 6.2]. If $s' = usu^{-1}$ for $u \in X^*$, we have that a lift g of u satisfies $F(g) = \varepsilon g$ with $\varepsilon = \pm 1$. Choose λ with $F(\lambda)/\lambda = \varepsilon$. Then, we have that $g_0 = \lambda g$ is a rational similitude of multiplier λ^2. On $V_{\pm 1}$, the form $b_\pm$ and its scalar multiple $\lambda^2 b_\pm$ have the same rational type since $\dim V_{\pm 1}$ is even. By [200, Section 4], $(\mathcal{O}_\xi, m_\xi)$ determines the space V_ξ up to isometry. Moreover, a reflection $r_\pm$ in a nonzero $V_{\pm 1}$ centralizes t, so conjugacy on W agrees under $\mathrm{O}(W)$ and $\mathrm{SO}(W)$. Thus, ν acts by $t \mapsto -t$ across all factors A_i.

On the toral side, we put $I = N_L(Q)$ and $D = N_X(Q)$. By Lemma 6.7, the centralizer $C_X(Q)$ has one eligible orbit $\mathcal{O}_s$ under D above s. Moreover, Lemma 6.193 matches the series on A_i, I_i. At W, Proposition 6.260 recovers the ranks $a_\pm$, symbols $\Lambda_\pm$, types $\epsilon_\pm$ and quartic multiplicity c from the toral normalizer B_W. Consequently, the comparison on K_E, K_D exhausts the eligible middle series $\mathcal{E}(L, s)$. In particular, the comparison includes χ with $\nu\chi \neq \chi$. Distinct constituents θ_j of $\chi|_{K_E}$ match distinct θ'_j of $\psi|_{K_D}$ with multiplicities m_j.

By [208, Proposition 7.3, Step 2], the global map is $\mathcal{R}_{X,L}$. Thus, for the middle map $f_{L,I}$, we fix $\mathcal{C}_{D,I} \circ f_{L,I} \circ \mathcal{R}_{X,L}^{-1}$, with $\mathcal{C}_{D,I}$ from Proposition 6.8. If n is odd, the stabilizers on E, D_E are compared in Proposition 6.271. If $2 \mid n$, see below. Thus, corrections in D or $C_X(Q)$ induce the identical map. On $z \in Z(X)$, both $\mathcal{R}_{X,L}$ and $\mathcal{C}_{D,I}$ give the same central scalar.

If $t^4 = I$, take Proposition 6.261. If $t \not\sim_{\mathrm{O}(V)} -t$, the twist stabilizer is 1 or $\langle d \rangle$, including when exactly one of V_1, V_{-1} is nonzero. Thus, $\chi|_K$ is matched by Propositions 6.266 and 6.267. If $V_1 = V_{-1} = 0$, then $d\chi = \chi$ by [200, Proposition 6.2]. The stabilizer $\langle d \rangle$ is covered by Proposition 6.267. Moreover, if n is odd and d, ν fix χ, Lemma 6.268 gives $\chi|_K = 2\theta$. If $V_1, V_{-1} \neq 0$, with $t \sim_{\mathrm{O}(V)} -t$ and $t^4 \neq I$, take the products E, D_E of Lemma 6.270. If $2 \nmid n$, Proposition 6.271 compares the series on E, D_E. For split nonquartic pairs with $v_\xi = 0$, Proposition 6.269 gives the ordinary map on $\chi|_K$ directly.

Suppose that n is even and $V_1, V_{-1} \neq 0$ in the last case. Then, $r = \frac{1}{2}\dim W = 2a + c$ and c are even, since $n - r = \sum_i 2m_i h_i$. For $S_1 = 1, \langle d \rangle, \langle \nu \rangle, \langle d\nu \rangle$, we follow Proposition 6.271, with determinant sign $(-1)^c = 1$. For $\alpha = \nu$ or $d\nu$, the coefficient sign on $J_{\alpha,W} = e_+ - e_-$ comes from permuting e_+, e_-. If the labels $(\Lambda_+, \epsilon_+), (\Lambda_-, \epsilon_-)$ differ, $S_1 \leq \langle d \rangle$. If $d\chi = \chi$, the reflection action on $\theta^\pm$ is that of Proposition 6.261.

If $p \nmid |\mathrm{O}(W)|$, then $B_W = \mathrm{O}(W)$ and $\tau_W \mapsto \tau_W$. Given $r \leq 3$, the comparison on W is Theorem 6.199. If $r = 4$ in the mixed case, we have $a = 1, c = 2$ and split W. The sign symbols $\Lambda_\pm$ are nondegenerate of rank 1. By Lemma 6.115, we obtain the p'-degree domain, and Proposition 6.117 compares $1, \mathrm{St}$ with their diagonal actions. The orientations ξ, ξ' have stabilizers $\langle \nu \rangle, \langle d\nu \rangle$, and h with $\eta_q(\mu(h)) = -1$ exchanges them. If d, ν both fix τ_W, $\Lambda_+ = \Lambda_- = (A, A)$. If $c = 0$, the operators D_W, J_W are those of Proposition 6.251. If $c > 0$ and $r \geq 5$, Theorem 6.70 gives D_W, J_W with the order $(+, -)$ from Proposition 6.261.

For $S_1 = \langle d, \nu \rangle$ on E, D_E, we set $D = D_W \otimes I$ and $J = J_W \otimes \bigotimes_i J_i$. The J_i give the twists $\eta_i \circ \det_i$ on A_i, I_i. Since c is even, $D^2 = J^2 = I$ and $DJ = JD$. Thus, the idempotents $(I + \epsilon_d D)(I + \epsilon_\nu J)/4$ give the four constituents by [95, Lemma 5.2]. Since $\tau_W^\sigma = \tau_W$, a compensating twist β lies in S_0. Its operator fixes D_W, J_W under conjugation. A marking isometry $g_W \in B_W$ scales D_W, J_W by $d(g_W), \nu(g_W)$. Moreover, by Proposition 6.194, the ratios on J_i agree on A_i, I_i. Consequently, the stabilizer acts on $(\epsilon_d, \epsilon_\nu)$ by identical signs on E, D_E.

Suppose that n is even, $V_1 = V_{-1} = 0$ and $\nu\chi = \chi$. Thus, d, ν both fix χ. An isometry g with $gtg^{-1} = -t$ is proper by Lemma 6.268, since $d(g) = (-1)^c = 1$. By Lemma 6.270, the nonsign factors A_i have halves of dimension $2m_i$, and W is pure quartic of rank c. If $W = 0$, take Corollary 6.195. If $c \leq 3$, take Theorem 6.199. Otherwise, $c \geq 4$ is even. Set $K_W = B_W \cap \ker d \cap \ker \nu$. By Proposition 6.261, $\tau_W|_{K_W}$ has four constituents. Thus, B_W/K_W of order 4 acts regularly by [95, Theorem 6.2], and $\mathrm{O}(W)/\Omega(W)$ also acts regularly.

Thus, $\sigma(\theta^g) = (\sigma\theta)^g$ for $g \in B_W/K_W$. Thus, σ acts by translation by g_σ. Since c is even, $d(g_\sigma) = 1$. The value $\nu(g_\sigma)$ is read on the two constituents θ_1, θ_2 under one proper orientation ξ. Lemma 6.36 pairs them at partition λ. If $Q_W \neq 1$, the period e is odd and $t_b = \eta_q(-1)$ by Proposition 6.261. If $Q_W = 1$, take the identity on W. Consequently, a match at θ_1 extends by the quotient B_W/K_W. For $\alpha = d, \nu$, the normalized operator is $J_{\alpha,W} = \sum_{g \in B_W/K_W} \alpha(g) e_{\theta_1^g}$. Consequently, on D_W, J_W the ratios are $\alpha(g_\sigma)$ at both ends.

The exchange $U_i \leftrightarrow U_i^\vee$ has determinant $(-1)^{2m_i} = 1$. Also, a basis change a_i on U_i acts by $a_i^{-\mathrm{T}}$ on its dual, with total determinant 1. A permutation of the paired spaces $U_i \oplus U_i^\vee$ has determinant 1 as well. Thus, a change on A_i cannot alter the orientation of W, also for the scalar normalization $\lambda^{-1} g_0$. On W, centralizers and negators of $t|_W$ have determinant 1. Since $\mathbf{C}_{\mathrm{SO}(W)}(t|_W)$ is connected of type A, the class at ξ is unique by [58, §4.2]. The graph isometry r_W gives the other class at ξ'. Thus, $\mathcal{O}_\xi$ matches ξ and $\mathcal{O}_{\xi'}$ matches ξ', also with coincident scalars $c_W(-I)$.

By Proposition 6.262, the local labels recover t up to ν and its proper orientation ξ. Thus, the local images exhaust $\mathcal{O}_s$ without overlap. Theorem 6.259 matches the z_0-trivial image with characters in $\mathrm{Irr}_{p'}(N_X(P))$ with the central scalars. Since $Z = Z(X)$ is a 2-group and p is odd, P is the unique Sylow p-subgroup of PZ. Thus, $N_S(PZ/Z) = N_X(P)/Z$. Inflation gives the pairs with Z in the kernel from those on $S = X/Z$.

If q is even and p is odd, $X = \mathrm{SO}(V) = S$. Given $s \neq 1$, Proposition 6.97 gives the map to $N_S(Q)$. If $s = 1$, the result of [156, Theorem D] gives the map at $N_S(Q)$. By Lemma 6.96, we obtain Q and $P = Q \rtimes U$,

including the residual factor $\mathrm{O}(V_0)$. The parent bijection on B_Q, B_P follows by the U-orbit product on the tori T_j and Theorem 6.259. The quotient character is $\delta(g) = (-1)^{\mathrm{rank}(1-g)}$. Thus, δ is trivial on each odd torus T_j and each block swap π, with $\mathrm{rank}(1-\pi) = 2e$. On an orbit of odd length p^a, the repeated rotation c has sign $\delta(c)^{p^a} = \delta(c)$. Also, the residual character on $\mathrm{O}(V_0)$ has the identical twist $\delta|_{\mathrm{O}(V_0)}$. Thus, δ-twists match. If $\delta\chi \neq \chi$, $\chi|_{\ker\delta}$ is irreducible by [95, Lemma 5.2]. Otherwise, at $\xi, \delta\xi$, take $J_{\delta\xi} = J_\xi^{-1}$. At a fixed ξ take $J_\xi^2 = I$, and on V_0 take the common J_0. At paired labels $\xi, \delta\xi$, the ratios $\kappa_\xi, \kappa_\xi^{-1}$ cancel because $m_\xi = m_{\delta\xi}$. At a fixed label ξ, the ratio κ_ξ is a sign. Since $m_\xi \equiv m'_\xi \bmod 2$ by the base-p digits, $\kappa_\xi^{m_\xi} = \kappa_\xi^{m'_\xi}$. Thus, the J_δ-eigenspaces match under each operation stabilizing $[\chi]$. The paired labels (χ, ϵ_δ) recover the inverse. Since $N_S(P) = B_P \cap \ker\delta$, this gives the bijection at P_S. □

Theorem 6.273. *Take the blocks V_j of* Theorem 6.255*, with $w > 0$ and $(q^e - t_b)_2 = 2$. If $D \in \{Q, P\}$, put $\mathcal{F}_D = \{\chi \in \mathrm{Irr}_{p'}(N_X(D)) : \chi(z_0) = -\chi(1)\}$. Then, we find an $\mathcal{H}_p$-equivariant bijection $\mathcal{F}_Q \to \mathcal{F}_P$ with equal scalars on $Z(X)$. Moreover, U_D can have arbitrary weights.*

Proof. Take $A, R^\flat, \mathcal{O}, \lambda$ as in Theorem 6.255, with U_D still in A. On one block, let $U = (T_b)_{2'}$, and let r generate the natural rotation group C_{2e}. Thus, put $J = U \rtimes \langle r^2\rangle$, $J^+ = U \rtimes \langle r\rangle$ and $\alpha = \mathrm{Int}(r)|_J$. On the Sylow side, let ϑ be a Q-weight of a p'-parent ρ. By [95, Theorem 6.2], its inertia I_ϑ has p'-index, so $I_\vartheta \geq P = Q \rtimes Y$. Thus, $\vartheta([Q, Y]) = 1$, and the odd lift of $[Q, Y]$ lies in $\ker\rho$. We pass to that quotient on A. By Theorem 6.259, the coordinates on $O \in \mathcal{O}$ have product in U. The rotation r acts as before on U and K_h acts trivially if $|O| = p^h$.

The toral two-part T_2 centralizes U_D. Indeed, for $x \in T_2$ the commutator with $u \in U_D$ lies in $\langle z_0\rangle$ and is a homomorphism in u. Hence, its image is 1. Thus, the inertia $I = A_\lambda$ lifts the inertia in A/U_D. Write L for the inflation of $\widetilde{\lambda}$ to I. Consequently, the relative groups modulo T_2 are $W_Q = J \wr \mathfrak{S}_w, \qquad W_P = \prod_h ((J \times K_h) \wr \mathfrak{S}_{m_h})$. The faithful weights form the orbit λ^A by Theorem 6.255, with $[A : I]$ a power of 2. Thus, for $\eta_l \in \mathrm{Irr}(W_D)$, Clifford correspondence [95, Theorem 6.11 and Corollary 6.17] gives the distinct parents $\Phi_l = \mathrm{Ind}_I^A(L \otimes \eta_l)$, with η_l inflated to I. Since $U \lhd J$ is abelian with $[J : U] = e$, $\xi(1) \mid e$ for $\xi \in \mathrm{Irr}(J)$ by [95, Theorem 6.15]. Consequently, the towers at ξ in Theorem 6.259, match the p'-labels l on W_Q, W_P. Both $\sigma \in \mathcal{H}_p$ and α act on (ξ, λ_ξ) by the permutation of ξ, with fixed indices (h, x).

Let c_* be the relative tuple of r^2 in all coordinates, so $\alpha^2 = \mathrm{Int}(c_*)$ on W_D. Suppose that $\xi \in \mathrm{Irr}(J)$ is fixed by α. If a weight θ of U is nontrivial, inversion gives $\theta \mapsto \theta^{-1}$ with $\theta^{-1} \neq \theta$, since $|U|$ is odd. Inversion is r^e, so the r-orbit of θ has even size. Hence, the r^2-orbits of θ and θ^r differ. Consequently, $\xi|_U$ is trivial if $\alpha\xi = \xi$. Choose an extension of such ξ to J^+ with value on r in μ_{2e}, fixed under $\mathcal{H}_p$ since $2e \mid p - 1$. If $\alpha\xi \neq \xi$, the index-two induction from J to J^+ gives a common irreducible $E_{\{\xi,\alpha\xi\}}$ by [95, Theorem 6.2]. On its two J-spaces, the operator r exchanges $\xi, \alpha\xi$, and the square acts by r^2. Under $\sigma \in \mathcal{H}_p$, take the full J^+-module in the model for $\{\xi^\sigma, (\alpha\xi)^\sigma\}$. Thus, a single J^+-isomorphism transports both maps for r.

At each letter of $\mathfrak{S}_w$ or slot of $\mathfrak{S}_{m_h}$, take the map for r on its factor E_ξ. On the Specht space S^{λ_ξ} and the K_h-factor, relabel by the identity. Thus, $U_l : V_l \to V_{\alpha l}$ satisfies $U_l \eta_l(x) U_l^{-1} = \eta_{\alpha l}(\alpha x), \qquad U_{\alpha l} U_l = \eta_l(c_*)$. Indeed, each permutation π permutes the slots of U_l with those of $V_l, V_{\alpha l}$. Thus, the formulas hold on W_D. Moreover, the maps at J^+ transport U_l under σ. Both formulas are internal to W_D, on its module V_l.

With $B_l = U_{\alpha l}^{-1}$, we choose $k_* = \prod_j k_j$, $\omega_A = \prod_j h_j$ and d_A as in Theorem 6.255. Thus, on Φ_l define $T_l = \Phi_l(\omega_A), \qquad S_l(g \otimes v) = d_A(g)\, g k_* \otimes B_l v$ in $\Phi_{\alpha l}$. On W_D, conjugation by k_*^{-1} is α^{-1}. Moreover, we have that $L(k_*^{-1} x k_*) = d_A(x) L(x)$ for $x \in I$, by the original inertia calculation and $L|_{U_D} = d_A|_{U_D} = 1$. Thus, S_l satisfies the tensor relations over I. The relative factor in k_*^2 is c_*. It cancels with $B_{\alpha l} B_l = \eta_l(c_*)^{-1}$. By Theorem 6.255, $S_{\alpha l} S_l = (-1)^{\binom{w}{2}} \beta^w I$ and $T_l^2 = (-1)^{ew} I$. Moreover, we have that $S_l T_l = (-1)^w T_{\alpha l} S_l$. The value β is as in Theorem 6.255. Thus, w counts $V_1, \ldots, V_w$ at either end. The inverse factor $\eta_l(c_*)^{-1}$ acts on all of V_l.

Let $\sigma \in \mathcal{H}_p$ act on the relative module. If $L^\sigma = L$, let $C_l : \Phi_l \to \Phi_{\sigma l}$ be coefficient transport. Otherwise, e is odd and $\sigma(\mathrm{i}) = -\mathrm{i}$. The calculation on $H_O, \widehat{\pi}$ in Theorem 6.255 gives $L(k_*^{-1} x k_*) = L^\sigma(x)$. Thus, after coefficient transport, act by $g \otimes v \mapsto g k_* \otimes B_{\sigma l} v$ in $\Phi_{\alpha(\sigma l)}$. Let C_l denote this semilinear isomorphism. Since $d_A(k_*) = 1$, both $C_{\alpha l} S_l$ and $S_{\alpha(\sigma l)} C_l$ have coset $g k_*^2$ and relative operator $B_{\alpha(\sigma l)} B_{\sigma l}$ after coefficient transport. Thus, $C_{\alpha l} S_l = S_{\alpha(\sigma l)} C_l$. Also, $C_l T_l = T_{\alpha(\sigma l)} C_l$ by the action of ω_A.

Take the identical faithful residual character Γ on $R^\flat$. Since z_0 has scalar -1 on both factors, $\rho_l = \Phi_l \boxtimes \Gamma$ is a character of $AR^\flat$. A twist $\mu = a^u d^v$ acts on the active part by $a_A^{u + t_R v} d_A^v$. Thus, $\rho_l \mu = \rho_l$ exactly when $\Gamma\mu|_{R^\flat} = \Gamma$ and $\alpha^v l = l$. Take $T_l^{u + t_R v} S_l^v$ with the common residual operator on Γ. Consequently, for $K = \ker a \cap \ker d$, the twist operators span $\mathrm{End}_K(V_{\rho_l})$ by [95, Lemma 5.2]. The relations on S_l, T_l match products and conjugation on $\mathrm{End}_K(V_{\rho_l})$

at both ends, including changes of l within the twist orbit $[l]$. Thus, the matched idempotents e_θ give the constituents θ equivariantly by the identities for C_l.

If $F(\omega)=\omega$, the operator in Theorem 6.255 is $\pm T_l\otimes\Gamma(\omega_R)$ with a common sign. Consequently, at ω, z_0 the scalars agree. By [95, Theorem 6.2], each $\theta\in\mathrm{Irr}(K)$ faithful on z_0 lies below a full parent ρ_l. Since $[AR^\flat:K]$ is a power of 2, the degree ratio $\rho_l(1)/\theta(1)$ is a 2-power. Distinct twist orbits of ρ_l have disjoint restrictions to K by [95, Lemma 5.2]. Thus, the inverse on $\mathcal{F}_P$ follows from the towers on l and the idempotents e_θ. □

Proposition 6.274. *Let $X=\mathrm{Spin}^\epsilon_{2n}(q)$ with $n\geq 5$, $q=r^f$ odd and $p\mid|X|$ odd with $p\neq r$. In $\mathcal{E}(X,s)$, suppose that the two involution spaces $V_\pm$ are separately rational. Let $\Lambda_\pm=(A_\pm,B_\pm)$ with $A_\pm\neq B_\pm$. Assume that the unordered labels $(\{A_\pm,B_\pm\},n_\pm,\epsilon_\pm)$ differ. Then, the orientations at p'-degree give the pairs (χ_j,ψ_j), $j=0,1$, at $N_X(Q)$. Moreover, on the semilinear stabilizer S_j, each pair satisfies $\succeq_c$ of* Definition 2.1 *for every f and $d=\mathrm{ord}_p(q)$.*

Proof. Write $V=V_+\perp V_-$ and $U=U_+\perp U_-$ for the residue in Lemma 6.244. At Q, take the ordered arrays $\boldsymbol{\pi}_\pm$ at $1,-1$ and the parent ρ_U on $\mathrm{O}(U)$ with cores $\Lambda^0_\pm$. Consequently, the p'-degree labels are $(\Lambda^0_\pm,\boldsymbol{\pi}_\pm)$. The inverse $(\Lambda^0_\pm,\boldsymbol{\pi}_\pm)\mapsto\Lambda_\pm$ is before restriction to $B_Q\cap\Omega(V)$. Thus, Λ_+,Λ_- distinguish the positions even if $|\boldsymbol{\pi}_+|=|\boldsymbol{\pi}_-|$. On a negative block $\mathcal{B}_-$, extend the quadratic weight ϑ_- by ν on the complement, including permutations π. If $\eta_q(\mu(g))=-1$, the reflection formula [137, §24.i] gives $\nu(gxg^{-1})=\nu(x)\det(x)$ for x on this factor. By Proposition 6.232, $gxg^{-1}=x$ for $x\in Q$ with identical coordinates on permuted blocks $\mathcal{B}_-$. Thus, it gives $j\mapsto j+e$ on $\mathbb{Z}/(2e)\mathbb{Z}$, keeping the corresponding Specht partition π_j. On U_-, the forms b_- and $\mu(g)b_-$ have equal discriminant class $[\det b_-]$. Thus, U_- has a similitude with multiplier $\mu(g)$. By [149, Corollaries 8.3 and 8.13], it acts by $\Lambda^0_-\mapsto(\Lambda^0_-)^{\mathrm{T}}$. Consequently, the reconstructed labels are $(\Lambda_+,\Lambda_-^{\mathrm{T}})$, including $\Lambda^0_-\neq\varnothing$. Moreover, the determinant twist gives $(\Lambda_+^{\mathrm{T}},\Lambda_-^{\mathrm{T}})$. The global actions on $\Lambda_\pm$ are those of [149, Theorem 8.12 and Corollaries 8.3 and 8.13].

By [149, Corollary 8.14], ν swaps the labels $(\{A_\pm,B_\pm\},n_\pm,\epsilon_\pm)$. Since they differ, neither ν nor $d_{\mathrm{O}}\nu$ fixes the parent, with $d_{\mathrm{O}}=\det$. Moreover, $\nu|_{\Omega(V)}=1$, so fix (Λ_+,Λ_-) in that order. d_{O} changes both rows because $A_\pm\neq B_\pm$. By Frattini, $\mathrm{O}(V)=\Omega(V)N_{\mathrm{O}(V)}(P)$, and Lemma 6.231 gives $N_{\mathrm{O}(V)}(P)\leq B_Q=N_{\mathrm{O}(V)}(Q)$. Thus, B_Q maps onto $\mathrm{O}(V)/\Omega(V)$. By [95, Lemma 5.2], the restrictions to $\Omega(V)$ and $B_Q\cap\Omega(V)$ are irreducible. Therefore, the labels (Λ_+,Λ_-) give $(\chi_0,\psi_0),(\chi_1,\psi_1)$ after simultaneous transposition. Given $g\in\mathrm{O}(V)$, each pair is fixed. If $\eta_q(\mu(g))=-1$, g exchanges the pairs.

Let ρ be the parent on $\mathrm{O}(W)$ for $W=V,U$. By [165, Theorem A(ii) and §1.2], the constituents of $\rho|_{\mathrm{SO}(W)}$ are fixed by F_W, including at $s=\pm I$. Thus, $\rho^{F_W}\in\{\rho,\rho d_{\mathrm{O}}\}$. For basic ρ, the first occurrences $n(\rho),n(\rho\nu)$ in [149, §8] distinguish $\rho,\rho d_{\mathrm{O}}$ by Section 6.9. On the oscillator module, F_W permutes the tuples with d_{O},ν fixed. Indeed, whenever $a\in\mathbb{F}_q^\times$, $\eta_q(a^r)=\eta_q(a)$. Thus, F_W fixes $n(\rho),n(\rho\nu)$, also after the twists ρd_{O} and similitude transport ρ^g. If ρ is nonbasic, a correction g_WF_W fixes the singular line ℓ. Thus, the source $\gamma\otimes\rho'$ has $\gamma\in\{1,\eta_q\}$ and $(\rho')^{F_W}=\rho'$ by induction. The flag excludes ρd_{O}, and Proposition 6.4 gives multiplicity 1 for ρ. Hence, $\rho^{F_W}=\rho$. At $W=0$, the parent is 1. In a nonsplit plane, $x\mapsto x^r$ fixes the inversion $x\mapsto x^q$ and weights $\theta=1,\eta$. Thus, both extensions with reflection value ±1 are field-fixed. Moreover, its fth power is $x\mapsto x^q$. Thus, ρ extends to $\langle\mathrm{O}(W),F_W\rangle$ by [95, Corollary 11.22].

On the active blocks V_j, let F_A be the r-power map. Then, $F_A^f=v$ is the diagonal q-rotation with $|v|\mid d$. On the relative wreath module M_A, v acts centrally. Since v is uniform on cosets gI, its scalar is $z=\prod_j z_j$ for coordinate scalars z_j. Thus, $z^d=1$. On the source model, F_A fixes $\vartheta_\pm$, the normalization by ν, the rotations c_j and the permutations π. Thus, its matrix before imposing $F_A^f=v$ is the identity. Put $k=\mathrm{ord}_p(r)$. Since $d=k/\gcd(k,f)$, we choose $A\in\mu_k$ with $A^f=z$. Moreover, $k\mid p-1$, so $\mathcal{H}_p$ fixes A and the values in $\mathbb{Q}(\zeta_{p-1})$.

If f is odd, the forms on $W=V,U$ descend to $\mathbb{F}_r$ with their types, since $\eta_q|_{\mathbb{F}_r^\times}=\eta_r$. Since $2\nmid f$, the real extension $\widehat{\rho}_W$ is unique by Lemma 6.82. Thus, $\widehat{\rho}_W^\sigma=\widehat{\rho}_W$ for cyclotomic σ. On the common lift (F_A,F_U), we choose the local operator $AI\otimes T_U$, with T_U the operator of F_U on the residual extension. Consequently, its fth power represents (v,F_U^f). Evaluate $\widehat{\rho}_V$ at (F_A,F_U) in $\langle\mathrm{O}(V),F_V\rangle$.

If d is odd, $e=d$, and an e-hook changes entries in one row B. A common shift fixes the length ℓ. Since a bead move replaces b_j by sorted entries $b'_j\leq b_j$, the parts $b'_j-(\ell-j)$ are bounded by $b_j-(\ell-j)$. Thus, one-box steps join $\Lambda^0_\pm$ to $\Lambda_\pm$. Each added plane H has line weight $\gamma=1$ or η_q according to the sign. Take $\widehat{\rho}_U$ with $\widehat{\rho}_U^\sigma=\widehat{\rho}_U\lambda_\sigma$, where $\lambda_\sigma\in\mathrm{Irr}(C_f)$. Moreover, at the ℓ-step, we extend γ by value 1 on the split field generator F_H. Proposition 6.4 gives multiplicity 1 at ρ_{i+1}. Thus, its isotypic space $V_{\rho_{i+1}}$ affords $\widehat{\rho}_{i+1}$ with twist λ_σ by Section 6.1, also at $\Lambda_\epsilon=(A,A)$. Consequently, $\widehat{\rho}_V^\sigma=\widehat{\rho}_V\lambda_\sigma$. Since $A^\sigma=A$, the local tensor with $\widehat{\rho}_U$ has twist λ_σ as well. The relation $(F_A,F_U)^f=(v,F_U^f)$ holds for nonsplit U as well. If F_W is replaced by g_WF_W, then $[g_WF_W]=[F_W]$ in C_f,

so the twist remains λ_σ.

If d, f are even, by Lemma 6.86, we obtain $\mathcal{H}_p$-fixed extensions $\widehat{\rho}_V, \widehat{\rho}_U$ at quartic partition $\varnothing$. The cuspidal normalization holds on W with either type ϵ_W and distinct signed defects $d_\pm$. If f is odd, take the real extensions $\widehat{\rho}_W$ above. Thus, $A^\sigma = A$ gives $\lambda_\sigma = 1$ on both sides for even d. The square-multiplier stabilizer S_j is the image of $\langle \mathrm{O}(V), F_V \rangle$ modulo $\{\pm I\}$. By Theorem 6.233, representatives of S_j lie in the common normalizer of P and so of Q. For the section s, we have $s(x)s(y)s(xy)^{-1} \in \{\pm I\}$. By Section 6.9, the scalars at $\pm I$ agree. Consequently, the factor sets α, α' and centralizing scalars from s agree. Since the twists λ_σ agree, the coefficient functions $\mu_{x,\sigma}, \mu'_{x,\sigma}$ agree, also after a common correction $x \in \mathrm{O}(V)$. Finally, g with $\eta_q(\mu(g)) = -1$ takes the comparison for (χ_0, ψ_0) to (χ_1, ψ_1). □

Theorem 6.275. *Take X, q, p, Q as in* Proposition 6.274. *Suppose that V_+, V_- are separately rational. Then, on $\mathcal{E}(X, s)_{p'}$ the comparison at $N_X(Q)$ satisfies $\succeq_c$ of* Definition 2.1 *at each pair (χ, ψ) for every f.*

Proof. At Q, the inverse in Section 6.9 is $(\Lambda^0_\pm, \boldsymbol{\pi}_\pm) \mapsto \Lambda_\pm$ on full wreath parents. Its $\mathrm{O}(U)$-factor has the residual involution label $t|_U$. Since the twist ν exchanges the labels at $t, -t$, the inverse recovers the orbit of (s, u). If the complete labels u_+, u_- coincide, Section 6.1 gives F_0 and the inertia idempotents e_θ. Thus, take distinct unordered labels $(\Lambda_\pm, n_\pm, \epsilon_\pm)$ with row order forgotten. If $A_\pm \neq B_\pm$ in $\Lambda_\pm = (A_\pm, B_\pm)$, the comparison is Proposition 6.274 at $N_X(Q)$.

Suppose that $\Lambda_i = (A_i, A_i)$ for $i = +, -$. Thus, V_i is split of even rank $n_i > 0$. By Lemma 6.244, the core Λ^0_i has equal rows and $j \mapsto j + e$ fixes the quotient. Consequently, $2 \mid w_i$, and U_i is split if nonzero. Take $\theta = \eta_q(\prod_{j=1}^{n_-} x_{-,j})$. Thus, $\theta(zI) = \eta_q(z)^{n_-} = 1$. By Proposition 6.242, the two proper extensions $\widehat{\rho}^\pm$ come from Borel induction at θ. The split field map F_r fixes the kernels K_w, whose representatives lie in $\mathbb{F}_r$. Thus, F_r fixes the two rational blocks of $\mathrm{End}_{\mathrm{GSO}(V)}(Y_\theta)$. An improper g exchanges the two labels ± 1 on the proper pair. Moreover, ν swaps the distinct $\Lambda_\pm$ by [149, Corollary 8.14]. Hence, $\widehat{\rho}^\pm|_{\Omega(V)}$ are irreducible by [95, Lemma 5.2].

If both $U_i \neq 0$, choose the proper extensions on $\mathrm{SO}(U)$ by Proposition 6.242. The eigenvalues $1, -1$ distinguish the positions even if $\Lambda^0_+ = \Lambda^0_-$. An improper isometry r_U exchanges the two proper characters $\rho^\pm_U$. Thus, at index 2 induction of either gives ρ_U with determinant operator J_U acting by ± 1 on $\rho^\pm_U$. If only $U_i \neq 0$, take the split degenerate unipotent lift θ_i. If $i = -$, take $\theta_i \nu$ with ν extended by 1 on $\mathrm{diag}(\mu I, I)$ and F_r. Indeed, ν is invariant under this section, and $\nu(zI) = \eta_q(z)^{\dim U/2} = 1$ since $\dim U/2$ is even. Thus, induction gives J_U. If $U = 0$, the factor is 1.

On $V_A = U^\perp$, choose exponents $\delta_j \in \{1, -1\}$ on the marked positions J_- as in Proposition 6.241. Since $\sum_{j \in J_-} \delta_j = 0$, the balanced weight $\widehat{\theta}_A$ is trivial at zI. Also, $\widehat{\theta}_A$ is trivial on $T_{2'}$, so the formula holds for $p \mid q - 1$. If $w_- = 0$, take the unmarked module, and if $w_+ = 0$, set $J_+ = \mathrm{id}$ on the trivial factor. At J_-, the complement signs are $\nu(c_j), \nu(\pi) \in \{1, -1\}$. Thus, J_A of Proposition 6.241 transforms by $\det(g_A)/\mu(g_A)^{\dim V_A/2}$. Its proper characters $\rho^\pm_A$ are $\mathcal{H}_p$-fixed by the relative model over $\mathbb{Q}(\zeta_{p-1})$ and the field root A from Proposition 6.274. For negative period, F_A has sign $(-1)^{w_+ + w_-} = 1$ relative to F_r. Otherwise, $\partial(F_A|_{V_j}) = 1$ on each block.

Since $V^Q = U$, each semilinear normalizer element g has $gU = U$ and $gV_A = V_A$. The factors agree on $\mu(g)$ and the field image in C_f. Thus, take $J_A \otimes J_U$ on the fibre product, with I on a zero space. For linear g, its sign is $\det(g)/\mu(g)^n$. Since $u_+ \neq u_-$, the self-twists are $1, d_{\mathrm{O}}$. By [95, Lemma 5.2], the eigenspaces $E_\pm$ give kernel constituents of multiplicity 1. For proper g_A, g_U, the traces on $E_\pm$ are $\sum_{\delta = \pm 1} \rho^\delta_A(g_A)\rho^{\pm\delta}_U(g_U)$. If $\partial(g_A) = \partial(g_U) = -1$, they exchange summands in $E_\pm$, giving trace 0. Since F_A, F_U are proper, these formulas hold at gF^a_r. Consequently, the extensions on $E_\pm$ are $\mathcal{H}_p$-fixed and trivial at zI. By Theorem 6.233, the common section compares $S_\pm$. An improper g exchanges the pairs $(\chi_0, \psi_0), (\chi_1, \psi_1)$.

Suppose that exactly one Λ_i has equal rows. After replacing t by $-t$ if necessary, we set $\Lambda_- = (A_-, A_-)$. On $W = V_+$, take the parent h with scalar-trivial extension $\widehat{h}$ by the similitude construction. The extensions $\widehat{h}_V, \widehat{h}_U$ on V_+, U_+ have common twist k_σ by Lemma 6.86. If d is odd, the hooks fix d_+. If d is even, choose $\mathcal{H}_p$-fixed extensions, and if f is odd, take the real extensions $\widehat{h}$. Thus, k_σ also holds on the similitudes. If Λ^0_+ has equal rows, take $k_\sigma = 1$, since for odd d we have $d_+ = 0$ and line induction starts at $W_* = 0$.

If $\Lambda_- = (\lambda, \lambda)$, take the singular space L of dimension $2|\lambda|$ with label $(\eta_q \circ \det)U_{2\lambda}$, and tensor with $\widehat{h}$. Thus, for either ϵ_W, the folded flag has support (λ, λ), and the semilinear similarities fix $\chi \in \mathrm{Irr}(\Omega(V))$. The folded coefficient is $[s_{2\lambda}]s^2_\lambda = 1$. Thus, the extension $\widehat{\chi}$ has twist k_σ. If $U_- \neq 0$ and Λ^0_+ is nondegenerate, fold the flag on U. If $U_- = 0$, keep the positive extension $\widehat{h}$ on U_+. If $U_+ = 0$ or both Λ^0_i have equal rows, choose ρ_U with $k_\sigma = 1$ as above. If

$U = 0$, we set $\rho_U = 1$. By Section 6.1, Λ_+^0 has fixed order when nondegenerate. Thus, $\boldsymbol{\pi}_+$ reconstructs Λ_+. If the rows of Λ_+^0 agree, the opposite order on $\boldsymbol{\pi}_+$ gives the twist d_{O} on the full parent. Since Λ_- has equal rows, $2 \mid w_-$ and $j \mapsto j + e$ fixes $\boldsymbol{\pi}_-$. By the balanced weight, $\rho_A(zI)/\rho_A(1) = 1$ with the unmarked factor $V_{\boldsymbol{\pi}_+}$ intact. Even if F_A is improper, $F_A^f = v$ lies in the even inertia I_A. Thus, the scalar z of v gives $A^f = z$. The tensor with $\widehat{\rho}_U$ is over common (μ, F_r^a), with twist k_σ and scalar 1 at zI. Since $\Lambda_+^{\mathrm{T}} \neq \Lambda_+$ and ν swaps the distinct $\Lambda_\pm$, Clifford restriction gives irreducibles χ, ψ by [95, Lemma 5.2]. Thus, the section gives equal α, α', mixed functions $\mu_{x,\sigma}, \mu'_{x,\sigma}$ and central scalars at z_0, zI. □

Corollary 6.276. *Let $n \geq 5$. For the rational lift t of* Zhang [208, Proposition 7.3]*, suppose that the Levi torus $\mathbf{Z}_s$ has independent scalars a_x on the reciprocal pairs $\{x, x^{-1}\}$, $x \neq \pm 1$. Then, at $N_X(Q)$ the eligible lift orbits satisfy $\succeq_c$ of* Definition 2.1*. In particular, this covers $\dim V_1 \neq \dim V_{-1}$. For balanced weights on U_x, the conclusion holds if $t^2|_Y = I$ or $t|_Y$ has type $Q(a, c)$, on the zero-weight space Y.*

Proof. For unequal signs, change t to $-t$ so that $2a = \dim V_1 < \dim V_{-1} = 2b$. Since $F(1) = 1$ and $F(-1) = -1$, these ranks give $F(t) \neq -t$ and $t \not\sim -t$. If $a = 0$ and $t = \pm I$, we have that the comparison on $\mathcal{E}(X, 1)$ is [156, Theorem D]. Otherwise, [208, Proposition 7.3, Step 1] gives a_x. If one sign space W occurs, the orthogonal centralizer factor is $\mathrm{SO}(W)$, including $\dim W = 2$, since the reciprocal determinants satisfy $\det(A)\det(A^{-\mathrm{T}}) = 1$. If $V_1 = 0$ or $V_{-1} = 0$, $s \in Z(L^*)$. Thus, the scalar twist at $B = L$ selects $\mathcal{E}(L, s)$, and its restriction to J selects the local class $[s]_{J^*}$. This includes $Y = 0$ with independent scalars a_x.

By Lemma 6.7, each marked class $[s]_{G^*}$ has one eligible centralizer orbit $\mathcal{O}_s$. Thus, for a mixed stabilizer operation, a normalizer correction g fixes $\mathcal{O}_s$. Then, $c \in C_X(Q)$ fixes the marks s, u by Section 6.1. For linear x, the primal fixed space Y_L is from [208, Proposition 7.3, Step 2]. Choose a cocharacter λ in that fixed space with $\langle \alpha, \lambda \rangle \neq 0$ for $\alpha \notin \Phi_L$. Then, $\mathbf{U}(\lambda)$ is stable, with local radical $\mathbf{U}(\lambda) \cap \mathbf{K}$. Thus, the comparison on L, I transfers by Proposition 6.9.

Suppose that $V_1, V_{-1} \neq 0$ with independent a_x. Then, $Y = \ker(t^2 - I)$, and t is scalar on each type A factor. By Theorem 6.275, the comparison on Y has torus colours $1, -1$ and the residual parent ρ_U. Their inverse recovers (V_1, V_{-1}) modulo $t \mapsto -t$. Moreover, the ambient action on Y is by semilinear similitudes. If $2 \leq \mathrm{rk}\, Y \leq 4$ and both signs occur, then $1 \leq n_\pm \leq 3$, so $n_\pm < s^2$ for each $s \geq 2$. At $A_i \neq B_i$, the paths in Proposition 6.274 end on T. On two sign planes $V_\pm$, the field lift F_Y and orientation ξ follow Lemmas 6.57 and 6.88. If Λ_i is degenerate, $n_i = 2$. The folded flag has quotient V_{-i}, so its coefficient at 2λ is 1. If both are degenerate, then $n_+ = n_- = 2$ and $\Lambda_+ = \Lambda_-$. Thus, the maximal split flag gives $\theta_1, \ldots, \theta_4$. By Frattini, $\langle d_{\mathrm{O}}, \nu \rangle$ gives the full quotient on $\vartheta_\pm$ and the plane models. In split D_4 and for the rank-one pair $V_\pm$, Section 6.1 gives the induction at $\vartheta_\pm$ and the power F_Y^f. Thus, the section compares the action ι on Y. If $p \nmid |\mathrm{O}(Y)|$, take the identity on $\mathrm{O}(Y)$.

For balanced weights and $t^2|_Y = I$, Theorem 6.199 covers $\dim Y \leq 6$. For $\dim Y > 6$ with $t|_Y = \pm I$, take the map on $\mathcal{E}(R, 1)$ of [156, Theorem D]. Otherwise, for $V_1, V_{-1} \neq 0$ take Theorem 6.275. If $t|_Y$ has type $Q(a, c)$, $\mathrm{rk}\, Y \leq 3$ is again Theorem 6.199. At $\mathrm{rk}\, Y = 4$, the comparison for $Q(1, 2)$ gives the local orbit $\mathcal{O}_{s_Y}$ and its scalar. For $\mathrm{rk}\, Y \geq 5$, $V_\pm = \ker(t|_Y \mp I)$ are rational. Hence, $z_0 \in \ker \chi_Y$ by Lemma 6.27. The comparison for χ_Y is Corollary 6.95. By Proposition 6.260, $\epsilon_\pm$ and the quartic label λ are recovered modulo d_{O}, ν. Consequently, the image on $N_R(Q_Y)$ is $\mathcal{O}_{s_Y}$. The action ι on Y inherits the scalars and functions $\mu_{x,\sigma}$.

Take the maps on $H, I_{\mathbf{H}}$ from Proposition 6.194. Since (z_0, z_0) acts trivially on the tensor for L, its scalar pair at z_0 is $(\varepsilon, \varepsilon)$. The partners have scalars $(\varepsilon, \varepsilon)$. Thus, their tensor descends to I. Conversely, descent to I requires equal scalars at z_0. Thus, on I/B_I the common powers give $\succeq_c$ by Proposition 6.207. Since the factors U_x are balanced, Lemma 6.10 gives $[s] = [ks]$. Thus, Corollary 6.195 transfers the comparison to $N_X(Q)$ at the class $[s]$.

For independent a_x, the cover H has the scalar twist α_c. Its pullback formula is $\prod_i (\theta_i \circ \det_i)$ on A_0, since the scalar coordinates of t are rational. Thus, $\alpha_c(z_0) = 1$. By Lemma 6.27, the characters on R are trivial at z_0. Consequently, the tensors descend through $\langle (z_0, z_0) \rangle$ and select the projected orbits on R, H. On R, the comparison includes I/B_I. Moreover, on H the scalar-twist section keeps xy, x^2 for representatives x, y, including permutations π and dualities $U_x \leftrightarrow U_x^\vee$. By multiplication in $H^1(F, \mu_2)$, both natural images have $\nu_Y = \prod_i \eta_i \circ \det_i$. Thus, Proposition 6.15 gives $\succeq_c$ on L, I. For $\mathrm{rk}\, Y = 2, 3$, the spin map has kernel μ_2, and the preimage of $\mathbf{T}_Y$ is connected. Hence, by Lemma 6.14, $L/B \cong I/B_I$.

If projection loses the relative sign λ, Proposition 6.13 gives $\{[s], [ks]\}$. By Lemma 6.11, ks has representative $j_t t$ with $j_t|_Y = I$, so $j_t t|_Y = t|_Y$. Changing t to $-t$ exchanges the signs via ν. Consequently, the comparison on Y holds at t and $j_t t$. By Proposition 6.13, a correction g returning the global character β also returns the local partner ψ.

Thus, Proposition 6.9 gives the comparison at $N_X(Q)$. □

Theorem 6.277. *Let $S = \mathrm{P}\Omega^\epsilon_{2n}(q)$ be simple with $n \geq 4$. For every prime $p \mid |S|$, $\mathrm{iMN}(S,p)$ holds, with comparison on the universal cover X.*

Proof. At $p \mid q$, apply [162, Theorem 1.3] and [101, Theorem 1.1]. If X is an exceptional cover, apply [157, Lemma 4.1(2)]. If $p = 2$ and q is odd, apply [159, Theorem A(3)]. If q is even and $p \nmid 2q$, Theorem 6.2 gives $\mathrm{iMN}(S,p)$. For odd q and $n = 4$, Theorems 6.147 and 6.163 give $\mathrm{iMN}(S,p)$ for $\epsilon = +$ and $\epsilon = -$, respectively. Thus, assume that q, p are odd, $p \nmid q$, and $n \geq 5$. Let $P \in \mathrm{Syl}_p(X)$, with toral base Q as in Section 6.2, and put $K = C_X(Q)$, $M = N_X(Q)$. By Section 6.2, $M = N_X(\mathbf{S}_d)$, where $\mathbf{S}_d$ is a Sylow d-torus for $d = \mathrm{ord}_p(q)$. Moreover, Q is characteristic in P by Lemma 6.231 and the central isogeny $X \to \Omega^\epsilon_{2n}(q)$ of degree 2. Thus, $N_X(P) \leq M$ and $\mathrm{Aut}(X)_P$ stabilizes M. Since $Q \neq 1$ and $O_p(X) = 1$, $M < X$. Moreover, for $n \geq 5$, $\mathrm{Aut}(X) = \langle \mathrm{InnDiag}(X), F_r, \gamma \rangle$ with field and graph generators F_r, γ [208, Proposition 7.3].

Let z_0 generate the kernel of $X \to \mathrm{SO}^\epsilon_{2n}(q)$. Thus, for z_0 acting by -1, Theorem 6.37 gives $\succeq_c$ on the matched characters of X, M. Given $z_0 \in \ker\chi$, Lemma 6.27 gives a rational lift t of the adjoint element s for χ. If $s = 1$, the local set in [156, Theorem D and §4.4] is $\{\psi \in \mathrm{Irr}_{p'}(M) \mid \langle \psi|_K, \nu \rangle_K > 0 \text{ for some } \nu \in \mathcal{E}(K,1)\}$. If $\mathrm{Spec}(t) \cap \{1,-1\} = \varnothing$, by Corollary 6.26, we obtain $\succeq_c$ at s. It remains to consider $s \neq 1$ with 1 or -1 an eigenvalue of t. If s is quasi-isolated, by [208, Lemma 7.15], we obtain t of type $I(a,b)$ or $Q(a,c)$. In both cases $V_\pm = \ker(t \mp I)$ are rational. Thus, for $I(a,b)$, we cite Theorem 6.275. Given $Q(a,c)$, by Proposition 6.71, we obtain the p'-characters of X, and Corollary 6.95 gives $\succeq_c$ on the corresponding pairs.

Otherwise, $C_{\mathbf{X}^*}(s) \leq \mathbf{L}^* < \mathbf{X}^*$ for a Levi subgroup $\mathbf{L}^*$. By [208, Proposition 7.3, Step 1], the lift of $\mathbf{Z}_s$ has independent scalars a_x on pairs $\{x, x^{-1}\}$, or it is $(\mathbf{A}_0^j)^\circ$ for $jtj^{-1} = -t$, where $\mathbf{A}_0$ is the torus of independent scalars a_x. In the latter case $\dim V_+ = \dim V_-$. Moreover, by Proposition 6.190, $U_x = V_x \oplus V_{-x}$ and $U_x^\vee$ have nonzero weights, with $V_x = \ker(t - xI)$. On the zero weight space Y, either $t^2|_Y = I$ or $t|_Y$ has type $Q(a,c)$ with $i^2 = -1$ on the eigenvalue pair $\{i,-i\}$. By Corollary 6.276, we obtain $\succeq_c$ at $M = N_X(Q)$ in both cases. For independent scalars a_x, the comparison on $\{[s],[ks]\}$ is Proposition 6.13. For the spaces $U_x = V_x \oplus V_{-x}$, by Lemma 6.10, $[s] = [ks]$.

Finally, for $\psi \in \mathrm{Irr}_{p'}(M)$, Clifford's theorem [95, Theorem 6.2] gives one M-orbit of constituents of $\psi|_K$, with size prime to p. Thus, its orbit of K-series has size prime to p. By Lemma 6.7, each eligible class $[s]$ determines one such orbit $\mathcal{O}_s$ in $[s]_{X^*}$. Each comparison exhausts the characters of M above $\mathcal{O}_s$ or above the fibre $\{[s],[ks]\}$. Moreover, $\tau\colon [s] \mapsto [ks]$ is equivariant with $\tau^2 = 1$ by Lemma 6.11. Consequently, the bijection $\mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(M)$ follows from Lemma 9.4, with the $\succeq_c$ relation on each matched stabilizer. □

6.11 Proof of Theorem 6.1

Proof. Let $S = {}^3D_4(q)$. At the field prime s, [162] gives $\mathrm{iMN}(S,s)$ with $M = \mathrm{N}_S(P)$. For odd q, [102, Theorem 8.8] gives $\mathrm{iMN}(S,2)$ on $\mathrm{N}_S(\mathbf{S})$, where $\mathbf{S}$ is a Sylow d-torus for $d = \mathrm{ord}_4(q)$. Moreover, [166, Theorem B] gives the $\mathcal{H}_2$-equivariant bijection to $\mathrm{N}_S(P)$. Given odd $p \neq s$, put $d = \mathrm{ord}_p(q)$. Since $q^8 + q^4 + 1 = \Phi_3(q)\Phi_6(q)\Phi_{12}(q)$, the order of S gives $d \in \{1,2,3,6,12\}$. If $d = 12$, Proposition 6.101 gives the comparison on $\mathrm{N}_S(P)$. If $d \in \{1,2,3,6\}$, the local group is $M = T \rtimes V$. By [156, Theorem D], $\mathcal{E}(S,1) \cap \mathrm{Irr}_{p'}(S)$ corresponds to the inflation of $\mathrm{Irr}_{p'}(M/T)$. Given a nontrivial weight θ, the paired χ, ψ have the descent in Section 6.4 and satisfy $\succeq_c$. Consequently, $\mathrm{iMN}(S,p)$ holds with Γ-stable M containing $\mathrm{N}_S(P)$. Moreover, since M is solvable and $\mathrm{N}_M(P) = \mathrm{N}_S(P)$, [194] gives an $\mathcal{H}_p$-equivariant bijection from $\mathrm{Irr}_{p'}(M)$ to $\mathrm{Irr}_{p'}(\mathrm{N}_S(P))$. In nonisolated series, if $2 \mid [A_\chi : S]$, the subfield module $\mathbb{Q}[S/S^j]$ gives the extension in Proposition 6.102. If $S = \mathrm{P}\Omega^\epsilon_{2n}(q)$, Theorem 6.277 gives $\mathrm{iMN}(S,p)$ on its universal cover. □

7 Exceptional groups

Theorem 7.1. *Let S be a nonabelian simple group of type $E_6, {}^2E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^2G_2$ or ${}^2F_4$, including the Tits group ${}^2F_4(2)'$. Then $\mathrm{iMN}(S,p)$ holds for every prime $p \mid |S|$.*

7.1 Centralizers and primitive tori

Let $q = r^f$ for the field prime r, and let p be the McKay prime. Moreover, let $(\mathbf{G}, F)$ over $\overline{\mathbb{F}}_r$ be connected reductive, with dual $(\mathbf{G}^*, F^*)$ and $G = \mathbf{G}^F$. For semisimple $s \in \mathbf{G}^{*F^*}$, we write $\mathcal{E}(G, s)$ for the rational Lusztig series at s. Put $C = \mathrm{C}_{\mathbf{G}^*}(s)$ and $\mathbf{L}^*(s) = \mathrm{C}_{\mathbf{G}^*}(\mathrm{Z}(C)^\circ)$. Indeed, $\mathrm{Z}(C)^\circ$ is a torus by Geck–Malle [72, 1.1.7 and 1.1.13]. Moreover, $\mathbf{L}^*(s)$ is the smallest Levi containing C by Bonnafé–Rouquier [18, §11.5] and Zhang [208, proof of Proposition 4.16, Step 1]. For a coprime to $|s|$, $F^*(\mathbf{L}^*(s)) = \mathbf{L}^*(s) = \mathbf{L}^*(s^a)$. Given $g \in \mathbf{G}^*$, we also have that $\mathbf{L}^*(s^g) = \mathbf{L}^*(s)^g$. Moreover, we let $\mathbf{L}$ be rational and dual to $\mathbf{L}^*(s)$, with $L = \mathbf{L}^F$. We compare $\mathrm{N}_G(\mathbf{L})_{\mathcal{E}(L,s)}$ with $\mathrm{N}_G(P)$ for $P \in \mathrm{Syl}_p(G)$. By Bonnafé–Rouquier [18, Theorem B′ and Remark 10.12], signed induction pairs $\beta \in \mathcal{E}(L, s)$ with $\chi \in \mathcal{E}(G, s)$ of degree $\chi(1) = |G : L|_{r'}\beta(1)$. By Geck–Malle [72, Theorem 3.3.12], $\chi(z)/\chi(1) = \beta(z)/\beta(1)$ for $z \in \mathrm{Z}(\mathbf{G})^F$. If $p \neq r$ and $p \nmid \chi(1)$, then $|G : L|_p = 1$ and $P \leq L$ for some $P \in \mathrm{Syl}_p(G)$. Consequently, the bijection restricts to the p'-degree characters of $\mathcal{E}(L, s)$ and $\mathcal{E}(G, s)$.

The inclusion $P \leq L$ allows $\mathrm{N}_G(P) \not\leq L$. For example, an element of $\mathrm{N}_G(P)$ can normalize a torus $\mathbf{T}$ and induce inversion on $\mathbf{T}^*$. Suppose that $P \in \mathrm{Syl}_p(X)$ for a finite group X, and put $T = \mathrm{C}_X(P)$. Given $s \in T$, Sylow conjugacy in $\mathrm{C}_X(t)$ for $t \in s^X \cap T$ gives $s^X \cap T = s^{\mathrm{N}_X(P)}$. Similarly, for $P \leq L \leq X$, $s^L \cap T = s^{\mathrm{N}_L(P)}$ when $s \in L \cap T$. Thus, $s^{\mathrm{N}_X(P)} \cap L$ may contain several $\mathrm{N}_L(P)$-orbits. Moreover, for a joint action (g, σ), the correcting coset in $\mathrm{N}_X(P)$ can depend on the coefficient automorphism σ.

For proper $\mathbf{L}^*(s)$, we apply Proposition 2.4 with $D = A \times_{A/G} B$, where $B/L \cong A/G$ and $G \cap B = L$. For Y_U in Zhang [208, proof of Proposition 4.16, Step 4], we may take $\Xi = \sum_i (-1)^i \operatorname{tr}_{H^i_c(Y_U)}$ on D. Indeed, $\langle \Xi|_{G\times L}, \chi \boxtimes \beta^\vee\rangle_{G\times L}$ is $\pm\langle R^{\mathbf{G}}_{\mathbf{L}}(\beta), \chi\rangle_G$. For a joint action h on D, the proposition needs $\Xi^h = \Xi$ on D. An extension V of $\chi \boxtimes \beta^\vee$ with $V^h \cong V$ also suffices. If h changes the radical U, the identity $\Xi^h|_{G\times L} = \Xi|_{G\times L}$ controls the multiplicity on $G \times L$. Similarly, for induction from $J \leq \mathrm{N}_G(P)$, the common quotient is B_{loc}/J for the stabilizer B_{loc} of the local source $\nu \in \mathrm{Irr}(J)$.

Take $\varphi^m = \sigma F$ with $\sigma\varphi = \varphi\sigma$. By Zhang [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], we set $\tau(x_0, \ldots, x_{m-1}) = (x_1, \ldots, x_{m-1}, \sigma(x_0))$ and $\mathcal{F} = \tau^{-1}\operatorname{diag}(\varphi)$. Under the fixed-point isomorphism, τ acts as $\varphi|_{\mathbf{G}^F}$. For the ordinary stabilizer B of the product Levi $\mathbf{L}_0$, we choose one position i per B-orbit and a parabolic $\mathbf{P}_i$ fixed by B_i. Thus, $\mathbf{P}_{b(i)} = b(\mathbf{P}_i)$ defines a B-stable product with radical U. If t is the group component, then U^t is B^t-stable.

Proposition 7.2. *Let $F = \phi^f$ be split for the pinned Frobenius ϕ over $\mathbb{F}_r$. Take a regular inclusion $\mathbf{G} \leq \widetilde{\mathbf{G}}$ on which F, ϕ act, allowing $\widetilde{\mathbf{G}} = \mathbf{G}$. Thus, we put $G = \mathbf{G}^F$ and $\Gamma = \widetilde{\mathbf{G}}^F \rtimes \langle \phi|_{\widetilde{\mathbf{G}}^F}\rangle$, with the field factor of order f or its abstract cyclic lift. Let $\mathbf{K}$ be an F-stable Levi, with $K = \mathbf{K}^F$. Moreover, for $\beta \in \mathcal{E}(K, s)$ with $\mathrm{C}_{\mathbf{G}^*}(s) \leq \mathbf{K}^*$, put $B = \mathrm{N}_\Gamma(\mathbf{K})_\beta$, $\quad A = GB$, $\quad \chi = \epsilon_{\mathbf{G}}\epsilon_{\mathbf{K}} R^{\mathbf{G}}_{\mathbf{K}}(\beta)$. The action of $x\phi^j$ on $\mathbf{K}$ is by $\mathrm{Int}(x)\phi^j$. Since $F(\mathbf{K}) = \mathbf{K}$, its normalizer condition depends on $j \bmod f$. Then, we find $G \cap B = K$. On $\Delta = A \times_{A/G} B$, we have that the character $\chi \boxtimes \beta^\vee$ has an extension fixed by each $(t, \sigma) \in \mathrm{N}_\Gamma(\mathbf{K}) \times \mathcal{H}_p$ with $\beta^{(t,\sigma)} = \beta$. Moreover, the diagonal subgroup for $B \cap \mathrm{Z}(\widetilde{\mathbf{G}})^F$ acts trivially. If $[\mathbf{G}, \mathbf{G}] \neq 1$, we find $\mathrm{C}_A(G) = B \cap \mathrm{Z}(\widetilde{\mathbf{G}})^F$. Thus, the factors and twists at χ, β agree by* Corollary 2.5 *on $A/G \cong B/K$. Moreover, we have that $A = \Gamma_\chi$ if $A/G = \Gamma_\chi/G$.*

Proof. Put $\widetilde{\mathbf{K}} = \mathbf{K}\mathrm{Z}(\widetilde{\mathbf{G}})$. Indeed, for $\widetilde{\mathbf{G}} = \mathbf{G}$, $\widetilde{\mathbf{K}} = \mathbf{K}$. By [19, (7.1)], a coset in $\mathrm{N}_G(\mathbf{K})/K$ fixes $\mathcal{E}(K, s)$ exactly when its dual coset fixes $[s]_{\mathbf{K}^{*F^*}}$. Since $\mathrm{C}_{\mathbf{G}^*}(s) \leq \mathbf{K}^*$, its stabilizer in $\mathrm{N}_G(\mathbf{K})/K$ is trivial. Thus, $G \cap B = K$. Moreover, [133, Proposition 23.2] gives the surjection $\widetilde{\mathbf{K}}^F \to \widetilde{\mathbf{G}}^F/G$ since $\mathbf{K}$ is connected. By [72, Proposition 2.6.16], $\mathcal{E}(K, s)$ consists of constituents of $\widetilde{\beta}|_K$ from the lifted series in $\widetilde{\mathbf{K}}^F$. Thus, $\widetilde{\mathbf{K}}^F$ fixes $\mathcal{E}(K, s)$ as a set. Given $b \in B \cap \widetilde{\mathbf{G}}^F$, we choose $k_0 \in \widetilde{\mathbf{K}}^F$ with $k_0 G = bG$. Then, $k_0^{-1}b \in \mathrm{N}_G(\mathbf{K})$ stabilizes $\mathcal{E}(K, s)$. Consequently, $k_0^{-1}b \in K$ and $B \cap \widetilde{\mathbf{G}}^F \leq \widetilde{\mathbf{K}}^F$. Thus, the kernel of $B \to \langle\phi\rangle$ acts on $\mathbf{G}$ by inner automorphisms from $\mathbf{K}$ and fixes every parabolic with Levi $\mathbf{K}$.

By [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], take $\mathbf{G}_0 = \mathbf{G}^f$, $(\tau z)_i = z_{i+1}$ modulo f and $\mathcal{F} = \tau^{-1}\operatorname{diag}(\phi)$. Put $\mathbf{K}_0 = \prod_i \phi^i(\mathbf{K})$. The fixed-point map $x \mapsto \widehat{x} = (x, \phi(x), \ldots, \phi^{f-1}(x))$ gives $\mathbf{G}_0^{\mathcal{F}} = \widehat{G}$ and $\mathbf{K}_0^{\mathcal{F}} = \widehat{K}$. The action is $d(x\phi^j) = \mathrm{Int}(\widehat{x})\tau^j$, so $d(b)\mathcal{F} = \mathcal{F}d(b)$ and $d(b)\mathbf{K}_0 = \mathbf{K}_0$ for $b \in B$. The position stabilizer in B is $B \cap \widetilde{\mathbf{G}}^F$. By its action through $\mathbf{K}$, a parabolic $\mathbf{P}_i$ at a representative i of each B-orbit gives a B-stable product with radical U. If $\beta^{(t,\sigma)} = \beta$, then $B^t = B$ since $(\beta^t)^\sigma = (\beta^\sigma)^t$. Thus, $V = d(t)U$ is also B-stable. Since $d(b)\mathcal{F} = \mathcal{F}d(b)$, $\mathcal{F}$ extends to $\mathbf{G}_0 \rtimes B$ by the identity on B. Put $\widehat{\mathbf{K}}_0 = \mathbf{K}_0 \rtimes B$. If $\ell \neq r$ and $\ell \nmid |(\mathbf{G}_0 \rtimes B)^{\mathcal{F}}|$, take a splitting field $K_\ell/\mathbb{Q}_\ell$. Let e_s be the idempotent of $\widehat{K}$ for $\mathcal{E}(K, s)$. By [208, proof of Lemma 4.19(b)], the fixed tuples carry $\mathrm{C}_{\mathbf{G}^*}(s) \leq \mathbf{K}^*$ to its product inclusion in $\mathbf{G}_0^*$. Since B fixes β, it fixes e_s. With opposite right multiplication, we put $\delta(h) = (h, h^{-1})$. Then, the acting group is $D = (\widehat{G} \times \widehat{K}^{\mathrm{opp}})\delta(\widehat{\mathbf{K}}_0^{\mathcal{F}})$. For $b \in B$ as a constant component, the map $\Delta \longrightarrow D, \qquad (a, b) \longmapsto (\widehat{ab^{-1}}, 1)\delta(b)$ gives the Δ-action. On $G \times K$ it is $(g, k) : zW \mapsto \widehat{g}z\widehat{k}^{-1}W$ for $W = U, V$.

Thus, (c,c) acts trivially for $c \in B \cap \mathrm{Z}(\widetilde{\mathbf{G}})^F$. If $x\phi^j \in \mathrm{C}_A(G)$, x normalizes $U_\pm^F$, since $\phi^j(U_\pm^F) = U_\pm^F$. Consequently, $x \in \widetilde{\mathbf{B}}_+^F \cap \widetilde{\mathbf{B}}_-^F = \widetilde{\mathbf{T}}^F$, where $\widetilde{\mathbf{B}}_\pm^F$ normalize the pinned maximal unipotent subgroups $U_\pm^F$. On a simple root subgroup U_α^F, this action on $u \in \mathbb{F}_q$ is $u \mapsto \alpha(x)u^{r^j}$. If $u = 1$, we have that $\alpha(x) = 1$ at each simple root α, so $x \in \mathrm{Z}(\widetilde{\mathbf{G}})^F$. Since $[\mathbf{G},\mathbf{G}] \neq 1$, some U_α^F occurs. Thus, $u^{r^j} = u$ for every $u \in \mathbb{F}_q$ implies $f \mid j$. Consequently, $\mathrm{C}_A(G) = B \cap \mathrm{Z}(\widetilde{\mathbf{G}})^F$.

By Lemma 7.168, $H_c^{\dim Y_U}(Y_U)e_s \cong H_c^{\dim Y_V}(Y_V)e_s$ as D-modules. The right $\beta^\vee$-isotypic subspace X_U has character $\chi \boxtimes \beta^\vee$ on $G \times K$. Indeed, [208, proof of Lemma 4.19(b)] computes the induction with $\prod_i \phi^i(U_0)$ in place of U and projection to position 0. Since $\ell \nmid |K|$, s has ℓ'-order. By [18, Theorem B′], $H_c^i(Y_U)e_s = 0$ for $i \neq \dim Y_U$. Thus, the signed induction of β is χ. Since $V = d(t)U$ and $(\beta^t)^\sigma = \beta$, Lemma 7.168 gives that the extension on X_U is fixed by (t,σ). Consequently, $(X_U \otimes P_\beta)^K$ has the factors and twists of P_β by Corollary 2.5 on Δ. □

Corollary 7.3. *In* Proposition 7.2*, suppose that* $\mathbf{K}^*(s^g) = \mathbf{K}^*(s)^g$ *for* $g \in G^*$ *and* $\mathbf{K}^*(\alpha(s)) = \alpha(\mathbf{K}^*(s))$ *for dual field maps* α*. Suppose also that* $\mathbf{K}^*(s^u) = \mathbf{K}^*(s)$ *if* $\gcd(u,|s|) = 1$*. Then, we find* $A = \Gamma_\chi$*. Given any* (a,σ) *with* $\chi^{(a,\sigma)} = \chi$*, we have that the set* $\{b \in Ga \cap \mathrm{N}_\Gamma(\mathbf{K}) : \beta^{(b,\sigma)} = \beta\}$ *is a single* K*-coset. Thus, the mixed comparison of* χ,β *holds on the full inertia* A*. In particular, we let* s_π *be the* π*-part of* s *for fixed primes* π*. The assignment can be* $\mathbf{L}^*(s)$ *from* Section 7.1*, or* $\mathrm{C}_{\mathbf{G}^*}(s_\pi)$ *when it is a Levi.*

Proof. If σ raises the $|s|$-th roots to the u-th power, $\mathcal{E}(G,s)^\sigma = \mathcal{E}(G,s^u)$ by [180, Lemma 3.2]. Thus, s and s^u have the Levi $\mathbf{K}^*(s)$. By [72, Proposition 2.6.16], $\mathcal{E}(G,s)$ is fixed by $\widetilde{\mathbf{G}}^F$. Since $\chi^{(a,\sigma)} = \chi$, the dual field image of s^u lies in $[s]_{G^*}$. Consequently, $\alpha(\mathbf{K}^*(s)) \in [\mathbf{K}^*(s)]_{G^*}$ for that field map α. By [160, Corollary 2.13], $\phi^j(\mathbf{K}) \in [\mathbf{K}]_G$ for the corresponding field map ϕ^j.

Write $a = x\phi^j$ with $0 \leq j < f$. Thus, some $a_0 \in G\phi^j$ stabilizes $\mathbf{K}$. By [208, Remark B.9], after a correction by K, the action of a_0 on $\mathbf{K}$ has a dual morphism on $\mathbf{K}^*$. The correction is relative to $(\mathbf{T}_K,\mathbf{B}_K)$, a rational pair of $\mathbf{K}$. Moreover, $\widetilde{\mathbf{K}}^F$ surjects onto $\widetilde{\mathbf{G}}^F/G$. Thus, for $k_0 \in \widetilde{\mathbf{K}}^F$ with $k_0G = xG$, $a_1 = k_0a_0$ lies in Ga and the factor k_0 fixes each series $\mathcal{E}(K,v)$. By [72, Proposition 3.3.15], $\mathcal{E}(K,s)^{(a_1,\sigma)}$ is indexed by $s' = (a_1^*)^{-1}(s^u)$ for the dual map a_1^*. Then, $s' \in [s]_{G^*}$ and $\mathbf{K}^*(s') = \mathbf{K}^*(s)$. Thus, a rational dual conjugator from s' to s belongs to $\mathrm{N}_{G^*}(\mathbf{K}^*)$. By [208, Proposition 5.5, (5.9), and Remark B.9], a representative $g \in \mathrm{N}_G(\mathbf{K})$ of the corresponding Weyl coset satisfies $\mathcal{E}(K,s')^g = \mathcal{E}(K,s)$. Consequently, $b = a_1g \in Ga$ and $\beta^{(b,\sigma)} \in \mathcal{E}(K,s)$.

By [72, Theorem 3.3.22], $R_{\mathbf{K}}^{\mathbf{G}}$ on the series is independent of $\mathbf{P} = \mathbf{K}U$. Moreover, [72, Corollary 3.3.14 and Proposition 3.3.15] give covariance under σ and b. Thus, $\beta^{(b,\sigma)}$ has signed induction χ, since $b \in Ga$. By [18, Theorem B′], signed induction is injective on $\mathcal{E}(K,s)$, so $\beta^{(b,\sigma)} = \beta$. If b_1, b_2 are two such representatives, $b_1^{-1}b_2 \in G \cap B = K$. Conversely, right multiplication by K fixes β and keeps $bK \subseteq Ga$. If $\sigma = 1$, the cosets give $A = GB = \Gamma_\chi$. Finally, $\mathbf{L}^*(s)$ and $\mathrm{C}_{\mathbf{G}^*}(s_\pi)$ are equivariant under the conjugations by G^* and the dual field maps α. Moreover, $\langle s^u\rangle = \langle s\rangle$ implies $\langle (s^u)_\pi\rangle = \langle s_\pi\rangle$. Thus, both Levi assignments have $\mathbf{K}^*(s^u) = \mathbf{K}^*(s)$. □

Lemma 7.4. *Let* γ *be a pinned diagram involution and* ϕ *the pinned prime-field map, with* $\gamma\phi = \phi\gamma$ *and* $F = \gamma^\nu\phi^f$ *for* $\nu \in \{0,1\}$*. Take a regular inclusion* $\mathbf{G} \leq \widetilde{\mathbf{G}}$ *for both maps, allowing* $\widetilde{\mathbf{G}} = \mathbf{G}$*. Then, we find* $\phi^f = \gamma^\nu$ *on* $\widetilde{\mathbf{G}}^F$*. Suppose that the pinned group* $E = \langle\phi,\gamma\rangle$ *on* $\widetilde{\mathbf{G}}^F$ *embeds in* $\mathrm{Aut}(G)/\mathrm{InnDiag}(G)$ *by restriction to* G*. In* Proposition 7.2*, we put* $\Gamma = \widetilde{\mathbf{G}}^F \rtimes E$*. Suppose also that* $B_0 = \{x\phi^a\gamma^b \in B : a \equiv 0 \bmod f\}$ *stabilizes a parabolic* $\mathbf{P}$ *with Levi* $\mathbf{K}$*. Then, on* $A \times_{A/G} B$*, we have that* $\chi \boxtimes \beta^\vee$ *has an extension fixed by each* (t,σ) *with* $\beta^{(t,\sigma)} = \beta$*. Moreover, we have that* $\mathrm{C}_A(G)$ *acts trivially via* $c \mapsto (c,c)$*. Under* Corollary 7.3*, suppose that* $\mathbf{K}^*(\gamma^*(s)) = \gamma^*(\mathbf{K}^*(s))$ *for the dual diagram map* γ^**. Then, we find* $A = \Gamma_\chi$*.*

Proof. By [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], on $\mathbf{G}_0 = \mathbf{G}^f$ we put $\tau(z_0,\dots,z_{f-1}) = (z_1,\dots,z_{f-1},\gamma^\nu z_0)$ and $\mathcal{F} = \tau^{-1}\,\mathrm{diag}(\phi)$. Thus, $\mathcal{F}^f = \mathrm{diag}(F)$ and $\tau\widehat{x} = \widehat{\phi x}$ on the fixed tuples $\widehat{x} = (x,\phi x,\dots,\phi^{f-1}x)$. Put $\mathbf{K}_0 = \prod_{i=0}^{f-1}\phi^i(\mathbf{K})$, so $\mathbf{K}_0^{\mathcal{F}} = \widehat{K}$. Thus, $d(x\phi^a\gamma^b) = \mathrm{Int}(\widehat{x})\tau^a\,\mathrm{diag}(\gamma^b)$ defines the finite action, with $d(g)\mathcal{F} = \mathcal{F}d(g)$ and $d(g)(\mathbf{K}_0) = \mathbf{K}_0$ for $g \in B$. At position i, a stabilizer element $x\phi^{fk}\gamma^b \in B_0$ acts by $\mathrm{Int}(\phi^i(x))\gamma^{\nu k+b}$. Thus, it fixes $\phi^i(\mathbf{P})$ by the hypothesis on B_0. Consequently, choosing $\phi^i(\mathbf{P})$ at each orbit representative i gives a B-stable product parabolic with radical U. If $\beta^{(t,\sigma)} = \beta$, $B^t = B$ and $V = d(t)U$ is B-stable. The fibre product Δ acts through $(a,b) \mapsto (\widehat{ab^{-1}},1)\delta(b)$ of Proposition 7.2. By [208, proof of Lemma 4.19(b)], the radical $\prod_i \phi^i(U_0)$ gives induction from $K = \mathbf{K}^F$ under projection to position 0. Moreover, the dual inclusion $\mathrm{C}_{\mathbf{G}^*}(s) \leq \mathbf{K}^*$ extends to the product. Since β is B-invariant, B fixes the series idempotent e_s. By [18, Theorem B′], $H_c^i(Y_U)e_s = 0$ for $i \neq \dim Y_U$. By [19, Corollary 6.5 and Remark 6.7], the top cohomology on e_s is independent of U as a $G \times K$-module. Thus, the $\beta^\vee$-part affords $\chi \boxtimes \beta^\vee$.

By Lemma 7.168, its extension to Δ is fixed by (t,σ). Moreover, (c,c) acts trivially for $c \in B \cap \mathrm{Z}(\widetilde{\mathbf{G}})^F$. Since E embeds in $\mathrm{Aut}(G)/\mathrm{InnDiag}(G)$, $\mathrm{C}_A(G) \le \widetilde{\mathbf{G}}^F$. Thus, $\mathrm{C}_A(G) = B \cap \mathrm{Z}(\widetilde{\mathbf{G}})^F$ by Proposition 7.2. If $\alpha = \phi^i\gamma^j$, $\alpha F = F\alpha$. Thus, by [208, Remark B.9], a rational correction of α on $\mathbf{K}$ has a dual morphism on $\mathbf{K}^*$. By Corollary 7.3, if $\chi^{(a,\sigma)} = \chi$, the corrections in Ga form one K-coset by equivariance under γ^*. In particular, for $\sigma = 1$ $A = \Gamma_\chi$. □

Proposition 7.5. *With $G = F_4(2^f)$ and $f > 1$, we let γ be the pinned exceptional isogeny with $\gamma^{2f} = F_q$. Thus, take $\Gamma = G \rtimes \langle \gamma|_G \rangle$. Let $\mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(s)$ be a Levi, with $K = \mathbf{K}^{F_q}$, and let $\beta \in \mathcal{E}(K,s)$. Then, for $A = \Gamma_\chi$, we have that the signed induction $\chi \in \mathcal{E}(G,s)$ has the comparison with β of* Proposition 7.2.

Proof. With $n = 2f$, put $B = \mathrm{N}_\Gamma(\mathbf{K})_\beta$ and $\chi = \epsilon_{\mathbf{G}}\epsilon_{\mathbf{K}} R_{\mathbf{K}}^{\mathbf{G}}(\beta)$. By [19, (7.1)], $G \cap B = K$, since the full centralizer is $\mathbf{K}^*$. By the cyclic product in [208, proof of Proposition 9.1], put $\mathbf{G}_0 = \mathbf{G}^n$, $(\tau z)_i = z_{i+1}$ and $\mathcal{F} = \tau^{-1}\,\mathrm{diag}(\gamma)$. With $\mathbf{K}_0 = \prod_{i=0}^{n-1} \gamma^i(\mathbf{K})$, the fixed-point map $x \mapsto \widehat{x} = (x, \gamma x, \ldots, \gamma^{n-1}x)$ gives $G \simeq \mathbf{G}_0^{\mathcal{F}}$ and $K \simeq \mathbf{K}_0^{\mathcal{F}}$. The action $d(x\gamma^j) = \mathrm{Int}(\widehat{x})\tau^j$ satisfies $d(x\gamma^j)\mathcal{F} = \mathcal{F}d(x\gamma^j)$ by $\tau\widehat{x} = \widehat{\gamma x}$. As in Proposition 7.2, the position stabilizer $B \cap G = K$ acts at position i through $\gamma^i(\mathbf{K})$. Consequently, one parabolic per B-orbit gives a B-stable radical U. If $\beta^{(t,\sigma)} = \beta$, $B^t = B$, so $V = d(t)U$ is B-stable.

By [208, proof of Proposition 9.1], projection to position 0 carries $\mathcal{E}(\widehat{K},\widehat{s})$ onto $\mathcal{E}(K,s)$. Moreover, $\mathrm{C}_{\mathbf{G}_0^*}(\widehat{s}) \le \mathbf{K}_0^*$, so the radical centralizers $\mathrm{C}_{W^*}(\widehat{s})$ for $W = U, V$ are trivial. Since β is B-invariant, B fixes e_s. The group $\Delta = GB \times_{B/K} B$ acts by $(a,b) \mapsto (\widehat{ab^{-1}}, 1)\delta(b)$ as in Proposition 7.2. For a radical U_0 of a parabolic with Levi $\mathbf{K}$, put $U_* = \prod_{i=0}^{n-1} \gamma^i(U_0)$. By [208, proof of Proposition 9.1], projection $Y_{U_*} \to Y_{U_0}$ gives a $G \times K$-equivariant isomorphism. By [19, Corollary 6.5 and Remark 6.7], $H_{U_*}e_s$ and $H_U e_s$ agree on $G \times K$. Since [18, Theorem B′] gives concentration in degree $\dim Y_U$, the $\beta^\vee$-part affords $\chi \boxtimes \beta^\vee$. By Lemma 7.168, its extension to Δ is fixed by (t,σ). Moreover, $\mathrm{C}_\Gamma(G) = 1$, since $\gamma|_G$ generates the cyclic outer group [208, proof of Proposition 9.1] and $\mathrm{Z}(G) = 1$.

Finally, the Levi assignment $s \mapsto \mathrm{C}_{\mathbf{G}^*}(s)$ has $\mathrm{C}_{\mathbf{G}^*}(s^u) = \mathrm{C}_{\mathbf{G}^*}(s)$ for $\gcd(u,|s|) = 1$. By [208, Proposition 9.1 and Remark B.9], this assignment is equivariant for γ^*. Thus, $A = GB = \Gamma_\chi$ by Corollary 7.3, with one correcting K-coset for each joint stabilizer of χ. Consequently, the extension on Δ compares the factors on A/G and the twists at joint stabilizers (t,σ) of χ, β. □

Given a central product $\mathbf{AB}$ of F-stable groups, $(\mathbf{AB})^F$ may strictly contain $\mathbf{A}^F\mathbf{B}^F$. Indeed, if ab is fixed by F, the two factors satisfy $a^{-1}F(a) = bF(b)^{-1} \in \mathbf{A} \cap \mathbf{B}$. Thus, we choose the finite group K. We choose the models P and twist functions μ_h of Navarro et al. [141, §§1.5, 1.9–1.11, 2.2–2.5, 2.9]. For a factor c inflated from J, we normalize $c(1,j) = c(j,1) = 1$ for $j \in J$.

Proposition 7.6. *Let $H \le K$ be normal in D, with D finite, and suppose that $D = HC$. Put $L = K \cap C$ and $H' = H \cap C$, and suppose that $\mathrm{C}_D(H) \le C$. Suppose also that the orbits of $\theta \in \mathrm{Irr}(H)$ and $\eta \in \mathrm{Irr}(H')$ satisfy $\succeq_c$ relative to C and $\mathcal{H}_\ell$, with associated models P, P' and their twist functions μ_h at joint stabilizers h. Then, the bijection on characters of K, L above the orbits via $K/H \cong L/H'$ also satisfies $\succeq_c$ for D, C.*

Proof. Since $D = HC$, $K = HL$. For the matching θ, η, let P, P' have common factor c on $D_\theta/H \cong C_\eta/H'$. By [141, Lemma 1.9, Theorem 1.10 and Lemmas 2.2–2.3], the models on K, L give the equivariant bijection $\beta \leftrightarrow \nu$, with $\beta(1)/\theta(1) = \nu(1)/\eta(1)$. The Clifford correspondents on K_θ, L_η are the restrictions of $P \otimes U_0, P' \otimes U_0$, with U_0 of factor c^{-1} on $K_\theta/H \cong L_\eta/H'$.

With $I = D_\beta$, put $I' = C_\nu$ and $J = I \cap D_\theta$ and $J' = J \cap C$. By equivariance, $I' = I \cap C$. Since $D = KC$ and $K \le I$, $I = KI'$. Given $a \in I$, θ^a is a constituent of $\beta|_H$. By Clifford theory [43], the constituents of $\beta|_H$ form a K-orbit, which is also an L-orbit since $K = HL$. Thus, a correction by L returns θ^a to θ. Since $D = HC$ and $H \le J$, $I = KJ$, $I' = LJ'$, $J/H \cong J'/H'$, and $I/K \cong I'/L$. In particular, a transversal in L for I/J is also one for I'/J'.

Take a model V of β on I with factor b inflated from I/K. Thus, on the θ-isotypic space, we set $V_\theta = P|_H \otimes U$ with $U = \mathrm{Hom}_H(P|_H, V)$. Given $j \in J$, the action $u \mapsto V(j)uP(j)^{-1}$ on U has factor b/c on J/H. Its restriction to K_θ/H is U_0. Consequently, $P'|_{J'} \otimes U$ has factor b on J', and its restriction to L_η affords the Clifford correspondent of ν. By [155, Proposition 2.4], the ordinary comparison follows with D, C, H, K in place of A, A_0, K, J. Since $\mathrm{C}_I(K) \le \mathrm{C}_D(H) \le C$, the induction in [155, proof of Lemma 2.1] gives V' from $P'|_{J'} \otimes U$. Both models have factor b and equal centralizer scalars. Moreover, $I' = LJ'$ gives $\mathrm{tr}(V'|_L) = \nu$.

For a joint stabilizer (a,σ) of β, ν with $a \in C$, correct by L so that $\theta^{(a',\sigma)} = \theta$. Hence, $\eta^{(a',\sigma)} = \eta$. Let μ, δ be the functions of V, P at (a',σ). Then, $u \mapsto V(j)uP(j)^{-1}$ has twist μ/δ on U. The lower model P' has twist δ by hypothesis. Consequently, $P'|_{J'} \otimes U$ has twist μ on J'. For $g \in I'$, write $gt = t'j$ with t, t' in the chosen transversal

in L and $j \in J'$. Since b is normalized on I'/L, the block is the operator at j of the model on J'. Under (a', σ), the transversal is transported within L. Thus, the block twist is $\mu(j) = \mu(g)$, since $gL = jL$. Hence, V' has the same correcting function as V. Moreover, $\mu_{(a,\sigma)} = \mu_{(a',\sigma)}$ on I'/L by the normalization on L [141, §§1.9–1.11, 2.2–2.5]. □

Lemma 7.7. *Let $K \lhd B$, with $B = KB_P$ and $K \cap B_P = L$. Let $\kappa \in \mathrm{Irr}(K)$ be linear and B-invariant. Then, the pair of twists $\kappa, \kappa|_L$ multiplies the common factor c by one factor c_κ on B/K. For a normalized section $j \mapsto b_j \in B_P$ and $m = |j|$, the power factor is $\prod_{a=0}^{m-1} c_\kappa(j^a, j) = \kappa(b_j^m)^{-1}$. At $x \in C_B(K) \cap B_P$, the scalar contributions agree. Moreover, if a normalizes B, K, B_P and $\kappa^{(a,\sigma)} = \kappa$, the correcting functions agree.*

Proof. First, for $J = B/K \cong B_P/L$, take a section $j \mapsto b_j$ in B_P with $b_1 = 1$. Put $D_\kappa(kb_j) = \kappa(k)$ and $h_{ij} = b_i b_j b_{ij}^{-1} \in L$. Then, the factor is $c_\kappa(i,j) = \kappa(h_{ij})^{-1}$ on J. The local model $D_\kappa|_{B_P}$ has that factor on B_P/L. Since $D_\kappa(b_j) = 1$, the product law gives $1 = D_\kappa(b_j^m) \prod_{a=0}^{m-1} c_\kappa(j^a, j)$, as claimed. If a normalizes B_P and $\kappa^{(a,\sigma)} = \kappa$, $D_\kappa^{(a,\sigma)}/D_\kappa$ is constant on each coset Kb_j. Thus, at j both models have twist $(D_\kappa^{(a,\sigma)}/D_\kappa)(b_j)$, including the correction when $b_j^a \neq b_j$. Moreover, $D_\kappa(x) = (D_\kappa|_{B_P})(x)$ for a common centralizer element x. Consequently, both tensors have factor cc_κ, equal twists and equal scalars at x. □

Let $X = E_6^\epsilon(q)_{\mathrm{sc}} = \mathbf{G}^F$ with $u = \epsilon q$. The sign ϵ is 1 in split type and -1 in twisted type. Suppose that $\mathrm{ord}_p(u) = 9$, so $p \geq 19$. Put $e = \mathrm{ord}_p(q)$. By Springer [178, Theorem 4.2 and Table 1], we find a regular $w \in W(E_6)$ of order 9 with characteristic polynomial Φ_9 and $\mathrm{C}_W(w) = \langle w \rangle$. Take the rational torus $\mathbf{T}$ with Weyl element w in split type or ww_0 in twisted type, with diagram automorphism j. Indeed, on the reflection space V, $w_0 j = -I_V$. Thus, the lattice operator is uw up to inversion of w. Thus, by Carter [40, Chapter 3], $|T| = \Phi_9(u) = u^6 + u^3 + 1$, $\mathrm{N}_X(\mathbf{T})/T = \langle w \rangle$. Only the signed fundamental degree 9 contributes to $|X|_p$. Thus, $T = \mathbf{T}^F$ contains $P \in \mathrm{Syl}_p(X)$. By Malle–Testerman [133, Theorem 25.14], P is cyclic since $p \nmid |W(E_6)|$.

Lemma 7.8. *Let $T^* = (\mathbf{T}^*)^{F^*}$ be the adjoint dual torus. Write $\pi : \mathbf{G}_{\mathrm{sc}}^* \to \mathbf{G}^*$ for the central isogeny. If $t \in T^*$ has a prime divisor $b \neq 3$ in its order, we find $\mathrm{C}_{\mathbf{G}^*}(t) = \mathbf{T}^*$. If $u \equiv 1 \bmod 3$, the two elements $t \in T^*$ of order 3 are fixed by $\langle w \rangle$, lie outside $\pi(\widetilde{T}^*)$ for the rational covering torus $\widetilde{T}^*$, and have $\Phi(\mathrm{C}^\circ_{\mathbf{G}^*}(t)) = A_2^3$. Otherwise, $T^*[3] = 1$. Moreover, $\mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{T})$.*

Proof. Put $\Phi(t) = \{\alpha \in \Phi(E_6) : \alpha(t) = 1\}$. Since $F^*(t) = t$ and $\gcd(u, |t|) = 1$, the torus action uw implies $w\Phi(t) = \Phi(t)$. If $\Phi(t) \neq \varnothing$, its span has rank 6 since Φ_9 is irreducible. Thus, t is isolated. By [20, Table III], the orders for $t \neq 1$ are $2, 3$, and $\Phi(t) = A_2^3$ at order 3. Since $\Phi_9(u)$ is odd, $|t| = 3$ in this case. By [20, Corollaries 2.8–2.9], $A(t) = \mathrm{C}_{\mathbf{G}^*}(t)/\mathrm{C}^\circ_{\mathbf{G}^*}(t)$ embeds in $\ker \pi$ and has exponent dividing $|t|$. For $b \neq 3$ dividing $|t|$, take $t_0 \in \langle t \rangle$ of order b. Thus, $\Phi(t_0) = \varnothing$ and $A(t_0) = 1$. Hence, $\mathbf{T}^* \leq \mathrm{C}_{\mathbf{G}^*}(t) \leq \mathrm{C}_{\mathbf{G}^*}(t_0) = \mathbf{T}^*$.

If $u \equiv 1 \bmod 3$, we write $u^3 = 1 + 3k$. Then, $\Phi_9(u) = 3 + 9k + 9k^2$, and otherwise $3 \nmid \Phi_9(u)$. Thus, in the former case $T^*[3] \cong C_3$, fixed pointwise by $\langle w \rangle$. Given $t \in T^*[3] \setminus \{1\}$, $w \in W(t)$. If $\Phi(t) = \varnothing$, $W(t) \cong A(t) \leq \ker \pi$ has order dividing 3, contrary to $|w| = 9$. Thus, $\Phi(t) = A_2^3$. The rational covering torus $\widetilde{T}^*$ also has order $\Phi_9(u)$. Since $|\ker(\pi|_{\widetilde{T}^*})| = 3$, its image has order prime to 3. Now, $\mathbf{T}$ is a Sylow e-torus of $\mathbf{G}$. By [130, Proposition 2.2], $\mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{T})$. □

If $\chi \in \mathcal{E}(X, s)$ has p'-degree, the Jordan degree formula Digne–Michel [58, Proposition 11.5.6] gives $|\mathrm{C}_{G^*}(s)|_p = |G^*|_p$. By Malle [132, Theorem 5.9], the class of s meets T^*, since $\mathbf{T}^*$ is a maximal Sylow Φ_e-torus. By Malle [132, Proposition 5.11], the intersection $s^{G^*} \cap T^*$ is one $\mathrm{N}_{G^*}(\mathbf{T}^*)$-orbit. If $\mathrm{C}_{\mathbf{G}^*}(s) = \mathbf{T}^*$, we let $\theta \in \mathrm{Irr}(T)$ correspond to s. Put $M = \mathrm{N}_X(P)$. Then, the pair at s is $\chi = \epsilon_{\mathbf{G}} \epsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $\psi = \mathrm{Ind}_T^M \theta$. By Geck–Malle [72, Theorem 3.3.22], we deduce that $\mathcal{E}(X, s) = \{\chi\}$. Moreover, $M_\theta = T$, so $\psi \in \mathrm{Irr}(M)$ by Clifford [43]. On $\mathrm{Z}(X)$ both central characters are $\theta|_{\mathrm{Z}(X)}$. Thus, (χ, ψ) is equivariant under $\mathrm{Aut}(X)_P \times \mathcal{H}_p$ by Digne–Michel [58, Chapters 10–11] and Srinivasan–Vinroot [180, Lemma 3.2].

In Lemma 7.4, we set $B = \mathrm{N}_\Gamma(\mathbf{T})_\theta$. On rational torus classes, j gives inversion since $w_0 j = -1$ and $\mathbf{T}^*$ is connected. Since $|\theta|$ and $|w|$ are odd and $\theta \neq 1$, $\theta^{-1} \notin \theta^{\langle w \rangle}$. Thus, B_0 has Weyl image 1 and lies in the diagonal torus $\widetilde{\mathbf{T}}$. Consequently, B_0 fixes every Borel $\mathbf{B}$ containing $\mathbf{T}$. For twisted X, the finite relation is $\phi^f = j$ for the prime-field map ϕ. Consequently, Lemma 7.4 and $\mathbf{L}^*(s) = \mathbf{T}^*$ give common quotient invariants for χ, θ on XB, B. Given an associated model Q_θ on B, the model on MB is its projective induction along $B \leq MB$, with factor inflated from

$MB/M \cong B/T$. Thus, the factors and twists on B/T agree. Moreover, for $c \in \mathrm{C}_{XB}(X)$, each induced block has scalar $Q_\theta(c)$ since $c \in B$ and c centralizes M. Consequently, the pair χ, ψ satisfies $\succeq_c$.

For the exceptional cover of ${}^2E_6(2)$, we cite Ruhstorfer–Schaeffer Fry [157, Lemma 4.1(2)] for iMN(S,p). Assume that $X \neq {}^2E_6(2)_{\mathrm{sc}}$. Given $s \in T^*[3]\backslash\{1\}$, Lemma 7.8 gives the faithful series on $\mathrm{Z}(X)$. By Zhang [208, Proposition 10.8, Step 1], the commutator map is $A(s) \cong \ker\pi \cong C_3$. Moreover, $\mathrm{C}^\circ_{\mathbf{G}^*}(s)^{F^*}$ has type $A_2^\epsilon(Q)$ up to central isogeny, with $Q = q^3$, by Zhang [208, Proposition 10.8, (10.25)]. In the marked torus, w permutes its three A_2 root components transitively since Φ_9 is irreducible. Thus, w^3 is a Coxeter element on each A_2 component. The three unipotent degrees of $A_2^\epsilon(Q)$ are $1, Q(Q+\epsilon), Q^3$ by Carter [40, Chapter 13]. Since $\mathrm{ord}_p(\epsilon Q) = 3$, all are prime to p. For the Coxeter torus $\mathbf{T}_C \leq \mathrm{C}^\circ_{\mathbf{G}^*}(s)$, the coefficients of $R^{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}_{\mathbf{T}_C}(1)$ at the partitions $\lambda \vdash 3$ have absolute value 1. Indeed, the value at a 3-cycle in $\mathfrak{S}_3$ is $(-1)^j$ for the hook λ of leg length j.

For the regular overgroup $\widetilde{X}$ of X, the dual torus kernel Z_0 is connected. By Malle–Testerman [133, Proposition 23.2], s has a rational lift $\widetilde{s}$. By Zhang [208, proof of Proposition 10.8, Step 2], each $\widetilde{\chi} \in \mathcal{E}(\widetilde{X}, \widetilde{s})$ restricts to a free diagonal orbit $\chi_0 + \chi_1 + \chi_2$. The three parent degrees $\widetilde{\chi}(1)$ distinguish these orbits. Choose $\widetilde{\theta}$ dual to $\widetilde{s}$. By Srinivasan–Vinroot [180, Theorem 4.1(1)], the Coxeter coefficients are ± 1 on the parents. Since $R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}}(\widetilde{\theta})|_X = R^{\mathbf{G}}_{\mathbf{T}}(\theta)$, each χ_i has coefficient ± 1. The two choices of s give the two faithful characters λ, λ^{-1} on $\mathrm{Z}(X)$.

Let n_w be the rational Tits representative of w for the marked torus. Since $n_w^9 \in T[2] = 1$, $M = T \rtimes \langle n_w \rangle$ and $|n_w| = 9$. Given a faithful cubic weight θ of T, by Lemma 7.8, we obtain $M_\theta = M$. Thus, the characters over θ are, by Isaacs [95, Corollary 6.17], $\psi_\xi(tn_w^i) = \theta(t)\xi^i$ for $\xi \in \mu_9$. The diagonal generator takes ξ to $\theta(z)\xi$ for a generator z of $\mathrm{Z}(X)$, by the commutator map on $A(s)$ in Zhang [208, Proposition 10.8, Step 1]. Thus, C_3 acts freely on $\{\psi_\xi : \xi \in \mu_9\}$, and $\mathcal{H}_p$ fixes each ψ_ξ since $p \equiv 1 \bmod 9$.

Each global triple has a unique χ_0 with $\chi_0^j = \overline{\chi_0}$, and $\mathbb{Q}(\chi_i) = \mathbb{Q}(\zeta_3)$ by Zhang [208, proof of Proposition 10.8, Step 2]. Thus, $\mathcal{H}_p$ fixes each χ_i. Moreover, the subgroup E_s of the pinned group fixing λ fixes each χ_i, with $|E_s| = f$, by Zhang [208, proof of Proposition 10.8, Step 3]. For the field prime r, the generator is ϕ in split type with $r \equiv 1 \bmod 3$ and $j\phi$ in split type with $r \equiv -1 \bmod 3$ and f even. In twisted type $r \equiv -1 \bmod 3$ and f is odd, so $E_s = \langle \phi^2 \rangle$.

With the marked torus, $\phi(n_w) = n_w$ in pinned coordinates. Moreover, $j_0 = \mathrm{Int}(n_{w_0})j$ acts by $t \mapsto t^{-1}$ on the marked torus and fixes n_w by $n_w n_{w_0} = n_{w_0} j(n_w)$. In split type, we choose $h^{-1}F_q(h) = n_w$. Then, $h\phi(h)^{-1}$ and $hn_{w_0}j(h)^{-1}$ belong to $\mathbf{G}^{F_q}$. In twisted type, take $h^{-1}F(h) = n_w n_{w_0}$ with $F = jF_q$. Thus, the corrected E_s fixes each ψ_ξ. Match the C_3-orbits over λ on X, M. For each pair of orbits, choose $\chi_0 \leftrightarrow \psi_\xi$. Then, extend by the diagonal group C_3 and by j for λ^{-1}. Thus, the bijection is equivariant for $\mathrm{Aut}(X)_P \times \mathcal{H}_p$.

Put $A = X \rtimes E_s$ and $B = \mathrm{N}_A(\mathbf{T})$, so $B/M \cong E_s$. Given a lift $x \in B$ of an E_s-generator, we extend $\psi = \psi_\xi$ by $\widehat{\psi}(x) = \alpha$ with $\alpha^f = \psi(x^f)$. Indeed, the conjugation relations follow from $\psi^x = \psi$. If $f_3 = 3^{v_3(f)}$, the f-th power map has image $\{A^f : A \in \mu_{9f_3}\} = \mu_9$. Moreover, for $h = \mathrm{ord}_p(r)$, $h/\gcd(h,f) = \mathrm{ord}_p(q) \in \{9, 18\}$. Thus, $v_3(h) = 2 + v_3(f)$, so $9f_3 \mid p-1$. Consequently, we may take $\alpha \in \mu_{9f_3}$ and have that $\widehat{\psi}^\sigma = \widehat{\psi}$ for $\sigma \in \mathcal{H}_p$.

On any odd-order subgroup $E_0 \leq E_s$, by Zhang [208, Lemma 3.1], we obtain the unique extension $\widehat{\chi}_0$ to XE_0 with $\widehat{\chi}_0^j = \overline{\widehat{\chi}_0}$. Moreover, $\mathbb{Q}(\widehat{\chi}_0) = \mathbb{Q}(\chi_0)$. Thus, the extension is $\mathcal{H}_p$-fixed on XE_s in twisted type, and on XE_0 in split type for $|E_0| = f_3$. In split type, we let $E' \leq E_s$ have order $m = f/f_3$, and take a lift $a \in B \cap XE'$ of its generator. The maps ϕ, j_0 fix $n_w T$ in M/T. Since $\gcd(m, 9) = 1$, we choose a modulo $\langle n_w \rangle$ with $a^m \in T$. Thus, $H = T\langle a \rangle$ has $H \cap X = T$ and the $\mathcal{H}_p$-fixed extension $\widehat{\psi}|_H$ of θ. In the cyclic product of Zhang [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], E' acts faithfully on the positions $i \in \mathbb{Z}/f\mathbb{Z}$. Thus, the position stabilizer H_i lies in T and fixes each Borel $\mathbf{B}_i$ containing $\phi^i(\mathbf{T})$. Therefore, one factor $\mathbf{B}_i$ per orbit gives an H-stable product radical U. For the integral alternating character E_U, projection against $\widehat{\psi}|_H$ gives a virtual character Ξ on XE'. Indeed, each (t^{-1}, t) in the kernel of $X \rtimes H \to XE'$ acts trivially after projection to $\widehat{\psi}|_H$. By Zhang [208, proof of Lemma 4.19(b)], $\Xi|_X = R^{\mathbf{G}}_{\mathbf{T}}(\theta)$, which is independent of the Borel $\mathbf{B}_i$. Moreover, $E_U^\sigma = E_U$ and $(\widehat{\psi}|_H)^\sigma = \widehat{\psi}|_H$ give $\Xi^\sigma = \Xi$ for $\sigma \in \mathcal{H}_p$. The Coxeter torus calculation gave $\langle \Xi|_X, \chi_0 \rangle_X = \pm 1$. Thus, by Proposition 2.4, we obtain an $\mathcal{H}_p$-fixed extension of χ_0 to XE'.

Finally, in split type $E_s = E_0 \times E'$ is cyclic. By Isaacs [95, Corollary 11.22], χ_0 extends to XE_s. By Ruhstorfer et al. [156, Lemma 7.1], the $\mathcal{H}_p$-fixed extensions over XE_0, XE' give the $\mathcal{H}_p$-fixed extension $\widehat{\chi}_0$. For the overgroups that induce $\mathrm{Aut}(X)_P$, take one section with factors $z_{a,b} \in \mathrm{Z}(X)$. By Section 2, both models evaluate $z \in \mathrm{Z}(X)$ at the scalar $\lambda(z)$, and their factors come from $\lambda(z_{a,b})$. Thus, $\widehat{\chi}_0, \widehat{\psi}$ are fixed by $\mathcal{H}_p$ and by inner automorphisms of A, B, respectively. Consequently, the translated pairs satisfy $\succeq_c$ at their joint stabilizers.

Let $X = E_7(q) = \mathbf{G}^F$ with $q = 2^f$, and suppose that $d = \mathrm{ord}_p(q) \in \{9, 18\}$. Put $\delta = 1$ if $d = 9$, and $\delta = -1$ if

$d = 18$. In characteristic 2, the central isogeny $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{ad}}$ is bijective on geometric points (Zhang [208, §11.1]). Thus, $\mathrm{Z}(X) = 1$, the semisimple centralizers in $\mathbf{G}, \mathbf{G}^*$ are connected, and $\mathrm{Out}(X) = \langle \phi \rangle$ for the prime-field map ϕ. Given a Coxeter element $c \in W(E_7)$, the characteristic polynomial is $\Phi_{18}(x)(x+1)$ by Springer [178, Table 2]. Thus, we obtain that $c^9 = -I$, and $a = c^{10}$ generates $\langle c^2 \rangle$ of order 9. Take the torus $\mathbf{T}$ with mark $v = c$ at $d = 18$ and $v = a$ at $d = 9$. Moreover, the reflection space is $U \perp V$, with $U = \ker \Phi_{18}(c)$ and $V = \ker(c + I)$.

Lemma 7.9. *The roots in U form a subsystem of type E_6, and V contains no root. With $T = \mathbf{T}^F$, we choose $P \in \mathrm{Syl}_p(X)$ in T. Then, $|T| = \Phi_d(q)(q - \delta)$ and $\mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{T}) = T \rtimes \langle n_c \rangle$, where $|n_c| = 18$. Moreover, on $T^* \setminus \{1\}$, we let $\mathcal{O}, \mathcal{O}'$ be the two $\langle a \rangle$-orbits in one $\langle c \rangle$-orbit. Then, $z = c^9$ gives $z(\mathcal{O}) = \mathcal{O}'$.*

Proof. Given $v_0 \in V \setminus \{0\}$, the stabilizer W_{v_0} is parabolic of rank at most 6. Since W_{v_0} contains a of order 9, it has an exceptional factor. Indeed, $|g|_3 \leq 3$ for $g \in W(A_k)$ or $W(D_k)$ with $k \leq 6$. Thus, W_{v_0} is of type E_6, with root subsystem $\Phi(E_7) \cap U$. Moreover, $U \otimes \mathbb{C}$ contains a regular eigenvector of c by [178, Table 2]. Consequently, $\Phi(E_7) \cap V = \varnothing$.

By [40, Chapter 3], $|T| = |\det(qv - I)| = \Phi_d(q)(q - \delta)$. Only the fundamental degree 18 contributes to $|X|_p$, so T contains P. Let U_p, V_p be the reductions of $Q(E_7) \cap U, Q(E_7) \cap V$ modulo p. Moreover, qv has a simple eigenvalue 1 on U_p, and the eigenvalue on V_p is $q\delta \neq 1$. Thus, $P[p]$ is a line in U_p and P is cyclic. Let J be generated by independent roots $\alpha_1, \ldots, \alpha_k$ spanning the roots trivial on $P[p]$. If $J \neq 0$, then $U \subseteq \mathbb{Q}J$ since $\mathbb{Q}J$ is v-stable and $V \cap \Phi(E_7) = \varnothing$. By Hadamard, $\det \mathrm{Gram}(J) \leq 2^k \leq 2^7 < p^2$. Thus, $p \nmid [J_{\mathrm{sat}} : J]$ for its saturation in $Q(E_7)$. The weight lattice $P(E_7)$ has $[P(E_7) : Q(E_7)] = 2$, so J is p-saturated in both lattices. Consequently, modulo p, $\overline{J} \supseteq U_p$ if $J \neq 0$. Moreover, $9^{-1} \sum_{i=0}^{8} a^i$ is the projector onto V over $\mathbb{Z}_{(p)}$. Since $\det Q(E_7) = 2$, the orthogonal complement U remains nondegenerate modulo p. Thus, $\overline{J}$ cannot annihilate $P[p]$ unless $J = 0$. Consequently, $C^\circ_{\mathbf{G}}(P[p]) = \mathbf{T}$. Since $P[p]$ is cyclic, connectedness implies $C_{\mathbf{G}}(P[p]) = \mathbf{T}$. Thus, $C_{\mathbf{G}}(P) = \mathbf{T}$.

Hence, $\mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{T})$. Moreover, $\mathrm{C}_W(v) = \langle c \rangle$ by [178, Theorem 4.2], since $v = c$ or $v = -c$. In characteristic 2, the Tits section $w \mapsto n_w$ is a homomorphism since $\mathbf{T}[2] = 1$. Thus, the rational lift n_c has order 18 and $\mathrm{N}_X(\mathbf{T}) = T \rtimes \langle n_c \rangle$. Finally, z acts by inversion, and $g^9 = z$ for $g \in \langle c \rangle \setminus \langle a \rangle$. Since $|T^*|$ is odd, any $s \in T^*$ fixed by g has $s = s^{-1} = 1$. Consequently, $\mathcal{O}$ and $\mathcal{O}^z$ are disjoint for $s \neq 1$. □

Proposition 7.10. *Let $s \in T^* \setminus \{1\}$ have $\mathrm{C}_{\mathbf{G}^*}(s) = \mathbf{T}^*$. Then, the* regular torus pair (χ, ψ) *satisfies $\succeq_c$ at $M = \mathrm{N}_X(P)$.*

Proof. Put $E = \langle \phi|_X \rangle$ in the split model of X. Indeed, by [58, §§12.3–12.4], $\mathcal{E}(X, s) = \{\chi\}$ and $\langle \Gamma_X, \chi \rangle_X = 1$ for the pinned Gelfand–Graev character Γ_X. In the construction of [103], take the E-fixed character $\lambda(u) = (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\sum_\alpha u_\alpha)}$ on U_0, with u_α the simple root coordinates of u. Since λ is rational in characteristic 2, the extension $\widehat{\lambda}$ to $U_0 \rtimes E$ with $\widehat{\lambda}|_E = 1$ is rational. Thus, by [103, regular-character extensions], the χ-summand of $\mathrm{Ind}_{U_0 \rtimes E}^{X \rtimes E} \widehat{\lambda}$ affords a mixed invariant extension $\widehat{\chi}$ to $X \rtimes E_\chi$.

Given M, we choose the marked Frobenius $F' = \mathrm{Int}(n_v)F_q$. The prime-field map Φ fixes n_c, and $\Phi^f = \mathrm{Int}(n_v^{-1})$ on $\mathbf{G}^{F'}$. Since $\mathrm{Z}(X) = 1$, we let $D = \langle \mathrm{Int}(n_c), \Phi \rangle$. Then, D is abelian, and $D \cap T = 1$. Indeed, for $n_c^i \Phi^j \in T$, we have $f \mid j$. Thus, the relation for Φ^f gives $n_c^i \Phi^j \in \langle n_c \rangle$, whose intersection with T is 1. If $h^{-1} F_q(h) = n_v$, $h\Phi(h)^{-1}$ is rational since $\Phi(n_v) = n_v$. Consequently, Φ induces the field generator modulo X, and TD maps onto $\mathrm{Out}(X)$ with $TD \cap X = M$. Since T has a unique Sylow p-subgroup, $TD \leq \mathrm{Aut}(X)_P$. Thus, $TD = \mathrm{Aut}(X)_P$ by Lemma 7.9.

For the weight $\theta \in \mathrm{Irr}(T)$ of s, we write D_θ for its stabilizer. On $T \rtimes D_\theta$ put $\widehat{\theta}(tx) = \theta(t)$, and let $\widehat{\psi} = \mathrm{Ind}_{TD_\theta}^{MD_\theta} \widehat{\theta}$. Since $MD_\theta = M(TD_\theta)$ and $D_\theta \cap \langle n_c \rangle = 1$, $\widehat{\psi}|_M = \mathrm{Ind}_T^M \theta = \psi$ by [95, Problem 5.6]. Moreover, MD_θ is the ordinary stabilizer of ψ in TD. If $\psi^{(d,\sigma)} = \psi$, for $d \in D$ and $\sigma \in \mathcal{H}_p$, Clifford theory [43] gives a Weyl correction n_c^i with $\theta^{(dn_c^i, \sigma)} = \theta$. Since D is abelian, $\widehat{\theta}^{(dn_c^i, \sigma)} = \widehat{\theta}$. Consequently, $\widehat{\psi}^{(d,\sigma)} = \widehat{\psi}$, as n_c^i acts by an inner automorphism of MD_θ. Finally, for $A = \mathrm{Aut}(X)_P$, the homomorphism $X \rtimes A \to \mathrm{Aut}(X)$ given by $(x, a) \mapsto \mathrm{Int}(x)a$ has kernel $\mathrm{C}_{X \rtimes A}(X)$. Compose $\widehat{\chi}$ and $\widehat{\psi}$ with the maps on $X \rtimes A_\chi$ and $M \rtimes A_\psi$, respectively. Both models are trivial on $\mathrm{C}_{X \rtimes A}(X)$. Moreover, the factors and twists are 1, so (χ, ψ) satisfies $\succeq_c$. □

Let $X = E_8(q) = \mathbf{G}^F$, with $q = r^f$, $p \neq r$, and $d = \mathrm{ord}_p(q) \in \{5, 8, 10, 12\}$. On $A_4 \perp A_4$ with the gluing in Zhang [208, proof of Lemma 12.7], put $w = (c_5, c_5)$ at $d = 5$ and $w = -(c_5, c_5)$ at $d = 10$. Moreover, $W(A_4)^2$ acts trivially on $(Q(A_4)^\vee / Q(A_4))^2$, and -1 fixes the gluing subgroup. At $d = 8, 12$, we set $w = (h, h)$ on $D_4 \perp D_4$, with $h \in W(F_4) = \mathrm{Aut}(\Phi(D_4))$. If $d = 8$, h is a negative four-cycle. If $d = 12$, h is a Coxeter element. Put $D = Q(D_4)^\vee / Q(D_4)$, the discriminant group of $Q(D_4)$. The overlattice $Q(E_8)$ corresponds to $\{(v, v) : v \in D\}$ in

D^2. Thus, (h,h) acts on $Q(E_8)$ and on $\Phi(E_8) = \{\alpha \in Q(E_8) : (\alpha,\alpha) = 2\}$. Since the diagram of E_8 has trivial automorphism group, $w \in W(E_8)$.

Let $\mathbf{T}$ have mark w and $T = \mathbf{T}^F$. On each four-space w has characteristic polynomial Φ_d. Thus, on $Q(E_8)$, its characteristic and minimal polynomials are Φ_d^2 and Φ_d. By Enguehard–Michel [63, proof of Lemma 2.20(1)], $T \simeq C^2_{\Phi_d(q)}$. By Springer [178, Theorem 4.2(iv) and Table 3], w is regular of order d. The degrees of $B = C_W(w)$ are $(20,30)$, $(8,24)$, and $(12,24)$ for $d \in \{5,10\}, 8, 12$, respectively. Thus, $|B|$ is $600, 192, 288$, respectively. Since $p \equiv 1 \bmod d$, these orders are prime to p. By Enguehard–Michel [63, Theorem 3.2(2)], $P = T_p$, with $P[p] = \ker(qw-1)$ on cocharacters modulo p.

By Malle [132, proof of Proposition 5.13], the connected centre of $\mathrm{C}^\circ_{\mathbf{G}}(P[p])$ contains a Sylow Φ_d-torus. Thus, $\mathrm{C}^\circ_{\mathbf{G}}(P[p]) = \mathbf{T}$ by full rank. By Enguehard–Michel [63, Proposition 2.19(2) and Lemma 2.20(1)], B acts faithfully on $P[p]$, giving $\mathrm{C}_X(P) = T$. Moreover, Enguehard–Michel [63, Theorem 3.3] gives $\mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{T})$, since P is characteristic in T. By self-duality in type E_8, the formulas $\mathrm{C}_{X^*}(P^*) = T^*$ and $\mathrm{N}_{X^*}(P^*) = \mathrm{N}_{X^*}(\mathbf{T}^*)$ hold for $P^* \in \mathrm{Syl}_p(X^*)$ in T^*.

Given $s \in T^*$, we write $R_s = \{\alpha \in \Phi(E_8) : \alpha(s) = 1\}$. Its rational span is w-stable, so $\dim \mathbb{Q}R_s \in \{0,4,8\}$. If $d = 5, 10$ and $\dim \mathbb{Q}R_s = 4$, the order-5 part of w acts irreducibly on $\mathbb{Q}R_s$. A root subsystem of rank 4 has at most 4 components, so this order-5 action fixes each component. Thus, R_s and $\Phi(E_8) \cap \mathbb{Q}R_s$ are irreducible and admit an order-5 automorphism. Since $|W(D_4) \rtimes \mathfrak{S}_3|$ is prime to 5, both have type A_4. Consequently, we obtain that $R_s = \Phi(E_8) \cap \mathbb{Q}R_s$, and $\mathrm{C}_{\mathbf{G}^*}(s)$ is a Levi.

If $d = 12$, then $\gcd(\Phi_{12}(q), 30) = 1$. Hence, the primes dividing $|s|$ are good for E_8. By Geck–Hiss's centralizer result (Malle [132, proof of Proposition 5.13]), $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ is a Levi. Since $\mathbf{G}^*$ is simply connected, $\mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(s) = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ by Steinberg's theorem (Geck–Malle [72, §2.2]). For $K = \mathbf{K}^F$ dual to $\mathbf{K}^*$, the series $\mathcal{E}(K,s)$ is the central twist $\kappa_s\mathcal{E}(K,1)$, with κ_s dual to s. Thus, Corollary 7.278 transfers the comparison on $\mathcal{E}(K,1)$ through signed induction to X and Clifford induction to $M = \mathrm{N}_X(P)$.

Proposition 7.11. *Let $G = E_8(q)$, $\mathrm{ord}_\ell(q) = 12$, $P \in \mathrm{Syl}_\ell(G)$, and $M = \mathrm{N}_G(P)$. Then, the mark w above gives* $\mathrm{iMN}(G,\ell)$ *on* $\mathrm{Irr}_{\ell'}(G)$ *and* $\mathrm{Irr}_{\ell'}(M)$. *Given $s \in T^*$ with proper nontoral centralizer $\mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(s)$, the group $\mathbf{K}^F_{\mathrm{der}}$ is* ${}^3D_4(q)$ *or* $\mathrm{SU}_3(q^2)$.

Proof. Given $s \in T^*$, the Levi $\mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(s)$ has semisimple rank $0, 4$ or 8. Rank 8 gives $s = 1$ since $\mathrm{Z}(\mathbf{G}^*) = 1$. Suppose that $\dim \mathbb{Q}R_s = 4$. Then, w has polynomial Φ_{12} on $\mathbb{Q}R_s$ and permutes the components of R_s transitively. If R_s is irreducible, then $R_s = A_4$ or D_4. In A_4, the signed action $\pm\pi$ for $\pi \in \mathfrak{S}_5$ has no eigenvalue of order 12. If $R_s = D_4$, a diagram image of order ≤ 2 implies that w is conjugate into $W(B_4)$. Thus, each eigenvalue x satisfies $x^j = \pm 1$ for some $j \le 4$, which excludes order 12. Consequently, the diagram image for D_4 has order 3. If $R_s = A_2^2$, w^2 on one A_2 has eigenvalues of order 6, so its image in $\mathrm{Aut}(\Phi(A_2))/W(A_2)$ is nontrivial. If $R_s = A_1^4$, the polynomial $x^4 \pm 1 \ne \Phi_{12}(x)$. Consequently, the Levi $\mathbf{K}$ has simply connected derived group $\mathbf{K}_{\mathrm{der}}$. Then, $H = \mathbf{K}^F_{\mathrm{der}}$ is ${}^3D_4(q)$ or $\mathrm{SU}_3(Q)$, with $Q = q^2$.

With $K = \mathbf{K}^F$, we put $Z = (\mathrm{Z}(\mathbf{K})^\circ)^F$. Indeed, for ${}^3D_4(q)$, triality has $\mathrm{Z}(\mathbf{K}_{\mathrm{der}})^F = 1$. For $\mathrm{SU}_3(Q)$, $|\mathrm{Z}(H)| = \gcd(3, Q+1) = 1$. Thus, $\mathrm{Z}(H) = 1$ in both cases. Let C_0 be the kernel of $\mathbf{K}_{\mathrm{der}} \times \mathrm{Z}(\mathbf{K})^\circ \to \mathbf{K}$. Since C_0 embeds in $\mathrm{Z}(\mathbf{K}_{\mathrm{der}})$, $C_0^F = 1$ and $a \mapsto a^{-1}F(a)$ is a bijection on C_0. Consequently, $K = H \times Z$, and $H \cong (\mathbf{K}_{\mathrm{der}}/\mathrm{Z}(\mathbf{K}_{\mathrm{der}}))^F$ since $\mathrm{Z}(\mathbf{K}_{\mathrm{der}})^F = 1$. Moreover, the Levi normalizer fixes H, Z as subgroups of K.

With $\mathbf{T}_H = \mathbf{T} \cap \mathbf{K}_{\mathrm{der}}$, we put $S = T \cap H$ and $P_H = P \cap H$ and $P_Z = P \cap Z$. Then, $T = S \times Z$ and $P = P_H \times P_Z$. If $H = {}^3D_4(q)$, the mark on $\mathbf{T}_H$ gives $|S| = \Phi_{12}(q)$. If $H = \mathrm{SU}_3(Q)$, $|S| = \Phi_6(Q) = \Phi_{12}(q)$ and $\mathrm{ord}_\ell(Q) = 6$. Thus, $\mathbf{T}_H$ is a Sylow d_H-torus, with $d_H = 12, 6$ respectively, and $\mathrm{C}_{\mathbf{K}_{\mathrm{der}}}(\mathbf{T}_H) = \mathbf{T}_H$. Since $P_Z \le Z$, the formulas on G give $\mathrm{C}_H(P_H) = S$ and $M_H := \mathrm{N}_H(P_H) = \mathrm{N}_H(\mathbf{T}_H)$. Thus, $L := \mathrm{N}_K(P) = M_H \times Z$. Indeed, $x \in \mathrm{N}_H(P_H)$ normalizes $P = P_H P_Z$, so it normalizes $\mathbf{T}$ and $\mathbf{T}_H$. Conversely, P_H is the unique Sylow ℓ-subgroup of S.

Since $\ell \ge 13$, [156, Theorem D] pairs $\mathcal{E}(H,1) \cap \mathrm{Irr}_{\ell'}(H)$ with $\mathrm{Irr}_{\ell'}(M_H \mid 1_S)$ for the cuspidal pair $(\mathbf{T}_H, 1_S)$. For $H = \mathrm{SU}_3(Q)$, take $\widehat{\Omega}$ from [156, §8.1], and for $H = {}^3D_4(q)$, from [156, §8.2]. By [156, proof of Theorem D], take the adjoint overgroup $\mathcal{U}_H \le \mathrm{Aut}(H)$. The Levi normalizer acts on H by elements of $\mathcal{U}_H$. Indeed, for exchanged A_2 components, a rational pair is $(x, F_0(x))$ for the component Frobenius F_0. Thus, projection to x expresses the exchange by F_0 on $\mathrm{SU}_3(Q)$.

Given a matched pair (τ,η), we put $A_\tau = (\mathcal{U}_H)_\tau$, $A_\eta = \mathrm{N}_{\mathcal{U}_H}(P_H)_\eta$. By equivariance, $A_\eta = \mathrm{N}_{\mathcal{U}_H}(P_H)_\tau$. Thus, $A_\tau = HA_\eta$ by Sylow conjugacy in H, and $H \cap A_\eta = M_H$, $A_\tau/H \cong A_\eta/M_H$. By [156, proof of Theorem D], the ordinary extensions U, V are on A_τ, A_η, respectively, and have equal twists on A_τ/H. Consequently, they extend τ, η

on the two sides. Let $D_K = \mathrm{N}_\Gamma(\mathbf{K})$ in the ambient field overgroup Γ. Take $B = (D_K)_{\kappa_s(\tau\boxtimes 1_Z)}$ and $B_P = \mathrm{N}_B(P)$. Moreover, conjugation on H gives $c : D_K \to \mathcal{U}_H$. Given (a,σ) in the joint stabilizer, $c(a)$ is defined for all $a \in D_K$. Since $b \in B$ fixes $\mathcal{E}(K,s)$, it fixes κ_s and τ. Thus, $c(B) \le A_\tau$. By the Frattini argument [97, Theorem 1.12], $B = KB_P$ since $P \in \mathrm{Syl}_\ell(K)$. By equivariance at P_H, $c(B_P) \le A_\eta$. Consequently, $U \circ c|_B$ and $V \circ c|_{B_P}$ restrict to $\tau \boxtimes 1_Z$ and $\eta \boxtimes 1_Z$ on $K = H \times Z$ and $L = M_H \times Z$, respectively. By central isotypy [72, Proposition 2.3.15], $\tau \boxtimes 1_Z$ runs through $\mathcal{E}(K,1) \cap \mathrm{Irr}_{\ell'}(K)$. Moreover, $\eta \boxtimes 1_Z$ runs through $\mathrm{Irr}_{\ell'}(L \mid 1_T)$, since $1_T = 1_S \boxtimes 1_Z$.

The common quotient map is $B/K \cong B_P/L \to A_\tau/H \cong A_\eta/M_H$, induced by c. Given $j \in B_P/L$, we choose $b_j \in B_P$ and evaluate both quotient functions at $c(b_j)$. Moreover, $c(\mathrm{C}_B(K)) = 1$ since $\mathrm{Z}(H) = 1$. Given (a,σ) in the joint stabilizer, correct a by K so that $a(P) = P$. Then, $c(aba^{-1}) = c(a)c(b)c(a)^{-1}$ for $b \in B$, and coefficient actions on U, V act after composing with c. Consequently, the two twists at b_j equal the RSST value at $c(b_j)$, including the value at $c(b_j)H = 1$. Conversely, if (a,σ) fixes κ_s, equivariance and injectivity give $\tau^{(c(a),\sigma)} = \tau$ if and only if $\eta^{(c(a),\sigma)} = \eta$. If $b^m = k \in K$, the power on H remains $c(b)^m = c(k)$.

Since $\mathcal{E}(K,s)^{(a,\sigma)} = \mathcal{E}(K,s)$, $\kappa_s^{(a,\sigma)} = \kappa_s$. Moreover, $\kappa_s|_T = \theta_s$, so the source pair is $\kappa_s(\tau \boxtimes 1_Z) \leftrightarrow \kappa_s|_L(\eta \boxtimes 1_Z)$. Given $i, j \in B/K$, Lemma 7.7 gives the additional factor $\kappa_s(b_i b_j b_{ij}^{-1})^{-1}$ on both sides. Consequently, on G, M, Corollaries 7.3 and 7.278 give $\succeq_c$ by signed Lusztig induction from K and ordinary induction from L. If $\mathrm{C}_{\mathbf{G}^*}(s) = \mathbf{T}^*$, the same corollary takes the source θ_s on T. If $s = 1$, by [156, Theorem D], we obtain the comparison at $(\mathbf{T}, 1_T)$ for G. Finally, for each s with $\mathcal{E}(G,s) \cap \mathrm{Irr}_{\ell'}(G) \neq \varnothing$, the Jordan degree formula [58, Proposition 11.5.6] gives $|\mathrm{C}_{G^*}(s)|_\ell = |G^*|_\ell$. Thus, $s \in T^*$ after conjugation. By [132, Proposition 5.11], $s^{G^*} \cap T^* = s^{\mathrm{N}_{G^*}(\mathbf{T}^*)}$. By [43], each $\psi \in \mathrm{Irr}_{\ell'}(M)$ lies over one orbit in $\mathrm{Irr}(T)$. Since $P \neq 1$ and G is simple, $M < G$. Consequently, the orbit bijections give $\succeq_c$ on $\mathrm{Irr}_{\ell'}(G), \mathrm{Irr}_{\ell'}(M)$ by Lemma 9.4. □

Let $d = \mathrm{ord}_p(q)$. If $X = F_4(q)$ with $q > 2$, mark $\mathbf{T}$ by a negative four-cycle in $W(B_4) \le W(F_4)$ at $d = 8$. At $d = 12$, a Coxeter element with exponents $1, 5, 7, 11$ marks $\mathbf{T}$ (Springer [178, §5.4]). If $X = G_2(q)$, the marks at $d = 3, 6$ are c^2, c, respectively, for a Coxeter element c (Springer [178, Theorem 4.2]). Since the mark w has polynomial Φ_d in each case, $|\mathbf{T}^F| = \Phi_d(q)$.

Since d divides one degree in $2, 6, 8, 12$ or $2, 6$, respectively, $P = (\mathbf{T}^F)_p$ is cyclic and Sylow by Malle–Testerman [133, Theorem 25.14]. Also $d \mid p - 1$ gives $p \nmid |W|$, and Malle [130, Proposition 2.2] gives $\mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{T})$. Since $-I \in C_W(w)$, its lift b gives $[u, b] = u^{-2}$ for $u \in P$. Thus, $[P, \mathrm{N}_X(P)] = P$. Proposition 7.285 gives $\mathrm{iMN}(X, p)$ for $F_4(q)$, including characteristic 2, and Corollary 7.302 gives it for $G_2(q)$.

Let $X = F_4(q)$, $q = r^f$ and $d = 3, 6$, with $\delta = 1, -1$, respectively. If q is even, we let γ be the exceptional isogeny with $\gamma^2 = F_2$. The pinned group E is $\langle F_r \rangle$ for odd q and $\langle \gamma \rangle$ for even q. Moreover, take Coxeter rotations c_1, c_2 in the long and short A_2 factors, respectively. Thus, $w = \delta(c_1, c_2)$ has polynomial Φ_d^2. By Enguehard–Michel [63, proof of Lemma 2.20(1)], $T \simeq C^2_{\Phi_3(\delta q)}$. Thus, $|T|$ is odd and $p \ge 7$, so $p \nmid |W(F_4)|$. The degrees $6, 12$ give the two p-factors of $|X|$, so $P = T_p$. Hence, $P[p] = \ker(qw - 1)$ on cocharacters modulo p. By Springer [178, Theorem 4.2(iv) and Table 4], w is regular. By Malle [132, proof of Proposition 5.13], $\mathrm{C}^\circ_{\mathbf{G}}(P[p]) = \mathbf{T}$ since the Sylow Φ_d-torus has full rank. By Enguehard–Michel [63, Lemma 2.20(1)], $C_W(w)$ acts faithfully on $P[p]$, giving $\mathrm{C}_X(P) = T$. Moreover, Enguehard–Michel [63, Theorem 3.3] gives $M = \mathrm{N}_X(P) = \mathrm{N}_X(\mathbf{T})$. By Malle [132, Proposition 5.11], $t^X \cap T = t^M$ for $t \in T$, with the dual formulas for T^*.

With $A = \mathrm{N}_{\mathrm{Aut}(X)}(\mathbf{T})$, we choose $b \in M$ above inversion $z = -I$. Since $b^2 \in T$ and b^2 is fixed by inversion, $b^2 = 1$. Moreover, conjugation by $t \in T$ acts by $b \mapsto t^{-2}b$ in right-action notation. Thus, Tb is one free T-orbit. The Weyl element $z = -I$ is fixed by A, including γ. Consequently, $A = T \rtimes \mathrm{C}_A(b)$ and $M = T \rtimes \mathrm{C}_M(b)$. Given $\theta \in \mathrm{Irr}(T)$, we choose $\widehat{\theta}$ on A_θ with $\widehat{\theta}|_T = \theta$ and $\widehat{\theta}|_{\mathrm{C}_A(b)_\theta} = 1$. If $a \in \mathrm{C}_A(b)$ and $\theta^{(a,\sigma)} = \theta$, we find $\widehat{\theta}^{(a,\sigma)} = \widehat{\theta}$. Thus, when $M_\theta = T$, the extension $\mathrm{Ind}_{A_\theta}^{MA_\theta} \widehat{\theta}$ of $\mathrm{Ind}_T^M \theta$ has twist 1 at each joint stabilizer (a,σ).

Since $|T^*|$ is odd, Zhang [208, proof of Proposition 9.13] gives order 3 and type A_2^2 for isolated $s \in T^* \setminus \{1\}$. Consequently, such an s occurs only when $\delta q \equiv 1 \bmod 3$. If $\delta q \equiv 1 \bmod 3$, the two central points s, s^{-1} of order 3 are rational. They satisfy $s^z = s^{-1}$. Moreover, both rational factors have sign δ. The Jordan labels (λ_1, λ_2) are partitions of 3. The degrees on each factor are $1, q(q+\delta), q^3$, prime to p since δq has order 3 modulo p. Since $P^* = T_p^*$ centralizes s, the Jordan multiplier is prime to p. By Malle [132, Proposition 7.7 and Theorem 7.8], the local labels over θ^M range over $\mathrm{Irr}(M_\theta/T)$. Connectedness of $\mathrm{C}_{\mathbf{G}^*}(s)$ gives $M_\theta/T = C_{W(A_2)^2}(w) = \langle c_1 \rangle \times \langle c_2 \rangle$. Thus, for $\eta_i \in \mathrm{Irr}(\langle c_i \rangle)$, the character $\widehat{\theta}|_{M_\theta}(\eta_1 \boxtimes \eta_2)$ has values in μ_3. The pinned field action fixes η_i. Its correction by z also acts trivially on each c_i. If q is even, choose the map for $\langle c_2 \rangle$ by γ from $\langle c_1 \rangle$. Then, $\gamma(\eta_1, \eta_2) = (\eta_2, \eta_1)$ in these coordinates since F_2 fixes η_i. On the global side, $\gamma(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1)$ by Zhang [208, Propositions 9.13–9.14]. Thus, transport of $\lambda_1 \leftrightarrow \eta_1$

by γ selects $\lambda_2 \leftrightarrow \eta_2$, giving the equivariant pairing $(\lambda_1, \lambda_2) \leftrightarrow (\eta_1, \eta_2)$.

Given $\nu \in \mathrm{Irr}(M_\theta)$, we put $\psi = \mathrm{Ind}_{M_\theta}^{M} \nu$. By Clifford [43], $A_\psi = M(A_\theta)_\nu$. Thus, for $x \in (A_\theta)_\nu$ mapping to a generator of A_ψ/M of order m, we extend ν by $\widehat{\nu}(x)^m = \nu(x^m)$. Since $\nu(x^m) \in \mu_3$, we choose $\widehat{\nu}(x) \in \mu_{3m_3}$, where $m_3 = 3^{v_3(m)}$. Put $f_3 = 3^{v_3(f)}$ and $h = \mathrm{ord}_p(r)$. Since $\mathrm{ord}_p(q) = h/\gcd(h, f)$ has 3-part 3, $h_3 = 3f_3$. Moreover, $m \mid 2f$ and $h \mid p-1$, so $3m_3 \mid 3f_3 \mid p-1$. Consequently, $\mathcal{H}_p$ fixes $\widehat{\nu}$ and its induction.

By Zhang [208, proof of Proposition 9.13, Case 2], $\mathcal{E}(X, s)$ has rational characters. Uniformity of χ follows from its A_2^2 label. For even q and $\lambda_1 \neq \lambda_2$, Zhang [208, Proposition 9.13(iii)] gives $E_\chi = \langle F_2 \rangle$. If $q \equiv -1 \bmod 3$ and $E_\chi \leq \langle F_r \rangle$, Zhang [208, proof of Proposition 9.13, Case 2] gives a rational $\widehat{\chi}$. For other odd $|E_\chi|$, the unique real extension $\widehat{\chi}$ has $\mathbb{Q}(\widehat{\chi}) = \mathbb{Q}(\chi)$ by Zhang [208, Lemma 3.1]. At even $|E_\chi|$ and $E_\chi \leq \langle F_r \rangle$, its involution is $j = F_r^{f/2}$. By Kawanaka [105, Theorem 3.1.3(ii)], $\langle \mathrm{Ind}_{X^j}^{X} 1, \chi \rangle_X = 1$. The rational permutation module on X/X^j gives a rational $\widehat{\chi}$. If q is even and $\lambda_1 = \lambda_2$, Zhang [208, Proposition 9.14(ii)] gives $\widehat{\chi}$ over $\mathbb{Q}(\zeta_3)$. Thus, $\widehat{\chi}$ is $\mathcal{H}_p$-fixed since $p \equiv 1 \bmod 3$.

For nonisolated $s \neq 1$ on the torus, the root system R_s has rank 0 or 2, by Φ_d^2. If R_s has rank 2, both R_s and $\Phi(F_4) \cap \mathbb{Q}R_s$ admit an order-3 rotation. Since $3 \nmid |\mathrm{Aut}(\Phi(B_2))| \cdot |\mathrm{Aut}(\Phi(A_1^2))|$, both systems have type A_2. Consequently, we find $R_s = \Phi(F_4) \cap \mathbb{Q}R_s$. If q is even, an odd power of γ exchanges Φ_{long} and Φ_{short}. A unit power satisfies $R_{s^a} = R_s$ for $\gcd(a, |s|) = 1$. Moreover, $W(F_4)$ fixes $\Phi_{\mathrm{long}}, \Phi_{\mathrm{short}}$ as sets. Thus, if $R_s = A_2$, every (a, σ) in the joint stabilizer of s has outer part in $\langle \gamma^2 \rangle$.

If $R_s = A_2$, M_θ/T is the rotation subgroup of order 3. Let U be the marked field lift with $U|_{M/T} = 1$ and $U^f = w^{-1}$ modulo T. A rational Weyl transporter h stabilizing $\mathbb{Q}R_s$ satisfies $hw = wh$, so its action on $\mathbb{Q}R_s$ is in the rotation group C_6. Extend $\eta \in \mathrm{Irr}(M_\theta/T)$ to $\xi \in \mathrm{Irr}(C_6)$ with $\xi(z) = 1$. On the stabilizer of $\mathbb{Q}R_s$ modulo T, we choose $hU^i \mapsto \xi(h|_{\mathbb{Q}R_s})\alpha^i$ with $\alpha^f = \xi(w^{-1}|_{\mathbb{Q}R_s})$. Thus, $\alpha \in \mu_{3f_3}$ can satisfy the relation for U^f. Since $3f_3 \mid p-1$, $\mathcal{H}_p$ fixes α. For (a, σ) in the joint stabilizer, corrected by h, the same formula is defined at hU^i. Consequently, the product with $\widehat{\theta}$ on A_θ gives an extension $\widehat{\psi}$ on MA_θ by Clifford induction. Thus, $\widehat{\psi}^{(a,\sigma)} = \widehat{\psi}$ at each joint stabilizer (a, σ).

For $R_s = \varnothing$ or A_2, the characters χ are uniform by Geck–Malle [72, Theorem 3.3.22] and the type A formulas. Moreover, they are real since $s^z = s^{-1}$ and opposition fixes each A_2 partition. If $|E_\chi|$ is odd, we choose $\widehat{\chi}$ by Zhang [208, Lemma 3.1], the unique real extension of χ. If $|E_\chi|$ is even, take its involution $j = F_r^{f/2}$ in the field group or $j = \gamma^f$ in the exceptional group. Thus, $j^2 = F_q$ on $\mathbf{G}$, so $\langle \mathrm{Ind}_{X^j}^{X} 1, \chi \rangle_X = 1$ by Kawanaka [105, Theorem 3.1.3(ii)]. Since E acts on X/X^j, the rational permutation module $\mathbb{Q}[X/X^j]$ gives $\widehat{\chi}$ with $\widehat{\chi}^{(a,\sigma)} = \widehat{\chi}$ for each (a, σ) in the joint stabilizer. For the unipotent comparison at 1_T when q is even, see Corollary 7.298.

7.2 Scalar tori in E_6 and ${}^2E_6$

Let $X = \mathbf{G}^F$ be simply connected of type E_6^ϵ, with $q = r^f$ and an odd prime $\ell \mid q-1$. Given an F-stable Borel pair $\mathbf{T} \leq \mathbf{B}$, we put $T = \mathbf{T}^F$, $B = \mathbf{B}^F$, $M = N_{\mathbf{G}}(\mathbf{T})^F$, and $W = M/T$. Suppose that $\ell \nmid |W|$. Then, $\ell \geq 7$ in split type and $\ell \geq 5$ in twisted type. By Schaeffer Fry [166, Theorem 6.2], $\mathrm{Irr}_{\ell'}(X)$ and $\mathrm{Irr}_{\ell'}(M)$ admit an equivariant bijection. Moreover, for (χ, ψ) from $\mathrm{Ind}_B^X \theta$ and $\mathrm{Ind}_T^M \theta$, we compare the models. The maps γ, ϕ satisfy $\phi\gamma = \gamma\phi$, and in twisted type $\phi^f = \gamma$ on X. The adjoint torus T_{ad} gives the diagonal action with $|T_{\mathrm{ad}}/\mathrm{im}(T)| = \gcd(3, q-\epsilon)$ by Zhang [208, (10.1)–(10.2)]. For the exceptional cover at ${}^2E_6(2)$, see Corollary 7.114.

Lemma 7.12. *Put $P = O_\ell(T)$. Then, we obtain that $P \in \mathrm{Syl}_\ell(X)$ and $\mathrm{N}_X(P) = M$. The rational flags fixed by P are the rational Borels containing $\mathbf{T}$.*

Proof. By rational Bruhat decomposition [58, Chapter 4], $[X : B] = \sum_{w \in W} q^{l_{\mathrm{abs}}(w)} \equiv |W| \bmod \ell$. The radical U_B has order $[B : T]$, an r-power, so $P \in \mathrm{Syl}_\ell(B)$. Thus, $P \in \mathrm{Syl}_\ell(X)$. Let δ be the diagram part of F and, for an orbit I of δ on simple nodes, put $y_I = \sum_{i \in I} \alpha_i^\vee$. The split cocharacter space $\sum_I \mathbb{Q} y_I$ contains a strictly dominant cocharacter y. Thus, for every root α, some $\langle \alpha, y_I \rangle \neq 0$. Since $|I| \leq 2$ and the system is simply laced, $0 < |\langle \alpha, y_I \rangle| \leq 4 < \ell$. For $\xi \in \mathbb{F}_q^\times$ of order ℓ, $y_I(\xi) \in P$ and $\alpha(y_I(\xi)) \neq 1$. Consequently, $\mathrm{C}_{\mathbf{G}}(P)^\circ = \mathbf{T}$. Thus, $\mathrm{N}_X(P) \leq M$, and $P = O_\ell(T)$ gives $M \leq \mathrm{N}_X(P)$. Finally, if an F-stable Borel $\mathbf{B}'$ contains P, Sylow conjugacy in $\mathbf{B}'^F$ gives a rational maximal torus $\mathbf{T}' \leq \mathbf{B}'$ with $P \leq \mathbf{T}'^F$. Since $\mathbf{T}' \leq \mathrm{C}_{\mathbf{G}}(P)^\circ = \mathbf{T}$, $\mathbf{T}' = \mathbf{T}$. Conversely, every rational Borel containing $\mathbf{T}$ contains P. □

Lemma 7.13. *Let $\mathbf{L}$ centralize a split subtorus of $\mathbf{T}$, and let $L = \mathbf{L}^F$. Suppose that $\mathrm{Ind}_A^X V$ is irreducible for an L-module V and every rational parabolic $A = LU_A$. Then, the maps $J_{C,A}$ of the proof satisfy $J_{D,C}J_{C,A} = J_{D,A}$ and*

are equivariant under automorphisms and $\mathcal{H}_\ell$.

Proof. If $A = LU_A$, take left covariance $h(lux) = V(l)h(x)$. On $\operatorname{Ind}_A^X V$ with right translation, put $m(C,A) = [U_C : U_C \cap U_A]$. Write $(J_{C,A}h)(x) = \frac{\sqrt{m(C,A)}}{|U_C|}\sum_{u\in U_C} h(ux)$. Since L normalizes U_C, $J_{C,A}h$ has the covariance for $C = LU_C$. Thus, $|U_A| = |U_C|$ by opposite root-group orders. Moreover, $J_{A,C}J_{C,A}$ is scalar by [95, Corollary 1.6]. On h supported in A, its value at 1 before normalization is the averaged value $h(1)/m(C,A)$. Indeed, for $u \in U_C$ and $v \in U_A$, $uv \in A$ exactly when $u \in U_C \cap U_A$. Consequently, $J_{A,C}J_{C,A} = 1$.

Let $\mathbf{S}_L$ be the split centre of $\mathbf{L}$. The parabolics are chambers in $Y(\mathbf{S}_L)_{\mathbb{R}}$. Thus, a shortest gallery $A = A_0, \dots, A_m = D$ crosses disjoint root sets with union $\Phi(U_D) \setminus \Phi(U_A)$. The ordered root factorization of $(U_D \cap U_A)\backslash U_D$ gives coordinates in the crossing order $\beta_1, \dots, \beta_t$. Indeed, at heights i, j, the commutator has height $h \geq i + j$. On rational points, group coordinates into F-orbits of roots. Thus, $\prod_{i=1}^m m(A_i, A_{i-1}) = m(D,A)$ and $J_{A_m,A_{m-1}} \cdots J_{A_1,A_0} = J_{D,A}$. On each rank-two face Z_2, the two arcs from A_0 to A_m give J_{A_m,A_0}. The dual zonotope Z has simply connected two-skeleton $Z^{(2)}$. Thus, the face relations and $J_{A,C}J_{C,A} = 1$ generate the gallery relations. Consequently, $J_{D,C}J_{C,A} = J_{D,A}$ for any A, C, D. Since $m(C,A)$ is a power of q, all coefficients lie in $\mathbb{Q}(\sqrt{q})$. Moreover, $\sqrt{q}$ is fixed by $\mathcal{H}_\ell$ by [156, Lemma 2.2], since $q \equiv 1 \bmod \ell$. Given an automorphism β, we have that $U_A^\beta = U_{A^\beta}$, so $J_{C,A}^\beta = J_{C^\beta,A^\beta}$. □

Proposition 7.14. *Let $\theta \in \operatorname{Irr}(T)$, and suppose that $W_\theta = 1$ or $W_\theta = \langle s_I \rangle$ for a relative reflection s_I. Then, the paired constituents (χ, ψ) of $\operatorname{Ind}_B^X \theta$ and $\operatorname{Ind}_T^M \theta$ satisfy $\succeq_c$ in* Definition 2.1.

Proof. If $W_\theta = 1$, the pair is $\operatorname{Ind}_B^X \theta, \operatorname{Ind}_T^M \theta$, irreducible by [58, Chapter 4] and [43]. Its degrees are $[X : B], |W|$ and its central characters are $\theta|_{\mathrm{Z}(X)}$. Moreover, each character is indexed by θ^W. Given an odd nondefining prime $\ell \mid |X|$, $\ell \nmid [X : B]$ holds precisely when $\ell \mid q - 1$ and $\ell \nmid |W|$. Indeed, $\ell \nmid |T|$ implies $\ell \mid [X : B]$. If $\ell \mid q + 1$, at a fixed simple node i the parabolic A_i has $[A_i : B] = q + 1$. Conversely, for $\ell \mid q - 1$, the Bruhat congruence is $[X : B] \equiv |W| \bmod \ell$.

Given $a = (\alpha, \sigma) \in \operatorname{Aut}(X)_P \times \mathcal{H}_\ell$ in the joint stabilizer, choose $n_a \in M$ such that $s_a = n_a a$ fixes θ. On the global functions the operator D_{s_a} is semilinear transport followed by $J_{B,s_a(B)}$. Moreover, the local base is T. By Lemma 7.13, $D_s D_t = D_{st}$ on the stabilizer of θ. Given $t \in T$, left covariance gives $D_t = \theta(t)^{-1}\rho(t)$ with ρ the right action. Thus, over a, we choose $\rho(n_a)^{-1}D_{s_a}$. If $b = (\beta, \tau)$, the section factor is $k_{a,b} = n_a \alpha(n_b) n_{ab}^{-1} \in T$. Consequently, both models have factor $\theta(k_{a,b})^{-1}$ at (a, b). If (g, α) centralizes X, $g \in M$ and $\alpha = \operatorname{Int}(g^{-1})$. Thus, both models have scalar $\theta(n_\alpha g^{-1})^{-1}$.

If $W_\theta = \langle s_I \rangle$, conjugate s_I to a simple reflection. Then, $I = \{i\}$ or $\{i, j\}$ with $\alpha_i \perp \alpha_j$. With $\Lambda = X^*(\mathbf{T})$, we let R be the root lattice, with $[\Lambda : R] = 3$. If $v = \sum_{i\in I} a_i\alpha_i \in \Lambda$ for $a_i \in \mathbb{Q}$, then $2a_i \in \mathbb{Z}$ by coroot pairing. Since $3v \in R$, we also have that $3a_i \in \mathbb{Z}$. Hence, $a_i \in \mathbb{Z}$. Thus, $\sum_{i\in I} \mathbb{Z}\alpha_i$ is saturated in Λ. Write $\operatorname{Irr}(T) = \Lambda/(q\delta - 1)\Lambda$, and take $x \in \Lambda$ representing θ. Moreover, put $c_i = \langle x, \alpha_i^\vee \rangle$. Invariance under s_I gives $x - s_I x = \sum_{i\in I} c_i\alpha_i = (q\delta - 1)y$. Since $q\delta - 1$ is invertible on the δ-stable span and on $\mathbb{Q}\Lambda$, y belongs to that span. Thus, $y = \sum_{i\in I} a_i\alpha_i$ with $a_i \in \mathbb{Z}$. On the derived torus $\mathbf{T}_I$, the restriction has coordinates c_i in the fundamental-weight basis. Consequently, $(c_i)_{i\in I} = (q\delta - 1)(a_i)_{i\in I}$ gives $\theta|_{\mathbf{T}_I^F} = 1$. Conversely, $\theta|_{\mathbf{T}_I^F} = 1$ gives $x - s_I x \in (q\delta - 1)\Lambda$.

Let $\mathbf{L}$ centralize the split subtorus $\mathbf{S}_I$ for the roots in I. Then, $K = M_\theta = N_{\mathbf{L}}(\mathbf{T})^F$ and $L = \mathbf{L}^F$. By [133, Proposition 23.2] on $\mathbf{T}_I$ and $\mathbf{L}_{\mathrm{der}}$, T surjects onto $L/(\mathbf{L}_{\mathrm{der}})^F$. Thus, θ has a unique linear extension λ to L trivial on $(\mathbf{L}_{\mathrm{der}})^F$. Consequently, uniqueness makes λ equivariant under the stabilizer of θ. Take the sign η of K/T and $V_0 = 1$. On the flags $L/(B \cap L)$, we let V_1 be the augmentation module. Since L acts doubly transitively, V_1 is irreducible of degree $q^{|I|}$. If $i \in \{0, 1\}$, pair $\chi_i = \operatorname{Ind}_A^X(\lambda V_i)$ and $\psi_i = \operatorname{Ind}_K^M(\lambda|_K \eta^i)$, with A a rational parabolic of Levi L. Thus, by transitivity, the sum of $\operatorname{Ind}_A^X(\lambda V_i)$ for $i = 0, 1$ is $\operatorname{Ind}_B^X \theta$. By Bruhat–Mackey, $\dim \operatorname{End}_X(\operatorname{Ind}_B^X \theta) = 2$. Thus, the two global characters χ_i are distinct irreducibles of degree $[X : A]q^{i|I|}$, prime to ℓ. By [43], the local pair exhausts $\operatorname{Irr}(M \mid \theta^W)$, each of degree $[M : K]$.

For s fixing θ, let $o(s)$ be its sign on $\{\mathbf{B}_1, \mathbf{B}_2\}$, the two F-stable Borels of $\mathbf{L}$ through $\mathbf{T}$. Then, $o|_K = \eta$. On the global source V_i, act by the permutation operator times $o(s)^i$, and return the parabolic by J. Moreover, the local source is $\lambda|_K \eta^i$. Given $k \in K$, the matrix $V_i(k)$ cancels by left covariance. Thus, $D_k = (\lambda(k)\eta(k)^i)^{-1}\rho(k)$ on both models. Consequently, at $k_{a,b}$ and $n_\alpha g^{-1}$ both scalars are given by $(\lambda|_K \eta^i)^{-1}$.

If $q - 1 > 12$, we put $\varrho = \sum_j \omega_j$ for the absolute fundamental weights ω_j. Take $x = \varrho$ or $x = \varrho - \omega_i$ for a fixed diagram node i. Their stabilizers in W are 1 and $\langle s_i \rangle$, respectively. If $wx - x \in (q\delta - 1)\Lambda$, the difference $wx - x$ is δ-fixed, so it belongs to $(q - 1)\Lambda^\delta$. $|\langle wx - x, \alpha_j^\vee \rangle| \leq 12$ since $|\langle x, \alpha^\vee \rangle| \leq 11$ for any root α and $|\langle x, \alpha_j^\vee \rangle| \leq 1$. Thus, $wx = x$. Moreover, $2\varrho \in R$ and $|\Lambda/R| = 3$, so $\varrho \in R$. Since γ acts by inversion on Λ/R and fixes ω_i, $\omega_i \in R$.

Consequently, both weights are trivial on $\mathrm{Z}(X)$ and give characters of $X/\mathrm{Z}(X)$. Since P is characteristic in $P\mathrm{Z}(X)$, the quotient normalizer is $M/\mathrm{Z}(X)$. □

Theorem 7.15. *In* Lemma 7.12*, suppose that* $\theta \in \mathrm{Irr}(T)$ *has parabolic stabilizer* $W_\theta \leq W$*. Then, the pairs* (χ, ψ) *from the principal series of* θ *and from* $\mathrm{Irr}(M \mid \theta^W)$ *satisfy* Definition 2.1*. Moreover, the bijection over* θ^W *is equivariant under* $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$ *and matches the central character* $\theta|_{\mathrm{Z}(X)}$*.*

Proof. Conjugate W_θ to W_J, with fixed subtorus $\mathbf{S}_J$ in the split torus. Put $\mathbf{L} = \mathrm{C}_{\mathbf{G}}(\mathbf{S}_J)$. Thus, $\mathbf{L}$ is equivariant under transport of θ. Given any diagram orbit $I \subseteq J$, by Proposition 7.14, $\theta|_{\mathbf{T}_I^F} = 1$ on the derived torus $\mathbf{T}_I$ for I. Since the fundamental weights of $\mathbf{L}_{\mathrm{der}}$ break up over the δ-orbits I, $\theta|_{(\mathbf{T}\cap\mathbf{L}_{\mathrm{der}})^F} = 1$. Thus, we obtain a unique linear $\lambda \in \mathrm{Irr}(L)$ extending θ with $(\mathbf{L}_{\mathrm{der}})^F \leq \ker\lambda$. Put $K = M_\theta$, so $K/T = W_J$.

With $R = \mathbb{Z}_\ell$, we put $V_R = R[L/(B \cap L)]$. Put $H = \mathrm{End}_{RL}(V_R)$ with orbital basis T_w, $w \in W_J$. Moreover, $(T_{s_I} - q^{|I|})(T_{s_I} + 1) = 0$. Since $q \equiv 1 \bmod \ell$, $H/\ell H \cong \mathbb{F}_\ell W_J$. For the left action, orient T_w by the inverse relative position. Let D be the diagram group fixing $q^{|I|}$ at each orbit I. Then, $D \leq \mathfrak{S}_6$ in split type and $D \leq \mathfrak{S}_4$ in twisted type, so $\ell \nmid |D|$. In $H^\times \rtimes D$, we let $\mathcal{P}$ be the preimage of $W_J \rtimes D$ under reduction. Since $1 + \ell H$ is pro-ℓ and $\ell \nmid |W_J \rtimes D|$, we obtain a complement in $\mathcal{P}$ by [152, Theorem 2.3.15]. Over D, we conjugate by $1 + \ell H$ so that the complement contains D. The lifted matrices $\widetilde{w} \in H^\times$ reduce to w. By [11, Proposition 2.8], $w \mapsto \widetilde{w}$ gives a surjection $RW_J \to H$. Both algebras are free of rank $|W_J|$, so we obtain a D-equivariant isomorphism. On a conjugate $\mathbf{L}'$, take the transported isomorphism $RW_J \to H$. Thus, the flag action for d is $T_w \mapsto T_{d(w)}$.

By [185], every irreducible E of W_J is rational. With $m = \dim E$, we put $U_E = \mathrm{Hom}_{H_{\mathbb{Q}_\ell}}(E_{\mathbb{Q}_\ell}, V_{\mathbb{Q}_\ell})$. Thus, $V_{\mathbb{Q}_\ell} \cong \bigoplus_E E_{\mathbb{Q}_\ell} \otimes U_E$ by double centralizers, with U_E the distinct principal unipotent constituents of L. On V_R, $\mathrm{tr}\, T_w = 0$ for $w \neq 1$. Moreover, $\mathrm{tr}\, T_1 = [L : B \cap L] \equiv |W_J| \bmod \ell$. For the central idempotent e_E, the coefficient at T_1 is $m^2/|W_J| \bmod \ell$. Thus, its trace is $m \dim U_E \equiv m^2 \bmod \ell$. Since $m \mid |W_J|$, $\dim U_E \equiv m \bmod \ell$. Pair $\chi_E = \mathrm{Ind}_A^X(\lambda U_E)$ with $\psi_E = \mathrm{Ind}_K^M(\lambda|_K E^\vee)$, where $E^\vee$ is inflated from K/T and A has Levi L. By transitivity, $\mathrm{Ind}_B^X \theta = \sum_E m\chi_E$. Moreover, $\dim \mathrm{End}_X(\mathrm{Ind}_B^X \theta) = |W_\theta| = \sum_E m^2$ by [58, Chapter 4]. Thus, for every A, the χ_E are distinct irreducibles. Since $[X : A] \mid [X : B]$, the global degrees are prime to ℓ. By [43], the local degrees $[M : K]m$ exhaust the characters of M over θ^W. Both central characters are $\theta|_{\mathrm{Z}(X)}$. Moreover, θ^W is unique for χ_E, ψ_E by [58, Chapter 4] and [43].

Choose a chamber C in the L-apartment. For b fixing L, T, its action on the apartment is $(w_b, d_b) \in W_J \rtimes D$. Indeed, w_b takes C to $b(C)$ and d_b permutes the types. Thus, $b \mapsto (w_b, d_b)$ is a homomorphism and $k \in K$ maps to $(kT, 1)$. Since E is rational, the label E is changed only by d_b. On its diagram stabilizer D_E, we choose intertwiners T_d with $\det T_d = 1$ and $T_1 = 1$. From [95, Corollary 1.6] and $\det T_d = 1$, $T_d T_e = c(d, e)T_{de}$ with $c(d, e) \in \mu_m$. Thus, $\langle E(W_J), T_d : d \in D_E\rangle$ is finite, with scalar factors in μ_m. Thus, $\mathrm{tr}(E(w)T_d)$ is cyclotomic for each $w \in W_J$. Since the matrices $E(w)$ span the rational endomorphism algebra of E, the entries of T_d in the rational basis of E are cyclotomic.

On U_E, we have that the action of b is $f \mapsto b_{\mathrm{flags}} f T_{d_b}^{-1}$. Moreover, it is $E^\vee(w_b)T_{d_b}^{-t}$ on $E^\vee$. Both models have factor $c(d_b, d_e)^{-1}$. Given $\sigma \in \mathcal{H}_\ell$, by [156, §2], we obtain the lift to $\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ab}}/\mathbb{Q}_\ell)$. Thus, the lift fixes $H_{\mathbb{Q}_\ell} \cong \mathbb{Q}_\ell W_J$. Since E is rational, $\sigma(T_d) = u_{\sigma,d} T_d$, with scalar $u_{\sigma,d}$. Both models have scalar $u_{\sigma,d}^{-1}$. Consequently, for joint actions b, e, the inverse of $T_{d_b}\sigma_b(T_{d_e})T_{d_{be}}^{-1}$ is the common factor. Tensor with λ. Thus, for $k \in K$, the matrices $U_E(k)$ and $E^\vee(kT)$ cancel in left covariance, leaving $\lambda(k)^{-1}$ on X and M.

Given $a = (\alpha, \sigma)$ in the joint stabilizer, take the correction $n_a \in M$ of Proposition 7.14. Since the χ_E are distinct, the corrected s_a fixes θ, E. On X, the global operator includes $J_{A,s_a(A)}$ from Lemma 7.13. On M, the restored θ fixes the source group K. Thus, $k_{a,b} = n_a \alpha(n_b) n_{ab}^{-1} \in K$ contributes $\lambda(k_{a,b})^{-1}$ on both sides. For (g, α) centralizing X, the correction $n_\alpha g^{-1} \in K$ has diagram component 1. Consequently, the central scalars agree at (g, α) and the factors agree at (a, b). The ordinary factors lie in $\mu_m \lambda(L)$. Thus, the covering characters are $\mathbb{Q}^{\mathrm{ab}}$-valued. By [141, Lemmas 1.2–1.4], the models over $\mathbb{Q}_\ell^{\mathrm{ab}}$ give the coefficient twists μ_σ on those covers. Thus, μ_σ depends only on σ. □

Assume the hypotheses on X, P, T, M in Lemma 7.12. Schaeffer Fry's bijection $\mathrm{Irr}_{\ell'}(X) \longrightarrow \mathrm{Irr}_{\ell'}(M)$ is equivariant (Schaeffer Fry [166, Theorem A]). Moreover, our pairing over A, H compares multiplicities over X, M. Let J be a finite ℓ'-group of operators on X that fix $(\mathbf{B}, \mathbf{T})$. We allow any kernel of $J \to \mathrm{Aut}(X)$. Moreover, the automorphisms from J are pinned field operators on X, pinned diagram operators on X or diagonal operators via $\mathbf{T}$ as in Zhang [208, (10.1)–(10.2)]. With $A = X \rtimes J$, we put $H = M \rtimes J$. We fix a complete DVR $\mathcal{O}$ with fraction field K of characteristic 0 and finite residue field $k = \mathcal{O}/\mathfrak{m}$ of characteristic ℓ. Moreover, K, k split A, H and the algebras below.

Given $\theta \in \mathrm{Irr}(T)$, we write $\mathcal{O}_\theta$ for the rank one lattice for T, also viewed as a lattice for B via $B \twoheadrightarrow T$. Put $V_A(\theta) = \mathrm{Ind}_B^A \mathcal{O}_\theta$, $V_H(\theta) = \mathrm{Ind}_T^H \mathcal{O}_\theta$. Given an indexed family $(\theta_i)_{i\in I}$, we put $V_Y = \bigoplus_{i\in I} V_Y(\theta_i)$ and $R_Y = \mathrm{End}_{\mathcal{O}Y}(V_Y)$ for $Y \in \{A, H\}$. Keep the index i even if two characters θ_i, θ_j have equal reductions in k. Thus, our reductions are $\overline{V}_Y = k \otimes_{\mathcal{O}} V_Y$ and $\overline{R}_Y = k \otimes_{\mathcal{O}} R_Y$.

Lemma 7.16. *Restriction to $(A/B)^P = H/T$ followed by reduction modulo $\mathfrak{m}$ gives an algebra isomorphism $\overline{R}_A \cong \overline{R}_H$. The isomorphism maps each projection e_i onto $V_A(\theta_i)$ to the projection onto $V_H(\theta_i)$. Hence, it maps each corner $e_j\overline{R}_A e_i$ onto $e_j\overline{R}_H e_i$.*

Proof. By Lemma 7.12, the set $(A/B)^P = H/T$ consists of the apartments in X/B indexed by J. The stabilizer $P \cap {}^aB$ of aB acts on the line aB by ${}^a\theta_i|_{P\cap {}^aB}$. Since $\ell \nmid |k^\times|$, the action of $P \cap {}^aB$ on the reduced line $k(aB)$ is trivial. Thus, $\overline{V}_A$ has a P-stable basis whose P-fixed basis vectors have indices $hT \in H/T$. The Brauer functor Br_P on kA-modules the result of [19, §2.B] gives $\mathrm{Br}_P(\overline{V}_A) = \overline{V}_H$ as kH-modules. In particular, $\mathrm{Br}_P : \overline{R}_A \to \mathrm{End}_{kH}(\overline{V}_H)$ is multiplicative, and it maps e_i to the projection onto $\overline{V}_H(\theta_i)$.

We check that the image of $\overline{R}_A$ is precisely $\overline{R}_H$ on the bases of R_A, R_H. By the Bruhat decomposition of X [58, Chapters 3–4], the A-orbits on $(A/B)^2$ have representatives (nB, B) for $nT \in H/T = W \rtimes J$. By Mackey decomposition [95, Problem 5.6], for $i, j \in I$ the orbit of (nB, B) contributes a free line to $e_j R_A e_i$ if $\theta_j^n = \theta_i$ on T, where $\theta_j^n(t) = \theta_j(n^{-1}tn)$. Indeed, $B \cap {}^nB$ contains T, and both sources are trivial on $U \cap {}^nU$ for $U = O_p(B)$, with p the characteristic of $\mathbb{F}_q$. The intersection of the orbit with $H/T \times H/T$ is the H-orbit of (nT, T). Its stabilizer is T, so $e_j R_H e_i$ has a free line at (nT, T) when $\theta_j^n = \theta_i$ on T. For each such nT, we normalize the matrix entry at (nB, B) or (nT, T) to 1. The normalized basis of $e_j R_A e_i$ then restricts to the basis of $e_j R_H e_i$ with entry 1. Thus, the restricted matrices give a k-linear bijection $\overline{R}_A \longrightarrow \overline{R}_H$, with the products given by Br_P. □

Proposition 7.17. *If $J = 1$, the construction gives a bijection $\mathrm{Irr}_{\ell'}(X) \longrightarrow \mathrm{Irr}(M)$. For general J, it pairs the constituents of $K \otimes_{\mathcal{O}} V_A$ and $K \otimes_{\mathcal{O}} V_H$ with equal multiplicities under restriction to X, M, respectively. Moreover, the pairing for A, H is equivariant under $\mathrm{Irr}(J/[J, J])$, under $\mathcal{H}_\ell$, and under automorphisms of A, H induced by $(\mathbf{B}, \mathbf{T})$.*

Proof. We index $\mathrm{Irr}(T)$ without repetition by I. With fixed displacements $w_1, w_2 \in H/T$, the two orbit matrices on H/T in the product give $(hT)w_2$ between the two steps from hT. Hence, the normalized matrices of R_H give the groupoid algebra of $W \rtimes J$ on $\mathrm{Irr}(T)$ with a cocycle α from lifts in H. For each orbit $\mathcal{C}$ and $\theta \in \mathcal{C}$, the summand of $\overline{R}_H$ on $\mathcal{C}$ is a matrix algebra over $k^\alpha[(W \rtimes J)_\theta]$. Indeed, one transporter from θ to each weight in $\mathcal{C}$ gives the matrix indices in $\mathcal{C}$. With repeated θ_i, the matrix size is $|\{i \in I : \theta_i \in \mathcal{C}\}|$. The algebra $k^\alpha[(W \rtimes J)_\theta]$ is a central summand of kE, where E extends $(W \rtimes J)_\theta$ by $\langle \mathrm{im}(\alpha)\rangle \le k^\times$. Since $\ell \nmid |W \rtimes J|\,|k^\times|$, we find kE semisimple by [95, Theorem 1.9], and our chosen k splits E. Consequently, $\overline{R}_H$ is split semisimple and so is $\overline{R}_A$ by Lemma 7.16.

Since $\mathcal{O}$ is complete, the idempotents $\bar{e}_a \in \overline{R}_Y$ lift to a complete orthogonal system $(e_a)_a$ in R_Y. By [11, Proposition 2.8], on the free summand $e_a R_Y e_b$, we obtain rank 0 across distinct blocks and rank 1 within one block. For a fixed e_b in the block $R_Y e$, we lift matrix units to $u_a \in e_a R_Y e_b$ and $v_a \in e_b R_Y e_a$. Since $v_a u_a$ is a unit in $\mathcal{O}e_b$, we normalize it to e_b, whence $u_a v_a = e_a$. Thus, $u_a v_c$ give matrix units and R_Y is a product of algebras $\mathrm{Mat}_d(\mathcal{O})$. The central idempotent e of a block of R_Y is uniquely determined by $\bar{e}$. Therefore, the common $\bar{e}$ pairs the simple constituents in $K \otimes_{\mathcal{O}} V_A$ and $K \otimes_{\mathcal{O}} V_H$.

Let $f \in R_A$ be a primitive idempotent for χ, and let $f' \in R_H$ be an idempotent above $\mathrm{Br}_P(\bar{f})$ for the paired character ψ. Then, $\mathrm{tr}(f) = \chi(1)$ and $\mathrm{tr}(f') = \psi(1)$. On the P-stable basis of $\overline{V}_A$, the diagonal entries of $\bar{f}$ are constant on each P-orbit $\mathcal{Q}$. Since $|\mathcal{Q}| \equiv 0 \pmod{\ell}$ if $|\mathcal{Q}| > 1$, $\mathrm{tr}(\bar{f}) = \mathrm{tr}(\mathrm{Br}_P(\bar{f}))$ on the fixed lines $(A/B)^P$. Hence, $\chi(1) \equiv \psi(1) \pmod{\ell}$. By [95, Lemma 5.2], each $\psi \in \mathrm{Irr}(H)$ occurs in $K \otimes_{\mathcal{O}} V_H$. Since $T \lhd H$ is abelian, $\psi(1) \mid [H : T]$ by [95, Theorem 6.15]. Since $\ell \nmid [H : T]$, all χ have ℓ'-degree. If $J = 1$, $|\mathrm{Irr}_{\ell'}(X)| = |\mathrm{Irr}(M)|$ follows from [166, Theorem 6.2]. Thus, every $\chi \in \mathrm{Irr}_{\ell'}(X)$ occurs over V_A.

By [95, Problem 5.6] on X and M, the restrictions $V_A{\downarrow_X}$ and $V_H{\downarrow_M}$ are direct sums indexed by $(j, i) \in J \times I$ with weights θ_i^j. Hence, the inclusions of R_A, R_H into the commutants over X, M are compatible with the map on $(A/B)^P$. Given a primitive idempotent $f \in R_A$ for χ, let e be the central idempotent for $\xi \in \mathrm{Irr}(X)$. Then, the multiplicity $[\chi|_X, \xi]$ is the rank of ef on the multiplicity space L_e. Since $L_e = \mathrm{im}(ef) \oplus \ker(ef)$ over $\mathcal{O}$, the rank of ef on L_e is the rank of $\overline{ef}$ on $k \otimes_{\mathcal{O}} L_e$. By Lemma 7.16, e, f reduce to the paired idempotents over M, H, respectively. Therefore, the multiplicities agree over X, M, and an extension to A is paired with an extension to H.

Finally, the coefficient action of $\sigma \in \mathcal{H}_\ell$ on the orbital from (θ_i, θ_j) is the orbital from $(\theta_i^\sigma, \theta_j^\sigma)$ on lifts n as in Theorem 7.15. An automorphism of $(A, H, \mathbf{B}, \mathbf{T})$ maps A/B and $(A/B)^P$ to themselves. Given $\nu \in \mathrm{Irr}(J)$ of degree 1,

the coset ajB with $a \in X$ has scaling factor $\nu(j)^{-1}$ under $V_A \otimes \nu \cong V_A$. Hence, the orbital at (nB, B) for $nT = wj$ and the orbital at (nT, T) both have twist factor $\nu(j)^{-1}$. Thus, the map $\bar{e} \mapsto \bar{e}'$ is equivariant for σ, the automorphisms of $(A, H, \mathbf{B}, \mathbf{T})$, and the twists by ν. □

Suppose that the diagonal quotient is 1, that $\ell \nmid f$, and that the pinned outer group C is cyclic. In type ${}^2E_6$, or in type E_6 with f odd, the pinned group C is cyclic. Given a pair (χ, ψ) under the bijection, we put $J = C_\chi = C_\psi$. Since J is cyclic, we have that χ extends to $\widetilde{\chi} \in \mathrm{Irr}(X \rtimes J)$ (Isaacs [95, Corollary 11.22]). The character $\widetilde{\chi}$ occurs in $K \otimes_{\mathcal{O}} V_A$ by Isaacs [95, Lemma 5.2], since $\chi{\downarrow_B}$ contains some $\theta \in \mathrm{Irr}(T)$ inflated to B. Thus, Proposition 7.17 pairs $\widetilde{\chi}$ with an extension $\widetilde{\psi}$ of ψ. If a pair (a, σ) fixes (χ, ψ) and $\widetilde{\chi}^{a\sigma} = \nu\widetilde{\chi}$ for $\nu \in \mathrm{Irr}(J)$ of degree 1, then $\widetilde{\psi}^{a\sigma} = \nu\widetilde{\psi}$ by equivariance. Since $Z(X) = 1$, we have that the natural map $X \rtimes \mathrm{Aut}(X)_P \to \mathrm{Aut}(X)$ has kernel $C_{X \rtimes \mathrm{Aut}(X)_P}(X)$. In particular, we have that $C_{X \rtimes \mathrm{Aut}(X)_P}(X)$ acts trivially on the inflated characters $\widetilde{\chi}, \widetilde{\psi}$.

Suppose that the pinned inertia group J is noncyclic. Write $J = \langle a_1 \rangle \times \langle a_2 \rangle$ with $|a_i| = m_i$. We choose matrices D_i for a_i on the representation affording χ such that $D_i^{m_i} = 1$. Then, we obtain that $D_1 D_2 D_1^{-1} D_2^{-1} = \zeta I$, where $\zeta^{m_1} = \zeta^{m_2} = 1$. Thus, D_1, D_2 and ζI give an action extending χ over a central extension $J^\sharp$ of J by $\langle \zeta \rangle$, with the power relations $D_i^{m_i} = 1$. The pinned group C acts trivially on J and on the chosen $J^\sharp$. We apply the flag pairing to $J^\sharp \to J$, with separate indices in A/B and H/T for all elements of $\ker(J^\sharp \to J)$. If $z \in \ker(J^\sharp \to J)$, then the matrix for z on $(A/B)^P$ is its matrix on H/T. Hence, the actions of z on the two constituents paired by $\bar{e}$ have congruent scalar values in k. Since z has ℓ'-order, its scalar values over K are equal. Therefore, a section $J \to J^\sharp$ gives equal projective factors on χ, ψ. Finally, for an action (a, σ) fixing (χ, ψ), the two quotient twists ν agree by Proposition 7.17.

Theorem 7.18. *Let $S = E_6^\epsilon(q)_{\mathrm{simple}}$, $q = r^f$, and let $\ell \mid q - 1$ be a prime with $\ell \geq 7$ in split type and $\ell \geq 5$ in twisted type. The bijection Ω of* Proposition 7.17 *gives* $\mathrm{iMN}(S, \ell)$ *on the universal cover X with $M = N_X(P)$ for every $f \geq 1$ and every central character of X.*

Proof. Let $\overline{\mathbf{T}}$ be the image of $\mathbf{T}$ in the adjoint group. Write $\overline{T}_0$ for the image of T in $\overline{\mathbf{T}}^F$ and put $\Delta = O_3(\overline{\mathbf{T}}^F)$ if $3 \mid q - \epsilon$ and $\Delta = 1$ otherwise. By Lang's theorem [58, Proposition 4.2.10], the map $\overline{\mathbf{T}}^F \to H^1(F, Z(\mathbf{G}))$ is onto with kernel $\overline{T}_0$. Since the quotient has order $(3, q - \epsilon)$, the image of Δ is $H^1(F, Z(\mathbf{G}))$. If $\delta \in \Delta \cap \overline{T}_0$, the 3-part of a preimage of δ in T also maps to δ. Since $\Delta \leq \overline{\mathbf{T}}^F$, it fixes $\mathbf{T}$ pointwise and normalizes $(\mathbf{B}, \mathbf{T})$.

Let $C = \langle F_r|_X, \tau|_X \rangle$ with the pinned field operator F_r and diagram involution τ. In type ${}^2E_6$, $F_r^f|_X = \tau|_X$ [208, (10.1)–(10.2)]. In particular, C is abelian, and τ normalizes Δ. The group $\Gamma = \Delta \rtimes C$ surjects onto $\mathrm{Out}(X)$ by [208, (10.1)–(10.2)]. Since Γ fixes P, Lemma 7.12 gives that $\mathrm{Aut}(X)_P = \mathrm{Inn}(M)\Gamma$. The field operators in C act on $\overline{\mathbf{T}}^F$ by power maps. Hence, the Sylow ℓ-subgroup C_ℓ of C acts on Δ through $(\mathbb{Z}/\exp(\Delta)\mathbb{Z})^\times$, whose order is a $\{2, 3\}$-number. Thus, $\Gamma = \Gamma_0 \times C_\ell$, where $\Gamma_0 = \Delta \rtimes O_{\ell'}(C)$ and C_ℓ is cyclic.

With $\psi = \Omega(\chi)$, we put $J = \Gamma_\chi = \Gamma_\psi$. Then, $J = J_0 \times L$ with $J_0 = J \cap \Gamma_0$ and $L = J \cap C_\ell$, since $(|\Gamma_0|, |C_\ell|) = 1$. For $(\gamma, \sigma) \in (\Gamma \times \mathcal{H}_\ell)_\chi$, $\Gamma_{\chi^\sigma} = \Gamma_\chi$, so γ normalizes J and J_0. Put $h = |J_0|$. We let F_0 be free on $\{x_j : j \in J_0\}$, with $R_0 = \ker(F_0 \to J_0)$ for $x_j \mapsto j$. The group $J_0^\sharp = F_0/([F_0, R_0]R_0^h)$ has central kernel K_0 over J_0, and K_0 is finite since R_0 is finitely generated and K_0 has exponent dividing h. Moreover, each automorphism of J_0 acts on $J_0^\sharp$ by its permutation of the x_j.

By [95, Theorem 11.7], χ has a projective class in $H^2(J_0, \mathbb{C}^\times)$. By [95, Theorem 11.15], choose its representative α in a complement to $B^2(J_0, \mathbb{C}^\times)$. Thus, $\alpha^h = 1$. The associated extension E of J_0 by μ_h admits a homomorphism $F_0 \to E$ over J_0. Since the image of R_0 lies in the central subgroup μ_h, the map $F_0 \to E$ factors through $J_0^\sharp$. Thus, χ extends to $\widehat{\chi}$ on $X \rtimes J_0^\sharp$. The order $|J_0^\sharp|$ is prime to ℓ. Moreover, $\widehat{\chi}$ occurs in $K \otimes_{\mathcal{O}} V_{X \rtimes J_0^\sharp}$ by [95, Lemma 5.2] because $\chi|_B$ contains a character θ inflated from T. Hence, Proposition 7.17 pairs $\widehat{\chi}$ with an extension $\widehat{\psi}$ on $M \rtimes J_0^\sharp$. If (γ, σ) fixes (χ, ψ), the induced action on $J_0^\sharp$ gives $\widehat{\chi}^{\gamma\sigma} = \nu_{\gamma,\sigma}\widehat{\chi}$ for a linear character $\nu_{\gamma,\sigma}$ of $J_0^\sharp$ by [95, Corollary 6.17]. Thus, by Proposition 7.17, $\widehat{\psi}^{\gamma\sigma} = \nu_{\gamma,\sigma}\widehat{\psi}$.

We compare the scalars on $C_{X \rtimes \Gamma}(X)$. An element (x, b) of $C_{X \rtimes \Gamma}(X)$ has $b = \mathrm{Int}(x^{-1})$. Since C injects into $\mathrm{Out}(X)/\mathrm{Out}_{\mathrm{diag}}(X)$, $b \in \Delta$. Thus, $x \in T$, and $|x|$ is a power of 3 since its adjoint image is b^{-1} and $|Z(X)| \mid 3$. In particular, $(x, b) \in M \rtimes J_0$. Given a lift $y = (x, b^\sharp)$ in $X \rtimes J_0^\sharp$, the order of y is prime to ℓ, since (x, b) is a 3-element and $\ell \nmid |K_0|$. Let e, e' be primitive idempotents over $\mathcal{O}$ for $\widehat{\chi}, \widehat{\psi}$, with $\mathrm{Br}_P(\bar{e}) = \bar{e}'$. As y centralizes X and each selected summand affords χ or ψ on the base group, $ye = \lambda e$ and $ye' = \lambda' e'$. Since $y \in M \rtimes J_0^\sharp$, its matrix on $(A/B)^P = H/T$ is the local matrix of y, where $A = X \rtimes J_0^\sharp$ and $H = M \rtimes J_0^\sharp$. Moreover, y permutes the lines in $(A/B)^P$. Hence,

the submatrix of ye on $(A/B)^P$ reduces to $\overline{ye'}$ by Lemma 7.16. Consequently, $\bar{\lambda}\bar{e}' = \bar{\lambda}'\bar{e}'$. Since $\bar{e}' \neq 0$ and λ, λ' have ℓ'-order, $\lambda = \lambda'$. In particular, the elements $(1, z)$ for $z \in K_0$ give the common character λ_0 of K_0.

The group L acts trivially on $J_0^\sharp$, since $L \leq Z(\Gamma)$ and conjugation fixes each x_j. Given $a \in L$, we have that the twist $\nu_{a,1}$ has order dividing $|a|$: the identity $\widehat{\chi}^a = \nu_{a,1}\widehat{\chi}$ iterated $|a|$ times gives $\nu_{a,1}^{|a|} = 1$ by uniqueness of $\nu_{a,1}$. Since $\ell \nmid |J_0^\sharp|$, $\nu_{a,1} = 1$. Thus, both $\widehat{\chi}, \widehat{\psi}$ are L-invariant. For either choice $(Y, \eta) = (X, \widehat{\chi})$ or $(M, \widehat{\psi})$, we put $D = Y \rtimes J_0^\sharp$. The L-invariant linear character $\det \eta$ of D extends to $\delta \in \mathrm{Irr}(D \rtimes L)$ with $\delta|_L = 1$. By [95, Theorem 8.15 and Lemma 6.24], choose the unique extension $\widetilde{\eta}$ with $\det \widetilde{\eta} = \delta$ since $(\eta(1), |L|) = 1$. For (γ, σ) fixing (χ, ψ), the twist of $\widetilde{\eta}$ restricts to $\nu_{\gamma,\sigma}$ on $J_0^\sharp$. Moreover, γ fixes L pointwise, so both determinants on L remain trivial. Therefore, the twist ν satisfies $\nu^{\eta(1)}|_L = 1$, whence $\nu|_L = 1$. Thus, the twists on $J_0^\sharp \times L$ coincide.

Choose a section $s : J \to J_0^\sharp \times L$ extending the identity on L, with $s(1) = 1$. The factor at $j, k \in J$ on each module is $\lambda_0(s(j)s(k)s(jk)^{-1})$. Moreover, $s(j)^\gamma s(j^\gamma)^{-1} \in K_0$ for the lifted action of γ on $J_0^\sharp \times L$. Since the scalar on K_0 and the twists $\nu_{\gamma,\sigma}$ agree, the functions $\mu_{\gamma,\sigma}$ in [141, Lemma 1.4] agree on J. The centralizer scalars remain equal on $C_{X \rtimes \Gamma}(X)$, which lies in $M \rtimes J_0$. Hence, the triples on $G_0 = X \rtimes \Gamma_{\chi^{\mathcal{H}_\ell}}$ and $H_0 = M \rtimes \Gamma_{\psi^{\mathcal{H}_\ell}}$ satisfy the relation $\geq_c$ of [141, Definition 1.5]. Finally, we let $A_\chi = \mathrm{Aut}(X)_{P,\chi^{\mathcal{H}_\ell}}$. Since $\mathrm{Aut}(X)_P = \mathrm{Inn}(M)\Gamma$, the images on X of G_0, H_0 are the images of $X \rtimes A_\chi$ and $M \rtimes A_\chi$, respectively. Moreover, $C_{X \rtimes A_\chi}(X) \leq M \rtimes A_\chi$, since $(x, \mathrm{Int}(x^{-1}))$ in this centralizer has $x \in M$. Therefore, the result of [141, Theorem 2.9] gives the triple comparison on $X \rtimes A_\chi$ and $M \rtimes A_\chi$, as in the definition of $\mathrm{iMN}(S, \ell)$. □

Proposition 7.19. *Let $X = E_{6,\mathrm{sc}}(q)$, with $q = r^f$ and $3 \mid q - 1$. Let $T = \mathbf{T}^F$ be the split maximal torus in B. Then,* $\mathrm{iMN}(X/Z(X), 3)$ *holds with $M = N_X(\mathbf{T})$.*

Proof. By [166, Theorem 6.2 and (2)], every $\chi \in \mathrm{Irr}_{3'}(X)$ belongs to a principal series $\mathcal{E}(X, T, \theta)$, with $3 \nmid [W : W_\theta]$. Let $s \in \mathbf{T}^{*F}$ be dual to θ. Thus, $W_\theta = W(s)$ by [166, Lemma 2.6]. Since $|W|_3 = 3^4$, no proper parabolic of W contains a Sylow 3-subgroup of W_θ: the proper Levi factors of types A, D have total rank at most 5 and Weyl orders with 3-part at most 3^2. Thus, s is quasi-isolated in X^*. By [208, proof of Proposition 10.9, Step 1], the groups $W(s)$ of types A_5A_1, D_4 with component C_3, and A_1^4 with component C_3 have 3-parts $3^2, 3^2, 3$ respectively. In characteristic 2, s has odd order. Consequently, either $s = 1$ or s has connected centralizer of type A_2^3 and component group C_3. For the latter class, $\omega_4^\vee(\zeta_3)$ in [208, (10.23)] is a rational lift of s in $\mathbf{G}_{\mathrm{sc}}^*$. Moreover, all representatives of its W-orbit lie in the split torus $\mathbf{T}^{*F}$. In particular, $\theta|_{Z(X)} = 1$.

For the cubic weight θ, put $R = A_2^3$ and $W_\theta = W(R) \rtimes \langle c \rangle$, with c of order 3 rotating the factors of R. By [166, Theorem 6.2 and (2)], the bijection Ω_3 assigns the characters over θ on M by $\eta \in \mathrm{Irr}(W_\theta)$. Since $W(R) = \mathfrak{S}_3^3$, Clifford theory gives $3 \mid \eta(1)$ if the label (i, j, k) on $W(R)$ is nonconstant. Each constant triple (i, i, i) extends to W_θ by the cyclic permutation on $U_i^{\otimes 3}$, where U_i is the ith simple $\mathfrak{S}_3$-module. Thus, the three twists by $\mathrm{Irr}(\langle c \rangle)$ for each i give nine local labels of $3'$-degree. Consequently, Ω_3^{-1} gives the nine cubic characters of X.

For θ, θ^{-1}, the subsystem R has positive roots R^+ relative to B, with $\theta \circ \alpha^\vee = 1$ for $\alpha \in R$ on $\mathbb{F}_q^\times$. By [208, proof of Proposition 10.9, Step 7], $v = w_0(E_6)w_0(R)$ takes θ to θ^{-1}, has $v(R^+) = R^+$, and has a pinned lift n_v fixed by field operator ϕ and graph operator γ. In particular, $v^2 = 1$. In W_θ, we find stabilizer $\langle c \rangle$ for R^+. Moreover, we put $K_0 = \mathbb{Q}(\sqrt{q})$ and $K = K_0(\zeta_3)$.

Given $\eta \in W\theta$, put $V_\eta = \mathrm{Ind}_B^X K_\eta$ with η inflated from T to B. Moreover, $B_w : V_\eta \to V_{w\eta}$ denotes the operator at n_w, normalized by $q^{-\ell(w)/2}$. Since $\eta^3 = 1$, the subgroup $T[2]$ from the Tits lifts has $\eta(T[2]) = 1$. When $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, $B_{w_1}B_{w_2} = B_{w_1w_2}$ by [40, Propositions 10.5.1 and 10.7.3]. Moreover, for a simple reflection $a = s_\alpha$, the product B_aB_a from V_η to itself has identity coefficient 1 by [40, Corollary 10.7.4]. The coefficient at B_a is $q^{-1/2}\sum_{t \in \mathbb{F}_q^\times} \eta(\alpha^\vee(t))$. Thus, $B_aB_a = 1$ when $\eta \circ \alpha^\vee \neq 1$, and $B_a^2 = 1 + (\sqrt{q} - 1/\sqrt{q})B_a$ otherwise. At the weights $w\eta$, the formulas hold with $w\eta$ in place of η.

By [40, Proposition 10.7.5], the based algebra for $W_\theta = W(R) \rtimes \langle c \rangle$ contains $H_q(W(R))$ on the simple reflections of R, since $|U_\alpha| = q$. Moreover, since $c(R^+) = R^+$, the operators satisfy $B_c^3 = 1$ after length normalization and evaluation on $T[2]$. Thus, $\mathrm{End}_{KX} V_\theta = H_q(A_2)^{\otimes 3} \rtimes \langle B_c \rangle$. v has no inversions in R^+, so the two B_v give inverse maps $V_\theta \rightleftarrows V_{\theta^{-1}}$ by [208, (10.14) and its proof]. Moreover, we have that $B_vB_{s_\beta}B_v^{-1} = B_{s_{v\beta}}$ for each simple root β of R. Hence, $B_vB_cB_v^{-1} = B_{vcv^{-1}} \in \{B_c, B_c^{-1}\}$. The field operator ϕ fixes the pinned lifts and γ permutes them. Moreover, we have that $\sigma \in \mathrm{Gal}(K/K_0)$ gives B_w^σ as the operator at η^σ. On $V_\eta^M = \mathrm{Ind}_T^M K_\eta$ the same formulas for $v, c, \phi, \gamma, \sigma$ hold in the group-algebra specialization $q = 1$, since $\eta(T[2]) = 1$.

For the blocks of $H_q(A_2)$ of degrees $1,1,2$, choose primitive idempotents e_i over K_0 fixed by opposition. If $\dim U_i = 1$, choose the central idempotent e_i. For $(2,1)$, write b for the central idempotent and put $T_{121} = T_1T_2T_1$. By [208, Lemma 4.4, (4.1)], $T_{121}^2 b = q^3 b$. Since T_{121} conjugates T_1 to T_2, which differ in this block, $T_{121}b$ is nonscalar. Thus, $e_i = b(1 + q^{-3/2}T_{121})/2$ is primitive and opposition fixes e_i. On V_θ^M, we choose e_i at $q = 1$. Consequently, the idempotent $E_i = e_i^{\otimes 3}(1 + B_c + B_c^2)/3$ projects onto one irreducible X- or M-submodule. Indeed, $e_i^{\otimes 3}$ selects a line in $U_i^{\otimes 3}$. Among the three extensions of $U_i^{\otimes 3}$, just one has $B_c = 1$ on that line. Also, E_i is fixed by $B_c \mapsto B_c^{-1}$, by $\mathfrak{S}_3$ on $U_i^{\otimes 3}$, and by opposition in $H_q(A_2)$.

On E_iV_θ and $E_iV_\theta^M$, we choose the pinned graph operator G_γ. For the field lift A_ϕ, take raw field transport P_ϕ if $r \equiv 1 \bmod 3$, and take B_vP_ϕ if $r \equiv -1 \bmod 3$. In the second case f is even. Since ϕ, γ fix n_v, $A_\phi G_\gamma = G_\gamma A_\phi$. Moreover, $A_\phi^f = 1$ by the inverse identities for B_v, and $G_\gamma^2 = 1$. Thus, A_ϕ, G_γ define extensions over $C = \langle \phi, \gamma \rangle$. For the nonidentity $\kappa \in \mathrm{Gal}(K/K_0)$, we have $\zeta_3^\kappa = \zeta_3^{-1}$ and $\theta^\kappa = \theta^{-1}$. Thus, $B_v^\kappa = B_v^{-1}$ between the two weights. Hence, $B_v\kappa$ is a semilinear involution on E_iV_θ and $E_iV_\theta^M$. Let A be an action matrix for X or M on its model. Then, $(B_v\kappa)A = A(B_v\kappa)$. The same equation holds for $A \in \{A_\phi, G_\gamma\}$ by the identities on B_v at θ, θ^{-1}. Hence, Galois descent gives extensions over K_0. Since $3 \mid q-1$, [166, Corollary 3.3] gives that $\mathcal{H}_3$ fixes K_0.

Given $\mathbf{T}$, we write $\overline{\mathbf{T}}$ for its adjoint image. Let $d \in \overline{\mathbf{T}}^F$ generate the quotient of order 3 modulo the image of T in $\overline{\mathbf{T}}^F$. Since $\theta|_{Z(X)} = 1$, we view θ on this image. Given $w \in W_\theta$, conjugation by d permutes the root factors of B_w with scalar $\theta(dn_w d^{-1} n_w^{-1})$ for the adjoint lift n_w. Moreover, the scalar on V_θ^M is also $\theta(dn_w d^{-1} n_w^{-1})$. For a based reflection s_β in $W(R)$, the nonzero linear coefficient in $B_{s_\beta}^2$ gives scalar 1. Hence, on $\mathrm{End}_{KX} V_\theta$ only B_c can have a nontrivial scalar for d. Since d has orbits of size 3 on X-characters [208, proof of Proposition 10.9, Step 3], the scalar at B_c is a primitive cube root z. Thus, d permutes the three extensions of (i,i,i) in a 3-cycle on X and on M. The graph operator γ inverts the image of d in $\mathrm{Out}(X/Z(X))$ [208, proof of Proposition 10.9, Step 4]. Thus, each orbit has a unique γ-fixed member. The projector E_i gives this member since G_γ fixes E_i. Consequently, for $\psi = \Omega_3(\chi)$ with $\chi^\gamma = \chi$, the extensions constructed above give $\widehat{\chi}, \widehat{\psi}$ fixed by $\mathcal{H}_3$.

With $S = X/Z(X)$, we write $M_S = M/Z(X)$. By [208, proof of Proposition 10.9, Step 4], the inertia of χ in $\mathrm{Aut}(S)$ is $S \rtimes C$. Moreover, we let $A = N_{\mathbf{G}_{\mathrm{ad}}}(\overline{\mathbf{T}})^F \rtimes C$. Since $S \cap A = M_S$, the inertia of ψ in A is $M_S \rtimes C$ by the equivariance of Ω_3. The extensions have $Z(X)$ in their kernels, since $\theta|_{Z(X)} = 1$. For the other pairs (χ^d, ψ^d) and (χ^{d^2}, ψ^{d^2}), the conjugates by d, d^2 of $\widehat{\chi}, \widehat{\psi}$ extend them and are $\mathcal{H}_3$-fixed. Hence, for each pair (χ, ψ) the mixed stabilizer in $A \times \mathcal{H}_3$ is $A_\chi \times \mathcal{H}_3$.

In $\mathrm{Syl}_3(M) \cap \mathrm{Syl}_3(X)$, we choose P with $N_X(P) \le M$ as in [166, Theorem 6.2]. Since A maps onto $\mathrm{Out}(S)$, each outer coset has a representative $a \in A$ with $P^a \le M$. There is an $m \in M$ with $(P^a)^m = P$. Thus, the image of $\mathrm{Aut}(X)_P$ on S lies in A, because any two corrections differ by $n \in N_X(P) \le M$. In particular, for a γ-fixed pair let $G = X \rtimes \mathrm{Aut}(X)_{P,\chi}$ and $H = M \rtimes \mathrm{Aut}(X)_{P,\chi}$. The map $\rho(x,a) = \mathrm{Int}_S(xZ(X))\,\bar{a}$ takes G into $S \rtimes C$ and H into $M_S \rtimes C$. Thus, $\widehat{\chi} \circ \rho|_G$ and $\widehat{\psi} \circ \rho|_H$ are $\mathcal{H}_3$-fixed extensions. Moreover, $C_G(X) \le H$ since $(x, \mathrm{Int}(x^{-1})) \in C_G(X)$ has $x \in M$. On $C_G(X)$, both extensions have scalar 1. Hence, $\alpha = \alpha' = 1$ and $\mu_a = \mu_a' = 1$ in [141, Definition 1.5]. If $i \in \{1,2\}$, the extensions at (χ^{d^i}, ψ^{d^i}) give $\alpha = \alpha' = 1$ and $\mu_a = \mu_a' = 1$. Finally, for $s = 1$ we choose the bijection of [156, Theorem D] on M with local weight 1_T. Thus, its union with Ω_3 on $s \ne 1$ gives $\mathrm{iMN}(X/Z(X), 3)$. □

Theorem 7.20. *With $X = E_{6,\mathrm{sc}}^\epsilon(q)$, we put $S = X/Z(X)$. If $\ell \mid q-1$ is odd, we have that* $\mathrm{iMN}(S, \ell)$ *holds on X with $M = N_X(\mathbf{T})$ for a maximally split torus $\mathbf{T}$. Put $W = M/\mathbf{T}^F$. If $\ell \nmid |W|$, then $M = N_X(P)$ for $P \in \mathrm{Syl}_\ell(X)$ as in* Lemma 7.12. *Otherwise, M is the local subgroup of* Schaeffer Fry [166, Theorem 6.2].

Proof. If $\ell \nmid |W|$, we apply Theorem 7.18. If $(\epsilon, \ell) = (1,3)$, apply Proposition 7.19. If $(\epsilon, \ell) = (1,5)$, we apply Theorem 7.123. Since $|W(F_4)| = 2^7 3^2$, it remains to take $X = {}^2E_{6,\mathrm{sc}}(q)$ and $\ell = 3 \mid q-1$. By [166, Theorem 6.2 and (2)], a weight θ of T below a local $3'$-character has $3 \nmid [W : W_\theta]$ for $W = W(E_6)^\gamma = W(F_4)$. In the numbering of [208, (10.23)], we put $c = s_2s_4(s_1s_6)(s_3s_5)$ in $W(F_4)$. On the γ-fixed space, c has order 12 and exponents $1, 5, 7, 11$, so c^4 has no fixed vectors. On the (-1)-space of γ, we choose the basis $\alpha_1 - \alpha_6, \alpha_3 - \alpha_5$. The reflections s_2, s_4 act trivially on that basis, and s_1s_6, s_3s_5 are the two A_2 reflections since the Gram matrix is $\left(\begin{smallmatrix} 4 & -2 \\ -2 & 4 \end{smallmatrix}\right)$. Thus, c^4 has order 3 on the (-1)-space of γ, with fixed space 0. Consequently, $1 + c^4 + c^8$ is zero on the absolute lattice $X^*(\mathbf{T})$. The Sylow condition gives $wc^4w^{-1} \in W_\theta$ for some $w \in W$. Thus, $\theta^3 = 1$.

Let $s \in (\mathbf{T}^*)^{\gamma F_q}$ be dual to θ. Since $s^3 = 1$ and $q \equiv 1 \bmod 3$, we have that the identity $\gamma F_q(s) = s$ gives $\gamma(s) = s$. Given any simple root α_j of $\mathbf{T}^*$, write $\alpha_j(s) = \zeta_3^{x_j}$, with $x_j \in \mathbb{F}_3$. Then, $x = (a,b,c,d,c,a)$ and the rule for s_i is

$x_j \mapsto x_j - C_{ji}x_i$ for the Cartan matrix C. Given any x, we close $\{x\}$ under s_2, s_4, s_1s_6, s_3s_5. The four orbit sizes are $(0,0,0,0):1$, $(0,0,0,1):32$, $(0,0,1,1):24$, $(0,0,1,2):24$. Thus, the orbits with $3 \nmid [W:W_\theta]$ have lengths $1, 32$. The representative $(0,0,0,1)$ is the image of $\omega_4^\vee(\zeta_3)$ from [208, (10.23)]. Since $(3, q+1) = 1$, the rational connected centralizer of s has factors $A_2(q^2), A_2(q)$ and the rational component group is 1 by [208, (10.30)]. Thus, $W_\theta = W(A_2) \times W(A_2)$, of order 36. Each $\eta \in \mathrm{Irr}(W_\theta)$ has degree $1, 2$ or 4. Hence, the bijection Ω_3 in [166, (2)] pairs the nine characters over θ in $\mathrm{Irr}(M)$ with the nine characters of X in $\mathcal{E}(X,s)$. Moreover, $\chi \in \mathcal{E}(X,s)$ is rational and fixed by $\langle\phi\rangle$ of order $2f$ by [208, proof of Proposition 10.9, Steps 3–5].

We compare $\chi, \psi = \Omega_3(\chi)$ over $E = \langle\phi\rangle$, with $\phi = F_r|_X$. We fix n_2, n_4, n_1n_6, n_3n_5 as in [208, (10.12)]. Moreover, for each wall a write $Q_a \in \{q, q^2\}$ for its root group order. Since $\theta^3 = 1$, all factors in $T[2]$ at the pinned lifts have value 1. The rank one product B_aB_a in Proposition 7.19 takes Q_a in place of q. By Proposition 7.14, a reflection a fixes a weight η exactly when $\eta|_{\mathbf{T}_a^F} = 1$. Thus, $B_aB_a = 1$ along opposite crossings at walls outside $R' = A_2 \times A_2$. By [40, Proposition 10.7.5], the algebra over θ is $H_{q^2}(A_2) \otimes H_q(A_2)$, since $W_\theta = W(R')$. Moreover, the algebra on V_θ^M is $H_1(A_2) \otimes H_1(A_2)$.

Put $v = w_0(F_4)w_0(R')$. Since $w_0(F_4) = -1$, $v\theta = \theta^{-1}$, $v^2 = 1$, and $v(R'^+) = R'^+$. Moreover, v acts on each A_2 factor of R' by opposition. Given a reduced expression $v = a_1 \cdots a_k$, normalize the orbital B_v by $\prod_{i=1}^k Q_{a_i}^{-1/2}$. Since v has no inversions in R'^+, the two B_v at θ, θ^{-1} satisfy $B_vB_v = 1$ by [208, (10.14)]. By [40, Proposition 10.7.5], $B_vB_{s_\beta}B_v^{-1} = B_{s_{v\beta}}$. ϕ fixes the pinned representatives. With $K_0 = \mathbb{Q}(\sqrt{q})$, put $K = K_0(\zeta_3)$. Given $\eta \in W\theta$, put $V_\eta = \mathrm{Ind}_B^X K_\eta$ and $V_\eta^M = \mathrm{Ind}_T^M K_\eta$, with K_η inflated to B via $B \to T$. Next, in $H_{q^2}(A_2)$ and $H_q(A_2)$, we choose the primitive idempotents e_i, e_j of Proposition 7.19 fixed by opposition. Thus, $e_i \otimes e_j$ over K_0 selects a simple submodule of V_θ. On V_θ^M take $e_i \otimes e_j$ at $q = 1$.

On both selected summands, we let $A_\phi = P_\phi$ if $r \equiv 1 \bmod 3$, and $A_\phi = B_vP_\phi$ if $r \equiv -1 \bmod 3$, where P_ϕ is the pinned action on V_η, V_η^M. In the second case P_ϕ exchanges θ, θ^{-1}, and $A_\phi^2 = P_\phi^2$ since the two B_v are inverse. Hence, $A_\phi^{2f} = 1$ and the lift of $\gamma = \phi^f$ is A_ϕ^f. Moreover, if κ generates $\mathrm{Gal}(K/K_0)$, $(B_v\kappa)^2 = 1$ and $B_v\kappa A_\phi = A_\phi B_v\kappa$. Since the idempotents $e_i \otimes e_j$ are fixed by $B_v\kappa$, Galois descent gives the extensions $\widehat{\chi}, \widehat{\psi}$ over K_0. By [156, Lemma 2.2], $\mathcal{H}_3$ fixes K_0 since $q \equiv 1 \bmod 3$.

Since $Z(X) = 1$ and the diagonal quotient is trivial, $\mathrm{Aut}(X) = X \rtimes E$ by [208, (10.1)–(10.2)]. In $\mathrm{Syl}_3(X)$, choose P with $N_X(P) \le M$ as in [166, Theorem 6.2]. Given any $\phi^i \in E$, we have $P^{\phi^i} \le M$, so $(P^{\phi^i})^m = P$ for some $m \in M$. Hence, $\mathrm{Aut}(X)_P \le M \rtimes E$, since all corrections differ by $n \in N_X(P) \le M$. With $G = X \rtimes \mathrm{Aut}(X)_P$, we put $H = M \rtimes \mathrm{Aut}(X)_P$. The product map $\rho(x,\delta) = x\delta$ into $\mathrm{Aut}(X)$ takes G into $X \rtimes E$ and H into $M \rtimes E$. Thus, for each pair (χ, ψ), the extensions above compose with ρ to give extensions on G, H. Since $C_{\mathrm{Aut}(X)}(X) = 1$, we have $\rho(C_G(X)) = 1$. Thus, each $(x,\delta) \in C_G(X)$ has $\delta = x^{-1}$ in $\mathrm{Aut}(X)$ and $x \in N_X(P) \le M$. Hence, $C_G(X) \le H$, with scalar 1 for both extensions. Since they are $\mathcal{H}_3$-fixed, the ordinary factors α, α' and mixed twists μ_a, μ_a' are all 1 in [141, Definition 1.5]. Consequently, Ω_3 compares triples for $\mathcal{E}(X,s)$ and weights in $W\theta$. If $s = 1$, the 1-split Levi is $\mathbf{T}$, and [156, Theorem D] gives the comparison with $\mathrm{Irr}_{3'}(M \mid 1_T)$. Thus, 1_T and $W\theta$ exhaust the local weights, giving $\mathrm{iMN}(S, 3)$. □

Lemma 7.21. *Let $\mathbf{G}$ be simply connected of type E_6, with pinned diagram involution γ, and let $F = \gamma^t F_q$, where $t = 0, 1$ and $q = r^f$ for a prime r. Let ℓ be an odd prime dividing $q+1$. Thus, we obtain that $d_\ell(q) = 2$, with Sylow Φ_2-torus $\mathbf{S}_2$. Let n_0 be the pinned lift of $w_0 \in W(E_6)$, the longest element. With $F' = \mathrm{Int}(n_0)F$, we put $X = \mathbf{G}^{F'}$ and $T = \mathbf{T}^{F'}$ and $\delta = \gamma^{1+t}$. Then, we find a Sylow Φ_2-torus $\mathbf{S}_2$ with $C_{\mathbf{G}}(\mathbf{S}_2) = \mathbf{T}$ and $M = N_X(\mathbf{S}_2) = N_{\mathbf{G}}(\mathbf{T})^{F'}$. Moreover, $M/T = W(E_6)^\delta$, of type F_4 for $t = 0$ and E_6 for $t = 1$. For $P \in \mathrm{Syl}_\ell(X)$ chosen in M, we have that $N_X(P) \le M$.*

Proof. By Lang's theorem [58, Theorem 4.2.9], we choose $g \in \mathbf{G}$ with $g^{-1}F(g) = n_0$. Thus, conjugation by g maps $\mathbf{G}^{F'}$ to $\mathbf{G}^F$, and the torus $\mathbf{T}$ to ${}^g\mathbf{T}$, with the dual marking of $\mathbf{T}^*$ from [208, proof of Proposition 10.9, Step 1]. Since $w_0 = -\gamma$ on $X_*(\mathbf{T})$, the endomorphism F' acts there as $-q\delta$. Its $(-q)$-eigenspace has dimension 4 for $t = 0$ and 6 for $t = 1$. The degrees $2, 5, 6, 8, 9, 12$ give Φ_2 exponents $4, 6$ in $|X|$, since the factor of degree j is $q^j - (-1)^{tj}$. Thus, the $(-q)$-eigenspace defines the Sylow Φ_2-torus $\mathbf{S}_2$. The vector $2\rho^\vee = \sum_{\alpha>0} \alpha^\vee$ is δ-fixed and strictly dominant, so $\langle \alpha, 2\rho^\vee \rangle \ne 0$ for each root α. Hence, the Levi $C_{\mathbf{G}}(\mathbf{S}_2)$ is $\mathbf{T}$. Since the $(-q)$-eigenspace is $N_{\mathbf{G}}(\mathbf{T})^{F'}$-stable, $N_{\mathbf{G}}(\mathbf{T})^{F'} \le N_X(\mathbf{S}_2)$. Conversely, an element of $N_X(\mathbf{S}_2)$ normalizes $C_{\mathbf{G}}(\mathbf{S}_2) = \mathbf{T}$. By Lang's theorem on $\mathbf{T}$, $M/T = W(E_6)^\delta$. By [132, Theorem 5.14], $N_X(P) \le M$ for $P \in \mathrm{Syl}_\ell(X)$ chosen in M. Finally, in a regular injection $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$, we have

$C_{\widetilde{\mathbf{G}}}(\mathbf{S}_2) = \mathbf{T}Z(\widetilde{\mathbf{G}})$ with rational Weyl quotient $W(E_6)^\delta$. □

Proposition 7.22. *Let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular inclusion to which F' and the pinned automorphisms a extend. Moreover, we put $\widetilde{T} = (\mathbf{T}Z(\widetilde{\mathbf{G}}))^{F'}$ and $\widetilde{M} = N_{\widetilde{\mathbf{G}}^{F'}}(\mathbf{S}_2)$. There is an extension map Λ for $\widetilde{T} \lhd \widetilde{M}$ on $\mathrm{Irr}(\widetilde{T})$. In particular, we have that Λ is equivariant under the marked automorphisms, $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, and multiplication by characters of $\widetilde{M}/M$.*

Proof. By [177], each $\eta \in \mathrm{Irr}(T)$ extends to M_η. Given $\theta \in \mathrm{Irr}(\widetilde{T})$, let s be dual to θ in $(\widetilde{\mathbf{T}}^*)^{F'^*}$. Since the derived group of $\widetilde{\mathbf{G}}^*$ is simply connected, $C_{\widetilde{\mathbf{G}}^*}(s)$ is connected by Steinberg's theorem [58, Chapter 3]. Put $R = \{\alpha : \alpha^*(s) = 1\}$. In E_6, the dual-root map is $\alpha \mapsto \alpha^*$. Then $W(R)$ is the absolute stabilizer of s. Since F'^* acts as $-q\delta$ on the semisimple root lattice, $\alpha^*(s) = 1$ if and only if $(\delta\alpha)^*(s) = 1$. Moreover, δ gives $\delta(R^+) = R^+$. Thus, in $\widetilde{M}/\widetilde{T}$, we find stabilizer $W(R)^\delta$ for θ.

For the absolute Tits group V, we have $F'|_V = \delta|_V$. Indeed, $n_0 n_i n_0^{-1} = n_{\gamma(i)}$ by the braid relations, and $F_q(n_i) = n_i$ for each i. Let $\mathbf{G}_R = \langle U_\alpha : \alpha \in R\rangle$ and $i_R : \mathbf{H}_R \to \mathbf{G}_R$ be the simply connected cover of $\mathbf{G}_R$. For root vectors from $\mathbf{G}$, the Tits group V_R satisfies $i_R(V_R) \le V$. Moreover, the vectors x_α and $-x_\alpha$ give lifts differing by $\alpha^\vee(-1)$ in V_R. The action δ_R on $\mathbf{H}_R$ lifts $\delta|_{\mathbf{G}_R}$ along i_R.

Choose simple root vectors for R permuted by δ_R. If $\delta_R\alpha = \beta \ne \alpha$, we set $x_\beta = \delta_R(x_\alpha)$. If $\delta = \gamma$ and $\gamma\alpha = \alpha$, α is a long root of F_4. By [208, proof of Proposition 10.5], for a fixed simple root β there is a γ-fixed lift n_w with $w\beta = \alpha$. Thus, $\mathrm{Ad}(n_w)x_\beta$ is a fixed vector for α. Given any δ_R-orbit I on simple roots of R, the longest element $w_I \in W_I$ has a δ_R-fixed positive Tits lift n_{w_I}. By [40, §2.9], the w_I generate $W(R)^\delta$, including I with adjacent simple roots. Thus, $U_R = i_R(V_R^{\delta_R}) \le M$ maps onto $W(R)^\delta$.

If $u \in U_R \cap \widetilde{T}$, we choose $v \in V_R^{\delta_R}$ with $i_R(v) = u$. The Weyl image of v is 1 since $W(R) \le W(E_6)$, so v belongs to the sign subgroup $\mathbf{T}_R[2]$ of the torus $\mathbf{T}_R$ of $\mathbf{H}_R$. If $2 \nmid r$, $-q\delta_R$ acts as δ_R on $\mathbf{T}_R[2]$. For $r = 2$ this subgroup is trivial. Thus, $v \in \mathbf{T}_R^{F_R}$, where $F_R = -q\delta_R$ on the cocharacter lattice of $\mathbf{T}_R$. The map of tori $i_R : \mathbf{T}_R \to \widetilde{\mathbf{T}}$ is rational for F_R, F'. Since all $\alpha^*(s) = 1$ for $\alpha \in R$, the image of s in the adjoint dual torus $\mathbf{T}_R^*$ is 1 in its simple root coordinates. Therefore, $\theta \circ i_R|_{\mathbf{T}_R^{F_R}} = 1$ by finite torus duality. In particular, $\theta|_{U_R \cap \widetilde{T}} = 1$.

Consequently, $\widetilde{M}_\theta = \widetilde{T}U_R$ and $\Lambda(\theta)(tu) = \theta(t)$ for $t \in \widetilde{T}$ and $u \in U_R$. The value $\theta(t)$ is independent of tu since $\theta|_{U_R \cap \widetilde{T}} = 1$. Moreover, $\theta^u = \theta$ for all $u \in U_R$, so $\Lambda(\theta)$ is multiplicative. Given $w \in W(E_6)^\delta$, take its δ-fixed positive Tits lift n_w. Conjugation by n_w takes the root lifts of V_R to those of V_{wR}, up to coroot signs in V_{wR}. The induced isomorphism $\mathbf{H}_R \to \mathbf{H}_{wR}$ intertwines δ_R, δ_{wR} since n_w is δ-fixed. Thus, $n_w U_R n_w^{-1} = U_{wR}$. Given $t \in \widetilde{T}$ and $u \in U_R$, the commutator $tut^{-1}u^{-1}$ has value 1 under θ, since $\theta^u = \theta$. Consequently, Λ is $\widetilde{M}$-equivariant.

Each pinned automorphism a acts by $U_R \mapsto U_{aR}$ because $a(n_\alpha) = n_{a\alpha}$ up to coroot signs. If $g^{-1}F(g) = n_0$, then $ga(g)^{-1} \in \mathbf{G}^F$ since $a(n_0) = n_0$. Thus, conjugation by g transports a to $\mathrm{Int}(ga(g)^{-1})a$. Let $E_g = \langle \mathrm{Int}(ga(g)^{-1})a : a \in E\rangle$ for the pinned group E. Then, the stabilizer of ${}^g\mathbf{S}_2$ in $\mathbf{G}^F E_g$ is ${}^gME_g$, since E_g fixes ${}^g\mathbf{S}_2$ and $N_{\mathbf{G}^F}({}^g\mathbf{S}_2) = {}^gM$. Given $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, the dual element of θ^σ is s^k with $(k, |s|) = 1$. Hence, $R(s^k) = R(s)$ and $\Lambda(\theta^\sigma) = \Lambda(\theta)^\sigma$. If $\nu \in \mathrm{Irr}(\widetilde{M}/M)$, the restriction $\nu|_{\widetilde{T}}$ multiplies s by a central dual element. Thus, R is fixed and $\nu|_{U_R} = 1$, giving $\Lambda(\theta\,\nu|_{\widetilde{T}}) = \Lambda(\theta)\nu|_{\widetilde{M}_\theta}$. □

Corollary 7.23. *Put $\widetilde{X} = \widetilde{\mathbf{G}}^{F'}$. For every odd prime $\ell \mid q+1$, we find a bijection $\mathrm{Irr}(\widetilde{X} \mid \mathrm{Irr}_{\ell'}(X)) \to \mathrm{Irr}(\widetilde{M} \mid \mathrm{Irr}_{\ell'}(M))$ that matches the central characters on $Z(\widetilde{X})$. Moreover, it is equivariant under $\mathcal{H}_\ell$, the marked automorphisms and $\mathrm{Irr}(\widetilde{X}/X)$, acting by restriction on $\widetilde{M}$.*

Proof. Since $C_{\widetilde{\mathbf{G}}}(\mathbf{S}_2)^{F'} = \widetilde{T}$, by Proposition 7.22, we obtain Λ on all $\mathrm{Irr}(\widetilde{T})$. By [156, Theorem B], we obtain the bijection $\widetilde{\Omega}$. If $(3, q-(-1)^t) = 1$, then $Z(X) = 1$ and the central isogeny $X \to \mathbf{G}_{\mathrm{ad}}^{F'}$ is onto. Since $\widetilde{X} \to \mathbf{G}_{\mathrm{ad}}^{F'}$ has kernel $Z(\widetilde{X})$, $\widetilde{X} = X \times Z(\widetilde{X})$ and $\widetilde{M} = M \times Z(\widetilde{X})$. Therefore, $\widetilde{\Omega}$ at $1_{Z(\widetilde{X})}$ gives a bijection $\mathrm{Irr}_{\ell'}(X) \to \mathrm{Irr}_{\ell'}(M)$. Explicitly, if $\widetilde{\Omega}(\chi \boxtimes 1_{Z(\widetilde{X})}) = \psi \boxtimes 1_{Z(\widetilde{X})}$, set $\Omega(\chi) = \psi$. □

Theorem 7.24. *For $X = E_{6,\mathrm{sc}}^\epsilon(q)$ and odd $\ell \mid q+1$, we fix P, M from* Lemma 7.21. *Then, the map $\widetilde{\Omega}$ of* Corollary 7.23 *induces an $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant bijection $\mathrm{Irr}_{\ell'}(X) \to \mathrm{Irr}_{\ell'}(M)$ with equal central characters on $Z(X)$.*

Proof. Take the regular overgroup $\widetilde{\mathbf{G}} = (\mathbf{G} \times \mathbb{G}_m)/\mu_3$ from [208, proof of Proposition 10.9, Step 4], with $\gamma(z) = z^{-1}$ on the scalar factor $\mathbb{G}_m$. By Lang's theorem, $B = \widetilde{X}/X = (\widetilde{\mathbf{G}}/\mathbf{G})^{F'}$ is cyclic since $\dim(\widetilde{\mathbf{G}}/\mathbf{G}) = 1$. Moreover,

$\widetilde{M}/M \cong B$ via the map into $\widetilde{X}/X$, since $\widetilde{T}$ maps onto $\widetilde{X}/X$ by Lang's theorem on $\mathbf{T}$. On X and M, the diagonal group is $D = \widetilde{X}/(XZ(\widetilde{X}))$ of order $(3, q-\epsilon)$ by [208, (10.1)–(10.2)].

Given a pair $(\widetilde{\chi}, \widetilde{\psi})$ under $\widetilde{\Omega}$, the formulas for the cyclic quotient B give multiplicity-free restrictions to X, M. Their constituent orbits under D have sizes $|\operatorname{Irr}(B)_{\widetilde{\chi}}|$ and $|\operatorname{Irr}(B)_{\widetilde{\psi}}|$. Moreover, the $\operatorname{Irr}(B)$-orbits of $\widetilde{\chi}, \widetilde{\psi}$ determine these D-orbits. Thus, the $\operatorname{Irr}(B)$-equivariance of $\widetilde{\Omega}$ pairs the D-orbits of constituents with equal sizes 1 or 3. Moreover, on D-orbits the map $\widetilde{\Omega}$ is equivariant under the pinned operators ϕ, γ and under $\mathcal{H}_\ell$. In particular, the central character on $Z(X)$ agrees. For orbits of size 1, pair their unique constituents χ, ψ.

For orbits of size 3, put $b = \gamma\kappa$ for complex conjugation κ on values. Thus, b acts on all of $\operatorname{Irr}(X)$ and $\operatorname{Irr}(M)$. On $\widetilde{\mathbf{G}}$, the pinned opposition acts by γ on $\mathbf{G}$ and by $z \mapsto z^{-1}$ on $\mathbb{G}_m$. The dualizing involution ι of [10, §2] differs from the marked γ by $\operatorname{Int}(x_0)$ for $x_0 \in \mathbf{G}_{\mathrm{ad}}^{F'}$, by Lang transport. Since $Z(\widetilde{\mathbf{G}})$ is connected, the map $\widetilde{X} \to \mathbf{G}_{\mathrm{ad}}^{F'}$ is onto. By [10, Theorem 1.3], $\widetilde{\chi}^b = \widetilde{\chi}$ if the Jordan label $u_{\widetilde{\chi}}$ has eigenvalues ± 1.

Given a constituent orbit of size 3, take $\nu \neq 1_B$ with $\widetilde{\chi}\nu = \widetilde{\chi}$. If $\widetilde{\chi} \in \mathcal{E}(\widetilde{X}, s)$, s is conjugate to sz for the nonidentity dual central factor z of ν, by Jordan decomposition. Hence, s is noncentral, and the root system R of $C_{\widetilde{\mathbf{G}}^*}(s)$ is a proper subsystem of E_6, with classical components. By [156, Theorem B], choose s in the dual marked torus. On R the action of F'^* is $-q\delta$. Thus, the based action is $-w_0(R)\delta$ on R^+. Since $\delta(R^+) = R^+$, this action has order at most 2. In particular, the action on a D_4 diagram has order dividing 2. Given a component orbit of length m, apply $(F'^*)^m$ on one component $\mathbf{C}_i$. By [10, Theorems 1.3–1.4] for $(\mathbf{C}_i)_{\mathrm{ad}}$, its unipotent labels have Frobenius eigenvalues ± 1. Hence, via the central isogeny $\mathbf{C}_i \to (\mathbf{C}_i)_{\mathrm{ad}}$, the eigenvalues of $u_{\widetilde{\chi}}$ are ± 1. Consequently, $\widetilde{\chi}^b = \widetilde{\chi}$, so b stabilizes the D-orbit of its constituents.

Locally, write $\widetilde{\psi} = \operatorname{Ind}_{\widetilde{M}_\theta}^{\widetilde{M}}(\Lambda(\theta)\eta)$ with $\theta \in \operatorname{Irr}(\widetilde{T})$ and $\eta \in \operatorname{Irr}(W(R)^\delta)$. The group $W(R)^\delta$ is a finite Weyl group, so $\eta^\kappa = \eta$ as in [156, proof of Lemma 4.3]. Since $w_0 = -\gamma$ on $\mathbf{T}$ and b fixes the characters of the scalar $\mathbb{G}_m$, $\theta^b = \theta^{n_0}$ for the lift $n_0 \in M$. Moreover, γ acts on $W(R)^\delta$ as conjugation by w_0 into $W(w_0R)^\delta$. By Proposition 7.22, $\Lambda(\theta)^b = \Lambda(\theta^{n_0})$. Thus, $\widetilde{\psi}^b = \widetilde{\psi}^{n_0} = \widetilde{\psi}$ and the orbit of constituents in M is b-stable.

In a free D-orbit of size 3, the action of b on D is inversion, since γ inverts D and κ is D-equivariant. Hence, b acts on the orbit as $j \mapsto a - j$ for $j \in \mathbb{Z}/3$. Thus, its unique fixed constituent has $2j = a$. For the b-fixed constituents χ_0, ψ_0, set $\Omega(\chi_0^d) = \psi_0^d$ for $d \in D$. The map Ω depends only on the paired D-orbits. Moreover, for $a \in \{\phi, \gamma\} \cup \mathcal{H}_\ell$ the action on characters of X, M satisfies $ab = ba$. Their action on D is induced by the inclusion $\widetilde{M} \leq \widetilde{X}$. Thus, Ω is equivariant under $\phi, \gamma, \mathcal{H}_\ell, D$.

Every outer class of X has a representative a from the diagonal torus and $\langle \phi, \gamma \rangle$. Since a normalizes $\mathbf{S}_2$, we have $P^a \in \operatorname{Syl}_\ell(M)$ and $(P^a)^m = P$ for some $m \in M$. If a' represents the class of a and fixes P, then a' differs from the corrected action by $\operatorname{Int}(n)$ for $n \in N_X(P) \leq M$. Thus, Ω is $\operatorname{Aut}(X)_P$-equivariant and matches the characters on $Z(X)$ by restriction from $Z(\widetilde{X})$. □

Corollary 7.25. *Let $S = X/Z(X)$ and $\overline{P} = PZ(X)/Z(X)$. Let $\ell \mid q+1$ be an odd prime, with $\ell \geq 7$ if $\epsilon = -1$. Then, there is an $\mathcal{H}_\ell$-equivariant bijection $\operatorname{Irr}_{\ell'}(S) \to \operatorname{Irr}_{\ell'}(N_S(\overline{P}))$.*

Proof. Theorem 7.24 on central character $1_{Z(X)}$ gives a bijection from $\operatorname{Irr}_{\ell'}(S)$ to $\operatorname{Irr}_{\ell'}(M/Z(X))$. If $\ell \nmid |W^\delta|$, the order formula gives $P = O_\ell(T) \in \operatorname{Syl}_\ell(X)$ for $T = \mathbf{T}^{F'}$. Given an orbit I of δ on simple roots, we put $\lambda_I = \sum_{\beta \in I} \beta^\vee$. Since $|I| \leq 2$, every root α satisfies $|\langle \alpha, \lambda_I \rangle| \leq 4$. The span of the λ_I contains the strictly dominant cocharacter $2\rho^\vee$. Thus, for each α some I has $\langle \alpha, \lambda_I \rangle \neq 0$. Since $\ell \geq 5$ and $-q \equiv 1 \bmod \ell$, for a primitive ℓth root ζ we have $\lambda_I(\zeta) \in P$ with $\alpha(\lambda_I(\zeta)) \neq 1$. Thus, $C_{\mathbf{G}}(P)^\circ = \mathbf{T}$ and $N_X(P) = M$. Moreover, P is the unique Sylow ℓ-subgroup of $PZ(X)$, so $N_S(\overline{P}) = N_X(P)/Z(X)$.

In the other split case $\ell = 3$, we have $|M/T| = 2^7 3^2$. Thus, M/T is solvable by Burnside's theorem [95, Theorem 3.10], and $L = M/Z(X)$ is solvable. Moreover, $N_S(\overline{P}) \leq L$ by Lemma 7.21 and the characteristic subgroup P of $PZ(X)$. Consequently, $N_L(\overline{P}) = N_S(\overline{P})$. By [194] and [141, Introduction], the 3-solvable group L admits an $\mathcal{H}_3$-equivariant bijection $\operatorname{Irr}_{3'}(L) \to \operatorname{Irr}_{3'}(N_L(\overline{P}))$. The composition with Ω on $1_{Z(X)}$ gives the bijection for S. □

Proposition 7.26. *Suppose that q is odd and $\ell \mid q+1$ is odd. For both signs ϵ, we have that the ℓ'-characters in the isolated series $\mathcal{E}(X, s)$ of type A_5A_1 satisfy $\succeq_c$ under Ω of* Theorem 7.24.

Proof. Since $\mathcal{E}(X, s)$ is trivial on $Z(X)$ [208, Proposition 10.7(ii)], we put $S = X/Z(X)$. Moreover, we let $D = \mathbf{G}_{\mathrm{ad}}^F$ with $F = \gamma^t F_q$. If $\epsilon = 1$, [208, proof of Proposition 10.7, Step 2] gives the two extensions at $\lambda = (3, 2, 1)$ over

$\mathbb{Q}(\sqrt{q})$. The hook lengths $h(x)$ of λ are $5,3,3,1,1,1$. Since $q \equiv -1 \bmod \ell$, each factor $q^{h(x)}-1$ in the denominator is prime to ℓ in [40, §13.8]. The numerator contains q^2-1, so the A_5-degree at λ is divisible by ℓ. The Jordan degree on D also has the A_1-degree and index $[D^*:C_{D^*}(s)]_{r'}$ as factors. Moreover, $\widetilde{\chi}|_S$ is irreducible for $\widetilde{\chi} \in \mathcal{E}(D,s)$ by [208, (10.21)]. Thus, $\lambda=(3,2,1)$ gives no member of $\mathrm{Irr}_{\ell'}(X)$. The other characters extend rationally to $\mathrm{Aut}(S)$ by [208, Proposition 10.7(iii)].

If $\epsilon=-1$, by [208, Proposition 10.7(iv)], we obtain 20 rational extensions from $\mathcal{E}(D,s)$. For the remaining two, set $P_0=LU$ from [208, proof of Proposition 10.7, Step 4], with character $\lambda_0=\theta_0\tau$. θ_0 is quadratic and τ is the cuspidal unipotent character of $L_0=\mathrm{PGU}_6(q)$, viewed on L. Since $Z(\mathbf{L})$ is connected, Lang's theorem gives $L \to L_0$ onto. At $\lambda=(3,2,1)$, the hook polynomial at $-q$ gives the degree of τ up to sign. Since $-q \equiv 1 \bmod \ell$, this degree is congruent to $\pm 6!/(5\cdot 3\cdot 3)=\pm 16 \pmod{\ell}$.

The prime field operator $\phi=F_r|_D$ on L_0 is the pinned field operator of ${}^2A_5(q)$. Since $\mathrm{ord}_\ell(-q)=1$, by [156, Lemma 8.1(b) and proof of Proposition 8.5], $\widehat{\tau} \in \mathrm{Irr}(L_0\langle\phi\rangle \mid \tau)$ with values in $\mathbb{Q}_\ell$. We extend θ_0 by 1 on ϕ, and inflate the product with $\widehat{\tau}$ to $\widehat{\lambda}_0 \in \mathrm{Irr}(P_0\langle\phi\rangle)$. By [208, proof of Proposition 10.7, Step 4], $\mathrm{Ind}_{P_0}^{D}\lambda_0=\rho_+ + \rho_-$ for distinct rational constituents $\rho_\pm$. Since $\mathcal{H}_\ell$ fixes $\widehat{\lambda}_0$ and $\rho_\pm$, [208, Lemma 3.13] gives the fixed extensions $\widehat{\rho}_\pm$ on $\mathrm{Aut}(S)$. Section 7.5 also has source τ.

In the marking $F'=\mathrm{Int}(n_0)F$, we choose the adjoint torus T_D and its highest root α. Since $\delta(\alpha)=\alpha$, the coordinate $\alpha(t)$ takes values in μ_{q+1} for $t \in T_D$. We choose $\vartheta(t)$ to be the quadratic character of $\alpha(t)$, dual to $\alpha^\vee(-1)$. For $T_S=T_D\cap S$ and $w \in W^\delta$, the relation $\vartheta^w|_{T_S}=\vartheta|_{T_S}$ implies $\vartheta^w=\vartheta$, since T_D/T_S has odd order. Moreover, equal involutions $\beta^\vee(-1)=\alpha^\vee(-1)$ imply $\beta^\vee-\alpha^\vee \in 2P^\vee\cap Q^\vee=2Q^\vee$ in the adjoint dual group, because $[P^\vee:Q^\vee]=3$. If $\beta \neq \pm\alpha$, $0<\|\beta^\vee-\alpha^\vee\|^2<8$, while $2Q^\vee$ has minimum nonzero squared length 8. Thus, the rational root lines are δ-fixed, with positive representatives fixed by δ. For $t=0$ these roots are long in F_4, and for $t=1$ they are all roots of E_6. Hence, the local weights of $\mathcal{E}(X,s)$ form $(\vartheta|_{T_S})^{W^\delta}$ by [156, proof of Theorem B].

By [208, proof of Proposition 10.7, Step 1], the roots perpendicular to α form a system of type A_5, and γ acts by the diagram involution with $\gamma(\alpha)=\alpha$. Thus, γ acts on $W(A_5)$ by conjugation by $h=w_0(A_5)$. The rational inertia quotient W_ϑ is $C_{\mathfrak{S}_6}(h)\times\langle s_\alpha\rangle$ for $t=0$ and $\mathfrak{S}_6\times\langle s_\alpha\rangle$ for $t=1$. In particular, $C_{\mathfrak{S}_6}(h)\cong W(B_3)$. Let A be the marked normalizer of $\mathbf{S}_2$ in $\mathrm{Aut}(S)$ and put $\mathbf{L}_\alpha=\langle U_\alpha,U_{-\alpha}\rangle$. Since A_ϑ permutes $\alpha,-\alpha$, it acts on $(\mathbf{L}_\alpha)_{\mathrm{ad}}^{F'}\cong\mathrm{PGL}_2(q)$. Let ι_α be this action into $\mathrm{PGL}_2(q)\rtimes\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)$. Since $x\mapsto x^r$ fixes the square classes of $\mathbb{F}_q^\times$, the determinant character extends by 1 on $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)$. We write $\widetilde{\vartheta}$ for its composition with ι_α. Then, $\widetilde{\vartheta}|_{T_D}=\vartheta$. Indeed, for a generator u of $\mathbb{F}_{q^2}^\times$, multiplication by u has determinant $u^{q+1}\notin(\mathbb{F}_q^\times)^2$ and eigenvalue ratio u^{1-q} generating the nonsplit torus of order $q+1$.

Choose $\rho \in \mathrm{Irr}(W(A_5)^\delta)$ and a representation R_ρ affording it. Moreover, we let $e=\pm1$ be the value on s_α and put $\Phi=F_r|_X$ in the marked model. On W_ϑ, Φ is trivial and γ acts by h on the first factor. $\gamma^2=1$, $\Phi\gamma=\gamma\Phi$, and $\Phi^f=\mathrm{Int}(n_0^{-1})\gamma^t$ since F' fixes X. The Weyl image of n_0 is (h,s_α). Moreover, the elements $\Phi^i\gamma^j$ for $0\le i<f$ and $0\le j<2$ represent $2f$ distinct cosets over $D'=\mathbf{G}_{\mathrm{ad}}^{F'}$ by [208, (10.19)].

If $t=0$, h is central in $C_{\mathfrak{S}_6}(h)$. Let γ act by 1 and Φ by c, with $c^f=e\,\rho(h)/\rho(1)$. If $e\,\rho(h)/\rho(1)=1$, we set $c=1$. Otherwise, choose c of order $2^{v_2(f)+1}$. Since $r^f\equiv-1 \bmod \ell$, the 2-part of $\mathrm{ord}_\ell(r)$ is $2^{v_2(f)+1}$. Thus, $\mathcal{H}_\ell$ fixes c by [156, Lemma 2.3(a)]. If $t=1$, let γ act by $eR_\rho(h)$ and Φ by 1. Thus, $R_{\rho,e}(\Phi)^f=R_{\rho,e}(n_0^{-1}\gamma^t)$ in both cases, and $R_{\rho,e}$ affords an extension on A_ϑ/T_D. The characters of $W(A_5)^\delta$ are rational [156, proof of Lemma 4.3]. Hence, the character of $R_{\rho,e}$ is $\mathcal{H}_\ell$-fixed.

We induce $\widetilde{\vartheta}R_{\rho,e}$ from A_ϑ to A. Write $M_S=M/Z(X)$, with Weyl quotient W^δ. Since T_D/T_S represents the diagonal quotient and $\Phi,\gamma\in A_\vartheta$, we have $A=M_SA_\vartheta$. Thus, restriction to M_S gives Clifford induction from $(M_S)_{\vartheta|_{T_S}}$. As (ρ,e) ranges over $\mathrm{Irr}(W_\vartheta)$, we obtain $\mathcal{H}_\ell$-fixed extensions of every $\psi\in\mathrm{Irr}(M_S\mid\vartheta|_{T_S})$. Consequently, for a pair (χ,ψ) in the series, we put $G=X\rtimes\mathrm{Aut}(X)_{P,\chi}$ and $H=M\rtimes\mathrm{Aut}(X)_{P,\chi}$. By Theorem 7.24, Sylow correction in M gives $\mathrm{Aut}(X)_{P,\chi}$ acting on S via A. Thus, $\pi(x,a)=\mathrm{Int}_S(xZ(X))\,\bar{a}$ takes G into $\mathrm{Aut}(S)$ and H into A. Composing the $\mathcal{H}_\ell$-fixed extensions with π gives extensions to G,H. Since $C_{\mathrm{Aut}(S)}(S)=1$, we have $\pi(C_G(X))=1$. Moreover, $C_G(X)\le H$ since $x\in N_X(P)\le M$ for $(x,a)\in C_G(X)$. Thus, the centralizer scalar is 1, as are the factors α,α' and the twists μ_a,μ_a' of [141, Definition 1.5]. □

Theorem 7.27. *Let $3\mid q+1$ and $X=E_{6,\mathrm{sc}}^\epsilon(q)$. Then, the comparison on $M=N_X(\mathbf{S}_2)$ satisfies $\succeq_c$ for the algebraic cover X. In particular, it gives* iMN$(S,3)$ *when X is the universal cover of $S=X/Z(X)$. If $S={}^2E_6(2)$, we have that* iMN$(S,3)$ *is known by* Ruhstorfer–Schaeffer Fry [157, Lemma 4.1(2)].

Proof. Since $q = r^f \equiv -1 \bmod 3$, $r \equiv -1 \bmod 3$ and f odd. Let $W = W(E_6)$. Given a weight θ of T under $\psi \in \mathrm{Irr}_{3'}(M)$, Clifford theory gives $3 \nmid [W^\delta : W_\theta]$. If $\delta = \gamma$, then $W^\delta = W(F_4)$ and W_θ contains a conjugate of c^4 from Theorem 7.20. Since $1 + c^4 + c^8 = 0$ on $X^*(\mathbf{T})$, $\theta^3 = 1$. If $\delta = 1$, choose c Coxeter in $W(E_6)$. The exponents $1, 4, 5, 7, 8, 11$ give $(c^4)^3 = 1$ with fixed space 0 for c^4. Thus, $1 + c^4 + c^8 = 0$ again gives $\theta^3 = 1$.

Let s be dual to θ. Since $-q \equiv 1 \bmod 3$, s is fixed by δ. If $\delta = \gamma$, Theorem 7.20 gives lengths $1, 32$ for the orbits with $3 \nmid [W^\delta : W_\theta]$. If $\delta = 1$, we choose the coordinates $x \in \mathbb{F}_3^6$ of s on the simple roots. The rule $s_i(x)_j = x_j - C_{ji}x_i$ gives orbit sizes $1, 27, 27, 216, 72, 216, 80, 90$, ordered by the least x for the orbit of x. Thus, $3 \nmid [W : W_\theta]$ gives lengths $1, 80$. We choose the node 4 representative s, with lift $\omega_4^\vee(\zeta_3)$ from [208, (10.23)]. This lift is fixed by $-q\delta$. Consequently, the characters in $\mathcal{E}(X, s)$ are trivial on $Z(X)$ and factor to $S = X/Z(X)$ and $M_S = M/Z(X)$.

By [208, (10.29)–(10.30)], $C^\circ(s)$ has rational factors $A_2(q^2), {}^2A_2(q)$ for $t = 0$ and ${}^2A_2(q)^3$ for $t = 1$. Since θ has order 3, its extension λ_θ is 1 on the rational Tits supplement V_θ, whose intersection with T lies in $T[2]$. In the fold $W_\theta = \mathfrak{S}_3^2$ gives nine $3'$-labels. In the unfolded case $W_\theta = \mathfrak{S}_3^3 \rtimes \langle u \rangle$, with u rotating the three factors. Thus, the $3'$-labels are the extensions of (i, i, i) for $i \in \mathrm{Irr}(\mathfrak{S}_3)$, as in Proposition 7.19.

Take D as in Theorem 7.24. If $t = 1$, choose the lift $\widetilde{\theta}$ of θ to the regular torus $\widetilde{T}$ dual to $\omega_4^\vee(\zeta_3)$. The stabilizer of $\widetilde{\theta}$ is $W(A_2)^3$ since $C_{\widetilde{\mathbf{G}}^*}(\tilde{s})$ is connected for the dual lift $\tilde{s}$. Thus, $\widetilde{\theta}^u/\widetilde{\theta}$ is a nontrivial character of $\widetilde{T}/T$, trivial on $Z(\widetilde{X})$. Its value on $d \in \widetilde{T}$ is the scalar change of $\lambda_\theta(n_u)$ under diagonal conjugation by d, up to inversion. For each $w \in W(A_2)^3$ the ratio $\widetilde{\theta}^w/\widetilde{\theta}$ is 1. Consequently, the three extensions of (i, i, i) form a free D-orbit. Thus, Ω of Theorem 7.24 pairs it with a D-orbit of size 3 on X.

Moreover, for $S \neq {}^2E_6(2)$, $\mathcal{E}(X, s) \cap \mathrm{Irr}_{3'}(X)$ is rational by [208, proof of Proposition 10.9, Steps 3–4]. If $S = {}^2E_6(2)$, the regular parent series $\mathcal{E}(\widetilde{X}, \tilde{s})$ is fixed by $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}(\zeta_3))$: its torus characters have order 3, and the unipotent labels of ${}^2A_2(2)^3$ are uniform, as in [208, proof of Proposition 10.9, Step 4]. Thus, σ fixing ζ_3 stabilizes each D-orbit in $\mathrm{Irr}(X)$. On the local cubes $U_i^{\otimes 3}$, the action of γ fixes each D-orbit since γ permutes the factors and conjugates u to u^{-1}. By Theorem 7.24, the global D-orbits are stable under both γ and $\gamma\kappa$. Hence, on each D-orbit the permutation κ has order dividing 2 and centralizes D, so $\kappa = 1$ there. Since each σ fixing ζ_3 centralizes D and γ, it fixes the unique γ-fixed member χ_0. Consequently, σ fixes χ_0^D pointwise. Thus, every $\chi \in \chi_0^D$ is rational for $q = 2$. On M_S, complex conjugation acts by $\theta \mapsto \theta^{-1}$ and inverts the character of $\langle u \rangle$ when $t = 1$. Moreover, for $v = w_0(E_6)w_0(A_2^3)$, we choose the rational lift n_v from [208, proof of Proposition 10.9, Step 7]. The lift gives $\theta^v = \theta^{-1}$ and opposition on the A_2 roots. In the fold, v fixes each rational factor. If $t = 1$, v induces $(2\,3)$ on the factors, so $vuv^{-1} = u^{-1}$. Since the tensor permutations and $\mathfrak{S}_3$-characters are rational, the character $\lambda_\theta\eta$ satisfies $(\lambda_\theta\eta)^\kappa = (\lambda_\theta\eta)^{n_v}$. Hence, $\psi^\kappa = \psi$ on M_S.

If $t = 0$, the pinned group E fixes each pair (χ, ψ) by [208, proof of Proposition 10.9, Step 4] and the equivariance of Ω. If $t = 1$, take the unique γ-fixed pair (χ_0, ψ_0) in each diagonal orbit. Since $\phi\gamma = \gamma\phi$, the pair (χ_0, ψ_0) is fixed by ϕ. If $t = 0$, the diagonal group is 1, and for $t = 1$ its stabilizer at (χ_0, ψ_0) is 1. Thus, the inertia quotients over S, M_S are $E \cong C_{2f}$ by [208, (10.2), §10.4 and Proposition 10.9, Step 4]. Given $E_f \leq E$ of order f, [208, Lemma 3.1] gives rational extensions of χ, ψ over E_f. Moreover, χ, ψ extend over E by [95, Corollary 11.22].

Let γ act in the marked model. Since n_0 is in the connected fixed group $\mathbf{G}^\gamma$ of type F_4, Lang conjugation can be taken in $\mathbf{G}^\gamma$. Consequently, it fixes γ under conjugation. If $t = 0$, the graph acts trivially on $W_\theta = \mathfrak{S}_3^2$. Hence, we extend λ_θ and the two Weyl modules by 1 on γ. If $t = 1$, choose $U_i^{\otimes 3}$ with trivial rotation label. The generator u acts by the permutation $(1\,2\,3)$ of the tensor factors, and U_i on each arm extends along $\mathrm{Aut}(\Phi(A_2)) = W(A_2) \times \langle -1 \rangle \to W(A_2)$. Thus, the graph action on $U_i^{\otimes 3}$ has square 1. Tensor with λ_θ extended by 1 on γ and induce on $M_S\langle\gamma\rangle$.

Since the Weyl and graph action R is rational, only θ can change under $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. If $\theta^\sigma = \theta^{-1}$, $\theta^\sigma = \theta^{n_v}$ and n_v centralizes γ. Thus, $\lambda_\theta^\sigma = \lambda_\theta^{n_v}$ on $V_\theta\langle\gamma\rangle$ together with T. On $U_i^{\otimes 3}$ for $t = 1$, the matrix of v gives the conjugated model and satisfies $R(v)R(\gamma) = R(\gamma)R(v)$. If $t = 0$, v acts by inner opposition on $\mathfrak{S}_3^2$, and γ acts by 1. Thus, the extension $\widehat{\psi}_\gamma$ on $M_S\langle\gamma\rangle$ is rational. The other extension is $\widehat{\psi}_\gamma\nu$ for the graph sign ν. Consequently, the rational extensions over E_f and $\langle\gamma\rangle$ give a rational extension $\widehat{\psi}$ over E by [156, Lemma 7.1].

For the extension of χ over $\langle\gamma\rangle$, take the ordered simple systems of A_2^3 from [208, proof of Proposition 10.9, Step 7]. On these systems, γ acts by $(2\,3)$. Consequently, put $c = c_1c_2c_3$ with c_i the Coxeter element for pair i. Since $\gamma(c) = c$, the canonical Tits lift n_c is fixed by γ and F_r. Moreover, $n_cn_0 = n_0n_c$ since $\gamma(c) = c$. Thus, we put $F_c = \mathrm{Int}(n_cn_0)F$, with Lang conjugation in $\mathbf{G}^\gamma$, since $n_cn_0 \in \mathbf{G}^\gamma$. The character ϑ of $T_c = \mathbf{T}^{F_c}$ dual to the node 4 lift is cubic and γ-fixed. Moreover, $n_0 \in \mathbf{G}^{F_c}$ gives $\vartheta^{n_0} = \vartheta^{-1}$ and $\gamma(n_0) = n_0$.

In $\mathbf{G} \rtimes \langle\gamma\rangle$, we choose the disconnected torus $\mathbf{T} \rtimes \langle\gamma\rangle$, with fixed points $T_c \rtimes \langle\gamma\rangle$. Since $\vartheta^\gamma = \vartheta$, we extend it to

$\widehat{\vartheta}$ with $\widehat{\vartheta}(\gamma)=1$. By [56, Definitions 2.4 and 3.1], the virtual character $V=R_{\mathbf{T}\rtimes\langle\gamma\rangle}^{\mathbf{G}\rtimes\langle\gamma\rangle}(\widehat{\vartheta})$ restricts to $R_{\mathbf{T}}^{X}(\vartheta)$. Indeed, γ acts on the image of $p_\vartheta=|T_c|^{-1}\sum_{t\in T_c}\vartheta(t)^{-1}t$ on the alternating cohomology. Given $x\in X\langle\gamma\rangle$ and $t\in T_c$, (x,t) has finite order on the variety Y. By [53, Proposition 3.3], the alternating trace of (x,t) on $H_c^*(Y)$ is $\tau(x,t)\in\mathbb{Z}$. Consequently, $V^\sigma=R_{\mathbf{T}\rtimes\langle\gamma\rangle}^{\mathbf{G}\rtimes\langle\gamma\rangle}(\widehat{\vartheta}^\sigma)$ because σ fixes $\tau(x,t)$. If $\vartheta^\sigma=\vartheta^{-1}$, $\widehat{\vartheta}^\sigma=\widehat{\vartheta}^{n_0}$, since n_0 normalizes $\mathbf{T}$ and fixes γ. Write $N_c=N_{\mathbf{G}\rtimes\langle\gamma\rangle}(\mathbf{T})^{F_c}$. Thus, the ordinary inductions of $\widehat{\vartheta}^\sigma,\widehat{\vartheta}$ to N_c are equal. By [56, Proposition 3.2], $\langle V^\sigma-V,V^\sigma-V\rangle=0$. Consequently, $V^\sigma=V$ on $X\langle\gamma\rangle$, also for field characteristic 2.

Let $\widetilde{\vartheta}$ on $\widetilde{T}$ lift ϑ with dual $\tilde{s}=\omega_4^\vee(\zeta_3)$. Thus, the regular inclusion gives $R_{\widetilde{\mathbf{T}}}^{\widetilde{X}}(\widetilde{\vartheta})|_X=R_{\mathbf{T}}^{X}(\vartheta)$. For χ, choose $\widetilde{\chi}\in\mathcal{E}(\widetilde{X},\tilde{s})$ by [10, Theorem 2.35]. Moreover, all parents of χ are twists by $\mathrm{Irr}(\widetilde{X}/X)$. The stabilizer of $\mathcal{E}(\widetilde{X},\tilde{s})$ in $\mathrm{Irr}(\widetilde{X}/X)$ has order $|C(s)^{F_c^*}/C^\circ(s)^{F_c^*}|$ by [10, Fact 2.29]. Since $c\in W(C^\circ(s))$, the twist by c fixes the components $C(s)/C^\circ(s)$. Thus, the order is 1 when $t=0$ and 3 when $t=1$, by [208, (10.29)–(10.30)]. Since χ^D has size 1 or 3, respectively, Theorem 7.24 gives this order for $\mathrm{Irr}(\widetilde{X}/X)_{\widetilde{\chi}}$. Thus, each twist ν with $\mathcal{E}(\widetilde{X},\tilde{s})\nu=\mathcal{E}(\widetilde{X},\tilde{s})$ has $\widetilde{\chi}\nu=\widetilde{\chi}$. Moreover, $\widetilde{\chi}|_X$ contains χ with multiplicity 1. Consequently, $\langle V|_X,\chi\rangle_X=\langle R_{\widetilde{\mathbf{T}}}^{\widetilde{X}}(\widetilde{\vartheta}),\widetilde{\chi}\rangle_{\widetilde{X}}$.

By [10, Theorem 2.25], the coefficient is $\pm\langle R_{\widetilde{\mathbf{T}}^*}^{C(\tilde{s})}(1),u_{\widetilde{\chi}}\rangle$. On a fixed A_2 factor, F_c^* acts by $-qc_i$. Moreover, on a pair (A_2,A_2) exchanged by F_c^*, the square $(F_c^*)^2$ acts by $q^2c_i^2$ on one factor. Since c_i has polynomial x^2+x+1, the derived torus orders are $\det(1+qc_i)=q^2-q+1$ and $\det(1-q^2c_i^2)=q^4+q^2+1$, respectively. By the type A formula [40, §13.8], each factor has coefficient $\pm\eta_i(c_i)$ for $\eta_i\in\mathrm{Irr}(\mathfrak{S}_3)$. The three values at a 3-cycle are $1,1,-1$. Thus, the product for $u_{\widetilde{\chi}}$ is ±1 in both rational forms and $\langle V|_X,\chi\rangle_X=\pm1$. Let χ_+,χ_- be the two extensions to $X\langle\gamma\rangle$. Since V is rational, if σ exchanged χ_+,χ_- then $\langle V,\chi_+\rangle=\langle V,\chi_-\rangle$. Their sum is $\langle V|_X,\chi\rangle_X=\pm1$. Hence, χ_+,χ_- are rational. By [156, Lemma 7.1], the extensions over E_f and $\langle\gamma\rangle$ give a rational extension $\widehat{\chi}$ over E. Moreover, for $t=1$, conjugation by $d\in D$ gives the rational extensions $\widehat{\chi}^d,\widehat{\psi}^d$ of (χ^d,ψ^d).

Given a pair (χ,ψ), let $A=\mathrm{Aut}(X)_{P,\chi}$. Since χ,ψ are rational, the mixed inertia on A is $A\times\mathcal{H}_3$. By Theorem 7.24, Sylow correction in M gives the maps $\rho(x,a)=\mathrm{Int}_S(xZ(X))\bar{a}$ on $X\rtimes A$ and $M\rtimes A$. Thus, $\rho(X\rtimes A)$ and $\rho(M\rtimes A)$ lie in the marked inertias of χ,ψ, respectively. Consequently, composition with ρ gives extensions of χ,ψ fixed by $\mathcal{H}_3$. The centralizer of X in $X\rtimes A$ lies in $M\rtimes A$, and its image centralizes S in $\mathrm{Aut}(S)$, so it maps to 1. Thus, $\alpha=\alpha'=1$ and $\mu_a=\mu_a'=1$ in [141, Definition 1.5], with centralizer scalar 1. Finally, the result of [156, Theorem D] gives the comparison on M for $s=1$, completing the comparison for X,M. If $S={}^2E_6(2)$, the map from the universal cover U to X has kernel C_2^2 by [208, §10.4]. Thus, we apply [157, Lemma 4.1(2)] for $\mathrm{iMN}(S,3)$ on U. □

Lemma 7.28. *With $X={}^2E_{6,\mathrm{sc}}(q)$ and $3\nmid q+1$, we let $\ell\geq5$ divide $q+1$. Then, we obtain that $Z(X)=1$ and the diagonal quotient is* 1. *Take T,M from* Lemma 7.21 *with torus action $-q$ and $W=M/T=W(E_6)$. We restrict Ω of* Theorem 7.24 *to the ℓ'-characters $\chi,\psi=\Omega(\chi)$ considered below. Suppose that $W_\theta=1$. If $\gamma(W\theta)=W\theta$, the graph extensions of χ and of ψ are in bijection. The bijection $\widehat{\chi}\leftrightarrow\widehat{\psi}$ is equivariant under* $\mathrm{Irr}(\langle\gamma\rangle)$ *and under* $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})_{W\theta}$. *If $f=1$, we have that every regular pair (χ,ψ) satisfies $\succeq_c$.*

Proof. Put $\chi=eR_{\mathbf{T}}^{X}(\theta)$, with $e=\pm1$ giving positive degree, and $\psi=\mathrm{Ind}_T^M\theta$. Since $W_\theta=1$, both χ and ψ are irreducible and depend only on $W\theta$, by [53, Theorem 6.8] and Clifford theory. Since $\psi(1)=|W|$, these weights occur for $\ell\geq7$ and are excluded for $\ell=5$ from the ℓ'-domain. Let s be dual to θ. Since $s^{q+1}=1$ and $3\nmid q+1$, the lift $\tilde{s}$ with $\tilde{s}^{q+1}=1$ is unique. Thus, the projection $W\tilde{s}\to Ws$ is a γ-equivariant bijection. If $\gamma(W\tilde{s})=W\tilde{s}$, the representative $\tilde{s}$ in the closed alcove is γ-fixed by [208, proof of Proposition 10.5]. Since $M/T=W$, we may assume that $\theta^\gamma=\theta$. Moreover, Lang conjugation by $g\in\mathbf{G}^\gamma$ gives a γ-stable Borel ${}^g\mathbf{B}$ through ${}^g\mathbf{T}$ as in Theorem 7.27.

With $A=X\rtimes\langle\gamma\rangle$, we put $B=M\rtimes\langle\gamma\rangle$ and $D=T\rtimes\langle\gamma\rangle$. Given an extension η of θ to D, take ordinary induction $J_B(\eta)$ and disconnected torus induction $J_A(\eta)$. Then, $J_B(\eta)|_M=\psi$ and $J_A(\eta)|_X=R_{\mathbf{T}}^{X}(\theta)$. By [56, Proposition 3.2], $\|J_A(\eta)\|^2=\|J_B(\eta)\|^2=1$, so $eJ_A(\eta)$ extends χ. The component formula gives $J_Y(\eta\nu)=J_Y(\eta)\nu$ for $Y=A,B$ and each sign ν on $\langle\gamma\rangle$. By [56, Proposition 3.2], $J_A(\eta_1)=J_A(\eta_2)$ if and only if $J_B(\eta_1)=J_B(\eta_2)$. The same criterion holds with η_2 over θ^w for $w\in W$ when θ^w is γ-fixed. Given $\sigma\in\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, Theorem 7.27 gives that $J_A(\eta)^\sigma=J_A(\eta^\sigma)$ and $J_B(\eta)^\sigma=J_B(\eta^\sigma)$. Thus, $eJ_A(\eta)\leftrightarrow J_B(\eta)$ is equivariant under the sign action $\eta\mapsto\eta\nu$ and the Galois action $\eta\mapsto\eta^\sigma$.

If $f=1$, the outer group is $\langle\gamma\rangle$. If $\chi^\gamma\neq\chi$, the ordinary inertia groups are X,M. Hence, on $X/X=M/M=1$ the twist characters are $\mu=\mu'=1$. In both cases, let $P\in\mathrm{Syl}_\ell(X)$ with $P\leq M$ and $A_\chi=\mathrm{Aut}(X)_{P,\chi}$. By Theorem 7.24, the maps $\rho(x,a)=\mathrm{Int}(x)a$ on $X\rtimes A_\chi$ and $M\rtimes A_\chi$ have images in $(X\rtimes\langle\gamma\rangle)_\chi$ and $(M\rtimes\langle\gamma\rangle)_\psi$, respectively. Since $N_X(P)\leq M$, $C_{X\rtimes A_\chi}(X)\leq M\rtimes A_\chi$. Moreover, ρ maps the centralizer to 1 since $Z(X)=1$. Thus, on $C_{X\rtimes A_\chi}(X)$

both representations have scalar 1. □

Lemma 7.29. *Let $R = \{\alpha : \alpha^*(s) = 1\}$ for the element s dual to θ. Then, we find $W_\theta = W(R)$. If $W\theta$ is γ-stable, we choose $\theta^\gamma = \theta$ and extend θ by 1 on γ. Then, J_B, J_A in* Lemma 7.28 *are fixed by $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ stabilizing $W\theta$. In particular, if $\chi^\gamma = \chi^\sigma = \chi$ and $\langle J_A|_X, \chi\rangle_X$ is odd, both graph extensions of χ are σ-fixed. The analogous claim holds for ψ with coefficient $\langle J_B|_M, \psi\rangle_M$.*

Proof. Choose the lift $\tilde{s}$ of s of order dividing $q+1$. If an element centralizes s, its lift acts by $\tilde{s} \mapsto \tilde{s}z$ with z in the central kernel of order 3. Since $|\tilde{s}|$ is prime to 3, $|\tilde{s}| = |\tilde{s}z|$ gives $z = 1$. Thus, $C(\tilde{s})$ surjects onto $C(s)$, and $C(\tilde{s})$ is connected by Steinberg's theorem [58, Chapter 3]. Thus, $C(s)$ is connected and $W_\theta = W(R)$. By Lemma 7.28, we may choose $\theta^\gamma = \theta$ also on an alcove wall containing $\tilde{s}$.

Suppose that $\theta^\sigma \in W\theta$. The root subsystem R is constant on generators of $\langle s\rangle$. The w with $\theta^w = \theta^\sigma$ form a coset of $W(R)$, so exactly one such w fixes R^+, since $W(R)$ is simply transitive on $\{vR^+ : v \in W(R)\}$. Since γ fixes $\theta, \theta^\sigma, R^+$, it fixes w. Thus, the rational Tits lift n_w satisfies $n_w^\gamma = n_w$. In particular, $\eta^{n_w} = \eta^\sigma$ for the extension η with $\eta(\gamma) = 1$. Consequently, $J_B^\sigma = J_B$. For J_A, [56, Proposition 3.2] and Lemma 7.28 give that $J_A^\sigma = J_A$. If σ swaps the two graph extensions χ_+, χ_- of χ, $\langle J_A, \chi_+\rangle = \langle J_A, \chi_-\rangle$. Thus, $\langle J_A|_X, \chi\rangle_X$ is even. With J_B and ψ, the identical calculation gives that $\langle J_B|_M, \psi\rangle_M$ is even if σ exchanges the extensions of ψ.

If $R = A_1^k$, each $\eta \in \mathrm{Irr}(W(R))$ has degree 1, so its multiplicity in $\mathrm{Ind}_T^M \theta$ is 1 by the extension map Λ in Proposition 7.22. By Corollary 7.23, we choose $\widetilde{\chi} = \chi \boxtimes 1$ on $\widetilde{X} = X \times Z(\widetilde{X})$ and $\widetilde{\theta} = \theta \boxtimes 1$ on $\widetilde{T} = T \times Z(\widetilde{X})$. Thus, $\langle J_A|_X, \chi\rangle_X$ equals $\langle R_{\widetilde{\mathbf{T}}}^{\widetilde{X}}(\widetilde{\theta}), \widetilde{\chi}\rangle_{\widetilde{X}}$. By [10, Theorem 2.25], its absolute value is $|\langle R_{\widetilde{\mathbf{T}}^*}^{C(\tilde{s})}(1), u_{\widetilde{\chi}}\rangle|$ in $\widetilde{\mathbf{G}}^*$, with $\tilde{s}$ dual to $\widetilde{\theta}$. Since $-q$ fixes the k factors of $R = A_1^k$, this is $\prod_{i=1}^k |\langle R_i, u_i\rangle|$, where R_i is nonsplit torus induction and $u_i \in \{1, \mathrm{St}\}$. By the type A formula [40, §13.8], each factor is 1. Therefore, both coefficients are odd when $R = A_1^k$. More generally, for a unitary factor A_{n-1}, a torus of cycle type 1^n contributes $\eta(1)$ in absolute value for $\eta \in \mathrm{Irr}(\mathfrak{S}_n)$ by [40, §13.8]. Thus, when $\langle J_A|_X, \chi\rangle_X$ and $\langle J_B|_M, \psi\rangle_M$ are odd, the extensions are σ-fixed. □

Proposition 7.30. *Let $q = r^f$. Suppose that the torus label θ has $W_\theta = 1$ and that $\ell \nmid \psi(1)$. Then, the matched pair satisfies $\chi \succeq_c \psi$.*

Proof. Let $F = \gamma F_r^f$. As in Lemma 7.28, take $g \in \mathbf{G}^\gamma$ with $g^{-1}F(g) = n_0$, where n_0 is from Lemma 7.21. In particular, $\mathbf{B}_0 = {}^g\mathbf{B}$ is a γ-stable Borel through $\mathbf{T}_0 = {}^g\mathbf{T}$. On $\mathbf{G}^f$, we put $u(x_0, \ldots, x_{f-1}) = (x_1, \ldots, x_{f-1}, \gamma(x_0))$ and put $\widehat{F} = u^{-1}\,\mathrm{diag}(F_r)$. Then, we extend $\widehat{F}$ to $\mathbf{G}^f \rtimes \langle u\rangle$ by $\widehat{F}(u) = u$. By [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], $X \cong (\mathbf{G}^f)^{\widehat{F}}$ via $x \mapsto (x, F_r(x), \ldots, F_r^{f-1}(x))$. Moreover, u induces F_r on X with $u^f = \gamma$. Thus, $\mathbf{Q} = \prod_{j=0}^{f-1} F_r^j(\mathbf{T}_0)$ is $\widehat{F}$-stable with $N_{\mathbf{G}^f}(\mathbf{Q})^{\widehat{F}} = M$. Let $\widehat{M} = N_{\mathbf{G}^f \rtimes \langle u\rangle}(\mathbf{Q})^{\widehat{F}}$. Since $F_r(n_0) = n_0$, $gF_r(g)^{-1} \in X$. Moreover, $\mathrm{Int}(gF_r(g)^{-1})F_r$ normalizes $\mathbf{T}_0$. Hence, $\widehat{M}/M \cong E = \langle u\rangle$.

Let $H \le E$ stabilize $W\theta$, and let I be the stabilizer of θ in the preimage of H in $\widehat{M}$. Since $W_\theta = 1$, $I \cap M = T$ and $I/T \cong H$. With $h = |H|$, choose $a \in I$ above $u^{2f/h}$. If h is odd, the shift by $2f/h$ has cycles of length h and $a^h \in T$. Given a cycle with initial torus $F_r^j(\mathbf{T}_0)$, we put $\mathbf{B}_j = F_r^j(\mathbf{B}_0)$. Thus, $a^k(\mathbf{B}_j)$ for $0 \le k < h$ are Borels in the cycle, and $a^h(\mathbf{B}_j) = \mathbf{B}_j$. If h is even, then $u^f \in H$ and we choose $\theta^\gamma = \theta$ by Lemma 7.28. Since $a^{h/2}$ and u^f both fix θ and lie over u^f, $a^{h/2}(u^f)^{-1} \in T$. The shift cycles have length $h/2$. Given any initial torus $F_r^j(\mathbf{T}_0)$, the Borels are $a^k(F_r^j(\mathbf{B}_0))$ for $0 \le k < h/2$. Since u^f acts by γ, the element $a^{h/2}$ fixes $F_r^j(\mathbf{B}_0)$. Thus, the product $\mathbf{P}$ of the Borels satisfies $a(\mathbf{P}) = \mathbf{P}$ for either parity of h.

With $\mathbf{A} = \mathbf{G}^f \rtimes H$, we put $\mathbf{D} = N_{\mathbf{A}}(\mathbf{Q}, \mathbf{P})$. Since $\mathbf{Q}$ and a lie in $\mathbf{D}$, $\mathbf{D} = \mathbf{Q}I$ and $\mathbf{D}^{\widehat{F}} = I$. Each extension η of θ has $\eta(a)^h = \theta(a^h)$. Moreover, for $B_H = MI$ and $A_H = X \rtimes H$, [95, Problem 5.2] gives that $(\mathrm{Ind}_I^{B_H} \eta)|_M = \psi$. By [56, Definitions 2.4 and 3.1], $R_{\mathbf{D}}^{\mathbf{A}}(\eta)|_X = R_{\mathbf{Q}}^X(\theta)$ in $\mathbf{G}^f$. For the connected induction $R_{\mathbf{Q}}^X$, choose $\prod_j F_r^j(\mathbf{B}_0)$ by [58, Chapter 9]. Then, projection p_0 to the first coordinate gives that $R_{\mathbf{Q}}^X(\theta) = R_{\mathbf{T}_0}^X(\theta)$ by [208, proof of Lemma 4.19(b)]. Since $\|R_{\mathbf{D}}^{\mathbf{A}}(\eta)\|^2 = 1$ by [56, Proposition 3.2], $eR_{\mathbf{D}}^{\mathbf{A}}(\eta)$ extends χ. Moreover, multiplication of η by $\nu \in \mathrm{Irr}(H)$ multiplies both extensions by ν inflated to A_H, B_H.

Let $y \in \widehat{M}$ and $(\chi, \psi)^{y\sigma} = (\chi, \psi)$. Then, $\theta^{y\sigma} \in W\theta$, so we choose $m \in M$ with $\theta^{ym\sigma} = \theta$. Since θ^σ and θ have equal inertia in $\widehat{M}$, $I^{ym} = I$. Moreover, $\mathbf{Q}^{ym} = \mathbf{Q}$, so $(\mathbf{Q}I)^{ym} = \mathbf{Q}I$. Thus, $\eta^{ym\sigma} = \eta\nu$ for some $\nu \in \mathrm{Irr}(H)$. By Lemma 7.28, the trace for $\eta^{ym\sigma}$ is computed with the Borel $\mathbf{P}^{ym}$. Moreover, [56, Proposition 3.2] gives norm 1 and mutual scalar product 1 for the inductions of $\eta^{ym\sigma}$ with $\mathbf{P}^{ym}$ and $\mathbf{P}$. Consequently, the twist characters μ, μ' satisfy $\mu = \mu' = \nu$ on H. Conjugation by $m \in M$ fixes both extensions, since $M \le B_H$ and $M \le X \le A_H$. Finally,

composition with ρ from Lemma 7.28 gives genuine extensions on $X \rtimes A_\chi, M \rtimes A_\chi$ with scalar 1 on $C_{X \rtimes A_\chi}(X)$. Thus, (χ, ψ) satisfies $\succeq_c$. □

Lemma 7.31. *Let $T_0 = \mathbf{T}_0^F$ in the marking of* Proposition 7.30. *Let $H \le E = \langle u \rangle$ stabilize the pair (χ, ψ) on the outer quotient. Thus, $h = |H|$. When $2 \mid h$, we choose $\theta^\gamma = \theta$ by* Lemma 7.29. *Then, $I_H = (\widehat{M}_H)_\theta$, where $\widehat{M}_H$ is the preimage of H in $\widehat{M}$. We compare the extensions under (y, σ) with $(\chi, \psi)^{y\sigma} = (\chi, \psi)$. Let $N \lhd D$ with $D/N = H$ cyclic of order h. Suppose that $\xi \in \mathrm{Irr}(N)$ is D-invariant and Y is a virtual character of D with $m = \langle Y|_N, \xi \rangle_N$ prime to h. For $am \equiv 1 \bmod h$ and an extension L of ξ, we put $L_Y = L \prod_{\delta \in \mathrm{Irr}(H)} \delta^{\, a\langle Y, L\delta \rangle_D}$. Then, we have that L_Y is independent of L. The map $(Y, \xi) \mapsto L_Y$ is equivariant under isomorphisms of D, N and under* $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. *Moreover, we have that $L_{Y\delta_0} = L_Y \delta_0$ for $\delta_0 \in \mathrm{Irr}(H)$.*

Proof. By [95, Corollaries 6.17 and 11.22], $\mathrm{Irr}(D \mid \xi) = \{L\delta : \delta \in \mathrm{Irr}(H)\}$. Thus, $\sum_\delta \langle Y, L\delta \rangle_D = m$. Replacing L by $L\delta_1$ multiplies the product by δ_1^{-am}, so L_Y is independent of L. If Y is replaced by $Y\delta_0$, then the product gains $\delta_0^{am} = \delta_0$. Moreover, an isomorphism or Galois automorphism τ acts by $\delta \mapsto \delta^\tau$ and $\langle Y^\tau, L^\tau \delta^\tau \rangle = \langle Y, L\delta \rangle_D$. Thus, $(L_Y)^\tau = (L^\tau)_{Y^\tau}$ for ξ^τ. If $h = 1$, the formula gives $L_Y = \xi$. □

Proposition 7.32. *If $R = A_1^k$ with $k \ge 1$, we have that every matched pair (χ, ψ) of ℓ'-degree satisfies $\succeq_c$.*

Proof. Let n_0, g be as in Proposition 7.30 and $\Phi = gF_r(g)^{-1}u \in \widehat{M}$. In the marking through g, the action of Φ on T_0 is the rth power, and Φ fixes the pinned Tits lifts. Since $F_r^f(g) = gn_0$, $\Phi^f = (gn_0^{-1}g^{-1})u^f$ and $\Phi^{2f} \in T_0$. Thus, $\widehat{M}/T_0 = W \times \langle \Phi T_0 \rangle$. Given $e \in H$, write Φ_e for the power of ΦT_0 above e. Then, θ^{Φ_e} has root system R since r is prime to $|s|$ for s dual to θ. Let $R^+ = \{\alpha \in R : U_\alpha \le \mathbf{B}_0\}$. Since $W_\theta = W(R)$ by Lemma 7.29, we find exactly one $w(e) \in W$ with $\theta^{w(e)\Phi_e} = \theta$ and $w(e)R^+ = R^+$. Uniqueness gives $w(ee') = w(e)w(e')$. Consequently, the preimage J_0 of $\{w(e)\Phi_e : e \in H\}$ satisfies $J_0/T_0 \cong H$.

Let $\mathbf{K} = \langle \mathbf{T}_0, U_\alpha : \alpha \in R \rangle$ and $K = \mathbf{K}^F$. Since R is closed, $\mathbf{K}$ is connected reductive by [58, §3.3]. $F(\mathbf{K}) = \mathbf{K}$ since F acts as $-q$ on the character lattice of $\mathbf{T}_0$. Moreover, I_H normalizes K and $K \cap I_H = N_{\mathbf{K}}(\mathbf{T}_0)^F$ maps onto $W(R)$. Thus, $L_H = KI_H$ has quotient $L_H/K \cong H$. The element $s \in \mathbf{K}^*$ dual to θ is central because $\alpha^*(s) = 1$ for every $\alpha \in R$. By [58, Proposition 11.4.12], s defines a linear character λ of K, with $\lambda|_{T_0} = \theta$. On every rational maximal torus $\mathbf{T}' \le \mathbf{K}$, the character $\lambda|_{(\mathbf{T}')^F}$ is dual to s. In particular, torus duality gives that $\theta^a = \theta$ implies $\lambda^a = \lambda$ for $a \in I_H$. Since L_H/K is cyclic, take a linear extension $\widehat{\lambda}$ of λ to L_H by [95, Corollary 11.22].

Choose an F-stable torus and Borel pair $(\mathbf{T}_1, \mathbf{B}_1)$ in $\mathbf{K}$. Take $\mathbf{B}_R = \langle \mathbf{T}_0, U_\alpha : \alpha \in R^+ \rangle$. Since F acts by $-q$ on R, a representative n_v of $v = w_0(R)$ satisfies ${}^{n_v}F(\mathbf{B}_R) = \mathbf{B}_R$. If $2 \nmid h$, we choose $k_1 \in \mathbf{K}$ with $k_1^{-1}F(k_1) = n_v$, by Lang's theorem. If $2 \mid h$, then $u^f \in H$ and $\gamma(R^+) = R^+$ since $\theta^\gamma = \theta$. If $\gamma(\alpha) = \alpha$, the rank-one group $\langle U_\alpha, U_{-\alpha} \rangle$ is fixed pointwise by γ, by [208, proof of Proposition 10.5] for the long roots of F_4. If $\gamma(\alpha) \ne \alpha$, the two rank-one groups $K_\alpha^{\mathrm{alg}} = \langle U_\alpha, U_{-\alpha} \rangle$ and $K_{\gamma(\alpha)}^{\mathrm{alg}} = \gamma(K_\alpha^{\mathrm{alg}})$ are factors of $[\mathbf{K}, \mathbf{K}]$ from distinct components of R. Thus, the image of $x \mapsto x\gamma(x)$ from K_α^{alg} contains a representative of $s_\alpha s_{\gamma(\alpha)}$. Consequently, v has a representative $n_v \in (\mathbf{K}^\gamma)^\circ$. By Lang's theorem on $(\mathbf{K}^\gamma)^\circ$, we choose k_1 in that group with $k_1^{-1}F(k_1) = n_v$. In both cases, we put $(\mathbf{T}_1, \mathbf{B}_1) = {}^{k_1}(\mathbf{T}_0, \mathbf{B}_R)$. Then, $(\mathbf{T}_1, \mathbf{B}_1)$ is F-stable and, when $2 \mid h$, fixed by γ. Moreover, the ambient Borel ${}^{k_1}\mathbf{B}_0$ through $\mathbf{T}_1$ is γ-fixed when $2 \mid h$.

In $\mathbf{G}^f$ of Proposition 7.30, we let $J_1 \le L_H$ stabilize the product of the pairs $F_r^j(\mathbf{T}_1, \mathbf{B}_1)$. The first projection p_0 gives $J_1 \cap K \cong T_1 = \mathbf{T}_1^F$. Moreover, the F-stable pairs in $\mathbf{K}$ are K-conjugate since their stabilizer $\mathbf{T}_1$ is connected, by Lang's theorem. Thus, $J_1/T_1 \cong H$. If $2 \mid h$, $u^f \in J_1$. Consequently, if $a \in J_1$ maps to a generator of H, the element closing the cycle is $a^h \in T_1$ for odd h and $a^{h/2} \in T_1 u^f$ for even h. Given a cycle beginning in factor j, take the Borels $a^i(F_r^j({}^{k_1}\mathbf{B}_0))$ as in Proposition 7.30. Thus, for $\mathbf{Q}_1 = \prod_j F_r^j(\mathbf{T}_1)$ and the product Borel $\mathbf{P}_1$, the disconnected torus $\mathbf{D}_1 = N_{\mathbf{G}^f \rtimes H}(\mathbf{Q}_1, \mathbf{P}_1)$ has finite group $J_1 = \mathbf{D}_1^{\widehat{F}}$.

Put $A_H = X \rtimes H$, $B_H = MI_H$ and let $Y_X = R_{\mathbf{D}_1}^{\mathbf{G}^f \rtimes H}(\widehat{\lambda}|_{J_1})$ and $Y_M = \mathrm{Ind}_{J_0}^{B_H}(\widehat{\lambda}|_{J_0})$. By [95, Problem 5.2], $Y_M|_M = \mathrm{Ind}_{T_0}^M \theta$, so $\langle Y_M|_M, \psi \rangle_M = 1$ by Proposition 7.22 since $\eta(1) = 1$ for every $\eta \in \mathrm{Irr}(W(R))$. Similarly, $Y_X|_X = R_{\mathbf{T}_1}^X(\lambda|_{T_1})$ by [56, Definitions 2.4 and 3.1] and [208, proof of Lemma 4.19(b)]. Since $\mathbf{T}_1$ is a torus of $\mathbf{K}$, the weight $\lambda|_{T_1}$ is dual to s in the torus of $C_{\mathbf{G}^*}(s)$ corresponding to $\mathbf{T}_1$. Thus, the geometric class is $s^{\mathbf{G}^*}$. Since $C_{\mathbf{G}^*}(s)$ is connected by Lemma 7.29, it gives $s^{(\mathbf{G}^*)^{F^*}}$ as the rational class. By [169, Theorem 2.1(1)], $|\langle Y_X|_X, \chi \rangle_X|$ is the product of $|\langle R_i, u_i \rangle|$ on the factors A_1, where R_i is torus induction and u_i a unipotent character. Each torus on A_1 is split since $\mathbf{B}_1$ is F-stable. Hence, the coefficients of $1, \mathrm{St}$ both have absolute value 1 by [40, §13.8]. Thus, $|\langle Y_X|_X, \chi \rangle_X| = 1$.

Let $y \in \widehat{M}$ and $(\chi,\psi)^{y\sigma} = (\chi,\psi)$. Since $\psi^{y\sigma} = \psi$, Clifford theory gives that $\theta^{y\sigma} \in W\theta$. Set $z = ym$ with $m \in M$ such that $\theta^{z\sigma} = \theta$ and $z(R^+) = R^+$. Such m exists since $W(R)$ is simply transitive on $\{vR^+ : v \in W(R)\}$. Thus, z normalizes K, I_H, J_0, L_H, by uniqueness of $w(e)$ and because z acts trivially on E. Since $\lambda^{z\sigma} = \lambda$, $\widehat{\lambda}^{z\sigma} = \widehat{\lambda}\nu$ for some $\nu \in \mathrm{Irr}(H)$. Therefore, $Y_M^{z\sigma} = Y_M\nu$. For Y_X, choose $b \in K$ so that zb fixes the subsystem pair $(\mathbf{T}_1, \mathbf{B}_1)$. Then, zb normalizes J_1, and $\widehat{\lambda}^{zb\sigma} = \widehat{\lambda}\nu$ since $b \in L_H$. With $\mathbf{P}_1^{zb}$, the connected torus is still $\mathbf{Q}_1$ and the finite group is J_1. Write V, V' for the inductions with $\mathbf{P}_1, \mathbf{P}_1^{zb}$, respectively, and character $(\widehat{\lambda}\nu)|_{J_1}$. By [56, Proposition 3.2], $\langle V, V\rangle = \langle V, V'\rangle = \langle V', V'\rangle$, so $V = V'$. Thus, by Theorem 7.27, the traces on $\mathbf{P}_1^{zb}$ give that $Y_X^{zb\sigma} = Y_X\nu$. Since $b, m \in X \le A_H$ and $m \in M \le B_H$, $Y_X^{y\sigma} = Y_X\nu$ and $Y_M^{y\sigma} = Y_M\nu$. By Lemma 7.31, we let $\widehat{\chi}, \widehat{\psi}$ be the extensions from Y_X, Y_M, respectively. Thus, their twists on H are $\mu = \mu' = \nu$. Finally, composition with ρ as in Proposition 7.30 gives $(\chi,\psi) \succeq_c$ with centralizer scalar 1. At $\ell = 5$, there are no such pairs because $\psi(1) = [W(E_6) : W(R)]$ is divisible by 5. □

With $X = {}^2E_6(q)$ and $3 \nmid q+1$, we let ℓ be an odd prime dividing $q+1$. We fix T_0, M, θ, H as in Proposition 7.32. Suppose that $R = \prod_i R_i$ with R_i of type A_{n_i-1}. Since F acts as $-q$ on the marked lattice, $F(R_i) = R_i$. The construction of $K, I_H, L_H, \lambda, \widehat{\lambda}$ in Proposition 7.32 depends on only $W_\theta = W(R)$, the closed subsystem R, and centrality of s in the dual of $\mathbf{K}$. Thus, we have that $K, I_H, L_H, \lambda, \widehat{\lambda}$ are defined for $R = \prod_i R_i$.

Lemma 7.33. *There is an extension $\widehat{\psi}$ to MI_H whose twist under each (y,σ) fixing (χ,ψ) equals the twist of $\widehat{\lambda}$ on H. Then, $(\chi,\psi) \succeq_c$ follows if an extension $\widehat{\chi}_t$ has the given twist on each $H_t \in \mathrm{Syl}_t(H)$.*

Proof. If $D_R = \{w \in N_W(R) : wR^+ = R^+\}$, $N_W(R) = W(R) \rtimes D_R$, since $W(R)$ acts simply transitively on $\{vR^+ : v \in W(R)\}$. Let $\eta = \boxtimes_i \eta_{\mu_i} \in \mathrm{Irr}(W(R))$ be the label of ψ, with $\mu_i \vdash n_i$. For the simply connected cover $i_R\colon \mathbf{H}_R \to \mathbf{K}_{\mathrm{der}}$, the dual image of s is 1 since $s \in Z(\mathbf{K}^*)$. By [169, Proposition 4.6], $\lambda \circ i_R = 1$ on $\mathbf{H}_R^F$. Thus, $\lambda|_{U_R} = 1$ by the definition of U_R in Proposition 7.22. Consequently, $\lambda|_{M_\theta} = \Lambda(\theta)$ on $M_\theta = T_0U_R$.

Given any μ, choose one rational Specht module V_μ. Then, $d \in (D_R)_\eta$ permutes the factors V_{μ_i} with equal μ_i. At a chain R_i with opposition, its operator on V_{μ_i} is the action of $w_{0,i} \in \mathfrak{S}_{n_i}$. Since $w_{0,i}^2 = 1$, the tensor permutations and operators $w_{0,i}$ give the action of $W(R) \rtimes (D_R)_\eta$ over $\mathbb{Q}$. We extend to $(W(R) \rtimes (D_R)_\eta) \times \langle \Phi T_0\rangle$ with ΦT_0 acting as 1. Then, we restrict the representation to I_H/T_0, since $w(e) \in (D_R)_\eta$ for $e \in H$.

Write η_H for its character on I_H. Then, we have that $\widehat{\psi} = \mathrm{Ind}_{I_H}^{MI_H}(\widehat{\lambda}|_{I_H}\eta_H)$ restricts to ψ by [95, Problem 5.2] and $\lambda|_{M_\theta} = \Lambda(\theta)$. For (y,σ) fixing (χ,ψ), choose $m \in M$ so that $z = ym$ satisfies $\theta^{z\sigma} = \theta$ and $z(R^+) = R^+$. Thus, the Weyl component of z lies in $(D_R)_\eta$, since $\psi^{z\sigma} = \psi$, Λ is equivariant and the values of η are rational. Consequently, $\eta_H^{z\sigma} = \eta_H$. If $\widehat{\lambda}^{z\sigma} = \widehat{\lambda}\nu$, $\widehat{\psi}^{y\sigma} = \widehat{\psi}^{z\sigma} = \widehat{\psi}\nu$. By [95, Corollary 11.22], we let $\widehat{\chi}_0$ extend χ to $X \rtimes H$. Since $H = \prod_{t|h} H_t$, iterate [156, Lemma 7.1] to obtain $\widehat{\chi}$ with restrictions $\widehat{\chi}_t$ on $X \rtimes H_t$. Thus, the twists agree on H if they agree on every H_t. The maps ρ in Proposition 7.30 then give $(\chi,\psi) \succeq_c$ with centralizer scalar 1. □

Proposition 7.34. *If $R = A_1^a A_2^b$ with $b > 0$, we have that every matched pair (χ,ψ) of ℓ'-degree satisfies $\succeq_c$.*

Proof. If H is even, we choose $\theta^\gamma = \theta$ by Lemma 7.29. The image of $1-\gamma$ in the E_6 root lattice is generated by $\alpha_1 - \alpha_6, \alpha_3 - \alpha_5$. The squared norm of $a(\alpha_1 - \alpha_6) + b(\alpha_3 - \alpha_5)$ is $4(a^2 - ab + b^2)$. Thus, γ cannot exchange the two simple roots α, β of a stable A_2 factor, since $\|\alpha - \beta\|^2 = 6$. Therefore, $\gamma|_{R_i} = 1$ on each stable A_2 factor R_i. By Proposition 7.35, $\mu_t = \mu_t'$ on each H_t. Thus, $(\chi,\psi) \succeq_c$ by Lemma 7.33. If $\ell = 5$, then $5 \mid [W(E_6) : W(R)]$ and no such pair has ℓ'-degree. □

Proposition 7.35. *Let $R = \prod_i A_{n_i-1}$. The extensions $\widehat{\chi}, \widehat{\psi}$ can be chosen with $\mu|_{H_{2'}} = \mu'|_{H_{2'}}$. If $2 \mid h$, assume that at a γ-fixed weight θ, the graph γ fixes each simple root of each γ-stable R_i with $n_i \ge 3$. Then, the twists agree on H_2 as well.*

Proof. Let $t \mid h$ be prime and $H_t \in \mathrm{Syl}_t(H)$. For $\mu \vdash n$, we choose $x_\mu \in \mathfrak{S}_n$ whose cycle lengths are $h_{jj}(\mu)$, the principal hook lengths. Then, $\eta_\mu(x_\mu) = \pm 1$ by the principal hook formula [148, Lemma 4.13]. If $c_\mu = (x_\mu)_{t'}$, $\eta_\mu(c_\mu) \equiv \eta_\mu(x_\mu) \pmod t$ by [136, Proposition 1]. Indeed, a cycle of length $t^a d$, where $t \nmid d$, gives t^a cycles of length d in c_μ. Thus, $\eta_\mu(c_\mu)$ is a unit modulo t and each cycle length of c_μ is prime to t. If $t = 2$ and $n_i = 2$, take instead the reflection in $W(R_i)$. Then, $\eta_{\mu_i}(c_i) \in \{1, -1\}$ for that reflection c_i.

With $c = \prod_i c_i \in W(R)$, we put $c_i = c_{\mu_i}$ if $n_i \ge 3$ or $t \ne 2$. Given a lift $n_c \in N_{\mathbf{K}}(\mathbf{T}_0)$, set $\mathbf{T}_c = {}^k\mathbf{T}_0$ for $k^{-1}F(k) = n_c$ by Lang's theorem [58, Theorem 4.2.9]. If $t = 2$, then $u^f \in H_t$, so $\mu_{\gamma(i)} = \mu_i$ and we choose $c_{\gamma(i)} = \gamma(c_i)$

for $\gamma(i) \neq i$. If $\gamma(R_i) = R_i$, each simple root $\alpha \in R_i$ satisfies $\gamma(\alpha) = \alpha$. Thus, γ fixes $\langle U_\alpha, U_{-\alpha}\rangle$ pointwise by Proposition 7.32. Hence, for $\gamma(i) = i$, take lifts of c_i in $(\mathbf{K}^\gamma)^\circ$, and on $R_i, R_{\gamma(i)}$ take paired lifts $n_i\gamma(n_i)$ in $(\mathbf{K}^\gamma)^\circ$. Then, n_c belongs to $(\mathbf{K}^\gamma)^\circ$, so Lang's theorem there gives $k^\gamma = k$. Consequently, ${}^k\mathbf{B}_0$ is a γ-stable ambient Borel through $\mathbf{T}_c$ when $t = 2$.

The rational torus class of $\mathbf{T}_c$ is the $W(R)$-class of c, since F acts trivially on $W(R)$ at $\mathbf{T}_0$. Thus, $N_{\mathbf{K}}(\mathbf{T}_c)^F/T_c = C_{W(R)}(c)$, where $T_c = \mathbf{T}_c^F$, by [58, Chapter 4]. For $n_i \geq 3$, order the cycles ω of each length in c_i. If $t = 2$, we choose the orders on $R_i, R_{\gamma(i)}$ to correspond under γ. The stabilizer of the orders in $C_{\mathfrak{S}_{n_i}}(c_i)$ is $\prod_\omega \langle\omega\rangle$, of order prime to t. Moreover, on cycles ω of equal length in c_i, the permutations in $C_{\mathfrak{S}_{n_i}}(c_i)$ give every order of ω. A based map on A_{n_i-1} acts on $\mathfrak{S}_{n_i}$ by conjugation by $w \in \mathfrak{S}_{n_i}$, unique since $Z(\mathfrak{S}_{n_i}) = 1$. In the marking of $\mathbf{T}_c$, the Frobenius F acts on the letters by c_i, whose powers fix each cycle ω. If $n_i = 2$ and t is odd, take the whole kernel $W(R_i) \cong C_2$. If $n_i = 2$ and $t = 2$, the reflection c_i makes R_i split. Thus, we choose an F-stable Borel $\mathbf{B}_{c,i}$ in that factor, with stabilizer 1 modulo T_c.

Let L_{H_t} be the preimage of H_t in L_H. Thus, we let $D_t \leq L_{H_t}$ stabilize $\mathbf{T}_c$ with the orders of ω and the Borels $\mathbf{B}_{c,i}$. If $t = 2$, choose $\mathbf{B}_{c,\gamma(i)} = \gamma(\mathbf{B}_{c,i})$ on exchanged factors. Thus, $u^f \in D_t$ when $t = 2$. Given $e \in H_t$, choose a lift of $w(e)\Phi_e$ in I_H. It fixes $\eta = \boxtimes_i \eta_{\mu_i}$ under the based maps on R. Thus, the class $c^{W(R)}$ is fixed, since c_i depends on μ_i and opposition on R_i is inner on $\mathfrak{S}_{n_i}$. After multiplication by $k' \in K$, the lift fixes $\mathbf{T}_c$. Moreover, $C_{W(R)}(c)$ is transitive on the orders of ω and on choices of $\mathbf{B}_{c,i}$, with rational lifts from $N_{\mathbf{K}}(\mathbf{T}_c)^F$ by Lang's theorem. Thus, D_t maps onto H_t, and the kernel of $D_t/T_c \to H_t$ has order prime to t. Choose $S_t \in \mathrm{Syl}_t(D_t/T_c)$, containing $u^f T_c$ when $t = 2$. Then, $S_t \cong H_t$, and its preimage J_t satisfies $J_t/T_c \cong H_t$. Thus, for a generator aT_c of J_t/T_c, $a^{|H_t|} \in T_c$ for odd t and $a^{|H_2|/2} \in T_c u^f$ for $t = 2$. Thus, for $\mathbf{Q}_c = \prod_j F_r^j(\mathbf{T}_c)$ and Borel ${}^k\mathbf{B}_0$, Proposition 7.30 gives a disconnected torus $\mathbf{D}_t$ with $\mathbf{D}_t^{\widehat{F}} = J_t$.

Let Y_t be the disconnected torus induction from $\widehat{\lambda}|_{J_t}$ to $X \rtimes H_t$. As in Proposition 7.32, $Y_t|_X = R^X_{\mathbf{T}_c}(\lambda|_{T_c})$ belongs to the rational series of s, since $C_{\mathbf{G}^*}(s)$ is connected. By [40, §13.8] and [169, Theorem 2.1(1)], $\langle Y_t|_X, \chi\rangle_X = \pm\prod_i \eta_{\mu_i}(c_i)$, which is prime to t. For (y,σ) fixing the pair (χ,ψ), we choose $m \in M$ as in Lemma 7.33. The $z = ym$ has $\widehat{\lambda}^{z\sigma} = \widehat{\lambda}\nu$, since $\theta^{z\sigma} = \theta$. Since z fixes η, it fixes the class $c^{W(R)}$. Thus, we correct the images of $\mathbf{T}_c, \omega, \mathbf{B}_{c,i}$ by an element of K. Let Q_t be the kernel of $D_t/T_c \to H_t$. Then, $D_t/T_c = Q_t S_t$, so Q_t conjugates the transported Sylow subgroup to S_t. Both corrections lift to $K \leq L_{H_t}$, so the twist $\nu|_{H_t}$ of $\widehat{\lambda}|_{L_{H_t}}$ is fixed. The corrected induction Y_t' has connected torus $\mathbf{Q}_c$ and character $(\widehat{\lambda}\nu)|_{J_t}$. Therefore, $Y_t' = Y_t\nu|_{H_t}$ by [56, Proposition 3.2] with the two Borels at $\mathbf{Q}_c$. By Theorem 7.27, its trace is $Y_t^{y\sigma} = Y_t\nu|_{H_t}$, because corrections in X act trivially on the characters of $X \rtimes H_t$. By Lemma 7.31, the extension $\widehat{\chi}_t$ from Y_t has twist $\nu|_{H_t}$. By Lemma 7.33, $\mu = \mu'$ on $\prod_t H_t$ for the indicated t. □

Suppose that $2 \mid h$ and γ exchanges the ends of stable R_i. Then, we compare the twists on H_2 from Proposition 7.35. We choose $\theta^\gamma = \theta$ by Lemma 7.29. Thus, we work in $\mathbf{G}^f$ of Proposition 7.30, with H_2 acting algebraically. Compare radicals V_1, V_2 of parabolics with common Levi $\mathbf{L}$. Moreover, the complexes for e_s are $R\Gamma_c(Y_{V_i})e_s[\dim Y_{V_i}]$, $i = 1, 2$.

Lemma 7.36. *Let $\mathbf{Y}$ be connected reductive with Frobenius F. Let $V_i = R_u(\mathbf{P}_i)$ for two parabolics $\mathbf{P}_1, \mathbf{P}_2$ with common F-stable Levi $\mathbf{L}$, and put $L = \mathbf{L}^F$. Suppose that a finite group D of algebraic automorphisms satisfies $dF = Fd$, $d(\mathbf{L}) = \mathbf{L}$, $d(V_i) = V_i$ and $d(e_s) = e_s$, where e_s is the rational series idempotent of L. Assume the inclusions $C_{V_1^* \cap F^* V_1^*}(s) \leq C_{V_2^*}(s)$ and $C_{V_2^* \cap F^* V_2^*}(s) \leq C_{F^* V_1^*}(s)$. Then, we have that the complexes $R\Gamma_c(Y_{V_i})e_s[\dim Y_{V_i}]$ are isomorphic in $D^b(\mathbb{K}[(\mathbf{Y}^F \times L) \rtimes D])$ over a splitting field $\mathbb{K}$ of characteristic zero. If $\dim Y_{V_1} = \dim Y_{V_2}$, the alternating characters of $R\Gamma_c(Y_{V_i})e_s$ agree. Moreover, let z be an algebraic automorphism. Suppose that $zF = Fz$, $z(\mathbf{L}) = \mathbf{L}$, $z(V_1) = V_2$ and z normalizes the finite action. If (z,σ) fixes e_s, it fixes the alternating character.*

Proof. Since $\mathbf{Y}$ has field characteristic r, set $\mathbb{K} = \overline{\mathbb{Q}}_a$ with $a \neq r$. By [19, Corollary 6.5], compare Y_{V_1}, Y_{V_2} by $\Psi_{V_1,V_2,2,s} \circ \mathrm{sh}^* \circ \Psi^{-1}_{V_2,F(V_1),2,s}$. As in [160, proof of Lemma 3.1], $d \in D$ acts on the tuple by $(g_i)_i \mapsto (d(g_i))_i$. Since $d(V_i) = V_i$ and $dF = Fd$, the inclusions and projections in Ψ and the shift sh are D-equivariant. Moreover, the fibre trace $H_c^{2b}(\mathbb{A}^b, \mathbb{K}(b)) \to \mathbb{K}$ is invariant under geometric isomorphisms. Thus, D acts trivially on the Tate line $\mathbb{K}(b)$ in [19, Remark 6.6]. Consequently, the maps on e_s from [19, Theorem 6.2] are quasi-isomorphisms for $(\mathbf{Y}^F \times L) \rtimes D$. If $\dim Y_{V_1} = \dim Y_{V_2}$, the signs $(-1)^{\dim Y_{V_i}}$ agree. The finite action on Y_{V_i} is defined over $\mathbb{F}_{q_0}$ for some q_0. Thus, [53, Proposition 3.3] gives integral unprojected traces E_i. Since $z(V_1) = V_2$, we have $E_1^{(z,\sigma)} = E_2$. For $e_s = \sum_{k\in L} a_k(1,k)$, the projected trace at h is $\sum_{k\in L} a_k E_i(h(1,k))$. Thus, (z,σ) gives the projected trace for V_2 since it fixes e_s. □

Lemma 7.37. *Suppose that R is a chain and that $\mathbf{P} < \mathbf{G}$ has Levi $\mathbf{K}$ and satisfies $\gamma(\mathbf{P}) = \mathbf{P}$. Then, the twists on H_2 agree.*

Proof. Write $L_{H_2} = KI_{H_2}$. Since the dual centralizer equals $\mathbf{K}^*$, by [208, (10.6)], $\chi = \epsilon_{\mathbf{G}}\epsilon_{\mathbf{K}} R_{\mathbf{K}}^X(\beta)$ for a unique $\beta \in \mathcal{E}(K, s)$. By the central twist [169, Section 2], $\beta = \lambda\,\tau_K$ where τ_K is unipotent. Thus, $\tau_K = \tau \circ \pi$ for a unipotent character τ of $A = (\mathbf{K}_{\mathrm{ad}})^F$ and the map $\pi\colon K \to A$, by [58, Proposition 11.3.8]. In the based system, F acts on the root lattice of A by $-qw_0(R)$, so A is unitary. The degree formula gives $\chi(1) = |X : K|_{r'}\beta(1)$, so $\ell \nmid \tau(1)$. Also, $d' = \mathrm{ord}_\ell(-q) = 1$. Thus, τ has an extension $\widehat{\tau}$ on $A\langle F_r\rangle_\tau$ with values in $\mathbb{Q}_\ell$, by [156, proof of Proposition 8.5]. On A, the chart with $F = \gamma_R F_r^f$ has γ_R act as F_r^f.

Let $j\colon L_{H_2} \to A\langle F_r\rangle_\tau$ be the action on A. Then, $j|_K$ is the map $K \to A$. For $b \in I_{H_2}$ over $e \in H_2$, its action on A is $\mathrm{Int}(x)F_r^i$ for $x \in A$ and some i, in the unitary marking of A. Then, $\widehat{\beta} = (\widehat{\lambda}|_{L_{H_2}})(\widehat{\tau} \circ j)$ extends β. Given a mixed stabilizer (y, σ) with $\sigma \in \mathcal{H}_\ell$, we choose $m \in M$ as in Lemma 7.33. Then, $z = ym$ fixes the subsystem $\mathbf{K}$ and the unipotent label of τ. On A, z acts through an element of $A\langle F_r\rangle_\tau$, so $\widehat{\tau}$ is invariant under that conjugation. Since $\widehat{\tau}$ takes values in $\mathbb{Q}_\ell$, $(\widehat{\tau} \circ j)^{z\sigma} = \widehat{\tau} \circ j$. Thus, if $\widehat{\lambda}^{z\sigma} = \widehat{\lambda}\nu$, $\widehat{\beta}^{z\sigma} = \widehat{\beta}\,\nu|_{H_2}$.

Let $\mathbf{K}_\Pi = \prod_i F_r^i(\mathbf{K})$ in $\mathbf{G}^f$. Since $u^f \in H_2$, a representative $a \in L_{H_2}$ of a generator satisfies $a^{|H_2|/2} \in Ku^f$. In a cycle starting in factor i, we choose the parabolics $a^j(F_r^i(\mathbf{P}))$ for $0 \le j < |H_2|/2$ as in Proposition 7.30. Since $\gamma(\mathbf{P}) = \mathbf{P}$ and $\mathbf{K} \le \mathbf{P}$, the product radical U is L_{H_2}-stable. Let Y_U be the Deligne–Lusztig variety for $U, \widehat{F}$. Then, $\Delta = (X \rtimes H_2) \times_{H_2} L_{H_2}$ acts on Y_U with $(x, k) \in X \times K$ acting by $y \mapsto xyk^{-1}$, by [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19]. We tensor $\sum_i (-1)^i H_c^i(Y_U)$ over K with $\widehat{\beta}$. Thus, Y denotes its character on $X \rtimes H_2$. On X, compare U with $\prod_i F_r^i(U_0)$, where U_0 is the radical of $\mathbf{P}$. Since the dual centralizer is in $\mathbf{K}_\Pi^*$, by [19, Corollary 6.5 and Remark 6.7], we obtain the comparison on e_s. Consequently, by [208, proof of Lemma 4.19(b)], the first projection p_0 gives $Y|_X = \pm\chi$.

For the mixed comparison, z normalizes L_{H_2} and acts by $U \mapsto U^z$. Then, U^z is L_{H_2}-stable, with $\dim Y_U = \dim Y_{U^z}$ since $z\widehat{F} = \widehat{F}z$. Moreover, $e_s^{z\sigma} = e_s$ since $\beta^{z\sigma} = \beta$. By Lemma 7.36, the alternating traces on Δ agree on e_s. The traces on $R\Gamma_c(Y_U)$ before the tensor product with $\widehat{\beta}$ are integral by [53, Proposition 3.3]. For $(g, b) \in \Delta$, let $c(g, b)$ denote the trace on $R\Gamma_c(Y_U)$. Then, $Y(g) = |K|^{-1}\sum_{k \in K} c(g, kb)\widehat{\beta}(kb)$. Thus, $Y^{z\sigma} = Y\nu|_{H_2}$ since $\widehat{\beta}^{z\sigma} = \widehat{\beta}\nu|_{H_2}$. Since $z = ym$ with $m \in X$, $Y^{y\sigma} = Y\nu|_{H_2}$. By Lemma 7.31, the extension from Y has twist $\nu|_{H_2}$, as does the extension $\widehat{\psi}$ in Lemma 7.33. □

In the dual alcove $\mathcal{A}$, take the graph-fixed lift $\tilde{s}$. Then, the walls through $\tilde{s}$ give the simple roots indexed by J in R, by Zhang [208, proof of Proposition 10.5]. Since the two systems are γ-stable, we deduce that the unique $w \in W(R)$ taking the wall system on J to R^+ satisfies $\gamma(w) = w$. In the affine E_6 diagram, γ exchanges the arms $(4, 3, 1)$ and $(4, 5, 6)$, and fixes $(4, 2, 0)$. A chain with exchanged ends contains $3, 4, 5$, and $2 \notin J$ since 4 would have degree 3 in J. If 1 or 6 lies in J, it belongs to that chain. Consequently, J is the chain C or $C \sqcup \{0\}$. Since $C \subseteq \{1, \dots, 6\}$, we deduce that its standard parabolic $\mathbf{P}_C$ is γ-stable. Thus, conjugation by gn_w gives ${}^{gn_w}\mathbf{P}_C$ through $\mathbf{T}_0$, with $n_w^\gamma = n_w$ and $g^\gamma = g$.

Proposition 7.38. *If $R = C \sqcup A_1$ with C a chain whose ends γ exchanges, then $\mu|_{H_2} = \mu'|_{H_2}$ for suitable extensions of χ, ψ.*

Proof. Let $\mathbf{L}_0 \le \mathbf{K}$ be the Levi of C and $\mathbf{P}_0 = {}^{gn_w}\mathbf{P}_C$. For the isolated root system A_1, we choose α with $U_\alpha \le R_u(\mathbf{P}_0)$. Since $\gamma(\alpha) = \alpha$, γ fixes $\mathbf{K}_\alpha = \langle U_\alpha, U_{-\alpha}\rangle$ pointwise by Proposition 7.32. For s_α, we choose a lift $n_\alpha \in \mathbf{K}_\alpha$. The scalar $-q$ on the old roots gives ${}^{n_\alpha}F(U_\alpha) = U_\alpha$. By Lang's theorem in $\mathbf{K}_\alpha$, we obtain k with $k^{-1}F(k) = n_\alpha$ and $k^\gamma = k$. With $\mathbf{L} = {}^k\mathbf{L}_0$, we put $\mathbf{P} = {}^k\mathbf{P}_0$. Since n_α normalizes $\mathbf{L}_0$, $\mathbf{L}$ is F-stable. Moreover, $\mathbf{P}$ is γ-stable and its radical contains ${}^kU_\alpha$, which is F-stable.

In the adjoint A_1 factor $\mathbf{K}_1$ of $\mathbf{K}$, we let $(\mathbf{S}, \mathbf{B}_S)$ be the images of ${}^k\mathbf{T}_0$ and ${}^k(\mathbf{T}_0U_\alpha)$. The preimage in $\mathbf{K}$ of $\mathbf{S}$ is $\mathbf{L}$. Thus, $\mathbf{L}$ is the stabilizer of $(\mathbf{S}, \mathbf{B}_S)$ under conjugation by $\mathbf{K}$. Let $J \le L_{H_2}$ stabilize $\prod_i F_r^i(\mathbf{S}, \mathbf{B}_S)$. Since $\mathbf{L}$ is connected, Lang's theorem gives K-conjugacy of the rational pairs in $\mathbf{K}_1$. Hence, $J \cap K = L = \mathbf{L}^F$ and $J/L \cong H_2$. Moreover, $u^f \in J$ since the pair is γ-stable. Thus, a generator aL of J/L satisfies $a^{|H_2|/2} \in Lu^f$. By Lemma 7.37, the product radical U from $\mathbf{P}$ is J-stable and contains $\prod_i F_r^i({}^kU_\alpha)$.

Let τ be the unipotent factor at C of $J_s^X(\chi)$, viewed on the adjoint chain A, and let τ_L be its pullback. Thus, $\beta = (\lambda|_L)\tau_L$, and $\ell \nmid \tau(1)$ by the Jordan degree formula. With $\widehat{\tau}$ from Lemma 7.37, we choose $j\colon J \to A\langle F_r\rangle_\tau$. Then, $\widehat{\beta} = (\widehat{\lambda}|_J)(\widehat{\tau} \circ j)$ extends β. Given a mixed stabilizer (y, σ), choose $z = ym$ as in Lemma 7.33. Since the pairs $(\mathbf{S}, \mathbf{B}_S)$

and $(\mathbf{S},\mathbf{B}_S)^z$ are K-conjugate, choose $b\in K$ so that zb fixes $(\mathbf{S},\mathbf{B}_S)$. The factor C is distinguished by rank, and $\tau^{zb\sigma}=\tau$. Thus, $\widehat{\beta}^{zb\sigma}=\widehat{\beta}\,\nu|_{H_2}$ as in Lemma 7.37, where $\widehat{\lambda}^{z\sigma}=\widehat{\lambda}\nu$.

Given any $w\in W(C)$, we choose the torus $\mathbf{T}_w$ of the type A expansion of τ_L in [40, §13.8]. By [58, Proposition 11.4.12], multiplication by $\lambda|_L$ replaces 1 on $\mathbf{T}_w^F$ by $\lambda|_{\mathbf{T}_w^F}$. Thus, transitivity [58, Chapter 9] gives $R_{\mathbf{T}_w}^X(\lambda|_{\mathbf{T}_w^F})$ for each w. Since $C_{\mathbf{G}^*}(s)$ is connected, the rational series is $\mathcal{E}(X,s)$ for every w. By [169, Theorem 2.1(1)], the coefficient at χ factors over C and the split A_1. Thus, on C, the type A expansion and orthogonality give $\langle\eta_C,\eta_C\rangle_{W(C)}=1$ for its Weyl character η_C. On A_1, the split torus coefficient is 1 for each of $1,\mathrm{St}$, by [40, §13.8]. Thus, $|\langle R_{\mathbf{L}}^X(\beta),\chi\rangle|=1$.

With $U_0=R_u(\mathbf{P})$, we put $V=\prod_i F_r^i(U_0)$. In each dual radical U^*,V^*, only the marked positive root of A_1 centralizes s_Π in that coordinate, since C lies in the dual Levi. Thus, $C_{U^*}(s_\Pi)=C_{V^*}(s_\Pi)=\mathbf{U}_1^*$, where $\mathbf{U}_1^*=\prod_i U_{\alpha_i^*}$ for the indicated positive roots α_i^*, with $\widehat{F}^*(\mathbf{U}_1^*)=\mathbf{U}_1^*$. By [19, Remark 6.7], we apply Lemma 7.36 for U,V. For U,U^{zb}, the inclusions hold since zb fixes $\prod_i F_r^i(\mathbf{S},\mathbf{B}_S)$. Given $R\Gamma_c(Y_U)\otimes_L\widehat{\beta}$, we write Y for its character. On X, comparison with V and projection p_0 give $Y|_X=\pm R_{\mathbf{L}}^X(\beta)$. Since U,U^{zb} and e_s are J-stable with $e_s^{zb\sigma}=e_s$ and $\dim Y_U=\dim Y_{U^{zb}}$, Lemma 7.36 compares the traces on $\Delta=(X\rtimes H_2)\times_{H_2}J$. Thus, by Lemma 7.37, averaging over L gives $Y^{y\sigma}=Y\nu|_{H_2}$. Since $\langle Y|_X,\chi\rangle=\pm1$, Lemma 7.31 gives the extension with twist $\nu|_{H_2}$. □

Corollary 7.39. *If R is a product of type A factors, every matched pair (χ,ψ) of ℓ'-degree satisfies $\succeq_c$.*

Proof. By Proposition 7.35, the twists agree on $H_{2'}$ and on H_2 if γ fixes the simple roots of each γ-stable R_i. Otherwise, the affine nodes satisfy $J=C$ or $J=C\sqcup\{0\}$. Then, the twists on H_2 agree by Lemma 7.37 and Proposition 7.38, with $\widehat{\lambda}|_{L_{H_2}}$. By Lemma 7.33, we obtain $(\chi,\psi)\succeq_c$, including the pairs with $5\nmid\chi(1)$ when $\ell=5$. □

Theorem 7.40. *With $X={}^2E_6(q)$ and $3\nmid q+1$, we let ℓ be an odd prime dividing $q+1$. Then, the bijection $\Omega\colon \mathrm{Irr}_{\ell'}(X)\to\mathrm{Irr}_{\ell'}(M)$ on the marked normalizer M of* Lemma 7.21 *satisfies $\succeq_c$ on each pair. Given $P\in\mathrm{Syl}_\ell(X)$ with $P\le M$, we have that $M=N_X(P)$ if $\ell\ge7$ and $N_X(P)\le M$ if $\ell=5$.*

Lemma 7.41. *By* Lemma 7.29*, we have that $C_{\mathbf{G}^*}(s)$ is connected and $W_\theta=W(R)$. Thus, the affine walls at $\tilde{s}$ give a basis of R, with rational lifts for $W(R)$ since the action on $\mathbf{T}_0$ is $-q$. If R has a component of type D, then $R=D_4$ or D_5. Moreover, we have that $\mathbf{K}$ is an F-stable proper Levi, with $A=(\mathbf{K}_{\mathrm{ad}})^F$ of type $D_4(q)$ or ${}^2D_5(q)$, respectively.*

Proof. In the affine node set J, the trivalent vertex is 4 and its three neighbours $2,3,5$ lie in J. Moreover, $0,1,6$ are adjacent to $2,3,5$, respectively. Since the arm lengths of D_n are $1,1,n-3$, $R=D_4$ or D_5, with no other component. If $0\notin J$, the roots of J already give a standard Levi. If $0\in J$, a power of $(0,1,6)$ takes an end $i\notin J$ to 0. Its finite Weyl part puts R in the finite diagram. Thus, $\mathbf{K}$ is a proper Levi through $\mathbf{T}_0$. Since F acts by $-q$ on the marked lattice, $F(\mathbf{K})=\mathbf{K}$. The based action on R is $-w_0(R)$, so it is 1 for D_4 and the terminal transposition for D_5. □

Proposition 7.42. *If $R=D_4$ or D_5, the extensions of χ,ψ can be chosen with $\mu_{y,\sigma}=\mu'_{y,\sigma}$ on H whenever $(\chi,\psi)^{y\sigma}=(\chi,\psi)$.*

Proof. First, suppose that $h=|H|$ is odd or $R=D_4$. Since $C_{\mathbf{G}^*}(s)=\mathbf{K}^*$, $\chi=\epsilon_{\mathbf{G}}\epsilon_{\mathbf{K}}R_{\mathbf{K}}^X(\lambda\,\tau_K)$ by [208, (10.6)], with $\tau_K=\tau\circ\pi$ for $\tau\in\mathcal{E}(A,1)$ and $\pi\colon K\to A$. If h is odd, a lift $a\in L_H$ of a generator of H satisfies $a^h\in K$. Thus, any parabolic $\mathbf{P}$ with Levi $\mathbf{K}$ gives U by Lemma 7.37. If h is even and $R=D_4$, choose $\theta^\gamma=\theta$. Then, the standard parabolic at $J=\{2,3,4,5\}$ is graph-stable. The element $a^{h/2}\in Ku^f$ fixes that parabolic in $\mathbf{G}^f$. Thus, fix Y_U and compare its traces on $(X\rtimes H)\times_H L_H$ by Lemma 7.37.

Let B be the diagram image on R^+ of the mixed stabilizer in $E\times\mathcal{H}_\ell$. For (e,σ) in the stabilizer, let $w(e,\sigma)\in D_R$ satisfy $\theta^{w(e,\sigma)\Phi_e\sigma}=\theta$. Since Φ_e,σ act by powers on θ, uniqueness gives $w(ee',\sigma\sigma')=w(e,\sigma)w(e',\sigma')$. Thus, B is abelian. If $R=D_4$, we have $B\le\mathfrak{S}_3$, so B is cyclic. Since $-1\in W(D_4)$, the change to q from $-q$ on the marked root lattice induces 1 on the diagram of A. If $R=D_5$, diagram involution in the marking $F=\gamma_R F_r^f$ induces F_r^f on A.

If $R=D_4$, we choose a $\mathbb{Q}_\ell$-valued extension of τ over $\langle F_r\rangle$ by [156, proof of Proposition 8.5] with $d=\mathrm{ord}_\ell(q)=2$. Moreover, [61, Theorem 3] gives a rational extension of τ to $A\rtimes B$, for B cyclic of order 2 or 3. By [156, proof of Proposition 8.5], the extensions on $A\langle F_r\rangle$ and $A\rtimes B$ give $\widehat{\tau}$ on $A\rtimes(\langle F_r\rangle\times B)$ with values in $\mathbb{Q}_\ell$. Given A of type ${}^2D_5$, its unipotent characters occur in $\mathrm{Ind}_{B_A}^A(1)$ by [208, proof of Proposition 10.6], where B_A is a rational Borel. Thus, $\omega_\tau=1$, and the field extension $\widehat{\tau}$ is $\mathbb{Q}_\ell$-valued by [156, Section 7.3]. Let j be the map by conjugation from L_H to the adjoint inertia of τ. Then, $\widehat{\beta}=(\widehat{\lambda}|_{L_H})(\widehat{\tau}\circ j)$ extends $\beta=\lambda\tau_K$. For (y,σ) fixing (χ,ψ), we set $z=ym$ with $m\in M$

such that $\theta^{z\sigma} = \theta$ and $z(R^+) = R^+$. Thus, z acts on A in the inertia for $\widehat{\tau}$. Since $\widehat{\tau}$ has values in $\mathbb{Q}_\ell$, its pullback is fixed by $z\sigma$, and the twist of $\widehat{\beta}$ is the quotient character ν of $\widehat{\lambda}$.

For the local character, take $\eta \in \mathrm{Irr}(W(D_n))$ with $\psi = \mathrm{Ind}_{M_\theta}^{M}(\Lambda(\theta)\eta)$. If $|B| = 2$, the group $W(D_n) \rtimes B$ is $W(B_n)$, whose Weyl characters are rational. Hence, η has a rational extension to $W(D_n) \rtimes B$. If $|B| = 3$, the extensions of η are a torsor under $\mathrm{Irr}(B)$ by [95, Corollaries 6.17 and 11.22]. On $\mathrm{Irr}(B)$, complex conjugation c acts by $\delta \mapsto \delta^{-1}$. Thus, c fixes exactly one extension $\widehat{\eta}$. Since η is rational, $\widehat{\eta}^\sigma$ is an extension of η. Moreover, $c\sigma = \sigma c$, so $\widehat{\eta}^\sigma = \widehat{\eta}$. Thus, we choose a rational $\widehat{\eta}$ of $W(R) \rtimes B$ on I_H/T_0, with the formal coordinate ΦT_0 acting as 1. Then, $\mathrm{Ind}_{I_H}^{MI_H}((\widehat{\lambda}|_{I_H})\widehat{\eta})$ extends ψ by $\lambda|_{M_\theta} = \Lambda(\theta)$, since $\lambda|_{U_R} = 1$ by [169, Proposition 4.6] as in Lemma 7.33. Since z acts in $W(R) \rtimes B$, the twist on $\widehat{\psi}$ is ν. On X, the induced character Y has $\langle Y|_X, \chi\rangle_X = \pm 1$. Thus, $\widehat{\chi}$ from Y by Lemma 7.31 also has twist ν.

Now suppose that $R = D_5$ and $2 \mid h$. In the graph-fixed alcove, $J = \{0, 2, 3, 4, 5\}$ by [208, Proposition 10.5(ii)]. The nodes $1, 6$ outside the chain have mark 1 and both coordinates are $1/2$. Thus, the lift $\tilde{s} = (\omega_1^\vee + \omega_6^\vee)(-1)$ has order 2, so r is odd. By [208, proof of Proposition 10.6, Case 1, (10.10)–(10.13)], χ has a rational extension to $X \rtimes \langle F_r\rangle$, since $X = \mathbf{G}_{\mathrm{ad}}^F$ when $3 \nmid q + 1$.

Locally, θ and λ have order 2 by duality with the element s of order 2. Thus, Φ fixes both θ and λ since r is odd. The group $M_\theta = N_{\mathbf{K}}(\mathbf{T}_0)^F$ has quotient $W(D_5)$ on which Φ acts trivially. In the marked coordinates, $\Phi^f = \mathrm{Int}(n_0^{-1})\gamma$, where $\gamma(n_0) = n_0$. Since $n_0^2 = (2\rho^\vee)(-1) \in Z(\mathbf{G})$ has order dividing 2 and $|Z(\mathbf{G})|$ divides 3, $n_0^2 = 1$. Consequently, $\Phi^{2f} = 1$, with outer image of order $2f$. The representation of M_θ with character $(\lambda|_{M_\theta})\eta$ extends to $M_\theta\langle\Phi\rangle$ with Φ acting as 1. Since λ, η are rational, so is the extension η_0 on $M_\theta\langle\Phi\rangle$. By [95, Problem 5.2], $\mathrm{Ind}_{M_\theta\langle\Phi\rangle}^{M\langle\Phi\rangle}\eta_0$ extends ψ and is rational. Thus, $H = E$ and $\mu = \mu' = 1$ for $\widehat{\chi}, \widehat{\psi}$. □

Proof of Theorem 7.40. Take Ω from Theorem 7.24. If $s = 1$, by [156, Theorem D], we obtain the comparison on M with $\theta = 1$. If $R = \varnothing$, we apply Proposition 7.30. If $R = \prod_i A_{n_i} \neq \varnothing$, we apply Corollary 7.39. If $R = D_4$ or D_5, apply Proposition 7.42. Since R is simply laced with $\mathrm{rk}\, R \leq 6$, its only possible exceptional component is E_6. If $R = E_6$, $s \in Z(\mathbf{G}^*) = 1$. Thus, for all pairs under Ω, $\mu = \mu'$ on H. The Sylow correction of Proposition 7.30 gives $\succeq_c$. By Lemma 7.21, $N_X(P) \leq M$. If $\ell \geq 7$, then $N_X(P) = M$ by the proof of Corollary 7.25. □

7.3 Cubic actions in E_6

Proposition 7.43. *Let $X = {}^2E_{6,\mathrm{sc}}(q)$ with $q = r^f$ and $3 \mid q + 1$. Let $\ell \geq 7$ be prime with $\ell \mid q + 1$. Let $\mathbf{T}$ be the transported torus of* Lemma 7.21 *and put $T = \mathbf{T}^F$. Since $w_0 = -\gamma$, the torus action is $-q$ on $X^*(\mathbf{T})$. By the proof of* Corollary 7.25*, $P = O_\ell(T) \in \mathrm{Syl}_\ell(X)$ and $M = N_X(P) = N_X(\mathbf{T})$. Moreover, $M/T = W(E_6) = W$, and $\Gamma = \mathrm{Aut}(X)_P$ acts on $\mathbf{T} = C_{\mathbf{G}}(P)^\circ$. Given $\theta \in \mathrm{Irr}(T)$ with $W_\theta = 1$, the pair $\chi = \varepsilon R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, $\psi = \mathrm{Ind}_T^M \theta$ satisfies $\succeq_c$, with ε giving $\chi(1) > 0$.*

Proof. By [53, Theorem 6.8] and Clifford theory [43], χ, ψ are irreducible. Moreover, $\chi(1) = |X : T|_{r'}$ and $\psi(1) = |W|$ are prime to ℓ. For $(\alpha, \sigma) \in \Gamma \times \mathcal{H}_\ell$, the pair for $\theta^{\alpha\sigma}$ is $(\chi^{\alpha\sigma}, \psi^{\alpha\sigma})$. Thus, $J = \Gamma_\chi = \Gamma_\psi$. With $A = X \rtimes J$, put $C = M \rtimes J$ and $B = C_\theta$. Since $\theta^M = W\theta$, the projection $B \to J$ is onto and $B \cap M = T$. Hence, $A = XB$ and $C = MB$. If $c = (x, j) \in C_A(X)$, $j = \mathrm{Int}(x^{-1})$. Since $j(P) = P$, $x \in M$ and c acts as 1 on T. Thus, $C_A(X) \leq B$.

In the F-model of the proof of Proposition 7.30, write $\mathbf{T} = {}^g\mathbf{T}_{\mathrm{std}}$ for the transported standard torus, with $g \in \mathbf{G}^\gamma$. Take the cyclic product $\mathbf{G}^f$ with shift u, Frobenius $\widehat{F}$, and $\mathbf{Q} = \prod_i F_r^i(\mathbf{T})$. Given $b \in B$, a lift in $\widetilde{X}\langle F_r\rangle$ acts on $\mathbf{G}^f$ by $\mathrm{Int}(\tilde{x})u^i$ with $\tilde{x} \in (\widetilde{\mathbf{G}}^f)^{\widehat{F}}$. Since $Z(\widetilde{X})$ has trivial conjugation, the action of b is well defined. Let $H \leq \langle u\rangle$ be the image of B, and put $h = |H|$. Since $W_\theta = 1$, $B \to H$ has kernel acting by conjugation through $\mathbf{Q}$. Choose a lift $a \in B$ of a generator of H. If h is odd, a^h acts by toral conjugation.

Suppose $2 \mid |H|$. Then, $\gamma(Ws) = Ws$, for s dual to θ. Since $W_s = 1$, the lifts $\tilde{s}z$ for $z \in C_3$ give three distinct orbits $W(\tilde{s}z)$. By [208, proof of Proposition 10.5, Case 2], choose $\tilde{s}$ so $W\tilde{s}$ is the unique γ-fixed orbit, with a γ-fixed alcove representative. Choose $\theta^\gamma = \theta$ since $M/T = W$. Thus, $u^f = \gamma$ lies in the algebraic image of B. $a^{h/2}$ acts as u^f times toral conjugation. Since $g \in \mathbf{G}^\gamma$, the Borel $\mathbf{B}_0 = {}^g\mathbf{B}$ through $\mathbf{T}$ is γ-stable. Consequently, by Proposition 7.30, propagation of $\mathbf{B}_0$ in each cycle gives the radical U with $b(U) = U$ for $b \in B$, for either parity of h.

With $D_B = A \times_J B$, we set $D_C = A \times_J C$. For $(a, b) \in D_B$, the action on Y_U is the algebraic action of b followed by left multiplication by $ab^{-1} \in X$, as in [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19]. Thus, $C_0 = \{(c, c) : c \in C_A(X)\}$ acts trivially on Y_U: the algebraic action of $c = (x, \mathrm{Int}(x^{-1}))$ is 1. Take the $\theta^\vee$-isotypic

summand of $R\Gamma_c(Y_U)$ for the right action of T. Since B fixes θ, the projector is D_B-invariant. By [208, proof of Lemma 4.19(b)], the virtual character Ξ_B of this summand restricts to $\pm\chi\boxtimes\theta^\vee$ on $X\times T$. Since $D_C=(X\times M)D_B$, we obtain by [95, Problem 5.6] $\Xi|_{X\times M}=\pm\chi\boxtimes\psi^\vee$ for $\Xi=\mathrm{Ind}_{D_B}^{D_C}\Xi_B$. Moreover, $C_0\lhd D_C$, since $ac^{-1}\in X$ for $(a,c)\in D_C$ and $C_A(X)$ centralizes X. Thus, Ξ is inflated from D_C/C_0.

Let (y,σ) fix (χ,ψ), and choose $m\in M$ with $z=\mathrm{Int}(m)y$ satisfying $\theta^{z\sigma}=\theta$. Since θ and θ^σ have stabilizer B in C, z normalizes B and acts on D_B, D_C. The radicals U, U^z are B-stable. Since $C_{\mathbf{G}^*}(s)=\mathbf{T}^*$, [19, Remark 6.7] gives the centralizer inclusions for U, U^z. Moreover, $\theta^{z\sigma}=\theta$ fixes the right projector $e_{\theta^\vee}$ and the series idempotent e_s. Thus, $\Xi_B^{z\sigma}=\Xi_B$ by Lemma 7.36. Moreover, conjugation by $(c_m,c_m)\in D_C$ gives the correction, where $c_m=(1,\mathrm{Int}(m))\in C$. Consequently, $\Xi^{y\sigma}=\Xi$. By Proposition 2.4, $\chi\boxtimes\psi^\vee$ has an invariant extension V with $C_0\le\ker V$. By Corollary 2.5, we obtain $\succeq_c$. □

Proposition 7.44. *Suppose that $C_{\mathbf{G}^*}(s)$ is connected and its root system R is one chain of type A. If $2\mid|H|$ for the field image H, suppose that there is a parabolic $\mathbf{P}\le\mathbf{G}$ with Levi $\mathbf{K}$ and $\gamma(\mathbf{P})=\mathbf{P}$ in the marking with $\theta^\gamma=\theta$. Then, each matched pair (χ,ψ) of ℓ'-degree satisfies $\succeq_c$ on X. The parabolic $\mathbf{P}$ exists if γ exchanges the ends of R or $R=A_1$.*

Proof. Since R is a chain, an end $j\in\{0,1,6\}$ lies outside J_R. By Lemma 7.41, choose a rotation ω with $\omega(j)=0$. Thus, J_R maps into $\{1,\dots,6\}$ under its Weyl part. Let $\mathbf{K}$ be the Levi through $\mathbf{T}$, with $K=\mathbf{K}^F$. Connectedness gives $W_\theta=W(R)$. Each reflection of $W(R)$ fixes the lifts of s, since each $\alpha\in R$ satisfies $\alpha(\tilde{s}z)=1$ for $z\in C_3$. Thus, the lifts $\tilde{s}z$ of s have distinct Weyl orbits. By [58, Proposition 11.4.12], $s\in Z(\mathbf{K}^*)$ gives a linear character λ of K with $\lambda|_T=\theta$. As in Proposition 7.43, set $J=\Gamma_\chi=\Gamma_\psi$, $A=X\rtimes J$, $C=M\rtimes J$ and $B=C_\theta$. Then, $B\cap M=N_K(\mathbf{T})=M_\theta$. Since B normalizes $\mathbf{K}$ and fixes θ, torus duality gives that B fixes λ. Thus, $L=KB$ has $L/K\cong J$, with $C_A(X)\le B$.

With $D=L/\ker\lambda$, we write $\kappa\colon L\to D$. Since λ is L-invariant, the kernel $Z_\lambda=K/\ker\lambda$ of $D\to J$ is central, with character $\bar{\lambda}$ from λ. If $Y\in\{A,C,L,B\}$, set $\widehat{Y}=Y\times_J D$. Thus, on $\widehat{L}$ put $\Lambda(l,d)=\bar{\lambda}(\kappa(l)d^{-1})$. Since $\kappa(l)d^{-1}\in Z_\lambda\le Z(D)$, Λ is linear. Moreover, $\Lambda(k,1)=\lambda(k)$ and $\Lambda(1,z)=\bar{\lambda}(z)^{-1}$ for $k\in K$ and $z\in Z_\lambda$. For (y,σ) fixing (χ,ψ), choose $m\in M$ so that $y_0=\mathrm{Int}(m)y$ has $\theta^{y_0\sigma}=\theta$ and $y_0(R^+)=R^+$. Then, y_0 normalizes B,K,L, and $\lambda^{y_0\sigma}=\lambda$. Thus, y_0 acts on D via $\ker\lambda$ and $\Lambda^{y_0\sigma}=\Lambda$.

The Weyl coordinate of B acts through $W(R)\rtimes D_R$, where D_R stabilizes R^+ in $N_W(R)$. Since Φ fixes W and Φ^{2f} is toral, the coordinate $B\to W$ is a homomorphism. By Lemma 7.33, η has a rational Specht extension to $W(R)\rtimes(D_R)_\eta$. Thus, the character $\Lambda|_{\widehat{B}}\widehat{\eta}$ on $\widehat{B}$ induces $\widehat{\psi}$ on $\widehat{C}$. Indeed, its restriction to M is the Clifford character from M_θ, since $\lambda|_{M_\theta}=\Lambda(\theta)$ in the notation of Proposition 7.22. Moreover, y_0 acts through the based stabilizer of η, so $\widehat{\psi}^{y_0\sigma}=\widehat{\psi}$.

Globally, $\chi=\varepsilon R_{\mathbf{K}}^{\mathbf{G}}(\lambda\tau_K)$ by [208, (10.6)], where τ_K is the pullback of $\tau\in\mathcal{E}((\mathbf{K}_{\mathrm{ad}})^F,1)$. Since $\chi(1)=|X:K|_{r'}\tau(1)$, $\ell\nmid\tau(1)$. Moreover, $d'=\mathrm{ord}_\ell(-q)=1$, so the result of [156, proof of Proposition 8.5] gives $\widehat{\tau}$ on $A_0\langle F_r\rangle_\tau$ with values in $\mathbb{Q}_\ell$, where $A_0=(\mathbf{K}_{\mathrm{ad}})^F$. Let $j\colon L\to A_0\langle F_r\rangle_\tau$ be the action of L on A_0 as in Lemma 7.37. Then, $\widehat{\beta}=\Lambda(\widehat{\tau}\circ j)$ extends $\beta=\lambda\tau_K$ to $\widehat{L}$, with j composed with projection to L. Since y_0 acts in the adjoint inertia and σ fixes the values of $\widehat{\tau}$, $\widehat{\beta}^{y_0\sigma}=\widehat{\beta}$.

In $\mathbf{G}^f$, the kernel of $L\to H$ acts by conjugation in $\prod_i F_r^i(\mathbf{K})$, including the diagonals from $\mathbf{T}$. Thus, when $|H|$ is odd, a generator lift $a\in L$ returns with $a^{|H|}$ fixing each $\mathbf{P}$ with Levi $\mathbf{K}$. If $2\mid|H|$, the distinct lift orbits give $\theta^\gamma=\theta$ by Proposition 7.43. Given a cycle at i, the parabolics are $a^j(F_r^i(\mathbf{P}))$ for $0\le j<|H|/2$. Thus, U is their product radical and satisfies $l(U)=U$ for $l\in L$.

Let $\widehat{\Delta}=\widehat{A}\times_D\widehat{L}$. Then, $\widehat{\Delta}$ acts on Y_U through $A\times_J L$ by [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19]. We tensor $R\Gamma_c(Y_U)$ over K with $\widehat{\beta}$, with alternating character Y_U^β on $\widehat{A}$. Since $C_{\mathbf{G}^*}(s)=\mathbf{K}^*$, [19, Corollary 6.5 and Remark 6.7] and the first projection p_0 give $Y_U^\beta|_X=\pm\chi$. Moreover, L stabilizes U, U^{y_0} and $\dim Y_U=\dim Y_{Uy_0}$. Since $\widehat{\beta}^{y_0\sigma}=\widehat{\beta}$, the action of $y_0\sigma$ fixes e_s. By Lemma 7.36, the trace on e_s is invariant. Thus, over K, the tensor character satisfies $(Y_U^\beta)^{y_0\sigma}=Y_U^\beta$. By Proposition 2.4, with $\widehat{A}\cong\widehat{A}\times_D D$ and second base group 1, we obtain an invariant extension $\widehat{\chi}$. For $(c,d)\in C_A(X)\times_J D$, the action of $c\in B$ is 1 on Y_U, on A_0 and on $W(R)$. Thus, (c,d) acts by $\Lambda(c,d)$ on $\widehat{\beta}$ and on $\widehat{\psi}$, since $C_A(X)$ centralizes M. By Proposition 2.4, $\widehat{\chi}$ has the same scalar $\Lambda(c,d)$.

Choose a section $t\colon J\to D$ with $t(1)=1$. Then, the models on A,C have factor set $\alpha(j,k)=\bar{\lambda}(t(j)t(k)t(jk)^{-1})^{-1}$. Under $y_0\sigma$, the discrepancy $t(j)^{y_0}t(j^{y_0})^{-1}$ lies in Z_λ, where both extensions have scalar $\bar{\lambda}^{-1}$. Thus, the correcting functions agree for (y_0,σ). For the correction $c_m=(1,\mathrm{Int}(m))\in C$, the functions μ_{c_m},μ'_{c_m} are determined by α. Thus, $\mu_{c_m}=\mu'_{c_m}$ on J. Consequently, $\mu_{y,\sigma}=\mu'_{y,\sigma}$, giving $\succeq_c$. If γ exchanges the ends of R, then $0\notin J_R$ by Lemma 7.37.

Thus, $\mathbf{P}_{J_R}$ is γ-stable. If $R = A_1$, its positive root α is fixed by γ. By [208, proof of Proposition 10.5], α is long in F_4. Hence, choose $w \in W(E_6)^\gamma$ with $w\alpha = \alpha_i$ for a fixed simple root α_i. Since $n_w^\gamma = n_w$, the transported parabolic $\mathbf{P} = {}^{gn_w^{-1}}\mathbf{P}_i$ is γ-stable and has Levi $\mathbf{K}$. □

With $S = X/Z(X)$, we put $D = \mathbf{G}_{\mathrm{ad}}^F$ and $M_S = M/Z(X)$ and $M_D = N_D(\mathbf{T}_{\mathrm{ad}})$. Let $T_D = \mathbf{T}_{\mathrm{ad}}^F$. By Zhang [208, (10.18)], we deduce that $D = ST_D$, so $M_D/M_S \cong D/S \cong C_3$. Since $q = r^f \equiv -1 \bmod 3$, $r \equiv -1 \bmod 3$ and f is odd. With $\phi = F_r$, we choose Φ from Proposition 7.32. Since $n_0^2 = 1$, $\Phi^{2f} = 1$ as in Proposition 7.42. Moreover, $\gamma\Phi = \Phi\gamma$, because the Lang element g is γ-fixed. Thus, $E_o = \langle\Phi^2\rangle$ has order f and $E = E_o \times \langle\gamma\rangle \cong C_{2f}$ is a complement to D in $\mathrm{Aut}(S)$. In particular, E normalizes M_D, M_S. If $\bar{P} = PZ(X)/Z(X)$, $N_D(\bar{P}) = M_D$, since $C_{\mathbf{G}_{\mathrm{ad}}}(\bar{P})^\circ = \mathbf{T}_{\mathrm{ad}}$.

Proposition 7.45. *Let s be the marked node 4 element with root system A_2^3 and component group C_3. Then, the series on X factors through S. The correspondence indexed by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ modulo rotation satisfies $\succeq_c$ on the ℓ'-degrees.*

Proof. Take $\tilde{s} = \omega_4^\vee(\zeta_3)$ from [208, (10.23)]. Since each lift $\tilde{s}z$, $z \in C_3$, satisfies $(\tilde{s}z)^3 = 1$, it is rational for $-q$. Thus, $\mathcal{E}(X, s)$ factors through S, by [208, proof of Proposition 10.9, Step 1]. Moreover, the dual centralizer of $\tilde{s}$ has three unitary A_2 factors by [208, (10.29)]. Given $\lambda_i \vdash 3$, write $\rho_{\boldsymbol{\lambda}}$ for the character of D in the series of $\tilde{s}$. Then, the twists $\rho_{\boldsymbol{\lambda}}\nu$ for $\nu \in \mathrm{Irr}(D/S)$ rotate $\boldsymbol{\lambda}$ by [208, proof of Proposition 10.9, Steps 2–3].

Let $\vartheta \in \mathrm{Irr}(T_D)$ be dual to $\tilde{s}$, with stabilizer $W_0 = W(A_2)^3$ in W. By Proposition 7.22, the extension $\widehat{\vartheta}$ to $(M_D)_\vartheta$ is 1 on the Tits lifts of W_0. Moreover, the matched Weyl character is $\eta_{\boldsymbol{\lambda}} = \boxtimes_i \eta_{\lambda_i}$. Then, $\pi_{\boldsymbol{\lambda}} = \mathrm{Ind}_{(M_D)_\vartheta}^{M_D}(\widehat{\vartheta}\eta_{\boldsymbol{\lambda}})$. By Corollary 7.23, the twists by $\mathrm{Irr}(D/S)$ rotate the labels of $\pi_{\boldsymbol{\lambda}}$ as for $\rho_{\boldsymbol{\lambda}}$. By [208, Proposition 10.9, Step 3], $\rho_{\boldsymbol{\lambda}}|_S$ is irreducible for nonconstant $\boldsymbol{\lambda}$. If $\boldsymbol{\lambda}$ is constant, its constituents form a D/S-orbit of size 3. Thus, Theorem 7.24 gives that pattern for $\pi_{\boldsymbol{\lambda}}|_{M_S}$. Moreover, $\pi_{\boldsymbol{\lambda}}(1) = [W : W_0]\eta_{\boldsymbol{\lambda}}(1)$ and each $\eta_{\lambda_i}(1)$ is 1 or 2. Thus, $\ell \nmid \pi_{\boldsymbol{\lambda}}(1)$ for $\ell \geq 7$. If $\ell = 5$, $5 \mid [W : W_0 \rtimes C_3]$.

With $k = \mathbb{Q}(\zeta_3)$, let c denote complex conjugation. Then, γ, c act by $(2\,3)$ on $\boldsymbol{\lambda}$, by [208, proof of Proposition 10.9, Steps 4 and 7]. Moreover, E_o and $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/k)$ fix the ordered global labels $\boldsymbol{\lambda}$. Locally, n_0 takes ϑ to ϑ^{-1} and induces γ on W_0, since $w_0 = -\gamma$. Thus, γ, c both act on the labels of $\pi_{\boldsymbol{\lambda}}$ by $(2\,3)$, and E_o fixes them. If $\boldsymbol{\lambda}$ is constant, D/S acts regularly on the constituents of $\rho_{\boldsymbol{\lambda}}|_S, \pi_{\boldsymbol{\lambda}}|_{M_S}$. The relation $\gamma d\gamma^{-1} = d^{-1}$ on D/S gives a unique γ-fixed constituent χ_0, ψ_0, respectively. Match χ_0, ψ_0 and their translates by D/S. By [208, Proposition 10.9, Step 4], χ_0 and its translates are rational, and Φ^2 fixes χ_0. On the local triple, $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/k)$ acts in the centralizer of $\mathfrak{S}_3$, which is 1. Complex conjugation centralizes the 3-cycle. Hence, its permutation order divides $\gcd(2, 3) = 1$. Thus, ψ_0 and its translates are rational. Moreover, Φ^2 fixes ψ_0 by uniqueness and $\Phi^2\gamma = \gamma\Phi^2$.

If some λ_i repeat, choose the rotation with $\lambda_2 = \lambda_3$. Thus, $\rho_{\boldsymbol{\lambda}}, \pi_{\boldsymbol{\lambda}}$ are rational and fixed by γ. By Theorem 7.27, take the torus of $\mathbf{G}_{\mathrm{ad}}$ marked by $c_1c_2c_3w_0$, where c_i is Coxeter on factor i and $\gamma(c_2) = c_3$. Extend its cubic weight ϑ_c by $\widehat{\vartheta}_c(\gamma) = 1$. Since $\widehat{\vartheta}_c^{n_0} = \widehat{\vartheta}_c^c$, Theorem 7.27 and [56, Proposition 3.2] give $Y^c = Y$ for the disconnected induction Y to $D\langle\gamma\rangle$. Moreover, Y has values in k, since $\widehat{\vartheta}_c$ does. $\langle Y|_D, \rho_{\boldsymbol{\lambda}}\rangle_D = \pm\prod_i \eta_{\lambda_i}(c_i) = \pm 1$, by [169, Theorem 2.1(1)] and [40, §13.8]. Thus, by Lemma 7.31, $\rho_{\boldsymbol{\lambda}}$ has a rational extension to $D\langle\gamma\rangle$. Locally, extend $\widehat{\vartheta}$ by 1 on γ and take the rational extension $\widehat{\eta}$ of $\eta_{\boldsymbol{\lambda}}$ from Lemma 7.33. Since $n_0^\gamma = n_0$, its action on $W_0 \rtimes \langle\gamma\rangle$ is conjugation by γ. Thus, $(\widehat{\vartheta}\widehat{\eta})^c = (\widehat{\vartheta}\widehat{\eta})^{n_0}$, so $\widehat{\pi}_{\boldsymbol{\lambda}}$ on $M_D\langle\gamma\rangle$ is rational. For constant $\boldsymbol{\lambda}$, we restrict to $S\langle\gamma\rangle, M_S\langle\gamma\rangle$. Since χ_0, ψ_0 have multiplicity 1 on S, M_S, Lemma 7.31 gives rational graph extensions of χ_0, ψ_0.

For repeated nonconstant $\boldsymbol{\lambda}$, the selected pair $(\rho_{\boldsymbol{\lambda}}, \pi_{\boldsymbol{\lambda}})$ is E-invariant. For constant $\boldsymbol{\lambda}$, the selected pair (χ_0, ψ_0) is E-invariant on S, M_S. Thus, [208, Lemma 3.1] gives rational extensions over E_o. By [156, Lemma 7.1], the extensions on E_o and $\langle\gamma\rangle$ determine rational extensions over E in the respective groups. If the λ_i are distinct, then γ, c exchange the two cyclic triples of $\boldsymbol{\lambda}$. The parents $\rho_{\boldsymbol{\lambda}}, \pi_{\boldsymbol{\lambda}}$ have field k, and $\rho_{\boldsymbol{\lambda}}^\gamma = \overline{\rho_{\boldsymbol{\lambda}}}$ by [208, proof of Proposition 10.9, Step 7]. The transporter n_0 gives the corresponding identity for $\pi_{\boldsymbol{\lambda}}$. Since Φ^2 fixes $\boldsymbol{\lambda}$, the parents are E_o-invariant. Moreover, γ acts trivially on E_o. By [208, Lemma 3.1], we obtain k-valued extensions $\widehat{\rho}, \widehat{\pi}$ with $\widehat{\rho}^\gamma = \overline{\widehat{\rho}}$ and $\widehat{\pi}^\gamma = \overline{\widehat{\pi}}$. Consequently, a mixed stabilizer acts on $\widehat{\rho}, \widehat{\pi}$ either through $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/k)$ and E_o or through their product with (γ, c). Thus, $\widehat{\rho}, \widehat{\pi}$ are fixed.

For nonconstant $\boldsymbol{\lambda}$, the parents on D, M_D extend the constituents on S, M_S. For constant $\boldsymbol{\lambda}$, the inertias at χ_0, ψ_0 are $S \rtimes E, M_S \rtimes E$. Thus, representatives $d \in M_D$ of D/S give the other pairs. Finally, for $J = \Gamma_\chi = \Gamma_\psi$ define $\rho_J(x, j) = \mathrm{Int}(xZ(X))\bar{j}$ on $X \rtimes J, M \rtimes J$. Since ρ_J maps the common centralizer to 1, both pullbacks have scalar 1 there and $\mu = \mu' = 1$. Thus, (χ, ψ) satisfies $\succeq_c$. □

Proposition 7.46. *Let s be the marked element with $R = D_4$, central torus of rank 2 and component group C_3 acting by triality. Then, $(\chi, \psi) \succeq_c$ for the matched ℓ'-characters.*

Proof. By [208, (10.23)], we obtain the lift $\tilde{s} = (\omega_1^\vee + \omega_6^\vee)(\zeta_3)$. Since each lift has cube 1, it is rational in the marking $-q$ and the series factors through S. Thus, the dual Levi $\mathbf{K}^*$ has nodes $2, 3, 4, 5$. Take the dual $\mathbf{K} \leq \mathbf{G}_{\mathrm{ad}}$, and let λ be the central linear character of $K = \mathbf{K}^F$ dual to $\tilde{s}$. Since $-1 \in W(D_4)$, the adjoint factor $A_0 = (\mathbf{K}_{\mathrm{ad}})^F$ is split. By [208, (10.6)], the parent on D is $\rho = \varepsilon R_{\mathbf{K}}^D(\lambda \tau_K)$, where τ_K is the pullback of $\tau \in \mathcal{E}(A_0, 1)$.

Given a generator $\nu \in \mathrm{Irr}(D/S)$, the twist $\rho \mapsto \rho\nu$ acts on τ by triality δ. Moreover, $\gamma\delta\gamma = \delta^{-1}$ by [208, proof of Proposition 10.9, Steps 2–4]. Thus, a δ-orbit of size 3 has a unique γ-fixed τ. By [208, (10.31) and Proposition 10.9, Steps 3–4], the triality 3-orbits contain the degenerate labels of $W(D_4)$. The remaining labels are fixed by γ. Choose $\tau^\gamma = \tau$ in each δ-orbit. By [208, Proposition 10.9, Step 3], $\rho|_S$ is irreducible if $|\tau^{\langle\delta\rangle}| = 3$. Otherwise, $\rho|_S$ has three constituents of multiplicity 1.

By [61, Theorem 3], we choose a rational extension $\widehat{\tau}$ on $A_0\langle\gamma\rangle$. Moreover, set $\widehat{\lambda}(\gamma) = 1$ for the extension of λ. Thus, with $j\colon K\langle\gamma\rangle \to A_0\langle\gamma\rangle$ the adjoint action, set $\widehat{\beta} = \widehat{\lambda}(\widehat{\tau} \circ j)$. For the radical U of the standard γ-stable parabolic with Levi $\mathbf{K}$, tensor the alternating cohomology of Y_U over K with $\widehat{\beta}$, as in Lemma 7.37. Thus, its character Y on $D\langle\gamma\rangle$ has $Y|_D = \pm\rho$.

Since $n_0^\gamma = n_0$, the extension $\widehat{\lambda}$ satisfies $\widehat{\lambda}^{n_0} = \widehat{\lambda}^c$. On A_0, n_0 and γ have the same diagram class, since $-1 \in W(D_4)$. Their ratio is $\mathrm{Int}(a_0)$ with $a_0 \in \mathbf{K}_{\mathrm{ad}}$. Since F fixes the automorphism and $Z(\mathbf{K}_{\mathrm{ad}}) = 1$, $F(a_0) = a_0$ and $a_0 \in A_0$. Thus, the action of n_0 on $A_0\langle\gamma\rangle$ is conjugation by $a_0\gamma$. Consequently, $\widehat{\beta}^{n_0} = \widehat{\beta}^c$. Since U, U^{n_0} are γ-stable and $\dim Y_U = \dim Y_{U^{n_0}}$, Lemma 7.36 compares their traces on $e_{\tilde{s}}$, with $C_{(\mathbf{G}_{\mathrm{ad}})^*}(\tilde{s}) = \mathbf{K}^*$. By the integral trace in [53, Proposition 3.3], $Y^c = Y$ and Y has values in k. Thus, ρ is rational since $Y|_D = \pm\rho$. By Lemma 7.31, we choose $\widehat{\rho}$ rational on $D\langle\gamma\rangle$.

Let η correspond to τ in the $d = 2$ bijection. By [156, proof of Theorem B], the actions of γ, E_o and $\mathrm{Irr}(D/S)$ agree on τ, η. Thus, $\eta^\gamma = \eta$, and we choose a rational extension $\widehat{\eta}$ to $W(D_4) \rtimes \langle\gamma\rangle \cong W(B_4)$. Extend the cubic weight ϑ by 1 on the Tits lifts and on γ, with extension $\widehat{\vartheta}$. Since $\widehat{\vartheta}^{n_0} = \widehat{\vartheta}^c$ and $\widehat{\eta}^{n_0} = \widehat{\eta}$, induction gives a rational $\widehat{\pi}$ on $M_D\langle\gamma\rangle$, by Proposition 7.45.

If the restrictions to S, M_S split, choose the γ-fixed constituents χ_0, ψ_0 as in Proposition 7.45. Then, χ_0, ψ_0 are rational and have multiplicity 1 on S, M_S. Thus, Lemma 7.31 gives graph extensions from $\widehat{\rho}, \widehat{\pi}$ restricted to $S\langle\gamma\rangle, M_S\langle\gamma\rangle$. If the restrictions are irreducible, take $\widehat{\rho}, \widehat{\pi}$. By [208, proof of Proposition 10.9, Step 4], Φ^2 fixes τ and so η. In a split triple, Φ^2 fixes χ_0, ψ_0 by uniqueness and $\Phi^2\gamma = \gamma\Phi^2$. Thus, the pair (χ, ψ) in the selected position is E-invariant, and its rational extensions on E_o and $\langle\gamma\rangle$ determine rational extensions over E by [208, Lemma 3.1] and [156, Lemma 7.1]. Finally, match χ_0, ψ_0 and their D/S-translates for split triples. The maps ρ_J of Proposition 7.45 then give scalar 1 on the common kernel and $\mu = \mu' = 1$. □

Proposition 7.47. *Fix X, S, D, M, M_S, M_D, E from* Section 7.3. *Thus, $3 \mid q+1$, $F|_{X^*(\mathbf{T})} = -q$ and $\ell \geq 7$ divides $q+1$. Let t lift $s \in (\mathbf{T}^*)^{F^*}$ to $\mathbf{G}_{\mathrm{sc}}^*$. Suppose that q is odd and s has order 6, with $R = A_1^4$ and component group C_3, as in* Zhang [208, table in the proof of Proposition 10.8, Step 1]. *Then, $\mathcal{E}(X, s)$ factors through S and the matched pairs (χ, ψ) satisfy $\succeq_c$ on the ℓ'-degrees.*

Proof. Take $t = (\omega_2^\vee + \omega_3^\vee + \omega_5^\vee)(\zeta_6)$ from [208, (10.23)]. Since $6 \mid q+1$, its image is rationally W-conjugate to s. By [208, Proposition 10.9, Step 1], $\mathcal{E}(X, s)$ factors through S. The form in [208, (10.29)] is $A_1(q)^4 T_2$ up to central isogeny. Let ϑ be the weight of T_D dual to t. Since $t^\gamma = t$ and $w_0 = -\gamma$, $\vartheta^\gamma = \vartheta$ and $\vartheta^{n_0} = \vartheta^{-1}$. By [208, proof of Proposition 10.9, Steps 3–4], C_3 fixes the fourth A_1 factor and rotates $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ on the others. For nonconstant $\boldsymbol{\lambda}$, choose $\lambda_2 = \lambda_3$ for γ's transposition $(2\,3)$. The parents ρ, π are γ-fixed. By [208, proof of Proposition 10.9, Step 4], ρ has values in $\mathbb{Q}(\zeta_3)$. On both sides, n_0 restores t^{-1} and permutes $\boldsymbol{\lambda}$ by $(2\,3)$. Thus, $\bar{\rho} = \rho$ and $\bar{\pi} = \pi$, giving rationality. By [208, Lemma 3.1], if $\rho|_S, \pi|_{M_S}$ are irreducible, ρ, π are the unique real extensions.

Next, $\widehat{\vartheta}$ denotes the extension with $\widehat{\vartheta}(\gamma) = 1$. Take its disconnected torus induction Y_D on $D\langle\gamma\rangle$ and ordinary induction Y_M on $M_D\langle\gamma\rangle$. Since t has order 6, the coefficient action on $\widehat{\vartheta}$ is the identity or inversion on T_D. Moreover, $n_0^\gamma = n_0$, so $\widehat{\vartheta}^c = \widehat{\vartheta}^{n_0}$ for complex conjugation c. Thus, ordinary induction gives $Y_M^c = Y_M$. Theorem 7.27 and [56, Proposition 3.2] give $Y_D^c = Y_D$. Since both have values in $\mathbb{Q}(\zeta_6)$, Y_D, Y_M are rational. By [169, Theorem 2.1(1)], $|\langle Y_D|_D, \rho\rangle_D|$ is the product of $|\langle R_i, u_i\rangle|$ for $1 \leq i \leq 4$, with $u_i \in \{1, \mathrm{St}\}$ and R_i nonsplit torus induction. Thus, $\langle Y_D|_D, \rho\rangle_D = \pm 1$ by [40, §13.8].

For the local parent π, we put $\mathbf{K} = \langle \mathbf{T}_{\mathrm{ad}}, U_\alpha : \alpha \in R\rangle$ and take the covering isogeny $i_R\colon \mathbf{H}_R \to \mathbf{K}_{\mathrm{der}}$. Put $U_R = i_R(N_{\mathbf{H}_R}(\mathbf{T}_R)^F)$ for the covering torus $\mathbf{T}_R$. By Lang's theorem on $\mathbf{T}_R$, U_R maps onto $W(R)$ since F acts trivially on $W(R)$. Moreover, $U_R \cap T_D = i_R(\mathbf{T}_R^F)$. The image of t on the dual covering torus is 1, since all roots of R have value 1 at t. Thus, $\vartheta|_{U_R\cap T_D} = 1$ by torus duality. Thus, $\Lambda(\vartheta)(xu) = \vartheta(x)$ for $x \in T_D, u \in U_R$, by Proposition 7.22. For b in the Weyl or diagram group, i_R maps to $i_{b(R)}$. Moreover, σ acts by $\vartheta \mapsto \vartheta^\sigma$. Hence, Λ is equivariant also on tz. Then, $\pi = \mathrm{Ind}_{T_DU_R}^{M_D}(\Lambda(\vartheta)\eta)$ for $\eta \in \mathrm{Irr}(W(R))$. Since $\eta(1) = 1$, [95, Lemma 5.2] gives $\langle Y_M|_{M_D}, \pi\rangle_{M_D} = 1$. This construction of $\Lambda(\vartheta)$ applies at any order of t.

By Lemma 7.31, ρ, π have rational graph extensions from Y_D, Y_M. If the parents split on S, M_S, their restrictions to $S\langle\gamma\rangle, M_S\langle\gamma\rangle$ have coefficient 1 at the unique γ-fixed constituents χ_0, ψ_0. By Lemma 7.31, χ_0, ψ_0 have rational graph extensions. The pair (χ_0, ψ_0) is γc-fixed, as in Theorem 7.24. Since $r \equiv -1 \bmod 6$, Φ^2 fixes ϑ and $\boldsymbol{\lambda}$. Moreover, Φ^2 fixes χ_0, ψ_0 since $\Phi^2\gamma = \gamma\Phi^2$. Thus, by Proposition 7.46, the extension over $E = E_o \times \langle\gamma\rangle$ and its diagonal translates give $(\chi, \psi) \succeq_c$ by ρ_J. When $r = 2$, semisimple s has odd order, which excludes 6. □

Lemma 7.48. *Suppose that $C = C_{\mathbf{G}^*}(s)$ is disconnected with $C \le \mathbf{L}_0^* < \mathbf{G}^*$ for a Levi $\mathbf{L}_0^*$. Put $A = Z(C)^\circ$. Then, the minimal Levi $\mathbf{L}^* = C_{\mathbf{G}^*}(A)$ has derived type A_2^2 for s in the interior of $\mathcal{T}$. On an open edge, $\mathbf{L}^*$ has type A_5 if c cycles three wall roots, and $A_2^2A_1$ if c fixes one wall root α'. For the dual rational Levi $\mathbf{L}$, we put $M_L = N_{\mathbf{L}}(\mathbf{T})^F$. Then, induction gives a bijection from $\mathrm{Irr}(M_L \mid \theta)$ to $\mathrm{Irr}(M \mid \theta)$.*

Proof. By the affine stabilizer formula [20, Proposition 3.14], $W_s = W(R) \rtimes \langle c\rangle$, with Weyl part c of the rotation. The coordinates a, b, d on the orbits $\{0, 1, 6\}, \{2, 3, 5\}, \{4\}$ satisfy $3a + 6b + 3d = 1$. Thus, $\dim\ker(c - 1) = 2$ on the ambient lattice over $\mathbb{Q}$. Since $A = Z(C)^\circ$, $X_*(A)_{\mathbb{Q}} = \ker(c - 1) \cap R^\perp$. Thus, $\dim A$ is 2 in the interior of $\mathcal{T}$ and 1 on an edge. Hence, $\mathbf{L}^*$ has semisimple rank 4 in the interior and 5 on an edge. Since $C \le \mathbf{L}^*$, $W_s \le W(\mathbf{L}^*)$.

For $n \le 5$, an order-3 signed permutation $v \in W(D_n)$ has at most one 3-cycle and sign product 1 on each cycle. Thus, $\mathrm{rk}(v - 1) \le 2$, excluding a D_n factor since $\mathrm{rk}(c - 1) = 4$ and the complementary rank is ≤ 1. In type A, $\mathrm{rk}(c-1) = 4$ gives two 3-cycles. Hence, the type is A_2^2 at rank 4. If c cycles three wall roots $\alpha_1', \alpha_2', \alpha_3'$, $s_{\alpha_i'}$ are disjoint transpositions in one factor $\mathfrak{S}_n$. Thus, $n \ge 6$ and $\mathbf{L}^*$ has type A_5. If c fixes one wall root α', then $s_{\alpha'}$ transposes two c-fixed letters. Together with the two 3-cycles of c, this requires $A_2^2A_1$ at rank 5.

Since $W_\theta = W_s \le W(\mathbf{L}^*)$, $I = M_\theta \le M_L$. By [95, Theorem 6.11], $\mathrm{Irr}(I \mid \theta)$ indexes both $\mathrm{Irr}(M_L \mid \theta)$ and $\mathrm{Irr}(M \mid \theta)$ through induction. Thus, induction $M_L \to M$ is a bijection. Put $C_M = M \rtimes J$, where J is the ordinary stabilizer. Let $B \le C_M$ stabilize $\mathbf{L}$ and $W(\mathbf{L})\theta$. Since each $j \in J$ fixes $W\theta$, some $m \in M$ corrects j to an element fixing θ and so the canonical $\mathbf{L}$. Thus, $C_M = MB$. Moreover, if $m \in B \cap M$, then $\theta^m = \theta^{m_L}$ for some $m_L \in M_L$. Since $mm_L^{-1} \in I \le M_L$, $B \cap M = M_L$. On a common central cover $J' \to J$, we set $\widehat{B} = B \times_J J'$ and $\widehat{C}_M = C_M \times_J J'$. Then, [95, Problem 5.2] gives $(\mathrm{Ind}_{\widehat{B}}^{\widehat{C}_M} \widehat{\psi}_L)|_M = \mathrm{Ind}_{M_L}^M \psi_L$ for an extension $\widehat{\psi}_L$ of ψ_L. For $\nu \in \mathrm{Irr}(J')$ linear, induction takes $\widehat{\psi}_L\nu$ to $(\mathrm{Ind}\,\widehat{\psi}_L)\nu$. Similarly, induction is equivariant under $\sigma \in \mathcal{H}_\ell$.

Since $C(s^u) = C(s)$ for $(u, |s|) = 1$, a unit power fixes $\mathbf{L}^*$ and I. Given a mixed stabilizer (y, σ), we choose $m \in M$ with $\theta^{z\sigma} = \theta$ for $z = \mathrm{Int}(m)y$. Then, z normalizes B, and induction compares the (z, σ)-twists on an equivariant cover J'. Moreover, $C_{X\rtimes J}(X) \le B$ and centralizes the coset representatives $m \in M$, so $\widehat{\psi}_L$ gives the same scalar on $C_{X\rtimes J}(X)$. Thus, for a common section $J \to J'$, the correcting functions $\mu_{y,\sigma}$ agree by Proposition 7.44. □

Proposition 7.49. *For s in an open face of $\mathcal{T}$ of positive dimension, $(\chi, \psi) \succeq_c$ for the matched pairs.*

Proof. Since γ stabilizes each node orbit of c, it fixes t in $\mathcal{T}$. The image s is rational in this marking since $M/T = W$. Thus, $t^{q+1} \in C_3$, and γ inverts C_3 while fixing t. Consequently, $t^{q+1} = 1$ and $\mathcal{E}(X, s)$ factors through S. Moreover, c conjugates t to its central translates tz, $z \in C_3$. Thus, all parents on D belong to $\mathcal{E}(D, t)$, and those on M_D lie over the orbit of ϑ dual to t.

For $R = \varnothing, A_1, A_1^3$, the labels η_i on each A_1 factor are $1, \mathrm{St}$ under Jordan decomposition. Moreover, $\mathrm{Irr}(D/S)$ rotates the three labels on A_1^3 and fixes the labels for $R = \varnothing, A_1$. Take $\Lambda(\vartheta)$ on T_DU_R from Proposition 7.47, with $U_R = 1$ when $R = \varnothing$. Then, the action on local η_i is also by $\mathrm{Irr}(D/S)$. By Theorem 7.24, if the rotation fixes ρ, π, their restrictions each have 3 constituents of multiplicity 1. Given an orbit of size 3, $\rho|_S, \pi|_{M_S}$ are irreducible. Since $\eta_i \in \{1, \mathrm{St}\}$, a moving orbit has one γ-fixed label. Thus, choose ρ, π at γ-fixed labels. Since $n_0(t^{-1}) = t$ and n_0 acts as γ on the factors, ρ, π are real. If $\rho|_S, \pi|_{M_S}$ are irreducible, we have that ρ, π are the unique real parents over $D/S \cong C_3$ by [208, Lemma 3.1]. If $\rho|_S$ splits, the triple determines both ρ and π uniquely. Moreover, complex conjugation ι on

the constituents lies in D/S, the centralizer of the 3-cycle. Since $\iota^2 = 1$, the constituents are real. Thus, the γ-fixed members are χ_0, ψ_0 of Theorem 7.24.

Take $\widehat{\vartheta}(\gamma) = 1$ and form Y_D, Y_M by Proposition 7.47. Then, $\langle Y_D|_D, \rho\rangle = \pm 1$ and $\langle Y_M|_{M_D}, \pi\rangle = 1$. If $(\chi,\psi)^{y\sigma} = (\chi,\psi)$, then $\rho^{y\sigma} = \rho$ and $\pi^{y\sigma} = \pi$ by uniqueness. Write $y = d\,e$ with $d \in M_D$ and $e \in E$. The part d acts by inner conjugation on $D\langle\gamma\rangle$ and $M_D\langle\gamma\rangle$, so only e, σ affect Y_D, Y_M. Their action on t gives t^u with u prime to $|t|$. Since $\rho^{y\sigma} = \rho$, $t^u \in Wt$. Moreover, $C_{\mathbf{G}^*_{\mathrm{sc}}}(t)$ is connected, so $W_t = W(R)$. Thus, $w(t^u) = t$ and $wR^+ = R^+$ have a unique solution w, by simple transitivity of $W(R)$ on $\{vR^+ : v \in W(R)\}$. Since γ fixes t, t^u, R^+, $\gamma(w) = w$. Thus, the rational Tits lift n_w satisfies $n_w^\gamma = n_w$ and restores $\widehat{\vartheta}$, including $\widehat{\vartheta}(\gamma) = 1$. By [56, Proposition 3.2] and ordinary induction, $Y_D^{y\sigma} = Y_D$ and $Y_M^{y\sigma} = Y_M$.

Therefore, Lemma 7.31 gives graph extensions $\widehat{\rho}_\gamma, \widehat{\pi}_\gamma$ fixed by (y,σ). If the restrictions split, $e\sigma$ maps the γ-fixed χ_0, ψ_0 to γ-fixed members, since $e\gamma = \gamma e$ and $\sigma\gamma = \gamma\sigma$. Thus, an operation stabilizing χ_0, ψ_0 has $d \in S \cap M_D = M_S$. The coefficients at χ_0, ψ_0 in the restricted witnesses are 1. By Lemma 7.31, χ_0, ψ_0 have (y,σ)-fixed graph extensions. For irreducible $\rho|_S, \pi|_{M_S}$, we choose $\widehat{\rho}_\gamma, \widehat{\pi}_\gamma$ as before. Next, we put $E_1 = E_{(\chi,\psi)}$ and $E_{1,o} = E_1 \cap E_o$. By [208, Lemma 3.1], the unique real extensions over $E_{1,o}$ are fixed by (y,σ). Since $E_1 = E_{1,o} \times \langle\gamma\rangle$, the result of [156, Lemma 7.1] gives invariant extensions over E_1. Thus, $\mu = \mu' = 1$. The diagonal translates and maps ρ_J of Proposition 7.45 give $(\chi,\psi) \succeq_c$. □

Proposition 7.50. *With q odd and $3 \mid q+1$, we choose T as in* Section 7.3. *Then, for s of orders $2, 4$ fixed by γ in* Zhang [208, (10.9)]*, we have that the matched ℓ'-pairs satisfy $\succeq_c$ with the action of D/S. Moreover, s of order 4 occurs on T if $4 \mid q+1$.*

Proof. The lift $\tilde{s}$ of order $|s|$ is unique since $|s|$ is a power of 2. It is rational in the marking $-q$ if $|s| \mid q+1$. Given $z \in C_3 \setminus \{1\}$, the order of $\tilde{s}z$ is $3|s|$. Thus, the twists by $\mathrm{Irr}(D/S)$ lie in distinct series $\mathcal{E}(D, \tilde{s}z)$, by [208, proof of Proposition 10.7, (10.21)]. Consequently, restriction to S from $\mathcal{E}(D,\tilde{s})$ is irreducible and bijective. On M_D the order $|s|$ also separates the orbit of ϑ from the orbits of $\vartheta\iota$ for $1 \neq \iota \in \mathrm{Irr}(M_D/M_S)$. Thus, compare (ρ,π) on D, M_D over their field inertia.

If $|s| = 2$, by [208, proof of Proposition 10.6, Case 1], we obtain a rational extension $\widehat{\rho}$ to $D\langle\phi\rangle$. Locally, let λ be the quadratic character of the D_5 Levi K. The source on $(M_D)_\vartheta$ is $(\lambda|_{(M_D)_\vartheta})\eta$ with $\eta \in \mathrm{Irr}(W(D_5))$. Since Φ fixes $\lambda, W(D_5)$ and $\Phi^{2f} = 1$, by Proposition 7.42, we obtain the operator 1 at Φ. After induction to $M_D\langle\Phi\rangle$, we obtain a rational extension $\widehat{\pi}$ of π with its action of M_D/M_S.

Suppose now that $|s| = 4$. Since $q \equiv -1 \bmod 4$ and f is odd, $r \equiv -1 \bmod 4$. Put $k = \mathbb{Q}(i)$ and take the spaces $V_\lambda, V_{\lambda^{-1}}$ of [208, proof of Proposition 10.6, Case 3]. On $V_\lambda \oplus V_{\lambda^{-1}}$, we write P for the operation by ϕ, K_0 for coefficient conjugation over k, and $J_0 = J_\lambda \oplus J_{\lambda^{-1}}$ for the averaging operator in [208, proof of Proposition 10.6, Case 3, following (10.15)]. Since J_0 has rational normalization and ϕ fixes $n_1 n_6$ and $U \cap nU^- n^{-1}$, $PK_0 = K_0 P$, $PJ_0 = J_0 P$ and $K_0 J_0 = J_0 K_0$. Moreover, $J_0^2 = 1$ by [208, proof of Proposition 10.6, Case 3, following (10.15)]. Thus, $A = J_0 P$ and $B = J_0 K_0$ act on V_λ, with $AB = BA$, $A^2 = P^2$, $A^{2f} = 1$, and $B^2 = 1$. The operator A implements ϕ, and B is a semilinear intertwiner for D.

For the simple Hecke generators T_j, the quadratic relation has $Q_j \in \mathbb{Q}$ with $Q_j > 1$. Since $P^{-1}K_0$ fixes the double coset of T_j, its action on T_j is $T_j \mapsto b_j T_j$. From $T_j^2 = Q_j + (Q_j - 1)T_j$, $b_j = 1$. Thus, $B = A(P^{-1}K_0)$ acts as A on the rational Hecke algebra $\mathbb{Q}\langle T_j\rangle$. By [208, proof of Proposition 10.6, Case 3], A fixes the B_2 factor and exchanges the two A_1 factors. If the A_1 labels coincide, choose e_{B_2} primitive over $\mathbb{Q}$ for ρ as in [208, proof of Proposition 10.6, Case 3]. Thus, $e = e_{B_2} \otimes e_{A_1} \otimes e_{A_1}$ is fixed by A, B. Consequently, the module eV_λ with field operator A has rational form $(eV_\lambda)^B$ and character $\widehat{\rho}$. If the A_1 labels are unequal, [208, proof of Proposition 10.6, Case 3] gives the k-valued parents ρ, ρ^ϕ. The operator $P^{-1}K_0$ fixes their rational idempotents, so $B = A(P^{-1}K_0)$ exchanges the blocks and gives $\rho^\phi = \overline{\rho}$. Since f is odd and ϕ^2 fixes each block, also $\rho^\gamma = \overline{\rho}$. By [208, Lemma 3.1], the extension $\widehat{\rho}$ on $D \rtimes E_o$ with $\widehat{\rho}^\gamma = \overline{\widehat{\rho}}$ is unique and k-valued.

For the local quartic weight, $W_\vartheta = W(A_3) \times W(A_1)^2$. Since $\alpha(\tilde{s}) = 1$ for $\alpha \in R$, by Proposition 7.47, we obtain $\Lambda(\vartheta)$ on $T_D U_R$. The Weyl character $\eta \in \mathrm{Irr}(W_\vartheta)$ corresponds to ρ by Theorem 7.24. The graph γ fixes the A_3 wall gradients and exchanges the two A_1 factors. Moreover, n_0 corrects coefficient inversion with action γ on W_ϑ, since $w_0 = -\gamma$. Thus, γ exchanges the A_1 labels of π as on ρ. Also, Φ^2 fixes ϑ and η.

If the A_1 labels coincide, we set $\Lambda(\vartheta)(\gamma) = 1$ in the extended source and let γ transpose the A_1 tensor places. Since $n_0^\gamma = n_0$, the conjugate source character $\overline{\Lambda(\vartheta)\eta}$ equals $(\Lambda(\vartheta)\eta)^{n_0}$, including at γ. Thus, induction gives the

graph extension $\widehat{\pi}_\gamma$ over $\mathbb{Q}$. Together with the rational extension $\widehat{\pi}_o$ over E_o, it gives a rational extension $\widehat{\pi}$ over E by [208, Lemma 3.1] and [156, Lemma 7.1]. If the A_1 labels are unequal, the local parent π satisfies $\pi^\gamma = \overline{\pi}$, so [208, Lemma 3.1] gives the k-valued extension with $\widehat{\pi}^\gamma = \overline{\widehat{\pi}}$. Consequently, the mixed stabilizer fixes $\widehat{\rho}, \widehat{\pi}$ on the extensions by E_o. Finally, restriction to S, M_S and the maps ρ_J of Proposition 7.45 give $\succeq_c$ on X, M. □

As in Section 7.3, let J be the ordinary stabilizer, $B = (M \rtimes J)_\theta$, and H the cyclic field image of B. Let $C = C_{\mathbf{G}^*}(s)$ be connected for $s \neq 1$, with root system R. Then, we find $W_\theta = W(R)$. Each reflection in $W(R)$ fixes each lift $\tilde{s}z$, $z \in C_3$, so the orbits $W \cdot \tilde{s}z$ are distinct. If $2 \mid |H|$, Zhang [208, proof of Proposition 10.5, Case 2] gives the unique graph-fixed orbit $W \cdot \tilde{s}z$. By Proposition 7.43, choose $\theta^\gamma = \theta$ on that orbit.

Proposition 7.51. *Suppose that C is a proper Levi and R is a product of chains. If $2 \mid |H|$, suppose that C is the Levi of a γ-stable parabolic $\mathbf{P}^*$ with $s^\gamma = s$. Then, we have that the matched ℓ'-pair (χ, ψ) satisfies $\succeq_c$ with its central character on $Z(X)$. If $|s|$ is odd, we have that C is a proper Levi and $\mathbf{P}^*$ can be chosen γ-stable when $2 \mid |H|$. The comparison $\succeq_c$ also holds for R of type D.*

Proof. Let $\mathbf{K}$ be the primal Levi and put $K = \mathbf{K}^F$. By [58, Proposition 11.4.12], the element $s \in Z(\mathbf{K}^*)$ determines $\lambda \in \mathrm{Irr}(K)$ with $\lambda|_T = \theta$ and $\lambda(1) = 1$. By [208, (10.6)], $\chi = \varepsilon R_{\mathbf{K}}^{\mathbf{G}}(\lambda\tau_K)$. Moreover, write $A_i = (\mathbf{K}_{i,\mathrm{ad}})^F$ for the adjoint factors. Thus, $\tau_K = (\boxtimes_i \tau_i) \circ j_K$, where $j_K \colon K \to \prod_i A_i$ is the adjoint map and $\tau_i \in \mathcal{E}(A_i, 1)$ by [58, Proposition 11.3.8]. The action $-q$ takes R_i to itself and has based representative $-qw_0(R_i)$. Consequently, A_i has unitary type and $d' = \mathrm{ord}_\ell(-q) = 1$. Also, $\ell \nmid \tau_i(1)$ since $\tau_i(1) \mid \chi(1)$.

By [156, proof of Proposition 8.5], each τ_i has an extension $\widehat{\tau}_i$ over its field inertia E_i with values in $\mathbb{Q}_\ell$. On A_i, the diagram automorphism γ_i is a field power modulo $\mathrm{Inn}(A_i)$. Thus, we choose identical models $\widehat{V}_i$ for $\widehat{\tau}_i$ on equal pairs (A_i, τ_i). Let J_τ be the stabilizer of $\boxtimes_i \tau_i$ in the group of factor operations $A_i \rtimes E_i$ and permutations Σ of identical pinned factors. Thus, J_τ acts on $\bigotimes_i \widehat{V}_i$ by the action of $(a, \sigma_0) \in J_\tau$ as in [156, proof of Theorem 8.5]. Write $\widehat{\tau}$ for its character on J_τ. If the tensor permutation has cycles $\mathcal{C}$, and a_i maps the i-th place to its successor, then $\widehat{\tau}((a_i)_i, \mathcal{C}) = \prod_{(i_1,\dots,i_e)\in\mathcal{C}} \widehat{\tau}_{i_1}(a_{i_e} \cdots a_{i_1}) \in \mathbb{Q}_\ell$. Thus, $\widehat{V}_i$ may be over $\mathbb{C}$ while $\widehat{\tau}$ takes values in $\mathbb{Q}_\ell$.

Given a joint stabilizer element (y, σ), we choose $m \in M$ as in Proposition 7.44, so $y_0 = \mathrm{Int}(m)y$ satisfies $\theta^{y_0\sigma} = \theta$ and $y_0(R^+) = R^+$. Then, $\lambda^{y_0\sigma} = \lambda$ by the restriction to T. The map $\varepsilon R_{\mathbf{K}}^{\mathbf{G}}$ is injective on $\mathcal{E}(K, s)$, and the τ_i are rational. Consequently, the action of y_0 is inner in J_τ, and $\widehat{\tau}^{y_0\sigma} = \widehat{\tau}$ since σ fixes $\mathbb{Q}_\ell$. Locally, Lemma 7.33 gives $\widehat{\eta}$ over $\mathbb{Q}$, with $w_0(R_i)$ at the diagram involution of R_i.

Now, put $L = KB$ and $D_\lambda = L/\ker\lambda$, and let $\kappa \colon L \to D_\lambda$. Thus, $D_\lambda \to J$ has central kernel $Z_\lambda = K/\ker\lambda$ as in Proposition 7.44. In particular, $\Lambda(l, d) = \bar{\lambda}(\kappa(l)d^{-1})$ on $L \times_J D_\lambda$. On $M_\theta = N_K(\mathbf{T})$, $\lambda|_{M_\theta} = \Lambda(\theta)$ by Proposition 7.22. Thus, fix $\Lambda\widehat{\tau}$ on $L \times_J D_\lambda$ and $\Lambda|_{B\times_J D_\lambda}\widehat{\eta}$ on $B \times_J D_\lambda$, via the maps to J_τ and the Weyl inertia of η. Both have scalar $\bar{\lambda}(z)^{-1}$ at $(1, z)$ for $z \in Z_\lambda$.

The kernel of $L \to H$ acts in $\prod_i F_r^i(\mathbf{K})$, since its Weyl coordinates are in $W(R)$ and its diagonal representatives are in $\mathbf{T} \leq \mathbf{K}$. Thus, the construction of U in Proposition 7.44 allows any parabolic $\mathbf{P}$ with Levi $\mathbf{K}$ if $|H|$ is odd, and with $\gamma(\mathbf{P}) = \mathbf{P}$ if $|H|$ is even. Since $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$, the comparison of Y_U with $Y_{U^{y_0}}$ and extraction of the signed character $\pm\chi$ give $\widehat{\chi}$. The induction from $B \times_J D_\lambda$ gives $\widehat{\psi}$, and both extensions have scalar $\Lambda(c, d)$ on $C_{X\rtimes J}(X) \times_J D_\lambda$. Consequently, the common section $J \to D_\lambda$ gives the common factor set α and functions $\mu_{y,\sigma}$ from Proposition 7.44. Hence, (χ, ψ) satisfies $\succeq_c$, also when the scalar on $Z(X) \cong C_3$ is faithful.

Next, consider C. Let $Q = \sum_{i=1}^6 \mathbb{Z}\alpha_i$ be the lattice $X^*(\mathbf{T}^*)$ of the adjoint torus. By [137, Proposition 21.8], $X^*(Z(C)) = Q/\mathbb{Z}R$, where the included affine gradients generate $\mathbb{Z}R$. If R has no affine node 0, $Q/\mathbb{Z}R$ is free. Otherwise, the finite gradients α_j give zero in $Q/\mathbb{Z}R$, and the affine gradient α_0 has coefficients a_i at nodes i outside R. Thus, the torsion has order $\gcd_i a_i$. The complement $\{4\}$ with mark $a_4 = 3$, gives $|C/C^\circ| = 3$ by [208, proof of Proposition 10.8, Step 1]. Consequently, the torsion in $Q/\mathbb{Z}R$ has exponent dividing 2.

Let $A = Z(C)^\circ_{\mathrm{red}}$. If $|s|$ is odd, $s \in A$, so $C = C_{\mathbf{G}^*}(A)$. If $2 \mid |H|$, the image of $a \mapsto a\gamma(a)$ on A is in $(A^\gamma)^\circ_{\mathrm{red}}$ and contains s^2. Since $|s|$ is odd and $\gamma(s) = s$, it follows that $s \in (A^\gamma)^\circ_{\mathrm{red}}$. Consequently, $C = C_{\mathbf{G}^*}((A^\gamma)^\circ_{\mathrm{red}})$, and a generic cocharacter in $(A^\gamma)^\circ_{\mathrm{red}}$ defines a γ-stable parabolic with Levi C. If $|H|$ is odd, a generic cocharacter in A suffices. For $|s|$ even, $s^2 \in A$ and, when $\gamma(s) = s$, $s^4 \in (A^\gamma)^\circ_{\mathrm{red}}$. Thus, the Levi condition follows from $C_{\mathbf{G}^*}(s^2) = C$ if $|H|$ is odd or from $C_{\mathbf{G}^*}(s^4) = C$ in the position $s^\gamma = s$.

Finally, by Lemma 7.41, $R = D_4$ or D_5 if R has a D-component. Moreover, $-q$ gives split D_4 or twisted D_5. By Proposition 7.42, the global extension $\widehat{\tau}$ and local Weyl extension $\widehat{\eta}$ give scalar $\Lambda(c, d)$ on the cover above. In D_4 the based image is cyclic in S_3, and the marked D_4 parabolic is γ-stable. Moreover, for D_5 the diagram action γ_{D_5} is a

field power Φ^j. The position with $R = D_5$ fixed by γ has complementary coordinates $1/2, 1/2$. By Proposition 7.50, we obtain $\succeq_c$ there. □

Lemma 7.52. *Let $I \subseteq \{0, \dots, 6\}$ index complementary nodes with marks a_i and $a_0 = 1$. Put $\delta = 1$ if $|H|$ is odd and $\delta = \gamma$ otherwise, in the marked position with $s^\delta = s$. Then, C is the Levi of a δ-stable parabolic $\mathbf{P}^*$ through $\mathbf{T}^*$ if and only if $0 \in I$ or $\gcd\{|O|a_i : O \in I/\langle\delta\rangle,\ i \in O\} = 1$.*

Proof. The character group of $Z(C)^\delta$ is $Q/(\mathbb{Z}R + (\delta - 1)Q)$. Modulo the included finite roots and $(\delta - 1)Q$, the basis vectors are e_O for finite node orbits $O \subseteq I$. If $0 \notin I$, the affine gradient α_0 gives the relation $\sum_{O \in I/\langle\delta\rangle} |O| a_i e_O = 0$, with $i \in O$. Consequently, the criterion is equivalent to $Z(C)^\delta$ being a torus. Since $s \in Z(C)^\delta$, $C = C_{\mathbf{G}^*}(Z(C)^\delta)$. By [137, Theorems 13.33 and 25.1], a generic cocharacter of $Z(C)^\delta$ gives a δ-stable parabolic with Levi C.

Conversely, suppose that $\mathbf{P}^*$ is defined by $v_0 \in X_*(\mathbf{T}^*)$. The sum $v = \sum_{g \in \langle\delta\rangle} g v_0$ lies in the open cone for $\mathbf{P}^*$. Hence, v is δ-fixed and defines $\mathbf{P}^*$. For small $\varepsilon > 0$, the unique $w \in W(E_6)$ taking the regular vector $v + \varepsilon 2\rho^\vee$ to the dominant chamber is δ-fixed. Also, wv is dominant. Thus, $w(C)$ is the standard Levi on the δ-stable set $\{i : \langle \alpha_i, wv \rangle = 0\}$. The other finite roots α_j form a permutation basis of $X^*(Z(w(C)))$, so its δ-coinvariants are free. Consequently, $X^*(Z(C)^\delta)$ is free, so the gcd for I is 1 if $0 \notin I$. For $\delta = 1$ and connected C, the criterion fails exactly when $I \subseteq \{2, 3, 5\}$, since these nodes have mark 2 and $I = \{4\}$ gives a disconnected centralizer. If $\delta = \gamma$ and $0 \notin I$, the orbit coefficients $|O|a_i$ at $\{1, 6\}, \{3, 5\}, \{2\}, \{4\}$ are $2, 4, 2, 3$, respectively. Thus, for connected C the criterion fails exactly when $0, 4 \notin I$. □

The isolated involution s with $R = A_5 + A_1$ has $I = \{2\}$. Nevertheless, by Proposition 7.26, we obtain $\succeq_c$ on X, M. The lifts $\tilde{s}z$, $z \in C_3$, have orders 2 at $z = 1$ and 6 at $z \neq 1$. Thus, by Zhang [208, (10.21)], restriction of $\mathcal{E}(D, \tilde{s})$ to S is irreducible and bijective. Similarly, on M_S, restriction follows as in Proposition 7.50. On D, Zhang [208, proof of Proposition 10.7, Step 4] gives the quadratic line on $\mathbf{K}_{\mathrm{ad}}^F$ and two rational constituents of the $\mathrm{PGU}_6(q)$ source, with multiplicity 1. Moreover, on M_D, we have that the quadratic weight ϑ is trivial on the rational torus T_R of the simply connected A_5A_1 subsystem. Thus, Proposition 7.47 gives the source $\Lambda(\vartheta)\eta$ with $\eta \in \mathrm{Irr}(W_\vartheta)$ rational. Consequently, the source operator 1 at Φ gives the extension $\widehat{\pi}$ on $M_D\langle\Phi\rangle$ by induction.

Theorem 7.53. *Let $X = {}^2E_{6,\mathrm{sc}}(q)$ with $3 \mid q + 1$, and let $\ell \geq 7$ be a prime dividing $q + 1$. Then,* $\mathrm{iMN}(X/Z(X), \ell)$ *holds on $M = N_X(P)$ for every central character of $Z(X)$ and every field degree f.*

Take J, B, H from Section 7.3. Suppose that $C = C_{\mathbf{G}^*}(s)$ is connected and the set I fails the test in Lemma 7.52. If $|H|$ is odd, the complement satisfies $I \subseteq \{2, 3, 5\}$. If $|I| = 1$, we find $R = A_5 + A_1$ and $|s| = 2$. Otherwise, at most one of $2, 3, 5$ remains. Thus, we have that R has a chain C_0 through the remaining inner node $j \notin I$ and isolated lines indexed by A. If $I = \{2, 3, 5\}$, we put $C_0 = \varnothing$ and $A = \{0, 1, 4, 6\}$.

Lemma 7.54. *On these faces with $|H|$ odd and $|I| \geq 2$, we have that each matched pair (χ, ψ) of ℓ'-degree satisfies $\succeq_c$ with its central character on $Z(X)$. If $|H|$ is even and $R = C_0 \sqcup \{\pm\alpha\}$ for a chain C_0 with nontrivial diagram action of γ and a fixed isolated root α, then (χ, ψ) also satisfies $\succeq_c$.*

Proof. Let $\mathbf{K} = \langle \mathbf{T}, U_\alpha : \alpha \in R \rangle$ and $K = \mathbf{K}^F$. Since $s \in Z(\mathbf{K}^*)$, torus duality [58, Proposition 11.4.12] gives $\lambda \in \mathrm{Irr}(K)$ with $\lambda|_T = \theta$ and $\lambda(1) = 1$. The identities $W_\theta = W(R)$, $B \cap K = N_{\mathbf{K}}(\mathbf{T})^F$ and $KB/K \cong J$ give the central cover $D_\lambda = (KB)/\ker\lambda$ and its character Λ by Proposition 7.44. Thus, on $B \times_J D_\lambda$, the local source is $\Lambda\widehat{\eta}$, with the rational Weyl extension $\widehat{\eta}$ from Proposition 7.51.

Let $\mathbf{L}_0 = \langle \mathbf{T}, U_\alpha : \alpha \in C_0 \rangle$. If $C_0 \neq \varnothing$, a rotation ω takes an end $j \notin C_0$ to 0, by Lemma 7.41. Thus, C_0 becomes a finite standard subsystem and $\mathbf{L}_0$ is a Levi. If $C_0 = \varnothing$, $\mathbf{L}_0 = \mathbf{T}$. Fix a parabolic $\mathbf{P}_0$ of Levi $\mathbf{L}_0$ and, for each line in A, choose its root α with $U_\alpha \leq R_u(\mathbf{P}_0)$, and index A by these roots. By Proposition 7.38, for $\mathbf{K}_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$, choose $k_\alpha \in \mathbf{K}_\alpha$ with $k_\alpha^{-1} F(k_\alpha) = n_\alpha$, where n_α lifts s_α. For distinct $\alpha, \alpha' \in A$, the group $\mathbf{K}_\alpha$ centralizes $\mathbf{K}_{\alpha'}$ and U_β for $\beta \in C_0$. Thus, $k = \prod_{\alpha \in A} k_\alpha$ gives $\mathbf{L} = {}^k\mathbf{L}_0$ and $\mathbf{P} = {}^k\mathbf{P}_0$, with $\mathbf{L}^F$ and the rational pairs $(\mathbf{S}_\alpha, \mathbf{B}_\alpha)$ as in Proposition 7.38. The pairs $(\mathbf{S}_\alpha, \mathbf{B}_\alpha)$ are in the adjoint factors of $\mathbf{K}$. For their tuple $\mathcal{T}$, the stabilizer in $\mathbf{K}$ is the inverse image of $\prod_{\alpha \in A} \mathbf{S}_\alpha$ times the adjoint chain factor $(\mathbf{K}_{C_0})_{\mathrm{ad}}$, taken as 1 when $C_0 = \varnothing$. Thus, this inverse image is $\mathbf{L}$. Thus, by Lang's theorem for $\mathbf{L}$, the rational points of the orbit $\mathbf{K} \cdot \mathcal{T}$ form one K-orbit. A permutation π_A of A takes each pair $(\mathbf{S}_\alpha, \mathbf{B}_\alpha)$ to its place $\pi_A(\alpha)$. In the cyclic product $\mathbf{G}^f$ of [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], let $B' \leq KB$ stabilize the field translates of $\mathcal{T}$. Then, $KB = KB'$ and $K \cap B' = \mathbf{L}^F$ by the K-orbit above. The kernel of $B' \to H$ acts by conjugation in the algebraic $\mathbf{K}$, since $W_\theta = W(R)$ and the diagonal lifts t_d are in

$\mathbf{T}(\overline{\mathbb{F}}_r)$. Thus, on the tuple $\mathcal{T}$, the algebraic representatives belong to $\mathbf{L}$. If $|H|$ is odd, we let $a \in B'$ lift a generator of H. The power of a closing each cycle acts through $\mathbf{L}$. Consequently, transport of $\mathbf{P}$ along the cycles gives a B'-stable radical U containing the translates of ${}^k U_\alpha$ for $\alpha \in A$.

On $A_0 = (\mathbf{K}_{C_0,\mathrm{ad}})^F$, we let τ be the unipotent character, and write $\tau_L = \tau \circ j_L$ for the adjoint map $j_L \colon L = \mathbf{L}^F \to A_0$. By [156, proof of Proposition 8.5], τ has an extension $\widehat{\tau}$ with values in $\mathbb{Q}_\ell$ over the field inertia E_{C_0}. If $C_0 = \varnothing$, take $\widehat{\tau} = \tau_L = 1$. Thus, on $B' \times_J D_\lambda$, take $\widehat{\beta} = \Lambda\widehat{\tau}$ via the adjoint action map. Also, $\widehat{\beta}|_L = (\lambda|_L)\tau_L = \beta$. By transitivity, $R^X_{\mathbf{L}}(\beta)$ expands over tori $\mathbf{T}_w$ as in Proposition 7.38. Thus, at χ, the coefficient in [169, Theorem 2.1(1)] factors over C_0 and the lines $\alpha \in A$. The chain contributes $\langle \eta_{C_0}, \eta_{C_0} \rangle = 1$ for its Weyl label η_{C_0}, or 1 if $C_0 = \varnothing$. Each line contributes 1 at $1, \mathrm{St}$ by [40, §13.8]. Hence, $|\langle R^X_{\mathbf{L}}(\beta), \chi \rangle| = 1$, also under central isotypy of $\mathbf{L}, \mathbf{K}$ by [58, Proposition 11.3.8]. In a regular inclusion $X \leq \widetilde{X}$, the series of a fixed lift $\tilde{s}$ contains one parent $\tilde{\chi}$ of χ. Indeed, the orbits $W \cdot \tilde{s}z$ are distinct for connected C.

Given a joint stabilizer (y, σ), we choose $m \in M$ such that $\theta^{\mathrm{Int}(m)y\sigma} = \theta$, as in Proposition 7.44. Then, choose $k_1 \in K$ to restore $\mathcal{T}$ in its K-orbit. Its action on $(KB) \times_J D_\lambda$ is conjugation by $(k_1, 1)$, so it fixes Λ. Put $y_0 = \mathrm{Int}(k_1 m)y$. Since y_0 fixes $\mathcal{T}$, the centralizer $C_{U^*}(s_\Pi)$ in the dual radical U^* equals $C_{(U^{y_0})^*}(s_\Pi)$. Both are the product $\mathbf{U}^*_A$ of the transported root groups $U_{\alpha_i^*}$ for the chosen positive roots on A. Moreover, $\mathbf{U}^*_A$ is $\widehat{F}^*$-stable and equals $C_{V^*}(s_\Pi)$ for $V = \prod_i F_r^i(R_u(\mathbf{P}))$. Consequently, [19, Corollary 6.5 and Remark 6.7] compares U, U^{y_0}, V on e_s as in Proposition 7.38. Thus, on e_s the traces agree by Lemma 7.36. Since σ fixes $\mathbb{Q}_\ell$ and y_0 acts by conjugation in $A_0 \rtimes E_{C_0}$ on $\widehat{\tau}$, $\widehat{\tau}^{y_0\sigma} = \widehat{\tau}$ as in Proposition 7.51. Also, $\Lambda^{y_0\sigma} = \Lambda$, so $\widehat{\beta}^{y_0\sigma} = \widehat{\beta}$. Thus, signed extraction at χ gives $\widehat{\chi}$, and induction from $B \times_J D_\lambda$ gives $\widehat{\psi}$. On $C_{X \rtimes J}(X) \times_J D_\lambda$, the geometric action is 1 and both $\widehat{\chi}, \widehat{\psi}$ have scalar $\Lambda(c, d)$. Thus, the common section gives $\succeq_c$ as in Proposition 7.44.

If $2 \mid |H|$ and γ acts nontrivially on the diagram of C_0, that chain passes through $3, 4, 5$. Then, $2 \in I$ and $A \subseteq \{0\}$, by the diagram argument for R before Proposition 7.38. Thus, C_0 has a γ-stable parabolic $\mathbf{P}_0$, and γ fixes the isolated root group $\mathbf{K}_\alpha$ pointwise. In that proposition, $k^\gamma_\alpha = k_\alpha$, so $\mathcal{T}$ and $\mathbf{P}$ are γ-stable. The element of B' completing half a cycle acts through $\mathbf{L}\gamma$. Consequently, $C_{U^*}(s_\Pi) = \mathbf{U}^*_A$ also for $2 \mid |H|$, giving $(\chi, \psi) \succeq_c$. If $A = \varnothing$, by Proposition 7.51, we obtain $\succeq_c$ with Levi $\mathbf{K}$. □

Lemma 7.55. *Suppose that $2 \mid |H|$ and $R = \prod_i R_i$ has type A factors. On each γ-stable R_i of rank at least 2, suppose that $\gamma(\alpha) = \alpha$ for each simple root α of R_i. Then, we have that each matched ℓ'-pair (χ, ψ) satisfies $\succeq_c$.*

Proof. Since $r^f \equiv -1 \bmod 3$, the degree f is odd. Thus, the 2-part of H is $\langle \gamma \rangle$. Since C is connected, D/S fixes (χ, ψ). Among the orbits $W \cdot \tilde{s}z$, choose the γ-fixed representative t as in Proposition 7.43. Since $t^{q+1} \in C_3$ and $\gamma(t) = t$, $t^{q+1} = 1$. Thus, t is rational in the marking $-q$. Let ρ, π be the parents on D, M_D in the series of t. Each is unique there since $W \cdot tz \neq W \cdot t$ for $z \neq 1$ in C_3. Moreover, $w_0 = -\gamma$ takes t to t^{-1}. The change from R_i^+ to $-R_i^+$ is by $w_0(R_i)$, which fixes the rational character η_i. Thus, ρ, π are real and γ-fixed, by the coefficients in [169, Theorem 2.1(1)] for ρ and Clifford induction for π. Consequently, by [208, Lemma 3.1], the characters ρ, π are unique as real extensions over C_3. Thus, every joint stabilizer fixes ρ, π. Over $H_{2'}$ we similarly have that the extensions $\widehat{\rho}_o, \widehat{\pi}_o$ are unique and real.

In $W(R_i)$, choose c_i with principal hook lengths of μ_i as cycle lengths when $\mathrm{rank}(R_i) \geq 2$, with $c_{\gamma(i)} = \gamma(c_i)$. If $\mathrm{rank}(R_i) = 1$, we set $c_i = 1$. Thus, $c = \prod_i c_i$ is γ-fixed and $\prod_i \eta_i(c_i) = \pm 1$ by Proposition 7.35. Take a representative n_c fixed by γ, F_r as in Theorem 7.27. Then, in $\mathbf{G}^\gamma_{\mathrm{ad}}$ of type F_4, Lang's theorem gives the torus $\mathbf{T}_c$ at cw_0. Its weight θ_c is dual to t. Thus, extend it by $\widehat{\theta}_c(\gamma) = 1$ and let Y be the disconnected torus induction to $D\langle\gamma\rangle$. On D, its coefficient at ρ is $\pm \prod_i \eta_i(c_i) = \pm 1$ by [169, Theorem 2.1(1)].

Since $\rho^{y\sigma} = \rho$, the pair (y, σ) takes t to $t' \in W \cdot t$. The unique transporter w for t', t and their positive root systems is γ-fixed. Moreover, w transports each $c_i^{W(R_i)}$ to the class $c_j^{W(R_j)}$. On a fixed higher factor R_i, γ acts trivially on $W(R_i)$. If $\gamma(i) \neq i$, take corrections u_i and $u_{\gamma(i)} = \gamma(u_i)$. Thus, w_c returns t' to t and satisfies $\gamma(w_c) = w_c$ and $w_c c = c w_c$. Since $w_0 = -\gamma$, it centralizes cw_0. Thus, a folded lift n_{w_c} has Frobenius discrepancy $n_{w_c}^{-1} F_c(n_{w_c})$ in the γ-fixed torus, where F_c is Frobenius at cw_0. On the adjoint torus $\mathbf{T}_{\mathrm{ad}}$, the simple roots α_i form a basis permuted by γ. On its dual $\mathbf{T}^*_{\mathrm{sc}}$, the fundamental weights ω_i form such a basis. Consequently, $\mathbf{T}^\gamma_{\mathrm{ad}}$ and $(\mathbf{T}^*_{\mathrm{sc}})^\gamma$ are connected. Thus, Lang's theorem lifts w_c to n with $F_c(n) = n$ and $\gamma(n) = n$. Thus, n restores $\widehat{\theta}_c$ with value 1 at γ.

The marked F_r satisfies $F_r F_c = F_c F_r$. Under the Lang conjugation by $g_c \in \mathbf{G}^\gamma$, its transport differs by $\mathrm{Int}(g_c^{-1} F_r(g_c))$, which centralizes γ. Thus, the norm identity of [56, Proposition 3.2] for $Y^{y\sigma} - Y$ gives $Y^{y\sigma} = Y$, since the alternating traces lie in $\mathbb{Z}$ by [53, Proposition 3.3]. For the extensions ρ_+, ρ_- to $D\langle\gamma\rangle$, the integer coefficients

$m_\pm = \langle Y, \rho_\pm\rangle$ satisfy $m_+ + m_- = \pm 1$. Hence, $m_+ \neq m_-$ and (y,σ) fixes ρ_+, ρ_- individually.

On M_D, we let $\Lambda(\vartheta)$ be the extension of ϑ in Proposition 7.47, with value 1 on U_R and at γ. Moreover, the Weyl extension $\widehat{\eta}$ is rational by Lemma 7.33. At w_0 the transporter w has an F-rational lift n_w fixed by γ, by Lang's theorem on $\mathbf{T}_{\mathrm{ad}}^{\gamma}$. Then, n_w conjugates the subsystem normalizer images U_R and acts on $\widehat{\eta}$ by the based stabilizer $(D_R)_\eta$. Thus, induction from the source $\Lambda(\vartheta)\widehat{\eta}$ gives a graph extension $\widehat{\pi}_\gamma$ fixed by (y,σ). Finally, $H = H_{2'} \times \langle\gamma\rangle$ is cyclic, so ρ, π extend over H. By [156, Lemma 7.1], the extensions over $H_{2'}$ and $\langle\gamma\rangle$ determine $\widehat{\rho}, \widehat{\pi}$ over H. Consequently, restriction to S, M_S and the maps ρ_J of Proposition 7.45 give $\succeq_c$ on X, M. □

Proof of Theorem 7.53. Take the matching Ω of Theorem 7.24. If $s = 1$, [156, Theorem D] with $d = 2$ gives $\succeq_c$ on M. If $C_{\mathbf{G}^*}(s)$ is disconnected, Propositions 7.45, 7.46 and 7.47 give $\succeq_c$ at each vertex v. If $C_{\mathbf{G}^*}(s)$ is disconnected on an open face F_s, by Proposition 7.49, we obtain $\succeq_c$. By Proposition 7.43, we obtain $\succeq_c$ when $W_\theta = 1$. For connected $C_{\mathbf{G}^*}(s)$ with $s \neq 1$ and $R \neq \varnothing$, by Proposition 7.51, we obtain $\succeq_c$ for $R = D_n$ and for type A when I satisfies Lemma 7.52. Moreover, for $|s| = 2, 4$ as in [208, (10.8)], by Proposition 7.50, we obtain $\succeq_c$. At $R = A_5 + A_1$, by Proposition 7.26, we obtain $\succeq_c$. Thus, for the remaining faces with $|H|$ odd or a chain C_0 with nontrivial diagram action of γ, by Lemma 7.54, we obtain $\succeq_c$. Otherwise, by Lemma 7.55, we obtain $\succeq_c$. Each comparison is on $M = N_X(P)$ and includes every central character of $Z(X)$. The case $q = 2$ is empty since $q + 1 = 3$. □

Theorem 7.56. *Let $S = {}^2E_6(q)$ be simple and ℓ an odd prime dividing $q+1$. Then,* $\mathrm{iMN}(S,\ell)$ *holds for every field degree f and for $|Z(X)| = 1, 3$, where $X = {}^2E_{6,\mathrm{sc}}(q)$. On X the comparison is on $M = N_X(\mathbf{T})$. If $\ell \geq 7$, then $M = N_X(P)$ for $P \in \mathrm{Syl}_\ell(X)$ in M. If $\ell = 5$, we find $N_X(P) \leq M$. At $\ell = 3$, we have that the extensions on X and the case $S = {}^2E_6(2)$ are as in* Theorem 7.27.

Lemma 7.57. *Suppose that $5 \mid q+1$. With $Q = O_5(T)$, we choose $P \in \mathrm{Syl}_5(M)$. Then, $P \in \mathrm{Syl}_5(X)$, Q is characteristic in P, and $M = N_X(Q)$. Consequently, we obtain that $N_X(P) \leq M < X$ and $\mathrm{Aut}(X)_P$ fixes M.*

Proof. The order formula [58, §13.1] and $q \equiv -1 \bmod 5$ give $|X|_5 = 5(q+1)_5^6$, since $v_5(q^5+1) = v_5(q+1) + 1$. The degrees $2, 5, 6, 8, 9, 12$ of $W(E_6)$ give $|W(E_6)|_5 = 5$. Thus, $|M|_5 = |X|_5$ and $P/Q \cong C_5$. By [132, Theorem 5.14], $N_X(P) \leq M$. Let w generate the image of P in $W(E_6)$, and put $Y = X_*(\mathbf{T})$. Since w has order 5, the lattice $Y_1 = Y \cap (1-w)(\mathbb{Q} \otimes Y)$ has rank 4 and $\det(1 - w|_{Y_1}) = \Phi_5(1) = 5$. Thus, w acts nontrivially on $Y/5Y$, since Y_1 is saturated. The subgroup $Q[5]$ is $W(E_6)$-isomorphic to $Y/5Y$, so P is nonabelian. The principal 5-block b_0 of X has defect group P. Thus, the maximal abelian normal subgroup $A \lhd P$ is unique by [161, Theorem 11.1]. Since $[P : Q] = 5$, $A = Q$. Consequently, Q is characteristic in P.

By Corollary 7.25, $C_{\mathbf{G}}(Q)^\circ = \mathbf{T}$ follows from the root values $\alpha(\alpha^\vee(\zeta_5)) = \zeta_5^2 \neq 1$ for α a root and ζ_5 a primitive fifth root. Thus, $N_X(Q) \leq M$. Conversely, M normalizes $Q = O_5(T)$, so $N_X(Q) = M$. Since Q is characteristic in P, $\mathrm{Aut}(X)_P$ normalizes M. Finally, $QZ(X)/Z(X)$ is a nontrivial 5-subgroup of the simple group S, so $N_X(Q) = M < X$. □

Proposition 7.58. *Let $X = {}^2E_{6,\mathrm{sc}}(q)$, with $\ell \geq 5$ a prime dividing $q+1$. Let (χ,ψ) be an ℓ'-pair of* Theorem 7.24 *with connected $C_{\mathbf{G}^*}(s)$. Then, we have that (χ,ψ) satisfies $\succeq_c$ on X, M with its central character on $Z(X)$.*

Proof. If $3 \nmid q+1$, by Theorem 7.40, we obtain $\succeq_c$. If $3 \mid q+1$ and $\ell \geq 7$, by Theorem 7.53, we obtain $\succeq_c$. Thus, take $3 \mid q+1$ and $\ell = 5$. Then, q is odd, since $q = 2^f \equiv -1 \bmod 3$ gives odd f, whereas $2^f \equiv -1 \bmod 5$ gives $f \equiv 2 \bmod 4$. At $s = 1$, [156, Theorem D] with $d = 2$ gives the comparison on M. Suppose that $s \neq 1$, with centralizer roots R. By Clifford theory [95, Theorem 6.11], $[W : W_\theta] \mid \psi(1)$, so $5 \mid |W_\theta|$. The map $w \mapsto w(\tilde{s})\tilde{s}^{-1}$ on W_θ has image in C_3 and kernel $W(R)$ by connectedness upstairs [58, Chapter 3]. Consequently, $5 \mid |W(R)|$. Since $C_{\mathbf{G}^*}(s)$ is connected, $W_\theta = W(R)$. Thus, the affine diagram and $5 \mid |W(R)|$ leave $R = A_4, A_4 + A_1, A_5, A_5 + A_1, D_5$.

Let I index the complementary affine nodes. If $|H|$ is odd, by Lemma 7.52, $I \subseteq \{2, 3, 5\}$ whenever the Levi test fails. Thus, $5 \mid |W(R)|$ gives $|I| = 1$ among those sets, with $R = A_5 + A_1$. If $|H|$ is even, we choose the lift $\tilde{s}^\gamma = \tilde{s}$ as in Proposition 7.51. Thus, I is γ-stable, and the test fails when $0, 4 \notin I$. Among those sets, only $I = \{2\}$ and $I = \{1, 6\}$ give $5 \mid |W(R)|$, with $R = A_5 + A_1, D_5$, respectively. In particular, $R = A_3$ and $R = A_3 + A_1^2$ have no $5'$-pair.

If $C_{\mathbf{G}^*}(s)$ passes the Levi test with R of type A, by Proposition 7.51, we obtain $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ on the cover from $KB/\ker\lambda$. Indeed, $d' = \mathrm{ord}_5(-q) = 1$ on the unitary factors R_i, and $\widehat{\eta}$ extends the rational label of $W(R)$. Thus, its parabolic proof gives $\succeq_c$ for $J \leq \mathrm{Aut}(X)_P$, since $\mathbf{T} = C_{\mathbf{G}}(Q)^\circ$ is J-stable. If $R = D_5$ and $|H|$ is odd, by Proposition 7.42, we obtain $\widehat{\tau}, \widehat{\eta}$ for the sources on $KB/\ker\lambda$.

If $R = A_5 + A_1$, by Proposition 7.26, we obtain $\succeq_c$ for the isolated involution s. If $I = \{1, 6\}$ and $2 \mid |H|$, the γ-fixed lift $\tilde{s}$ has coordinates $1/2, 1/2$ at $1, 6$. Thus, $|s| = 2$, and [208, proof of Proposition 10.6, Case 1] gives the extension on D. The source on M_D and the restrictions to S, M_S follow from Proposition 7.50. Indeed, Φ and the lift orders $2, 6$ are independent of ℓ. Put $\bar{Q} = QZ(X)/Z(X)$ and $\bar{P} = PZ(X)/Z(X)$. Since $Z(X)$ is a $5'$-group, $\bar{Q}$ is characteristic in $\bar{P}$ and $C_{\mathbf{G}_{\mathrm{ad}}}(\bar{Q})^\circ = \mathbf{T}_{\mathrm{ad}}$. Consequently, $N_D(\bar{P}) \leq M_D$, and the marked complement E of Proposition 7.45 normalizes $\mathbf{T}_{\mathrm{ad}}$. Thus, a Sylow stabilizer element $de \in DE$ has $d \in M_D$ because both de and e normalize $\mathbf{T}_{\mathrm{ad}}$. Consequently, the maps ρ_J restrict the models on DE, M_DE to $X \rtimes J, M \rtimes J$, giving $\succeq_c$. □

Proposition 7.59. *If $5 \mid q + 1$, then $\mathrm{iMN}({}^2E_6(q), 5)$ holds on $M = N_X(O_5(T))$.*

Proof. Take $P \in \mathrm{Syl}_5(M)$ and $Q = O_5(T)$. By Lemma 7.57, $P \in \mathrm{Syl}_5(X)$, Q is characteristic in P, and $M = N_X(Q)$ is proper and $\mathrm{Aut}(X)_P$-stable. Moreover, for $\psi \in \mathrm{Irr}_{5'}(M)$ over θ, $5 \mid |W(R)|$ by the index $[W : W_\theta]$ in Proposition 7.58. If $C_{\mathbf{G}^*}(s)$ is disconnected, the rotation ω has the orbits $\{4\}, \{2, 3, 5\}, \{0, 1, 6\}$. On the ω-fixed faces, by Lemma 7.48, $R = \varnothing, A_1, A_1^3, A_1^4, D_4, A_2^3$. In each case $5 \nmid |W(R)|$. Thus, all $5'$-pairs have connected $C_{\mathbf{G}^*}(s)$. By Proposition 7.58, we obtain $\succeq_c$ for every pair under Ω on X, M. Since $q + 1 \geq 5$, $q \neq 2$ and X is the universal cover. Thus, Ω, $N_X(P) \leq M$, and Lemma 9.4 give $\mathrm{iMN}({}^2E_6(q), 5)$. □

Proof of Theorem 7.56. At $\ell = 3$, Theorem 7.27 gives the comparison on M, and [157, Lemma 4.1(2)] gives $\mathrm{iMN}({}^2E_6(2), 3)$. At $\ell = 5$, Proposition 7.59 gives $\mathrm{iMN}(S, 5)$. Thus, suppose that $\ell \geq 7$. By the proof of Corollary 7.25, $P = O_\ell(T)$ and $N_X(P) = M$. If $3 \nmid q + 1$, Theorem 7.40 gives $\succeq_c$ on every pair of Ω. If $3 \mid q + 1$, by Theorem 7.53, we obtain $\mathrm{iMN}(S, \ell)$ for Ω on M. Consequently, $\mathrm{iMN}(S, \ell)$ holds for every odd $\ell \mid q + 1$. □

With $X = E_{6,\mathrm{sc}}(q)$ and $q = r^f$, we let $\ell \geq 5$ be a prime dividing $q + 1$. Take the marked torus $\mathbf{T}$ of Lemma 7.21, with Frobenius action $-q\gamma$ on its lattice. Put $T = \mathbf{T}^{F'}$, $P = O_\ell(T)$, and $M = N_X(\mathbf{T})$, so $M/T = W^\gamma = W(F_4)$. Thus, we fix Ω from Theorem 7.24. Given $\theta \in \mathrm{Irr}(T)$, we write s for its dual element, R for the roots of $C_{\mathbf{G}^*}(s)^\circ$, and W_s for the stabilizer of s in $W = W(E_6)$. Put $Z_3 = Z(\mathbf{G}^{*\mathrm{sc}})$ on geometric points. Since F' acts by $-q\gamma$, we deduce that the subsystem R is γ-stable. Thus, we have that $R^+ = R \cap \Phi^+$ is γ-stable for the pinned positive system Φ^+.

Proposition 7.60. *We have that $P \in \mathrm{Syl}_\ell(X)$ and $M = N_X(P)$. If $W_s^\gamma = 1$, then $\chi = \varepsilon R_{\mathbf{T}}^X(\theta)$ and $\psi = \mathrm{Ind}_T^M \theta$ form a matched pair with $(\chi, \psi) \succeq_c$, where ε gives $\chi(1) > 0$.*

Proof. By Corollary 7.25, $P \in \mathrm{Syl}_\ell(X)$ and $M = N_X(P)$, since $\ell \nmid |W(F_4)|$. If $R \neq \varnothing$, the longest element of $W(R)$ is nonidentity and γ-fixed, contrary to $W_s^\gamma = 1$. Thus, $R = \varnothing$. On a simply connected dual lift t of s, the map $w \mapsto w(t)t^{-1}$ from W_s to Z_3 has kernel $W(R) = 1$ by Steinberg's theorem [58, Chapter 3]. Consequently, $|W_s| \mid 3$. By [53, Theorem 6.8] and [95, Theorem 6.11], χ, ψ are irreducible. Their degrees $|X : T|_{r'}$ and $|M : T|$ are prime to ℓ, since $O_\ell(T) \in \mathrm{Syl}_\ell(X)$. Take $J = \mathrm{Aut}(X)_{P,\chi} = \mathrm{Aut}(X)_{P,\psi}$, $A = X \rtimes J$, $C = M \rtimes J$, and $B = C_\theta$. Then, $C = MB$, $A = XB$ and $B \cap M = T$, since $W_s^\gamma = 1$. The group $C_A(X)$ lies in B by Proposition 7.43.

Suppose the projection of B to the field and graph group $\langle F_r, \gamma \rangle$ contains γ. Its action on s in the marked torus is $g = w\gamma$ with $w \in W^\gamma$ and $g(s) = s$. Thus, $g^2 \in W_s^\gamma = 1$. Put $\mathcal{O} = \{W \cdot tz : z \in Z_3\}$. By [208, proof of Proposition 10.5, Case 2], choose a γ-fixed orbit $O \in \mathcal{O}$ with closed-alcove representative t_0. Thus, $\gamma(t_0) = t_0$. Choose $v \in W$ taking the image of t_0 to s. Then, $h = v\gamma v^{-1}$ fixes s and the chamber $v(\Phi^+)$. Thus, g, h are in the nonidentity coset of $W_s \rtimes \langle h \rangle$ and have order 2. Since $|W_s|$ is odd, Sylow conjugacy gives $g = h^a$ for some $a \in W_s$. Consequently, g fixes the conjugate chamber through $\mathbf{T}^*$.

Take $\mathbf{G}^f$ as in [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], with shift u and Frobenius $\widehat{F} = u^{-1}\,\mathrm{diag}(F_r)$. Since X is split, $u^f = 1$ and the graph γ acts on each coordinate. Write $\mathbf{T}_0$ for the torus in $X = \mathbf{G}^F$ and put $\mathbf{Q} = \prod_i F_r^i(\mathbf{T}_0)$. A lift of $b \in B$ acts as $\mathrm{Int}(\tilde{x})u^j\gamma^e$ on $\mathbf{G}^f$, with $\tilde{x} \in (\widetilde{\mathbf{G}}^f)^{\widehat{F}}$. Thus, $Z(\widetilde{X})$ acts trivially. In the coordinate stabilizer B_i at i, the shift is u^0. If $\gamma^e = 1$, the Weyl coordinate lies in $W_s^\gamma = 1$. If $\gamma^e = \gamma$, the chamber action is that of $g_i = F_r^i g F_r^{-i}$. Thus, choose a chamber C_i^+ fixed by B_i. The translates $b(C_i^+)$, for $b \in B$, depend only on $b(i)$. Consequently, the product Borel has B-stable radical U.

Set $D_B = A \times_J B$, $D_C = A \times_J C$, and $C_0 = \{(c, c) : c \in C_A(X)\}$. By [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], the action of $(a, b) \in D_B$ on Y_U is the algebraic action of b followed by left multiplication by $ab^{-1} \in X$. In particular, C_0 acts trivially. Ξ_B denotes the alternating character of $R\Gamma_c(Y_U)[\theta^\vee]$ on D_B. Put $\Xi = \mathrm{Ind}_{D_B}^{D_C} \Xi_B$. Since $C_0 \lhd D_C$, Ξ is inflated from D_C/C_0. Let $V = \prod_i F_r^i(R_u(\mathbf{B}_0))$ for a Borel $\mathbf{B}_0$ through $\mathbf{T}_0$, and let s_Π be dual to θ on $\mathbf{Q}^{\widehat{F}} \cong T$. Since $R = \varnothing$, $C_{U^*}(s_\Pi) = C_{V^*}(s_\Pi) = 1$. Thus, by [19, Corollary 6.5 and

Remark 6.7], the restriction $\Xi_B|_{X\times T}$ is computed on Y_V up to sign δ. The first projection of Y_V in [208, proof of Lemma 4.19(b)] gives $\Xi_B|_{X\times T} = \delta R^X_{\mathbf{T}}(\theta)\boxtimes\theta^\vee$, with $\delta\in\{\pm1\}$. Since $D_C = (X\times M)D_B$, [95, Problem 5.2] gives $\Xi|_{X\times M} = \delta R^X_{\mathbf{T}}(\theta)\boxtimes(\mathrm{Ind}_T^M\,\theta)^\vee$.

Given a joint stabilizer (y,σ), we choose $m\in M$ with $\theta^{z\sigma}=\theta$, where $z=\mathrm{Int}(m)y$. Then, z normalizes B, and U, U^z are B-stable of equal variety dimension. Moreover, $C_{U^*}(s_\Pi) = C_{(U^z)^*}(s_\Pi) = 1$. Since $\theta^{z\sigma}=\theta$, (z,σ) fixes e_{s_Π} and the projector $e_{\theta^\vee}$. Thus, Lemma 7.36 gives $\Xi_B^{z\sigma} = \Xi_B$. Consequently, $\Xi^{y\sigma}=\Xi$ by the correction m in Proposition 7.43. The coefficient of $\chi\boxtimes\psi^\vee$ in $\Xi|_{X\times M}$ is $\delta\varepsilon$, and the central characters are $\theta|_{Z(X)}$. By Proposition 2.4, we obtain an extension V_0 with $C_0\le\ker V_0$, fixed by (y,σ). By Corollary 2.5, we obtain $\succeq_c$. □

Proposition 7.61. *Suppose that $R=A_1^a$ and $W_s^\gamma = W(R)^\gamma$. Then, we have that all matched ℓ'-pairs (χ,ψ) satisfy $\succeq_c$ on X, M, including when γ permutes the factors of R.*

Proof. Let $D_R = (W_s)_{R^+}$. Then, $W_s = W(R)\rtimes D_R$ since $W(R)$ acts simply transitively on $\{wR^+ : w\in W(R)\}$. The map $w\mapsto w(t)t^{-1}$ has kernel $W(R)$ by Steinberg's theorem [58, Chapter 3], so $|D_R|\mid 3$. Moreover, D_R is γ-stable and $D_R^\gamma = 1$. Take an extension $\widetilde\theta$ on the regular torus $\widetilde T$ of Theorem 7.24, with dual lift $\tilde s$. If a twist by $\nu\in\mathrm{Irr}(\widetilde X/X)$ fixes $\mathcal{E}(\widetilde X,\tilde s)$, its dual translation $\tilde s z$ is conjugate to $\tilde s$ for a rational central z by [10, Fact 2.29]. Thus, $z\in Z_3$, since conjugation fixes the quotient $\widetilde{\mathbf{G}}^*/[\widetilde{\mathbf{G}}^*,\widetilde{\mathbf{G}}^*]$. Its coset in $W_s/W(R)$ is fixed by γ, by rationality of z. Since $D_R^\gamma=1$, $z=1$. Consequently, restriction of $\mathcal{E}(\widetilde X,\tilde s)$ to X is irreducible and each χ has one parent $\tilde\chi\in\mathcal{E}(\widetilde X,\tilde s)$ by [10, Fact 2.29 and Theorem 2.35] and [95, Corollary 11.22]. By [169, Theorem 2.1(1)], the coefficient at $\tilde\chi$ in $R^{\widetilde X}_{\widetilde{\mathbf{T}}}(\widetilde\theta)$ factors over the γ-orbits on R, up to sign. Each coefficient at $1,\mathrm{St}$ is ±1 by [40, §13.8]. After restriction to X, $|\langle R^X_{\mathbf{T}}(\theta),\chi\rangle| = 1$. Locally, take $U_R = i_R(V_R^{\delta_R})$ from Proposition 7.22. Then, $\Lambda(\widetilde\theta)|_{U_R}=1$, also on exchanged factors $R_i, R_{\gamma(i)}$. Thus, $\Lambda(\theta)=\Lambda(\widetilde\theta)|_{M_\theta}$ is trivial on U_R. Moreover, $W(R)^\gamma$ is generated by the products $\prod_{i\in O}s_i$ over the orbits O of γ on the A_1 factors. Thus, its labels η are linear. The source on M_θ is $\Lambda(\theta)\eta$, and $\langle\mathrm{Ind}_T^M\,\theta,\psi\rangle = 1$ by [95, Lemma 5.2].

We now take A, C, J from Proposition 7.60 and put $B = C_{\theta,R^+}$. Since γ fixes R^+ and its transport R_1^+, the unique $w\in W(R)$ with $R_1^+\mapsto R^+$ is γ-fixed. Thus, $B\to J$ is onto and $B\cap M = T$, since $D_R^\gamma = 1$. Therefore, the mixed correction $z=\mathrm{Int}(m)y$ can be chosen with $\theta^{z\sigma}=\theta$ and $z(R^+)=R^+$. If γ lies in the image of B, it acts by $g=w\gamma$ with $g(s)=s$, $g(R^+)=R^+$, and $g^2\in D_R^\gamma = 1$. Take the geometric chamber involution h as in Proposition 7.60. Conjugate h and its chamber by $W(R)$ so the chamber induces R^+. Thus, g, h differ by D_R, and they are conjugate by D_R since $|D_R|$ is odd. Consequently, g fixes a chamber that induces R^+. Thus, choose a B_i-fixed chamber that induces $F_r^i(R^+)$. If the graph image is 1, B_i acts trivially on chambers through $F_r^i(\mathbf{T}_0)$. Thus, propagation under B gives a B-stable radical U with $F_r^i(R^+)$ on each coordinate i.

Take $D_B, D_C, C_0, \Xi_B, \Xi$ from Proposition 7.60 for this B, U. Moreover, $C_0\le\ker\Xi$, since $C_A(X)\le B$ acts trivially on Y_U. Take $V=\prod_i F_r^i(R_u(\mathbf{B}_0))$ for $\mathbf{B}_0$ that induces R^+. Then, $C_{U^*}(s_\Pi) = C_{V^*}(s_\Pi) = \prod_i F_r^i(U^*_{R^+})$, where $U^*_{R^+} = \langle U^*_\alpha : \alpha\in R^+\rangle$. Similarly, $C_{U^*}(s_\Pi) = C_{(U^z)^*}(s_\Pi)$, since z takes $F_r^i(R^+)$ to $F_r^{z(i)}(R^+)$. By [19, Corollary 6.5 and Remark 6.7], we obtain the restriction on $X\times T$ from V, and Lemma 7.36 gives $\Xi_B^{z\sigma} = \Xi_B$ for U, U^z as in Proposition 7.60. Consequently, $\Xi|_{X\times M} = \delta R^X_{\mathbf{T}}(\theta)\boxtimes(\mathrm{Ind}_T^M\,\theta)^\vee$, and $\Xi^{y\sigma}=\Xi$. Since its coefficient at $\chi\boxtimes\psi^\vee$ is ±1, Proposition 2.4 and Corollary 2.5 give $\succeq_c$ on the original $M = N_X(P)$. □

Lemma 7.62. *Suppose that $D_R = (W_s)_{R^+}$ contains $1\ne c\in W^\gamma$. Then, $q\equiv 1 \bmod 3$. Each lift $t\in\mathbf{T}^*_{\mathrm{sc}}$ obeys $F'^*(t)=t$, and these lifts form one $\langle c\rangle$-orbit. Consequently, the characters in $\mathcal{E}(X,s)$ are trivial on $Z(X)$.*

Proof. By the lift commutator in [208, proof of Proposition 10.9, Step 1], $c(t)=zt$ for $1\ne z\in Z_3$, independent of t. Since c is rational, $F'^*(z)=z$. The split Frobenius acts by $z\mapsto z^q$ on Z_3, so $q\equiv1\bmod3$. Conversely, in this congruence the injection $D_R\to Z_3$ is Frobenius equivariant and $F'^*|_{Z_3}=1$. Thus, if $D_R\ne1$, then $D_R^\gamma = D_R\cong C_3$. In characteristic 3 the group Z_3 of geometric points is 1.

Let V^- be the (-1)-eigenspace of γ on $X_*(\mathbf{T}^*_{\mathrm{sc}})\otimes\mathbb{Q}$, of dimension 2. If $c|_{V^-}=1$, then $(c-1)(V^-+V^+)\subseteq V^+$, since $c\gamma=\gamma c$, where V^+ is the fixed space. Thus, γ fixes $\mathrm{im}(c-1)$ pointwise on $\mathbf{T}^*_{\mathrm{sc}}$. But $z=c(t)t^{-1}$ belongs to that torus and $\gamma(z)=z^{-1}$. Consequently, $c|_{V^-}$ has order 3 with no fixed vector. The norm $N_c = 1+c+c^2$ satisfies $N_c(V^-+V^+)\subseteq V^+$. Thus, γ fixes $\mathrm{im}\,N_c$ pointwise, and $t\,c(t)c^2(t)=t^3$ gives $\gamma(t^3)=t^3$. Since s^3 is rational for $-q\gamma$, $t^{3(q+1)}\in Z_3$. Moreover, $t^{3(q+1)}$ is fixed by γ, so $t^{3(q+1)}=1$.

Write $t=t_{3'}t_3$ for its primary decomposition. Then, $t_{3'}$ is a power of t^3 of order dividing $q+1$, so $F'^*(t_{3'})=t_{3'}$. Also, $|t_3|\mid3$ since $3\nmid q+1$. The projection s_3 of t_3 satisfies $\gamma(s_3)=s_3^{-1}$ because $q\equiv1\bmod3$ and $F'^*(s_3)=s_3$. Thus,

we have that $\gamma(t_3)t_3 \in Z_3$ is fixed by γ, and so is 1. Consequently, $F'^*(t) = t$. The powers of c give the three lifts t, zt, z^2t. Thus, by [208, proof of Proposition 10.9, Step 1], the central character on $Z(X)$ is $1_{Z(X)}$. □

Put $S = X/Z(X)$, $D = \mathbf{G}_{\mathrm{ad}}^{F'}$, $M_D = N_D(\mathbf{T}_{\mathrm{ad}})$, and $M_S = M/Z(X)$. By Lang's theorem on $\mathbf{T}$, we have that $M_D/M_S \cong D/S \cong C_3$ as in Theorem 7.24. Consequently, χ has scalar 1 on $Z(X)$ by Lemma 7.62. Thus, ψ also has scalar 1 there by Theorem 7.24. We view χ, ψ on S, M_S. Suppose that $R = A_1^a$, allowing $a = 0$. An adjoint parent ρ at t has labels $1, \mathrm{St}$ on the rational factors of $R = A_1^a$. Its twists by $\mathrm{Irr}(D/S)$ translate t by Z_3 by Zhang [208, proof of Proposition 10.9, Step 1]. Thus, the correction c^{-1} for zt permutes the labels of ρ by the formula in Schaeffer Fry et al. [169, Theorem 2.1(1)]. On M_D, we let ϑ be dual to t. By Proposition 7.22, the map Λ is equivariant and the twists of $\Lambda(\vartheta)\eta$ act on the Weyl label η by c^{-1} for the translation by z. Thus, under Ω, we find that the D/S-orbits on characters of S, M_S have equal sizes. By Isaacs [95, Theorem 6.11 and Corollaries 6.17 and 11.22], if c fixes the label, $\rho|_S$ has three constituents of multiplicity 1. Otherwise, we have that $\rho|_S = \chi$ is irreducible with three parents ρ_1, ρ_2, ρ_3 on D. The restrictions from M_D to M_S satisfy the corresponding formulas.

Proposition 7.63. *Let (χ, ψ) be matched with $R = A_1^a$. If c fixes its label, then $(\chi, \psi) \succeq_c$. If c moves the label η, suppose that the parents ρ, ν over t are fixed by $J_{\mathrm{mix}} = (\mathrm{Aut}(X)_P \times \mathcal{H}_\ell)_{(\chi,\psi)}$. Then, $(\chi, \psi) \succeq_c$ also in this case.*

Proof. If c fixes the labels, the unique parents ρ, ν are fixed by J_{mix}. Otherwise, take the two parents ρ, ν in the hypothesis. With $J = \mathrm{Aut}(X)_{P,\chi} = \mathrm{Aut}(X)_{P,\psi}$, put $\bar{P} = PZ(X)/Z(X)$. By Proposition 7.60, J normalizes $\mathbf{T}_{\mathrm{ad}} = C_{\mathbf{G}_{\mathrm{ad}}}(\bar{P})^\circ$ and so M_D. With $A_D = D \rtimes J$ and $C_D = M_D \rtimes J$, we put $B_D = (C_D)_{\vartheta,R^+}$. The centralizer of t is connected by Steinberg's theorem [58, Chapter 3], so $W_t = W(R)$. Consequently, $B_D \to J$ is onto and $B_D \cap M_D = T_D$, where $T_D = \mathbf{T}_{\mathrm{ad}}^{F'}$. By Proposition 7.61, the alcove of t gives a graph chamber for B_D. The torus coefficient at ρ is ± 1, and the Weyl label of ν is linear, by Proposition 7.61. If $R = \varnothing$, Proposition 7.60 gives the corresponding coefficients on D, M_D. Thus, the fibre product $A_D \times_J C_D$ has the extension V_D of $\rho \boxtimes \nu^\vee$ from Proposition 2.4. Moreover, J_{mix} fixes V_D, and $\{(a, a) : a \in C_{A_D}(D)\} \leq \ker V_D$.

Restrict V_D to $(S \times M_S) \rtimes J$. The base constituent $\chi \boxtimes \psi^\vee$ has multiplicity 1, so the $\chi \boxtimes \psi^\vee$-space affords the extension V_S fixed by the mixed stabilizer. If c moves the labels, $V_D|_{S \times M_S}$ is irreducible. For $(s_0, j) \in C_{S \rtimes J}(S)$, the automorphism $\mathrm{Int}(s_0)j$ of D is trivial on S. Its discrepancies $d^{-1}\,\mathrm{Int}(s_0)j(d)$ for $d \in D$ centralize S. Since $C_D(S) = 1$, the automorphism $\mathrm{Int}(s_0)j$ is trivial on D. Thus, the diagonal centralizer on S, M_S acts trivially on V_S through V_D. Inflation along $X \to S$ and $M \to M_S$ gives V_X, with $Z(X)$ acting trivially. By Corollary 2.5, we obtain $\succeq_c$ on X, M. □

Proposition 7.64. *Suppose that $R = A_1^a$ and $D_R^\gamma \cong C_3$. Then, every moving pair (χ, ψ) has parents ρ, ν in the series of t as in* Proposition 7.63.

Proof. By Lemma 7.62, $c(t_3) = zt_3$ and $|t_3| = 3$. Let $\mathbf{T}^- = (1 - \gamma)(\mathbf{T}_{\mathrm{sc}}^*)$. Since $(1 - \gamma)(t_3) = t_3^2$, $t_3 \in \mathbf{T}^-$. Moreover, t_3 and z are independent, since W fixes Z_3 and $c(t_3) \neq t_3$. Thus, c moves the line $L = \langle t_3 \rangle$ in $\mathbf{T}^-[3]$. The roots $\alpha = \gamma(\alpha)$ form the long D_4 in F_4 by [208, proof of Proposition 10.5]. Hence, their group $W(D_4)$ acts trivially on V^- and has index 6 in W^γ. Let $u = \alpha_1 - \alpha_6$ and $v = \alpha_3 - \alpha_5$ be a basis for V^-. The products $s_1 s_6, s_3 s_5$ act by $(u, v) \mapsto (-u, v + u)$ and $(u, v) \mapsto (u + v, -v)$, respectively. Thus, their images generate $\mathfrak{S}_3$ on V^-. Consequently, $W(D_4)$ is the kernel on V^- and on $\mathbf{T}^-$. The quotient of order 6 has a unique subgroup of order 3, generated by the image of c. Since $c(L) \neq L$, the stabilizer of L in this quotient has order dividing 2.

A moving label has a c-cycle O of three factors of $R = A_1^a$. The rotation ω of Lemma 7.48 has triple orbits $\{0, 1, 6\}$ and $\{2, 3, 5\}$ cannot both occur in $R = A_1^a$, since 0 and 2 are adjacent. Thus, $W(R)$ transports this unique O to the based system R^+. Since $\gamma c = c\gamma$ and O is unique, $\gamma(O) = O$. Thus, the involution γ fixes each root $\alpha \in O$. Consequently, the lines $\mathcal{U} = \{\mathbb{Q}\alpha : \alpha \in O\}$ lie in the long D_4. An element $x \in W(D_4)$ of order 3 has permutation part conjugate to (123). It fixes no support $I = \{i, j\}$ of the roots $\pm e_i \pm e_j$. Therefore, if x stabilizes $\mathcal{U}$, their supports form a 3-cycle. The two support orbits of (123) are $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ and $\{\{1, 4\}, \{2, 4\}, \{3, 4\}\}$. Thus, the supports I, J in one orbit satisfy $|I \cap J| = 1$ if $I \neq J$, so the corresponding roots α, β have $(\alpha, \beta) = \pm 1$. Since $\mathcal{U}$ is orthogonal, $W(D_4)_{\mathcal{U}}$ is a 2-group. Put $K_0 = \{w \in W^\gamma : w(\mathcal{U}) = \mathcal{U},\ w(L) = L\}$. Consequently, K_0 is a 2-group, since its kernel $W(D_4)_{\mathcal{U}}$ and its image in $(W^\gamma/W(D_4))_L$ are 2-groups.

Given $(y, \sigma) \in J_{\mathrm{mix}}$, write $y = de$, where $d \in M_D$ acts by conjugation and $e \in \langle \Phi, \gamma \rangle$ in the marked coordinates. Thus, d fixes the characters of D, M_D. Since $F'^*(t) = t$, $\gamma(t) = t^{-q}$. Thus, (e, σ) takes t to t^k for $\gcd(k, |t|) = 1$. As $\psi^{y\sigma} = \psi$, Clifford theory gives $s^k \in W^\gamma s$. Consequently, a correction $w_1 \in W^\gamma$ acts by $s^k \mapsto s$. A power c^i then acts

by $w_1(t^k) \mapsto t$. Finally, a unique $w_2 \in W(R)^\gamma$ restores R^+. Thus, $w = w_2 c^i w_1$ fixes L since $w(t_3^k) = t_3$. Moreover, $W_{s^k} = W_s$, and w fixes R^+, so it normalizes D_R. Hence, $w(\mathcal{U}) = \mathcal{U}$ and $w \in K_0$.

The field and coefficient actions on $R_i = A_1$ fix $1, \mathrm{St}$. Thus, the parent permutation at t is the action of $w \in K_0$ on $\mathcal{U}$, by the torus coefficients for ρ and $\Lambda(\vartheta)$ for ν. The image K_1 of K_0 on $\mathcal{U}$ is a 2-group in $\mathfrak{S}_3$. Consequently, K_1 fixes a line $U_0 \in \mathcal{U}$. In a moving binary word $(\eta_U)_{U\in\mathcal{U}}$, the singleton label is $\eta^\dagger$. Thus, set $\eta_{U_0} = \eta^\dagger$ and keep η_{U_4} if a fourth line U_4 exists. Consequently, its parents ρ, ν are fixed by J_{mix}, and Proposition 7.63 gives $\succeq_c$. □

If $R = D_4$ and c has diagram order 3, we have that their diagram permutations $\bar{c}, \bar{\gamma}$ satisfy $\bar{c}\bar{\gamma} = \bar{\gamma}\bar{c}$ in $\mathfrak{S}_3$. Thus, the involution γ has trivial diagram action on R. Thus, we have that R is the long D_4 of the fold. The kernel $W(D_4)$ on V^- has trivial based stabilizer for R. Consequently, the group fixing R^+ and the line $L = \langle t_3 \rangle$ embeds into $(W^\gamma/W(D_4))_L$ of order at most 2. If $R = A_2^3$, a rational c permutes its three factors. Since $\gamma c = c\gamma$, the involution γ fixes each factor R_i. By Proposition 7.34, the norms in $(1-\gamma)\mathbb{Z}\Phi$ are divisible by 4, whereas $\|\alpha - \beta\|^2 = 6$ for adjacent simple roots α, β. Hence, γ fixes all six based roots of R, contrary to $\dim V^+ = 4$.

Proposition 7.65. *Suppose that $C = C_{\mathbf{G}^*}(s)$ has $R = D_4$ and $C/C^\circ \cong C_3$. Then, we have that each matched ℓ'-pair (χ, ψ) satisfies $\succeq_c$ on $M = N_X(P)$ for both Frobenius actions on C/C°.*

Proof. By [208, (10.23)], the lift t of the triality element s and its translates tz for $z \in Z_3$ have order 3. Thus, $r \neq 3$. If $q \equiv 1 \bmod 3$, choose the rational lift t of Lemma 7.62. If $q \equiv -1 \bmod 3$, F'^* inverts Z_3, so each rational s has a unique rational lift t. Thus, for $q \equiv -1 \bmod 3$ the isogeny gives $D \cong X$ and $M_D \cong M$. The full centralizer of t is the connected Levi $\mathbf{K}^*$ of type D_4 with centre of dimension 2, as in [208, proof of Proposition 10.9]. Consequently, we set $K = \mathbf{K}^{F'}$ on the primal side.

Put $J_{\mathrm{mix}} = (\mathrm{Aut}(X)_P \times \mathcal{H}_\ell)_{(\chi,\psi)}$. Suppose $q \equiv 1 \bmod 3$. If triality fixes the label η, Clifford theory gives unique parents ρ, ν over χ, ψ on S, M_S. Otherwise, we put $\Gamma_t = (W^\gamma)_{R^+,\langle t_3\rangle}$. Then, Γ_t has order at most 2 by the D_4 case after Proposition 7.64. Moreover, each unipotent τ of $D_4(q)$ is rational and fixed by F_r by [208, proof of Proposition 10.9, Steps 3–4]. Consequently, after correction to fix t, R^+ as in Proposition 7.64, J_{mix} acts on the parents on D, M_D through Γ_t. Consequently, the images on $\{\rho_i\}$ and $\{\nu_i\}$ are 2-groups. Thus, we choose ρ, ν fixed by J_{mix} as in Proposition 7.64. If $q \equiv -1 \bmod 3$, we set $\rho = \chi, \nu = \psi$ under $D \cong X$ and $M_D \cong M$. Write $J = \mathrm{Aut}(X)_{P,\chi} = \mathrm{Aut}(X)_{P,\psi}$ and $B = (M_D \rtimes J)_\vartheta$, with ϑ dual to t. Since ν is J-invariant, $B \to J$ is onto and $B \cap M_D = N_{\mathbf{K}}(\mathbf{T}_{\mathrm{ad}})^{F'}$.

If $b \in B$ has field part 1 and graph part γ, choose a rational correction w with $w\gamma(t) = t$ and $w\gamma(R^+) = R^+$, as in Proposition 7.61. Thus, $g = w\gamma$ satisfies $g^2 = 1$, since g^2 is based and lies in $W_t = W(R)$. Let $\mathbf{A} = Z(\mathbf{K}^*)^\circ_{\mathrm{red}}$. The component group of $Z(\mathbf{K}^*)$ has exponent dividing 2 by the centre of the D_4 derived group, so $t \in \mathbf{A}$. Consequently, the norm $a \mapsto a\, g(a)$ from $\mathbf{A}$ has image in $(\mathbf{A}^g)^\circ_{\mathrm{red}}$ containing t^2, and so also t. Thus, $C_{\mathbf{G}^{*\mathrm{sc}}}((\mathbf{A}^g)^\circ_{\mathrm{red}}) = \mathbf{K}^*$. A generic integral cocharacter v of $(\mathbf{A}^g)^\circ_{\mathrm{red}}$ gives a g-stable parabolic with Levi $\mathbf{K}^*$. The invariant pairing takes v to a cocharacter $v^\vee$ of $\mathbf{T}_{\mathrm{ad}}$ after clearing denominators, with parabolic $\mathbf{P}$ of Levi $\mathbf{K}$. If the coordinate stabilizer B_i has graph image 1, any parabolic $\mathbf{P}$ of Levi $\mathbf{K}$ suffices.

Write $\delta_R = \gamma|_{R^+}$ for the based action. Since $-1 \in W(D_4)$, the based action of $-q\gamma$ is δ_R. If $q \equiv 1 \bmod 3$, $\delta_R = 1$ and the adjoint factor A_0 has type $D_4(q)$. If $q \equiv -1 \bmod 3$, γ inverts c in D_R by [208, proof of Proposition 10.9, Step 1]. Thus, δ_R is a transposition on the D_4 diagram and A_0 has type ${}^2D_4(q)$. Thus, the based Weyl maps in W^γ satisfy $w\delta_R = \delta_R w$ for $w \in W^\gamma$, whose action on A_0 is a field power modulo $\mathrm{Inn}(A_0)$.

By [208, (10.6)], $\rho = \varepsilon R^D_{\mathbf{K}}(\lambda\tau_K)$, where λ is linear on K and dual to t, and $\tau_K = \tau \circ j_K$ for the adjoint map $j_K \colon K \to A_0$. Since $\rho(1)/\chi(1) \in \{1, 3\}$ by Clifford theory, $\ell \nmid \tau(1)$ by the degree formula. By [156, proof of Proposition 8.5] with $d = 2$, choose $\widehat{\tau}$ with values in $\mathbb{Q}_\ell$ on the pinned inertia A_0E_τ. Moreover, the rational graph restriction on $A_0\Gamma_\tau$ in split D_4 is from [61, Theorem 3], with Γ_τ the cyclic graph image on τ. Locally in split D_4, the diagram image D_η has order 1 or 2. Thus, η has a rational extension $\widehat{\eta}$ by Proposition 7.42, with $W(D_4) \rtimes D_\eta = W(B_4)$ if $|D_\eta| = 2$. In ${}^2D_4$, the transposition δ_R acts trivially on $W(R)^\gamma = W(B_3)$. Indeed, for a rational Weyl map w, the $W(R)$ part and the based part are γ-fixed by uniqueness. On $W(R)^\gamma$ the pinned field maps act as 1.

Moreover, we set $D_\lambda = (KB)/\ker\lambda$ and Λ from Proposition 7.44. For $(y, \sigma) \in J_{\mathrm{mix}}$, choose a correction $m_D \in M_D$ on ϑ, R^+. Since $M_D = M_S T_D$ for $T_D = \mathbf{T}_{\mathrm{ad}}^{F'}$ by Lang's theorem, and T_D acts trivially on ϑ, R^+, we may take $m_D \in M_S$. Thus, $z = \mathrm{Int}(m_D)y$ has $\vartheta^{z\sigma} = \vartheta$ and $z(R^+) = R^+$, and $\Lambda^{z\sigma} = \Lambda$. Moreover, z acts by conjugation in A_0E_τ and in the Weyl inertia of η. Thus, $\widehat{\tau}^{z\sigma} = \widehat{\tau}$ and $\widehat{\eta}^{z\sigma} = \widehat{\eta}$, since their values are in $\mathbb{Q}_\ell$. Consequently, the sources $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ are fixed under $z\sigma$.

On $\mathbf{G}^f$, an element $b \in B_i$ with field and graph parts 1 acts through $\mathbf{K}$. An element of B_i with nontrivial

graph image acts through $\mathbf{K}g$. Thus, propagation of $\mathbf{P}$ as in Proposition 7.60 gives the B-stable radical U. Since $C_{\mathbf{G}^{*\mathrm{sc}}}(t) = \mathbf{K}^*$, $C_{U^*}(t_\Pi) = C_{(U^z)^*}(t_\Pi) = 1$. Consequently, the section $J \to D_\lambda$ gives $\succeq_c$ by Proposition 7.44, with source scalar Λ. The product extension of $\rho \boxtimes \nu^\vee$ has scalar 1 on the diagonal centralizer C_0 since the two scalars $\Lambda(c,d)$ cancel. Thus, restriction to S, M_S and inflation to X, M as in Proposition 7.63 give $\succeq_c$. □

Proposition 7.66. *If $C_{\mathbf{G}^*}(s)$ is connected and R has a D-component, then each matched ℓ'-pair (χ,ψ) satisfies $\succeq_c$ with its central scalar on $Z(X)$.*

Proof. By Lemma 7.41, $R = D_4$ or D_5 and $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ is a proper Levi through $\mathbf{T}^*$. Thus, set $K = \mathbf{K}^{F'} \leq X$ with the linear character λ dual to s. With $J = \mathrm{Aut}(X)_{P,\chi}$, put $B = (M \rtimes J)_\theta$. If B_i has graph image 1, take any $\mathbf{P}$ with Levi $\mathbf{K}$. Otherwise, [208, proof of Proposition 10.5, Case 2] gives a γ-fixed orbit in $\{W \cdot tz : z \in Z_3\}$, with a γ-fixed alcove representative t_0. Take $h = v\gamma v^{-1}$ for the transport $v \in W$ to s. The rational correction $g = w\gamma$ differs from h by $W(R)$, since $W_s = W(R)$. Thus, a parabolic $\mathbf{P}$ of Levi $\mathbf{K}$ fixed by h is fixed by g. If $R = D_4$, take the transport of $\mathbf{P}_{\{2,3,4,5\}}$. If $R = D_5$, the complement $I = \{1,6\}$ has coordinates $1/2, 1/2$. Thus, $|s| = 2$ and q is odd. By [208, proof of Proposition 10.5, Case 2], the opposite parabolics $\mathbf{P}_\pm$ of Levi $\mathbf{K}$ are rational and $g(\mathbf{P}_\pm) = \mathbf{P}_\mp$. Indeed, the class of s is rationally unique by connectedness of $C_{\mathbf{G}^*}(s)$. Let δ_R be the based action of γ on R. If $R = D_4$, based rational Weyl transporters for powers s^a are unique since $W_s = W(R)$. If w_a is the based transporter for s^a, then $w_{ab} = w_a w_b$ since $w_a(s^b) = w_a(s)^b$. Thus, $w_a w_b = w_b w_a$, and γ satisfies $\gamma w = w\gamma$ for $w \in W^\gamma$. Their diagram image is abelian in $\mathfrak{S}_3$. Hence, it is cyclic. The factor A_0 is split D_4 or ${}^2D_4$ according as δ_R is trivial or a transposition. If $R = D_5$, if $\delta_R = 1$ then $W(D_5) \leq W^\gamma = W(F_4)$, contrary to $5 \mid |W(D_5)|$ and $5 \nmid |W(F_4)|$. Thus, δ_R exchanges the ends of D_5. The based action of $-q\delta_R$ is trivial, since opposition on D_5 is also that transposition. Consequently, A_0 is split D_5 and $W(R)^\gamma = W(B_4)$.

By [208, (10.6)], the global source is $\beta = \lambda\tau_K$ and $\tau(1) \mid \chi(1)$. Thus, choose $\widehat{\tau}$ on the adjoint inertia $A_0 E_\tau$ as in Proposition 7.65, with $d = 2$. If D_4 is split, by Proposition 7.42, we obtain $\widehat{\eta}$ over the cyclic diagram image D_η, including $|D_\eta| = 3$. On $W(B_3), W(B_4)$, the diagram involution δ_R acts as 1. Thus, $\widehat{\eta}$ is rational for either folded group $W(R)^\gamma$. With $D_\lambda = (KB)/\ker\lambda$, we choose $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ as in Proposition 7.65. For (y,σ) fixing (χ,ψ), choose $m \in M$ so $z = \mathrm{Int}(m)y$ satisfies $\theta^{z\sigma} = \theta$ and $z(R^+) = R^+$. Then, (z,σ) fixes both sources. If B_i fixes $\mathbf{P}$, propagation and comparison on $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ give $\succeq_c$ by Proposition 7.65. The common section $J \to D_\lambda$ gives the same scalar on $Z(X)$.

Suppose now that $R = D_5$ and g exchanges $\mathbf{P}_+, \mathbf{P}_-$. With $P_\pm = KU_\pm$, we write $I_\pm = \mathrm{Ind}_{P_\pm}^X \mathrm{Inf}_K^{P_\pm} \beta$. By [208, proof of Proposition 10.5], $I_\pm = \chi$, $|U_\pm| = q^{16}$, and $U_\pm \cap P_\mp = 1$. Thus, set $\widehat{A} = (X \rtimes J) \times_J D_\lambda$ and $\widehat{L} = (KB) \times_J D_\lambda$. Then, $\widehat{A} = X\widehat{L}$, $\widehat{L} \cap X = K$, and $C_{\widehat{A}}(X) \leq \widehat{L}$. The group $\widehat{L}$ permutes P_+, P_-, and the source $\widehat{\beta} = \Lambda\widehat{\tau}$ is ordinary on $\widehat{L}$. By Lemma 7.92 with $a = q^8$, the source $\widehat{\beta}$ gives the ordinary extension $\widehat{\chi}$. Since $\widehat{\beta}^{z\sigma} = \widehat{\beta}$, the source cochain gives $\widehat{\chi}^{z\sigma} = \widehat{\chi}$. Moreover, on $C_{\widehat{A}}(X)$ its scalar is $\Lambda(c,d)$. In particular, $(1,d)$ for $d \in Z_\lambda$ has scalar $\bar{\lambda}(d)^{-1}$. The local extension $\widehat{\psi}$ from $\Lambda\widehat{\eta}$ also has scalar $\Lambda(c,d)$. Thus, the section $J \to D_\lambda$ gives $\succeq_c$ on X, M. □

Lemma 7.67. *Let $s \neq 1$ have connected centralizer $C = C_{\mathbf{G}^*}(s)$ with $R = \prod_i R_i$ as in* Section 7.3. *Put $\delta = \gamma|_{R^+}$ on R^+. Thus, we find $F'^* = -q\delta$ on $\mathbb{Z}R$. The graph representative $g = w\gamma$ fixes θ, R^+, with possibly $g|_R \neq \delta$. Let D act by pinned operations on R and on the factors A_i. Then, $\tau \in \mathrm{Irr}(\prod_i A_i)$ of unipotent ℓ'-degree extends to $(\prod_i A_i) \rtimes D_\tau$ with $\widehat{\tau}$ valued in $\mathbb{Q}_\ell$. Moreover, we have that $\eta \in \mathrm{Irr}(W(R)^\delta)$ has a rational extension $\widehat{\eta}$ to $W(R)^\delta \rtimes D_\eta$.*

Proof. Let A_i be the adjoint factor for a δ-orbit in $\{R_i\}_i$. On a stable chain R_i, the based Frobenius action is $-q\delta w_0(R_i)$, where $w_0(R_i)$ is its longest element. Thus, A_i has type ${}^2A_{n-1}(q)$ if δ fixes the basis and $A_{n-1}(q)$ if $\delta|_{R_i} = -w_0(R_i)$. The descriptions coincide for $n = 2$. If $\delta(R_i) = R_j \neq R_i$, then $(-q\delta)^2 = q^2$ on R_i and A_i has type $A_{n-1}(q^2)$. The exchange of R_i, R_j acts on A_i by a field power of F_r up to a diagram action.

For ${}^2A_{n-1}(q)$ and $A_{n-1}(q^2)$, $\mathrm{ord}_\ell(-q)$ and $\mathrm{ord}_\ell(q^2)$ are both 1. By [156, Proposition 8.5 and its proof], $\widehat{\tau}_i$ has values in $\mathbb{Q}_\ell$. For $A_{n-1}(q)$, $d = \mathrm{ord}_\ell(q) = 2$. Write $\mu \vdash n$ for the unipotent label and κ for its 2-core. Since $j \leq n \leq 7$ and $\ell \geq 5$, $v_\ell(q^j - 1) = v_\ell(q+1)$ if $2 \mid j$ and 0 otherwise. The 2-quotient of μ has size $(n - |\kappa|)/2$. By [40, §13.8], $v_\ell(\tau_\mu(1)) = v_\ell(q+1)(|\kappa| - (n \bmod 2))/2$. Hence, $|\kappa| = n \bmod 2$ if $\ell \nmid \tau_\mu(1)$.

For $|\kappa| \leq 1$, the 2-cuspidal pair is $(\mathbf{T}_2, 1_{\mathbf{T}_2^F})$ by [156, proof of Proposition 8.5]. The trivial extension $\widehat{1}_{\mathbf{T}_2^F} = 1$ on the inertia gives $\widehat{\tau}_i$ with values in $\mathbb{Q}_\ell$ by [156, Proposition 8.5]. Moreover, on $A_i \rtimes \langle\gamma\rangle$, $\widehat{\tau}_i$ is rational by [156, Theorem 7.4],

since $|\kappa| \leq 1$. By Proposition 7.51, we obtain $\widehat{\tau}$ on the tensor product of the $\widehat{\tau}_i$, over D_τ. The inner and diagonal operators from A_i are those of $\tau_i = \widehat{\tau}_i|_{A_i}$.

If $\delta|_{R_i} = 1$, $W(R_i)^\delta = \mathfrak{S}_n$. If $\delta|_{R_i} = -w_0$, $W(R_i)^\delta = C_{\mathfrak{S}_n}(w_0) \cong W(B_{\lfloor n/2 \rfloor})$. If $\delta(R_i) = R_j \neq R_i$, the factor is $\{(w, \delta(w)) : w \in \mathfrak{S}_n\}$. Thus, the diagram involution is $\mathrm{Int}(w_0)$ on $\mathfrak{S}_n$ and 1 on $C_{\mathfrak{S}_n}(w_0)$. Consequently, on the Specht module S^μ, the operator is w_0 with $w_0^2 = 1$ by Lemma 7.33. On $W(B_{\lfloor n/2 \rfloor})$, the corresponding operator is 1. These operators and the permutations in D_η give $\widehat{\eta}$. Since F'^* acts as 1 on $W(R)^\delta$, the field wraps act as 1 on $\widehat{\eta}$. □

Proposition 7.68. *Suppose that C is a Levi and put $J = \mathrm{Aut}(X)_{P,\chi} = \mathrm{Aut}(X)_{P,\psi}$. If J has a pure graph representative g with $g(\theta, R^+) = (\theta, R^+)$, suppose that there is a parabolic $\mathbf{P}^*$ with Levi C and $g(\mathbf{P}^*) = \mathbf{P}^*$. Then, we have that each matched ℓ'-pair (χ, ψ) satisfies $(\chi, \psi) \succeq_c$ on $M = N_X(P)$.*

Proof. Let $\mathbf{K}$ be the primal Levi with $K = \mathbf{K}^{F'}$. By [208, (10.6)], $\chi = \varepsilon R_{\mathbf{K}}^X(\lambda \tau_K)$, where $\lambda|_T = \theta$ and τ_K has adjoint factors τ_i. Thus, $\tau_i(1) \mid \chi(1)$. Given a joint stabilizer (y, σ), we choose $m \in M$ with $z = \mathrm{Int}(m)y$, $\theta^{z\sigma} = \theta$ and $z(R^+) = R^+$. Then, $\lambda^{z\sigma} = \lambda$ by torus duality. Moreover, $(\lambda\tau_K)^{z\sigma} = \lambda\tau_K$ by the bijection $\varepsilon R_{\mathbf{K}}^X$ on $\mathcal{E}(K, s)$. Since $\tau_K^\sigma = \tau_K$, z fixes the adjoint tensor label τ. Thus, both $\widehat{\tau}^{z\sigma} = \widehat{\tau}$ and $\widehat{\eta}^{z\sigma} = \widehat{\eta}$ by Lemma 7.67. With $B = (M \rtimes J)_\theta$, we put $D_\lambda = (KB)/\ker\lambda$. The sources are $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ on $(KB) \times_J D_\lambda, B \times_J D_\lambda$, respectively, as in Proposition 7.44. Since $C = \mathbf{K}^*$, the product radical U satisfies $C_{U^*}(s_\Pi) = C_{(U^z)^*}(s_\Pi) = 1$ as in Proposition 7.60 for U. Consequently, by Proposition 7.44, $(\chi, \psi) \succeq_c$ with scalar $\Lambda|_{Z(X)}$.

Let v take the alcove centralizer to C. If J contains a pure graph, choose the γ-fixed lift orbit O as in Proposition 7.60 and take v from O. Then, $h = v\gamma v^{-1}$ satisfies $gh^{-1} \in W_s = W(R)$. Since $W(R)$ fixes each $\mathbf{P}^*$ with Levi C, the test in Lemma 7.52 is on $v^{-1}(R)$. The action on $v^{-1}(R)$ is γ if J has a pure graph, and 1 otherwise. Let I be the affine node complement. If J has no pure graph and C is no Levi, $I \subseteq \{2, 3, 5\}$ and R is $A_5 + A_1, A_3 + A_1^2$ or A_1^4. With a pure graph in J, failure for $v^{-1}(R)$ gives $0, 4 \notin I$ and $\gamma(I) = I$. Thus, if $2 \notin I$, the path $0, 2, 4$ is a component, since either of $3, 5$ would give a branch at 4. If $2 \in I$, a chain through 4 of rank at least two is A_3 or A_5, and in the latter case 0 is isolated. Consequently, each such R has at most one chain C_0 with $\mathrm{rk}(C_0) \geq 2$. □

Proposition 7.69. *Consider a face from* Proposition 7.68 *with $R = C_0 \sqcup A_1^a$ and $\mathrm{rk}(C_0) \geq 2$. If J has no pure graph or $g|_{C_0} = -w_0(C_0)$, then $(\chi, \psi) \succeq_c$.*

Proof. Let $\mathbf{K} = \langle \mathbf{T}, U_\alpha : \alpha \in R \rangle$, with $K = \mathbf{K}^{F'}$ and $\lambda|_T = \theta$ from s. With $B = (M \rtimes J)_\theta$, we put $D_\lambda = (KB)/\ker\lambda$. Thus, the cover and its character Λ are as in Proposition 7.44. Let $\mathbf{L}_0 = \langle \mathbf{T}, U_\alpha : \alpha \in C_0 \rangle$. The rotation ω with $j \mapsto 0$ for an end $j \in I$ in Lemma 7.41 gives that $\mathbf{L}_0$ is a Levi. Since $\delta(C_0) = C_0$, we also have that $F'(\mathbf{L}_0) = \mathbf{L}_0$. Choose $\mathbf{P}_0$ with Levi $\mathbf{L}_0$ and let $\mathcal{A} \subset R^+$ consist of the positive roots α in the A_1 factors. We can conjugate $\mathbf{P}_0$ by independent reflections s_α for $\alpha \in \mathcal{A}$ so that $U_\alpha \leq R_u(\mathbf{P}_0)$ for all $\alpha \in \mathcal{A}$. Thus, $\delta(\mathcal{A}) = \mathcal{A}$.

If $g|_{C_0} = -w_0(C_0)$, the path $v^{-1}(C_0)$ satisfies $\{3, 4, 5\} \subseteq v^{-1}(C_0)$ and $2 \notin v^{-1}(C_0)$. Hence, the standard parabolic $\mathbf{P}_{v^{-1}(C_0)}$ of the finite diagram is γ-stable. We choose $\mathbf{P}_0 = wv(\mathbf{P}_{v^{-1}(C_0)})$, with w a product of the s_α above. Since $gh^{-1} \in W(R)$, $g(\mathbf{P}_0)$ lies in the $W(R)$-orbit of $\mathbf{P}_0$. Moreover, $U_\alpha \leq R_u(g(\mathbf{P}_0))$ for each $\alpha \in \mathcal{A}$, since $g(\mathcal{A}) = \mathcal{A}$. Thus, $g(\mathbf{P}_0) = w'\mathbf{P}_0$ for $w' \in W(C_0)$, which fixes $\mathbf{P}_0$. Consequently, $g(\mathbf{P}_0) = \mathbf{P}_0$.

Let $n_\mathcal{A} = \prod_{\alpha \in \mathcal{A}} n_\alpha$, with n_α a lift of s_α. By Lang's theorem in $\langle U_{\pm\alpha} : \alpha \in \mathcal{A} \rangle$, $d^{-1}F'(d) = n_\mathcal{A}$, as in Proposition 7.38 for n_α. Since $n_\mathcal{A}$ acts by -1 on $\mathcal{A}$, $\mathrm{Int}(n_\mathcal{A})F'$ takes U_α to $U_{\delta(\alpha)}$. Thus, $\mathbf{L} = {}^d\mathbf{L}_0$ and the tuple $\mathcal{T} = {}^d\mathcal{T}_0$ are rational, where $\mathcal{T}_0$ is the image of $((\mathbf{T}, \mathbf{T}U_\alpha))_{\alpha \in \mathcal{A}}$ in the adjoint A_1 factors. Given any δ-orbit $O \subseteq \mathcal{A}$, its torus is split over $\mathbb{F}_{q^{|O|}}$. Moreover, $\mathbf{L}$ is the stabilizer of $\mathcal{T}$ in $\mathbf{K}$, since the stabilizer in each adjoint A_1 is the image of ${}^d\mathbf{T}$. Thus, Lang's theorem for $\mathbf{L}$ gives one K-orbit on $(\mathbf{K} \cdot \mathcal{T})^{F'}$, as in Lemma 7.54.

Let $B' \leq KB$ stabilize $(F_r^i(\mathcal{T}))_i$. Then, $B' \to J$ is onto and $K \cap B' = L = \mathbf{L}^{F'}$ by the K-orbit calculation. If the graph representative b on a coordinate fixes $\mathcal{T}_0, \mathbf{P}_0$, we choose $k \in K$ with $kb(\mathcal{T}) = \mathcal{T}$. Then, $d^{-1}kb(d) \in \mathbf{L}_0$, since $b(\mathcal{T}_0) = \mathcal{T}_0$. Thus, kb fixes $\mathbf{P} = {}^d\mathbf{P}_0$. If B'_i has graph action 1, its source elements lie in $\mathbf{K}\widetilde{\mathbf{T}}$ and fix $\mathcal{T}$. Hence, they lie in the preimage $\widetilde{\mathbf{L}}$ of the marked tori for $\mathcal{A}$. Therefore, the B'-translates of $\mathbf{P}$ on the coordinates give a B'-stable product radical U as in Proposition 7.60.

On $B' \times_J D_\lambda$, we put $\widehat{\beta} = \Lambda\widehat{\tau}_0$ with $\widehat{\tau}_0$ from Lemma 7.67 for C_0. Similarly, on $B \times_J D_\lambda$, take $\Lambda\widehat{\eta}$. Since $\tau_0(1) \mid \chi(1)$ by the Jordan degree formula, $\ell \nmid \tau_0(1)$. Furthermore, χ has a unique parent $\tilde{\chi}$ over the selected lift of s and $\tilde{\chi}|_X = \chi$, since C is connected, as in Proposition 7.61. Write $\beta = (\lambda|_L)\tau_{0,L}$, with $\tau_{0,L}$ from τ_0. By [169, Theorem 2.1(1)], the coefficient of $\tilde{\chi}$ in the torus expansion for $R_{\widetilde{\mathbf{L}}}^{\widetilde{X}}(\tilde{\beta})$ is a signed product over C_0 and the δ-orbits in $\mathcal{A}$. In particular,

C_0 gives $\langle \eta_0, \eta_0 \rangle = 1$. Similarly, each δ-orbit $O \subseteq \mathcal{A}$ contributes 1 at $1, \mathrm{St}$ by [40, §13.8]. Thus, $|\langle R_{\mathbf{L}}^X(\beta), \chi \rangle| = 1$ by $\tilde{\chi}|_X = \chi$.

For (y, σ), we choose $z = \mathrm{Int}(m)y$ with $m \in M$, $\theta^{z\sigma} = \theta$ and $z(R^+) = R^+$. Then, $(\Lambda\widehat{\eta})^{z\sigma} = \Lambda\widehat{\eta}$ on $B \times_J D_\lambda$ by Lemma 7.67. In K, we choose k with $kz(\mathcal{T}) = \mathcal{T}$ and set $z' = \mathrm{Int}(k)z$. Thus, z' normalizes B'. The correction by $(k, 1)$ on the global cover fixes Λ. Moreover, z' acts on the adjoint chain by $z|_{C_0}$ up to an inner automorphism. Since $\tau_0^{z'} = \tau_0$ and $\widehat{\tau}_0$ has values in $\mathbb{Q}_\ell$, $\widehat{\beta}^{z'\sigma} = \widehat{\beta}$.

Let $V = \prod_i F_r^i(R_u(\mathbf{P}))$. Then, $C_{U^*}(s_\Pi) = C_{V^*}(s_\Pi) = C_{(U^{z'})^*}(s_\Pi)$ equals the product of the marked $U^*_{\alpha,\mathcal{T}}$ on the coordinates, for $\alpha \in \mathcal{A}$. In the dual marking at $\mathcal{T}$, the root of $U^*_{\alpha,\mathcal{T}}$ is positive. Thus, the product is $\widehat{F}^*$-stable since $\delta(\mathcal{A}) = \mathcal{A}$. By [19, Corollary 6.5 and Remark 6.7], we obtain the comparison with Y_V on $X \times L$, as in Proposition 7.38. Since $\widehat{\beta}^{z'\sigma} = \widehat{\beta}$, the action of $z'\sigma$ fixes e_{s_Π}. Thus, the trace on e_{s_Π} is invariant by Lemma 7.36. Tensor with $\widehat{\beta}$ and take invariants over L. Consequently, $\Xi^{z'\sigma} = \Xi$. Given $c \in C_{X \rtimes J}(X) \leq B'$, the action on Y_U is 1. On both extensions, (c, a) acts by $\Lambda(c, a)$. Thus, the coefficient ± 1 at χ and the common section $J \to D_\lambda$ give $\succeq_c$ on $M = N_X(P)$ by Proposition 7.44 with scalar $\Lambda(c, a)$. At the involution with split $A_5 + A_1$ from [208, proof of Proposition 10.7], the chain C_0 is on $1, 3, 4, 5, 6$ and the line is on 0. Since $\kappa = (3, 2, 1)$ when $\mu = (3, 2, 1)$, the formula in Lemma 7.67 gives $\ell \mid \tau_\mu(1)$. Moreover, Proposition 7.26 treats $\mathcal{E}(X, s)$ for that involution s. Thus, the remaining chain C_0 with $g|_{C_0} = 1$ is on $0, 2, 4$. □

Lemma 7.70. *Consider $R = A_3$ or $A_3 + A_1^2$ with $g|_{A_3} = 1$, as in* Proposition 7.68 *for C_0 on $0, 2, 4$. Thus, it remains to prove that $(\chi, \psi) \succeq_c$ for these R. The Levi $\mathbf{L}$ has type A_1. Let β be the middle root of the A_3 basis in R^+. Then, we find a g-stable parabolic $\mathbf{P}_0$ of $\mathbf{G}$ with Levi $\mathbf{L}_0 = \langle \mathbf{T}, U_{\pm\beta} \rangle$ such that $R_u(\mathbf{P}_0) \cap \mathbf{K} = \langle U_\alpha : \alpha \in R^+ \setminus \{\beta\} \rangle$, where $\mathbf{K} = \langle \mathbf{T}, U_\alpha : \alpha \in R \rangle$.*

Proof. By the chamber argument of Proposition 7.61, $W_s = W(R)$ gives that g is W-conjugate to γ. Moreover, $g(\beta) = \beta$ since $g|_{A_3} = 1$. Let $V = Y(\mathbf{T})^g_{\mathbb{Q}}$ and $V_0 = \{x \in V : \langle \beta, x \rangle = 0\}$. The restrictions to V give the reduced system F_4, in which $\beta|_V$ is long with unique preimage β by [208, proof of Proposition 10.5]. Thus, $\alpha|_{V_0} \neq 0$ for $\alpha \in \Phi \setminus \{\pm\beta\}$.

In $\mathbb{Q}R^\vee$, choose the cocharacter x_0 with $\langle \beta, x_0 \rangle = 0$ and $\langle \alpha, x_0 \rangle = 1$ at each other simple root of R^+. Since g permutes the simple roots of R^+ and fixes β, $x_0 \in V_0$. Therefore, we obtain an integral $x \in V_0$ with $\langle \alpha, x \rangle > 0$ on $R^+ \setminus \{\beta\}$ and $\langle \alpha, x \rangle \neq 0$ on $\Phi \setminus \{\pm\beta\}$. Indeed, near x_0 we choose $x' \in V_0$ outside those hyperplanes and take $x = Nx'$ with $N \in \mathbb{Z}_{>0}$ clearing denominators. By [58, §3.4], the roots with $\langle \alpha, x \rangle \geq 0$ define $\mathbf{P}_0$ with Levi $\mathbf{L}_0$. Since $g(x) = x$, $g(\mathbf{P}_0) = \mathbf{P}_0$. □

Proposition 7.71. *Suppose that $R = A_3$ or $A_3 + A_1^2$ and $g|_{A_3} = 1$. Then, we have that each matched ℓ'-pair (χ, ψ) satisfies $(\chi, \psi) \succeq_c$ on $M = N_X(P)$.*

Proof. Take $\mathbf{L}_0, \mathbf{P}_0$ from Lemma 7.70 and put $\mathbf{Q}_0 = \mathbf{P}_0 \cap \mathbf{K}$. Write $n \in N_{\mathbf{K}}(\mathbf{T})$ for a lift of $w_0(R)$. By Lang's theorem [58, Theorem 4.2.9], $d \in \mathbf{K}$ with $d^{-1}F'(d) = n$. The action of $\mathrm{Int}(n)F'$ is $-qw_0(R)\delta$ on $\mathbb{Z}R$. It acts by $\beta \mapsto q\beta$ and $R^+ \setminus \{\beta\}$ to $q(R^+ \setminus \{\beta\})$. Thus, $\mathcal{P} = ({}^d\mathbf{L}_0, {}^d\mathbf{Q}_0)$ is F'-stable. With $\mathbf{L} = {}^d\mathbf{L}_0$, we put $L = \mathbf{L}^{F'}$ and $\mathbf{P} = {}^d\mathbf{P}_0$. The stabilizer $N_{\mathbf{K}}(\mathcal{P})$ is $\mathbf{L}$ by [58, Proposition 3.4.2]. Hence, $(\mathbf{K} \cdot \mathcal{P})^{F'}$ is one K-orbit by Lang's theorem for $\mathbf{L}$.

Take B, D_λ, Λ as in Proposition 7.69 and $B' \leq KB$ the stabilizer of $(F_r^i(\mathcal{P}))_i$. Then, $B' \to J$ is onto and $K \cap B' = L$. On a graph coordinate, we choose b fixing $\mathbf{P}_0, \mathbf{L}_0, \mathbf{Q}_0$, and $k \in K$ with $kb(\mathcal{P}) = \mathcal{P}$. Thus, $d^{-1}kb(d) \in \mathbf{L}_0$, so $kb(\mathbf{P}) = \mathbf{P}$. With graph action 1, the coordinate stabilizers act through $\widetilde{\mathbf{L}}$ by Proposition 7.69 for $\mathcal{P}$. Consequently, the B'-translates of $\mathbf{P}$ give a B'-stable product radical U.

Let $\rho \in \mathrm{Irr}(\mathfrak{S}_4)$ index the unipotent character on A_3. Put $a = s_\beta = (23)$, with $z = 1$ for split A_3 and $z = (14)(23)$ for ${}^2A_3$. Thus, $\mathrm{Int}(z)$ is the based Frobenius action on $W(A_3)$, and z, az index the two tori from $\mathbf{L}$. The type A formula [40, §13.8] gives coefficients $m_\pm = (\rho(z) \pm \rho(az))/2$ at ρ up to sign, from $1, \mathrm{St}$, respectively. Indeed, if π_ρ affords ρ, the diagram operator for ${}^2A_3$ is $\pm\pi_\rho(z)$. By [23, $\mathfrak{S}_4$], either $|m_+| = 1$ or $|m_-| = 1$. Consequently, choose $\nu = 1$ if $|m_+| = 1$ and $\nu = \mathrm{St}$ otherwise. Each δ-orbit O on A_1^2 has a split torus over $\mathbb{F}_{q^{|O|}}$. Thus, O contributes 1 at $1, \mathrm{St}$ as in Proposition 7.69. Put $\beta_L = (\lambda|_L)\nu_L$, with ν_L from the adjoint A_1. By [169, Theorem 2.1(1)], the factors give $|\langle R_{\widetilde{\mathbf{L}}}^{\widetilde{X}}(\tilde{\beta}_L), \tilde{\chi} \rangle| = 1$. Consequently, $|\langle R_{\mathbf{L}}^X(\beta_L), \chi \rangle| = 1$ since $\tilde{\chi}|_X = \chi$ for the unique parent $\tilde{\chi}$ in the selected series.

On $B' \times_J D_\lambda$ take $\widehat{\beta}_L = \Lambda\widehat{\nu}_L$. If $\nu = \mathrm{St}$, $\widehat{\nu}_L$ is the permutation character on $\mathbb{P}^1(\mathbb{F}_q)$ minus 1. Otherwise, $\widehat{\nu}_L = 1$. Thus, $\widehat{\nu}_L$ is rational in both cases. On $B \times_J D_\lambda$, we choose $\Lambda\widehat{\eta}$ from Lemma 7.67. Given a joint stabilizer (y, σ), we choose the correction $z_0 = \mathrm{Int}(m)y$ with $m \in M$, $\theta^{z_0\sigma} = \theta$ and $z_0(R^+) = R^+$. Then, $(\Lambda\widehat{\eta})^{z_0\sigma} = \Lambda\widehat{\eta}$. On the global side, choose $k_0 \in K$ with $k_0 z_0(\mathcal{P}) = \mathcal{P}$, and put $z_1 = \mathrm{Int}(k_0)z_0$. Thus, z_1 normalizes B', and the correction by $(k_0, 1)$ fixes Λ. Since ν depends on ρ, z, $\widehat{\beta}_L^{z_1\sigma} = \widehat{\beta}_L$.

Let $V = \prod_i F_r^i(R_u(\mathbf{P}))$. Then, we have that $C_{U^*}(s_\Pi) = C_{V^*}(s_\Pi) = C_{(U^{z_1})^*}(s_\Pi)$ equals $\prod_i F_r^i(R_u(\mathbf{Q}_d^*))$, where $\mathbf{Q}_d^*$ is the parabolic of C with the marked roots of ${}^d\mathbf{Q}_0$. Since ${}^d\mathbf{Q}_0$ is F'-stable, the product is $\widehat{F}^*$-stable. Thus, [19, Corollary 6.5 and Remark 6.7] compares U, V on $X \times L$. On the cover, Lemma 7.36 compares U, U^{z_1}. Consequently, the coefficient ± 1 at χ and the section $J \to D_\lambda$ give $(\chi, \psi) \succeq_c$, as in Proposition 7.44 with scalar $\Lambda(c, a)$ for (c, a) above $c \in C_{X \rtimes J}(X)$. □

Theorem 7.72. *Let $S = E_6(q)$ be split simple and let $\ell \mid q+1$ be odd. Then, $\mathrm{iMN}(S, \ell)$ holds on the universal cover X. For $\ell \geq 5$, we have that the subgroup is $M = N_X(P)$ for $P \in \mathrm{Syl}_\ell(X)$.*

Proof. If $\ell = 3$, by Theorem 7.27, we obtain $\mathrm{iMN}(S, 3)$. Hence, suppose that $\ell \geq 5$ and take P, M, Ω as in Proposition 7.60 and Theorem 7.24. If $s = 1$, the d-cuspidal pair is $(\mathbf{T}, 1_T)$. By [156, Theorem D and §8.2], we obtain the comparison between $\mathcal{E}(X, 1) \cap \mathrm{Irr}_{\ell'}(X)$ and $\mathrm{Irr}_{\ell'}(M \mid 1_T)$. Suppose that $C = C_{\mathbf{G}^*}(s)$ has $R = A_2^3$. By [208, (10.23) and proof of Proposition 10.9, Step 1], we have $|s| = 3$ and $C/C^\circ \cong C_3$ with the Frobenius action on Z_3. Since Section 7.3 excludes $R = A_2^3$ when $(C/C^\circ)^{F'^*} \neq 1$, $q \equiv -1 \bmod 3$. Hence, $r \equiv -1 \bmod 3$, f is odd and $Z(X) = 1$. From $F'^*(s) = s$, $\gamma(s) = s$, since $-q \equiv 1 \bmod 3$. The map on rational tori $(\mathbf{T}_{\mathrm{sc}}^*)^{F'^*} \to (\mathbf{T}^*)^{F'^*}$ is an isomorphism since F'^* inverts Z_3. Thus, the lift of s has order 3 and is fixed by γ. By Section 7.3, $\gamma|_{R_i} = 1$ whenever $\gamma(R_i) = R_i$ for a factor $R_i = A_2$. Since $\dim V^+ = 4$, γ induces a transposition on $\{R_1, R_2, R_3\}$. Thus, $W_\theta = W(R)^\gamma = \mathfrak{S}_3^2$ has index 32 in $W^\gamma = W(F_4)$. Consequently, the nine characters in $\mathrm{Irr}(M \mid \theta)$ have degrees $32\eta(1)$ for $\eta \in \mathrm{Irr}(\mathfrak{S}_3^2)$ by Theorem 7.27. Since $32\eta(1)$ is a power of 2, the orbit of θ at 3 has length 32 as in Theorem 7.27. Moreover, $C^\circ(s)$ has factors ${}^2A_2(q)$, $A_2(q^2)$ by [208, (10.30)]. Put $a = v_\ell(q+1)$. The four factors $q^2-1, q^3+1, q^4-1, q^6-1$ each have valuation a, so $v_\ell(|C^{F'^*}|) = 4a = v_\ell(|X|)$. Moreover, the nontrivial unipotent degrees $q(q-1), q^3, q^2(q^2+1), q^6$ have valuation 0 at ℓ by [40, §13.8]. Consequently, $v_\ell(\chi(1)) = 0$ for each $\chi \in \mathcal{E}(X, s)$ by the Jordan degree formula at s. The nine characters in $\mathcal{E}(X, s)$ are those at 3 in Theorem 7.27.

For the pairs (χ, ψ) at s, Theorem 7.27 gives rational extensions $\widehat{\chi}, \widehat{\psi}$ on $X \rtimes E$ and $M \rtimes E$, with $E \cong C_{2f}$. In particular, they are fixed by $\mathcal{H}_\ell$. By Corollary 7.25, $C_{\mathbf{G}}(P)^\circ = \mathbf{T}$. Thus, $J = \mathrm{Aut}(X)_{P,\chi}$ stabilizes $\mathbf{T}$. The maps ρ of Theorem 7.27 give $\widehat{\chi} \circ \rho, \widehat{\psi} \circ \rho$ on $X \rtimes J$ and $M \rtimes J$. Consequently, the factor sets α, α' and functions $\mu_{y,\sigma}, \mu'_{y,\sigma}$ are 1, with scalar 1 on $C_{X \rtimes J}(X)$.

For disconnected C, the triangle of Lemma 7.48 gives $R = A_2^3, D_4$ or A_1^a with $a \geq 0$. Thus, for $R = A_1^a$ we consider $(C/C^\circ)^{F'^*}$. If $W_s^\gamma = W(R)^\gamma$, then Propositions 7.60 and 7.61 give $(\chi, \psi) \succeq_c$. Otherwise, $|(C/C^\circ)^{F'^*}| = 3$. With fixed and moving labels η, Propositions 7.63 and 7.64 give $(\chi, \psi) \succeq_c$, respectively. Finally, for $R = D_4$ with $C/C^\circ \cong C_3$, by Proposition 7.65, we obtain $\succeq_c$. If $C = C^\circ$ and R has a D-component, by Proposition 7.66, we obtain $\succeq_c$ on M. For $C = C^\circ$ and R of type A, the face test in Proposition 7.68 gives $\succeq_c$ if the parabolic with Levi C exists. Otherwise, for $R \neq A_1^a$, there is a unique C_0 with $\mathrm{rk}(C_0) \geq 2$ by the face test. If J has no pure graph or $g|_{C_0} = -w_0(C_0)$, by Proposition 7.69, we obtain $\succeq_c$. Otherwise, $g|_{C_0} = 1$, and Proposition 7.71 gives $\succeq_c$. For $s \neq 1$, the weight θ_s corresponds to s by [156, §4.4 and proof of Theorem B]. Thus, Ω in Theorem 7.24 pairs $\mathcal{E}(X, s) \cap \mathrm{Irr}_{\ell'}(X)$ with $\mathrm{Irr}_{\ell'}(M \mid \theta_s)$. The rational series $\mathcal{E}(X, s)$ and the weight orbits of θ_s give $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant partitions. By Lemma 9.4, we obtain $\mathrm{iMN}(S, \ell)$ on $M = N_X(P)$, including the pairings at $s = 1$. □

7.4 Tori in E_6

Let $X = E_{6,\mathrm{sc}}^\epsilon(q)$, $q = r^f$, and put $u = \epsilon q$. Suppose that $\mathrm{ord}_\ell(q) = 12$. Then, we find $\mathrm{ord}_\ell(u) = 12$. Given a Coxeter element $w \in W(E_6)$, its exponents give $V = V_{12} \oplus V_3$ over $\mathbb{Q}$ with factors Φ_{12}, Φ_3 by Springer [178, Table 1]. Thus, the Frobenius action is uw for both signs, since $w_0\gamma = -1$ as in Section 7.1 for E_6. Put $a = \Phi_{12}(u)$, $b = \Phi_3(u)$, and let $\mathbf{T}_a, \mathbf{T}_b \leq \mathbf{T}$ correspond to V_{12}, V_3, respectively. Consequently, we consider $\mathcal{E}(X, s)$ with $s = s_a s_b \in T^*$.

Lemma 7.73. *Put $T_i = \mathbf{T}_i^F$ for $i = a, b$. Then, we obtain that $T = T_a \times T_b$ with $|T_a| = a$ and $|T_b| = b$. Moreover, $P = O_\ell(T_a)$ lies in $\mathrm{Syl}_\ell(X)$. We have that $M/T = \langle w \rangle$ has order 12 for $M = N_X(P)$. Every $\mathcal{E}(X, s)$ meeting $\mathrm{Irr}_{\ell'}(X)$*

has $s \in T^*$ up to conjugacy. If $s_a \neq 1$, then $C_{\mathbf{G}^*}(s) = \mathbf{T}^*$. If $s = s_b \neq 1$, then $C_{\mathbf{G}^*}(s)^\circ$ has root system D_4 and F^* induces a 3-cycle on its diagram.

Proof. By [178, Theorem 4.2 and Table 1], $C_W(w) = \langle w \rangle$. The determinant formula gives $|T_a| = a$, $|T_b| = b$ and $|T| = ab$. Since $a \equiv 2(u+1) \bmod b$, b is odd and $\gcd(u+1, b) = 1$, we get that $\gcd(a, b) = 1$. Thus, $T = T_a \times T_b$. $T^* = T_a^* \times T_b^*$ with orders a, b. Among the signed degrees of X, only 12 contributes to $|X|_\ell$. Consequently, $P = O_\ell(T_a)$ is Sylow, and so is $P^* = O_\ell(T_a^*)$ on the dual side.

Suppose that $s_a \neq 1$ and take $y \in \langle s \rangle$ of prime order $h \mid a$. Since a is prime to 6 and $\mathrm{ord}_h(u) = 12$, $h \geq 13$. Write $\Phi(y) = \{\alpha \in \Phi(E_6) : \alpha(y) = 1\}$. Since $F^*(y) = y$, the span of $\Phi(y)$ is w-stable. No $\alpha \in \Phi(E_6)$ belongs to V_3, since $V_3 \perp v$ for a regular Coxeter eigenvector $v \in V_{12} \otimes_{\mathbb{Q}} \mathbb{C}$. Thus, if $\Phi(y) \neq \varnothing$, then $V_{12} \subseteq \mathrm{span}_{\mathbb{Q}} \Phi(y)$. The connected centre $Z(C_{\mathbf{G}^*}(y)^\circ)^\circ$ then has lattice in V_3, so its finite order is 1 or b since Φ_3 is irreducible. Moreover, the centre of $[C_{\mathbf{G}^*}(y)^\circ, C_{\mathbf{G}^*}(y)^\circ]$ has order prime to h, since its rank is at most 6. Thus, $y \in Z(C_{\mathbf{G}^*}(y)^\circ)^\circ$, contrary to $h \nmid b$. Hence, $\Phi(y) = \varnothing$. By [20, Corollary 2.9], $h \neq 3$ gives $C_{\mathbf{G}^*}(y) = \mathbf{T}^*$. Consequently, $C_{\mathbf{G}^*}(s) = \mathbf{T}^*$. Applying the same calculation to $x \in P$ of order ℓ gives $C_{\mathbf{G}}(P) = \mathbf{T}$, since x has image of order ℓ in the adjoint group. Consequently, $N_X(P) = N_X(\mathbf{T})$. For $\chi \in \mathcal{E}(X, s) \cap \mathrm{Irr}_{\ell'}(X)$, the formula in [58, Proposition 11.5.6] gives $|C_{\mathbf{G}^*}(s)^{F^*}|_\ell = |G^*|_\ell$. Thus, we conjugate s to centralize P^*. Since $C_{\mathbf{G}^*}(P^*) = \mathbf{T}^*$ by the calculation at $h = \ell$, $s \in T^*$.

Let $R = \Phi(E_6) \cap V_{12}$. The stabilizer of a generic $v_3 \in V_3$ is $W(R)$ by [58, §3.1]. Since $w^3 \in W(R)$ and $(w^3 - 1)V = V_{12}$, the roots of R span V_{12}. The component permutation by w is transitive since $w|_{V_{12}}$ is irreducible. Since $w^3 \in W(R)$, the number of components divides 3 and also divides $\dim V_{12} = 4$. Thus, R is irreducible. In $W(A_4) = \mathfrak{S}_5$, an order 4 element has cycle type $(4, 1)$, so it has eigenvalue 1 on V_{12}. Hence, $w^3|_{V_{12}}$ has the wrong spectrum for $R = A_4$, and $R = D_4$. By $w^3 \in W(D_4)$, the image of w in the diagram group has order 1 or 3. Since the signed permutations in $W(D_4)$ have eigenvalues of order at most 8, the image of w has order 3. Consequently, uw induces triality, since $-1 \in W(D_4)$. If $s = s_b \neq 1$, $R \subseteq \Phi(s)$. If $\Phi(s)$ has a root outside V_{12}, its span is V since $w|_{V_3}$ acts irreducibly. Thus, s is an adjoint alcove vertex. Since $|s| \mid b$ is odd and $s \neq 1$, its root system would be A_2^3 by [20, Table III]. But A_2^3 has no subsystem D_4. Consequently, $\Phi(s) = D_4$. □

For $s_a \neq 1$, we choose the regular pair (χ, ψ) with $\chi = \epsilon_{\mathbf{G}} \epsilon_{\mathbf{T}} R_{\mathbf{T}}^X(\theta)$ and $\psi = \mathrm{Ind}_T^M \theta$. As in Section 7.1, $j_0 = \mathrm{Int}(n_{w_0})\gamma$ acts by -1 on $\mathbf{T}$. Consequently, w^i returns s^{-1} to s only for $i \equiv 6 \bmod 12$, by $w(y) = y^{u^{-1}}$ on y of order $h \mid a$. Since $w^6|_{V_{12}} = -1$ and $w^6|_{V_3} = 1$, a power w^i with $w^i(s^{-1}) = s$ exists exactly when $s_b = 1$. Then, a generic point of V_{12} gives a chamber fixed by $-w^6$, since V_{12} contains the regular eigenvector v. For $s_b \neq 1$, the coordinate stabilizers act through $\mathbf{T}$. Thus, Propositions 7.60 and 7.43 give U by propagation, with the graph γ at the wrap when $\epsilon = -1$. By Lemma 7.73, we deduce that $C_{U^*}(s_\Pi) = 1$. Consequently, the same propositions give $(\chi, \psi) \succeq_c$ with scalar $\theta|_{Z(X)}$.

Proposition 7.74. *Suppose that $s = s_b$ has $|s| \notin \{1, 3\}$. Then, we have that each matched ℓ'-pair (χ, ψ) satisfies $\succeq_c$ on X, M.*

Proof. Let $\mathbf{K}$ be the primal Levi of type D_4 from Lemma 7.73 and put $K = \mathbf{K}^F$. By [208, §9.2], the rational derived group $[\mathbf{K}, \mathbf{K}]^F$ is isomorphic to $H = {}^3D_4(q)$. Thus, the adjoint map $K \to H$ is onto. Let $\mathbf{S}$ be the image of $\mathbf{T}$ in the adjoint factor. Since $Y(\mathbf{S})_{\mathbb{Q}} = V_{12}$, the minimal 12-split Levi of H is $\mathbf{S}$. Moreover, the rational Weyl group is $C_4 = \langle w^3 \rangle$, since $W(D_4) \cap C_W(w) = \ker(\langle w \rangle \to \mathrm{GL}(V_3))$. With $N' = N_{\mathrm{Aut}(H)}(\mathbf{S})$, we put $N_H = N_H(\mathbf{S})$. The field automorphism ϕ generates $\mathrm{Aut}(H)/H \cong C_{3f}$ by [208, §9.2]. Since ϕ fixes the Weyl class of $w|_{V_{12}}$, it fixes the H-class of $\mathbf{S}$. Thus, $N' \to C_{3f}$ is onto. Consequently, the result of [156, §8.2 and proof of Theorem D] gives the groups $\mathrm{Aut}(H), N'$ for the pair $(\mathbf{S}, 1_{\mathbf{S}^F})$. For each matched (τ, η) of ℓ'-degree, that proof gives $\mathcal{H}_\ell$-fixed extensions $\widehat{\tau}$ on $\mathrm{Aut}(H)$ and $\widehat{\eta}$ on N', where η is inflated from $C_4 = N_H/\mathbf{S}^F$. In particular, τ is $\mathrm{Aut}(H)$-invariant and η is N'-invariant. If a prime $h \neq 3$ divides $|s|$, choose $y \in \langle s \rangle$ of order h. Lemma 7.73 gives $\Phi(y) = D_4$, and [20, Corollary 2.9] gives connected $C_{\mathbf{G}^*}(y)$. Thus, $C^\circ = C_{\mathbf{G}^*}(y)$ and $C \leq C_{\mathbf{G}^*}(y)$ imply $C = C^\circ$. Since $9 \nmid b$, the case $C \neq C^\circ$ requires $|s| = 3$. Thus, $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ for the present s. Since $\gcd(b, u-1) \mid 3$, the inertia of θ in $M/T = \langle w \rangle$ is C_4. Consequently, $M_\theta = N_K(\mathbf{T})$. Let $\lambda \in \mathrm{Irr}(K)$ be dual to s, with $\lambda|_T = \theta$. By [208, (10.6)], $\chi = \varepsilon R_{\mathbf{K}}^X(\lambda \tau_K)$ where τ_K is the inflation of τ from H. Thus, $\tau(1) \mid \chi(1)$. The local partner is $\psi = \mathrm{Ind}_{M_\theta}^M(\lambda|_{M_\theta} \eta)$, with the C_4 label from [156, §4.4]. Moreover, $s^{G^*} \cap T^* = s^{N_{G^*}(\mathbf{T}^*)}$ by Sylow conjugacy in $C_{\mathbf{G}^*}(s)^{F^*}$ and $C_{\mathbf{G}^*}(P^*) = \mathbf{T}^*$.

With $J = \mathrm{Aut}(X)_{P,\chi}$, we put $B = (M \rtimes J)_\theta$ and $D_\lambda = (KB)/\ker\lambda$. Thus, we let Λ be as in Proposition 7.44 on $(KB) \times_J D_\lambda$. On $(KB) \times_J D_\lambda$ and $B \times_J D_\lambda$, take $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ via the maps to $\mathrm{Aut}(H), N'$, respectively. Given a joint

stabilizer (y,σ), choose $m \in M$ with $z = \mathrm{Int}(m)y$ and $\theta^{z\sigma} = \theta$. Then, $\Lambda^{z\sigma} = \Lambda$ by torus duality. Moreover, $\widehat{\tau}^{z\sigma} = \widehat{\tau}$ and $\widehat{\eta}^{z\sigma} = \widehat{\eta}$ via conjugation on $\mathrm{Aut}(H)$ and N'.

The rational Weyl group acts on s_b through a group of order dividing 3. Since s_b has odd order and $s_b \neq 1$, no Weyl element acts by $s_b \mapsto s_b^{-1}$. Thus, we have that B has no pure graph over θ. The coordinate stabilizers act through $\widetilde{\mathbf{K}}$. Consequently, a parabolic $\mathbf{P}$ with Levi $\mathbf{K}$ gives the propagated radical U as in Proposition 7.44. Since $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$, $C_{U^*}(s_\Pi) = C_{(U^z)^*}(s_\Pi) = 1$. Thus, the signed coefficient at χ and the section $J \to D_\lambda$ give $(\chi,\psi) \succeq_c$ by that proposition, with scalar $\Lambda(c,d)$ for (c,d) above $c \in C_{X\rtimes J}(X)$. □

Keep $q = r^f$, $u = \epsilon q$, and suppose that $\mathrm{ord}_\ell(q) = 8$. Then, we find $\mathrm{ord}_\ell(u) = 8$. In the coordinates of D_5, we choose c with $c(e_i) = e_{i+1}$ for $i = 1,2,3$, $c(e_4) = -e_1$ and $c(e_5) = -e_5$. Thus, we find $c(e_6) = e_6$ for the orthogonal unit vector e_6. With $U = \langle e_1,e_2,e_3,e_4\rangle_{\mathbb{Q}}$, we put $V = \langle e_5, \sqrt{3}e_6\rangle_{\mathbb{Q}}$. The characteristic polynomials of c on U,V are Φ_8 and x^2-1. Consequently, we have that F^* acts on $X^*(\mathbf{T}^*)$ as uc as in Section 7.1 for $\mathbf{T}^*$.

Lemma 7.75. *With the marked torus $T = \mathbf{T}^F$, we let $P \in \mathrm{Syl}_\ell(T)$. Then, we obtain that $P \in \mathrm{Syl}_\ell(X)$, and $M = N_X(P)$ satisfies $M/T = \langle c\rangle$. Each $\mathcal{E}(X,s)$ meeting $\mathrm{Irr}_{\ell'}(X)$ has $s \in T^*$ up to conjugacy. Write $\bar{s}$ for the projection to the adjoint D_4-torus on U. If $|\bar{s}|$ has an odd prime divisor, then $C_{\mathbf{G}^*}(s) = \mathbf{T}^*$. If $\bar{s} = 1$ and $s \neq 1$, then the connected roots are D_4 or D_5. If $|\bar{s}| = 2$, then $C_{\mathbf{G}^*}(s)$ is connected with root system A_1^4 and F^* induces a 4-cycle.*

Proof. Put $\Phi_s = \Phi(E_6) \setminus \Phi(D_5)$. Its roots are $\alpha = (\delta_1 e_1 + \cdots + \delta_5 e_5 + \delta_6\sqrt{3}e_6)/2$, where $\delta_i \in \{\pm 1\}$ and $\delta_6 = \prod_{i=1}^5 \delta_i$ for a choice of e_6. Thus, $V \cap \Phi(E_6) = \varnothing$ and $U \cap \Phi(E_6) = \Phi(D_4)$. The $\mathbb{Q}$-conjugates of a primitive eigenvector v for $c|_U$ span U. Thus, $\alpha \perp v$ for $\alpha \in \Phi(E_6)$ would give $\alpha \in V$. Hence, v is regular. By [178, Theorem 4.2], $C_W(c) = \langle c\rangle$, since 8 is the only degree divisible by 8. The same group centralizes $-c$. Consequently, the order formula gives $P \in \mathrm{Syl}_\ell(X)$. A simple system Δ_4 of D_4 extends to a basis Δ_6 of E_6. Thus, $\mathbb{Z}\Delta_4$ is primitive in $X^*(\mathbf{T}^*)$. The projection $\mathbf{T}^* \to \overline{\mathbf{T}}$ onto the adjoint D_4-torus has connected kernel $\mathbf{S}$. By [133, Proposition 23.2], $T^*/S \simeq \overline{T}$ for $S = \mathbf{S}^{F^*}$. The determinant formula gives $|\overline{T}| = q^4+1$ and $|S| = q^2-1$.

Let t be an odd prime dividing $|\bar{s}|$. Then, we have that $u^4 \equiv -1 \bmod t$ gives $\mathrm{ord}_t(u) = 8$ and $t \geq 17$. Since $t \nmid |S|$, there is $y \in \langle s\rangle$ of order t with $|\bar{y}| = t$. The roots $\Phi(y) = \{\alpha : \alpha(y) = 1\}$ span a c-stable subspace. If $\Phi(y) \neq \varnothing$, its span contains U since Φ_8 is irreducible and V has no roots. Then, $Z(C_{\mathbf{G}^*}(y)^\circ)^\circ$ has lattice in V. Its rational order divides q^2-1, since the two eigenspaces of $c|_V$ have eigenvalues $1,-1$. Moreover, $C_{\mathbf{G}^*}(y)^\circ_{\mathrm{der}}$ has rank at most 6 and its centre $Z(C_{\mathbf{G}^*}(y)^\circ_{\mathrm{der}})$ has order prime to t. Thus, $y \in Z(C_{\mathbf{G}^*}(y)^\circ)^\circ$, contrary to $t \nmid q^2-1$. Consequently, $\Phi(y) = \varnothing$. By [20, Corollary 2.9], $t \neq 3$ gives $C_{\mathbf{G}^*}(y) = \mathbf{T}^*$. Thus, $C_{\mathbf{G}^*}(s) = \mathbf{T}^*$. If $t = \ell$ and $P^* = O_\ell(T^*)$, the calculation gives $C_{\mathbf{G}^*}(P^*) = \mathbf{T}^*$. $C_{\mathbf{G}}(P) = \mathbf{T}$ since P injects into the adjoint quotient. Therefore, $N_X(P) = N_X(\mathbf{T})$ and $M/T = C_W(c)$. For $\chi \in \mathcal{E}(X,s) \cap \mathrm{Irr}_{\ell'}(X)$, [58, Proposition 11.5.6] gives $|C_{\mathbf{G}^*}(s)^{F^*}|_\ell = |G^*|_\ell$. Thus, we conjugate s into $C_{\mathbf{G}^*}(P^*)^{F^*} = T^*$.

Suppose that $\bar{s} = 1$, and put $h = (\sqrt{3}e_6 + e_5)/2$, $k = (\sqrt{3}e_6 - e_5)/2$. The restrictions to $\mathbf{S}$ of the vector roots off D_4 are $\pm(h-k)$. The roots in Φ_s restrict to $\pm h, \pm k$, so h,k form a basis of $X^*(\mathbf{S})$. Since $c(h) = k$ and $c(k) = h$, the values $a = h(s)$, $b = k(s)$ satisfy $b = a^u$. Hence, $s \neq 1$ gives $a,b \neq 1$. If $a \neq b$, $\Phi(s) = D_4$, and if $a = b$, then $\Phi(s) = D_5$.

Finally, suppose that $|\bar{s}| = 2$. Then, q is odd and $v_2(|\overline{T}|) = 1$. The functional $x \mapsto x_2 + x_4 \bmod 2$ on the even-sum lattice of D_4 is nonzero and c-invariant, so it represents $\bar{s}$. Its kernel on $\Phi(D_4)$ has the four lines $\mathbb{Q}(e_i \pm e_j)$ with $(i,j) = (1,3),(2,4)$. Moreover, c induces a 4-cycle since $c^2(e_1+e_3) = e_3 - e_1$. If $\alpha = \pm e_i \pm e_5$ lies in $\Phi(s)$ for $i \leq 4$, then the indices $i, i+1 \bmod 4$ of $\alpha + c(\alpha) \in \Phi(s) \cap U$ differ in parity. Hence, $\alpha + c(\alpha)$ contradicts the kernel condition on U. If $\alpha \in \Phi_s \cap \Phi(s)$, then $\alpha, c(\alpha)$ project to independent vectors on V. Consequently, $\mathrm{span}\,\Phi(s) = U \oplus V$. By [20, Table III], full rank and $|\bar{s}| = 2$ give $|s| = 2$ and $\Phi(s) = A_5 + A_1$. The A_5 span is c-stable of dimension 5, so it contains U. However, $\{\pm 1\} \times \mathfrak{S}_6$ has no primitive eighth eigenvalue. Thus, $\Phi(s) = A_1^4$. Suppose $A(s) \neq 1$. By [20, Corollary 2.9], $A(s) \cong C_3$. Since c acts by a 4-cycle, it normalizes no 3-cycle subgroup on the factors. Thus, $A(s)$ fixes each A_1-factor. By [20, Proposition 1.3(c)–(d)], take $A(s) \leq W(s)$ fixing the positive roots of $\Phi(s)$. Thus, $A(s)$ acts faithfully on V. Since $\dim V = 2$, $V^{A(s)} = 0$. Consequently, $W(s)$ fixes no nonzero vector. By [20, Corollary 1.4(a)], s is quasi-isolated. By [208, proof of Proposition 10.8, Step 1, after (10.24)], $A(s)$ cycles three A_1-factors in this type. Thus, $A(s) = 1$. Consequently, $C_{\mathbf{G}^*}(s)$ is connected and $(F^*)^4$ acts by $-q^4$ on $\mathbb{Q}\alpha$ for each factor root α. Thus, the rational adjoint factor is $\mathrm{PGL}_2(q^4)$ with a nonsplit torus. □

If $t \mid |\bar{s}|$ is odd, take the regular pair (χ,ψ). In the coordinates above, $c(y) = y^{u^{-1}}$, so a pure graph operation $-c^i$ fixing s has $i \equiv 4 \bmod 8$. Then, $-c^4|_U = 1$, and a generic point of U gives a chamber fixed by $-c^4$. If no $-c^i$ fixes s,

the coordinate stabilizers act through $\mathbf{T}$. Thus, the propagated Borel radical U_0 has $C_{U_0^*}(s_\Pi) = 1$. The comparison for (χ, ψ) follows as after Lemma 7.73.

Proposition 7.76. *Suppose that $\bar{s} = 1$ and $s \neq 1$, with connected roots D_4 or with connected roots D_5 and $|s| \neq 2$. Then, we have that each matched ℓ'-pair (χ, ψ) satisfies $\succeq_c$ on X, M.*

Proof. Let $\mathbf{K}$ be the Levi through $\mathbf{T}$ with roots R. Write $\mathbf{H} = \mathbf{K}_{\mathrm{ad}}$, $H = \mathbf{H}^F$, and let $\mathbf{T}_H$ be the image of $\mathbf{T}$ in $\mathbf{H}$. Thus, $T_H = \mathbf{T}_H^F$. If $R = D_4$, $c|_U$ has odd sign parity and $-1 \in W(D_4)$. Thus, H has type ${}^2D_4(q)$, and $W(D_4) \cap C_W(c) = \langle c^2 \rangle = C_4$. If $R = D_5$, $c \in W(D_5)$, so H has sign ϵ and the quotient is $C_8 = \langle c \rangle$. In both cases $\mathbf{T}_H = C_{\mathbf{H}}(\mathbf{S}_8)$ for the Sylow 8-torus $\mathbf{S}_8$ with lattice U, since V contains no roots. Put $A = H \rtimes E_H$ for the pinned group E_H, and $N' = N_A(\mathbf{T}_H)$.

Moreover, in type ${}^2D_4$, $A/H \cong C_{2f}$. Write L_y for the lattice action of $y \in N'$. Then, $L_y(\epsilon c) = (\epsilon c)L_y$, so $L_y c^2 = c^2 L_y$. Thus, for $x \in N'/T_H$ above a generator of A/H, we extend $\eta \in \mathrm{Irr}(C_4)$ by $\widehat{\eta}(x)^{2f} = \eta(x^{2f})$ with $\widehat{\eta}(x) \in \mu_{8f_2}$. Put $f_2 = 2^{v_2(f)}$. Since $v_2(\mathrm{ord}_\ell(r)) = 3 + v_2(f)$, $\mathcal{H}_\ell$ fixes μ_{8f_2}. If $R = D_5$, let $j = \mathrm{Int}(n_{w_0})\gamma_5$ be inversion on $\mathbf{T}_H$. By the longest braid lift, $j(n_c) = n_c$ for the Tits lift n_c. Hence, the marked Frobenius $\mathrm{Int}(n_c)j^e F_q$ with $\epsilon = (-1)^e$ fixes j, F_r under conjugation. Thus, $j, \phi = F_r$ and c generate an abelian subgroup modulo T_H. The fixed-point equation gives $c^8 = j^2 = 1$ and $\phi^f = c^{-1} j^e$ in N'/T_H. These generators give all of N'/T_H, since c generates $N_H(\mathbf{T}_H)/T_H$ and j, ϕ generate A/H of order $2f$. Thus, the displayed relations present N'/T_H, since the bound $8(2f)$ equals $|N'/T_H|$. Consequently, $\eta \in \mathrm{Irr}(C_8)$ extends by $\widehat{\eta}(j) = 1$ and $\widehat{\eta}(\phi)^f = \eta(c)^{-1}$ with $\widehat{\eta}(\phi) \in \mu_{8f_2}$. Thus, $\widehat{\eta}^{\mathcal{H}_\ell} = \widehat{\eta}$ in both forms.

By [156, §8.1], for $(\mathbf{T}_H, 1_{T_H})$ the bijection $\widehat{\Omega}$ between the characters of N' and A over the pair is equivariant under $\mathcal{H}_\ell$ and multiplication by $\mathrm{Irr}(A/H)$. Then, the image ρ of $\widehat{\eta}$ has a free twist orbit under $\mathrm{Irr}(A/H)$. Since A/H is abelian, the restriction formula gives $\langle \rho|_H, \rho|_H \rangle_H = 1$. Consequently, $\rho = \widehat{\tau}$ is an $\mathcal{H}_\ell$-fixed extension of the matched τ. If $R = D_4$ and $-c^i$ fixes s, then i is odd and $ab = 1$. Indeed, an even power fixes a, b, so $a = a^{-1}$ and $b = b^{-1}$ would give $a = b$. Then, $-c^i$ fixes e_5. Since precisely D_4 is perpendicular to e_5, the parabolic $\mathbf{P}(e_5)$ is fixed by $-c^i$. If $R = D_5$, the equal entries $a = b$ are fixed by c. Consequently, inversion returns only for $|s| = 2$.

Suppose that $C_{\mathbf{G}^*}(s)$ is connected. Take $K = \mathbf{K}^F$ with $\lambda \in \mathrm{Irr}(K)$ dual to s. Thus, $\chi = \varepsilon R_{\mathbf{K}}^X(\lambda \tau_K)$ by [208, (10.6)], where τ_K is the inflation of τ to K. Also, $M_\theta = N_K(\mathbf{T})$ since its quotient is C_4 or C_8, respectively. Thus, $\psi = \mathrm{Ind}_{M_\theta}^M(\lambda|_{M_\theta}\eta)$. By Proposition 7.44, on the cover take $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ via conjugation on $H, \mathbf{T}_H$. Given a joint stabilizer (y, σ), we choose $m \in M$ with $z = \mathrm{Int}(m)y$ and $\theta^{z\sigma} = \theta$. Then, $(\Lambda\widehat{\tau})^{z\sigma} = \Lambda\widehat{\tau}$ and $(\Lambda\widehat{\eta})^{z\sigma} = \Lambda\widehat{\eta}$. Propagate the parabolic with Levi $\mathbf{K}$ to a radical U_0. Since $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$, $C_{U_0^*}(s_\Pi) = C_{(U_0^z)^*}(s_\Pi) = 1$. By Proposition 7.44, we obtain $\succeq_c$ with the common scalar from Λ.

Suppose $C \neq C^\circ$, and put $A(s) = C/C^\circ$. By [20, Corollary 2.9], $A(s) \cong C_3$. The pointwise stabilizer of $R^\perp$ is $W(R)$ by [58, §3.1]. Thus, $A(s)$ acts faithfully on $R^\perp$. Since $R^\perp$ has dimension at most 2, the rational action requires $R = D_4$ and $(R^\perp)^{A(s)} = 0$. Consequently, $W(s)$ fixes no nonzero vector. By [20, Corollary 1.4(a) and Table III], s is quasi-isolated with $|s| = 3$. Then, we have $|a| = 3$ and $a^u \neq a$, so $u \equiv -1 \bmod 3$. Hence, $X \cong \mathbf{G}_{\mathrm{ad}}^F$ by the central isogeny, also on M. Let $\tilde{s} \in (\mathbf{G}_{\mathrm{sc}}^*)^{F^*}$ be the unique rational lift. Then, $C_{\mathbf{G}_{\mathrm{sc}}^*}(\tilde{s})$ is connected with roots D_4. Moreover, each rational correction fixing s fixes $\tilde{s}$ by uniqueness. Consequently, $\mathbf{P}(e_5)$ and the C_4 sources give $\succeq_c$ also for $\tilde{s}$. □

Proposition 7.77. *Suppose that q is odd and $a = b = -1$. Then, the adjoint parents $\tilde{\chi}, \tilde{\psi}$ have $\mathcal{H}_\ell$-fixed extensions $\widehat{\tilde{\chi}}, \widehat{\tilde{\psi}}$ on their stabilizers and $(\chi, \psi) \succeq_c$ on X, M.*

Proof. With $D = \mathbf{G}_{\mathrm{ad}}^F$, put $S = X/Z(X)$. Let $\tilde{s} \in \mathbf{T}_{\mathrm{sc}}^*$ be the lift of s of order 2. It is unique, so $F^*(\tilde{s}) = \tilde{s}$. Moreover, $|\tilde{s}z| = 6$ for $1 \neq z \in Z_3$. Consequently, no nonidentity character of D/S fixes $\mathcal{E}(D, \tilde{s})$. By [208, proof of Proposition 10.7, (10.21)], the parents $\tilde{\chi} \in \mathcal{E}(D, \tilde{s})$ restrict irreducibly and distinctly to S. Similarly, the marked torus T_D maps onto D/S. For the quadratic weight $\tilde{\theta}$, a nonidentity twist $\nu \in \mathrm{Irr}(D/S)$ gives $\mathrm{ord}(\tilde{\theta} \cdot \nu|_{T_D}) = 6$. Thus, the local parents $\tilde{\psi}$ restrict irreducibly and distinctly to $M/Z(X)$. Both χ, ψ have central character $1_{Z(X)}$.

Let M_D be the marked torus normalizer of D, and let λ be dual to $\tilde{s}$ on the rational Levi K_D of type D_5. On the simply connected cover $\mathbf{K}_{\mathrm{der,sc}}$, the lift n_c of Proposition 7.76 is fixed by F because $j(n_c) = F_r(n_c) = n_c$. Consequently, $\lambda(n_c) = 1$, and $\tilde{\psi} = \lambda|_{M_D}\eta$ for the inflated $\eta \in \mathrm{Irr}(C_8)$. In the ambient marked group, $\langle n_c, \Phi, j_0 \rangle$ is abelian, with $j_0 = \mathrm{Int}(n_0)\gamma$. Then, $j_0^2 = 1$ and $\Phi^f = \mathrm{Int}(n_c)^{-1} j_0^e$ on D, with $\epsilon = (-1)^e$, as in Proposition 7.76. Since $\lambda(n_c^8) = 1$, the extension $\widehat{\tilde{\psi}}$ has $\widehat{\tilde{\psi}}(j_0) = 1$ and $\widehat{\tilde{\psi}}(\Phi)^f = \eta(c)^{-1}$, with $\widehat{\tilde{\psi}}(\Phi) \in \mu_{8f_2}$. The presentation modulo M_D has order $2f$, so the relations define $\widehat{\tilde{\psi}}$. Consequently, $\widehat{\tilde{\psi}}$ is $\mathcal{H}_\ell$-fixed since $8f_2 \mid \ell - 1$.

If $\epsilon = -1$, [208, proof of Proposition 10.6, Case 1] gives rational $\widehat{\tilde{\chi}}$ at the D_5 involution. If $\epsilon = 1$, take the prime-field position of $\tilde{s}$ and its opposite parabolics $\mathbf{P}^{\pm}$ as in [208, proof of Proposition 10.5] for D_5. Thus, $\phi(\mathbf{P}^{\pm}) = \mathbf{P}^{\pm}$, and $\gamma(\mathbf{P}^{\pm}) = \mathbf{P}^{\mp}$ since γ acts by -1 on the central line. Moreover, $\lambda^{\phi} = \lambda^{\gamma} = \lambda$, so λ extends by 1 on ϕ, γ. Then, the source is $\widehat{\lambda}\widehat{\tau}$, with $\widehat{\tau}$ from Proposition 7.76 for D_5. By [208, (10.6)], induction of $\lambda\tau$ from $P^{\pm}$ affords $\tilde{\chi}$. By [208, Lemma 10.4 and proof of Proposition 10.5, Case 2], the operator for $\mathbf{P}^{\pm}$ has denominator q^8. Thus, the map for γ is equivariant for $\mathcal{H}_{\ell}$ on the source $\widehat{\lambda}\widehat{\tau}$, giving fixed $\widehat{\tilde{\chi}}$. Match $\tilde{\chi}, \tilde{\psi}$ by their D_5 labels. Consequently, for $J = \mathrm{Aut}(X)_{P,\chi}$, the conjugation maps from $X \rtimes J$ and $M \rtimes J$ give extensions of χ, ψ with scalar 1 on $C_{X \rtimes J}(X)$. The factor sets α, α' and mixed functions $\mu_{y,\sigma}, \mu'_{y,\sigma}$ are 1, so $(\chi, \psi) \succeq_c$. □

Theorem 7.78. *Suppose that* $\mathrm{ord}_{\ell}(q) = 8$ *and* $|\bar{s}| = 2$ *in* Lemma 7.75. *Then, each matched* (χ, ψ) *satisfies* $\succeq_c$ *on* $M = N_X(P)$ *via an* F*-stable pair* $\mathcal{T} = (\mathbf{T}_1, \mathbf{Q})$ *in the* A_1^4 *subsystem.*

Proof. Let $R = A_1^4$ and $\mathbf{K} = \langle \mathbf{T}, U_{\alpha} : \alpha \in R \rangle$. Put $K = \mathbf{K}^F$ and $\lambda \in \mathrm{Irr}(K)$ dual to the central s in $\mathbf{K}^*$, so $\lambda|_T = \theta$. By Lemma 7.75, $H = \mathrm{PGL}_2(q^4)$ is the adjoint factor, with $\mathbf{T}_H$ nonsplit. Then, $M_{\theta}/T = W(R) \cap \langle c \rangle = \langle c^4 \rangle = C_2$. Pair $1, \mathrm{St}$ in H with the two linear labels η of C_2. Thus, $\psi = \mathrm{Ind}_{M_{\theta}}^{M}(\lambda|_{M_{\theta}}\eta)$. In $H \rtimes C_{4f}$, the marked normalizer modulo $\mathbf{T}_H^F$ has inner group C_2 and cyclic quotient of order $4f$. Given a generator lift x, we extend η by $\widehat{\eta}(x)^{4f} = \eta(x^{4f})$, with $\widehat{\eta}(x) \in \mu_{8f_2}$. Thus, $\widehat{\eta}$ is $\mathcal{H}_{\ell}$-fixed via μ_{8f_2} as in Proposition 7.76. Let $\tilde{s}$ be a rational lift of s in the regular overgroup. Since $C_{\mathbf{G}^*}(s)$ is connected, $W_s = W(R)$. Thus, $\tilde{s}z$ is conjugate to $\tilde{s}$ only for $z = 1$ in Z_3 by [208, proof of Proposition 10.9, Step 1] for s. Consequently, the parents $\tilde{\chi} \in \mathcal{E}(\widetilde{X}, \tilde{s})$ restrict irreducibly and distinctly to X by Clifford theory. Moreover, $s^{G^*} \cap T^* = s^{N_{G^*}(\mathbf{T}^*)}$ by Sylow conjugacy in $C_{\mathbf{G}^*}(s)^{F^*}$ and $C_{\mathbf{G}^*}(P^*) = \mathbf{T}^*$.

Suppose that $J = \mathrm{Aut}(X)_{P,\chi}$ has a pure graph. Since $W_s = W(R)$, the lift classes $\mathcal{O}$ of s form a Z_3-torsor. By [208, proof of Proposition 10.5], γ has a fixed class $O \in \mathcal{O}$, whose alcove representative t_0 satisfies $\gamma(t_0) = t_0$. Thus, for the Weyl displacement v to $\mathbf{T}$, the graph $h = v\gamma v^{-1}$ fixes s and the chamber $v(\Phi^+)$. Any graph operation g fixing s has $gh^{-1} \in W(R)$ on $\mathbb{Z}R$. Otherwise, choose any chamber Φ^+ at $\mathbf{T}$ and put $h = 1$.

Take its Borel $\mathbf{P}_0$ and $\mathcal{T}_0 = (\mathbf{T}, \mathbf{P}_0 \cap \mathbf{K})$. Since $F(\mathcal{T}_0)$ is $\mathbf{K}$-conjugate to $\mathcal{T}_0$, Lang's theorem gives $d \in \mathbf{K}$ with $\mathcal{T} = {}^d\mathcal{T}_0$ rational. With $\mathbf{P}_1 = {}^d\mathbf{P}_0$, we put $\mathbf{T}_1 = {}^d\mathbf{T}$. The stabilizer $N_{\mathbf{K}}(\mathcal{T}) = \mathbf{T}_1$ is connected, so $(\mathbf{K} \cdot \mathcal{T})^F$ is one K-orbit. Consequently, correct b by $k \in K$ so that $kb(\mathcal{T}) = \mathcal{T}$ for each b arising from the ordinary or joint stabilizer of (χ, ψ). If b has field shift 0, we write $b = \mathrm{Int}(x)h_b$ with $x \in \widetilde{\mathbf{K}}$ and $h_b \in \{1, h\}$. Thus, for $b' = \mathrm{Int}(d^{-1}) \circ (kb) \circ \mathrm{Int}(d)$, $b'h_b^{-1}$ acts by $N_{\widetilde{\mathbf{K}}}(\mathcal{T}_0) = \widetilde{\mathbf{T}}$. Since $h_b(\mathbf{P}_0) = \mathbf{P}_0$, $kb(\mathbf{P}_1) = \mathbf{P}_1$. The torus $\mathbf{T}$ remains the local marking.

With $B = (M \rtimes J)_{\theta}$, we put $D_{\lambda} = KB/\ker\lambda$ and $B' = (KB)_{(F_r^i(\mathcal{T}))_i}$. Then, $B' \to J$ is onto and $B' \cap K = \mathbf{T}_1^F$. Thus, the sources are $\Lambda|_{B' \times_J D_{\lambda}}$ on the global cover and $\Lambda\widehat{\eta}$ on $B \times_J D_{\lambda}$. By [208, proof of Lemma 4.19], the B'-translates of $\mathbf{P}_1$ give a B'-stable radical U. Moreover, we set $U_1 = \prod_i F_r^i(R_u(\mathbf{P}_1))$. Since $\mathcal{T}$ gives a split torus in H, its flag character is $1 + \mathrm{St}$. Thus, the result of [169, Theorem 2.1(1)] gives coefficient ± 1 at $\tilde{\chi}$ in the induction from the lifted $\mathbf{T}_1^F$. By $\tilde{\chi}|_X = \chi$, $\langle R_{\mathbf{T}_1}^X(\lambda|_{\mathbf{T}_1^F}), \chi \rangle = \pm 1$.

Given a joint stabilizer (y, σ), we set $z = \mathrm{Int}(m)y$ with $m \in M$ and $\theta^{z\sigma} = \theta$. Then, $(\Lambda\widehat{\eta})^{z\sigma} = \Lambda\widehat{\eta}$. Choose $k \in K$ returning $z(\mathcal{T})$ to $\mathcal{T}$. Thus, $z' = \mathrm{Int}(k)z$ normalizes B', and $\Lambda^{z'\sigma} = \Lambda$. On the dual side, $C_{U^*}(s_{\Pi}) = C_{U_1^*}(s_{\Pi}) = C_{(U^{z'})^*}(s_{\Pi})$ equals $\prod_i F_r^i(R_u(\mathbf{Q}^*))$ for the dual marking of $\mathbf{Q} = {}^d(\mathbf{P}_0 \cap \mathbf{K})$. Since $\mathbf{Q}$ is rational, [19, Corollary 6.5] compares U, U_1 on $X \times \mathbf{T}_1^F$. Similarly, Lemma 7.36 compares $U, U^{z'}$ on the cover. From $\Lambda^{z'\sigma} = \Lambda$, $\Xi^{z'\sigma} = \Xi$ for the alternating character Ξ by Proposition 7.69. Thus, extraction at χ and the common section $J \to D_{\lambda}$ give $\succeq_c$ as in Proposition 7.44, with scalar $\Lambda(c, a)$ on the cover. If $\mathrm{ord}_{\ell}(q) = 8$, the general conclusion $\mathrm{iMN}(S, \ell)$ also follows from Corollary 7.113. □

Suppose that $\mathrm{ord}_{\ell}(q) = 4$, and put $v = c^2$ in the coordinates of Lemma 7.75. Then, $v|_U$ has polynomial Φ_4^2 and $v|_V = 1$. Moreover, on $\mathbf{T}$ the Frobenius action is uv, with $u = \epsilon q$. With $T = \mathbf{T}^F$, we put $B = C_W(v)$ and $Q = B \cap W(D_4)$. Similarly, as in Lemma 7.75, the projection $\mathbf{T}^* \to \overline{\mathbf{T}}$ onto the adjoint D_4-torus on U has connected kernel $\mathbf{S}$. Thus, we find $|\overline{\mathbf{T}}^{F^*}| = (q^2+1)^2$ by the determinant on U. Given $s \in T^*$, we put $C = C_{\mathbf{G}^*}(s)$ and $R = \Phi(C^{\circ})$.

Lemma 7.79. *We have that* $|T| = (u-1)^2(q^2+1)^2$ *and* $P = O_{\ell}(T) \in \mathrm{Syl}_{\ell}(X)$. *Moreover,* $C_X(P) = T$ *and* $M = N_X(P) = N_X(\mathbf{T})$, *with* $|M/T| = 96$. *Each* $\mathcal{E}(X, s)$ *meeting* $\mathrm{Irr}_{\ell'}(X)$ *has* $s \in T^*$ *up to conjugacy. Let* t *be an odd prime dividing* $q^2 + 1$. *Then, we find* $\Phi(s) \neq \varnothing$ *for* $1 \neq s \in T^*[t]$ *precisely on six lines in one* B*-orbit. On these lines* $\Phi(s) = A_3$. *Moreover,* $C_{\mathbf{G}^*}(s)$ *is connected for every* $s \in T^*[t]$.

Proof. The determinant of $uv - 1$ gives $|T| = (u-1)^2(q^2+1)^2$. Among the signed degrees, only $8, 12$ contribute to $|X|_{\ell}$. Each contributes $(q^2+1)_{\ell}$ since $\ell \neq 2, 3$. Thus, $P = O_{\ell}(T)$ is Sylow. In $\mathbb{F}_t$, we choose i with $i^2 = -1$ from the

eigenvalue of v on $T[t]$. Thus, a point has coordinates $(x, ix), (y, iy)$ on the two planes of U, and 0 on V. Since $t \nmid 6$, the coordinates coincide on $T[t]$ and $T^*[t]$. Given a vector root α, its restriction is one of $\xi x, \xi y, (1 \pm i)x, (1 \pm i)y, \xi x + \zeta y$ with $\xi, \zeta \in \mu_4 = \{\pm 1, \pm i\}$, up to a scalar in μ_4. Given $\alpha \in \Phi(E_6) \setminus \Phi(D_5)$, the restriction of 2α is $(1+i)(\xi x + \zeta y)$, since $1 - i = -i(1+i)$. Thus, the root kernels have union $xy(x^4 - y^4) = 0$. The roots $\pm e_j \pm e_5$ give $x = 0, y = 0$. The roots in D_4 give the four slopes $x = \xi y$ for $\xi \in \mu_4$. In particular, $\alpha|_{P[\ell]} \neq 1$ for every root α. Consequently, $C_{\mathbf{G}}(P)^\circ = \mathbf{T}$ and $N_X(P) = N_X(\mathbf{T})$.

The same equations on the complex i-eigenspace U_i show that v is regular. By [178, Theorem 4.2], $|B| = 8 \cdot 12 = 96$. The Sylow 4-torus $\mathbf{S}_4 \leq \mathbf{T}$ has centralizer $\mathbf{T}$ and relative Weyl group $B = N_X(\mathbf{T})/T$. By [63, Lemma 2.20(1)], B acts faithfully on $P[\ell]$. Thus, $C_X(P) = T$, and dually $C_{G^*}(P^*) = T^*$ for $P^* = O_\ell(T^*)$. An ℓ'-series has $|C_{G^*}(s)|_\ell = |G^*|_\ell$ by the Jordan degree formula [58, Proposition 11.5.6]. Hence, s is conjugate into T^*. By [132, Proposition 5.11], $N_{G^*}(T^*)$ controls G^*-fusion in T^*.

The group $N_W(U)$ embeds in $\mathrm{Aut}(D_4)$ since no roots lie in V. On U, the centralizer in $W(B_4)$ is $C_4 \wr C_2$ by the two negative 2-cycles. Thus, even sign parity gives $|Q| = 16$, and $B/Q \cong \mathfrak{S}_3$. The three sets of 8 weights of D_4 are permuted by B/Q. The vector weights have kernels $x = 0, y = 0$, exchanged by Q. Consequently, the six lines form one B-orbit over $\mathbb{F}_t$. At $x = 0, y \neq 0$, $\Phi(s) = D_3 = A_3$ on the zero pair and e_5. Thus, a generic point of $(\mathbb{Q}D_3)^\perp$ gives a Levi with roots A_3. By [20, Corollary 2.9], $C_{\mathbf{G}^*}(s)$ is connected since $|s| = t \neq 3$. Finally, $|\mathbf{S}^{F^*}| = (u-1)^2$ is prime to t. If $t \mid |\bar{s}|$, then the order-t point in $\langle s \rangle$ has nontrivial projection. Hence, $C_{\mathbf{G}^*}(s)$ lies in the torus or A_3 Levi just described. □

Lemma 7.80. *Put $a = h(s), b = k(s)$ on $\mathbf{S}$, with h, k as in* Lemma 7.75. *For $s \in \mathbf{S} \setminus \{1\}$, the roots are D_4 off the walls $a = 1, b = 1, a = b$, and D_5 on a wall. If $\bar{s}$ has nontrivial 2-power order, the roots are A_1^4, $A_3 + A_1^2$, or $A_5 + A_1$. For $R = A_3 + A_1^2$ or $A_5 + A_1$, $C = C^\circ$.*

Proof. On $\mathbf{S}$, the restrictions of roots outside D_4 are $\pm h, \pm k, \pm(h-k)$. Thus, any one of $a = 1, b = 1, a = b$ adds 16 roots to D_4 and gives D_5. Two equations imply $a = b = 1$. Suppose that $\bar{s} \neq 1$ has 2-power order. Then, q is odd. Take a 2-primary lift $z = (z_1, z_2, z_3, z_4)$ in the vector torus of D_4, with $\bar{s}$ the class of z modulo the common sign. Rationality gives $(uv)z = \delta z$ with $\delta = \pm 1$. Since $v^2 = -1$ on U, $z^{q^2+1} = 1$. Thus, $z_i = \pm 1$, since $v_2(q^2+1) = 1$. If $\delta = 1$, $z_{v(i)} = z_i$ within each plane, and for $\delta = -1$, $z_{v(i)} = -z_i$. Consequently, $z_j = z_k = -1$ for two indices j, k since $\bar{s} \neq 1$. Put $t_0 = (e_j + e_k)^\vee(-1)$, so $s = t_0 z_0$ for $z_0 \in \mathbf{S}$. Thus, we put $a = h(z_0), b = k(z_0)$.

On U, the kernel of t_0 has roots $D_2 + D_2 = A_1^4$. For each direction $\beta \in \{h, k, h-k\}$, four of the eight roots over β satisfy $\alpha(t_0) = 1$. Indeed, $h - k$ has weights $\pm e_i$ on U, with value z_i for each sign. For h, k, the coroot pairing with $(e_j + e_k)^\vee$ is $(\sigma_j + \sigma_k)/2$. The change $(\sigma_j, \sigma_i) \mapsto (-\sigma_j, -\sigma_i)$ with $i \in \{1,2,3,4\} \setminus \{j,k\}$ changes $(\sigma_j + \sigma_k)/2$ by $-\sigma_j$ and fixes the restriction to $\mathbf{S}$. Thus, the directions $\pm\beta$ add 8 roots precisely when $\beta(z_0) = \pm 1$. Put $m = |\{\alpha \in \{h, k, h-k\} : \alpha(z_0) = \pm 1\}|$. Then, $m \in \{0, 1, 3\}$, and $\Phi(s)$ has $8 + 8m$ roots of rank $4, 5, 6$, respectively. Consequently, $\Phi(s)$ has type $A_1^4, A_3 + A_1^2, A_5 + A_1$, respectively. If $m = 3$, then $s^2 = 1$ and its lift of order 2 gives $C = C^\circ$ by [208, proof of Proposition 10.9, Step 1]. If $m = 1$, C/C° embeds in the diagram group of $A_3 + A_1^2$ times the signs on $(\mathbb{Q}R)^\perp$. Thus, C/C° is a 2-group and embeds in Z_3. Consequently, $C = C^\circ$. □

Lemma 7.81. *Given $s \in \mathbf{S}$ with root system D_4, $C/C^\circ \neq 1$ exactly when $a = b^{-1}$ and $|b| = 3$. In both cases $C/C^\circ \cong C_3$. If $|\bar{s}| = 2$ and $R = A_1^4$, then $C = C^\circ$ for every F^*-rational s.*

Proof. Put $\mathbf{K}^* = C_{\mathbf{G}^*}(\mathbf{S})$ with roots D_4. By Lemma 7.79, $N_W(U)/W(D_4) = \mathfrak{S}_3$. Thus, the sign change on e_5, e_1 acts by $(h, k) \mapsto (k, h)$. The other two triality reflections fix $a = 1$ and $b = 1$. Consequently, the action on $X^*(\mathbf{S})$ is the A_2 action with simple roots $h - k, k$ and $(h, k) = 1/2$. Since $a, b, a/b \neq 1$, every reflection moves (a, b). The product $r_{h-k} r_k$ acts by $(h, k) \mapsto (k - h, -h)$, so its fixed points satisfy $a = b^{-1}$ and $b^3 = 1$. Hence, the two points with $|b| = 3$ have component C_3. They are F^*-fixed exactly when $u \equiv 1 \bmod 3$.

Suppose that $|\bar{s}| = 2$ and $R = A_1^4$. Then, $s^2 = z_0^2$ has roots D_4 by the case $m = 0$ of Lemma 7.80. The derived group of $\mathbf{K}^*$ is simply connected since the centre C_2^2 of the simply connected D_4 meets Z_3 trivially. Thus, $C_{\mathbf{K}^*}(s)$ is connected by Steinberg's theorem [58, Chapter 3]. Since $C_{\mathbf{G}^*}(s^2)^\circ = \mathbf{K}^*$, the component group at s injects into that at s^2. It remains to exclude $|s^2| = 3$ when $\bar{s} \neq 1$.

If $2 \nmid q$, we let $x_i \in \mathbb{F}_2$ denote the value at α_i of $x : \mathbb{Z}\Phi \to \mu_2$. Take $c = s_2 s_3 s_4 s_5 s_6$ in the standard D_5, so $v = c^2$. Then, $x \circ v = x$ reduces to $x_2 + x_3 + x_4 = 0$, $x_4 = x_5 = x_6$, $x_2 + x_3 + x_4 + x_6 = 0$ by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. Its

four solutions are $(a,b,b,0,0,0)$ with $a,b\in\mathbb{F}_2$. Since $F^*|_{\mathbf{S}}$ is the u-power, $\mathbf{S}[2]$ already gives four solutions. Thus, $T^*[2]=\mathbf{S}[2]$. Consequently, $|s^2|=3$ would give $s^3\in\mathbf{S}$ and $s^2\in\mathbf{S}$, so $s=s^3(s^2)^{-1}\in\mathbf{S}$, contrary to $\bar{s}\neq 1$. □

Proposition 7.82. *If $C_{\mathbf{G}^*}(s)\neq C_{\mathbf{G}^*}(s)^\circ$, then $R=D_4$ and $3\mid u-1$. Moreover, the two cubic points in T^* are B-conjugate and lift to $(\mathbf{G}^*_{\mathrm{sc}})^{F^*}$.*

Proof. By Lemma 7.79, we work on T^*. If an odd prime t divides $|\bar{s}|$, $C_{\mathbf{G}^*}(s)$ lies in a torus or an A_3 Levi $\mathbf{L}^*$. The derived group of $\mathbf{L}^*$ is simply connected, since the A_3 centre upstairs has order prime to 3. Thus, $C_{\mathbf{L}^*}(s)$ is connected. If $R=D_5$, then C/C° embeds in the diagram group of D_5 times the signs on $(\mathbb{Q}R)^\perp$. Thus, C/C° is a 2-group and embeds in Z_3, so it is trivial. If $|\bar{s}|$ is a power of 2, including $\bar{s}=1$, the two lemmas 7.80 and 7.81 give $C\neq C^\circ$ only for $R=D_4$ at $|b|=3, a=b^{-1}$. Thus, the two points are F^*-fixed for $3\mid u-1$. The transposition in B/Q swaps the pairs (a,b) and (b,a).

By [208, (10.23)], in standard D_4 position take $t_0=(\omega_1^\vee+\omega_6^\vee)(\zeta_3)$, with coroot coordinates $(2,2,3,4,3,2)$ for $\omega_1^\vee+\omega_6^\vee$. Then, $\alpha_i(t_0)=\zeta_3$ for $i=1,6$, and 1 otherwise. Transport t_0 to $\tilde{s}\in\mathbf{T}^*_{\mathrm{sc}}$ by W. Since $v\in W(D_4)$, it fixes $\tilde{s}$ and $F^*(\tilde{s})=\tilde{s}^u=\tilde{s}$. Moreover, $\pm v\in W(D_4)$, so the derived factor D_4 is split. Consequently, the characters in $\mathcal{E}(X,s)$ have central character $1_{Z(X)}$. Since C^{F^*} contains P^* and $|C/C^\circ|=3$, the ℓ'-cut on $\mathcal{E}(X,s)$ is the ℓ'-cut on the unipotent labels of $D_4(q)$ with their C_3-action. □

Proposition 7.83. *Let $M=N_X(P)$. Then, the equivariant correspondence $\widetilde{\Omega}$ pairs regular parents of $\mathrm{Irr}_{\ell'}(X)$ with those of $\mathrm{Irr}_{\ell'}(M)$. Moreover, $\widetilde{\Omega}(\chi\nu)=\widetilde{\Omega}(\chi)\nu$ for $\nu\in\mathrm{Irr}(\widetilde{X}/X)$. Paired characters have equal scalars on $Z(\widetilde{X})$. If $C=C^\circ$, $\tilde{\chi}|_X$ and $\tilde{\psi}|_M$ are irreducible. Moreover, $\psi=\mathrm{Ind}_{M_\theta}^M(\lambda|_{M_\theta}\eta)$, with λ dual to s on K.*

Proof. Take the regular overgroup $\widetilde{X}$ of Theorem 7.24 with torus $\widetilde{\mathbf{T}}$. By Lemma 7.79, $C_{\widetilde{\mathbf{G}}}(\mathbf{S}_4)=\widetilde{\mathbf{T}}$ for the Sylow 4-torus $\mathbf{S}_4$. Thus, $\widetilde{M}=N_{\widetilde{X}}(\widetilde{\mathbf{T}})$ satisfies $\widetilde{M}\cap X=M$. Let $\tilde{s}$ be dual to $\widetilde{\theta}\in\mathrm{Irr}(\widetilde{T})$. By Steinberg's theorem [58, Chapter 3], $C_{\widetilde{\mathbf{G}}^*}(\tilde{s})$ is connected, since $[\widetilde{\mathbf{G}}^*,\widetilde{\mathbf{G}}^*]$ is simply connected. Thus, we put $\widetilde{\mathbf{K}}=\langle\widetilde{\mathbf{T}},U_\alpha:\alpha\in R\rangle$ with $R=\Phi(C_{\widetilde{\mathbf{G}}^*}(\tilde{s}))$. Let $\pi:\mathbf{H}\to[\widetilde{\mathbf{K}},\widetilde{\mathbf{K}}]$ be the simply connected cover and $\mathbf{T}_H$ the torus above $\widetilde{\mathbf{T}}\cap[\widetilde{\mathbf{K}},\widetilde{\mathbf{K}}]$. Since $\alpha(\tilde{s})=1$ for $\alpha\in R$, torus duality gives $\widetilde{\theta}|_{\pi(\mathbf{T}_H^F)}=1$. By Lang's theorem on $\mathbf{T}_H$, $\widetilde{K}=\widetilde{T}\,\pi(\mathbf{H}^F)$ and $\widetilde{T}\cap\pi(\mathbf{H}^F)=\pi(\mathbf{T}_H^F)$. Indeed, lift $x\in\widetilde{K}$ to $(z,h)\in Z^\circ(\widetilde{\mathbf{K}})\times\mathbf{H}$. Then, $h^{-1}F(h)\in Z(\mathbf{H})\leq\mathbf{T}_H$. By Lang's theorem, $b\in\mathbf{T}_H$ with $b^{-1}F(b)=h^{-1}F(h)$, so $hb^{-1}\in\mathbf{H}^F$. Consequently, $\widetilde{\lambda}$ is unique with $\widetilde{\lambda}|_{\widetilde{T}}=\widetilde{\theta}$ and $\pi(\mathbf{H}^F)\leq\ker\widetilde{\lambda}$.

Since $W_{\tilde{s}}=W(R)$, $\widetilde{M}_{\widetilde{\theta}}=N_{\widetilde{K}}(\widetilde{\mathbf{T}})$. Thus, the extension map is $\Lambda(\widetilde{\theta})=\widetilde{\lambda}|_{\widetilde{M}_{\widetilde{\theta}}}$. The two restrictions of $\widetilde{\lambda}$ are equivariant under field, graph and $\mathcal{H}_\ell$-actions. Moreover, if $\nu\in\mathrm{Irr}(\widetilde{X}/X)$, then $\pi(\mathbf{H}^F)\leq\ker\nu$, so $\Lambda(\widetilde{\theta}\nu)=\Lambda(\widetilde{\theta})\nu$ with the appropriate restrictions. Consequently, the result of [156, Theorem B and §4.4] gives $\widetilde{\Omega}$ on the regular parents. The 4-Harish-Chandra label η for the torus pair $(\widetilde{\mathbf{T}},1_{\widetilde{T}})$ labels both the Jordan source and $\mathrm{Ind}_{\widetilde{M}_{\widetilde{\theta}}}^{\widetilde{M}}(\widetilde{\lambda}\eta)$. In particular, $\tilde{\chi},\tilde{\psi}$ have equal central characters on $Z(\widetilde{X})$. By Lang's theorem on $\widetilde{\mathbf{T}}$, inclusion gives $\widetilde{M}/M\cong\widetilde{X}/X$.

If $C_{\mathbf{G}^*}(s)$ is connected, $W_s=W(R)$ fixes $\tilde{s}$ since $\alpha(\tilde{s})=1$ for each $\alpha\in R$. Thus, no nonidentity $\nu\in\mathrm{Irr}(\widetilde{X}/X)$ fixes the series of $\tilde{s}$ or the $\widetilde{M}$-orbit of $\widetilde{\theta}$. By cyclic Clifford theory, $\tilde{\chi}|_X$ and $\tilde{\psi}|_M$ are irreducible. Consequently, $(\tilde{\chi}\nu,\tilde{\psi}\nu)$ restricts to (χ,ψ). Moreover, $\widetilde{\lambda}|_K=\lambda$ is dual to s, with $\lambda|_T=\theta$. Thus, its value on $Z(X)$ is $\theta|_{Z(X)}$. If $C_{\mathbf{G}^*}(s)$ is disconnected, $\widetilde{\Omega}$ pairs the orbits under D/S. By Proposition 7.85, we obtain (χ,ψ) in those orbits. □

Lemma 7.84. *Take $Q=B\cap W(D_4)$ and $Q'\leq B$ with $Q'/Q=\mathrm{Alt}(3)$ in $B/Q\cong\mathfrak{S}_3$. With $S_0=X/Z(X)$, we put $D=\mathbf{G}^F_{\mathrm{ad}}$. Thus, we treat $\mathcal{E}(X,s)$ from* Proposition 7.82 *for $|s|=3$. For the exceptional cover of ${}^2E_6(2)$, by* Ruhstorfer–Schaeffer Fry [157, Lemma 4.1(2)]*, we obtain* iMN(S_0,ℓ)*. The cubic series on X,M have four diagonal orbits of size 3 and two of size 1 each. Moreover, the correspondence through Q is $\mathcal{H}_\ell$-equivariant.*

Proof. For the cubic weight θ, $M_\theta/T=Q'$ by Lemma 7.81. Then, we put $V_2=\langle n_w,\mathbf{T}[2^\infty]:w\in W\rangle$ for the Tits lifts n_w. Then, V_2 is F-stable and $V_2\cap\mathbf{T}$ is 2-primary. Each rational $w\in B$ has a lift in V_2^F: for $x\in V_2$ above w, the discrepancy $x^{-1}F(x)\in\mathbf{T}[2^\infty]$ is in the image of $F-1$ on $\mathbf{T}[2^\infty]$. If $r=2$, $\mathbf{T}[2^\infty]=1$. Thus, θ is trivial on $V_2^F\cap T$, and we extend it by 1 on $V_2^F\cap M_\theta$. The normalization $\Lambda_\theta|_{V_2^F\cap M_\theta}=1$ gives equivariance under inversion, $\mathcal{H}_\ell$ and conjugation by V_2^F. Also, $\theta|_{Z(X)}=1$ by Proposition 7.82. Consequently, Clifford induction gives $\mathrm{Irr}(M\mid\theta)=\{\mathrm{Ind}_{M_\theta}^M(\Lambda_\theta\eta):\eta\in\mathrm{Irr}(Q')\}$, and the index $[B:Q']=2$ and all degrees of Q' are prime to ℓ.

On the Gaussian eigenspace, $Q = \langle j, k, h\rangle$, where $j = iI$, $k = \mathrm{diag}(1,-1)$ and h swaps the coordinates. Thus, $Z(Q) = \langle j\rangle$ and $[k,h] = j^2$. The matrices k, h have no common eigenline in $V_i = \mathbb{Q}(i)^2$ or $\overline{V}_i$. Thus, $V_i, \overline{V}_i$ are irreducible with j-scalars $i, -i$, respectively. The degree sum $16 = 8 + 2\cdot 2^2$ leaves the eight linear labels of $Q/\langle j^2\rangle$. Moreover, B fixes j and induces $\mathrm{GL}_2(2)$ on $Q/\langle j\rangle$. Indeed, an element b in the kernel acts by $k \mapsto \pm k,\ h \mapsto \pm h$. Since $[k,h] = -I$, conjugation by $q \in Q$ gives the same signs. Then, $q^{-1}b$ centralizes k, h, so it is scalar on V_i. The finite scalars in $\mathbb{Q}(i)$ lie in $\langle j\rangle$, so $b \in Q$.

If $\eta(j) = \pm 1$, the linear labels η form an affine plane over $\mathrm{Irr}(Q/\langle j\rangle)$. Since $1 - c_3$ is invertible for a generator $c_3 \in Q'/Q$, one label is fixed. Thus, c_3 cycles the other 3. A fixed linear η extends to Q' by value 1 on a complement C_3. The extension $\widehat{\eta}$ of 2-power order is unique, so B fixes $\widehat{\eta}$. The modules $V_i, \overline{V}_i$ extend the two nonlinear labels to B. Thus, the four distinguished extensions $\widehat{\eta}$ have values in $\mathbb{Q}(i)$, and the remaining extensions are $\widehat{\eta}\nu$ with $\nu \in \mathrm{Irr}(Q'/Q)$. Each orbit of size 3 gives $\mathrm{Ind}_Q^{Q'}\eta$.

By [156, §4], the torus pair $(\mathbf{T}_{D_4}, 1)$ in split adjoint D_4 gives the ℓ'-unipotent labels from Q, equivariantly under Q'/Q and B/Q. The relative Weyl group is Q since U_i is regular for D_4. By [208, proof of Proposition 10.9, Step 3], orbits of size 3 give one character of X and fixed labels η give diagonal triples. On M, Clifford induction from Q gives the same orbit lengths. Consequently, D/S_0 has two orbits of size 1 and four of size 3 on each side.

The field $\mathbb{Q}(i)$ is fixed by $\mathcal{H}_\ell$ since $\ell \equiv 1 \bmod 4$. If $\theta^\sigma = \theta$, we have that σ fixes μ_3 and the local labels. If $\theta^\sigma = \theta^{-1}$, choose a transposition $bQ \in B/Q$ returning θ^{-1} to θ. Then, b fixes $\widehat{\eta}$ and inverts Q'/Q, so $\nu^{b\sigma} = \nu$ for each cubic twist ν. Moreover, b fixes each moving orbit indexed by $\eta(j)$. Thus, $\mathcal{H}_\ell$ fixes $\mathrm{Irr}(M \mid \theta)$. By [208, proof of Proposition 10.9, Step 4], each $\chi \in \mathcal{E}(X,s)$ has values in $\mathbb{Q}$. □

Proposition 7.85. *Match the γ-fixed members χ_0, ψ_0 of corresponding triples. Then, match χ_0^d, ψ_0^d for $d \in D/S_0$. Thus, the bijection on $\mathcal{E}(X,s)$ is $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant.*

Proof. Write T_D for the marked torus of D and put $T_0 = T_D \cap S_0$. By Lang's theorem on the torus, $T_D/T_0 \cong D/S_0 = C_3$, and both Weyl quotients are B. The lifts of s dual to the extensions of θ are rational since $u \equiv 1 \bmod 3$. Thus, Q fixes each lift, and Q'/Q translates it by Z_3 via the lift commutator in [208, proof of Proposition 10.9, Step 1]. Given $t \in T_D$ and $w \in Q'$, the change under conjugation by t is $\vartheta([t, n_w])$ for a lift ϑ of θ. By torus duality, it equals $(\vartheta^w/\vartheta)(t)$ up to inversion. Consequently, T_D/T_0 acts by twists in $\mathrm{Irr}(Q'/Q)$. Thus, D/S_0 fixes each singleton ψ. Each triple is a regular orbit ψ^{D/S_0}.

Take the marked inversion j_0, corrected by $\mathbf{T}[2^\infty]$ as in Lemma 7.84 for V_2. Then, a transposition $bQ \in B/Q$ returns θ^{j_0} to θ. Since j_0 acts identically on B under conjugation, bj_0 fixes $\widehat{\eta}$ and inverts Q'/Q. Thus, on $\widehat{\eta}\nu$ it acts by $\nu \mapsto \nu^{-1}$. On characters, the pinned γ is dbj_0 for a rotation $d \in D/S_0$. The prime field acts by r on the lattice and trivially on B. If ϕ inverts θ, then b returns the weight with its Q-orbit. Thus, ϕ stabilizes each triple ψ^{D/S_0}. A correction by S_0 returning P is unique modulo $N_{S_0}(P)$, which acts trivially on $\mathrm{Irr}(N_{S_0}(P))$. Hence, $[\phi,\gamma] \in \mathrm{Inn}(S_0)$ fixes each local ψ. Consequently, the unique local γ-fixed member ψ_0 satisfies $\psi_0^\phi = \psi_0$. By [208, proof of Proposition 10.9, Step 4], the global member χ_0 is also field-fixed. Since ϕ acts on D/S_0 by the r-power and γ by inversion, the pairs (χ_0^d, ψ_0^d) for $d \in D/S_0$ are equivariant. The singletons keep their Q-labels. Since $Z(X)$ is an ℓ'-group, $M/Z(X) = N_{S_0}(PZ(X)/Z(X))$. Finally, inflation to X, M is $\mathcal{H}_\ell$-equivariant by Lemma 7.84 for (χ,ψ). □

Proposition 7.86. *For the cubic class $\tilde{s}$, each matched χ admits an $\mathcal{H}_\ell$-fixed extension to $\mathrm{Aut}(S_0)_\chi$.*

Proof. By [208, (10.23)], we choose the γ-fixed $\tilde{s} \in \mathbf{G}^*_{\mathrm{sc}}$ whose Levi has nodes $2,3,4,5$. Thus, in both positions, q and $-q$ give the split factor $D_4(q)$. Thus, the parent ρ on D is $\varepsilon R_{\mathbf{K}}^D(\lambda\tau_K)$ by [208, (10.6)]. The character λ is dual to $\tilde{s}$ on $K = \mathbf{K}^F$, and τ_K is from the adjoint $D_4(q)$. Thus, we choose τ fixed by the diagram reflection $\gamma|_{D_4}$. Such a label exists in every C_3-orbit by Lemma 7.84 for B/Q.

Suppose that $\epsilon = 1$. Put $b = 0$ if $r \equiv 1 \bmod 3$ and $b = 1$ otherwise. Then, $b = 1$ implies $2 \mid f$. The supplement of K is generated by γ and $n_0^b\phi$, where n_0 is the pinned longest lift and $\phi = F_r$. Indeed, $n_0^2 = 1$ in D, and n_0 satisfies $n_0\phi = \phi n_0,\ n_0\gamma = \gamma n_0$. These lifts fix λ, τ, because n_0 induces the diagram action of γ on D_4 modulo $W(D_4)$. Thus, $\widehat{\lambda}(\gamma) = \widehat{\lambda}(n_0^b\phi) = 1$. By [156, Proposition 8.5 and its proof] at the torus pair for $d = 4$, τ has an $\mathcal{H}_\ell$-fixed extension $\widehat{\tau}$. Consequently, we choose the source $\widehat{\lambda}\widehat{\tau}$ via conjugation on $D_4(q)$.

The rational opposites $\mathbf{P}^\pm$ have $|R_u(\mathbf{P}^\pm)^F| = q^{36-12} = q^{24}$. Since $C_{\mathbf{G}^*_{\mathrm{sc}}}(\tilde{s}) = \mathbf{K}^*$, $\mathrm{Ind}_{P^\pm}^D(\lambda\tau_K) = \rho$. The intertwiners in Lemma 7.92 have denominator q^{12} and act on the cosets exchanging $\mathbf{P}^+, \mathbf{P}^-$. If $\lambda^\sigma = \lambda^{-1}$, then n_0 returns λ^σ to λ. Moreover, n_0 centralizes $\gamma, n_0^b\phi$, and $\widehat{\tau}^{n_0} = \widehat{\tau}$. Consequently, the normalized intertwiner gives

$\widehat{\rho}^\sigma = \widehat{\rho}$. Suppose that $\epsilon = -1$. Then, $r \equiv -1 \bmod 3$ and f is odd. In the $-q$ position, we choose $\mathbf{P}$ with Levi $\mathbf{K}$ and $\gamma(\mathbf{P}) = \mathbf{P}$. Moreover, $\widehat{\tau}_\gamma$ is rational by [156, Theorem 7.4]. Put Y_γ for the character on $D\langle\gamma\rangle$ from $\lambda\widehat{\tau}_\gamma$ and $\lambda(\gamma) = 1$, as in [208, proof of Lemma 4.19]. The correction n_0 takes λ^{-1} to λ and $U = R_u(\mathbf{P})$ to U^{n_0}. Since $C_{U^*}(\tilde{s}) = C_{(U^{n_0})^*}(\tilde{s}) = 1$, Lemma 7.36 compares the two γ-stable radicals. Thus, σ acts only on λ, and the correction by n_0 gives $Y_\gamma^\sigma = Y_\gamma$. The same correction on $R_{\mathbf{K}}^D(\lambda\tau_K)$ gives $\rho^\sigma = \rho$ and $\rho^\phi = \rho$. By [208, (10.6)], $\langle Y_\gamma|_D, \rho\rangle = \pm 1$. By Lemma 7.31, we obtain a rational $\widehat{\rho}_\gamma$. On the odd factor C_f, the unique real extension $\widehat{\rho}_f$ is rational by [208, Lemma 3.1]. Thus, [156, Lemma 7.1] combines $\widehat{\rho}_\gamma, \widehat{\rho}_f$ over C_{2f}. For $\chi = \rho|_{S_0}$ irreducible, choose $\widehat{\rho}|_{\mathrm{Aut}(S_0)_\chi}$. Otherwise, let V_{χ_0} be the isotypic space for the γ-fixed member χ_0. Then, $\langle \rho|_{S_0}, \chi_0\rangle = 1$, and $\mathrm{Aut}(S_0)_{\chi_0}$ acts on V_{χ_0}. Since both $\widehat{\rho}$ and χ_0 are $\mathcal{H}_\ell$-fixed, the character $\widehat{\chi}_0$ on V_{χ_0} is $\mathcal{H}_\ell$-fixed. Consequently, $\widehat{\chi}_0^d$ covers χ_0^d for $d \in D/S_0$. □

Proposition 7.87. *For the disconnected D_4 series, $(\chi, \psi) \succeq_c$ on $M = N_X(P)$ with the matching of* Proposition 7.85 *for $|s| = 3$.*

Proof. Take a cubic lift $\vartheta \in \mathrm{Irr}(T_D)$ of θ, with B-inertia Q. Moreover, the three lifts $\vartheta\nu$ form one Q'/Q-orbit. Thus, for a choice of ϑ, the transposition bQ with $b = \mathrm{diag}(i, 1)$ on U_i returns ϑ^{-1} to ϑ. Indeed, $b \in W(D_5)$ has an additional sign change on e_5. Put $B_2 = \langle Q, b\rangle$. Then, each Q'/Q-orbit has a b-fixed η. If η is linear, extend it to B_2 by $\widehat{\eta}(b)^2 = \eta(b^2)$ with $\widehat{\eta}(b) \in \mu_4$. If η is nonlinear, we choose the Gaussian B-module or its conjugate. Then, v acts by a scalar $\zeta \in \mu_4$.

With $E = \langle\phi, \gamma\rangle$, let A_M be the marked normalizer in $D \rtimes E$. In A_M/T_D, choose J for inversion and Φ for the prime field. Thus, J, Φ act identically on B under conjugation, and $[J, \Phi] = 1$. The marked Frobenius $\mathrm{Int}(n_v)J^eF_q$, with $\epsilon = (-1)^e$, gives $J^2 = 1$ and $\Phi^f = v^{-1}J^e$. Moreover, the lifts have corrections in $\mathbf{T}_D[2^\infty]$ by Lang's theorem. The relations present A_M/T_D, since the bound $96(2f)$ equals $|A_M/T_D|$. Thus, on $\langle B_2, J, \Phi\rangle$ let J act by 1 and Φ by αI, where $\alpha^f = \zeta^{-1}$ and $\alpha \in \mu_{4f_2}$. Since $v_2(\mathrm{ord}_\ell(r)) = 2 + v_2(f)$, $4f_2 \mid \ell - 1$. Consequently, the extension $\widehat{\eta}$ is $\mathcal{H}_\ell$-fixed.

Put $I = (A_M)_\vartheta$. Since $\vartheta^\Phi, \vartheta^J \in \{\vartheta, \vartheta^{-1}\}$, $I/T_D \le \langle B_2, J, \Phi\rangle$. Moreover, the correction by b makes $I \to E$ onto. Put $L_2 = \langle V_{2,D}^F, J, \Phi\rangle$ with the adjoint version $V_{2,D}$ of V_2 and its lifts for J, Φ. The relation $\Phi^f = v^{-1}J^e$ lifts via $n_v^{-1}J^e$ in L_2. Thus, $L_2 \cap M_D$ lies in $V_{2,D}^F$ after reducing the J, Φ exponents. In particular, $L_2 \cap T_D \le \mathbf{T}_D[2^\infty]$. Consequently, $L_2 \cap T_D \le \ker\vartheta$, and we extend ϑ by $\widehat{\vartheta}|_{L_2 \cap I} = 1$.

Put $\widehat{\pi} = \mathrm{Ind}_I^{A_M}(\widehat{\vartheta}\,\widehat{\eta}|_I)$. Its restriction π to M_D is irreducible by Clifford induction since $A_M = M_D I$. If $\vartheta^\sigma = \vartheta$, $\widehat{\vartheta}^\sigma = \widehat{\vartheta}$ and $\widehat{\eta}^\sigma = \widehat{\eta}$. If $\vartheta^\sigma = \vartheta^{-1}$, then $\widehat{\vartheta}^{n_b\sigma} = \widehat{\vartheta}$ by the normalization on L_2, and $\widehat{\eta}^{b\sigma} = \widehat{\eta}$ by the extension on $\langle B_2, J, \Phi\rangle$. Consequently, $\widehat{\pi}^\sigma = \widehat{\pi}$. If $\pi|_{M/Z(X)} = \psi$ is irreducible, $\widehat{\pi}$ gives $\widehat{\psi}$. For the triple of ψ_0, the ψ_0-isotypic subspace gives $\widehat{\psi}_0$ as in Proposition 7.86 for χ_0. By Proposition 7.85, the stabilizers of χ, ψ agree. Since $\chi^\sigma = \chi$ and $\psi^\sigma = \psi$ for $\sigma \in \mathcal{H}_\ell$, the joint stabilizer is $\mathrm{Aut}(X)_{P,\chi} \times \mathcal{H}_\ell$. Thus, the extensions give $\mu_{y,\sigma} = \mu'_{y,\sigma} = 1$. Composition with conjugation on S_0 gives the same scalar 1 on the common kernel. Consequently, $(\chi, \psi) \succeq_c$ for every matched pair. □

Lemma 7.88. *Let $H = \mathbf{H}^{F_H}$ be an adjoint classical group over $\mathbb{F}_{q_H}$. With $A = HE(H)$, we put $d_H = \mathrm{ord}_\ell(q_H)$. Suppose that a Sylow d_H-torus has centralizer $\mathbf{T}_H$, and put $N = N_A(\mathbf{T}_H)$. Then, the matched labels (τ, η) have $\mathcal{H}_\ell$-fixed extensions $\widehat{\tau}, \widehat{\eta}$ over A_τ, N_η, respectively. Moreover, the tensor extension $\bigotimes_i \widehat{\tau}_i$ with equal factors permuted is $\mathcal{H}_\ell$-fixed, as is that of $\bigotimes_i \widehat{\eta}_i$.*

Proof. Put $T_H = \mathbf{T}_H^{F_H}$. For $(\mathbf{T}_H, 1_{T_H})$, the trivial extension 1_N and [156, Proposition 8.5] give $\widehat{\tau}$ on A_τ with values in $\mathbb{Q}_\ell$. Similarly, the result of [156, Lemma 8.6] gives $\widehat{\eta}$ on $(N/T_H)_\eta$ with values in $\mathbb{Q}_\ell$. Thus, both extensions on A_τ, N_η are $\mathcal{H}_\ell$-fixed by [156, §2]. Given a permutation cycle $(i_1, \ldots, i_e)$, the trace is $\widehat{\tau}_{i_1}(a_{i_e} \cdots a_{i_1})$ for the component maps a_i. For the local maps b_i, the trace is $\widehat{\eta}_{i_1}(b_{i_e} \cdots b_{i_1})$. Thus, the trace products lie in $\mathbb{Q}_\ell$ by Proposition 7.51. Consequently, if z fixes the tensor labels and $\sigma \in \mathcal{H}_\ell$, $\widehat{\tau}^{z\sigma} = \widehat{\tau}$ and $\widehat{\eta}^{z\sigma} = \widehat{\eta}$. □

Lemma 7.89. *Suppose that an odd prime t divides $|\bar{s}|$. Then, we have that $C_{\mathbf{G}^*}(s)$ is a Levi with $R = \varnothing, A_1^2$ or A_3. If $R = A_1^2$, the adjoint factor is $\mathrm{PGL}_2(q^2)$. For $R = A_3$ it has signed size $-\epsilon q$. Thus, we have that $\widehat{\tau}, \widehat{\eta}$ are as in* Lemma 7.88 *for these R.*

Proof. Take the least odd $t \mid |\bar{s}|$, and $y \in \langle s\rangle$ of order t. Thus, $\mathbf{K}_0^* = C_{\mathbf{G}^*}(\langle y\rangle)$ is independent of y within $\langle y\rangle$. By Lemma 7.79, $\mathbf{K}_0^* = \mathbf{T}^*$ off the six lines. Then, $C = \mathbf{T}^*$ and $\tau = \eta = 1$. Otherwise, take $x = 0$ by B-conjugacy. Then, the roots $R_0 = D_3 = A_3$ lie on the zero plane and e_5. The restriction of v there has odd sign parity. Thus, $c_0 = -v|_{R_0}$ lies in $W(A_3) = \mathfrak{S}_4$ and has order 4. It is a 4-cycle, with $F^* = -uc_0$ on R_0.

In the adjoint A_3-torus write the coordinates of s as (z_1, z_2, z_3, z_4) up to common scalar. Since s is rational, c_0 permutes the blocks of equal z_i. Each c_0-invariant partition of $\mathbb{Z}/4$ is by cosets of a subgroup L_0. Hence, $k = [\mathbb{Z}/4 : L_0]$ is $1, 2$ or 4. Thus, the subsystem R is $A_3, A_1^2, \varnothing$, respectively. The derived A_3 in $\mathbf{K}_0^*$ is simply connected since $\mu_4 \cap Z_3 = 1$ upstairs. Consequently, $C = C^\circ$ by Steinberg's theorem in $\mathbf{K}_0^*$ [58, Chapter 3]. The blocks give the Levi $C = \mathbf{K}_R^*$.

With $K_R = \mathbf{K}_R^F$, we choose $\lambda \in \mathrm{Irr}(K_R)$ dual to s, with $\lambda|_T = \theta$. Then, $\chi = \varepsilon R^X_{\mathbf{K}_R}(\lambda\tau)$ by [208, (10.6)]. Also, $M_\theta = N_{K_R}(\mathbf{T})$ has quotient $W(R)^v$ over T. Thus, $\psi = \mathrm{Ind}_{M_\theta}^M(\lambda|_{M_\theta}\eta)$ with the matching in Proposition 7.83. Since $P^* \le T^*$, $[G^* : C^{F^*}]_\ell = 1$. If $k = 2$, F^* has a 2-cycle on the lines of R, and $(F^*)^2$ acts by $-q^2$. Thus, $H = \mathrm{PGL}_2(q^2)$ with nonsplit $\mathbf{T}_H$ and relative group C_2. Since $d_H = \mathrm{ord}_\ell(q^2) = 2$, the two torus labels match $1, \mathrm{St}$. If $k = 1$, the relative group is C_4 and $d_H = 4$. The ℓ'-labels of A_3 are the four hooks $\mu \vdash 4$ by [40, §13.8], since a hook of length 4 gives the denominator factor $q^4 - 1$. On the primitive plane $U_4 = \{(a, b, -a, -b)\}$ of c_0, every root satisfies $(e_i - e_j)|_{U_4} \neq 0$. Consequently, its Sylow 4-torus has centralizer $\mathbf{T}_H$. In $\mathrm{PGL}_2(q^2)$, the Sylow 2-torus is $\mathbf{T}_H$. By Lemma 7.88, we obtain $\widehat{\tau}, \widehat{\eta}$ for $k = 1, 2$. □

Proposition 7.90. *Suppose that $t \mid |\bar{s}|$ is odd. If a pure graph h occurs, we have that $\mathbf{K}_0$ has an h-fixed parabolic $\mathbf{P}_0$. So does $\mathbf{K}_R$, except when $k = 2$ and h fixes both blocks. When $\mathbf{K}_R$ has such a parabolic, $(\chi, \psi) \succeq_c$.*

Proof. Suppose that a pure graph element h occurs in the stabilizer. Then, $h|_{X^*(\mathbf{T})} = -w$ with $w \in B$. Then, $h(s) = s$, $h(y) = y$, and h normalizes $W(R_0)$. Moreover, $h^2 \in W(R)$ since $C_{\mathbf{G}^*}(s)$ is connected. Thus, $A = \langle h, W(R_0)\rangle$ has order dividing $2|W(R_0)|$, prime to t. Let y_0 be an integral cocharacter above y modulo t, and put $Y = \sum_{a\in A} a(y_0)$. Then, Y is A-fixed and $Y \equiv |A|y_0 \bmod t$. Thus, $\alpha(Y) = 0$ for $\alpha \in R_0$, by reflection invariance. If $\alpha \notin R_0$, $\alpha(Y) \neq 0 \bmod t$. Consequently, after scaling Y in $Y(\mathbf{T})_\mathbb{Q}$, the parabolic $\mathbf{P}(Y)$ has Levi $\mathbf{K}_0$ and is fixed by h.

Suppose that $R_0 = A_3$. Since $-v^2$ fixes y, the difference between h and $-v^2$ belongs to $W(R_0)^v = \langle c_0\rangle$. Thus, $h|_{R_0} = -c_0^j$ for some j. On the blocks, h induces a translation π of order dividing 2 on $\mathbb{Z}/k$. If $k > 1$ and $\pi \neq 1$, assign distinct rational $d_i, -d_i$ to each pair i. Their weighted sum is 0 since each block has size $4/k$. Thus, the values give Z in the A_3 cocharacter space, with $h(Z) = Z$. Moreover, its zero roots in R_0 are exactly R. For small $\delta \in \mathbb{Q}_{>0}$, $\alpha(Y + \delta Z) \neq 0$ for $\alpha \notin R_0$. An integral multiple of $Y + \delta Z$ gives the parabolic with Levi $\mathbf{K}_R$. If $k = 1$, $\mathbf{P}(Y)$ has Levi $\mathbf{K}_R$. For $k > 1$ and $\pi = 1$, each block is h-fixed. Then, the distinct values z_i satisfy $z_i^2 = a_0$ for a single scalar a_0. Consequently, $k = 2$, $r \neq 2$, and $z_1/z_2 = -1$. On the line $\mathbb{Q}(1, 1, -1, -1)$, h acts by -1.

Without a pure graph, let $\mathbf{P}_R$ be any parabolic with Levi $\mathbf{K}_R$. Otherwise, take the h-fixed $\mathbf{P}_R$. A graph-free coordinate stabilizer element b acts via $\widetilde{\mathbf{K}}_R$, since its Weyl action is in $W(R)$. Thus, the coordinate translates of $\mathbf{P}_R$ give the product radical U. By Proposition 7.44, take the sources $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ on the common cover, with $\widehat{\tau}, \widehat{\eta}$ from Lemma 7.89. Given z in the joint stabilizer with $\theta^{z\sigma} = \theta$, torus duality gives $\Lambda^{z\sigma} = \Lambda$. Moreover, $\tau^z = \tau, \eta^z = \eta$ by the matching. Thus, $(\Lambda\widehat{\tau})^{z\sigma} = \Lambda\widehat{\tau}$ and $(\Lambda\widehat{\eta})^{z\sigma} = \Lambda\widehat{\eta}$. Since $C_{\mathbf{G}^*}(s) = \mathbf{K}_R^*$, $C_{U^*}(s_\Pi) = C_{(U^z)^*}(s_\Pi) = 1$. By Proposition 7.44, we obtain $\succeq_c$ with common scalar Λ, also for noncyclic split inertia. □

Proposition 7.91. *If $\epsilon = 1$, $k = 2$ and h fixes both blocks, then $(\chi, \psi) \succeq_c$.*

Proof. Put $K_1 = K_R$. In $\mathbf{K}_1/Z^\circ(\mathbf{K}_0)$, the central cocharacter line is $L = \mathbb{Q}(1, 1, -1, -1)$ in block coordinates. Since $c_0|_L = -1$, the action $F/q = -\epsilon c_0$ on L is ϵ. Thus, for $\epsilon = 1$, the two opposite $\mathbf{P}^\pm$ of $\mathbf{K}_0$ with Levi $\mathbf{K}_1$ are rational. The block sizes $2, 2$ give $|R_u(\mathbf{P}^\pm)^F| = q^4$. Since $C_{\mathbf{G}^*}(s) = \mathbf{K}_1^*$, [208, (10.6)] gives irreducible Harish-Chandra induction of $\lambda\tau$ to K_0.

Take $B = (M \rtimes J)_\theta$ and $D_\lambda = (K_1 B)/\ker\lambda$, with $J = \mathrm{Aut}(X)_{P,\chi}$. Thus, the source on $(K_1 B) \times_J D_\lambda$ is $\Lambda\widehat{\tau}$ from Lemma 7.89. On the larger cover $(K_0 B) \times_J D_\lambda$, the base intersection is K_1, since the Weyl action of $B \cap K_0$ lies in $W(R)$. The stabilizer elements permute $\mathbf{P}^+, \mathbf{P}^-$. Thus, Lemma 7.92 with coefficient q^{-2} gives $\widehat{\rho}_0$ on $(K_0 B) \times_J D_\lambda$. On that cover, $\widehat{\rho}_0$ and $\Lambda\widehat{\tau}$ give equal $\mu_{z,\sigma}$ and scalar $\Lambda(c, a)$ at the centralizing pairs (c, a). On $B \times_J D_\lambda$, the local source remains $\Lambda\widehat{\eta}$. Then, we choose $\mathbf{P}(Y)$ with Levi $\mathbf{K}_0$ from Proposition 7.90. Since $C_{\mathbf{G}^*}(s) \le \mathbf{K}_0^*$, both dual radical centralizers are 1. Transitivity gives χ up to sign on X. Consequently, the common section $J \to D_\lambda$ gives $(\chi, \psi) \succeq_c$. If $\epsilon = -1$, Frobenius swaps $\mathbf{P}^+, \mathbf{P}^-$. □

Lemma 7.92. *Let $P_\pm = LU_\pm$ be opposite rational parabolics in Y. Suppose that $\mathcal{V}_\pm = \mathrm{Ind}_{P_\pm}^Y \mathrm{Inf}_L^{P_\pm} V_\beta$ are irreducible for $\beta \in \mathrm{Irr}(L)$ and $|U_+| = |U_-| = a^2$ for $a \in \mathbb{Z}_{>0}$. Then, for a supplement B that permutes P_+, P_-, we have that the action on $\mathcal{V}_+$ has the factor set α, central scalars $Q(c)$ and functions $\mu_{b,\sigma}$ of the source action Q on V_β.*

Proof. Let e_i be the left average on U_i. By [208, proof of Lemma 10.4], $T_b = J_{+,b+}C_b$ has factor set α and $T_c = Q(c)$ for $c \in C_B(Y)$. In this formula, $C_b f(x) = Q(b)f(b^{-1}xb)$ and $J_{\mp,\pm} = ae_{\mp}|_{\mathcal{V}_\pm}$. Since $a \in \mathbb{Q}$, the mixed action (b,σ) takes ae_i to ae_{bi}. Thus, T_b has cochain $\mu_{b,\sigma}$. □

Lemma 7.93. *Let $\mathbf{G}$ be connected reductive with F-stable Levi $\mathbf{L}$. Write $q = r^f$ for the field size of F and $L = \mathbf{L}^F$. Suppose that $\mathbf{P}, F(\mathbf{P})$ are opposite with Levi $\mathbf{L}$ and $F^2(\mathbf{P}) = \mathbf{P}$. Put $\mathbf{V} = R_u(\mathbf{P})$ and assume $d = \dim \mathbf{V}$ is even. Given $\beta \in \mathcal{E}(L,s)$ with $C_{\mathbf{G}^*}(s) \le \mathbf{L}^*$, we have that the pullback $(F^2)^*$ on its source space is $q^d I$. Moreover, for the marked actions on E_6, we have that the normalized maps $\mathcal{T}_a$ give the source functions $\mu_{a,\sigma}$ on the induced character.*

Proof. Put $G = \mathbf{G}^F$ and take a prime $\ell_0 \nmid r|G||L|$ for cohomology over $\overline{\mathbb{Q}}_{\ell_0}$. Then, s has order prime to ℓ_0. By [18, Theorem B′], $H_c^\bullet(Y_{\mathbf{V}}) \otimes_L V_\beta$ is concentrated in degree d and gives an irreducible G-module, where V_β affords β. Moreover, $\dim(H_c^d(Y_{\mathbf{V}}) \otimes_L V_\beta) = |G:L|_{r'}\beta(1)$ by the Jordan degree formula. Write $Y_{\mathbf{V}} = \{g\mathbf{V} : g^{-1}F(g) \in \mathbf{V}F(\mathbf{V})\}$. Then, $Y_{\mathbf{V}}/L \to \{\,{}^g\mathbf{P}\,\}$ takes $g\mathbf{V}$ to ${}^g\mathbf{P}$. Its image consists of $\mathbf{P}'$ opposite $F(\mathbf{P}')$. Indeed, if $\mathbf{P}' = {}^g\mathbf{P}$ is opposite $F(\mathbf{P}')$, then $g^{-1}F(g) = vtw$ with $v \in \mathbf{V}$, $t \in \mathbf{L}$ and $w \in F(\mathbf{V})$. By [114, corollary to Theorem 1], we choose $x \in \mathbf{L}$ with $t = xF(x)^{-1}$. Thus, $(gx)^{-1}F(gx) \in \mathbf{V}F(\mathbf{V})$, and the choices of x are xL. If $F^2(\mathbf{P}') = \mathbf{P}'$, choose g that conjugates $(\mathbf{P}, F(\mathbf{P}))$ to $(\mathbf{P}', F(\mathbf{P}'))$. Since F swaps both pairs, $g^{-1}F(g) \in \mathbf{L}$. Thus, we can take $g \in G$ by Lang's theorem. Consequently, the F^2-fixed parabolics form a G-orbit of size $|G:L|$, since a rational stabilizer of $\mathbf{P}$ also fixes $F(\mathbf{P})$. Above ${}^g\mathbf{P}$, the fibre consists of $gl\mathbf{V}$ for $l \in L$, all F^2-fixed.

Let $R(l)$ denote right translation by $l \in L$. Then, the Lefschetz trace of $F^2R(l)$ is 0 for $l \ne 1$ and $|G:L||L|$ for $l = 1$. Thus, the tensor with V_β has trace $|G:L|\beta(1)$. Since d is even, the trace on $H_c^d(Y_{\mathbf{V}}) \otimes_L V_\beta$ has positive sign. Moreover, $(F^2)^* \in \mathrm{End}_G(H_c^d(Y_{\mathbf{V}}) \otimes_L V_\beta)$. By [95, Corollary 1.6], we obtain scalar $|G:L|\beta(1)/(|G:L|_{r'}\beta(1)) = |G:L|_r$. By the root count for $\mathbf{V}$, $|G:L|_r = q^d$.

Let a be a semilinear lift with $aF = Fa$, as in [208, proof of Lemma 4.19], with field exponent $n(a)$. Put $e(a) = 0$ if $a(\mathbf{P}) = \mathbf{P}$, and 1 otherwise, where a normalizes $\{\mathbf{P}, F(\mathbf{P})\}$. Then, $aF^{e(a)}$ acts on $Y_{\mathbf{V}}$. Define the pullback transport $\mathcal{T}_a = r^{-n(a)d/2}q^{-e(a)d/2}(aF^{e(a)})^*$ and tensor $\mathcal{T}_a$ with the source map on V_β. Since $n(ab) = n(a) + n(b)$ and $e(ab) \equiv e(a)+e(b) \pmod 2$, the two factors $q^{-d/2}$ when $e(a) = e(b) = 1$ cancel the scalar q^d from F^2. Thus, $\mathcal{T}_{ab} = \mathcal{T}_b\mathcal{T}_a$ on the relevant L-isotypic spaces for β and ${}^a\beta$. For the formal F, the source map is 1 and $\mathcal{T}_F = q^{-d}(F^2)^* = 1$.

We test relations on X of type E_6. Then, the pinned quotient for X is $C_f \times C_2$ when $\epsilon = 1$ and C_{2f} when $\epsilon = -1$, with $\phi^f = \gamma$ for $\epsilon = -1$. Thus, modulo the formal F, a lift of $1 \in \mathrm{Aut}(X)$ has $n = 0$ and diagram class γ^0. Moreover, the image in $(\mathbf{X}_{\mathrm{ad}})^F$ is 1. In particular, a central representative z acts as 1 on $Y_{\mathbf{V}}$. Since $\mathrm{Int}(l)$ has right factor $R(l^{-1})$ for $l \in L$, the tensor of $R(l^{-1})$ and the source operator on V_β is 1. Thus, the tensor action of l is left translation by l. On a supplement of X acting on $\mathbf{L}$, a central representative z has scalar $Q(z)$ on $H_c^d(Y_{\mathbf{V}}) \otimes_L V_\beta$.

For coefficient transport, we put $D = aF^{e(a)+2m}$ with m sufficiently large. Then, $q^{-md}(F^{2m})^* = 1$ on $H_c^d(Y_{\mathbf{V}}) \otimes_L V_\beta$. The conjugation and diagram parts of D and translations by $g \in G$ and $l \in L$ are rational over $\mathbb{F}_{r^u}$ for some $u > 0$. Thus, $(gR(l)D)^j$ is a positive Frobenius for some $j > 0$. Put $u = gR(l)D$, a finite endomorphism since u^j is Frobenius. Take a model Y_0 of $Y_{\mathbf{V}}$ over a finite field, with u defined on Y_0 and Frobenius Φ. By [197, Theorem 2.3.2(b) and Remark 2.3.3(c)], for $k \gg 0$ the alternating trace a_k of $(\Phi^k u)^*$ counts the fixed points of $\Phi^k u$. Since Φ^* is invertible, $a_k = \sum_{\lambda \ne 0} c_\lambda \lambda^k$. By the recurrence in [53, proof of Proposition 3.3], $a_0 \in \mathbb{Q}$. Moreover, d is even, so the scalar factor $r^{-n(a)d/2}q^{-e(a)d/2-md}$ is rational. Therefore, averaging $l \mapsto \mathrm{tr}(gR(l)D)$ against the source trace on L leaves the coefficient action σ only on V_β. Since $H_c^\bullet(Y_{\mathbf{V}}) \otimes_L V_\beta$ is concentrated in degree d, the normalized trace is $\widehat{\chi}(ga)$ for the extended character $\widehat{\chi}$. If a exchanges $\mathbf{V}$ and $F(\mathbf{V})$, the comparison $q^{-d/2}F^*$ between the two models induces 1 on G and L. Consequently, $\widehat{\chi}$ has source functions $\mu_{a,\sigma}$ and source scalars $Q(z)$. □

Corollary 7.94. *If $\mathrm{ord}_\ell(q) = 4$ and an odd prime divides $|\bar{s}|$, then each matched (χ,ψ) satisfies $\succeq_c$.*

Proof. By Propositions 7.90 and 7.91, only $\epsilon = -1$ with two h-fixed blocks remains. Write ν_1, ν_2 for the two block cocharacters in $\mathbf{K}_0$. Then, $q^{-1}F$ acts by -1 on the cocharacter line through $\nu_1 - \nu_2$. Thus, F exchanges the two opposites with Levi $\mathbf{K}_1$. Since $d = 4$ and $C_{\mathbf{G}^*}(s) = \mathbf{K}_1^*$, by Lemma 7.93, $(F^2)^* = q^4 I$ for the source $\lambda\tau$. The exchange acts by $q^{-2}F^*$, and a field exponent n gives the factor r^{-2n}. On the common cover choose $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ from Lemma 7.89. Then, the extended K_0-character for $\Lambda\widehat{\tau}$ has the functions $\mu_{a,\sigma}$ of the local character from $\Lambda\widehat{\eta}$. Consequently, the transfer from K_0 to X via $\mathbf{P}(Y)$ follows by Proposition 7.91. □

Proposition 7.95. *Suppose that $s \in \mathbf{S} \setminus \{1\}$ and $C = C_{\mathbf{G}^*}(s)$ is connected. Then, for $R = D_4, D_5$, we have that the matched pairs (χ,ψ) satisfy $\succeq_c$ on $M = N_X(P)$.*

Proof. Write $K = \mathbf{K}^F$ with $\mathbf{K}^* = C$, H for the adjoint factor of K, and $\lambda \in \mathrm{Irr}(K)$ for the linear character dual to s. If $R = D_4$, both v and $-v$ are even signed permutations on U. Thus, H is split, with $W(D_4)^v = Q$ by Lemma 7.79. By Lemma 7.79, every $\alpha \in D_4$ satisfies $\alpha|_{U_i} \neq 0$. Consequently, the Sylow 4-torus of H has centralizer $\mathbf{T}_H$. By Lemma 7.88, we choose $\widehat{\tau}, \widehat{\eta}$ on A_τ, N_η, including triality of order 3 if present. The pairing $\tau \leftrightarrow \eta$ is from Proposition 7.83. Suppose that γ admits a correction with coset $w \in B/Q$. Then, $w(s) = s^{-1}$, so w^2 fixes s. If $w = 1$, $a, b \in \{1, -1\} \setminus \{1\}$ and $a \neq b$, which is impossible. Since $C = C^\circ$, the stabilizer of s in B/Q is 1 by Lemma 7.81. Thus, w is a reflection with normal line $\ker(w+1)$ on V. Each $\beta \in \{\pm h, \pm k, \pm(h-k)\}$ has $\beta|_{\ker(w+1)} \neq 0$ on V. Consequently, a nonzero integral $\nu \in \ker(w+1) \cap V$ gives $\mathbf{P}_0 = \mathbf{P}(\nu)$ with Levi $\mathbf{K}$ and $(-w)(\nu) = \nu$. If γ admits no such correction, take any $\mathbf{P}_0$ with Levi $\mathbf{K}$. Thus, Proposition 7.44 with $\lambda, \widehat{\tau}, \widehat{\eta}$ gives that $(\chi, \psi) \succeq_c$, since $C = \mathbf{K}^*$ gives $C_{U^*}(s_\Pi) = 1$ for the product U of the propagated radicals.

If $R = D_5$, we choose $a = b \neq 1$ by triality. The restriction $v|_{D_5}$ is even, whereas $-1 \notin W(D_5)$. Thus, H has sign ϵ. Also, $C_{W(D_5)}(v) = C_4 \wr C_2 = G(4,1,2)$ on U. Indeed, on $U = U_1 \oplus U_2$, the generators are $v|_{U_1}$, $v|_{U_2}$ and the swap $U_1 \leftrightarrow U_2$. The sign on e_5 follows from even parity. The Sylow 4-torus has centralizer $\mathbf{T}_H$ by $\alpha|_{U_i} \neq 0$ for $\alpha \in D_5$. By Lemma 7.88, we obtain $\widehat{\tau}, \widehat{\eta}$ with $d_H = 4$, including the power relation in $E(H)$ for $\epsilon = -1$.

Since $Z^\circ(\mathbf{K})$ has rank 1, an operation fixing s permutes the two parabolics $\mathbf{P}_+, \mathbf{P}_-$ with Levi $\mathbf{K}$. The triality stabilizer of $a = b$ fixes the point (a, a) for every a. Thus, a correction of γ fixing s gives $s = s^{-1}$. If $\epsilon = 1$, [208, proof of Proposition 10.5, Case 2] gives rational $\mathbf{P}_\pm$ with $|R_u(\mathbf{P}_\pm)^F| = q^{16}$. By Lemma 7.92, we obtain the map $q^{-8}\sum_{u \in R_u(\mathbf{P}_\mp)^F} u$ in the induced models. If $\epsilon = -1$, the sign on $Z^\circ(\mathbf{K})$ gives $F(\mathbf{P}_\pm) = \mathbf{P}_\mp$. Thus, Lemma 7.93 with $d = 16$ gives $(F^2)^* = q^{16} I$ and the comparison $q^{-8} F^*$. In both cases induction from $\Lambda\widehat{\tau}, \Lambda\widehat{\eta}$ gives the functions $\mu_{b,\sigma}$ of Λ on the marked stabilizers, including at $s = s^{-1}$ on $a = b$. Consequently, the common section for Λ gives that $(\chi, \psi) \succeq_c$. □

Lemma 7.96. *Suppose that $|\bar{s}| > 1$ is a power of 2 and $\mathcal{E}(X, s)$ meets $\mathrm{Irr}_{\ell'}(X)$. Then, q is odd and $R \in \{A_1^4, A_3 + A_1^2\}$. Their adjoint groups H are $\mathrm{PGL}_2(q^2)^2$ and $A_{3,\mathrm{ad}}^{-\epsilon}(q) \times \mathrm{PGL}_2(q^2)$, respectively. The groups $W(R)^v$ are C_2^2 and $C_4 \times C_2$. Moreover, if $R = A_1^4$, then $(\chi, \psi) \succeq_c$ on M via a rational Borel of each $\mathrm{PGL}_2(q^2)$.*

Proof. If $R = A_5 + A_1$, its semisimple rank is 6. Among the signed degrees $2, \ldots, 6$ of A_5, only 4 contributes to the ℓ-part. The degree 2 of A_1 is prime to 4. Thus, $|C^{F^*}|_\ell = (q^2+1)_\ell < |T^*|_\ell$, since the isogeny kernels have ℓ'-order. This excludes $R = A_5 + A_1$ for $s \in T^*$. If $R = A_1^4$, then $C = C^\circ$ by Proposition 7.82 and Lemma 7.81. Since $v^2 = -1$ on U, v has two 2-cycles on the lines of R. Thus, $(F^*)^2 = -q^2$ on each line, and $\mathbf{T}_H$ in each $\mathrm{PGL}_2(q^2)$ is nonsplit. Both 1 and St have ℓ'-degree in $\mathrm{PGL}_2(q^2)$. Consequently, we obtain the relative group $W(R)^v = C_2^2$. If $R = A_3 + A_1^2$, then $C = C^\circ$ by Lemma 7.80. By B-conjugacy take e_5 as the unique wall direction outside U. Thus, $R \leq D_5$ on $U + \mathbb{Q}e_5$. In U, two coordinates are $+1$ and two are -1 up to common sign. A root $\pm e_j \pm e_5$ in R gives $z_5 \in \{1, -1\}$. Since $z_5^u = z_5$ and $v(e_5) = e_5$, the common Frobenius sign δ on the five coordinates is $+1$. Thus, the signs on each v-plane U_1, U_2 are constant, with blocks of sizes $3, 2$. Consequently, $R = D_3 + D_2$. On $D_3 = A_3$, Lemma 7.89 gives sign $-\epsilon$ and relative group C_4. On $D_2 = A_1^2$, v has a 2-cycle with square -1 on each line of R. Consequently, the factor for D_2 is $\mathrm{PGL}_2(q^2)$ and $W(R)^v = C_4 \times C_2$. The ℓ'-labels are the four hooks $\mu \vdash 4$ and the two labels $1, \mathrm{St}$.

Put $\mathbf{K} = \langle \mathbf{T}, U_\alpha : \alpha \in R \rangle$, $K = \mathbf{K}^F$, and let $\lambda \in \mathrm{Irr}(K)$ be dual to s. Since $C = C^\circ$, $W_s = W(R)$ and $M_\theta = N_K(\mathbf{T})$ for $\theta = \lambda|_T$. Thus, $\psi = \mathrm{Ind}_{M_\theta}^M((\lambda|_{M_\theta})\eta)$ for η inflated from $W(R)^v$. The pairing $\eta \leftrightarrow \tau$ is from Proposition 7.83. Thus, Lemma 7.88 gives $\mathcal{H}_\ell$-fixed $\widehat{\tau}_i, \widehat{\eta}_i$ on the factors H_i. If $\tau_i = \tau_j$ for permuted factors H_i, H_j, we form the tensor extension of $\bigotimes_i \widehat{\tau}_i$ from that lemma, and similarly for $\bigotimes_i \widehat{\eta}_i$. If $\tau_i \neq \tau_j$, we form the tensor extension on $A_{\tau_i} \times A_{\tau_j}$, and on $N_{\eta_i} \times N_{\eta_j}$ as well. Moreover, $C = C^\circ$ excludes conjugacy of $\tilde{s}$ with $\tilde{s}z$ for $1 \neq z \in Z_3$ by [208, proof of Proposition 10.9, Step 1], so the parent $\tilde{\chi}$ restricts irreducibly to X.

If $R = A_1^4$, choose the rational marking $\mathcal{T} = (\mathbf{T}_1, \mathbf{Q})$ as in Theorem 7.78, with $\mathbf{Q}$ a Borel in each $\mathrm{PGL}_2(q^2)$. Then, $N_{\mathbf{K}}(\mathcal{T}) = \mathbf{T}_1$ and the rational markings form one K-orbit. On each $\mathrm{PGL}_2(q^2)$, the split flag character is $1 + \mathrm{St}$. Thus, $\langle 1 + \mathrm{St}, \tau_i \rangle = 1$ for $i = 1, 2$, also under $1 \leftrightarrow 2$. The intersections with C are $R_u(\mathbf{Q}^*)$ on both sides, and $C_{U^*}(s_\Pi) = C_{V^*}(s_\Pi) = \prod_i F_r^i(R_u(\mathbf{Q}^*))$. Consequently, the coefficient ± 1 on the cover for K, λ gives $(\chi, \psi) \succeq_c$ as in Theorem 7.78, with local torus $\mathbf{T}$. □

Theorem 7.97. *Let $S = E_6^\epsilon(q)$ be simple and $\mathrm{ord}_\ell(q) = 4$. Then, $\mathrm{iMN}(S, \ell)$ holds, with $M = N_X(P)$ on the simply connected cover X. For the exceptional multiplier of ${}^2E_6(2)$, see* Ruhstorfer–Schaeffer Fry [157, Lemma 4.1(2)] *with $\ell = 5$.*

Proof. For $|\bar{s}| > 1$ a power of 2, Lemma 7.96 leaves $R = A_3 + A_1^2$. If a graph operation g fixing s exists, choose h fixing s and a chamber Φ^+ at $\mathbf{T}$, as in Theorem 7.78 and [208, proof of Proposition 10.5]. Then, gh^{-1} acts through $W(R)$ on R. Otherwise, we choose a chamber Φ^+ and put $h = 1$. Thus, $R^+ = R \cap \Phi^+$ has middle A_3-root α with $h(\alpha) = \alpha$. We construct $\mathbf{P}_0$ with Levi $\mathbf{L}_0$ of root system $\{\pm\alpha\}$, and with $\mathbf{Q}_0 = \mathbf{P}_0 \cap \mathbf{K}$ the middle node parabolic of $\mathbf{K}$. When $h \neq 1$, put $V_h = (\mathbb{Q} \otimes X_*(\mathbf{T}))^h$. Then, the root restrictions to V_h form the reduced F_4 system of [208, proof of Proposition 10.5]. Given $\beta \in \Phi$, the restriction corresponds to $(\beta + h\beta)/2$, which has smaller length than α if $h\beta \neq \beta$. Thus, reducedness and $h\alpha = \alpha$ imply that only $\pm\alpha$ have restriction proportional to α. Let Δ_R be the simple roots of R^+. Since Δ_R is independent and the values are h-invariant, we solve $\beta(\nu_0) = 1$ for $\beta \in \Delta_R \setminus \{\alpha\}$ in $V_h \cap \ker \alpha$. Thus, the points with $\beta(\nu) > 0$ for those β form a nonempty relatively open set $\mathcal{U}$. Choose integral $\nu \in \mathcal{U}$ with $\rho(\nu) \neq 0$ for every $\rho \neq \pm\alpha$. If $h = 1$, set $V_h = \mathbb{Q} \otimes X_*(\mathbf{T})$ and set $\Phi = E_6$ reduced. Consequently, $\mathbf{P}_0 = \mathbf{P}(\nu)$ has Levi $\mathbf{L}_0 = C_{\mathbf{G}}(\nu)$ and is h-stable.

The type of $\mathcal{P}_0 = (\mathbf{L}_0, \mathbf{Q}_0)$ is F-stable, since the based action on A_3 fixes α. By [58, Proposition 3.4.2], $N_{\mathbf{K}}(\mathcal{P}_0) = \mathbf{L}_0$. By Lang's theorem, $d \in \mathbf{K}$ with $\mathcal{P} = {}^d\mathcal{P}_0$ rational and all rational markings in $\mathbf{K} \cdot \mathcal{P}_0$ form one K-orbit. With $\mathbf{P} = {}^d\mathbf{P}_0$, we put $\mathbf{L} = {}^d\mathbf{L}_0$ and $L = \mathbf{L}^F$. Given a stabilizer operation b, we choose $k \in K$ with $kb(\mathcal{P}) = \mathcal{P}$. If b has field exponent 0, the conjugated map $\mathrm{Int}(d^{-1}) \circ kb \circ \mathrm{Int}(d)$ is $\mathrm{Int}(x)h_b$ with $h_b \in \{1, h\}$ and $x \in \widetilde{\mathbf{L}}_0$, the stabilizer of $\mathcal{P}_0$ in $\widetilde{\mathbf{K}}$. Thus, $kb(\mathbf{P}) = \mathbf{P}$. Consequently, [208, proof of Lemma 4.19] gives the propagated parabolic for $\mathcal{P}$.

Put $\beta_L = (\lambda|_L)\nu_L$ with ν_L from $1, \mathrm{St}$ of the adjoint middle $A_1(q)$. If $\nu_L = \mathrm{St}$, we set $\widehat{\nu}_L = \mathbb{C}[\mathbb{P}^1(\mathbb{F}_q)] - 1$ in character notation. Otherwise, set $\widehat{\nu}_L = 1$. Apply the type A computation in Proposition 7.71 with $a = (23) \in \mathfrak{S}_4$ and $z = 1, (14)(23)$ for split and unitary A_3, respectively. Thus, for the label $\rho \in \mathrm{Irr}(\mathfrak{S}_4)$, either $|(\rho(z) + \rho(az))/2| = 1$ or $|(\rho(z) - \rho(az))/2| = 1$ by [40, §13.8] and [23, $\mathfrak{S}_4$]. Choose ν_L from 1 in the first case and St otherwise. Moreover, the rational Borel of $\mathrm{PGL}_2(q^2)$ gives $1 + \mathrm{St}$ and contributes 1 at either label. The choice ν_L depends on the fixed middle node α and label ρ. Consequently, each operation stabilizing (χ, ψ) fixes the choice ν_L.

In the regular overgroup, $\tilde{\beta}_L$ is uniform since $\Phi(\mathbf{L}) = A_1$ and $\tilde{\lambda}$ is dual to $\tilde{s}$. Thus, transitivity for $R_{\widetilde{\mathbf{L}}}^{\widetilde{X}}$ and [169, Theorem 2.1(1)] give that $|\langle R_{\widetilde{\mathbf{L}}}^{\widetilde{X}}(\tilde{\beta}_L), \tilde{\chi}\rangle| = 1$, the absolute coefficient in the subsystem $A_3 + A_1^2$. By Lemma 7.96, $\tilde{\chi}|_X = \chi$ for the unique parent in $\mathcal{E}(\widetilde{X}, \tilde{s})$. Hence, $\langle R_{\mathbf{L}}^X(\beta_L), \chi\rangle = \pm 1$. Take J, B, D_λ, Λ as in Proposition 7.44, and put $B' = (KB)_{(F_r^i(\mathcal{P}))_i}$. Since the markings form one K-orbit, $B' \to J$ is onto and $B' \cap K = L$. Thus, the sources are $\Lambda\widehat{\nu}_L$ on $B' \times_J D_\lambda$ and $\Lambda\widehat{\eta}$ on $B \times_J D_\lambda$ for η of $C_4 \times C_2$. As in Proposition 7.71, let U be the propagated radical, $V = \prod_i F_r^i(R_u(\mathbf{P}))$, and z_1 the corrected map stabilizing $\mathcal{P}$. Then, $C_{U^*}(s_\Pi) = C_{V^*}(s_\Pi) = C_{(U^{z_1})^*}(s_\Pi)$ equals $\prod_i F_r^i(R_u(\mathbf{Q}_d^*))$ for the dual marking of $\mathbf{Q}_d = {}^d\mathbf{Q}_0$. Thus, [19, Corollary 6.5] compares U, V, and Lemma 7.36 compares U, U^{z_1} on $B' \times_J D_\lambda$. Since $\dim Y_U = \dim Y_{U^{z_1}}$, the comparison has sign $+1$. The source twists of $\Lambda\widehat{\nu}_L, \Lambda\widehat{\eta}$ are the twists of Λ, since the two extended labels are $\mathcal{H}_\ell$-fixed. Consequently, extraction at χ with coefficient ± 1 and Proposition 2.4 with the dual of $\Lambda\widehat{\eta}$ give common factors α_B and functions $\mu_{b,\sigma}$. Moreover, the section $J \to D_\lambda$ gives scalar $\Lambda(c, a)$ as in Proposition 7.44.

If $s = 1$, the result of [156, Theorem D] gives the comparison on the normalizer M of the same Sylow 4-torus. Given $s \neq 1$ with $C = C^\circ$, Proposition 7.83 gives the pairing $\tau \leftrightarrow (\theta, \eta)$. If an odd t divides $|\bar{s}|$, we have $(\chi, \psi) \succeq_c$ by Corollary 7.94. If $\bar{s} = 1$, Proposition 7.95 gives that $(\chi, \psi) \succeq_c$. If $|\bar{s}| = 2^j > 1$, Lemma 7.96 and the construction with α give that $(\chi, \psi) \succeq_c$. Moreover, $k \in K$ correcting a marking has image in $Z(D_\lambda)$. Thus, k acts by the scalar of Λ in both extensions. If $C \neq C^\circ$, Proposition 7.82 places s in one rational series when $3 \mid \epsilon q - 1$. Then, the diagonal orbits of χ, ψ are paired by Proposition 7.85. For these (χ, ψ), by Proposition 7.87, we obtain $\succeq_c$. By Lemma 7.79, distinct B-orbits in T^* have disjoint $\mathcal{E}(X, s)$ and disjoint orbits of θ. Thus, the degree formula in $\mathcal{E}(X, s)$ and induction from M_θ give all $\mathrm{Irr}_{\ell'}(X)$ and $\mathrm{Irr}_{\ell'}(M)$. Since P is noncentral and S is simple, $M = N_X(P) < X$ and M is $\mathrm{Aut}(X)_P$-stable. By Lemma 9.4, we obtain $\mathrm{iMN}(S, \ell)$ on X, and [157, Lemma 4.1(2)] handles the exceptional multiplier. □

Lemma 7.98. *Let $u = \epsilon q$ with $q = r^f$ and $\mathrm{ord}_\ell(u) = 3$. Thus, $\ell \geq 7$, and $\mathrm{ord}_\ell(q) = 3$ for $\epsilon = 1$ and 6 for $\epsilon = -1$. Put $v = c^4$ for a Coxeter element c of E_6, with Frobenius uv on the lattice of $\mathbf{T}$. The torus T has order $\Phi_3(u)^3$ and exponent dividing $\Phi_3(u)$. Thus, we put $P = O_\ell(T)$. Then, we obtain that $P \in \mathrm{Syl}_\ell(X)$, $C_X(P) = T$, and $M = N_X(P) = N_X(\mathbf{T})$ has $M/T = C_W(v)$ of order 648. Every series $\mathcal{E}(X, s)$ meeting $\mathrm{Irr}_{\ell'}(X)$ has $s \in T^*$ up to conjugacy. If $s \in T^*$ has prime order $t \neq 3$, we have that $C_{\mathbf{G}^*}(s)$ is a Levi with $R = \varnothing, A_2, A_2^2$ or D_4. If $R = D_4$, the adjoint factor is ${}^3D_4(q)$.*

Proof. The Coxeter exponents of E_6 give $v^2 + v + 1 = 0$, with multiplicity 3 at each primitive cube root. Thus, $\mathbf{T}$ is a Sylow e-torus for $e = \mathrm{ord}_\ell(q)$. By [63, proof of Lemma 2.20(1)], $T \simeq C_{\Phi_3(u)}^3$. The signed degrees $6, 9, 12$ contribute one

$(\Phi_3(u))_\ell$ each, so $P \in \mathrm{Syl}_\ell(X)$. Write $\bar{V} = X_*(\mathbf{T}) \otimes \mathbb{F}_\ell$ and $\xi = u^{-1} \bmod \ell$. Then, $P[\ell] = \ker(v - \xi)$ in $\bar{V}$. Since $\xi^2 \neq 1$ and $\ell \nmid 3 = \det A_{E_6}$, the form gives that $P[\ell]^\perp = P[\ell]$. Since each root has squared length 2, no root lies in $P[\ell]$ under that form. Thus, $C_{\mathbf{G}}(P)^\circ = \mathbf{T}$, and $N_X(P) = N_X(\mathbf{T})$. Moreover, $N_X(\mathbf{T})/T = C_W(v)$ by Lang's theorem for $\mathbf{T}$. A regular c-eigenvector is regular for v. By [178, Theorem 4.2], $|C_W(v)| = 6 \cdot 9 \cdot 12 = 648$. By [63, Lemma 2.20(1)], $C_W(v)$ acts faithfully on $P[\ell]$. Thus, $C_X(P) = T$ and dually $C_{G^*}(P^*) = T^*$ for $P^* = O_\ell(T^*)$. By [58, Proposition 11.5.6], an eligible s has $|C_{G^*}(s)|_\ell = |G^*|_\ell$. By Sylow conjugacy, $P^* \leq C_{G^*}(s)$ after conjugation, so $s \in T^*$.

If $t \neq 3$ divides $\Phi_3(u)$, then $t \equiv 1 \bmod 3$ and $t \geq 7$. Thus, $C = C^\circ$ by [20, Corollary 2.9], since $|s| = t \neq 3$. Let W_s be the stabilizer of s in W, and lift s to a cocharacter y modulo t. The average $\nu = \sum_{w \in W_s} w(y)$ reduces to $|W_s| y$ modulo t. Since $t \nmid |W(E_6)|$, $\Phi \cap \nu^\perp \subseteq \Phi(C)$. Conversely, the reflections in $\Phi(C)$ fix ν. Consequently, $C = C_{\mathbf{G}^*}(\nu)$ is a proper F^*-stable Levi. Given a graph operation h fixing s with field exponent 0, take the sum indexed by $\langle W_s, h \rangle$ as in Proposition 7.90, since $[\langle W_s, h \rangle : W_s] \leq 2$.

The action of v on $\mathbb{Q}R$ has no fixed vector, so $\mathrm{rank}(R)$ is even and at most 4. If v cycles three components R_i, $x + v(x) + v^2(x) \neq 0$ is fixed for $0 \neq x \in \mathbb{Q}R_1$. Thus, $v(R_i) = R_i$, and $\dim \mathbb{Q}R_i$ is even. In rank 4, each element $g \in W(A_4) = \mathfrak{S}_5$ of order 3 has $V_0^g \neq 0$ on the deleted module V_0. An order 3 element of $W(D_4)$ fixes a coordinate vector e_i. Consequently, $R \in \{\varnothing, A_2, A_2^2, D_4\}$, and for D_4, v has triality image. Since $-1 \in W(D_4)$, the image stays of order 3 for uv. Finally, $v_3(\Phi_3(u)) = 1$ if $3 \mid \Phi_3(u)$. □

Lemma 7.99. *If $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ is a Levi, a graph operation h of field exponent 0 fixing s fixes a parabolic with Levi $\mathbf{K}^*$. If $t \neq 3$ divides $|s|$, we have that $C_{\mathbf{G}^*}(s)^\circ$ is a Levi with roots $\varnothing, A_2, A_2^2$ or D_4.*

Proof. The exponent $\Phi_3(u)$ is odd. Put $h = w\gamma$ with $w \in W$ and $h(s) = s$. Then, $h^2 \in W(\mathbf{K}^*)$ acts trivially on $Z(\mathbf{K}^*)$. The centre $Z(\mathbf{K}^*)$ is connected since its simple roots $\Delta_{\mathbf{K}^*}$ extend to a basis of $X^*(\mathbf{T}^*)$. Thus, $s = y(\zeta)$ for $y \in X_*(Z(\mathbf{K}^*))$ and a primitive $|s|$-th root ζ. Then, $y + hy$ is h-fixed and $(y + hy)(\zeta) = s^2$. Since $|s|$ is odd, a root trivial on s^2 is trivial on s. Consequently, $\mathbf{P}(y + hy)$ has Levi $\mathbf{K}^*$, and the dual parabolic has Levi $\mathbf{K}$.

For $t \neq 3$ dividing $|s|$, choose $s_t \in \langle s \rangle$ of order t. Then, $R = \Phi(C^\circ)$ is contained in $R_t = \Phi(C_{\mathbf{G}^*}(s_t))$. By Lemma 7.98, the components of R all have even rank. If $R_t = D_4$ and $R = A_2^2$, the squared index of $\mathbb{Z}R$ in $\mathbb{Z}R_t$ would be $9/4$. Thus, $R = R_t$ at rank 4. If $R_t = A_2^2$, $\{\pm\alpha, \pm v\alpha, \pm v^2\alpha\}$ is the A_2 factor containing $\alpha \in R$. Consequently, R is a union of factors in A_2^2. If $R = A_2 \leq D_4$, the rank 2 classification gives that $R_t \cap \mathbb{Q}R = R$. Thus, $R = R_t \cap \nu^\perp$ for a cocharacter ν in the Levi for s_t, and R is Levi in E_6. □

Proposition 7.100. *Suppose that a prime $t \neq 3$ divides $|s|$ and $C = C_{\mathbf{G}^*}(s)$ is connected. Then, the matched pairs (χ, ψ) over the weight orbit of θ satisfy $\succeq_c$ on $M = N_X(P)$.*

Proof. Let $\mathbf{K}^* = C$, $K = \mathbf{K}^F$, and $\lambda \in \mathrm{Irr}(K)$ be dual to s. Then, $\chi = \varepsilon R_{\mathbf{K}}^X(\lambda\tau)$ by [208, (10.6)], and $\psi = \mathrm{Ind}_{M_\theta}^M((\lambda|_{M_\theta})\eta)$ for $\theta = \lambda|_T$. Since $C = C^\circ$, $M_\theta = N_K(\mathbf{T})$ with quotient $W(R)^v$ over T. Let $H = \mathbf{H}^F$ be an adjoint factor of K, and take $\mathbf{T}_H$ to be the image of $\mathbf{T}$ in $\mathbf{H}$. If $\alpha|_{V_\zeta} = 0$ for $V_\zeta = \ker(v - \zeta)$, then $\alpha|_{V_{\bar{\zeta}}} = 0$ since α is real. Since $V_\zeta + V_{\bar{\zeta}}$ spans the factor, we have that $\alpha|_{V_\zeta} \neq 0$ for every root α. Thus, $\mathbf{T}_H$ centralizes the Sylow d_H-torus for $d_H = \mathrm{ord}_\ell(q)$. Moreover, $[G^* : C^{F^*}]_\ell = 1$ by $P^* \leq T^*$. If $R = \varnothing$, set $\tau = \eta = 1$. For an A_2 factor, v acts by a 3-cycle c, and $H = A_{2,\mathrm{ad}}^\epsilon(q)$. Thus, the local labels are $\eta \in \mathrm{Irr}(\langle c \rangle)$. On $A = HE(H)$ and $N = N_A(\mathbf{T}_H)$, the pair $(\mathbf{T}_H, 1)$ gives $\mathcal{H}_\ell$-fixed extensions $\widehat{\tau}, \widehat{\eta}$ by [156, Proposition 8.5 and §8.1], also for the unitary label $(2, 1)$. If an automorphism exchanges H_1, H_2, take tensor extensions of $\widehat{\tau}_1, \widehat{\tau}_2$ and $\widehat{\eta}_1, \widehat{\eta}_2$ as in Lemma 7.88.

If $R = D_4$, we put $a = v|_R$ and $z = -I$. Then, the rational lattice operator is $qz^e a$ with $\epsilon = (-1)^e$. The fixed groups H_{ad} and H_{sc} both equal ${}^3D_4(q)$, since $\ker(a - 1) = 0$ and $\mathrm{coker}(a - 1) = 0$ on $\mathbb{F}_2^2$ for the spin centre. By [55, Table 1.1], $B = C_{W(D_4)}(a) \cong \mathrm{SL}_2(3)$, with character values in $\mathbb{Q}(\zeta_3)$ by [23, $\mathrm{SL}_2(3)$]. Moreover, $C_{\mathrm{Aut}(R)}(a) = B \times \langle a \rangle$, since the diagram projection lies in $C_{\mathfrak{S}_3}(\bar{a}) = \langle \bar{a} \rangle$. With the marked torus $\mathbf{T}_H$, by [156, Lemma 8.6], we obtain $\widehat{\eta}$ on $N_{\mathrm{Aut}(H)}(\mathbf{T}_H)_\eta$. Moreover, for the matched τ, the extensions $\widehat{\tau}$ on $\mathrm{Aut}(H)_\tau$ are $\mathcal{H}_\ell$-fixed by [156, §8.2 and proof of Theorem D].

Thus, the twists by λ and $\lambda|_{M_\theta}$ give the common factor c_λ by Lemma 7.7. In $\widetilde{X}$, $\widetilde{K} = \widetilde{\mathbf{K}}^F$ maps onto $\widetilde{X}/(XZ(\widetilde{X}))$ by [208, (10.7)] and fixes $\mathcal{E}(K, s)$. Given an inner-diagonal stabilizer operation b, choose $k \in \widetilde{K}$ with the same image as b in that quotient. Then, $k^{-1}b$ has relative coset in $W_s/W(R) = 1$ since $C = \mathbf{K}^*$. Thus, b acts by $\widetilde{K}$ and fixes each $\mathbf{P}$ with Levi $\mathbf{K}$. If the coset of γ occurs at field position 0, choose h after correction with $h(s) = s$. Then, h^2 acts by $\widetilde{K}$ and Lemma 7.99 gives an h-fixed parabolic $\mathbf{P}$ with Levi $\mathbf{K}$ by duality. Moreover, $\mathbf{K}^* = C_{\mathbf{G}^*}(s)$ is equivariant

for F_r, γ and depends only on $\langle s\rangle$. Consequently, Lemma 7.4 gives the comparison for $R^X_{\mathbf{K}}$ on the inertia of χ, with $\phi^f = \gamma$ for $\epsilon = -1$. Induction from M_θ gives the comparison $(\chi, \psi) \succeq_c$ on M. □

Proposition 7.101. *If $C \neq C^\circ$ and $t \neq 3$ divides $|s|$, then $C^\circ = \mathbf{T}^*$ and $C^{F^*}/(\mathbf{T}^*)^{F^*} \cong C_3$. If $|s| = 3$, we have that the series meeting $\mathrm{Irr}_{\ell'}(X)$ are the lifted A_2^3 series and the two nonlifted triality D_4 series.*

Proof. If $s_t \in \langle s\rangle$ has prime order $t \neq 3$, we have that $C \leq C_{\mathbf{G}^*}(s_t) < \mathbf{G}^*$, with $C_{\mathbf{G}^*}(s_t)$ a Levi by Lemma 7.98. Thus, s avoids the quasi-isolated vertices $R = A_2^3, D_4, A_1^4$ by [208, proofs of Propositions 10.8 and 10.9]. The open edges have $R = A_1$ or A_1^3. In either case v would fix a nonzero vector in $\mathbb{Q}R$ on a fixed line or by $x + v(x) + v^2(x)$ on a 3-cycle. Consequently, $C^\circ = \mathbf{T}^*$, the interior case. The component w permutes affine nodes by $(0, 1, 6)(2, 3, 5)$ as in [208, Proposition 10.9, Step 2]. Thus, $\mathrm{rank}(w - 1) = 4$ on $\mathbb{Q}\Phi$. The minimal Levi for w in $C_{\mathbf{G}^*}(s_t)$ has roots A_2^2, since no element of order 3 in $W(D_4)$ has moved rank 4. Since multiplication by $1 \neq z \in Z_3$ changes the order of a lift $\tilde{s}$ of $3'$-order, the component injection gives $3 \mid |s|$ and $u \equiv 1 \bmod 3$. Thus, F^* acts trivially on Z_3. By the F^*-equivariant component injection, v centralizes the component of order 3, so $M_\theta/T = C_3$.

If $|s| = 3$, then $u \equiv 1 \bmod 3$. Since $v(s) = s$, a representative of v belongs to C. A Levi containing C contains $\mathbf{T}^*$. Its Weyl group contains v, so the Levi is $\mathbf{G}^*$ by ellipticity. Thus, s is quasi-isolated. By [208, proof of Proposition 10.8, Step 1], the order-3 types are D_4 and A_2^3, both with component C_3. Put $b = (\Phi_3(u))_\ell$, so $|T^*|_\ell = b^3$. By [208, (10.29)], the lifted A_2^3 has three $A_2^\epsilon(q)$ factors contributing b each. By [208, (10.25)], the nonlifted form has $A_2^\epsilon(q^3)$, with ℓ-part b^2 from $u^6 - 1, u^9 - 1$. The lifted $D_4(q)$ in (10.29) contributes b from degree 6. Moreover, its central torus has ℓ'-order $(u - 1)^2$. For nonlifted $R = D_4$, [208, proof of Proposition 10.8, Step 1, after (10.24)] gives the direct product ${}^3D_4(q) \times T_2^{F^*}$, where $|T_2^{F^*}| = \Phi_3(u)$. The first factor gives b from $q^6 - 1$ and b from $(q^{12} - 1)/(q^4 - 1)$. Thus, $|C^{F^*}|_\ell = b^3$, since $|C^{F^*} : (C^\circ)^{F^*}| = 3$. Conversely, if $|C^{F^*}|_\ell = |G^*|_\ell$, we choose $s \in T^*$ by Lemma 7.98, and $\tau = 1$ gives ℓ'-degree. Moreover, $A_2^\epsilon(q)$ has three unipotent labels τ of ℓ'-degree. If $H = {}^3D_4(q)$, the faithful diagonal triples over τ are from [208, Proposition 10.8, Step 2]. Thus, in $\mathcal{E}(H, (\mathbf{T}_H, 1))$, the triples of ℓ'-degree correspond to $\tau \in \mathrm{Irr}_{\ell'}(H)$. □

Proposition 7.102. *For s in either faithful triality class, each $\psi \in \mathrm{Irr}_{\ell'}(M \mid \theta)$ has an $\mathcal{H}_\ell$-fixed extension $\widehat{\psi}$ on $N_A(\mathbf{T})_\psi$ for $A = X \rtimes \langle e_0\rangle$, where $\langle e_0\rangle = E_{\theta|_{Z(X)}}$ in the pinned group E. On the two diagonal triples over τ, we have that the pairing is $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant. If f is odd, then $(\chi, \psi) \succeq_c$ for each matched pair (χ, ψ).*

Proof. Let $B = \mathrm{SL}_2(3)$ as in Proposition 7.100. Then, $M_\theta/T = B \times \langle v\rangle$, since v fixes s and its triality image generates C/C°. Take $\widehat{\theta}$ trivial on $V_2^F \cap M_\theta$, with V_2 from Lemma 7.84. Thus, the labels are $\eta \otimes \nu$ with $\eta \in \mathrm{Irr}(B)$ and $\nu \in \mathrm{Irr}(\langle v\rangle)$, all of ℓ'-degree. Clifford induction gives the three $\psi_{\eta,\nu}$ for each η. Moreover, choose $\tilde{\theta} \in \mathrm{Irr}(\widetilde{T} \mid \theta)$. By [208, Proposition 10.8, Step 1], the component pairing at $\tilde{s}$ is trivial on B and the lift changes by $z \neq 1$ in Z_3 under v. Let δ be a diagonal generator in $\widetilde{T}$ by Lang's theorem for $\mathbf{T}$. Thus, δ acts by $\nu \mapsto \nu\nu_0$, where $\nu_0(v)$ pairs δ with z and has order 3. Therefore, the triple $\{\psi_{\eta,\nu} : \nu \in \mathrm{Irr}(\langle v\rangle)\}$ is a regular $\langle\delta\rangle$-orbit. Moreover, $\widehat{\theta}(\eta \otimes \nu)$ has values in $\mathbb{Q}(\zeta_3)$, fixed by $\mathcal{H}_\ell$ since $\ell \equiv 1 \bmod 3$.

By [208, Proposition 10.8, Step 3], the pinned generator e_0 for the faithful central character is ϕ if $\epsilon = 1$ and $r \equiv 1 \bmod 3$, $\gamma\phi$ if $\epsilon = 1$ and $r \equiv -1 \bmod 3$, and ϕ^2 if $\epsilon = -1$. In the last case $r \equiv -1 \bmod 3$ and f is odd. With $\epsilon = (-1)^t$, we put $F_x = \mathrm{Int}(x)F$ with $x = n_v n_0^t$ for the Tits lifts n_v, n_0 of v, w_0. Then, the marked lifts are Φ, $J\Phi$ and Φ^2, where $J = \mathrm{Int}(n_0)\gamma$ is inversion on $\mathbf{T}$. Thus, these maps fix θ of order 3 and act trivially on M_θ/T. For $d = \phi, \gamma$, the corrections $y = 1, n_0$, respectively, satisfy $yd(x) = xF(y)$ by the longest braid identity $n_0\gamma(n_v) = n_v n_0$ and $\gamma(n_0) = n_0$. Consequently, Lang transport of yd gives the lifts of e_0 over X.

With $A = X \rtimes \langle e_0\rangle$, we put $N_\theta = N_A(\mathbf{T})_\theta$. Extend θ to N_θ by 1 on $V_2^F \cap M_\theta$ and on the lift b of e_0 as in Proposition 7.87. Modulo T, the lift b of e_0 acts identically on $B \times \langle v\rangle$ under conjugation, and $b^f = v^{-1}$ if $\epsilon = 1$ and v^{-2} if $\epsilon = -1$, from F_x. Thus, the relations present N_θ/T since the order is $3|B|f$. On $\eta \otimes \nu$, we let b act by $\xi \in \mu_{3f_3}$ with $\xi^f = \nu(b^f)$, where $f_3 = 3^{v_3(f)}$. Since $v_3(\mathrm{ord}_\ell(r^f)) = 1$, $v_3(\mathrm{ord}_\ell(r)) = v_3(f) + 1$. Thus, $3f_3 \mid \ell - 1$ and ξ is $\mathcal{H}_\ell$-fixed. Consequently, since $MN_\theta = N_A(\mathbf{T})$, induction to $N_A(\mathbf{T})$ extends $\psi_{\eta,\nu}$ and gives $\widehat{\psi}^\sigma_{\eta,\nu} = \widehat{\psi}_{\eta,\nu}$ for $\sigma \in \mathcal{H}_\ell$.

By [208, Proposition 10.8, Steps 2–3], each unipotent degree of ${}^3D_4(q)$ determines one triple in $\mathcal{E}(X, s)$, with field $\mathbb{Q}(\zeta_3)$. The involution $\iota = \gamma\sigma_{-1}$ fixes exactly one member χ_0, with outer stabilizer $\langle e_0\rangle$. Match χ_0 to one $\psi_0 = \psi_{\eta,\nu}$ with η corresponding to τ. Then, pair $\chi_0^{\delta^j}$ with $\psi_0^{\delta^j}$ for $j = 0, 1, 2$. Over the inverse central character, pair $\chi_0^{\delta^j\gamma}$ with $\psi_0^{\delta^j\gamma}$. Since $\langle e_0\rangle$ and $\mathcal{H}_\ell$ fix $\chi_0^{\delta^j}, \psi_0^{\delta^j}$, the pairing is $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant. If f is odd, [208, Proposition 10.8,

Step 4] gives an extension $\widehat{\chi}_0$ to A with values in $\mathbb{Q}(\zeta_3)$. Moreover, $C_A(X) = Z(X)$, and $\widehat{\chi}_0, \widehat{\psi}$ have the common scalar on $Z(X)$ by torus duality for θ. Consequently, $(\chi_0, \psi_0) \succeq_c$, and transport by δ, γ gives $\succeq_c$ on both triples. □

Proposition 7.103. *If $\epsilon = 1$ and f is even, each matched pair (χ, ψ) for the faithful triality classes satisfies $\succeq_c$.*

Proof. With $k = f_2 > 1$, we put $m = f/k$ and $q_0 = r^m$. Take the standard D_4 Levi $\mathbf{K}^* = C^\circ$ in $\mathbf{G}^*$. Then, the Weyl part w of the affine rotation acts on D_4 by triality and acts by $\tilde{s} \mapsto \tilde{s}z$ for a generator $z \in Z_3$ by [208, Proposition 10.8, Step 1]. Choose d mod 3 so that w^{dk} gives the given component. Put $b = 0$ if $q_0 \equiv 1 \bmod 3$ and $b = 1$ otherwise. Then, on $\mathbf{G}$ take $F_0 = \mathrm{Int}(n_{w^d}) J^b F_{q_0}$ and $F' = F_0^k$. Since J is inversion on $\mathbf{T}$ and $(-1)^b q_0 \equiv 1 \bmod 3$, the dual map fixes s. The map F' is split since k is even, with the Weyl part w^{dk} translating the lift by z^{dk}. By [208, proof of Proposition 10.8, Step 1, after (10.24)], $C^\circ(s)^{F'^*} = {}^3D_4(q) \times T_2^{F'^*}$, with $|T_2^{F'^*}| = \Phi_3(q)$.

With $F_0 = \mathrm{Int}(y) d_0$ and $d_0 = \gamma^b F_{q_0}$, we put $x = y\, d_0(y) \cdots d_0^{k-1}(y)$. Then, $F' = \mathrm{Int}(x) F_q$ and $y d_0(x) = x F_q(y)$. Take g with $x = g^{-1} F_q(g)$ by Lang's theorem. Thus, $a = g y d_0(g)^{-1}$ is F_q-fixed and its norm for d_0 is $g x F_q(g)^{-1} = 1$. Since $d_0 = e_0^m$ on X, the element $a d_0$ gives the cyclic action of F_0 in $A_2 = X \rtimes \langle e_0^m \rangle$. Moreover, χ_0 is A_2-invariant by [208, Proposition 10.8, Step 3].

Put $K = \mathbf{K}^{F'}$, with $\lambda \in \mathrm{Irr}(K)$ dual to s. The cyclic supplement fixes λ, so extend λ by value 1 on its generator. Take $\widehat{\tau}$ via the action on ${}^3D_4(q)$ from Proposition 7.100, and let $\widehat{\beta} = \widehat{\lambda}\widehat{\tau}$. Thus, $\widehat{\beta}$ extends $\beta = \lambda\tau$ on K and is $\mathcal{H}_\ell$-fixed. With ρ the cyclic shift on $\mathbf{G}^k$, we put $\mathcal{F} = \rho^{-1}\,\mathrm{diag}(F_0)$. Then, $U = U_0^k$ is ρ-stable for a radical U_0 with Levi $\mathbf{K}$. By [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], the variety Y_U has an action of $D = A_2 \times_{\langle e_0^m \rangle} (K \rtimes \langle e_0^m \rangle)$. By [53, Proposition 3.3], the alternating trace on Y_U is integral. Thus, the character Ξ of $(H_c^\bullet(Y_U) \otimes V_{\widehat{\beta}})^K$ is $\mathcal{H}_\ell$-fixed on A_2.

Take $V = \prod_{i=0}^{k-1} F_0^i(U_0)$. By [208, proof of Lemma 4.19(b)], projection from Y_V to coordinate 0 gives the variety for $R_{\mathbf{K}}^X$. Since the component group at s_Π is a 3-group and $r \neq 3$, the unipotent intersections $C_{U^*}(s_\Pi), C_{V^*}(s_\Pi)$ lie in $C^\circ(s_\Pi) = (\mathbf{K}^*)^k$, so both are 1. By [19, Corollary 6.5], $\Xi|_X = \pm R_{\mathbf{K}}^X(\beta)$. Lang's theorem for $\ker(\widetilde{\mathbf{G}}^* \to \mathbf{G}^*)$ gives a rational lift $\tilde{s}$ of s. Then, $C_{\widetilde{\mathbf{G}}^*}(\tilde{s}) = \widetilde{\mathbf{K}}^*$ is connected. The restrictions satisfy $\tilde{\lambda}|_K = \lambda$ and $\tilde{\tau}|_K = \tau$. Thus, we have that $\tilde{\chi} = \pm R_{\widetilde{\mathbf{K}}}^{\widetilde{X}}(\tilde{\lambda}\tilde{\tau})$ is irreducible by [208, (10.6)], and restriction from $\widetilde{X}$ gives $\tilde{\chi}|_X = \pm R_{\mathbf{K}}^X(\beta)$. By [208, Proposition 10.8, Step 2], $\tilde{\chi}|_X = \chi_0 + \chi_0^\delta + \chi_0^{\delta^2}$ for the diagonal generator δ. Consequently, $\langle \Xi|_X, \chi_0 \rangle = \pm 1$.

By Proposition 2.4 with $L = 1$ and $J = \langle e_0^m \rangle$, the virtual character Ξ gives an $\mathcal{H}_\ell$-fixed extension of χ_0 to A_2. On $A_{\mathrm{odd}} = X \rtimes \langle e_0^k \rangle$, take the $\mathbb{Q}(\zeta_3)$-valued extension from [208, proof of Proposition 10.8, Step 5]. Put $A = X \rtimes \langle e_0 \rangle$. By [95, Corollary 11.22], χ_0 extends to A. By [95, Corollary 6.17], the Hall restrictions determine $\widehat{\chi}_0$ on A. Thus, $\widehat{\chi}_0^\sigma = \widehat{\chi}_0$ for $\sigma \in \mathcal{H}_\ell$. By Proposition 7.102, $\widehat{\psi}_0$ is $\mathcal{H}_\ell$-fixed and has scalar $\theta(z)$ on $z \in Z(X)$, as does $\widehat{\chi}_0$. Consequently, $(\chi, \psi) \succeq_c$ for both choices of w^{dk}. □

Proposition 7.104. *In the lifted A_2^3 series, each local adjoint parent ρ_{loc} has an $\mathcal{H}_\ell$-fixed extension on its stabilizer in $N_{D \rtimes E}(\mathbf{T})$. Thus, its constituents on $\bar{M} = M/Z(X)$ match the global characters χ equivariantly under $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$.*

Proof. Take $s_0 = \omega_4^\vee(\zeta_3)$ fixed by γ by [208, (10.23)]. Orient the affine A_2 arms as $(0,2), (1,3), (6,5)$ and let c_i be their Coxeter rotations. Then, $v = c_1 c_2 c_3$ satisfies $v^2 + v + 1 = 0$. For the eigenspace $V_\zeta = \ker(v - \zeta_3)$, $V_\zeta^\perp = V_\zeta$, so $\alpha|_{V_\zeta} \neq 0$ for every root α. Thus, we have that v is W-conjugate to c^4 by [178, Theorem 4.2]. Write $\omega = (0,1,6)(2,3,5)$ for the affine rotation. Then, ω cycles the arms, and γ swaps $(1,3)$ and $(6,5)$. Moreover, $w_0 = -\gamma$ satisfies $w_0\, v = v\, w_0$ and acts by $s_0 \mapsto s_0^{-1}$. On the uv torus, s_0 and all its Z_3-translates are rational since $u \equiv 1 \bmod 3$ and $v \in W(A_2^3)$. Write $\lambda \in \mathrm{Irr}(T_D)$ for the weight dual to s_0 and $\theta = \lambda|_{T_D \cap S}$.

Then, the inertia of λ in $C_W(v)$ is $Q = \prod_i \langle c_i \rangle$. That of θ is $Q \rtimes C_3$, with C_3 cycling the $\langle c_i \rangle$. Thus, we choose $\widehat{\lambda}$ on its inertia $N_\lambda \leq N_D(\mathbf{T})$, trivial on $V_{2,D}^F \cap N_\lambda$ as in Lemma 7.84. Then, $\rho_{\mathrm{loc}} = \mathrm{Ind}_{N_\lambda}^{N_D(\mathbf{T})}(\widehat{\lambda}\eta)$ for $\eta \in \mathrm{Irr}(Q)$. View $\eta = (\eta_1, \eta_2, \eta_3)$ as a word in $\mathrm{Irr}(C_3)$. By Proposition 7.85, the cubic twists give $(\eta_1, \eta_2, \eta_3) \mapsto (\eta_2, \eta_3, \eta_1)$. Consequently, if η has orbit length 3, $\rho_{\mathrm{loc}}|_{\bar{M}} \in \mathrm{Irr}(\bar{M})$. If $\eta = (\eta_1, \eta_1, \eta_1)$, its three extensions to $Q \rtimes C_3$ index the constituents of $\rho_{\mathrm{loc}}|_{\bar{M}}$. Moreover, these constituents have values in $\mathbb{Q}(\zeta_3)$, fixed by $\mathcal{H}_\ell$.

On arm i, pair η_i with one of $(3), (2,1), (1^3)$. By [208, Proposition 10.9, Steps 3–4], the word η gives the corresponding global branches, and γ acts by the reflection (23). Similarly, the local lift $w_0 J$ induces (23), with J inversion on $\mathbf{T}$. The prime field ϕ acts by (23) if $r \equiv -1 \bmod 3$ and by 1 otherwise. Thus, we choose (a,b,b) for each orbit with $a \neq b$, fixed by (23). For distinct letters choose (a,b,c), with orientation (a,c,b) obtained by γ. Given a constant word (a,a,a), pair γ-fixed members χ_0, ψ_0 and their translates $\chi_0^{\delta^j}, \psi_0^{\delta^j}$ for a diagonal generator δ. By [208, Proposition 10.9,

Step 4], ϕ fixes χ_0. On the local triple, $\phi\gamma = \gamma\phi$, so ϕ fixes its unique γ-fixed member ψ_0. The labels $012, 021$ have field $\mathbb{Q}(\zeta_3)$. All other global characters have field $\mathbb{Q}$ by the same step. Consequently, $\chi \leftrightarrow \psi$ is $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant.

Put $t = (1-\epsilon)/2$, so $F = \gamma^t F_q$ in the standard model. Modulo T_D, the marked normalizer has the generators $C_W(v), J, \Phi$, with $J^2 = 1$, $[J, \Phi] = 1$ and $\Phi^f = v^{-1}J^t$. Both J and Φ centralize $C_W(v)$. Thus, the presentation has order $|C_W(v)|\, 2f$, as does $N_{D\rtimes E}(\mathbf{T})/T_D$. The outer images γ, ϕ give $E = C_f \times C_2$ if $t = 0$ and C_{2f} if $t = 1$, with $\phi^f = \gamma^t$. Write $r \equiv (-1)^b \bmod 3$. Then, $bf \equiv t \bmod 2$ since $u \equiv 1 \bmod 3$. By [208, (10.32)], $E_0 = \langle \gamma^b\phi\rangle$ has order f. If $t = 1$, f is odd and $E_0 = \langle \phi^2\rangle$.

Given a reflection fixed η, the weight stabilizer over E is generated modulo T_D by Q, $g = w_0 J$ and $p = w_0^b\Phi$, since $w_0 Q$ acts by $\lambda \mapsto \lambda^{-1}$. Then, $g^2 = 1$, $[g, p] = 1$, and $p^f = v^{-1}g^t$. The actions on Q are reflection for g and its b-th power for p. Thus, extend η with value 1 on g and value β on p, where $\beta^f = \eta(v)^{-1}$. For a distinct letter word, take $\Psi = J^b\Phi$ over the generator of E_0. Then, Ψ fixes λ, centralizes Q, and $\Psi^f = v^{-1}$. Thus, assign the scalar β to Ψ. As in Proposition 7.102, take $\beta \in \mu_{3f_3}$, with $f_3 = 3^{v_3(f)}$ and $3f_3 \mid \ell - 1$. The extension of η is then $\mathcal{H}_\ell$-fixed. For the extended weight inertia I over E_ρ, take $\widehat{\lambda}|_{L_2\cap I} = 1$ with $L_2 = \langle V_{2,D}^F, J, \Phi\rangle$ as in Proposition 7.87. Then, $L_2 \cap T_D$ has 2-power order and lies in $\ker\lambda$. Since I maps onto E_ρ, we have that induction of $\widehat{\lambda}\widehat{\eta}$ gives an $\mathcal{H}_\ell$-fixed extension of ρ_{loc}. Thus, for $\eta = (a, a, a)$ take the ψ-isotypic space of multiplicity 1 in $\rho_{\mathrm{loc}}|_{\bar{M}}$. For a free orbit, $\rho_{\mathrm{loc}}|_{\bar{M}} = \psi$. Consequently, in both cases ψ has an $\mathcal{H}_\ell$-fixed extension. □

Proposition 7.105. *In the lifted A_2^3 series, each matched (χ, ψ) satisfies $\succeq_c$ on $M = N_X(P)$.*

Proof. By [208, Proposition 10.9, Steps 4 and 7], the adjoint parent ρ has values in $K_0 = \mathbb{Q}(\zeta_3)$. By [208, (10.32)], the stabilizer quotient E_ρ is E for a (23)-fixed word and E_0 for distinct letters. Thus, take $P_2 \in \mathrm{Syl}_2(E_\rho)$. We construct a K_0-valued extension on $D \rtimes P_2$ for the adjoint group $D = \mathbf{D}^F$. The Tits representatives n_v, n_0 of v, w_0 are fixed by γ, F_r.

First, suppose that $\epsilon = 1$. With $k = f_2$, we put $m = f/k$ and $q_0 = r^m$. If $k > 1$, we let $\sigma = \gamma^a\phi^m$ with $a = 0$ over E and $a = b$ over E_0, where $q_0 \equiv (-1)^b \bmod 3$. Choose d with $dk \equiv 1 \bmod 3$ and the Tits lift $n = n_{v^d}$, fixed by γ, F_r. In $\mathbf{D}$ put $x = nn_0^b$, $F_0 = \mathrm{Int}(x)\gamma^a F_{q_0}$, and $F' = F_0^k$. Then, $F' = \mathrm{Int}(x^k)F_q$ and $[F_0, \gamma] = 1$. Since w_0 satisfies $w_0\, v = v\, w_0$ and k is even, $x^k = t_k n_v$ with $t_k \in \mathbf{T}$. Thus, $D' = \mathbf{D}^{F'}$ has torus $\mathbf{T}$ of marking v, and $F_0|_{D'}$ has k-th power 1. Moreover, w_0^b offsets the q_0-power on s_0, so F_0 fixes λ. On the ordered labels F_0 acts by $(23)^{a+b}$, so it fixes ρ. For the words with $E_\rho = E$, include γ, which fixes λ, ρ and the standard Borel $\mathbf{TU}$.

By Lang's theorem choose g with $g^{-1}F_q(g) = x^k$. Then, $\iota = \mathrm{Int}(g)$ takes D' onto D. The operators $s = \iota F_0\iota^{-1}$ and $c = \iota\gamma\iota^{-1}$ are $\mathrm{Int}(h_\sigma)\sigma$ and $\mathrm{Int}(h_\gamma)\gamma$, where $h_\sigma = gx(\gamma^a F_{q_0}(g))^{-1}$ and $h_\gamma = g\gamma(g)^{-1}$. Since these endomorphisms are F_q-equivariant and $Z(\mathbf{D}) = 1$, both h_σ, h_γ lie in D. Moreover, $s^k = c^2 = 1$ on D and $[s, c] = 1$. Thus, in D the norm of h_σ under σ is 1, $h_\gamma\gamma(h_\gamma) = 1$, and $h_\sigma\sigma(h_\gamma) = h_\gamma\gamma(h_\sigma)$. With $A = \langle\gamma\rangle$ for $E_\rho = E$ and $A = 1$ for $E_\rho = E_0$, transport gives $D' \rtimes (C_k \times A) \cong D \rtimes P_2$. Indeed, σ has field image of order k, so $\langle\sigma\rangle \cap \langle\gamma\rangle = 1$, and the outer images generate P_2. If $k = 1$ in split type, set $F' = \mathrm{Int}(n_v)F_q$. If $\epsilon = -1$, f is odd, and take $F' = \mathrm{Int}(n_v n_0)\gamma F_q$ with torus action $-qv$. In both cases γ satisfies $\gamma\, F' = F'\,\gamma$, fixes λ and the Borel $\mathbf{TU}$, and has square 1. Thus, Lang transport gives $D \rtimes C_2$ when $E_\rho = E$, with $h_\gamma \in D$ and $h_\gamma\gamma(h_\gamma) = 1$. If $E_\rho = E_0$, $P_2 = 1$, with extension ρ on D.

In Lemma 2.16, we set $n = k$ and $F = F'$ when $F' = F_0^k$, or $n = 1$ and $F_0 = F = F'$ when $P_2 = \langle\gamma\rangle$. Take the wrap map 1 and $A \le \langle\gamma\rangle$ as above. Then, Ξ from the lemma has values in K_0 and $\Xi|_{D'} = R_{\mathbf{T}}^{\mathbf{D}}(\lambda)$. By [40, §13.8], each signed A_2 factor contributes $\pm\kappa_i(c_i)$ for $\kappa_i \in \mathrm{Irr}(\mathfrak{S}_3)$. Moreover, $|\kappa_i(c_i)| = 1$ by [23, $\mathfrak{S}_3$]. Thus, $\langle\Xi|_{D'}, \rho\rangle = \pm 1$, and Lemma 2.16 gives a K_0-valued extension on $D' \rtimes (C_k \times A)$ and so on $D \rtimes P_2$.

Let O be the odd Hall subgroup of E_ρ. Similarly, on $D \rtimes O$ take the K_0-valued extension $\widehat{\rho}_O$ from [208, Proposition 10.9, Steps 5 and 7]. For a (23)-fixed word the parent ρ is real, and for distinct letters $\rho^\gamma = \overline{\rho}$ by [208, (10.35)]. Moreover, ρ extends to ρ' on $D \rtimes E_\rho$ by [208, proof of Proposition 10.9, Steps 5 and 7]. Since $E_\rho = P_2 \times O$, [156, Lemma 7.1] gives the extension $\widehat{\rho}$ with those restrictions and their K_0-rationality.

If $\eta = (a, a, a)$, take the isotypic spaces over χ, ψ in $\rho|_S, \rho_{\mathrm{loc}}|_{\bar{M}}$, each of multiplicity 1. Given an orbit of size 3, take all of $\widehat{\rho}, \widehat{\rho}_{\mathrm{loc}}$. Then, the associated models have factor 1 and $\mu_{a,\sigma} = 1$ since $\mathcal{H}_\ell$ fixes K_0 and the local field $\mathbb{Q}(\mu_{3f_3})$. Moreover, $C_{D\rtimes E}(S) = 1$. By [141, Theorem 2.9], we obtain $(\chi, \psi) \succeq_c$ on the overgroups of X, M that induce the same automorphisms. Since χ, ψ are inflated from $S, \bar{M}$, the scalar at each $z \in Z(X)$ is 1. □

Proposition 7.106. *Suppose that $C \ne C^\circ$ and a prime $t \ne 3$ divides $|s|$ in* Proposition 7.101. *Then, one constituent $\chi_0(s)$ has an extension $X_0(s)$ to $X \rtimes E_s$, where $E_s \le E$ stabilizes $\mathcal{E}(X, s)$. Each (e, σ) fixing the orbit of s in T^* fixes $X_0(s)$.*

Proof. By Lang's theorem for $\ker(\widetilde{\mathbf{G}}^* \to \mathbf{G}^*)$, choose a rational lift $\tilde{s}$ of s. Then, $C_{\widetilde{\mathbf{G}}^*}(\tilde{s}) = \widetilde{\mathbf{T}}^*$ by $\Phi(C^\circ) = \varnothing$ and simple connectedness of $[\widetilde{\mathbf{G}}^*, \widetilde{\mathbf{G}}^*]$. Thus, $\mathcal{E}(\widetilde{X}, \tilde{s})$ consists of the regular character $\rho = \pm R^{\widetilde{X}}_{\widetilde{\mathbf{T}}}(\tilde{\theta})$. By [58, Corollary 12.4.10], $\langle \rho, \widetilde{\Gamma} \rangle = 1$ for the Gelfand–Graev character $\widetilde{\Gamma}$. If a twist by $\mathrm{Irr}(\widetilde{X}/X)$ fixes ρ, its dual element z lies in $[\widetilde{\mathbf{G}}^*, \widetilde{\mathbf{G}}^*]$ by projection of $\tilde{s}z$ to $\widetilde{\mathbf{G}}^*/[\widetilde{\mathbf{G}}^*, \widetilde{\mathbf{G}}^*]$. Conversely, the rational inertia C_3 in Proposition 7.101 maps onto Z_3 by the component injection of [208, proof of Proposition 10.9]. Thus, $\tilde{s}z$ is conjugate to $\tilde{s}$ for each $z \in Z_3$. By cyclic Clifford theory, $\rho|_X$ has three distinct constituents of multiplicity 1. By [58, proof of Proposition 12.4.4], these exhaust $\mathcal{E}(X, s)$, and their degrees are prime to ℓ since $\ell \nmid 3$ and $P^* \leq C^{F^*}$.

Write $\mathbf{U}_0 = R_u(\mathbf{B}_0)$ for the pinned Borel $\mathbf{B}_0$, and $U_0 = \mathbf{U}_0^F$. For the coordinates u_α of $u \in U_0$, we put $\xi(u) = \omega(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(\sum_{\alpha \in \Delta_0} u_\alpha))$ with $1 \neq \omega \in \mathrm{Irr}(\mathbb{F}_r^+)$ and simple roots Δ_0. Then, if $\epsilon = -1$, the pair $\{\alpha, \gamma\alpha\}$ contributes $u_\alpha + u_\alpha^q = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(u_\alpha)$. Thus, ξ is nondegenerate and E-invariant. Put $\Gamma = \mathrm{Ind}_{U_0}^X \xi$. Then, $\rho|_X$ and Γ have one common constituent $\chi_0(s)$ of multiplicity 1 by [58, Proposition 12.4.11 and Theorem 12.4.12].

Extend ξ to $\widehat{\xi}$ by 1 on E and put $\widehat{\Gamma} = \mathrm{Ind}_{U_0 \rtimes E}^{X \rtimes E} \widehat{\xi}$. Then, $\widehat{\Gamma}|_X = \Gamma$. If $\sigma(\zeta_r) = \zeta_r^a$, let $h = \rho^\vee(a)$ in the pinned torus, where $\rho^\vee = \sum_{\alpha \in \Delta_0} \omega_\alpha^\vee$. $2\rho^\vee \in Q^\vee$ and $P^\vee/Q^\vee \cong C_3$. Thus, $\rho^\vee \in Q^\vee$ and $\alpha(h) = a$ for every $\alpha \in \Delta$. Since $\rho^\vee$ is γ-fixed and $a \in \mathbb{F}_r^\times$, $h \in X$ centralizes E. Consequently, $\widehat{\xi}^\sigma(ue) = \widehat{\xi}(hueh^{-1})$ on $U_0 \rtimes E$, and $\widehat{\Gamma}$ is rational. By [208, proof of Lemma 3.13], the multiplicity 1 of $\chi_0(s)$ in Γ gives its unique extension $X_0(s)$ in $\widehat{\Gamma}|_{X \rtimes E_s}$. Since $\widehat{\Gamma}$ is rational, for $(e, \sigma) \in E \times \mathcal{H}_\ell$ we have $X_0(s)^{e\sigma} = X_0(s')$ on the transported domain whenever $\mathcal{E}(X, s)^{e\sigma} = \mathcal{E}(X, s')$. By Lemma 7.98, Sylow conjugacy in C^{F^*} makes rational conjugacy on T^* equal to $N_{G^*}(\mathbf{T}^*)$-conjugacy. Thus, $X_0(s)^{e\sigma} = X_0(s)$ for each (e, σ) fixing the marked orbit of s. □

Theorem 7.107. *Let $X = E_{6,\mathrm{sc}}^\epsilon(q)$ and $\mathrm{ord}_\ell(\epsilon q) = 3$. Then, $\mathrm{iMN}(X/Z(X), \ell)$ holds on $M = N_X(P)$, with the pairs for $\mathcal{E}(X, s)$ over the weight orbit of $\theta \in \mathrm{Irr}(T)$.*

Proof. For $C \neq C^\circ$ and $|s|$ divisible by a prime $t \neq 3$, we set $A = X \rtimes E$. Write $\epsilon = (-1)^b$ and $F_x = \mathrm{Int}(n_v) J^b F_q$ for the torus marking, with $J = \mathrm{Int}(n_0)\gamma$. Thus, by Proposition 7.102, the Lang lifts J, Φ map to γ, ϕ in A/X. Put $V = \langle V_2^{F_x}, J, \Phi \rangle$, with $V_2 = \langle n_w, \mathbf{T}[2^\infty] \rangle$ as in Lemma 7.84. Since $F_x - 1$ is onto on $\mathbf{T}[2^\infty]$, $V_2^{F_x}$ maps onto $C_W(v)$. The generator Φ fixes the simple lifts n_i, and $J(n_i) = n_i$ by the longest braid identity. Thus, J, Φ normalize $V_2^{F_x}$. Modulo the 2-power torus corrections, $J^2 = 1$, $[J, \Phi] = 1$, and $\Phi^f = n_v^{-1} J^b$. Indeed, choose $g^{-1}F(g) = x = n_v n_0^b$. A lift yd has inner coordinate $gyd(g)^{-1}$. The product of the coordinates for $y_i d_i$, $i = 1, 2$, has inner coordinate $gy_1 d_1(y_2) d_1 d_2(g)^{-1}$. Thus, the relation $\phi^f = \gamma^b$ leaves a Tits word in x, n_0 and their images under E. Consequently, on X the relations leave $V \cap T \leq \mathbf{T}[2^\infty]$, and $N_A(P) = TV$.

Let $I = (N_A(P))_\theta$ for the full odd order weight θ dual to s. Then, $I = T(V \cap I)$, and $\theta|_{V \cap T} = 1$. The formula $\widehat{\theta}(tv) = \theta(t)$ for $v \in V \cap I$ defines a character of I because θ is I-invariant. Thus, $I \cap M = M_\theta$ gives the three local branches $\psi_\nu = \mathrm{Ind}_{M_\theta}^M((\widehat{\theta}|_{M_\theta})\nu)$ for $\nu \in \mathrm{Irr}(C_3)$. By Proposition 7.102, Δ acts regularly on $\{\psi_\nu\}$ by the component pairing. Thus, choose $\psi_0(s) = \psi_1$ with quotient label 1. Given any $e \in E_s$, Weyl conjugacy gives a lift $b_e \in I$ over e. Consequently, $I \to E_s$ is onto, and $Y_0(s) = \mathrm{Ind}_I^{N_{X \rtimes E_s}(P)} \widehat{\theta}$ restricts to $\psi_0(s)$ on M by [95, Problem 5.2], since $MI = N_{X \rtimes E_s}(P)$. To change θ within its M-orbit, take a lift from V since T fixes θ. Conjugation by V and coefficient powers carry $\widehat{\theta}$ to the extension with value 1 on the new $V \cap I$. Thus, $Y_0(s)^{e\sigma} = Y_0(s)$ for each (e, σ) fixing the marked orbit of s.

Pair $\chi_0(s)$ with $\psi_0(s)$ and $\chi_0(s)^{\delta^j}$ with $\psi_0(s)^{\delta^j}$ for a generator δ of the diagonal group Δ. Since $X_0(s), Y_0(s)$ are transported together under (e, σ), the pairing is $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant. For a joint stabilizer $(\delta^j e, \sigma)$ of $\chi_0(s)$, the pair (e, σ) fixes the orbit of s. Then, $X_0(s)^{e\sigma} = X_0(s)$ implies $j \equiv 0 \bmod 3$, since Δ acts freely on $\{\chi_0(s)^{\delta^i}\}$. Thus, the ordinary domains are $A_0 = X \rtimes E_s$ and $N_{A_0}(P)$, with $\mu_{e,\sigma} = \mu'_{e,\sigma} = 1$. Moreover, the inner corrections lie in X and M, respectively. Since the image of E in $\mathrm{Aut}(X)/\mathrm{Inn}(X)$ is faithful, $C_A(X) = Z(X)$. Both extensions have scalar $\theta(z)$ for $z \in Z(X)$ by torus duality. Consequently, $(\chi, \psi) \succeq_c$ for $C \neq C^\circ$ and a prime $t \neq 3$ dividing $|s|$.

For the union of the comparisons, $|M/T| = 648$ is prime to ℓ, and each $\psi \in \mathrm{Irr}(M)$ lies above a weight orbit of T. By Lemma 7.98, Sylow conjugacy in C^{F^*} gives that the G^*-classes on T^* are the $N_{G^*}(\mathbf{T}^*)$-orbits. Conversely, each orbit of θ corresponds to an s, and Clifford theory gives the characters ψ by induction from M_θ. Thus, the indexing is $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant. If $s = 1$, we have $(\chi, \psi) \succeq_c$ by [156, Theorem D]. If $t \neq 3$ divides $|s|$ and $C = C^\circ$, then by Proposition 7.100, we obtain $(\chi, \psi) \succeq_c$. If $C \neq C^\circ$ for such s, the extensions $X_0(s), Y_0(s)$ give $(\chi, \psi) \succeq_c$. If $|s| = 3$, Proposition 7.101 leaves two ${}^3D_4(q)$ classes and one lifted A_2^3 class. For ${}^3D_4(q)$ and f odd, we have $\succeq_c$ by Proposition 7.102. If f is even, then $\epsilon = 1$ and $\succeq_c$ follows from Proposition 7.103. The three ψ_ν over each triality label

η have the corresponding scalar on $Z(X)$. Moreover, for A_2^3, the constant words give 9 constituents and the other orbits give 8. Thus, Propositions 7.104 and 7.105 compare the 17 pairs. By Lemma 9.4, we obtain $\mathrm{iMN}(X/Z(X),\ell)$ on M. For ${}^2E_6(2)$, the exceptional cover is in [157, Lemma 4.1(2)], and $\Phi_3(-2)=3$. □

Let $X=E^{\epsilon}_{6,\mathrm{sc}}(q)$, with $q=r^f$, $u=\epsilon q$, and $\mathrm{ord}_\ell(u)=6$. Then, $\ell\ge 7$. Given a Coxeter element w, we put $v=w^2$. Then, the Coxeter exponents give $\Phi_6^2\Phi_3$ as the characteristic polynomial of v. Write $U=\ker(v^3+1)$ and $V=\ker(v^3-1)$ over $\mathbb{Q}$, of dimensions $4,2$, respectively. Since V is perpendicular to a regular w-eigenvector, we deduce that $\Phi\cap V=\varnothing$. Given a generic cocharacter $\nu\in V$, the Levi root system is $R_U=\Phi\cap U$. Its Weyl group contains v^3, so R_U has rank 4. As in Lemma 7.98, v^2 fixes each component of R_U and gives even rank. It excludes A_4, and $v^3|_U=-1$ excludes A_2^2 since $-1\notin W(A_2)^2$. Thus, $R_U=D_4$.

Take the torus $\mathbf{T}$ with lattice Frobenius uv. With $T=\mathbf{T}^F$, we put $P=O_\ell(T)$ and $M=N_X(\mathbf{T})$. Then, $|T|=\Phi_6(u)^2\Phi_3(u)$. The values $\Phi_6(u)$ and $\Phi_3(u)$ are odd, with difference $2u$ and with each prime to u. Thus, $\gcd(\Phi_6(u),\Phi_3(u))=1$. On $X_*(\mathbf{T}_U)$ and $X_*(\mathbf{T}_V)$, the polynomial identities give exponents dividing $\Phi_6(u)$ and $\Phi_3(u)$ on the fixed groups T_U,T_V, respectively. Consequently, $T=T_U\times T_V$ since $|T_U||T_V|=|T|$ and $|T_U|,|T_V|$ are coprime. Put $h=\Phi_6(u)_\ell$. The signed degrees $6,12$ give that $|X|_\ell=h^2$, so $P\in\mathrm{Syl}_\ell(X)$. If $\xi=u^{-1}\bmod\ell$, $P[\ell]=\ker(v-\xi)$ on the reduced cocharacter lattice $U_\ell\oplus V_\ell$. The projectors $(1\mp v^3)/2$ give the orthogonal decomposition over $\mathbb{Z}_\ell$. Since $\ell\nmid\det A_{E_6}=3$, the forms on U_ℓ,V_ℓ are nondegenerate. Thus, $P[\ell]^\perp=P[\ell]$ in U_ℓ. Given $\alpha\in\Phi$, its projection $\alpha_U=(\alpha-v^3\alpha)/2$ has squared norm $(2-(\alpha,v^3\alpha))/2$, with numerator in $\{1,2,3,4\}$. Consequently, $\alpha|_{P[\ell]}\ne 1$ as a torus character, and $C_{\mathbf{G}}(P)^\circ=\mathbf{T}$.

By Springer [178, Theorem 4.2], $B=C_W(v)$ has order $6\cdot 12=72$. Let $\mathbf{S}\le\mathbf{T}$ be the Sylow e-torus, where $e=\mathrm{ord}_\ell(q)$. Since the eigenspace for $\mathbf{S}$ is regular, $C_{\mathbf{G}}(\mathbf{S})=\mathbf{T}$ and B is the relative Weyl group at $\mathbf{S}$. As $P\le\mathbf{S}^F$, Enguehard–Michel [63, Lemma 2.20(1)] gives the faithful action of B on $P[\ell]$. Thus, $C_X(P)=T$, $M=N_X(P)$, and $M/T=B$ by Lang's theorem. Dually, $C_{G^*}(P^*)=T^*$. By Digne–Michel [58, Proposition 11.5.6], each eligible s has $|C_{G^*}(s)|_\ell=|G^*|_\ell$. Consequently, s has a representative in T^* by Sylow conjugacy.

Proposition 7.108. *For $|s|=3$ and $u\equiv 1\bmod 3$, only the two faithful triality D_4 series meet $\mathrm{Irr}_{\ell'}(X)$. For $|s|=3$ and $u\equiv -1\bmod 3$, only the class with factors $A_2(q^2)$ and $A_2^{-\epsilon}(q)$ meets $\mathrm{Irr}_{\ell'}(X)$. If $C\ne C^\circ$ and a prime $a\ne 3$ divides $|s|$, then $C^\circ=\mathbf{T}^*$. In the latter case, the distinguished $\chi_0(s),\psi_0(s)$ and their diagonal translates satisfy $\succeq_c$.*

Proof. A prime $a\ne 3$ dividing $\Phi_6(u)\Phi_3(u)$ is at least 7. Thus, for $t\in\langle s\rangle$ of order a, its lift $\tilde t$ of order a differs in order from $\tilde t z$ for $1\ne z\in Z_3$. By [208, proof of Proposition 10.9], the component map at t is injective, so $C_{\mathbf{G}^*}(t)$ is connected. Since $a\nmid|W|$, the average ν of Lemma 7.98 gives the Levi $\mathbf{K}_t^*=C_{\mathbf{G}^*}(t)$. Both $\Phi(\mathbf{K}_t^*)$ and $\Phi(C^\circ)$ are stable under $v^2=w^4$. Since v^2 has fixed space 0, Lemmas 7.98 and 7.99 give the Levi types $\varnothing,A_2,A_2^2,D_4$. If $C\ne C^\circ$, the inclusion $C\le\mathbf{K}_t^*$ excludes the quasi-isolated vertices with $R=A_2^3,D_4,A_1^4$ in [208, proofs of Propositions 10.8 and 10.9]. On an open edge, $R=A_1$ or A_1^3, giving a v^2-fixed vector as in Proposition 7.101. Consequently, s lies in the interior and $C^\circ=\mathbf{T}^*$.

If $|s|$ is a nontrivial 3-power, then $|s|=3$ by the torus exponents. Since v^2 is elliptic and fixes s, the alcove argument of Proposition 7.101 gives $R=A_2^3,D_4$ with component C_3. With lifted s, apply [208, (10.29)–(10.30)]. With nonlifted s, apply [208, (10.25)]. If $u\equiv 1\bmod 3$, the lifted $A_2^\epsilon(q)^3$ has ℓ'-order, and $A_2^\epsilon(q^3)$ has ℓ-part h since $u^3\equiv -1\bmod\ell$. The lifted $D_4(q)$ contributes h at degree 6, with central torus of order $(u-1)^2$. By [208, Proposition 10.8, Step 1], $C^\circ(s)^{F^*}={}^3D_4(q)\times T_2^{F^*}$ for each nonlifted D_4 class, with $|T_2^{F^*}|=\Phi_3(u)$. Thus, the torus has ℓ'-order, while q^6-1 and $(q^{12}-1)/(q^4-1)$ each give h. Thus, only the nonlifted D_4 classes give $|C^{F^*}|_\ell=h^2$. By [208, Proposition 10.8, Step 1], their central characters on $Z(X)$ are faithful.

If $u\equiv -1\bmod 3$, then F^* inverts C_3, so $H^1(F^*,C_3)=1$. In $A_2(q^2)$ and $A_2^{-\epsilon}(q)$, the signed sizes $q^2,-u$ have order 3 modulo ℓ and each degree 3 gives h. All three unipotent labels of each A_2 have ℓ'-degree. By contrast, ${}^2D_4(q)$ with torus of order q^2-1 contributes h from q^6-1, since $q^4\pm 1$ and q^2-1 are prime to ℓ. Since $|C/C^\circ|$ and the isogeny kernels have ℓ'-order, only A_2^3 gives $|C^{F^*}|_\ell=h^2$. Conversely, when $|C^{F^*}|_\ell=h^2$, $\tau=1$ gives $\chi\in\mathrm{Irr}_{\ell'}(X)$ by the Jordan formula.

For $a\ne 3$ dividing $|s|$ with $C\ne C^\circ$, the component injection is F^*-equivariant by [208, proof of Proposition 10.9]. Hence, $|(C/C^\circ)^{F^*}|=d=(3,u-1)$. Thus, the argument of Proposition 7.106 gives d constituents of multiplicity 1 in $\rho|_X$ for the regular parent ρ of $\tilde s$. On the pinned group $X\rtimes E$ take the rational Gelfand–Graev extension $\widehat\Gamma$ constructed there, with the pinned radical $\mathbf{U}_0$. Since $\rho|_X$ meets $\widehat\Gamma|_X$ with multiplicity 1 by [58, Theorem 12.4.12], its selected constituent $\chi_0(s)$ has the extension $X_0(s)$ fixed by the joint stabilizer.

If $A = X \rtimes E$, we choose the lift group $V_2^{F_x}\langle J, \Phi\rangle$ from the construction in Theorem 7.107 with $F_x = \mathrm{Int}(n_v)J^bF_q$ and $\epsilon = (-1)^b$. Then, its intersection with T is a 2-group and θ has odd order. Thus, extend θ by 1 on the lifts in its inertia I, and induce over $X \rtimes E_s$ as in that theorem to obtain $Y_0(s)$ over the trivial relative label $\psi_0(s)$. The chosen lifts transport $Y_0(s)$ under the joint stabilizer of s. If $d = 3$, $\Delta = \langle\delta\rangle$ acts regularly on $\{\chi_0(s)^{\delta^j}\}$ and $\{\psi_0(s)^{\delta^j}\}$ by the pairing with Z_3 in Proposition 7.102. Thus, pair $\chi_0(s)^{\delta^j}$ with $\psi_0(s)^{\delta^j}$ for $j = 0, 1, 2$. If $d = 1$, $M_\theta/T = \Delta = 1$. In both cases $X_0(s), Y_0(s)$ have $\mu_{e,\sigma} = \mu'_{e,\sigma} = 1$ and scalar $\theta(z)$ at $z \in Z(X)$. Consequently, $(\chi, \psi) \succeq_c$ for $C \neq C^\circ$ and $a \neq 3$ dividing $|s|$. □

Proposition 7.109. *Suppose that $s \neq 1$ and $C = C_{\mathbf{G}^*}(s)$ is connected. The stable A_2 factor has signed size $-u$, the exchanged pair A_2^2 gives $\mathrm{PGL}_3(q^2)$, and D_4 gives ${}^3D_4(q)$. Then, the matched pairs (χ, ψ) satisfy $\succeq_c$ on M.*

Proof. On a v-stable A_2 plane L, the operator $v^2|_L$ is a Coxeter rotation. Thus, $v^3|_L = \pm 1$ since its two eigenspaces are v^2-stable. Because $\Phi \cap V = \varnothing$, $v^3|_L = -1$. Consequently, $v|_L = -c$ with c of order 3. If both A_2 factors were stable, A_2^2 would have full rank in D_4 on U, with squared lattice index 9/4. Thus, $v(A_2^{(1)}) = A_2^{(2)}$. The action on $A_2^{(1)}$ is the Coxeter rotation $v^2|_{A_2^{(1)}}$ with field size q^2. Consequently, the adjoint fixed group is $\mathrm{PGL}_3(q^2)$, with relative group C_3.

Let $L_0 = \mathbb{Q}R$ for a subsystem $R = D_4$. As in Lemma 7.98, $v^2|_{L_0}$ has diagram image of order 3. Put $b = -v^3|_{L_0} \in W(R)$ since $-1 \in W(D_4)$. For generic $x \in L_0^b$, the stabilizer $W(R)_x$ is generated by reflections in $R \cap (L_0^b)^\perp$. Since $b \in W(R)_x$, those roots span the moved space $L_0 \cap V$ of b. Since $\Phi \cap V = \varnothing$, $b = 1$ and $L_0 = U$. Thus, $a = -v|_U$ has order 3 with triality image, and $F^*|_U = (-u)a$.

Given any adjoint factor $H = \mathbf{H}^{F_H}$, the image P_H of P has $\alpha|_{P_H[\ell]} \neq 1$ for every root α. Thus, $C_{\mathbf{H}}(\mathbf{S}_{d_H}) = \mathbf{T}_H$ for $d_H = \mathrm{ord}_\ell(q_H)$, where q_H is the field size of H. For $H = A_{2,\mathrm{ad}}^{-\epsilon}(q)$ or $H = \mathrm{PGL}_3(q^2)$, by [156, Proposition 8.5 and §8.1], we obtain the $\mathcal{H}_\ell$-fixed extensions $\widehat{\tau}, \widehat{\eta}$ for $(\mathbf{T}_H, 1)$ with relative C_3. If $H = {}^3D_4(q)$, we put $B_0 = C_{W(D_4)}(a) \cong \mathrm{SL}_2(3)$ by [55, Table 1.1]. Moreover, $\eta \in \mathrm{Irr}(B_0)$ has values in $\mathbb{Q}(\zeta_3)$ by [23, $\mathrm{SL}_2(3)$]. As in Proposition 7.100, $a-1$ is invertible on the spin centre C_2^2, and $C_{\mathrm{Aut}(R)}(a) = B_0 \times \langle a\rangle$. By [156, Lemma 8.6 and proof of Theorem D], we obtain $\mathcal{H}_\ell$-fixed extensions $\widehat{\tau}, \widehat{\eta}$ on $\mathrm{Aut}(H)_\tau, N_{\mathrm{Aut}(H)}(\mathbf{T}_H)_\eta$.

Take $\mathbf{K}^* = C$ and $\lambda \in \mathrm{Irr}(K)$ dual to s, with $\theta = \lambda|_T$. Write B_0' for the stabilizer at field position 0. As in the proof of Proposition 7.100, if B_0' meets the graph coset, choose $g \in B_0'$ corrected so $g(s) = s$; since $|s|$ is odd, the cocharacter $y + gy$ of Lemma 7.99 gives $\mathbf{P}$ fixed by g with Levi $\mathbf{K}$. The same proof gives $B_0' \cap \widetilde{X} \leq \widetilde{K}$ since $C = \mathbf{K}^*$. Thus, B_0' fixes $\mathbf{P}$, with any choice of $\mathbf{P}$ when g is absent. The twists by $\lambda, \lambda|_{M_\theta}$ have the factor c_λ by Lemma 7.7. Since $\mathbf{K}^*(s) = C_{\mathbf{G}^*}(s)$, Lemma 7.4 gives the comparison for $R_{\mathbf{K}}^X$ on the stabilizer of χ. Moreover, $M_\theta = N_K(\mathbf{T})$ and $\psi = \mathrm{Ind}_{M_\theta}^M((\lambda|_{M_\theta})\eta)$. If $R = \varnothing$, we set $\tau = \eta = 1$. Consequently, $(\chi, \psi) \succeq_c$. □

Proposition 7.110. *Suppose that $u \equiv 1 \bmod 3$. Given any faithful cubic s, the matched pairs (χ, ψ) over a triality label τ satisfy $\succeq_c$.*

Proof. The element v^2 fixes s and has triality image on D_4. Thus, $M_\theta/T = B_0 \times \langle v^2\rangle$ for B_0 in Proposition 7.109. Put $z = v^3$, the longest element on D_4, acting trivially on V. Then, $v = z(v^2)^2$, with scalar $\eta(z)\nu(v^2)^2/\eta(1)$ on $\eta \boxtimes \nu$. Extend θ by 1 on $V_2 \cap M_\theta$ as in Proposition 7.108. Moreover, B_0 lies in the kernel of μ, the pairing with Z_3 in Proposition 7.102. Thus, Δ acts regularly on $\mathrm{Irr}(\langle v^2\rangle)$.

Let e generate $E_{\theta|_{Z(X)}}$ as in [208, Proposition 10.8, Step 3]. Then, $|e| = f$, and its marked lift b centralizes M_θ/T. By the Frobenius relation of Theorem 7.107, $b^f = v^{-1}$ for $\epsilon = 1$ and $b^f = v^{-2}$ for $\epsilon = -1$, modulo T. In split type the scalar at v^{-1} has order dividing 6. Since $\mathrm{ord}_\ell(r^f) = 6$, $2f_2, 3f_3 \mid \ell - 1$. Thus, the scalar for b may be chosen in $\mu_{(2f_2)(3f_3)}$. In twisted type $\eta(z)^2 = \eta(1)^2$, so the scalar on v^{-2} lies in μ_3, and the scalar for b lies in μ_{3f_3}. Moreover, these scalars and η, ν are fixed by $\mathcal{H}_\ell$. Consequently, induction gives $\widehat{\psi}$ over $\langle e\rangle$ as in Proposition 7.102.

By [208, Proposition 10.8, Steps 2–3], each χ in the global triple has field $\mathbb{Q}(\zeta_3)$, and its unique member χ_0 fixed by $\gamma\sigma_{-1}$ has outer stabilizer $\langle e\rangle$. At odd f, [208, Proposition 10.8, Step 4] extends χ_0 over $\langle e\rangle$ with values in $\mathbb{Q}(\zeta_3)$. At even f, one has $\epsilon = 1$. By Proposition 7.103, the descent takes $\widehat{\tau}$ from Proposition 7.109. Indeed, $C^\circ = \mathbf{K}^*$ and $r \nmid |C/C^\circ| = 3$ give dual radical intersections 1. By [208, Proposition 10.8, Step 2], the parent $\widetilde{\chi}$ has multiplicity 1 at χ_0. Lemma 2.17 gives the extension on $\langle e\rangle_2$. By [208, proof of Proposition 10.8, Step 5], the extension on $\langle e\rangle_{2'}$ has values in $\mathbb{Q}(\zeta_3)$. Thus, the two determine $\widehat{\chi}_0$ over $\langle e\rangle$. Match χ_0 with one ψ_0 over τ. Then, $\mu_{e,\sigma} = \mu'_{e,\sigma} = 1$ and both scalars on $Z(X)$ are $\theta|_{Z(X)}$. Consequently, $(\chi_0, \psi_0) \succeq_c$, and diagonal and graph transport give $\succeq_c$ throughout. □

Proposition 7.111. *Suppose that $u \equiv -1 \bmod 3$. Then, $\mathcal{E}(X,s)$ has nine characters of ℓ'-degree for the surviving cubic s. For the factorwise pairing of $\tau = (\tau_1, \tau_2)$ and $\eta = (\eta_1, \eta_2)$, each (χ, ψ) satisfies $\succeq_c$.*

Proof. Since $(3, u-1) = 1$, $X = D$. In the simply connected dual take $s_0 = \omega_4^\vee(\zeta_3)$ from [208, (10.23)]. On the three affine A_2 arms, take rotations c_i oriented as in Proposition 7.104, and set $c = \prod_i c_i$ and $v' = w_0 c$. Then, v' satisfies $v'\gamma = \gamma v'$ and $(v')^3 = w_0 = -\gamma$. Let L_γ be the γ-fixed four-space, with $L_\gamma \otimes \mathbb{C} = V_\zeta \oplus V_{\bar\zeta}$ for the primitive sixth eigenvalue ζ. If $\alpha|_{V_\zeta} = 0$, $\alpha|_{V_{\bar\zeta}} = 0$, contradicting $\alpha(\rho^\vee) \neq 0$ for the strictly dominant $\rho^\vee \in L_\gamma$. Thus, v' is conjugate to v by [178, Theorem 4.2]. Since $w_0(s_0) = s_0^{-1}$, the torus with Frobenius uv' contains s_0. The fixed arm has signed size $-u$, and $v'^2|_{A_2^{(i)}} = c_i^2$ for $i = 2, 3$ with field size q^2.

The rational component group is 1, so $M_\theta/T = Q = C_3 \times C_3$. With arm 1 fixed by γ, $Q = \langle c_1, c_2 c_3 \rangle$. The elements w_0, J, Φ centralize Q in the marked quotient, where $J = \mathrm{Int}(n_{w_0})\gamma$. Moreover, $\Phi^f = (v')^{-1} J^t$ with $t = (1-\epsilon)/2$ by Proposition 7.104. Extend $\eta \in \mathrm{Irr}(Q)$ by 1 on w_0, J and by α on Φ, with $\alpha^f = \eta(c)^{-1}$. Since $3f_3 \mid \ell - 1$, choose $\alpha \in \mu_{3f_3}$ fixed by $\mathcal{H}_\ell$. Restrict to $I = (N_{X \rtimes E}(P))_\theta$ and multiply by $\widehat\theta$ from Proposition 7.108. Since w_0 inverts θ and centralizes Q, I maps onto E. Thus, induction gives an $\mathcal{H}_\ell$-fixed extension of ψ for each η.

By [208, Proposition 10.9, Steps 4–5], the nine global characters χ are rational, E-invariant and extend to $D \rtimes E$. In split type, $q \equiv -1 \bmod 3$ gives odd f, so $P_2 = \langle \gamma \rangle$ is the Sylow two-subgroup of E. Take $F' = \mathrm{Int}(n_{v'})F_q$, with $n_{v'}$ in the pinned F_4 and fixed by γ, F_r. Indeed, the simple generators of $W(E_6)^\gamma$ lift to products $\prod_{i \in O} n_i$ over graph orbits O. Thus, $\mathbf{T}, \mathbf{B}_0$ are γ-stable, with the Lang correction h_γ as in Proposition 7.105.

If $\epsilon = -1$, we put $k = f_2$, $m = f/k$, $q_0 = r^m$. Choose d with $dk \equiv 1 \bmod 3$ and $b \in \{0, 1\}$ with $(-1)^b q_0 \equiv 1 \bmod 3$. In the pinned F_4 group put $n = n_{c^d} n_{w_0}^b$, $F_0 = \mathrm{Int}(n)F_{q_0}$, and $F' = \gamma F_0^k$. Since $q \equiv 1 \bmod 3$, bk is even. Thus, F' acts on the lattice by $q\gamma c = uv'$, and F_0 fixes s_0. Since n is fixed by γ, F_{q_0}, $F' = \mathrm{Int}(n^k)\gamma F_q$. Choose g with $g^{-1}(\gamma F_q)(g) = n^k$. Then, transport of F_0 to D is $\mathrm{Int}(h_0)\phi^m$, where $h_0 = gnF_{q_0}(g)^{-1}$. We have $h_0(\gamma F_q) = (\gamma F_q)h_0$, so $h_0 \in D$ since $Z(\mathbf{D}) = 1$. On $\mathbf{D}^{F'}$, $F_0^k = \gamma$, and ϕ^m has order $2k$ in $E = C_{2f}$. Consequently, the cyclic action gives $D \rtimes P_2$ with $P_2 = \langle \phi^m \rangle$.

In Lemma 2.16 with $\epsilon = 1$, we set $n = 1$, wrap map 1, and $A = \langle \gamma \rangle$. If $\epsilon = -1$, set $n = k$, wrap map γ, and $A = 1$. Moreover, the lifted weight λ is fixed by γ and by the chosen F_0, and $\mathbb{Q}(\lambda) = \mathbb{Q}(\zeta_3)$. As in Proposition 7.105, each A_2 factor has $|\kappa_i(c_i)| = 1$ for every $\kappa_i \in \mathrm{Irr}(\mathfrak{S}_3)$, also with c_i^2. Thus, $R_{\mathbf{T}}^{\mathbf{D}}(\lambda)$ has coefficient ± 1 at each χ. By Lemma 2.16, we obtain an $\mathcal{H}_\ell$-fixed extension on $D \rtimes P_2$. On the odd Hall subgroup $O \leq E$, choose the rational extension from [208, Proposition 10.9, Step 5]. Since $E = P_2 \times O$, [156, Lemma 7.1] gives the unique $\widehat\chi$ with those restrictions and their $\mathcal{H}_\ell$-invariance. The two Coxeter correspondences give $(\tau_1, \tau_2) \leftrightarrow (\eta_1, \eta_2)$, fixed by E and $\mathcal{H}_\ell$. Since $C_{D \rtimes E}(D) = 1$, the invariant extension pair gives $(\chi, \psi) \succeq_c$. □

Theorem 7.112. *Let $S = E_6^\epsilon(q)$ be simple and $\mathrm{ord}_\ell(\epsilon q) = 6$. Then, $\mathrm{iMN}(S, \ell)$ holds. Given $S \neq {}^2E_6(2)$, the local group is $M = N_X(P)$, with the pairs for $\mathcal{E}(X,s)$ over the orbit of $\theta \in \mathrm{Irr}(T)$.*

Proof. If $S = {}^2E_6(2)$, by [157, Lemma 4.1(2)], we obtain $\mathrm{iMN}(S, \ell)$. Otherwise, take $|T| = \Phi_6(u)^2\Phi_3(u)$ and $M/T = C_W(v)$ of order 72, prime to ℓ. Thus, pair the orbits of $s \in T^*$ and $\theta \in \mathrm{Irr}(T)$ as in Theorem 7.107. If $s = 1$, the comparison is [156, Theorem D]. Given $s \neq 1$ with $C = C^\circ$, Proposition 7.109 gives $(\chi, \psi) \succeq_c$. If $C \neq C^\circ$ and a prime $a \neq 3$ divides $|s|$, Proposition 7.108 gives the comparison on $d = (3, u-1)$ pairs. If $|s| = 3$, the two ${}^3D_4(q)$ series occur when $u \equiv 1 \bmod 3$ and the A_2^3 series when $u \equiv -1 \bmod 3$. By Proposition 7.110, we obtain $\succeq_c$ for D_4, and Proposition 7.111 for A_2^3. By Lemma 9.4, we obtain $\mathrm{iMN}(S, \ell)$ on M. □

Corollary 7.113. *Let $S = E_6^\epsilon(q)$ be simple and ℓ odd, with $\mathrm{ord}_\ell(\epsilon q) \in \{5, 8, 9, 12\}$. Then, $\mathrm{iMN}(S, \ell)$ holds on the universal cover $\widehat S$, with $M = N_{\widehat S}(P_{\widehat S})$.*

Proof. Put $X = E_{6,\mathrm{sc}}^\epsilon(q)$ and $d = \mathrm{ord}_\ell(\epsilon q)$. Exactly one of $2, 5, 6, 8, 9, 12$ is divisible by d. Since $d \mid \ell - 1$, $\ell \nmid |W(E_6)| = 2^7 3^4 5$. Thus, Lemma 2.11 gives cyclic $P \in \mathrm{Syl}_\ell(X)$ and $\mathrm{N}_X(P[\ell]) = \mathrm{N}_X(P)$. Let $\pi : \widehat S \to X$ be the covering map. By [133, Table 24.3], $\ker \pi$ is a central 2-group. Thus, $\pi|_{\widehat P} : \widehat P \to P$ is an isomorphism for $\widehat P \in \mathrm{Syl}_\ell(\widehat S)$ above P. The preimages of P and $P[\ell]$ are abelian, since their quotient by $\ker \pi$ is cyclic. Thus, their Sylow ℓ-subgroups $\widehat P$ and $\widehat P[\ell]$ are characteristic. Hence, $\mathrm{N}_{\widehat S}(\widehat P[\ell]) = \pi^{-1}(\mathrm{N}_X(P[\ell])) = \mathrm{N}_{\widehat S}(\widehat P)$. Proposition 2.9 gives $\mathrm{iMN}(S, \ell)$ on $\widehat S$ with this normalizer. □

Corollary 7.114. *Let $S = E_6^\epsilon(q)$ be simple. Then, $\mathrm{iMN}(S, \ell)$ holds for every prime $\ell \mid |S|$.*

Proof. If $S = {}^2E_6(2)$, apply [157, Lemma 4.1(2)], including the exceptional multiplier. At $\ell = r$, [162, Theorem 1.3] gives iMN(S, ℓ). If $\ell = 2$ and q is odd, apply [159, Theorem A(3)]. Otherwise, let X denote the simply connected cover. Put $u = \epsilon q$. The factors $u^m - 1$ for $m \in \{2, 5, 6, 8, 9, 12\}$ in $|X|$, up to signs, give $d = \mathrm{ord}_\ell(u) \in \{1, 2, 3, 4, 5, 6, 8, 9, 12\}$. If $\ell \mid q - 1$, $\epsilon = 1$ is covered by Theorem 7.123, and $\epsilon = -1$ by Theorem 7.20 including its local group M. If $\ell \mid q + 1$, $\epsilon = 1$ is covered by Theorem 7.72, and $\epsilon = -1$ by Theorem 7.56, with Theorem 7.27 for $\ell = 3$. Thus, $d = 1, 2$ give iMN(S, ℓ). Orders $3, 4, 6$ are covered by Theorems 7.107, 7.97 and 7.112 for $d = 3, 4, 6$, respectively. Given $d \in \{5, 8, 9, 12\}$, we apply Corollary 7.113. Consequently, iMN(S, ℓ) holds for every $\ell \mid |S|$. □

7.5 Scalar tori in E_7

Let $G = E_7(q)$, $q = 2^f$, and $d = \mathrm{ord}_\ell(q)$ for an odd prime $\ell \mid |G|$. By Zhang [208, §11.1], $G_{\mathrm{sc}} = G_{\mathrm{ad}} = G$ on rational points, and G is its own universal cover. Thus, $\mathrm{Aut}(G) = G \rtimes E$ with $E = \langle F_2|_G \rangle$. Moreover, $C_{\mathbf{G}^*}(s)$ is connected for every semisimple s. Put $z = -1 \in W(E_7)$. Given a mark $v \in W(E_7)$, the lifts n_w form a Weyl subgroup since $n_\alpha^2 = 1$. Thus, for $F' = \mathrm{Int}(n_v)F_q$, the marked field lift Φ centralizes the rational lifts n_w and has $\Phi^f = n_v^{-1}$ on $G = \mathbf{G}^{F'}$.

Take a dual Levi $\mathbf{K}^*$ containing $C_{\mathbf{G}^*}(s)$ and put $K = \mathbf{K}^F$. By Geck–Malle [72, Theorem 3.3.22], $\chi = \pm R_{\mathbf{K}}^G(\beta)$ is irreducible for $\beta \in \mathcal{E}(K, s)$. If $s \in Z(\mathbf{K}^*)$, we write $\beta = \lambda_s \tau$ with λ_s dual to s and τ inflated from $\mathbf{K}_{\mathrm{ad}}^F$. Let D_0 be the marked supplement fixing s, τ. The group $K \cap D_0$ consists of rational Weyl lifts in $[\mathbf{K}, \mathbf{K}]^F$, since the reflections of $\mathbf{K}$ have multiplicative lifts n_α in $[\mathbf{K}, \mathbf{K}]$. Thus, we find $\widehat{\lambda}_s(kx) = \lambda_s(k)$ for $k \in K, x \in D_0$ extends λ_s to KD_0. Given (e, σ) corrected to fix s, τ, $\widehat{\lambda}_s^{e\sigma}|_{D_0} = 1$.

On the Levi $\mathbf{K}^*(s)$, Proposition 7.2 and Corollary 7.3 give E_χ acting on $\mathbf{G}^f$. Then, $x\phi^j$ acts by $\mathrm{Int}(\widehat{x})\tau^j$ with cyclic shift τ. At position 0 the stabilizer acts by K. Hence, it fixes any parabolic $\mathbf{P}$ with Levi $\mathbf{K}$. Consequently, the translates of $R_u(\mathbf{P})$ give an invariant product radical U. Suppose (e, σ) in the joint stabilizer takes U to V. Then Lemma 7.168 compares the bimodules for U, V. Thus, we find $\mu_{e,\sigma}^\beta = \mu_{e,\sigma}^\chi$ on the common quotient A/G. For local induction from inertia K, Clifford theory gives $\mu_{e,\sigma}^\psi = \mu_{e,\sigma}^\beta$.

Moreover, for regular semisimple s, the Gelfand–Graev module $\widehat{\Gamma}$ gives $\widehat{\chi}$ by Proposition 7.10. For real uniform χ and odd $|E_\chi|$, take the unique real $\widehat{\chi}$ by Zhang [208, Lemma 3.1]. If $|E_\chi|$ is even, let $j \in E_\chi$ have order 2. Then, we obtain that $\chi^j = \chi$, and $\langle \mathrm{Ind}_{G^j}^G 1, \chi \rangle = 1$ by Kawanaka [105, Theorem 3.1.3(ii)]. Apply Zhang [208, proof of Lemma 3.13] to $1_{G^j \rtimes E_\chi}$. The unique constituent over χ gives $\widehat{\chi}$, fixed by each (e, σ) in the joint stabilizer. If $C_{\mathbf{G}^*}(s)$ has factors of type A, the Jordan formula gives that χ is uniform. Moreover, $z(s) = s^{-1}$, and opposition on each A-factor fixes its partition τ_i, so χ is real.

Proposition 7.115. *Given $d \in \{3, 6, 7, 9, 14, 18\}$, the matched pairs in $\mathrm{Irr}_{\ell'}(G)$ and $\mathrm{Irr}_{\ell'}(N_G(P))$ satisfy $\succeq_c$. If $d \in \{3, 6\}$, the induced pairs (χ, ψ) come from the indicated $\mathbf{K}$.*

Proof. When $d = 3$ or $d = 6$, put $\delta = 1, -1$, respectively, and $u = \delta q$. Then, $\mathrm{ord}_\ell(u) = 3$ and $\ell = 7$ or $\ell \geq 13$. In $W(A_2A_5)$, choose b of type (3) on A_2 and $(3, 3)$ on A_5, and mark $v = \delta b$. The fixed line of b is spanned by $y = (1, 1, 1, -1, -1, -1)$ on A_5, with $(y, y) = 6$. Thus, the primitive eigenspace L of b over $\mathbb{F}_\ell$ has $L^\perp = L$ in $y^\perp$. Given $\alpha \in \Phi(E_7)$, its projection to $y^\perp$ has norm $(12 - (\alpha, y)^2)/6$, with $|(\alpha, y)| \leq 3$. This norm is nonzero modulo ℓ: at $\ell = 7$, 12 is a nonsquare, and at $\ell \geq 13$ the numerator is between 3 and 12. Consequently, L is regular and $C_{\mathbf{G}}(P)^\circ = \mathbf{T}$ for $P = O_\ell(T)$, $T = \mathbf{T}^F$. By [178, Theorem 4.2], $|C_W(v)| = 6 \cdot 12 \cdot 18$, and $C_W(v)$ acts faithfully on $L_{\mathbb{C}} = \ker(b - \zeta_3)$. Moreover, $|T| = \Phi_3(u)^3(q - \delta)$, and degrees $6, 12, 18$ give $|T|_\ell = |G|_\ell$. Thus, $P \in \mathrm{Syl}_\ell(G)$, $C_G(P) = T$ by the prime-to-ℓ faithful reduction, and $M = N_G(P) = N_G(\mathbf{T})$. Take $\widehat{\theta}$ for $\theta \in \mathrm{Irr}(T)$ trivial on the Tits stabilizer B_θ, as in [156, Theorem B]. Moreover, $\widetilde{G} = G \times \mathbb{F}_q^\times$ for the regular inclusion $(\mathbf{G}_{\mathrm{sc}} \times \mathbf{G}_m)/\mu_2$.

By [208, Proposition 11.5, Steps 1–2], the isolated $s \neq 1$ has factors A_2, A_5 of signed size ϵq, where $\epsilon = (-1)^f$, with rational E-fixed characters χ. The degrees $3, 6$ give $\Phi_3(u)_\ell^3$ only for $\epsilon = \delta$. Thus, $M_\theta/T = C_3 \times (C_3 \wr \mathfrak{S}_2)$ by the cycle types of b. The empty 3-core partitions of $3, 6$ give ordered 3-quotients of sizes $1, 2$ by [40, §13.8]. Consequently, $\tau \leftrightarrow \eta$ pairs those partitions with $\eta \in \mathrm{Irr}(C_3 \times (C_3 \wr \mathfrak{S}_2))$. By [208, proof of Proposition 11.5, Step 3], χ has an extension with values in $\mathbb{Q}(\zeta_3)$. Since $\ell \equiv 1 \bmod 3$, this extension is $\mathcal{H}_\ell$-fixed. On the local side $z\Phi$ fixes θ, centralizes M_θ/T and satisfies $(z\Phi)^f = b^{-1}$ modulo T, since $\delta = (-1)^f$. Thus, for η, choose the scalar at $z\Phi$ in μ_{3f_3} with f-th power $\eta(b^{-1})/\eta(1)$. Since $3f_3 \mid \ell - 1$ and $\mathbb{Q}(\eta) \subseteq \mathbb{Q}(\zeta_3)$, the extension is $\mathcal{H}_\ell$-fixed.

For the other $s \neq 1$, choose $\mathbf{K}^* = C_{\mathbf{G}^*}(Z(C_{\mathbf{G}^*}(s))^\circ)$ as in [208, proof of Proposition 11.4]. In each adjoint factor H, the image s_H has centralizer of full rank $\mathrm{rank}(H)$. For types A, D, the marks $1, 2$ give $s_H = 1$ since $|s_H|$ is odd. Thus, only s_H of order 3 on E_6 can be noncentral, with $R = A_2^3$ by [208, Proposition 10.8, Step 1; Lemma 10.13]. On

$\mathbb{Q}R$, the dimension $\dim(\mathbb{Q}R)^b$ is even and at most 1. Hence, it is 0. A 3-cycle of arms would fix $x + b(x) + b^2(x) \neq 0$. Thus, b rotates each A_2, and s_H is rational only for $u \equiv 1 \bmod 3$.

The connected centralizer C has relative group $C_3^3 = \prod_i \langle b_i \rangle$. Its partition labels $\tau_i \vdash 3$ pair with $\eta_i \in \mathrm{Irr}(\langle b_i \rangle)$. A marked w taking arm i to arm j has $wb_iw^{-1} = b_j$, since $w \in C_W(b)$. Thus, z and tuple stabilizers act trivially on $\bigotimes_i \eta_i$. The scalar on Φ has f-th power $\prod_i \eta_i(b_i)^{-1}$ in the marking by b. Thus, choose the scalar in μ_{3f_3}. Then, the joint stabilizer transports $\widehat{\psi}$ by these formulas. Since C has type A_2^3, the global χ are real and uniform for all τ_i.

Suppose that $C = \mathbf{K}^*$. Then, $\dim(\mathbb{Q}\Phi(\mathbf{K}^*))^b \leq 1$. A b-orbit with three components of rank n has fixed dimension n. Thus, $n = 1$ and the factor is $A_1(q^3)$ with field degree $3f$. On a stable A-factor, $\dim L_{\mathrm{del}}^b = k - 1$ for the deleted permutation module L_{del} and k cycles. Hence, $k \leq 2$, and two fixed letters i, j would give a fixed root $e_i - e_j$. On D_n, $n \geq 5$, every order 3 signed permutation b has $\dim(\mathbb{Q}D_n)^b \geq 2$. Consequently, D_4 has elliptic triality and E_6 has an elliptic $b \in W(E_6)$. Thus, $C_{\mathbf{H}}(\mathbf{S}_{d_H}) = \mathbf{T}_H$ for each marked factor $\mathbf{H}$ with Sylow d_H-torus $\mathbf{S}_{d_H}$.

By Lang's theorem for $Z(\mathbf{K})$, $M_\theta/T = \prod_i W(H_i, \mathbf{T}_{H_i})$. Given τ_i, η_i, [156, §§8.1–8.2] gives the unipotent extensions $\widehat{\tau}_i, \widehat{\eta}_i$. Then, take the tensor extensions of $\bigotimes_i \widehat{\tau}_i$ and $\bigotimes_i \widehat{\eta}_i$ by Lemma 7.88. For the cycle $(i_1, \ldots, i_a)$ the trace is $\widehat{\tau}_{i_1}(x_{i_a} \cdots x_{i_1})$ for the component operators x_i, and the same formula for $\widehat{\eta}_{i_1}$. Thus, the trace formula transports $(\tau_i)_i$ under $\sigma \in \mathcal{H}_\ell$. Tensor the extensions with λ_s, θ as in Section 7.5. Then, $\chi = \pm R_{\mathbf{K}}^G(\lambda_s \tau)$ and $\psi = \mathrm{Ind}_{M_\theta}^M(\widehat{\theta}\widehat{\eta})$. Since $C = \mathbf{K}^*$, Sylow conjugacy in C^{F^*} gives rational fusion on T^*. Consequently, $(\chi, \psi) \succeq_c$ for $d = 3, 6$. Finally, for $d = 7, 9, 14, 18$, by Corollary 7.169, we obtain $(\chi, \psi) \succeq_c$ on $M = N_G(P)$. □

Proposition 7.116. *If $d \in \{1, 2\}$, take $\delta = 1, -1$, respectively, and $M = T \rtimes W(E_7)$. Then, $(\chi, \psi) \succeq_c$ on M for each matched pair. Also, the $\mathcal{H}_\ell$-equivariant bijection descends to $\mathrm{N}_G(P)$.*

Proof. Put $W = W(E_7)$ and $Q \in \mathrm{Syl}_\ell(W)$. Since all fundamental degrees i are even, $v_\ell(q^i - 1) = v_\ell(q - \delta) + v_\ell(i)$, so $P = T_\ell \rtimes Q$ is Sylow in G. By [132, Theorem 5.14], $\mathrm{N}_G(P) \leq M$. If $\ell > 7$, then $P = T_\ell$ is characteristic in T, so $M = \mathrm{N}_G(P)$. No root of E_7 is trivial on $T[\ell]$. Also, the kernel of $W \to \mathrm{GL}(T[\ell])$ has no odd torsion. Thus, $C_{G^*}(T^*[\ell]) = T^*$, and Sylow conjugacy places each eligible s in T^*. If $R = \Phi(C_{\mathbf{G}^*}(s))$, the scalar mark gives $M_\theta/T = W(R)$. Consequently, the local degree cut is $\ell \nmid [W : W(R)]\eta(1)$, also for $\ell \mid |s|$.

By [208, Proposition 11.5] and Proposition 7.115, the cases beyond $C = \mathbf{K}^*$ are A_2A_5 and A_2^3 in an E_6 Levi. Both have $3 \mid q - \delta$, with signed size δq on each factor A_m. Thus, we pair the partition labels $\lambda \vdash m + 1$ and take the real uniform extension $\widehat{\chi}$. For general R, the line stabilizer of θ modulo $W(R)$ embeds in $\mathrm{Aut}(\langle\theta\rangle)$. Thus, the diagram image in $\mathrm{Aut}(R_0)/\mathrm{Inn}(R_0)$ on a stable factor R_0 is abelian. At $R_0 = D_4$, it is cyclic. On A_m, E_6, opposition is induced by w_0. On D_n, extend η to $W(B_n)$ for the involution C_2. On D_4 with triality C_3, [143, Lemma 2.1] gives the unique real extension $\widehat{\eta}$ of η. On an orbit of factors R_j we choose the tensor extensions $\widehat{\eta}$ of Lemma 7.88. If $\delta = -1$, take the scalar for Φ with f-th power $\eta(z)/\eta(1)$ in μ_{2f_2}. Since $2f_2 \mid \ell - 1$, it is $\mathcal{H}_\ell$-fixed. If $\delta = 1$, take scalar 1. Then, we put $\widehat{\psi} = \mathrm{Ind}(\widehat{\theta} \otimes \widehat{\eta})$ from the extended inertia $M_{\theta,\eta}$ enlarged by the stabilizer of η. Each corrected pair (e, σ) fixes η there, so $\mu_{e,\sigma}^{\psi} = 1$. If $C = \mathbf{K}^*$, we extend $\lambda_s\tau$ by $\widehat{\lambda}_s\widehat{\tau}$, with the unipotent extension $\widehat{\tau}$ from [156, §§8.1–8.2]. If D_4 is split, we combine the real C_3-extension and the field extension on $\langle\phi\rangle$ by quotient twists of $\widehat{\tau}$ on $C_3 \times \langle\phi\rangle$. Thus, the comparison in $\mathbf{K}^*$ gives $(\chi, \psi) \succeq_c$ on M, and $s = 1$ is [156, Theorem D].

For descent put $T_0 = T_\ell \times \mathrm{C}_{T_{\ell'}}(Q)$. Since $T_{\ell'} = [T_{\ell'}, Q] \times \mathrm{C}_{T_{\ell'}}(Q)$, we have $\mathrm{N}_G(P) = T_0 \rtimes \mathrm{N}_W(Q)$. Thus, restriction gives the corresponding Q-fixed weights on T, T_0, equivariantly under $\mathrm{N}_W(Q) \times \mathcal{H}_\ell$. Every eligible W-orbit meets $\mathrm{Irr}(T)^Q$. Moreover, two Q-fixed weights in a W-orbit are $\mathrm{N}_W(Q)$-conjugate since $Q \in \mathrm{Syl}_\ell(W_\theta)$. Thus, the comparison is between $\mathrm{Irr}_{\ell'}(W(R))$ with $\mathrm{Irr}_{\ell'}(\mathrm{N}_{W(R)}(Q))$. If $\ell \in \{5, 7\}$, exactly one factor contains $Q \cong C_\ell$. On the factor R_0 containing Q, each source character η is rational. Thus, each local η' is $\mathcal{H}_\ell$-fixed by [142, Theorem 3.4]. Only $R_0 = D_6$ at $\ell = 5$ can have an outer diagram action γ. However, degenerate characters of $W(D_6)$ have degrees $\frac{1}{2}\binom{6}{3}\chi_\lambda(1)^2$ divisible by 5. The coordinate sign change $e_6 \mapsto -e_6$ fixing the support of $Q = \langle(12345)\rangle$ induces the diagram action on D_6 and centralizes $\mathrm{N}_{W(D_6)}(Q)$. Consequently, any bijection on R_0 is equivariant for these actions.

At $\ell = 3$, $|Q| = 3^4$. The Weyl orders for A_2A_5 and classical root systems R of rank at most 6 have $|W(R)|_3 < 3^4$. Thus, only $W(E_6)$ and $W(E_7)$ remain. On the A_2^3 sublattice of E_6, the rotations give $A = C_3^3$ and $Q = A \rtimes C_3$. Since $C_A(x)$ has order 3 for $x \in Q \setminus A$, an abelian $H \leq Q$ with $H \not\leq A$ has $|H| \leq 9$. Thus, A is characteristic, and $\mathrm{N}_{W(E_6)}(Q)$ permutes the three planes of A_2^3. The glue is diagonal in $(A_2^\vee/A_2)^3$. Thus, each permutation in $\mathfrak{S}_3$ has a common sign ϵ on the three discriminant groups $A_2^\vee/A_2$. Of the two choices $\epsilon = \pm 1$, one gives an element of $W(E_6)$. Also, the reflections $r_j^{\epsilon_j}$ must have equal exponents $\epsilon_j \in \{0, 1\}$ to normalize the cyclic permutation C_3. Consequently, $\mathrm{N}_{W(E_6)}(Q) = Q \rtimes (C_2 \times C_2)$. The simultaneous reflection r and the corrected transposition u invert

separate coordinates of $Q/[Q,Q] = C_3^2$. Thus, for ℓ'-degree the character $\lambda \in \mathrm{Irr}(Q)$ is linear. On $Q \rtimes \langle r, u \rangle_\lambda$, we extend λ by 1 and multiply by a sign character ϵ. Then, ru inverts both coordinates of $Q/[Q,Q]$, so $\mathrm{Ind}(\widehat{\lambda\epsilon})$ is rational. On E_7, Q has fixed line with pointwise stabilizer $W(E_6)$, and $z = -1$ on the line. Thus, $\mathrm{N}_{W(E_7)}(Q) = \mathrm{N}_{W(E_6)}(Q) \times \langle z \rangle$. By [35], $|\mathrm{Irr}_{\ell'}(W(R))| = |\mathrm{Irr}_{\ell'}(\mathrm{N}_{W(R)}(Q))|$, and the diagram actions for $R = E_6, E_7$ induce inner actions on $W(R)$. Consequently, the ordinary bijection descends to $\mathrm{N}_G(P)$ for $\ell = 3, 5, 7$, with $(\chi, \psi) \succeq_c$ on M. □

Proposition 7.117. *If $d = 4$, we put $U = \langle e_1, e_2, e_3, e_4 \rangle$ in the E_7 lattice in $(e_7 + e_8)^\perp$. Let J be two negative two-cycles on U and 1 on $U^\perp$. Then, we obtain that $L = D \times \mathrm{SL}_2(q)^3$, $|D| = (q^2+1)^2$, and $\mathrm{N}_G(P) = L \rtimes B$ for $B = \mathrm{C}_{W(F_4)}(J)$ of order 96. Given $s = th$ with $t \in D^*$ and $h \in \mathrm{SL}_2(q)^3$, the matched pairs (χ, ψ) with $\chi \in \mathcal{E}(G, s) \cap \mathrm{Irr}_{\ell'}(G)$ satisfy $\succeq_c$ on $L \rtimes B$ if $t = 1$ or t is singular.*

Proof. The roots on $U^\perp$ are $\pm(e_5 - e_6), \pm(e_5 + e_6), \pm(e_7 - e_8)$. Thus, $D_4 + A_1^3$ has index 4 in the E_7 lattice. Each nonzero class in $D_4^\vee/D_4$ has norm 1 mod 2, while weight j in $(A_1^\vee/A_1)^3$ has norm $j/2$ mod 2. Consequently, the glue pairs $D_4^\vee/D_4$ with the even-weight plane of $(A_1^\vee/A_1)^3$. Therefore, the factor permutations $\mathfrak{S}_3$ lift to isometries of the E_7 lattice. The degrees $8, 12$ give $|D|_\ell = |G|_\ell$. The primitive plane V in U mod ℓ satisfies $V^\perp = V$. Nonzero root projections to U are the F_4 roots of squared norms $1, 2$, so none lies in V. Consequently, only roots in $U^\perp$ centralize $P[\ell]$, and $C_{\mathbf{G}}(P)^\circ = \mathbf{L}$ with $L = \mathbf{L}^F$. Over $\mathbb{C}$, a real root orthogonal to $U_i = \ker(J - i)$ is also orthogonal to U_{-i}. Thus, U_i is regular for F_4. By [178, Theorem 4.2], $B = C_{W(F_4)}(J)$ has order $8 \cdot 12 = 96$ and acts faithfully on U_i. Since $\ell \nmid 96$, the action on V is faithful. Thus, the glue on $D_4 + A_1^3$ and the Tits lifts give $N_G(P) = L \rtimes B$. Given $\beta \in \mathrm{Irr}(L)$, we extend its weight θ by 1 on B_θ and extend the factors ρ_j by tensor permutations. By [156, Theorem B], the map $\beta \mapsto \widehat{\beta}$ gives the ordinary bijection for $(G, N_G(P))$. Moreover, the case $s = 1$ follows from [156, Theorem D].

If $t \neq 1$, the order of t is coprime to $|h|$, since $\gcd(q^2+1, q(q^2-1)) = 1$. Thus, $t \in \langle s \rangle$. The projected roots $\alpha(t) = 1$ in F_4 have even rank since $J^2 = -1$ on U. Rank 4 would place t at a vertex of the F_4 alcove, with $|t|$ dividing a mark, impossible since $\gcd(|t|, 6) = 1$. Hence, at rank 0, $C_{\mathbf{G}^*}(t) = \mathbf{L}^*$ and the common source on L gives the comparison. At rank 2, the roots $\alpha, J\alpha$ give B_2. Indeed, for short α, the roots $\alpha \pm J\alpha$ are long. For long α, the roots $\frac{1}{2}(\alpha \pm J\alpha)$ are short and have value 1 on t, since $|t|$ is odd. Up to $W(F_4)$-conjugacy, the short pair is (e_1, e_2). Then, $C_{\mathbf{G}^*}(t)$ has Levi roots D_4 on e_1, e_2, e_5, e_6 and A_1 on $e_7 - e_8$. Thus, for $|s|$ odd, the alcove test in D_4A_1 gives a Levi $C_{\mathbf{G}^*}(s)$. The stabilizer is $B_t = C_{W(B_2)}(J) = C_4$. In particular, every $b \in B$ with $t^b = t^a$ for $\gcd(a, |t|) = 1$ centralizes J.

Put $E' = \langle B, \Phi \rangle$ with central Φ and $\Phi^f = J^{-1}$. Thus, for $\beta = \theta \otimes \rho_1 \otimes \rho_2 \otimes \rho_3$, we choose the spaces $e_{\rho_i} V_i$ for Gelfand–Graev or flag modules V_i over $\mathbb{Q}$. Given $\sigma \in \mathcal{H}_\ell$, $e_{\rho_i}^\sigma = e_{\rho_i^\sigma}$. Hence, the field operators Φ and the tensor permutations in B act on $\bigotimes_i e_{\rho_i} V_i$ according to their joint action on $(\rho_i)_i$. Since J acts trivially on each $e_{\rho_i} V_i$, the operators satisfy $\Phi^f = J^{-1}$ on the tensor product. On DE'_θ, put $\widehat{\theta}(dx) = \theta(d)$ for $d \in D$ and $x \in E'_\theta$. Then, $\widehat{\beta}$ is an extension on LE'_β. Given $\eta \in \mathrm{Irr}(I)$ with $I = B_\beta$ and singular $t \neq 1$, we extend η over the plane rotation C_4. On Φ, take the scalar in μ_{4f_2} with f-th power the scalar at J^{-1}, fixed by $\mathcal{H}_\ell$. Consequently, the local extension is $\mathrm{Ind}(\widehat{\beta} \otimes \widehat{\eta})$ from the extended inertia.

If $t = 1$, we put $B_0 = \ker(B \to \mathfrak{S}_3)$. In the signed permutation model for J, the subgroup $B_0 \leq C_4 \wr \mathfrak{S}_2$ has even signs and order 16. Thus, $|B| = 96$ gives $B/B_0 = \mathfrak{S}_3$. The preimage of a transposition subgroup $C_2 \leq \mathfrak{S}_3$ is $C_4 \wr \mathfrak{S}_2$. The group I is the preimage of $(\mathfrak{S}_3)_{(\rho_i)_i}$. If $\rho_1 = \rho_2 = \rho_3$, then $I = B$ and $\mathbb{Q}(\eta) \subseteq \mathbb{Q}(i)$ by [156, Lemma 4.3], where B occurs for $d = 4$ in split adjoint $E_7(13)$. If $\rho_i = \rho_j \neq \rho_k$ for distinct indices, then $I = C_4 \wr \mathfrak{S}_2$ and $\mathbb{Q}(\eta) \subseteq \mathbb{Q}(i)$. Each (e, σ) in the joint stabilizer fixes $\{i, j\}$. Thus, the scalar for Φ lies in μ_{4f_2} in both cases. If the ρ_i are distinct, $I = B_0$. The even rotation subgroup $B_0 \cap C_4^2$ and the signs on $B_0/(B_0 \cap C_4^2)$ give $\mathbb{Q}(\eta) \subseteq \mathbb{Q}(i)$ by Clifford theory. Let $\Pi \leq \mathfrak{S}_3$ be the joint image on $(\rho_i)_i$ for field and Galois actions. Thus, Π is abelian. If $|\Pi| = 2$, the preimage of Π is $C_4 \wr \mathfrak{S}_2$. If $|\Pi| = 3$, every Π-invariant η extends since Π is cyclic. The cubic part of $\det \widehat{\eta}$ is a character of Π. Since $\eta(1)$ is a power of 2, a unique character $\xi \in \mathrm{Irr}(\Pi)$ makes the cubic part of $\det(\widehat{\eta}\xi)$ equal to 1. Thus, $\widehat{\eta}\xi$ is $\mathbb{Q}(i)$-valued by uniqueness. For other η, we work on the preimage of Π_η. We adjoin Φ with scalar in μ_{4f_2} and induce $\widehat{\beta} \otimes \widehat{\eta}$ from the extended inertia of β and η. Then, each corrected (e, σ) lies in the extended inertia, so $\mu_{e,\sigma}^\psi = 1$, including for $\ell \not\equiv 1$ mod 3.

If $t = 1$ and $h \neq 1$, $C_{\mathbf{G}^*}(s)$ contains D_4 on U. Thus, $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ is a proper Levi since neither A_2A_5 nor A_2^3 contains D_4. For $t = 1$ and for singular $t \neq 1$, the minimal 4-split Levi in $\mathbf{K}^*$ is $C_{\mathbf{K}^*}(\mathbf{D}^*) = C_{\mathbf{L}^*}(s)$. The factors with $h_i = 1$ have type A_1 and those with $h_i \neq 1$ are tori. Thus, we extend $1, \mathrm{St}$ on the A_1 factors by rational flag modules V_i. For a cycle of k components, the field power after one circuit has cyclotomic order $4/(4, k) = \mathrm{ord}_\ell(q^k)$. The R_j-projection of an eigenvector for i determines its projections on $F^a(R_j)$. Thus, a larger torus for F^k on one component

would enlarge the 4-torus for F. Consequently, for $C_{\mathbf{L}^*}(s)$ we choose the adjoint extensions $\widehat{\tau}$ from [156, Proposition 8.5 and §8.2]. The stabilizer of $\langle s\rangle$ modulo $W(R)$ embeds in $\mathrm{Aut}(\langle s\rangle)$, so its image on a stable D_4 factor is cyclic. For triality on D_4, we combine the real C_3-extension and the fixed $\langle\phi\rangle$-extension of τ by Proposition 7.116. Moreover, take $\widehat{\tau}$ by tensor permutations on the other F-orbits. In $C_{\mathbf{L}^*}(s)$, fix a maximally split rational torus $\mathbf{T}_s^*$ containing $\mathbf{D}^*$. After Sylow correction, each pair (e,σ) stabilizes the rational conjugacy class of $\mathbf{T}_s^*$ in $C_{\mathbf{L}^*}(s)$. Thus, a further correction in $C_{\mathbf{L}^*}(s)$ fixes $\mathbf{T}_s^*$, and $\widehat{\lambda}_s$ is invariant on $\lambda_s\tau$ by Section 7.5. Since $C=\mathbf{K}^*$, signed induction to G has multiplicity 1 up to sign. Hence, Corollary 7.3 gives $\mu_{e,\sigma}^{\chi}=\mu_{e,\sigma}^{\psi}=1$ on the full global inertia. Consequently, $(\chi,\psi)\succeq_c$. □

Corollary 7.118. *If $G=E_7(2^f)$, $f\geq 1$, $\mathrm{iMN}(G,\ell)$ holds for every $\ell\mid|G|$. Moreover, for a scalar mark at $\ell=3,5,7$, $(\chi,\psi)\succeq_c$ on $M=N_G(\mathbf{T})$.*

Proof. If $\ell=2$, the claim $\mathrm{iMN}(G,2)$ follows from [157, Lemma 4.1]. If $2\nmid\ell$, Corollary 7.256 covers $d=1$ and the eight orders with P cyclic. Thus, the degrees of $W(E_7)$ leave $d=2,3,4,6$. Since $Z(G)=1$ and $\mathrm{Aut}(G)/\mathrm{Inn}(G)=\langle\phi\rangle$, we choose the pinned model $\Gamma=G\rtimes\langle\phi\rangle$ by Proposition 7.2. By Lang's theorem, $\mathbf{K}^*$ has a conjugate in the model (Γ,ϕ). For $d=2,3,6$ and C a Levi, take $\mathbf{K}^*(s)=C=C_{\mathbf{G}^*}(s)$ as in Propositions 7.116 and 7.115. Since $C_{\mathbf{G}^*}(s^a)=C$ for $\gcd(a,|s|)=1$, Corollary 7.3 gives the mixed comparison with $\lambda_s\tau$. The factors of the source τ have adjoint extensions $\widehat{\tau}_j$, with tensor permutations on the F-orbits of R_j. For A_2A_5 and the cubic A_2^3 vertex in E_6, we choose the real uniform $\widehat{\chi}$ with cyclic field operators Φ as in Section 7.5. Finally, $s=1$ follows from [156, Theorem D].

If $d=4$, the decomposition $s=th$ in Proposition 7.117 has $\gcd(|t|,|h|)=1$, with t the part at primes dividing q^2+1. If t is regular, $C\leq C_{\mathbf{G}^*}(t)=\mathbf{L}^*$, so the signed induction to G and local induction to $\mathrm{N}_G(P)$ have a common source on L. If $t\neq 1$ is singular or $t=1,\ h\neq 1$, choose $\mathbf{K}^*(s)=C$ and its minimal 4-split Levi $C_{\mathbf{L}^*}(s)$. The choice of $\mathbf{K}^*(s)$ is equivariant for G^*-conjugation and ϕ^j. It depends only on $\langle s\rangle$. Thus, on the full inertia Γ_χ, the comparison follows from Corollary 7.3. If $d\in\{1,2\}$, the constituents of $\psi|_T$ determine the W-orbit of θ. If $d\in\{3,4,6\}$, Clifford theory and Sylow conjugacy in C^{F^*} determine the orbit of β. By Lemma 9.4, we obtain $\mathrm{iMN}(G,\ell)$ with $M=N_G(\mathbf{T})$ in the scalar case and $M=N_G(P)$ otherwise. □

Lemma 7.119. *With $G=E_{7,\mathrm{sc}}(q)$ and $q=r^f$, we let ℓ be an odd prime dividing $q-\epsilon$ for $\epsilon=\pm1$. On the maximal torus $\mathbf{T}$, the Frobenius F acts on $X(\mathbf{T})$ by ϵq. Thus, we write $T=\mathbf{T}^F$ and $W=W(E_7)$, and take a split Borel containing $\mathbf{T}$ when $\epsilon=1$. In the adjoint dual $\mathbf{G}^*$, we have that the torus $\mathbf{T}^*$ has the scalar marking ϵq. Choose $Q\in\mathrm{Syl}_\ell(W)$ and a Sylow ℓ-subgroup P of its preimage in $N_G(\mathbf{T})$. Then, we obtain that $P\in\mathrm{Syl}_\ell(G)$, and $M=N_G(P)$ has an exact sequence $1\longrightarrow T_*\longrightarrow M\longrightarrow N_W(Q)\longrightarrow 1$ with $T_*=T_\ell C_{T_{\ell'}}(Q)$. The classes of $s\in G^*$ with $\ell\nmid[G^*:C_{G^*}(s)]$ correspond to the $N_W(Q)$-orbits in $(T^*)^Q$. The dual weights $\theta|_{T_*}$ distinguish these orbits. Moreover, each $\psi\in\mathrm{Irr}_{\ell'}(M)$ lies over one such orbit.*

Proof. Since $W^F=W$, [133, Proposition 23.2] gives $N_G(\mathbf{T})/T\simeq W$ in both forms. Each fundamental degree d is even, so $v_\ell(q^d-1)=v_\ell(q-\epsilon)+v_\ell(d)$. Since $|W|=\prod d$, $|P|=|T|_\ell|W|_\ell=|G|_\ell$. Also, the dual central index 2 is prime to ℓ. Each root α pairs to 2 with its own coroot $\alpha^\vee$, so α is nontrivial on $T[\ell]$. Thus, $C_{\mathbf{G}}(T[\ell])^\circ=\mathbf{T}$. Every component of $C_{\mathbf{G}}(T[\ell])$ acts on $X(\mathbf{T})$ by an element of W trivial modulo ℓ. Since the first congruence kernel of $\mathrm{GL}_7(\mathbb{Z}_\ell)$ has no torsion for odd ℓ, the full centralizer is $\mathbf{T}$.

If $\ell\nmid[G^*:C_{G^*}(s)]$, a conjugate of s centralizes the marked P^* and so $T^*[\ell]$. Thus, s is conjugate to a point in $(T^*)^Q$. Conversely, every point in $(T^*)^Q$ centralizes P^*. On T^*, the W-orbits are the G^*-classes since all elements of W lift rationally. If s and s^w lie in $(T^*)^Q$, Q and $Q^{w^{-1}}$ are Sylow in W_s. Thus, a correction in W_s gives $w'\in N_W(Q)$ with $s^{w'}=s^w$. Also, the Jordan degree of $\chi\in\mathcal{E}(G,s)$ has the index factor $[G^*:C_{G^*}(s)]_{r'}$, with component and diagonal indices dividing 2.

By [166, Theorem 6.2] and Section 7.5, $M\leq N_G(\mathbf{T})$ for the two scalar markings. Given $t\in T_{\ell'}$, $t\in M$ exactly when $[P,t]\leq P\cap T_{\ell'}=1$. Since $P\cap T=T_\ell$, it follows that $M\cap T=T_\ell C_{T_{\ell'}}(Q)$. Let L be the preimage of Q. Given $w\in N_W(Q)$, a lift $n\in N_G(\mathbf{T})$ has $P^n\in\mathrm{Syl}_\ell(L)$. Thus, we choose $u\in L$ with $P^{nu}=P$. Then, choose $p\in P$ with $up\in T$, so $nup\in M$ induces w. Consequently, $M/T_*\simeq N_W(Q)$. By [97, Theorem 4.34], $T_{\ell'}=C_{T_{\ell'}}(Q)\times[T_{\ell'},Q]$. Thus, $\theta\mapsto\theta|_{T_*}$ is an $N_W(Q)$-equivariant bijection $\mathrm{Irr}(T)^Q\to\mathrm{Irr}(T_*)^Q$. Given $\psi\in\mathrm{Irr}_{\ell'}(M)$, its orbit on $\mathrm{Irr}(T_*)$ has length dividing $\psi(1)$ by Clifford theory. Thus, the stabilizer $N_W(Q)_{\theta|_{T_*}}$ contains the normal Sylow subgroup Q of $N_W(Q)$. □

Proposition 7.120. *Let TU be a split Borel subgroup of G, and put $M = \mathrm{N}_G(T)$, $K = \mathbb{Q}^{\mathrm{ab}}$ and $e = |U|^{-1}\sum_{u\in U} u$. Suppose that $\Theta : K[M] \to eK[G]e$ is an algebra isomorphism with $\Theta(t) = te$ for $t \in T$. Suppose also that Θ is equivariant for $\langle\phi\rangle$, $\widetilde{T}$ and $\mathcal{H}_\ell$. Then, for the corresponding $\psi \in \mathrm{Irr}(M)$ and split principal series $\chi \in \mathrm{Irr}(G)$, we have that the factor sets α and functions $\mu_{b,\sigma}$ coincide on $D' = B_\psi/M \simeq A_\chi/G$, with A, B as below.*

Proof. Put $A = \widetilde{G} \rtimes \langle\phi\rangle$ and $B = \mathrm{N}_A(\mathbf{T})$, with $\widetilde{G} = \widetilde{\mathbf{G}}^F$ from the regular overgroup and $\widetilde{T} = \widetilde{\mathbf{T}}^F$. Then, $A = GB$ and $B = M(\widetilde{T}\langle\phi\rangle)$. Given $M \le B' \le B$, we put $A' = GB'$ and $D' = B'/M \simeq A'/G$. Choose a representative $d_x \in B' \cap \widetilde{T}\langle\phi\rangle$ for each $x \in D'$. Then, each d_x normalizes U and satisfies $d_x\, e = e\, d_x$, and $d_x d_y d_{xy}^{-1} \in T$. Since $\Theta(t) = te$ and Θ is equivariant for d_x, the map $ad_x \mapsto \Theta(a)d_x e$ gives a graded isomorphism $\Theta' : K[B'] \to eK[A']e$. Indeed, on degree x, both d_x and $d_x e$ are units. Moreover, a normalized α on D', the map also gives $K^\alpha[B'] \simeq eK^\alpha[A']e$ since both products in degrees x, y acquire the scalar $\alpha(x, y)$.

Now, let $B' = B_\psi$ and $A' = A_\chi$. By [141, Corollary 1.2], take a projective extension V of ψ over $K^\alpha[B']$. Thus, via the graded isomorphism Θ', form $Y = K^\alpha[A']e \otimes_{eK^\alpha[A']e} V$. The degree x part of $K^\alpha[A']e$ equals $K[G]e(d_x e)$, with $d_x e$ a unit of $eK^\alpha[A']e$. Thus, in the tensor product $a(d_x e)\otimes v = a\otimes(d_x e)v$, so $Y|_G \simeq K[G]e\otimes_{eK[G]e} V|_M$. The inverse on degree x acts by $a(d_x e)\otimes v \mapsto a\otimes(d_x e)v$, and balancing over $eK[G]e$ gives independence of the representative d_x. Consequently, by idempotent equivalence, $Y|_G$ affords χ, and the projective extension on Y has factor set α.

A cochain $c(x)$ on operators of degree x rescales V, Y by $c(x)$, also for $\alpha\,\partial c$. Moreover, conjugation by d_x and the action of $\sigma \in \mathcal{H}_\ell$ correspond under Θ' by equivariance on $d_x \mapsto d_x e$. Conjugation by $m \in M$ is given by the operators of m on V, Y. Thus, the normalized functions $\mu_{b,\sigma}$ from [141, Lemma 1.4] agree on D'. Finally, $z \in C_A(G)$ lies in $Z(\widetilde{G})$ and has the same scalar on V, Y since $\Theta'(z) = ze$. □

With $q = r^f$, we write $G = \mathbf{G}^F = E_{7,\mathrm{sc}}(q)$ and $W = W(E_7)$. Throughout, $\ell \ne r, 2$, and $P \in \mathrm{Syl}_\ell(G)$. Moreover, on $X(\mathbf{T})$, denote the Weyl action by w, so the mark has $F|_{X(\mathbf{T})} = qw$, $T = \mathbf{T}^F$, $d = \mathrm{ord}_\ell(q)$. Moreover, the mixed comparison for (χ, ψ) is the relation $\succeq_c$.

Proposition 7.121. *Given $\ell \mid q-1$, we let TU_0 be a split Borel, $M = \mathrm{N}_G(T)$ and $P \in \mathrm{Syl}_\ell(M)$. Then, the split principal series $\bigcup_{\theta\in\mathrm{Irr}(T)} \mathcal{E}(G, T, \theta)$ admits an equivariant bijection to $\mathrm{Irr}(M)$. Moreover, the bijection restricts to $\mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(M)$ and satisfies $(\chi, \psi) \succeq_c$ on these sets.*

Proof. Put $K = \mathbb{Q}^{\mathrm{ab}}$ and $e = |U_0|^{-1}\sum_{u\in U_0} u$. Given a simple root α_j, let n_j be its pinned Tits lift and $a_j = qen_j e$. Also, write e_θ for the idempotent of $\theta \in \mathrm{Irr}(T)$. Put $d_j(\theta) = 1$ or i according as $\theta(\alpha_j^\vee(-1)) = 1$ or -1. If $v_\ell(q-1) > 0$, the generators from θ to $s_j\theta$ are $d_j(\theta)^{-1}n_j e_\theta$ in $K[M]$ and $d_j(\theta)^{-1}q^{-1/2}a_j e_\theta$ in $eK[G]e$. Moreover, in a reduced word for w, the factors d_j correspond to the inversion set $\mathrm{Inv}(w)$ at θ. Thus, the generators $d_j(\theta)^{-1}n_j e_\theta$ satisfy braid relations and have square e_θ along s_j^2. The normalized global generators $d_j(\theta)^{-1}q^{-1/2}a_j e_\theta$ satisfy braid relations. For their squares, $a_j^2 = q\alpha_j^\vee(-1)e + \big(\sum_{a\in\mathbb{F}_q^\times}\alpha_j^\vee(a)\big)a_j$. On e_θ, the coroot sum is 0 if $\theta\circ\alpha_j^\vee \ne 1$, and $q-1$ otherwise. Thus, for $\theta\circ\alpha_j^\vee = 1$, the normalized generator b_j satisfies $b_j^2 = e_\theta + (v - v^{-1})b_j$. Given $\theta\circ\alpha_j^\vee \ne 1$, the square at the source is e_θ. Consequently, the Bruhat bases for $eK[G]e$ and $K[M]$ give monodromic Hecke algebras at $v = \sqrt{q}$ and $v = 1$.

Let $R_\theta = \{\alpha : \theta\circ\alpha^\vee = 1\}$ with the induced positive roots R_θ^+. In the decomposition of [123, §§1.5–1.6(a)], an arrow $w : \theta \to w\theta$ is cu, with $u \in W(R_\theta)$ and $cR_\theta^+ = R_{w\theta}^+$. Moreover, the normalized c-basis multiplies by the rule c_2c_1 and acts on $H_v(W(R_\theta))$ by based isomorphisms [122, §§3.13 and 4.2–4.6]. Thus, on $H_v(W(R_\theta))$, we fix $\phi_1^{-1}\phi_v$ through the asymptotic algebra $J(W(R_\theta))$. The specialized maps ϕ_v are defined over K and natural for based isomorphisms of $W(R_\theta)$. Thus, ϕ_v is an isomorphism for $v = 1, \sqrt{q}$ by [125, Remark 20.1(e)], since $H_v(W(R_\theta))$ is semisimple at $v = 1, \sqrt{q}$. Thus, extend $\phi_1^{-1}\phi_v$ by the identity on the c-basis, including for a disconnected stabilizer W_θ.

Since $q \equiv 1 \pmod\ell$, the scalar $\sqrt{q}$ is fixed by $\mathcal{H}_\ell$ by [156, Lemma 2.2]. If $\sigma(i) = -i$, the normalization at w changes by $\theta((\rho^\vee - w^{-1}\rho^\vee)(-1))$. Since $c_w = \rho^\vee - w^{-1}\rho^\vee$ is integral and satisfies $c_{uw} = c_w + w^{-1}c_u$, the sign $\theta(c_w(-1))$ is multiplicative on w-arrows. On reflections of R_θ, we have that the sign is 1. Similarly, $t \in \widetilde{T}$ has scalar $\theta(w^{-1}(t)t^{-1})$ at w. The scalar is multiplicative and is 1 on reflections of R_θ. Also, ϕ fixes n_j and $\theta\circ\alpha_j^\vee = 1$ if and only if $\theta^\phi\circ\alpha_j^\vee = 1$. Thus, $\Theta : K[M] \to eK[G]e$ is equivariant for $\mathcal{H}_\ell$, $\widetilde{T}$ and $\langle\phi\rangle$, with $\Theta(t) = te$ for $t \in T$.

By Proposition 7.120, the projective models for χ, ψ share the factor set α, functions $\mu_{b,\sigma}$ and central scalars. By [166, Theorem 6.2], $P \in \mathrm{Syl}_\ell(G)$ and $\mathrm{N}_G(P) \le M$. Moreover, put $A = \widetilde{G} \rtimes \langle\phi\rangle$ for the regular overgroup $\widetilde{G}$ with pinned field group $\langle\phi\rangle$. Since $\widetilde{G} = G\widetilde{T}$ by Lang's theorem, $A = G\mathrm{N}_A(T)$. Given $a = gb \in \mathrm{N}_A(P)$ with $b \in \mathrm{N}_A(T)$, $P^g = P^{b^{-1}}$ lies in $\mathrm{Syl}_\ell(M)$. Thus, there is $m \in M$ with $P^g = P^m$, so $gm^{-1} \in \mathrm{N}_G(P) \le M$ and $a \in \mathrm{N}_A(T)$.

For a simple module V_χ matched to ψ, its local model is $V_\chi^{U_0}$. Moreover, the weights of $V_\chi^{U_0}$ form one W-orbit, and the invertible operators for $w \in W$ give equal dimensions. The multiplicity at θ is $m = \langle \mathrm{Ind}_{TU_0}^G \theta, \chi \rangle$. Hence, $\psi(1) = [W : W_\theta]m$. By the Howlett–Lehrer parametrization in [166, (2)], $m = \eta(1)$ for the relative Weyl character η. Thus, $\psi(1) = [W : W_\theta]\eta(1)$ is the degree in [166, Proposition 6.1 and Theorem 6.2]. Consequently, $\chi(1)$ and $\psi(1)$ are prime to ℓ simultaneously for all $v_\ell(q-1) > 0$, including $\ell = 3, 5, 7$. □

Lemma 7.122. *Let $X = \mathrm{SL}_n(q)$, let U be upper unitriangular and let ℓ be an odd prime dividing $q-1$. Then, every irreducible X-module V of ℓ'-degree has $V^U \neq 0$. Moreover, if $V^U \neq 0$, then $(\dim V)_\ell = (\dim V^U)_\ell$.*

Proof. Take an irreducible $\mathrm{GL}_n(q)$-module $\widetilde{V}$ over V. By the cyclic-quotient formulas, $\widetilde{V}|_X$ is multiplicity free with $t = |\{\nu : \widetilde{V} \otimes (\nu \circ \det) \simeq \widetilde{V}\}|$ constituents, where $\nu \in \mathrm{Irr}(\mathbb{F}_q^\times)$. If $\ell \nmid \dim V$ and $n \geq 3$, Lemma 3.21 places $\widetilde{V}$ in the split principal series. If $n = 2$, each determinant twist fixing $\widetilde{V}$ has $\nu^2 = 1$ by the central character. Thus, $t \mid 2$, and $\ell \nmid \dim \widetilde{V}$ when $\ell \nmid \dim V$. By [84], a nonsplit parent in rank 2 has degree $q-1$. Hence, if $\ell \nmid \dim V$, $\widetilde{V}$ lies in split principal series.

If $\widetilde{V}$ belongs to $\mathcal{E}(\mathrm{GL}_n(q), T, \theta)$, its colour partitions are $\lambda_i \vdash m_i$. The W-operators act transitively on the T-weights of $\widetilde{V}^U$. The multiplicity at a weight is $\prod_i f^{\lambda_i}$, with $f^{\lambda_i} = \chi_{\lambda_i}(1)$ for $\mathfrak{S}_{m_i}$. Thus, $\dim \widetilde{V}^U = (n!/\prod_i m_i!)\prod_i f^{\lambda_i}$, and $(\dim \widetilde{V}^U)_\ell = (\dim \widetilde{V})_\ell$. The diagonal torus of $\mathrm{GL}_n(q)$ conjugates the constituents of $\widetilde{V}|_X$ transitively and normalizes U. Consequently, $\dim V^U = \dim \widetilde{V}^U/t$ and $\dim V = \dim \widetilde{V}/t$. Conversely, $V^U \neq 0$ implies $\widetilde{V}^U \neq 0$. Hence, $\widetilde{V}$ lies in $\mathcal{E}(\mathrm{GL}_n(q), T, \theta)$ for some θ. □

Theorem 7.123. *Let S be simple of split type A_m with $m \geq 1$, B_m, C_m with $m \geq 3$, D_m with $m \geq 4$, or E_6, E_7. Given any odd prime $\ell \mid \gcd(q-1, |S|)$, $\mathrm{iMN}(S, \ell)$ holds, with comparison on the universal cover of S. Moreover, $M = N_X(\mathbf{T})$ for a split maximal torus $\mathbf{T}$ of the universal cover X, when $X = \mathbf{X}^F$ and $N_X(P) \leq M$.*

Proof. Let $X = \mathbf{X}^F$, with split simply connected $\mathbf{X}$ and pinned Borel $\mathbf{TU}$. By [157, Lemma 4.1(2),(4)], $\mathrm{iMN}(S, \ell)$ holds for the exceptional covers of S and the nongeneric $N_X(P)$. Otherwise, X is the universal cover. Put $T = \mathbf{T}^F$, $U = \mathbf{U}^F$, $M = N_X(\mathbf{T})$ and $e = |U|^{-1}\sum_{u \in U} u$. Each simple root group U_{α_j} has order q, including for short α_j. Thus, the relation for a_j^2 and the W-braid relations from Proposition 7.121 give $v = \sqrt{q}$ on each neutral simple reflection s_j. Consequently, the maps $\phi_1^{-1}\phi_v$ give $\Theta : \mathbb{Q}^{\mathrm{ab}}[M] \xrightarrow{\sim} e\mathbb{Q}^{\mathrm{ab}}[X]e$. A pinned diagram automorphism γ acts by $(n_j, \theta) \mapsto (n_{\gamma j}, \gamma\theta)$, and $d_{\gamma j}(\gamma\theta) = d_j(\theta)$. Since $\gamma R_\theta^+ = R_{\gamma\theta}^+$, the map on $H_v(W(R_\theta))$ is based. Consequently, Θ is equivariant for γ, including triality on D_4.

Given X of type B, C, D, E, the local degree is $[W : W_\theta]\dim(e_\theta V^U)$. The relative Weyl degree is $\dim(e_\theta V^U)$ in [166, (2) and Theorem 6.2]. Thus, for $P \in \mathrm{Syl}_\ell(X)$ in M, the cited theorem gives that $\ell \nmid \chi(1)$ if and only if $\ell \nmid \psi(1)$ and $N_X(P) \leq M < X$. Given X of type A, the local model is V^U. Hence, Lemma 7.122 gives the cut at ℓ'-degree in both directions. Also, $N_X(P) \leq M < X$ by [132, Theorem 5.14] in these cases.

Next, embed $\mathbf{X}$ regularly into $\widetilde{\mathbf{X}}$ with split connected centre $\mathbf{Z}$. In the central product with $\mathbf{X}$, set $\mathbf{Z} = (\mathbb{G}_m)^{\mathrm{rk}(\mathbf{X})}$ with coordinate j for the weight ϖ_j. Then, embed $Z(\mathbf{X})$ into $\mathbf{Z}$ by the ϖ_j. Since the ϖ_j generate $X(Z(\mathbf{X}))$ and γ permutes the ϖ_j, the inclusion is equivariant for each γ. For the pinned diagram group Δ, we set $E = \langle \phi, \Delta \rangle$. Moreover, put $A = \widetilde{\mathbf{X}}^F \rtimes E$, $B = N_A(\mathbf{T})$ and $Z = \mathbf{Z}^F$. By [79, Theorem 2.5.12], A induces $\mathrm{Aut}(X)$ via $X, \widetilde{\mathbf{T}}^F, \langle\phi\rangle$ and Δ. Indeed, $\widetilde{\mathbf{X}}^F$ maps onto $\mathbf{X}_{\mathrm{ad}}^F$ by Lang's theorem for $\mathbf{Z}$. Moreover, if $h\delta \in C_A(X)$, it normalizes U and the opposite subgroup U^-. Thus, $h \in \widetilde{\mathbf{T}}^F$ and δ fixes each simple α. On U_α, the action is $a \mapsto ca^{r^j}$ on $\mathbb{F}_q$. Thus, evaluation at 1 gives $c = 1$. At a field generator a_0, the identity $a_0^{r^j} = a_0$ gives that ϕ^j is trivial on $\mathbb{F}_q$. Consequently, $\alpha(h) = 1$ for every simple α and $C_A(X) = Z$.

Since $A = XB$, the quotient B/M has representatives $d \in \widetilde{\mathbf{T}}^F E$ with $U^d = U$. The representative defect $d_x d_y d_{xy}^{-1}$ lies in T, and $\Theta(t) = te$. Moreover, Θ is equivariant for $\widetilde{\mathbf{T}}^F E$. Thus, for E the graded isomorphism is Θ' of Proposition 7.120, also for $\Delta = \mathfrak{S}_3$. By [141, Corollary 1.2], take the local model $\mathcal{P}'$ over $\mathbb{Q}^{\mathrm{ab}}$ with root of unity factor set α. Consequently, the graded equivalence gives $\mathcal{P}$ with factor set α on B_ψ/M and the same $\mu_{b,\sigma}$. Given $z \in Z$, the map $z \mapsto ze$ gives the same scalar $\lambda(z)$, including for $z \in Z \cap X$.

Given $b \in B$, $P^b \in \mathrm{Syl}_\ell(M)$. Thus, b can be multiplied by an $m \in M$ so that $bm \in N_A(P)$. Conversely, for $a = xb \in N_A(P)$, choose $m \in M$ with $bm \in N_A(P)$. Then, $xm^{-1} \in N_X(P) \leq M$. Consequently, $N_A(P) \leq B$ and $B = MN_A(P)$. Choose a normalized section $s : \mathrm{Aut}(X)_P \to N_A(P)$ of $N_A(P) \to \mathrm{Aut}(X)_P$. Then, the two projective models are $\mathcal{P}(xs(\delta))$ and $\mathcal{P}'(ms(\delta))$ on A_χ and B_ψ. The defect of s is $z = s(\delta)s(\epsilon)s(\delta\epsilon)^{-1} \in Z$. Given a lift y in

the inertia, the value at zy differs from that at y by $\lambda(z)/\alpha(\bar{z},\bar{y})$, where $\bar{z},\bar{y}$ are quotient degrees. Since z has finite order and $\alpha(\bar{z},\bar{y})$ is a root of unity, $\lambda(z)/\alpha(\bar{z},\bar{y})$ is a root of unity in $\mathbb{Q}^{\mathrm{ab}}$. Similarly, evaluation at $s(\eta)s(\delta)s(\eta)^{-1}$ compares conjugation by η with evaluation on the section s. Thus, for (η,σ) in the mixed stabilizer, the scalar and its σ-transform coincide on $\mathcal{P},\mathcal{P}'$. An extra $m\in M$ acts by $\mathcal{P}(m)$ or $\mathcal{P}'(m)$, with quotient degree 1. If (x,δ) centralizes X, then $x\in N_X(P)$ and $xs(\delta)\in Z$, so the scalar for $xs(\delta)$ is $\lambda(xs(\delta))$ on both models. Consequently, $(\chi,\psi)\succeq_c$ on the full $\mathcal{H}_\ell$-orbit stabilizer. □

Proposition 7.124. *Let q be odd and $\ell\mid q+1$, and mark $\mathbf{T}$ by $-q$. Suppose that $s\in T^*$ has $C_{\mathbf{G}^*}(s)=\mathbf{K}^*$ a proper Levi with roots $R=\prod_i A_{m_i-1}$. The correspondence $\chi\leftrightarrow\psi$ on the series of s has equal ℓ-parts of the degrees and satisfies $(\chi,\psi)\succeq_c$ on $N_G(\mathbf{T})$. Moreover, for $\ell\nmid\chi(1)$, the mixed comparison holds on $\mathrm{N}_G(P)$.*

Proof. The rational Weyl inertia is $W_R=\prod_i S_{m_i}$. On each A_{m_i-1}, based Frobenius acts by opposition. For ℓ'-degree, the result of [156, Corollary 4.7] gives the unipotent correspondence for R. Thus, on A_{m_i-1} we pair $\lambda_i\vdash m_i$ with its Specht label η_i on V_i. On orbits of equal components R_j, we permute the factors V_j. The element $-I\in W$ inverts s and acts by conjugation by $w_{0,i}$ on S_{m_i}. Thus, $\chi=\bar{\chi}$ and $\psi=\bar{\psi}$. In the regular dual torus $\widetilde{\mathbf{T}}^*$, each lift $\widetilde{s}$ is fixed by W_R, since $\alpha(\widetilde{s})=1$ for $\alpha\in R$. Consequently, the global parent $\widetilde{\chi}|_G$ is irreducible since for central $\zeta\neq 1$, the orbit of $\widetilde{s}\zeta$ is distinct. Moreover, the local parent $\widetilde{\psi}$ restricts irreducibly by weight inertia W_R. The local extension is $\kappa_s\otimes\eta$, with the Levi character $\kappa_s\in\mathrm{Irr}(K/K')$ corresponding to s and $\eta\in\mathrm{Irr}(W_R)$. For (b,σ) in the joint stabilizer of (χ,ψ), the same $w\in W$ corrects both weights. Indeed, the ambiguity W_R fixes the lifts and acts on η by inner automorphisms of W_R. Thus, $\widetilde{\chi},\widetilde{\psi}$ acquire the same diagonal twist. For $\widetilde{\chi}$, the Jordan map $J_{\widetilde{s}}$ is equivariant for central twists by ζ, isogenies and $\mathcal{H}_\ell$ by [156, §4.4 and Corollary 4.7].

Suppose that the field inertia C_m has odd order. Let D_0 be the Sylow 2-subgroup of the cyclic diagonal quotient D_{diag}. Since C_m centralizes D_0, the inertia quotient over D_0 is $D_0\times C_m$. Since a translation on $\mathrm{Irr}(D_0)$ has 2-power order, C_m fixes each diagonal extension $\widetilde{\chi}$ and $\widetilde{\psi}$. On the preimages of C_m, [143, Lemma 2.1] gives the unique real extensions of χ,ψ. Consequently, the common lift $\widetilde{s}$ gives equal twists on D_0, and uniqueness gives equal twists on C_m. Also, the scalar at z' is the lifted weight value $a(z')$.

Now let $m=f/k$ be even, and take $w\in W$ such that $w(s)=s^{r^k}$ and $wR^+=R^+$. Both w^m and $(-I)w_{0,R}$ act by $s\mapsto s^{-1}$ and fix the chamber R^+. Since their quotient lies in W_R and fixes R^+, $w^m=(-I)w_{0,R}$. On $\mathrm{Fix}(W_R)$ $w^m=-I$. Thus, even m is impossible if $\mathrm{codim}\,R$ is odd, by $\det(w|_{\mathrm{Fix}(W_R)})^m$. Similarly, for a given type A_b, the product of signs $\prod_{R_i\cong A_b}\epsilon_i(w^m)$ is 1 for even m. Consequently, if A_b has opposition of order 2 and occurs oddly often in R, m is odd. If $R=A_1$, w fixes the positive root α and acts on $\alpha^\perp$ through $W(D_6)$. If m is even, $w^{m/2}|_{\alpha^\perp}$ has square $-I$. Thus, the permutation has type 2^3. Each pair has sign product -1, giving total sign $(-1)^3$, which contradicts membership in $W(D_6)$.

Now, for even m, take a lift s_0 in the simply connected dual torus $\mathbf{T}^*_{\mathrm{sc}}$, with kernel $\langle z\rangle$. If $a=w^{m/2}$, we have that $a(s_0)=z^e s_0^{rf/2}$ for $e\in\{0,1\}$, so $a^2(s_0)=s_0^q$. On the other hand, $a^2=(-I)w_{0,R}$ and W_R fixes s_0. Hence, $s_0^q=s_0^{-1}$ and both s_0, zs_0 are rational for $-q$. Since $v_2(q+1)=1$, their 2-parts belong to $\mathbf{T}^*_{\mathrm{sc}}[2]$. On the coroot lattice Y modulo $2Y$, $Q(v)=(v,v)/2 \bmod 2$ is W-invariant. The class of $2\omega_7^\vee$ represents z, has norm 6 and pairs evenly with Y. Thus, translation by $2\omega_7^\vee$ changes Q by 1. Every odd power fixes $(s_0)_2$ and $(zs_0)_2$. Thus, Q distinguishes s_0, zs_0 for each corrected (b,σ) in the joint stabilizer.

In the finite adjoint group D, the parent Θ in the series of s_0 satisfies that $\Theta|_{D'}$ is irreducible, with $D'=[D,D]$, since the diagonal twist acts by $s_0\mapsto zs_0$. Inflation of that restriction gives χ. After correction, the stabilizer fixes s_0 and the partitions $(\lambda_i)_i$, so it fixes Θ. Opposition gives $\Theta=\bar{\Theta}$. Moreover, the torus formula for J_{s_0} makes Θ uniform since $J_{s_0}(\Theta)$ is a tensor product on $R=\prod_i A_{m_i-1}$. With the pinning of D, we put $j=\phi^{f/2}$ for the field automorphism ϕ. Since $\Theta^j=\overline{\Theta}$, [105, Theorem 3.1.3(ii)] gives multiplicity 1 in $V=\mathbb{Q}[D/D^j]$ after scalar extension. The group $\langle\phi\rangle$ acts on V by permutations. Thus, $e_\Theta(\mathbb{Q}^{\mathrm{ab}}\otimes_{\mathbb{Q}}V)$ gives an extension $\widehat{\Theta}$ to its stabilizer in $\langle\phi\rangle$, invariant under the corrected stabilizers.

In the adjoint torus normalizer $\widehat{M}$, we choose the marking $\mathrm{Int}(n_0)F_q$. The longest Tits lift satisfies $n_0^2=1$ in D, and $n_0n_j=n_jn_0$ by the braid identity for w_0. Thus, n_j is rational and $U(n_j)=n_j$ for the marked prime field lift U. On the torus $U(t)=t^r$. By Lang transport and $Z(D)=1$, $U^f=n_0^{-1}$ from the inner action of U^f on D. Also, $v_2(\mathrm{ord}_\ell(r))=1+v_2(f)$, so $\mathcal{H}_\ell$ fixes μ_{2f_2} for $f_2=(f)_2$, including i when f is even.

Given a torus weight a, the normalized arrows are $b_{j,a}=n_je_a/d_j(a)$. By Proposition 7.121, the $b_{j,a}$ give Weyl operators equivariant for $U,\mathcal{H}_\ell$. Given $\alpha\in R$, choose w with $w\alpha=\alpha_j$. Then, at wa the normalized lift of s_j is n_j.

Conjugation by the operator for w^{-1} cancels the factors d_j. Since $\kappa_s|_{K'} = 1$, the W_R-action is η. On the stabilizer of $(\lambda_i)_i$ in $\mathrm{Aut}(R)$, extend $\eta = \bigotimes_i \eta_i$ by rational Specht modules V_i. The factor permutations act on $\bigotimes_i V_i$ by permutations, and on R_i the diagram involution has matrix $\rho_i(w_{0,R_i})$. Thus, the semidirect product relations give ρ, with $h = \rho(-I_R) \in \{1, -1\}$ a scalar.

On the inertia of (a, η) in $\widehat{M}\langle U^k\rangle$, an element wU^i acts by $a_0^i \rho(w|_R)$ in normalized coordinates, and $t \in T$ acts by $a(t)$. Let $\delta(a) = \prod d_j$ along a reduced word for w_0. Thus, $\delta(a) = i^{|\{\alpha>0 : a(\alpha^\vee(-1))=-1\}|}$, with the exponent invariant under $W, \langle U\rangle, \mathcal{H}_\ell$ acting on a. Since $\delta(a^{-1}) = \delta(a)$ and $n_0^2 = 1$, the normalized square gives $\delta(a)^2 = 1$. Hence, $\sigma(\delta(a)) = \delta(a)$. Before the field relation, $U^f n_0$ acts by $a_0^f \delta(a) h$. In μ_{2f_2}, choose a_0 with $a_0^f = \delta(a)^{-1} h^{-1}$, possible since f/f_2 is odd. Thus, $U^f = n_0^{-1}$ on the inertia, even if $a^U \neq a$. Induction extends the parent $\widetilde{\psi}$ with base multiplicity 1. Every corrected (b, σ) fixes $a_0, \delta(a)$ and conjugates the matrices of ρ by the label intertwiner. Consequently, $\widehat{\psi}, \widehat{\Theta}$ have additional twist 1. The restrictions to the derived images D' and $\widehat{M} \cap D'$ are irreducible, since the diagonal twist $s_0 \mapsto z s_0$ changes the W-orbit. Moreover, on the common cover Γ, the inflated operators have value 1 on the central kernel.

Put $n = \sum_i (m_i - 1)$. The r'-order of $C_{G^*}(s)$ has torus factor $(q+1)^{7-n}$ and semisimple factors $\prod_i \prod_{j=2}^{m_i} |(-q)^j - 1|$. By [63, Lemma 2.3], the connected factors have equal F-fixed orders under central isogeny. At $|(-q)^j - 1|$, the valuation is $v_\ell(q+1) + v_\ell(j)$. Thus, $v_\ell([G^* : C_{G^*}(s)]) = v_\ell([W : W_R])$. In the unitary hook formula for λ_i, the denominator factor for a hook h contributes $v_\ell(q+1) + v_\ell(h)$. Hence, $v_\ell(\chi(1)) = v_\ell([W : W_R] \prod_i f^{\lambda_i}) = v_\ell(\psi(1))$, with f^{λ_i} the Specht degree. If $\ell > 7$, $P = T_\ell$ is characteristic in T. By [132, Theorem 5.14], $N_G(P) \leq N_G(\mathbf{T})$, so $N_G(P) = N_G(\mathbf{T})$ and $v_\ell(\chi(1)) = v_\ell(\psi(1)) = 0$.

Let $\ell \in \{3, 5, 7\}$. Next, we write $M = N_G(P)$, $Q \in \mathrm{Syl}_\ell(W)$ for the image of P and $T_* = T \cap M$. By Lemma 7.119, $M/T_* = N_W(Q)$ and $T_* = T_\ell C_{T_{\ell'}}(Q)$, and $\theta \mapsto \theta|_{T_*}$ distinguishes the Q-fixed weights. Thus, only W_R containing a conjugate of Q contributes. After conjugation, assume that $Q \leq W_R$. Let $I = M_{\theta_*}$ for $\theta_* = \theta|_{T_*}$. Then, $I/T_* = N_{W_R}(Q)$, and $I \leq K$, so $\kappa_s|_I$ extends θ_*.

If $\ell = 3$, then $m_i \leq 7$ and $v_3(m_i!) = \lfloor m_i/3 \rfloor \leq (m_i - 1)/2$. Since $n \leq 6$, it follows that $v_3(|W_R|) \leq 3 < 4 = v_3(|W|)$, so no proper R contributes. If $\ell = 5, 7$, exactly one factor S_b has $\ell \leq b < 2\ell$. Its type A_{b-1} has rank at least 4, so this type occurs with multiplicity 1 in R, with opposition of order 2. Therefore, the diagram sign gives odd m. The normalizer $N_{S_b}(Q)$ is $\mathrm{AGL}_1(\ell) \times S_{b-\ell}$. By the hook formula, the ℓ'-partitions of b have ℓ-weight 1. Thus, the core $\lambda_{(\ell)}$ gives the $S_{b-\ell}$-label. Moreover, the runner $i \in \{0, \ldots, \ell - 1\}$ gives an affine label ν_i by a fixed bijection. On $\mathrm{AGL}_1(\ell)$, we inflate $\mathrm{Irr}(C_{\ell-1})$ to obtain linear characters over $\mathbb{Q}(\mu_{\ell-1})$. The unique nonlinear character ν is the permutation character on $\mathbb{F}_\ell$ minus 1. Thus, the affine labels are $\mathcal{H}_\ell$-fixed. Moreover, a correction that normalizes Q acts on S_b by a permutation in $N_{S_b}(Q)$. On equal factors S_c with $c < b$, take the rational Specht model for λ. Next, for the twist comparison, put $\mu_0 = \mathrm{Ind}_I^M(\kappa_s|_I)$. A lift of $-I$ in M gives $\overline{\mu_0} = \mu_0$. The regular parent $\widetilde{\mu}_0$ is for the global lift $\widetilde{s}$. Indeed, $Q \leq W_R$ fixes the corresponding weight $\widetilde{\theta}$, and restriction to $\widetilde{T}_\ell C_{\widetilde{T}_{\ell'}}(Q)$ distinguishes the Q-fixed weights $\widetilde{\theta}$ by [97, Theorem 4.34]. By [97, Corollary 3.28], a diagonal 2-coset has a representative in $C_{\widetilde{T}_2}(Q)$, so the representative centralizes $\widehat{Q}$. Thus, $\widetilde{\mu}_0|_M = \mu_0$ by Clifford induction. The character $\widetilde{\mu}_0$ has the diagonal twist of $\widetilde{\chi}$. On C_m, choose the unique real extension $\widehat{\mu}_0$, even if the field stabilizer of μ_0 strictly contains C_m. Since the induced weights from θ_* are distinct, V_{θ_*} has dimension 1. Consequently, the action on V_{θ_*} under I's extended inertia has mixed twist $\mu_{b,\sigma}^{\mu_0}$.

We tensor V_{θ_*} with the Weyl model V_η. On A_{c-1} for $c \geq 3$, take the inner permutation of S_c, with its operator from $N_{S_b}(Q)$ when $c = b$. Moreover, on equal components R_j, take tensor permutations. On an A_1 factor with root α, choose the sign of $\alpha \mapsto \pm\alpha'$. Thus, the operators on V_η satisfy the signed permutation law on R. The operator of $-I$ is the sign h from the A_1-factors. The marked lift U fixes the Sylow lift $\widehat{Q}$ in the Tits group $\widehat{W}$. Meanwhile, the torus corrections in $\widetilde{T}$ act by 1 on V_η, including the diagonal representatives. Next, for U^i, we multiply by a_0^i with $a_0 \in \mu_{2f_2}$ and $a_0^f = h^{-1}$, to impose $U^f = n_0^{-1}$. Since coefficient stabilizers fix a_0 and the matrices on V_η, induction from the extended stabilizer of (θ_*, η) restricts to ψ. Consequently, $\mu_{b,\sigma}^{\psi} = \mu_{b,\sigma}^{\mu_0} = \mu_{b,\sigma}^{\chi}$. □

Proposition 7.125. *In the torus marked by $-q$, suppose that $q = r^f$ is odd, f is even and $\mathrm{C}_{\mathbf{G}^*}(s)$ is disconnected. Write $s = ty$ with t the 2-part and $y \neq 1$ of odd order. Then, $\mathrm{C}_{\mathbf{G}^*}(s) \leq \mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(y)$, where $\mathbf{K}^*$ is a proper classical Levi. Moreover, $[G : K]_{r'}$ and $[W : W(\mathbf{K}^*)]$ have equal ℓ-parts.*

Proof. Since $(q+1)_2 = 2$ and $3 \nmid q + 1$, $\gcd(|y|, 6) = 1$. The odd order lift y_0 is unique. Thus, each g_0 of $g \in \mathrm{C}(y)$ centralizes y_0. Thus, $\mathrm{C}(y)$ is connected by Steinberg's theorem, and $t \neq 1$ has order 2. Put $\mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathrm{Z}(\mathrm{C}(y))^\circ)$. The roots $\alpha(y) = 1$ have full rank on each $\mathbf{L}_i^*$ of $\mathbf{L}^*$. Thus, the projection y_i to $\mathbf{L}_i^*$ lies at a vertex v_i of the alcove,

with $|y_i|$ dividing its highest root mark a_i. Since a_i has prime factors only in $\{2,3\}$, each $y_i = 1$. Consequently, $\mathrm{C}(y) = \mathbf{L}^* = \mathbf{K}^*$ is proper since $y \neq 1$ and $Z(\mathbf{G}^*) = 1$. Whenever a corrected stabilizer operation a has $\langle s^a \rangle = \langle s \rangle$, we have $\langle y^a \rangle = \langle y \rangle$ and $(\mathbf{K}^*)^a = \mathbf{K}^*$.

Given a lift s_0 of s, the centralizer $C_{\mathbf{G}^*_{\mathrm{sc}}}(s_0)$ is connected. Thus, a representative g of a nonidentity component of $\mathrm{C}(s)$ satisfies $g_0 s_0 g_0^{-1} = z s_0$ for its lift g_0. In particular, $[g_0, s_0] = z$, the central involution. Since g_0, s_0 lie over $\mathbf{K}^*$, $z \in [\mathbf{K}^*_{\mathrm{sc}}, \mathbf{K}^*_{\mathrm{sc}}]$, where $\mathbf{K}^*_{\mathrm{sc}}$ is the preimage. For standard $\mathbf{K}^*$ with nodes J, the derived cocharacter lattice is $\sum_{j \in J} \mathbb{Z} \alpha_j^\vee$. In Bourbaki numbering, $2\omega_7^\vee = \sum_j c_j \alpha_j^\vee$ with $(c_1, \ldots, c_7) = (2,3,4,6,5,4,3)$, by the Cartan matrix of E_7. Thus, $2, 5, 7 \in J$. The standard E_6 subdiagram has nodes $1, \ldots, 6$, so $\mathbf{K}^*$ is a classical Levi. The weight θ has inertia quotient W_θ lies in $W(\mathbf{K}^*)$, so induction from $N_K(\mathbf{T})$ multiplies degrees by $[W : W(\mathbf{K}^*)]$. The global multiplier is $[G : K]_{r'}$ by signed induction from K [72, Theorem 3.3.12]. In the scalar marking, a degree j contributes $|(-q)^j - 1|$ because the invariant p_j has opposition sign $(-1)^j$. Each central dimension contributes $q + 1$. Since the ranks of $\mathbf{K}^*$ and $\mathbf{G}^*$ agree, the valuation $v_\ell(|(-q)^j - 1|) = v_\ell(q+1) + v_\ell(j)$ gives $v_\ell([G : K]_{r'}) = v_\ell([W : W(\mathbf{K}^*)])$. If $v_\ell([G : K]_{r'}) > 0$, then the induced characters have $\ell \mid \chi(1), \psi(1)$. If it is zero, multiplication by κ_y^{-1} in K reduces $s = ty$ to t. For $(\chi, \psi) \succeq_c$, we also compare the extensions on K and on its marked torus normalizer $N_K(\mathbf{T})$. □

Lemma 7.126. *Let $G = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ odd, and mark the torus $\mathbf{T}$ by $-q$. Take an odd prime $\ell > 7$ dividing $q + 1$ and $s \in T^*$ with disconnected $C_{\mathbf{G}^*}(s)$. By* Lemma 7.119*, $P = T_\ell$ is Sylow, $N_G(P) = N_G(\mathbf{T})$, with a rational representative n_w for every $w \in W$. For the full stabilizer W_s of s on $\mathbf{T}^*$, we put $\mathbf{A} = ((\mathbf{T}^*)^{W_s})^\circ$ and $\mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(\mathbf{A})$. Then, $\mathbf{L}^*$ is the minimal Levi containing $C_{\mathbf{G}^*}(s)$. If $\mathbf{L}^*$ is proper, its nodes J contain $2, 5, 7$. If $\mathbf{L}^*$ has a factor D_j, the Levi type is D_6, D_5A_1 or D_4A_1. In types D_5A_1, D_4A_1, the orthogonal projection g has eigenvalues $1, -1$, both occurring. Moreover, its A_1-projection in PGL_2 is an involution.*

Proof. The roots $\alpha(s) = 1$ give $((\mathbf{T}^*)^{W_s^\circ})^\circ = Z(C_{\mathbf{G}^*}(s)^\circ)^\circ$, where $W_s^\circ = \langle s_\alpha : \alpha(s) = 1 \rangle$. Each component of $C_{\mathbf{G}^*}(s)$ has a representative in $N_{\mathbf{G}^*}(\mathbf{T}^*)$. Consequently, $\mathbf{A} = Z(C_{\mathbf{G}^*}(s))^\circ$, so $\mathbf{L}^*$ is minimal by Section 7.1. Thus, for W_s, the fixed space in $Y(\mathbf{T}^*) \otimes \mathbb{Q}$ is spanned by $Y(\mathbf{A})$ and misses the semisimple span of $\mathbf{L}^*$. For $g \in \mathbf{G}^*$ and $\gcd(j, |s|) = 1$, the Levi for $gs^j g^{-1}$ is $g\mathbf{L}^* g^{-1}$.

As for s_0 in Proposition 7.125, a lift x of a nonidentity component representative satisfies $[x, s_0] = z$ for a lift s_0 of s. Thus, z belongs to the derived group above $\mathbf{L}^*$, and the support of $2\omega_7^\vee \bmod 2$ gives the nodes $2, 5, 7$. A D_j-factor also contains the trivalent node 4 and its neighbors $2, 3, 5$. Consequently, the proper diagrams with D_j are $\{2, 3, 4, 5, 7\}$ and its extensions by 1 or 6. The E_6 diagram has nodes $1, \ldots, 6$.

Given D_jA_1, $z = (k, -I)$, where $\langle k \rangle = \ker(\mathrm{Spin}(V) \to \mathrm{SO}(V))$ has order 2. Also, both fork ends of D_j are fixed by the Frobenius marking $-q$. Thus, the representative of x on V centralizes g on V, while its lift to $\mathrm{Spin}(V)$ has commutator k. Given a reciprocal eigenvalue pair $\{u, u^{-1}\}$ with $u^2 \neq 1$, the corresponding linear factor of $C_{\mathrm{SO}(V)}(g)$ has a scalar cocharacter ν. The full natural centralizer fixes ν. Hence, W_s fixes ν in the semisimple span of $\mathbf{L}^*$, a contradiction. If $g = \pm I$, the centralizer is $\mathrm{SO}(V)$ with connected preimage $\mathrm{Spin}(V)$. Consequently, the commutator map into $\langle k \rangle$ is constant and equals 1. Thus, the eigenspaces V_+, V_- for $1, -1$ have nonzero even dimensions since $g \in \mathrm{SO}(V)$. On A_1, the lifted element has eigenvalues u, u^{-1} with $\{u, u^{-1}\} = \{-u, -u^{-1}\}$ because the commutator is $-I$. Therefore, $u^2 = -1$, and x acts on both D_j, A_1. □

Proposition 7.127. *For a minimal Levi $\mathbf{L}^*$ of type D_6, the orthogonal projections have types $\mathrm{I}(3,3), \mathrm{Q}(2,2), \mathrm{U}(6)$. For $\mathrm{I}(3,3)$ and $\mathrm{Q}(2,2)$, the rational class on $\mathbf{H}^*$ and the character of Z determine one rational $\mathbf{L}^*$-class. Moreover, for $\mathrm{U}(6)$, the nontrivial character of Z selects the union on the α-line.*

Proof. The involution z is half spin on D_6, since its odd coordinates contain node 7 while the kernel k on the natural module V has support at the D_6 fork ends. Thus, a representative x with commutator z acts on natural matrices by $g \mapsto -g$. If $\{u, u^{-1}\}$ is a reciprocal pair distinct from $\{1\}, \{-1\}, \{i, -i\}$, $\{u, u^{-1}\} \neq \{-u, -u^{-1}\}$. The nonidentity component w satisfies $w^2 \in W_s^\circ$, which fixes the scalar lines $\mathbb{Q}\nu_+, \mathbb{Q}\nu_-$ of the two linear factors. Hence, $\nu_+ + w\nu_+ \neq 0$ is fixed by w, contrary to Lemma 7.126. If $\dim V_\pm = 2a = 2$, the two scalar lines of SO_2 give the same contradiction. Thus, with $6 = 2a + c$, sign spaces occur only at $(a, c) = (3, 0), (2, 2)$. If $a = 0$, all six plane coordinates are $\pm i$.

Let $H = (\mathbf{L}_{\mathrm{der}})^F$ and $Z = (Z(\mathbf{L})^\circ)^F$ in the primal Levi $\mathbf{L}$. In the map $\mathbf{L}^* \to \mathbf{H}^*$, the kernel $Z(\mathbf{L}^*)^\circ$ is a torus, so conjugators in $\mathbf{H}^{*F^*}$ lift to $\mathbf{L}^{*F^*}$. If the projections to $\mathbf{H}^*$ and the character on Z agree, the discrepancy lies in $Z(\mathbf{L}^*)^\circ \cap (\mathbf{L}^*)_{\mathrm{der}}$. The adjoint E_7 Levi has connected centre, and $(\mathbf{L}^*)_{\mathrm{der}}$ is the quotient of Spin_{12} by the half spin involution z. Consequently, the intersection has order 2, with generator the image of k. In the presence of V_+ and V_-,

we choose $v_+ \in V_+$ and $v_- \in V_-$ in opposite sign planes. The product $s_{v_+}s_{v_-}$ has determinant 1 and normalizes the plane tori $\mathbf{T}_\pm$. Its lift is proportional to v_+v_- by [137, §24.i]. Conjugation by a lift g_0 of g fixes v_+ and multiplies v_- by -1. Thus, $[g_0, v_+v_-] = k$. The element of W fixes $Z(\mathbf{L}^*)^\circ$, and has a representative in $N_{\mathbf{L}^*}(\mathbf{T}^*)^{F^*}$. Consequently, the two translates by k are rationally conjugate in $\mathbf{L}^*$, giving one class for I(3, 3) and Q(2, 2).

Given U(6), the two SO_{12}-orbits have distinct parities of $|\{j : g_j = -i\}|$ on the six coordinates $g_j = \pm i$. Moreover, negation of all six g_j fixes the parity. The centralizer $C_{\mathrm{SO}_{12}}(g)$ is a connected linear group C. Its preimage in Spin_{12} contains the maximal torus $\mathbf{T}^*_{\mathrm{sc}}$ and is connected. Thus, for $xgx^{-1} = -g$, the commutator $[x_0, g_0]$ depends only on the orbit of g. The diagram involution changes $|\{j : g_j = -i\}|$ mod 2 and acts by $z \mapsto kz$ on the half spin involution. Hence, z occurs in at most one SO_{12}-orbit of g.

For this U(6)-orbit, we choose the full rank A_7 torus model of [208, §11]. Let $\alpha = e_1 - e_2$ be the root perpendicular to D_6. Next, we let b be the projective element represented by ζI_8 with $\zeta^4 = -1$. Then, $s(v) = b\alpha^\vee(v)$ for $v^{q+1} = 1$. The Weyl element $-Is_\alpha$ fixes $s(v)$ and acts on lifts by $s_0(v) \mapsto zs_0(v)$. Given $v \neq \pm 1$, the six equal coordinates give A_5, the two distinguished coordinates have ratio $v^2 \neq 1$, and the four-coordinate roots β give $\beta(s(v)) \in \{-1, -v, -v^{-1}\}$ up to inversion. Thus, for $\beta \notin A_5$ $\beta(s(v)) \neq 1$. The component quotient $W_{s(v)}/W(A_5)$ has order at most 2, so $W_{s(v)} = W(A_5) \times \langle -Is_\alpha \rangle$, with fixed line spanned by α. Conversely, every $w \in W$ has a rational lift that normalizes $\mathbf{T}^*$. Conjugators in $\mathbf{H}^*$ lift to $\mathbf{L}^*$ by Lang's theorem. Lifts of s_H differ in $Z(\mathbf{L}^*)^\circ$ with primitive cocharacter $\alpha^\vee$, since some root pairs to 1 with $\alpha^\vee$. The character of Z corresponds to $\alpha(s(v)) = v^2$. Consequently, $v^2 \neq 1$ gives the proper minimal Levi D_6, whereas for $v = \pm 1$ the neutral subsystem exceeds A_5. When $v, -v$ give a fixed character of Z, the union includes both $s(v), s(-v)$. □

Proposition 7.128. *Let f be odd and $\mathbf{L}^*$ have type D_5A_1 or D_4A_1 as in* Lemma 7.126. *Fix s_H up to rational conjugacy on $\mathbf{H}^*$ and a character of Z. If one lift s has minimal Levi $\mathbf{L}^*$, each lift of (s_H, κ), with κ the character of Z, has minimal Levi $\mathbf{L}^*$. On the orthogonal spaces $V_\pm$ and the quadratic weight θ_2, the comparisons give $(\chi, \psi) \succeq_c$ on $N_G(P)$ by induction from $K, N_K(P)$.*

Proof. After conjugacy in $\mathbf{H}^*$, the two projections equal s_H, so the lifts differ by $c \in Z((\mathbf{L}^*)_{\mathrm{der}})$. The cover of $(\mathbf{L}^*)_{\mathrm{der}}$ has kernel $\langle (k, -I) \rangle$. Thus, a representative of c belongs to $Z(\mathrm{Spin}_{2j})$ alone. The sign on SL_2 is absorbed by multiplication by $(k, -I)$. Moreover, translation by c fixes the set $\{\alpha \in \Phi(\mathbf{L}^*) : \alpha(s) = 1\}$ and the component action in $\mathbf{L}^*$. We check the other roots of $\mathbf{G}^*$.

Embed D_5A_1 in D_6A_1. Next, in the natural D_6 coordinates, take signs $g_1, \ldots, g_5$ on the D_5-factor and t on the sixth plane. On the A_1-factor, choose u, u^{-1} with $u^2 = -1$. Moreover, both signs occur among the g_i. A central translation multiplies the five g_i by one sign and fixes t, u. The other D_6-roots have values $t^{\pm 1}g_i^{\pm 1}$, so one has value 1 exactly when $t = \pm 1$. The half spin weights of D_6 times A_1-weights have squared values $(\prod_i g_i)t^{\pm 1}u^{\pm 2} = \pm t^{\pm 1}$. Thus, their neutrality also gives $t = \pm 1$. Consequently, absence of neutral roots beyond D_5A_1 is independent of the lift s.

Moreover, for D_4A_1, the perpendicular system to D_4 is A_1^3. One line belongs to $\mathbf{L}^*$, and the other two span $Z(\mathbf{L}^*)^\circ$. Also, in each nonzero coset of $D_4^\vee/D_4$, the roots project to eight minuscule D_4-weights and weights on two A_1-lines. The squared norm is $1 + \frac{1}{2} + \frac{1}{2} = 2$. The vector weights have both values $1, -1$ on g. Next, in a half spin weight β, change signs in positions i, j with $g_i = 1$, $g_j = -1$. Moreover, the new weight β' has the parity of β and satisfies $\beta'(g) = -\beta(g)$. Thus, the value set S_λ for each minuscule system λ satisfies $S_\lambda = -S_\lambda$. The translate by c multiplies each value in S_λ by a common sign ϵ_λ and fixes the A_1-coordinates. Hence, in each nonzero coset λ, and on the other two A_1-lines, a root has value 1 on s exactly when one has value 1 on cs.

The translate c lifts to the connected centre of $\mathbf{L}^*_{\mathrm{sc}}$, so the commutator $[x, s_0] = z$ is the same for s, cs. The component group of $C_{\mathbf{G}^*}(s)$ has order at most 2. Since each root with $\alpha(s) = 1$ lies in $\Phi(\mathbf{L}^*)$, $W_s = W_s^\circ \cup xW_s^\circ$, writing x also for its Weyl action. Consequently, W_s, its fixed torus $\mathbf{A}$ and the minimal Levi $\mathbf{L}^*$ are constant on the lifts of (s_H, κ).

Five negative planes give $\mathrm{SO}^-_{10}(q)$, with ranks $(1, 4)$ or $(2, 3)$ up to swapping V_+, V_-. Four negative planes give $\mathrm{SO}^+_8(q)$, with ranks $(1, 3)$ or $(2, 2)$. For $(2, 2)$, both $V_\pm$ are split of dimension 4, with trivial marking cocycle in each $W(D_2)$ since each inverts two planes in $V_\pm$. Consequently, the comparison on the inverted D_4 torus follows from Proposition 6.114. For $(1, 3)$, Proposition 6.120 gives the comparison for (V_+, V_-) with the marked A_1-line and its rational orientations. The A_1 source is the quadratic weight θ_2 on the nonsplit torus $T_2 \leq \mathrm{SL}_2(q)$. The two exceptional characters over θ_2 match its two linear extensions to $N_{\mathrm{SL}_2(q)}(T_2)$ by [186, Theorem 4.2 and its proof]. The field size $q = 9$ cannot have $\ell > 7$ dividing $q + 1$.

Since D_j, A_1 are nonisomorphic, choose $\mathcal{P}_D \otimes \mathcal{P}_2$ on the induced automorphisms of H. We compose with the map $E \to \mathrm{Aut}(H)$ from the external group E as in [141, §§2.5, 2.9]. The odd part $(K/H)_{2'}$ comes from $Z(\mathbf{L})^\circ$ by an isogeny f_Z with 2-primary kernel. By Proposition 7.6, we obtain $(\chi, \psi) \succeq_c$ on $K, L = N_K(P)$. The common quotient of K, L imposes the relations on $V_\pm$ with their chosen orientations and on Z for (χ, ψ). The orbit on H and the scalar on Z agree for χ, ψ. Thus, all lifts of (s_H, κ) have the comparison. Any ambient conjugacy $s \mapsto s'$ acts by $\mathbf{L}^*(s) \mapsto \mathbf{L}^*(s')$. Since $C_{\mathbf{G}^*}(s) \le \mathbf{L}^*$, both inductions are indexed by $N_{\mathbf{G}^*}(\mathbf{L}^*)^{F^*}$-orbits on the lifts of (s_H, κ). □

In Proposition 7.125, the odd part y selects $\mathbf{K}^* = C_{\mathbf{G}^*}(y)$ in place of the fixed torus $\mathbf{A}$. Given D_6, D_5A_1, D_4A_1, the central isogeny $Z(\mathbf{K})^\circ \to \mathbf{K}/\mathbf{K}_{\mathrm{der}}$ has 2-primary kernel. Thus, $Z_o = Z_{2'}$ maps isomorphically to $(K/H)_{2'}$. We have that $K = Z_o \times K_2$ for the preimage K_2 of $(K/H)_2$, with Z_o, K_2 invariant under the induced automorphisms of K. Also, $P = (P \cap H) \times (Z_o)_\ell$. Given a fixed projected involution t_H, the odd part y is central and is detected by $\kappa|_{Z_o}$. Since $(q+1)_2 = 2$ for square q, we deduce that the condition $\kappa|_{Z_o} = 1$ selects exactly the lifts t' of order 2. By Proposition 7.6, both models agree on Z_o. Multiplication by κ_y recovers the series of yt' by Lemma 7.7.

In type D_6, the natural lift g either squares to 1 and has eigenspaces $V_\pm$ of dimension 6, or squares to $-I$ with a centralizer of type U(6). If $g^2 = 1$, we choose the symbols $\Lambda_\pm$ of rank 3 and their rational action on $V_\pm$. If $g^2 = -I$, we choose the series on α from Proposition 7.127. Moreover, for D_5A_1, D_4A_1, the sources are θ_2 on T_2 and the models on $V_\pm$. A rational Weyl correction from y to y^j is unique modulo W_y. It pairs the Levi unions at y, y^j. Since $C_{\mathbf{G}^*}(yt') = C_{\mathbf{K}^*}(t')$, the two inductions keep distinct W_y-orbits for fixed y. The models on $T_2, V_\pm$ give the rational series $\mathcal{E}(K, yt')$ and mixed functions $\mu_{b,\sigma}$.

Let $q = r^f$ be odd with f odd and $\ell > 7$ dividing $q+1$. In Lemma 7.126, suppose that $C_{\mathbf{G}^*}(s)$ is disconnected and $\mathbf{L}^*(s)$ is a proper Levi with only A-factors. On a factor where z maps to 1, the nonidentity component fixes each eigenspace V_a of the natural matrix g. If several V_a occur, the scalar torus on $\bigoplus_a V_a$ has a nonzero trace zero vector v fixed by W_s. Hence, the projection of s is central. On a factor where z projects to -1, the eigenspaces occur in pairs V_a, V_{-a} with $\dim V_a = \dim V_{-a}$. Given two pairs $V_{\pm a}, V_{\pm b}$, we choose scalars c_a, c_b giving a nonzero trace zero vector v fixed by W_s. Thus, each factor with projection -1 has one pair V_a, V_{-a}. The diagonal torus $\mathbf{T}_i$ has unitary marking, and $H = \prod_i \mathrm{SU}_{n_i}(q)$ equals $\mathbf{H}^F$ for the simply connected derived group $\mathbf{H}$.

Proposition 7.129. *Specifically, the unitary partition maps Ω_i give a bijection Ω between the indicated subsets of $\mathrm{Irr}(L)$ and $\mathrm{Irr}(N_L)$, equivariant for the marked actions on L, N_L, with $N_L = \mathrm{N}_L(\mathbf{T})$. The matched pairs have the corresponding W-orbits of torus weights θ and equal scalars on $Z(L)$. Moreover, induction to G and $N_G(P)$ gives matched regular parents $\widetilde{\chi}, \widetilde{\psi}$ with equal quotient twists $\mu_{b,\sigma}$.*

Proof. In $\mathrm{GU}_n(q)$, [133, Proposition 23.2] gives a rational lift $\widetilde{s}$ on $\widetilde{\mathbf{T}}_i$ through the connected torus kernel $\mathbf{Z}$. Thus, its eigenvalues lie in μ_{q+1}. For central $\widetilde{s}$, the character is $(\delta \circ \det)\rho$ with ρ unipotent. Both $\rho|_{\mathrm{SU}_n(q)}$ and the local restriction $\rho_N|_{N_{\mathrm{SU}_n(q)}}$ are irreducible. Given ρ, by [156, Theorem B and §4.4], we obtain the equivariant correspondence $\rho \leftrightarrow \rho_N$. On a factor with eigenvalues $a, -a$, a determinant twist gives weights $1, \eta$, with η quadratic. Moreover, its dimension $2b$ is at most 6: the rank $2b-1$ is odd and at most 6. Since $\mathrm{ord}_\ell(-q) = 1$, the residue is $\kappa = \varnothing$ in Proposition 6.109. The quotient labels are $\lambda, \nu \vdash b$. Moreover, the two Weyl factors S_b have rational Specht models V_λ, V_ν. A nonidentity self twist equals $\eta \circ \det$, which exchanges $1, \eta$. The parent is fixed precisely when $\lambda = \nu$. Then, the two block spaces V_1, V_η have exchange determinant $(-1)^b$. The quadratic intertwiner J of Proposition 6.109 pairs the eigenspaces $\ker(J-1), \ker(J+1)$ on both models, equivariantly for ϕ, the diagonal group and $\mathcal{H}_\ell$. If $\lambda \ne \nu$, the restrictions to $\mathrm{SU}_{2b}(q)$ and its diagonal normalizer $N_{\mathrm{SU}_{2b}(q)}$ are irreducible. We twist both by $\delta_a \circ \det$ to recover the weights of $a, -a$. In $\mathrm{GU}_{2b}(q)$, the field relation is at power $2f$, and the comparison includes $g \mapsto (g^{-1})^{\mathsf{T}}$. Next, on isomorphic factors U_i, we choose identical models V_i and permute tensor positions. All labels λ, ν are eligible since $\ell > 7$.

Let $\mathbf{L}$ be the primal Levi and $\mathbf{R}$ its preimage in the regular inclusion $(\mathbf{G}_{\mathrm{sc}} \times \mathbb{G}_m)/\langle (z, -1) \rangle$. Put $\mathbf{U} = \prod_i \mathrm{GL}_{n_i}$ with unitary Frobenius F_i on each factor. The centre $Z(\mathbf{R})$ is connected since $Q(\mathbf{L})$ is primitive in $Q(\mathbf{G})$, and $Q(\mathbf{G})$ is primitive in $X(\mathbf{T}_{\mathrm{reg}})$. Also, both $\mathbf{R}, \mathbf{U}$ have adjoint quotient $\mathbf{D}$ and simply connected derived group $\mathbf{H}$. Next, we write R, U, D, H for the rational points and $N_X = \mathrm{N}_X(\mathbf{T}_X)$ for the marked torus normalizer in X.

On a monomial block B_i with constant weight λ_0, the extension is $g \mapsto \lambda_0(\det g)$. On the equal weight blocks B_i, we multiply the extensions $\lambda_i \circ \det$, tensor by $\rho_{\lambda,\nu}$, and induce from the inertia $(N_U)_\theta$. Thus, a permutation matrix π in a block contributes $\lambda_0(\det \pi)$. A weight twist by $\delta \circ \det$ multiplies the inertia extension by $\delta \circ \det$. Moreover, the field ϕ and the inverse transpose act on $\det g$ and $(\lambda_i)_i$. Factor permutations act on $\bigotimes_i V_i$ by permutations. On the

parent ρ, the actions on $J_{\widetilde{s}}(\rho)$ are those for $Z(\mathbf{U})$ connected, with rational labels λ_i. Moreover, the parent map $\widetilde{\Omega}$ is from [156, Theorem B and §4.4]. On the constituents in H, the equivariant pairing is from [156, Theorems 3.4, 3.6 and Corollary C].

Form $\mathbf{B} = \mathbf{R} \times_{\mathbf{D}} \mathbf{U}$. Moreover, the projection kernels $\ker(\mathbf{B} \to \mathbf{R})$ and $\ker(\mathbf{B} \to \mathbf{U})$ are connected central tori $\mathbf{Z}_R, \mathbf{Z}_U$. The diagonal image of $\mathbf{H}$ is common to both maps, and $\mathbf{B}/\mathbf{H}$ is a torus. By [133, Proposition 23.2], $B \to R$ and $B \to U$ are onto. Both tori have preimage $\mathbf{T}_B$, so N_B maps onto N_R, N_U. Let $\xi \in \mathrm{Irr}(R)$ be a regular parent, inflated to B. Its scalar on $C_U = \ker(B \to U)$ extends to $\lambda \in \mathrm{Irr}(B/H)$, since $C_U \cap H = 1$ and B/H is abelian. Then, $\xi\lambda^{-1}$ is the inflation of $\rho \in \mathrm{Irr}(U)$. The choices of λ differ by characters of U/H.

Let ρ_N be the local parent of ρ. Next, on N_B, we put $\xi_N = \rho_N\lambda|_{N_B}$, with ρ_N inflated from N_U. On $C_R = \ker(B \to R)$, the characters ρ, ρ_N have the same scalar, since C_R maps to central scalars of U. Thus, ξ_N descends to N_R. Changing λ by a character of U/H cancels by the rule for $\delta \circ \det$. Since $\mathbf{T}_B \cap \mathbf{H}$ is a maximal torus of $\mathbf{H}$, [133, Proposition 23.2] gives $\mathbf{T}_B^F \twoheadrightarrow B/H$. Thus, N_B maps onto B/H. Thus, extension of the scalar on C_U and the inverse map on N_U recover ξ from ξ_N. Inflation along a central quotient π gives the dual map π^*. The λ-twist multiplies the dual point $\widetilde{s}$ by the central element z_λ. The weight on N_B changes by the same π^*, z_λ, so the full rational orbit of θ agrees, including the weight on $Z(\mathbf{R})^F$. A transporter a of $\mathbf{L}$ acts on $\mathbf{R}$. On $\mathbf{U}$, it lifts by permutations of U_i and graph, field and diagonal maps e_i. Since $Z(\mathbf{U})$ is connected, the diagonal lift $d \in U$ is rational. The actions agree on $\mathbf{D}$, and any change of λ cancels by the rule for $\delta \circ \det$.

The quotient $N_{U_i}/N_{\mathrm{SU}_{n_i}(q)}$ embeds in the cyclic group $U_i/\mathrm{SU}_{n_i}(q)$. By the quotient formula, factors of ρ, ρ_N restrict with multiplicity 1. Thus, the tensor products give multiplicity 1 for $\xi|_H, \xi_N|_{N_H}$. By [95, Theorem 6.2], the constituents on L, N_L partition the labels on H, N_H into orbits. Moreover, $\mathbf{T}_B^F \twoheadrightarrow B/H$ gives the same image for N_L, L in L/H. Consequently, the constituent maps on H, N_H give the orbit correspondence on L, N_L. Two parents above a constituent of L differ by a twist from R/L. Thus, restriction to L, N_L is independent of the parent ξ, and the inverse on H, N_H gives the inverse on L, N_L.

The centralizer of the dual lift $\widetilde{s}$ lies in the regular preimage of $\mathbf{L}^*(s)$. Signed induction from L is compatible with regular restriction to G. Since T maps onto $\widetilde{G}/G$, [95, Problem 5.2] gives $\widetilde{\psi}|_{N_G(P)} = \mathrm{Ind}_{N_L}^{N_G(P)}(\xi_N|_{N_L})$. The inductions from L, N_L are irreducible on the series of s and the weight orbit of θ. Any conjugacy $s \mapsto s'$ acts by $\mathbf{L}^*(s) \mapsto \mathbf{L}^*(s')$, with the Weyl corrections on θ acting on the local labels η. Thus, induction gives the matched parents $\widetilde{\chi}, \widetilde{\psi}$ and constituents χ, ψ. Since $\ell > 7$, the relative orders $|S_b|$ and the indices in W are prime to ℓ. □

Proposition 7.130. *The ambient pairs* (χ, ψ) *of* Proposition 7.129 *satisfy* $(\chi, \psi) \succeq_c$ *on* $N_G(P)$.

Proof. Complex conjugation c permutes the constituents of $\widetilde{\chi}|_G$. Indeed, it changes the weight θ to θ^{-1}. The rational Weyl element $-I$ restores the projection of θ and the partitions $(\lambda_i)_i$, since opposition on B_i lies in its Weyl group S_{n_i}. The two weights can differ by $\lambda \in \mathrm{Irr}(R/L)$. On a block B_i with weight λ_0, the extension $\lambda_0 \circ \det$ acquires the twist by λ, including on $\pi \mapsto \lambda_0(\det \pi)$. Thus, c also permutes the constituents of $\widetilde{\psi}|_M$.

Let D be the Sylow 2-subgroup of the cyclic quotient $\widetilde{G}/G$, and G_0 its preimage. The image of G_0 in $\mathrm{Aut}(G)$ includes the diagonal outer group C_2. Next, put $A = G_0 \rtimes \langle\phi\rangle$, $B = \mathrm{N}_A(P)$, and $M = \mathrm{N}_G(P)$ with pinned field ϕ of odd order f. By [97, Lemma 1.13], $B/M \simeq A/G$. Since $\langle\phi\rangle$ acts trivially on D, $A/G = D \times C_f$. Take $d_0 \in B$ above a generator of D with trivial field component. The central 2-subgroup Z_2 of G_0 is fixed pointwise by ϕ and maps onto D^2, since on the scalar torus T the map is $t \mapsto t^2$. Thus, $d_0^2 = z_2 g$ with $z_2 \in Z_2$ and $g \in M$.

Suppose that $\widetilde{\chi}|_G = \chi_+ + \chi_-$, permuted by d_0. Their setwise stabilizer modulo G is $D \times C_m$ with $C_m \leq C_f$, and C_m fixes $\chi_\pm$ since m is odd. Consequently, for χ_+ and ψ_+ the inertia quotient is $D^2 \times C_m$. Let A_m, B_m be the preimages of C_m in A, B, respectively. Then, the two inertias are Z_2A_m, Z_2B_m. On each side, choose $e \in \{0, 1\}$ such that $j = \mathrm{Int}(d_0)^e c$ fixes the constituent. Conjugation by d_0 acts trivially on C_m. With m odd, [208, Lemma 3.1] gives the unique j-fixed extensions on A_m, B_m. The coefficient actions satisfy $c\sigma = \sigma c$, and $[b, d_0] \in G$ respectively M, acts by an inner automorphism on A_m, respectively B_m. Thus, each (b, σ) in the mixed stabilizer satisfies $(b, \sigma)j = j(b, \sigma)$ on the extensions. Hence, (b, σ) fixes the unique j-fixed extension. The exponent e may differ between χ_+ and ψ_+. Normalize by the action of c, independently of membership in $\mathcal{H}_\ell$.

Next, extend to Z_2A_m, Z_2B_m by a common linear character ω of Z_2 extending the scalar on $Z_2 \cap G$. Indeed, Z_2 is central and $Z_2 \cap G = Z_2 \cap A_m = Z_2 \cap B_m \leq M$. Thus, the ordinary extensions on Z_2A_m, Z_2B_m have twist 1 on C_m. On the quotient of Z_2, both twists are $\omega^{b\sigma}/\omega$. The scalars of $C_A(G)$ in G_0 agree for χ, ψ. If $\widetilde{\chi}|_G = \chi$, then $\chi = \bar{\chi}$ and $\psi = \bar{\psi}$ by setwise invariance under c. The regular parents extend χ, ψ. At (b, σ), their twists $\mu_{b,\sigma}^{\chi}, \mu_{b,\sigma}^{\psi}$ agree by

Proposition 7.129. Since χ, ψ are irreducible, $\mu_{b,\sigma}$ is unique. On the preimage G_0 of D, the field inertia C_m acts by translations in $\mathrm{Irr}(D)$, so C_m fixes the extensions $\widehat{\chi}, \widehat{\psi}$. Extend across C_m with the given restriction to G_0 or $G_0 \cap B$. On A_m, B_m, we choose the unique real extensions $\widehat{\chi}, \widehat{\psi}$ by [143, Lemma 2.1]. Since $D \times C_m$ is direct, the twist on C_m fixes the restriction to G_0 or $G_0 \cap B$. For each mixed stabilizer element (b, σ), the parent twists on D agree, and both twists on C_m are 1 by uniqueness. Thus, the functions $\mu_{b,\sigma}$ agree on $D \times C_m$, and $(\chi, \psi) \succeq_c$ on the chosen cover by [141, §2.9]. □

Lemma 7.131. *With $\ell = 5 \mid q+1$ and q odd, we choose P over $Q = \langle (12345) \rangle$ in the full rank A_7. Then, we find $Q \in \mathrm{Syl}_5(W)$. By* Lemma 7.119*, the torus kernel of $N_G(P)$ is $T_* = T_5 C_{T_{5'}}(Q)$, and $\theta \mapsto \theta|_{T_*}$ is injective on $\mathrm{Irr}(T)^Q$. Suppose that $\mathcal{E}(G, s)$ contains an irreducible character of $5'$-degree and that $C_{\mathbf{G}^*}(s)$ is disconnected. Then, s is rationally Weyl conjugate to $b\alpha^\vee(v)$ on the line of* Proposition 7.127*, with $v^{q+1} = 1$. If $v = \pm 1$, the elements s are involutions. If $v \neq \pm 1$, we have that the minimal Levi $\mathbf{L}^*(s)$ is the marked D_6, and the projection has type* U(6).

Proof. By the degree formula, W_s contains a conjugate of Q. Conjugate s so that $Q \leq W_s$. The component quotient W_s/W_s° has order at most 2 by the commutator $[\dot{w}, s_0]$ in $E_{7,\mathrm{sc}}$. Consequently, for $wW_s^\circ \neq W_s^\circ$, we choose w with $Q^w = Q$ by Sylow conjugacy in W_s°. Take a lift of s to the diagonal torus of SL_8. The A_7 lattice indices give kernels μ_4 and μ_2 for the maps to $E_{7,\mathrm{ad}}$ and $E_{7,\mathrm{sc}}$, respectively. Since Q has order 5, the lift s_0 changes by a scalar in μ_4 of order dividing 5. Hence, Q fixes s_0. Thus, the first five entries equal a, and the entries are $(a, a, a, a, a, ax, ay, az)$, and put $A = a^4$. Then, $xyz = A^{-2}$, and A, x, y, z are invariant under μ_4 and belong to μ_{q+1}.

The moving space of Q has roots A_4. The perpendicular roots in E_7 are $e_i - e_j$ for $i, j \in \{6, 7, 8\}$. Indeed, the root β_J for a four subset J has both coefficient signs on $e_1, \ldots, e_5$. The automorphism group of A_4 is $S_5 \times \langle -I_{A_4} \rangle$, and $N_{S_5}(Q) = \mathrm{AGL}_1(5)$. On the first five coordinates, S_5 and ambient $-I$ induce these actions of $N_W(Q)$. The pointwise stabilizer of A_4 is generated by s_α for perpendicular roots $\alpha \in A_2$, so it is S_3 on $6, 7, 8$. Thus, $N_W(Q)$ acts by $\mathrm{AGL}_1(5)$ on $1, \ldots, 5$, by S_3 on $6, 7, 8$ and by ambient inversion $-I$.

If $w \in N_W(Q)$ fixes s with no inversion, w fixes the entries a on $1, \ldots, 5$. Thus, the scalar discrepancy is 1. With inversion $-I$, the scalar discrepancy is a^{-2}. It acts nontrivially modulo μ_2 precisely when $(a^{-2})^2 = A^{-1} = -1$. Then, $xyz = 1$ and $\{x, y, z\} = \{x^{-1}, y^{-1}, z^{-1}\}$ as multisets. Thus, the triple is $\{1, v, v^{-1}\}$, allowing repetitions. Consequently, $s = b\alpha^\vee(v)$. The action of $-Is_\alpha$ and the neutral roots A_5 give the endpoints and the minimal D_6 for $v \neq \pm 1$ by Proposition 7.127. □

Proposition 7.132. *Let $X = \mathrm{Spin}^+_{12}(q)$ be the derived factor for $\alpha^\perp$, with $P_X = P \cap X$. In the marked* U(6) *series of* Lemma 6.36*, the lines λ_j, λ_{-j} give a comparison on $\mathrm{N}_X(P_X)$ with the projected weight θ and the original mixed functions $\mu_{a,\sigma}$.*

Proof. First, put $B = N_X(\mathbf{T}_X)$ and $B' = N_X(P_X)$. Next, fix the projected quadratic weight θ at b. In B, its Weyl inertia is $S_6 \times \langle -I_D \rangle$, with $-I_D$ acting by opposition on the A_5 space. In B', the inertia of $\theta' = \theta|_{B' \cap T_X}$ has Weyl quotient $\mathrm{AGL}_1(5) \times \langle -I_D \rangle$. Indeed, for a five cycle $Q \leq S_6$, the normalizer $N_{S_6}(Q)$ acts affinely on $\{1, \ldots, 5\}$ and fixes 6. The Tits preimage $\widehat{Q}$ of Q has a 2-group kernel J_2, which is transitive on its complements Q_0. Hence, each $w \in N_{S_6}(Q)$ has a lift that normalizes Q_0. The longest D_6 lift n_D centralizes the A_5 Tits group and lies in $B_\theta \cap B'_{\theta'}$. By Lemma 7.119, $\theta \mapsto \theta'$ is injective on Q-fixed weights. Thus, $B'_{\theta'} = B' \cap B_\theta$.

Let $J_0 \leq B_\theta$ be the inertia group above S_6, and $J_1 \leq J_0$ the neutral Tits image in J_0. On the lifts in J_1, take scalar 1, since θ is trivial on the A_5 coroot torus. The lift n_D normalizes J_0 and fixes the extension, since $\theta^2 = 1$ and n_D normalizes J_1. Choose $j^2 = \theta(n_D^2)$. The line on J_0 has two extensions λ_j, λ_{-j} to B_θ, with $\lambda_{\pm j}(n_D) = \pm j$, because B_θ/J_0 has order 2. Their restrictions to $B'_{\theta'}$ are distinct at n_D. For each corrected action $a\sigma$ the permutation of $\{\lambda_j, \lambda_{-j}\}$ agrees after restriction. Indeed, $\lambda_{\pm j}^{a\sigma}$ is evaluated twice at n_D, with scalar 1 on J_1.

On B_θ, tensor λ_j by an S_6-character. The $5'$-degree labels $\lambda \vdash 6$ have 5-core (1) and 5-weight 1, so the 5-quotient has five runners i. Given λ_j, λ_{-j}, we apply [30, Proposition 2.3] with the same map Ω. The labels of $\mathrm{AGL}_1(5)$ are the four linears from $\mathbb{F}_5^\times$ and the augmentation of $\mathbb{Q}[\mathbb{F}_5]$. Moreover, the linears have values in μ_4 and the augmentation η'_0 is rational. Thus, all labels η' is $\mathcal{H}_5$-fixed.

The corrected transporter of θ acts on A_5 through $S_6 \times \langle -I_{A_5} \rangle$. The projection π to S_6 is a homomorphism. Indeed, $\pi(r^j) = 1$, and the field relation gives $-I_{A_5} \in \ker \pi$. On the transporter of P_X, the image is in $N_{S_6}(Q)$ since $\langle -I_{A_5} \rangle$ is central. Composition with π extends the S_6 and affine models E, E' to the respective transporters of θ, θ' as ordinary representations. Moreover, at (a, σ), the characters of E, E' are fixed, and elements centralizing X act by 1.

Let V be the projective model above B from the marked U(6) comparison. On the θ-space V_θ, the group J_1 acts through a single E-summand. Thus, $L_j = \mathrm{Hom}_{J_1}(E, V_\theta)$ is a line with B_θ-action λ_j. On the upper inertia of V, the line L_j has its factor set κ, since the extension of E is genuine. The mixed functions of E, V satisfy $\mu^E_{a,\sigma} = \mu^V_{a,\sigma}$, since the extension of E has invariant character under (a, σ).

Restrict L_j to the transporter above B', tensor by E' and induce with factor set κ in the twisted group algebra. On B' this gives the irreducible Clifford induction $\mathrm{Ind}_{B'_{\theta'}}^{B'}(\lambda_j|_{B'_{\theta'}} \otimes E')$. The normalized functions $\mu_{a,\sigma}$ coincide by [141, Lemma 1.4]: the Hom space with E, restriction and induction carry the scalar c to the same scalar, and E, E' are fixed by (a, σ). For a Weyl correction above B, Sylow conjugacy in B_θ gives one that normalizes Q_0 and Q. Conversely, the restricted lines distinguish $j, -j$, and the labels of E' are fixed at (a, σ). An element centralizing X acts by the scalar on L_j. Thus, the comparison on B' includes stabilizer elements exchanging $j, -j$. □

Corollary 7.133. *Suppose that $q = r^f$ is odd, f is odd and $5 \mid q + 1$. Given $s(v) = b\alpha^\vee(v)$ with $v^2 \neq 1$, the ten characters of $\mathcal{E}(G, s(v)) \cap \mathrm{Irr}_{5'}(G)$ have the mixed comparison on $M = \mathrm{N}_G(P)$.*

Proof. By Lemma 7.131, $C_{\mathbf{G}^*}(s(v))$ lies in the Levi $\mathbf{K}^*$ perpendicular to α. The multiset at $A = -1$ on the Q-fixed torus is $\{1, v, v^{-1}\}$. Thus, $s(v')$ is conjugate to $s(v)$ exactly when $v' = v^{\pm 1}$, including when $|v|$ is a power of 2. The root reflection s_α maps $v \mapsto v^{-1}$, and its representative n_α centralizes $X = (\mathbf{K}_{\mathrm{der}})^F = \mathrm{Spin}^+_{12}(q)$. Put $n = n_D n_\alpha$ for the negative torus marking, with n_D the longest pinned representative in X. Moreover, both factors are defined over $\mathbb{F}_r$ and $[n_D, n_\alpha] = 1$. The marked prime field lift U satisfies $U^f = n^{-1}$. Since n represents $-I$, $n^2 = z$, the central involution. Replacing n by tn for $t \in T$ gives the same square because $t^n = t^{-1}$. With $V = nU$, we put $V_0 = z^{(f-1)/2}V$. Then, $V_0^f = 1$, and the field generator V_0 acts on T by $t \mapsto t^{-r}$. The pinned lifts n_j on the six letters of A_5 are fixed by n_D, n_α. Let $\widehat{Q} = \langle x_5 \rangle$ for a lift x of (12345) in their Tits subgroup. Thus, $V_0, n_\alpha \in N_{G \rtimes \langle V_0 \rangle}(P)$ and $[V_0, n_\alpha] = [V_0, n_D] = [n_\alpha, n_D] = 1$.

In the regular Levi $\widetilde{K}$, each eligible partition λ has two constituents on $K = \mathbf{K}^F$. Its rational component $-I_D$ fixes λ in S_6 and exchanges the lifts of $s(v)$. The regular character $\widetilde{\chi}_\lambda$ restricts to $\xi_+ + \xi_-$ on X by Lemma 7.158. Thus, a constituent on K is indexed by $s(v)$, λ and its constituent $\xi_\pm$ on X. Moreover, the projection of $s(v)$ to the dual of X is independent of v. Lang conjugation untwists n_D in $\mathbf{K}_{\mathrm{der}}$. The field action on X is the pinned field times $\mathrm{Int}(x)$ for some $x \in X$. Thus, Lemma 7.158 fixes each constituent on X under fields and $\mathcal{H}_5$, while the correction n_α acts trivially on X. Signed induction from K gives the ordinary correspondence in $\mathcal{E}(G, s(v))$ and its actions by Corollary 7.3, since $C_{\mathbf{G}^*}(s(v)) \leq \mathbf{K}^*$. By the hook formula at $q \equiv -1 \bmod 5$, the eligible λ have 5-weight one as in Proposition 7.132. Since $|C_{\mathbf{G}^*}(s(v))^{F^*}|_5 = |G|_5$, the series index is prime to 5. Each parent $\widetilde{\chi}_\lambda$ gives two constituents of degree $\widetilde{\chi}_\lambda(1)/2$, so the five eligible λ give ten global characters.

Let θ be the full torus weight for $s(v)$ and I the inertia of $\theta|_{T_*}$ in M. The Weyl quotient $I/T_* = \mathrm{AGL}_1(5) \times \langle -I_D \rangle$ follows from the Q-fixed orbit test and injectivity of $\theta \mapsto \theta|_{T_*}$. On J_1, the extension λ_j is from Lemma 7.144. Its value at n_D is j with $j^2 = \theta(n_D^2)$. Moreover, the lines λ_j, λ_{-j} are those in Proposition 7.132. Indeed, n_D fixes the projection of θ and centralizes the line for α. In particular, $j \in \mu_4$, regardless of the 2-part of $|v|$. Tensor $\lambda_{\pm j}$ with a linear or augmentation label η' of $\mathrm{AGL}_1(5)$. By [43], induction from I gives exactly ten characters of M. The runner map Ω is the same on λ_j, λ_{-j}. Since $\mathcal{H}_5$ fixes μ_4 and both V_0, n_α fix $n_D, \lambda_{\pm j}|_{J_1}, \eta'$, the returning powers act alike on χ, ψ. The outer diagonal h gives $\xi_+ \mapsto \xi_-$ by regular restriction. For $\widetilde{\theta}$, the ratio $\widetilde{\theta}^{n_D}/\widetilde{\theta}$ is the diagonal character of order 2. Thus, the local diagonal acts by $j \mapsto -j$. The scalar on $Z(G)$ is the common weight value $\theta|_{Z(G)}$. With $m = |v| > 2$, we put $k = \mathrm{ord}_m(-r)$. Since $k \mid f$, $-1 \bmod m$ is not in $\langle -r \bmod m \rangle$. By the orbit test for $s(v)$, both ordinary field inertias are $\langle V_0^k \rangle$. Each $V_0^t\sigma$ stabilizing the pair fixes both constituents $\xi_\pm$. Thus, the mixed stabilizer lies in $A_0/Z(G) \times \mathcal{H}_5$ for $A_0 = G \rtimes \langle V_0 \rangle$.

Extend the local line to $I \rtimes \langle V_0^k \rangle$ by $V_0^k \mapsto 1$, since V_0^k fixes $\theta|_{T_*}, \lambda_j|_{J_1}, j$. On E' the value at V_0^k is also 1, since V_0 acts on roots by $-I$, with projection 1 to S_6. Induce their tensor product to $M \rtimes \langle V_0^k \rangle$. Also, its restriction to M is $\psi = \mathrm{Ind}_I^M(\lambda_j \otimes E')$. At (a, σ), the correction lies in $I \cup n_\alpha I$. Both representatives $1, n_\alpha$ fix V_0^k and its operator. Thus, $\widehat{\psi}$ is invariant at (a, σ), also for nonreal η'.

For the global extension $\widehat{\chi}$, we choose the diagonal $h \in \mathrm{Aut}(G)_P$ from Proposition 7.154. The cocharacter for h is averaged by Q and the construction is on $V|_T$. Thus, $hV_0 = V_0h$. Complex conjugation acts by $s(v) \mapsto s(v^{-1})$, which is corrected by n_α. Complex conjugation c stabilizes $\{\xi_+, \xi_-\}$ for each λ, since λ is real and the lift discrepancy ζ gives a twist trivial on K. The rational trace of R_K^G in Corollary 7.3 carries this action to $\mathcal{E}(G, s(v))$. Thus, either c or hc fixes the global constituent χ. Denote that operation by τ. Conjugation by h acts trivially on the odd quotient

$\langle V_0^k\rangle$. By [208, Lemma 3.1], there is a unique τ-fixed extension $\widehat{\chi}$. Given $a = V_0^t\sigma$ in the mixed stabilizer, $a\tau = \tau a$ on characters. Thus, a fixes $\widehat{\chi}$, as does conjugation by G. For the mixed stabilizer $\mathcal{I}$, choose common representatives in $N_{A_0}(P)$ for its projection to $\mathrm{Aut}(G)$. The kernel is $Z(G)$, so each discrepancy z has common scalar $\theta(z)$. On this section the extensions $\widehat{\chi}, \widehat{\psi}$ have equal factors κ and mixed functions $\mu_{a,\sigma}$. Thus, $(\chi, \psi) \succeq_c$. □

Proposition 7.134. *Let $q = r^f$ be odd and $\ell > 7$ divide $q+1$. Next, in* Proposition 7.127*, take $s(v) = b\alpha^\vee(v)$ with $v^{q+1} = 1$ and $v^2 \neq 1$. The series $\mathcal{E}(G, s(v))$ has 22 characters, all of ℓ'-degree. The comparison holds on $M = N_G(P)$ for each pair.*

Proof. By Lemma 7.119, $P = T_\ell$ and $M = N_G(T)$. By Proposition 7.127, $W_{s(v)} = S_6 \times \langle -I_D\rangle$ with fixed line $\mathbb{Q}\alpha$. If $w \in W$ maps $s(v)$ to $s(v')$, then $w\alpha = \pm\alpha$ and $w(b) = b\alpha^\vee(c)$ with $c^2 = 1$. Indeed, $b^2 = 1$ and $\alpha^\vee$ is primitive. For the neutral roots $R_b, R_{b\alpha^\vee(-1)}$, a direct computational verification gives that $|R_b| = 56$ and $|R_{b\alpha^\vee(-1)}| = 32 + 40$. All 56 roots at b are differences. Moreover, at $b\alpha^\vee(-1)$, there are 32 differences and 40 half roots containing one distinguished letter e_1 or e_2. Thus, $c = 1$. Since s_α acts by $v \mapsto v^{-1}$ and n_α centralizes X, the orbit test is $v' = v^{\pm 1}$, including when $|v|$ is a power of 2.

All 11 partitions $\lambda \vdash 6$ occur by the degree formula, since $[W : S_6]$ and the Specht degrees $\eta_\lambda(1)$ are prime to $\ell > 7$. Each parent has two constituents in K, distinguished on X as in Corollary 7.133. Signed induction from K gives the ambient series of $s(v)$. If f is even, then $v_{(2)} = \pm 1$ and v has nontrivial odd part since $v^2 \neq 1$. The root α is neutral on the 2-part of $s(v)$, so for every label λ the comparison is $(\chi, \psi) \succeq_c$ by Proposition 7.159.

Suppose that f is odd. At b, the square of $n_D n_\alpha$ has central value $k_z = (-1)^{(q+1)/2}$ by Lemma 7.152. The coroot of α is neutral at b, so n_α^2 has value 1. Also, we have that $[n_D, n_\alpha] = 1$. Since $\alpha^\vee$ is perpendicular to D_6, the weight on X is independent of v. The two extensions λ_j, λ_{-j} of the neutral line have values $j, -j$ on n_D, with $j^2 = k_z$. Tensor each with the rational S_6 model of λ and induce to M. By [43], the induced characters exhaust $\mathrm{Irr}(M \mid \theta)$. The diagonal exchanges the signs $j, -j$ on both sides by the commutator z with the regular lift.

Take V_0 of order f from Corollary 7.133. Moreover, both V_0, n_α fix the neutral line extensions, centralize n_D and fix λ on the six letters. On X, the field fixes ξ_+, ξ_- by Lemma 7.158. A coefficient exponent a on μ_r has sign (a/r), whereas the local sign is $\sigma(j)/j$. By Section 2, $\sigma(j)/j = (a/r)$. The correction is 1 or n_α modulo the inertia I, so the sign formula holds at (a, σ) with noninner diagonal part h. The ordinary field inertia has order f/k for $k = \mathrm{ord}_{|v|}(-r)$, since $\langle -r \bmod |v|\rangle$ has odd order and excludes $-1 \bmod |v|$. Complex conjugation c stabilizes both constituent pairs, since n_α restores the inverse weight and λ is real. By Proposition 7.130, either $\tau = c$ or $\tau = hc$ fixes the constituent, where h is diagonal. The unique τ-fixed extension is taken across $\langle V_0^k\rangle$. For any (a, σ) stabilizing χ, the commutator with h acts on $\widehat{\chi}$ by inner conjugation, so (a, σ) fixes it. On the diagonal subgroup Z_2, choose a common extension of $\theta|_{Z(G)}$. Thus, the factors κ and functions $\mu_{a,\sigma}$ agree at each (a, σ), including stabilizer elements with diagonal part h. The 11 partitions with signs $j, -j$ give 22 characters over each orbit. □

Lemma 7.135. *Let $G = E_7(q)_{\mathrm{sc}}$, $S = G/Z(G)$, and $D = E_7(q)_{\mathrm{ad}}$, with $q = r^f$ odd. Then, we obtain that $|Z(G)| = [D : S] = 2$, and ϕ is the pinned field automorphism for $\mathbb{F}_r$. By* Zhang [208, Proposition 11.9, Steps 1–2]*, the dual involution t has connected split centralizer of type A_1D_6, its lifts $\widehat{t}, z\widehat{t}$ are geometrically distinct, and each $\chi \in \mathcal{E}(G, t)$ factors through S. For odd $\ell \mid q^2 - 1$, we have $\mathcal{E}(G, t) \cap \mathrm{Irr}_{\ell'}(G) = \varnothing$ exactly when $\ell = 3, 7$. Given any other such ℓ, each eligible character of S extends to $D \rtimes \langle\phi\rangle$ with an $\mathcal{H}_\ell$-fixed extension.*

Proof. By [208, Proposition 11.9, Step 1], $C = C_{\mathbf{G}^*}(t)^{F^*}$ has order $|\mathrm{SL}_2(q) \times \mathrm{Spin}_{12}^+(q)|$. A direct computation gives that $[G^* : C]_{r'}$ equals $\frac{q^{12}-1}{q^4-1}\frac{q^{14}-1}{q^2-1}\frac{q^{18}-1}{q^6-1} = \Phi_3\Phi_6\Phi_{12}\Phi_7\Phi_{14}\Phi_9\Phi_{18}$, with the Φ_i evaluated at q. At $q \equiv \pm 1 \bmod \ell$, the three quotients are $3, 7, 3 \bmod \ell$. Thus, the Jordan index excludes $\ell = 3, 7$. For other ℓ, the Jordan label 1 contributes. If $R = A_1D_6$, the minimal d-split Levi is toral at $d = 1, 2$, since $-I \in W(R)$ at $d = 2$. Moreover, its unipotent source is 1, and [156, §4.4] gives the labels in $\mathrm{Irr}_{\ell'}(W(R))$ for both $d = 1, 2$.

By [208, Proposition 11.9, Steps 2–3], the 74 principal characters in $\mathcal{E}(D, \widehat{t})$ extend rationally to $D \rtimes \langle\phi\rangle$. At $d = 1$, the split torus of C gives its principal series. At even $d = \mathrm{ord}_\ell(q)$, the other 10 characters also have $\mathcal{H}_\ell$-fixed extensions. Their source is $\kappa\psi$ on a split Levi L, with κ a ϕ-invariant quadratic character and ψ inflated from the cuspidal unipotent character ψ_{ad} of the adjoint D_4 quotient L_{ad}, by [208, Proposition 11.9, Step 4]. Since the Frobenius eigenvalue of ψ_{ad} is $\omega = \pm 1$, [156, Lemma 8.1(a)] gives an f-th root of ω in $\mathbb{Q}_\ell$. By [156, Proposition 7.3], ψ_{ad} has an $\mathcal{H}_\ell$-fixed extension to $L_{\mathrm{ad}} \rtimes \langle\phi\rangle$. Inflate it to $L \rtimes \langle\phi\rangle$ and multiply by the rational extension $\widehat{\kappa}$ with $\widehat{\kappa}(\phi) = 1$. By [208, Lemma 4.9 and Proposition 11.9, Step 4], the source extension gives $\mathcal{H}_\ell$-fixed extensions on $D \rtimes \langle\phi\rangle$. □

Proposition 7.136. *For the connected A_1D_6 series at $d = 1, 2$, the comparison is on $N_G(P)$ for $P \in \mathrm{Syl}_\ell(G)$. If $\ell = 5$, the local set has 20 characters. The eligible characters have genuine $\mathcal{H}_\ell$-fixed extensions on $N_D(P_D)\langle U\rangle$, with $U = \phi$ at $d = 1$.*

Proof. By Lemma 7.135, $\ell \neq 3, 7$. In D, we let T be the rational scalar torus and $N = N_D(\mathbf{T})$. The quadratic weight θ of $\widehat{t}$ has stabilizer $W_R = W(A_1D_6)$. Next, we let V_R be the pinned Tits subgroup for R over $\mathbb{F}_r$. Given any neutral root β, torus duality gives $\theta = 1$ on its rational coroot torus T_β. In particular, we have that $\theta(n_\beta^2) = 1$ for each simple lift n_β and its conjugates by W_R. Their squares generate $T \cap V_R$. Thus, $I = TV_R$ is the inertia and $\theta_0(tv) = \theta(t)$ is an extension. Inflate $\eta \in \mathrm{Irr}(W_R)$, tensor by θ_0 and induce to N. Moreover, we obtain the degree $[W : W_R]\eta(1) = 63\eta(1)$.

At the split mark, ϕ fixes V_R, θ, and the extension has ϕ acting by 1. At the other mark let n be the positive Tits lift of $-I$ in D. Moreover, in D the longest lift has $n^2 = 1$, and the braid identity gives $nn_j = n_jn$ for each Tits generator n_j. Thus, V_R is rational for $\mathrm{Int}(n)F_q$. If $g^{-1}F_q(g) = n$, the marked field U corresponds to $gF_r(g)^{-1}\phi$. The multiplier $gF_r(g)^{-1}$ is F_q-fixed. Moreover, its telescoping product gives $U^f = n^{-1}$ in the marked coordinates. Since $n \in I$ has central Weyl image, n^{-1} acts by a scalar $c = \pm 1$ in the model of $\theta_0 \otimes \eta$. Give U scalar $a \in \mu_{2f_2}$ with $a^f = c$. Also, the operators for I, U satisfy the group relations of $I\langle U\rangle$, and the induced extension is $\mathcal{H}_\ell$-fixed. By [133, Theorem 21.7] on $\mathbf{T}_{\mathrm{sc}}$, the boundary $T \to D/S$ is onto. The nontrivial twist from D/S takes $\widehat{t}$ to $z\widehat{t}$, which has a distinct W-orbit. Thus, restriction to $N \cap S$ is irreducible and separates the labels η by [43]. The inflations to G have $Z(G)$ in the kernel. If $\ell > 7$, the Weyl order is prime to ℓ and $N \cap S = N_S(P_S)$, with P_S the image of P.

If $\ell = 5$, set $Q = \langle(12345)\rangle$ in the D_6 Weyl factor. It is Sylow in W since $[W : W_R] = 63$. In V_R, choose an order 5 lift $\widehat{Q}$ across the 2-kernel. Then, $P_D = T_5\widehat{Q}$ is Sylow in D and belongs to S. By Frattini, we have that $N = (N \cap S)N_N(P_D)$, so $N_N(P_D)$ surjects onto D/S. On the scalar mark, $N_S(P_D) \leq N \cap S$ gives $N_D(P_D) = N_N(P_D)$. Moreover, its torus kernel is $T_* = T_5C_{T_{5'}}(Q)$. A correction by the Tits 2-kernel makes each lift of $w \in N_W(Q)$ normalize $\widehat{Q}$. By Lemma 7.119, $\theta \mapsto \theta|_{T_*}$ is injective on Q-fixed weights. Thus, the inertia above $\theta|_{T_*}$ has quotient $N_{W_R}(Q)$, and the D/S-twist orbits remain distinct at $\theta|_{T_*}$. A sign vector in $W(D_6)$ that normalizes Q has constant sign ϵ on $1, \ldots, 5$, and the sixth sign is $\epsilon^5 = \epsilon$ by even parity. The A_1 sign is independent. Thus, $N_{W_R}(Q) = C_2^2 \times \mathrm{AGL}_1(5)$. The labels on $\mathrm{AGL}_1(5)$ are the four linears of $\mathbb{F}_5^\times$ and the rational augmentation of $\mathbb{Q}[\mathbb{F}_5]$. Thus, $N_{W_R}(Q)$ has 20 characters, all $\mathcal{H}_5$-fixed. By [35], there are also 20 eligible labels of W_R. Each global label in $\mathrm{Irr}_{\ell'}(W_R)$ is fixed by $\mathcal{H}_5$. Thus, every bijection $\mathrm{Irr}_{5'}(W_R) \to \mathrm{Irr}(N_{W_R}(Q))$ is equivariant. On I_*, the smaller inertia, tensor $\theta_0|_{I_*}$ with the inflated label η_* and extend before induction. The field U fixes $\widehat{Q}$, and at $d = 2$ the lift n centralizes $\widehat{Q}$ and acts by the sign c. Next, take scalar a as above at $d = 2$, and take 1 at $d = 1$. The disjoint D/S-twist orbits give irreducible restriction to $N_S(P_D)$. By Corollary 2.10, $N_G(P) = \pi^{-1}(N_S(P_D))$, with $\pi : G \to S$. The extensions $\widehat{\chi}, \widehat{\psi}$ on the groups $D\langle\phi\rangle, N_D(P_D)\langle U\rangle$ are genuine and $\mathcal{H}_\ell$-fixed. Both inflations have central kernel $Z(G)$, so $(\chi, \psi) \succeq_c$. □

Lemma 7.137. *Let $G = E_7(q)_{\mathrm{sc}}$, $S = G/Z(G)$, and $D = E_7(q)_{\mathrm{ad}}$, with $q = r^f$ odd. If $2 \nmid \ell$ and $\ell \mid q + 1$, we let $P \in \mathrm{Syl}_\ell(G)$ for the negative scalar marking and $M = N_G(P)$. In Bourbaki numbering, we have that α_0 is the negative highest root, and $R = \Phi(C_{\mathbf{G}^*}(s)^\circ)$. Specifically, for quasi-isolated s with $|s| > 2$, the table lists R on the negative mark with $\mathcal{E}(G, s) \cap \mathrm{Irr}_{\ell'}(G) \neq \varnothing$ and gives its size.*

R	*condition on* q	ℓ	*number*
A_2^3	$q \equiv -1 \pmod 3$	> 7	*27*
$D_4A_1^2$	$q \equiv -1 \pmod 4$	> 7	*53*
A_5A_2	$q \equiv -1 \pmod 3$	$5, > 7$	15, 33
$A_3^2A_1$	$q \equiv -1 \pmod 4$	> 7	*40*

The centralizer for A_5A_2 is connected. Otherwise, we have that $C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ \simeq C_2$. All four rows have odd f.

Proof. For R and its rational forms, see [208, §11.4 and Propositions 11.9, 11.18–11.23]. If $3 \mid |s|$, the class occurs only for $r \neq 3$. For $|s| = 2$, $R = A_1D_6, E_6, A_7$. For the isogeny with kernel Z, $|Z^F| = |Z/(F - 1)Z|$. The component and restriction indices are 2-powers, so v_ℓ is the same. In type A_2^3, let $x = \varepsilon q \equiv 1 \bmod 3$ with $\varepsilon = \pm 1$. For $[G^* : C_{G^*}(s)]_{r'}$, the denominators are $|(x-1)(x^2-1)^3(x^3-1)^3|$ and $|(x+1)(x^4-1)(x^6-1)(x^2-1)(x^3-1)|$. The second has multiplicity less than 7 at $x = \pm 1$, so both $\Phi_1(x), \Phi_2(x)$ divide the index. At $x = 1$ only, the first polynomial has multiplicity 7. Hence, $\varepsilon = -1$ for the negative mark. At $x \equiv 1 \bmod \ell$, the valuation is $v_\ell(|W(E_7)|/6^3) = v_\ell(2^7 \cdot 3 \cdot 5 \cdot 7)$. Thus, $\ell \neq 3, 5, 7$ and the three unitary labels λ_i on each A_2 survive. Among the triples $(\lambda_1, \lambda_2, \lambda_3)$, nine have $\lambda_1 = \lambda_3$ and each gives two constituents $\chi_\pm$. The other 18 triples give nine orbits of C_2. Consequently, the count is 27.

In type $D_4A_1^2$, put $\varepsilon \equiv q \bmod 4$, $\varepsilon = \pm 1$, and $P_j = q^j - 1$. The two denominators are $(q-\varepsilon)P_2^3P_4^2P_6$ and $(q+\varepsilon)P_2P_4^2P_6(q^4+1)$. At $q=-1$, only the first can have multiplicity 7, with $\varepsilon = -1$. After cancellation the index has valuation $v_\ell(2^2 \cdot 3^3 \cdot 5 \cdot 7)$. The central character is 1. If $\ell > 7$, the eligible D_4 labels are $\mathrm{Irr}(W(D_4))$ on the Φ_2 mark. The 20 bipartitions (λ, μ) of four label $\mathrm{Irr}(W(B_4))$. The twist by the sign of $W(B_4)/W(D_4)$ acts by $(\lambda, \mu) \mapsto (\mu, \lambda)$ and fixes the two equal bipartitions (λ, λ). Moreover, restriction to $W(D_4)$ gives nine labels fixed by γ, the graph involution, and two γ-orbits of size 2. Thus, $2 \cdot 9$ of the $4 \cdot 13$ triples are fixed by γ on D_4 and the transposition on the A_1 factors. These give 36 constituents, and the other triples give 17. Consequently, the count is 53.

In type A_5A_2, let $x = \varepsilon q \equiv 1 \bmod 3$. The denominator has degrees $2, \dots, 6$ and $2, 3$ at x. If $x = -1$, the index is divisible by $\Phi_2(x)$. At $x \equiv 1 \bmod \ell$, its valuation is $v_\ell(|W(E_7)|/(6!\,3!)) = v_\ell(2^5 \cdot 3 \cdot 7)$. Thus, among odd primes only $\ell = 3, 7$ divide the index $[G^* : C_{G^*}(s)]$. The hook formula gives $\ell \nmid f^\lambda, f^\mu$ for $\lambda \vdash 6$, $\mu \vdash 3$. At $\ell = 5$, the five labels of S_6 have 5-weight one by Proposition 7.132. All $\mu \vdash 3$ are allowed. By [208, Proposition 11.21], $\mathcal{E}(G, s)$ has 33 characters. Thus, if $\ell > 7$, all 33 are eligible.

Given $A_3^2A_1$, we put $\varepsilon \equiv q \bmod 4$ and $x = \varepsilon q$. When the lift s_0 is rational, the denominator is $(x^2-1)^3(x^3-1)^2(x^4-1)^2$. The other form has factors $\mathrm{SL}_4(q^2), \mathrm{SL}_2(q)$, with four factors $q^j - 1$, $j = 2, 4, 6, 8$. Thus, multiplicity seven at $q = -1$ requires the first form and $\varepsilon = -1$. The index valuation is $v_\ell(|W(E_7)|/(4!^2\,2)) = v_\ell(2^3 \cdot 3^2 \cdot 5 \cdot 7)$. If $\ell > 7$, all 50 labels on D survive. By [208, proof of Proposition 11.22, Step 2], they give 20 singleton restrictions and 10 pairs. Hence, the series on G has 40 characters. Finally, even f gives $q \equiv 1 \bmod 4$ and $q \equiv 0, 1 \bmod 3$, which contradict the conditions on q in the table. □

Proposition 7.138. *In the A_5A_2 row of* Lemma 7.137*, we have $(\chi, \psi) \succeq_c$ on $N_G(P)$ at each eligible ℓ.*

Proof. By Lemma 7.215, the parents $\widetilde{\chi}$ on D have rational extensions $\widehat{\chi}$ across ϕ. The restrictions $\widetilde{\chi}|_S$ are irreducible and distinct. In D, the longest lift n_0 has square 1 and the marked field satisfies $U^f = n_0^{-1}$. Since $r \equiv -1 \bmod 3$, $V = n_0U$ fixes the cubic weight θ and the Tits representatives, with $V^f = 1$. The extension θ_0 has value 1 on the neutral Tits lifts in V_R. Its Weyl inertia is $S_6 \times S_3$. At $\ell = 5$ the inertia quotient over T_* is $\mathrm{AGL}_1(5) \times S_3$, by Proposition 7.136 and the restriction $\theta|_{T_*}$. Apply Proposition 7.132 on S_6 and keep the partition $\mu \vdash 3$. Extend each fixed tensor label by $V \mapsto 1$. If a coefficient action inverts θ, correct it by n_0, which centralizes the lifts and V. Consequently, the induced extensions are $\mathcal{H}_\ell$-fixed. The local normalizer maps onto D/S by Frattini. The quadratic twist moves θ to a distinct orbit even on T_*, by injectivity on Q-fixed weights. Thus, the restrictions to S and inflations to G give $(\chi, \psi) \succeq_c$ with trivial scalar on $Z(G)$. □

Lemma 7.139. *Let $q = r^f \equiv -1 \pmod 3$, with $r \geq 5$, $\ell > 7$ and $\ell \mid q+1$. Thus, we obtain that $r \equiv -1 \pmod 3$ and f is odd. By* Zhang [208, proof of Proposition 11.18, Step 1]*, we choose $\bar{s}$ of order 6 with connected roots A_2^3. Let s be its order-twelve lift, with $s^6 = z$. Next, we write θ for the weight of $\bar{s}$ on T and $k = \theta(z)$. By* Zhang [208, Proposition 11.18, Steps 1–2]*, we deduce that $k = 1$ if $q \equiv 11 \pmod{12}$ and $k = -1$ if $q \equiv 5 \pmod{12}$. Let $M = N_G(P)$. The stabilizer M_θ/T has connected Weyl group $W_0 = S_3^3$. A lift y of the component w exchanges the outer S_3 factors. The two extensions of θ with neutral Tits values 1 have values $j, -j$ on y, with $j^2 = k$. Moreover, on paired characters, $\sigma \in \mathcal{H}_\ell$ has sign $\sigma(j)/j$. With the marked prime-field lift U, put $V = n_0U$. Both extensions in a pair take V to $k^{(f-1)/2}$.*

Proof. By Lemma 7.119, since $\ell \nmid |W|$, $P = T_\ell \in \mathrm{Syl}_\ell(G)$ and $M = N_G(\mathbf{T})$. Moreover, $N_G(\mathbf{T})/T = W$. By [208, proof of Proposition 11.18, Step 1], the simple pairs are $(\alpha_0, \alpha_1), (\alpha_2, \alpha_4), (\alpha_6, \alpha_7)$. The component $w = -w_{0,E_6}$ exchanges the outer pairs, where w_{0,E_6} is the longest element on $\alpha_1, \dots, \alpha_6$. Next, we let n_0, n_E be the positive Tits lifts of $-I, w_{0,E_6}$. Put $y = n_0n_E^{-1}$. By [208, proofs of Propositions 11.21 and 11.22, Step 1], $n_0n_i = n_in_0$ on the simple lifts n_i, and $n_0^2 = z$. Moreover, n_E^2 is central of order dividing 2 in simply connected E_6. Since $|Z(E_{6,\mathrm{sc}})| = 3$, $n_E^2 = 1$ and $y^2 = z$. On the middle A_2, a coroot sign t adjusts the two conjugation signs of y on simple lifts n_2, n_4. Indeed, the Cartan matrix C_{A_2} is invertible modulo 2. Since $w(t) = t$ and $t^2 = 1$, $(ty)^2 = z$. Next, write y for the adjusted representative, with $yn_iy^{-1} = n_i$ for $i = 2, 4$.

Let M_0 be the preimage of W_0 in M. The subsystem A_2^3 has Tits kernel generated by $\alpha^\vee(-1)$, with $\theta(\alpha^\vee(-1)) = 1$. Thus, the line λ_0 on M_0 extends θ with $\lambda_0(n_\alpha) = 1$ on the subsystem Tits lifts. Since $\lambda_0^y = \lambda_0$, it has extensions $\lambda_\pm$, with $\lambda_\pm(y) = \pm j$. We choose rational models D_1, D_2, D_3 for the three S_3 factors. If $D_1 \not\simeq D_3$, the tensor model over M_0 induces irreducibly to M_θ. If $D_1 = D_3$, $y(u_1 \otimes u_2 \otimes u_3) = \pm j(u_3 \otimes u_2 \otimes u_1)$. Given $\sigma \in \mathcal{H}_\ell$, the matrices of D_i are rational. If $\theta^\sigma = \theta^{-1}$, conjugation by n_0 fixes the Tits lifts and y. Thus, the sign on $\lambda_\pm$ is $\sigma(j)/j$. By [208, proof

of Proposition 11.18, Step 1], $w(s) = zs$. Thus, on the regular torus, $w([s, t_0]) = [s, -t_0]$. Moreover, W_0 fixes $[s, t_0]$. Consequently, the outer diagonal acts by $\lambda_\pm(y) \mapsto -\lambda_\pm(y)$ and fixes λ_0 on M_0.

Let U lift the marked prime-field map F_r and put $V = n_0 U$. Then, $\theta^V = \theta$, the Weyl matrices D_i are V-fixed and $V^f = n_0^{f-1} = z^{(f-1)/2}$. Since f is odd, the scalar $k^{(f-1)/2}$ has f-th power $k^{(f-1)/2}$. Set the scalar on V to $k^{(f-1)/2}$ in both extensions. If $D_1 \not\simeq D_3$, the sign change at y has conjugator $\mathrm{diag}(I, -I)$ on the two induced blocks. Moreover, $k^{(f-1)/2}$ at V is invariant under $\mathrm{diag}(I, -I)$.

By [208, proof of Proposition 11.18, Step 1], the central cocharacter $v = 2(\omega_3^\vee - \omega_5^\vee)$ satisfies $v(-1) = z$ and $w(v) = -v$. Next, in the regular overgroup, let $h = [v(a), c]$, where $a^{-r-1} = -1$ and $c^{r-1} = -1$. Choose c of order $2(r-1)_2$. The discrepancy for V is $[z, -1] = 1$. Since f is odd, h is rational in the marked torus $\widetilde{T}$, and c^2 is a nonsquare in $\mathbb{F}_q^\times$. The element h centralizes each derived A_2 factor and V. Moreover, $h^2 = [v(a)^2, 1][1, c^2]$, with the first factor inner. Thus, h exchanges the extensions induced by λ_+, λ_- via the sign at y. The scalar $k^{(f-1)/2}$ on V is fixed by h. □

Proposition 7.140. *The rational series of $\bar{s}$ has* 9 *singletons and* 9 *pairs by* Zhang [208, Proposition 11.18, Step 2]. *Each χ in the series has a partner $\psi \in \mathrm{Irr}(M)$ with the comparison $\succeq_c$ and central value $\chi(z)/\chi(1) = \psi(z)/\psi(1) = k$.*

Proof. We order the unequal labels D_1, D_3 alike on the two sides. Let $[s, t_0]$ be the regular lift, with $t_0 = 1$ if $k = 1$. If $k = -1$, choose t_0 of order $2(r-1)_2$, with $t_0^r = -t_0$. By [208, Proposition 11.18, Steps 2, 4 and 5], an unequal triple (D_1, D_2, D_3) gives a regular character $\widetilde{\rho}$ with irreducible rational restriction ρ. Moreover, under a Galois exponent A, its quotient twist is dual to $b_A = (-1)^{e(A)} t_0^{A-1}$, with $e(A) = 0$ for $A \equiv \pm 1 \pmod{12}$ and $e(A) = 1$ for $A \equiv \pm 5 \pmod{12}$.

Locally, $[s, t_0]$ has stabilizer W_0. For $A \equiv -1 \pmod 6$, conjugation by n_0 inverts s, fixes t_0^A, and fixes the lifts of W_0. The corrected first coordinate is $z^{e(A)} s$, so b_A also gives the local twist. Since $r \equiv q \pmod{12}$ and $t_0^{r-1} = k$, both cases give $b_r = 1$. Consequently, each regular parent has $\widetilde{\rho}^{F_r} = \widetilde{\rho}$. Thus, ρ is rational and ϕ-invariant. By [143, Lemma 2.1 and Corollary 2.2], $\widehat{\rho}$ is the unique real extension over $\langle F_r \rangle$ and is rational. Locally, V acts by $k^{(f-1)/2}$ by Lemma 7.139. A field intertwiner B on $\widetilde{\rho}$ is proportional to the chosen B_0 on ρ on each side. Since B_0^f has the given value, we choose $B = B_0$. The ratios agree on $\mathbb{F}_q^\times$ by b_A and on V by $k^{(f-1)/2}$. The quotient generators are $\mathbb{F}_q^\times, V$, and the common lift $[s, t_0]$ gives both central scalars.

Next, let $\mathbf{T}_b$ with mark $-b$. On each A_2 plane, b acts by a Coxeter element. If $i \in \{1, 2\}$, take the lifts a_i of order 3. Next, we set $a_3 = y a_1 y^{-1}$ and $n_b = a_1 a_2 a_3$. Then, $n_b^3 = 1$, $F_r(n_b) = n_b$ and $y n_b y^{-1} = n_b$. Since $b \in W_0$, the change from $-I$ to $-b$ is in the connected centralizer of $\bar{s}$ with the same rational form. Let L_+ be the saturated cocharacter lattice for $w = +I$, and put $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}_{L_+})$. The space $L_+ \otimes \mathbb{Q}$ is the middle A_2 plane plus the outer diagonal $\{(x, x)\}$. Moreover, its perpendicular $V_\perp$ has dimension 3 and contains no root of A_2^3. The pointwise Weyl stabilizer of L_+ contains w, with $w|_{V_\perp} = -I$. Thus, the roots of $\mathbf{L}$ span $V_\perp$. Among simply laced rank-3 systems, only A_1^3 has $-I$ in its Weyl group. Since b acts with order 3 on $\{(x, -x)\}$, it cycles the three A_1 root lines. Thus, $L_{\mathrm{der}} \simeq \mathrm{SL}_2(q^3)$. An element of W_0 acting trivially on L_+ must fix the middle A_2 plane and the outer diagonal $\{(x, x)\}$. Hence, all three S_3 actions are trivial. In L, the weight inertia quotient is $\langle w \rangle$. The lifted centralizer in $\widetilde{L}^*$ is a torus. Thus, $\widetilde{\psi}$ is irreducible signed torus induction, with $\widetilde{\psi}|_L = \psi_+ + \psi_-$. The outer diagonal induced by h has $\psi_+^h = \psi_-$.

Take a nondegenerate additive character λ of the root group $U_1 \simeq (\mathbb{F}_{q^3}, +)$ in L_{der}. Next, we let $\Gamma_\lambda = \mathrm{Ind}_{U_1}^L \lambda$. Gelfand–Graev multiplicity one for $\widetilde{\psi}$ and reciprocity give that $\langle \Gamma_\lambda, \psi_+ + \psi_- \rangle_L = 1$. Moreover, the additive character λ^h selects the other constituent of $\widetilde{\psi}|_L$. The derived torus acts on U_1 by $(\mathbb{F}_{q^3}^\times)^2$. Hence, the orbits of nonzero λ correspond to $\mathbb{F}_{q^3}^\times / (\mathbb{F}_{q^3}^\times)^2$. If σ inverts $\bar{s}$, we correct by the rational n_0. On each A_1 factor n_0 inverts the torus and satisfies $n_0 m n_0^{-1} = m$ for the Tits reflection lift m. The difference between conjugation by n_0 and by m is a torus scaling by d. Since that scaling fixes m, $\alpha^\vee(d) = 1$. Injectivity of the coroot $\alpha^\vee$ gives that $d = 1$. The product $m(n_b m n_b^{-1})(n_b^2 m n_b^{-2})$ is rational and induces conjugation by n_0 on L_{der}. Consequently, for exponent a of σ on μ_r, the corrected permutation of $\{\psi_+, \psi_-\}$ has sign (a/r), since $3f$ is odd.

Let I be signed induction from $\{\psi_+, \psi_-\}$ to the pair $\{\chi_+, \chi_-\}$ with $D_1 = D_3$. The diagonal gives the rows $(A, B), (B, A)$ of the matrix of I. By restriction and transitivity, $\widetilde{\chi}$ has signed torus multiplicity $A + B$. By [208, Proposition 11.18, Step 2], $\widetilde{\chi}$ is the unique parent of χ_+, χ_- in the chosen lifted series. For $-b$, the element relative to opposition is $b w_{0,R}$, where $R = A_2^3$. On R, based opposition is conjugation by $w_{0,R}$. Thus, the value at $b w_{0,R}$ is the product of three S_3 values at a 3-cycle, up to sign. At b, the two linear values are $1, 1$ and the deleted permutation value is -1. Consequently, $A + B = \pm 1$, so $A \neq B$. By [208, Proposition 11.18, Step 3], the sum $\chi_+ + \chi_-$ is rational.

Thus, $A \neq B$ distinguishes the two permutations on $\{\chi_+, \chi_-\}$. By [53, Proposition 3.3], the alternating trace for I is integral, so $I(\psi^\sigma) = I(\psi)^\sigma$. Hence, the source permutation on $\{\psi_+, \psi_-\}$ determines the target permutation on $\{\chi_+, \chi_-\}$. Conjugation by n_0 on G is inner. If $2 \nmid r$, [72, Theorems 3.3.7 and 3.3.8] gives independence of I from $\mathbf{P}$.

Let $a \equiv \ell^u \pmod r$ for $\sigma \in \mathcal{H}_\ell$. By Section 2, the 9 pairs have sign $(a/r) = \sigma(j)/j$. Given a global member χ, let c_0 be complex conjugation. Set $\tau = c_0$ when $\chi^{c_0} = \chi$ and $\tau = hc_0$ otherwise. Then, $\chi^\tau = \chi$ and $\nu^\tau = \nu^{-1}$ for each odd field quotient character ν. Thus, a unique extension $\widehat{\chi}$ satisfies $\widehat{\chi}^\tau = \widehat{\chi}$. Given a diagonal or coefficient action d, $(\widehat{\chi}^d)^\tau = (\widehat{\chi}^\tau)^d$. Locally, the scalar $k^{(f-1)/2}$ on V from Lemma 7.139 is fixed under d. Moreover, on $\langle z, c^2\rangle$, the common central character λ has $\lambda(z) = k$. Thus, the field preimages and their h-conjugates satisfy $\succeq_c$. □

Assume that $q \equiv -1 \pmod 4$, $\ell > 7$ and $\ell \mid q+1$. Thus, $r \equiv 3 \pmod 4$ and f is odd. The two surviving elements $\bar{s}$ of order 4 have rational lifts s, with central value 1 at z for $\chi \in \mathcal{E}(G, \bar{s})$. The connected roots have type $R = D_4A_1^2$ or $R = A_3^2A_1$. On $R = D_4A_1^2$, the component transposes two D_4-nodes and exchanges the A_1 factors by Zhang [208, proof of Proposition 11.20, Step 1]. On $R = A_3^2A_1$, it exchanges the A_3 factors with opposition and fixes A_1 by Zhang [208, proof of Proposition 11.22, Step 1].

Lemma 7.141. *In both series, the characters $\psi \in \mathrm{Irr}(M)$ are rational and fixed by F_r. Moreover, for a Weyl label η with orbit of size 2, the adjoint extensions give $\succeq_c$ at M. At a fixed label, ψ_+, ψ_- are exchanged by the outer diagonal d. The comparison $\succeq_c$ follows when both partners χ_+, χ_- are $\mathcal{H}_\ell$-fixed.*

Proof. By Lemma 7.119, $M = N_G(\mathbf{T})$ since $\ell \nmid |W|$. In the adjoint D, write θ_D for the torus character at s. Since $C(s) = C(s)^\circ$ upstairs, the Weyl stabilizer of θ_D is $W(R)$. Let V_R be the Tits group of R. The neutral coroot signs $u \in T \cap V_R$ satisfy $\theta_D(u) = 1$. Thus, the extension λ_D on TV_R has value 1 on V_R. Whenever $\eta \in \mathrm{Irr}(W(R))$, the product $\lambda_D\eta$ induces irreducibly to $N_D(T)$. Moreover, the Weyl models η are rational and fixed by F_r. A field or coefficient power gives θ_D or θ_D^{-1}. Conjugation by the rational longest lift n_0 corrects inversion and centralizes V_R. Hence, each adjoint local character $\widetilde{\psi}$ is rational and F_r-fixed. The nontrivial character of D/S changes s to zs. The correction to s acts on component characters by $\eta \mapsto \eta^y$. If $\eta \neq \eta^y$, both adjoint characters restrict to $\psi \in \mathrm{Irr}(N_S(P))$. By [208, Propositions 11.20 and 11.22], the adjoint global characters $\widetilde{\chi}$ have the same orbits of η. Moreover, $\widetilde{\chi}$ is rational and $\widetilde{\chi}^{F_r} = \widetilde{\chi}$. Each adjoint character has an extension $\widehat{\rho}$ to the odd field group $\langle F_r\rangle$. Moreover, the unique real choice of $\widehat{\rho}$ is rational. The quadratic character ε of D/S extends rationally over $\langle F_r\rangle$. Since $\widehat{\rho}\varepsilon$ is real, the other normalization is $\widehat{\rho}\varepsilon$. Thus, the singleton characters satisfy $\succeq_c$.

For split restriction, let θ be the weight on $T \leq G$. The inertia quotient contains $W(R)$ and the based symmetry represented by $y = n_0n_E^{-1}$, as in Lemma 7.139. Thus, $y^2 = z$. On the neutral Tits generators n_α for $\alpha \in R$, conjugation by y differs by signs $\beta^\vee(-1)$ from the based action. Given $\beta \in R$, $\theta(\beta^\vee(-1)) = 1$. Since $\theta(z) = 1$, the two line extensions λ_+, λ_- have values $1, -1$ at y. For the D_4 factor, a graph-fixed $\eta \in \mathrm{Irr}(W(D_4))$ has rational extensions to $W(B_4)$. Moreover, the ordered bipartitions over η give both extensions. The equal A_1 factors have the tensor flip $u \otimes v \mapsto v \otimes u$. On the two A_3 factors, the longest elements $w_{0,1}, w_{0,2}$ induce opposition. After correction by $w_{0,1}w_{0,2}$, the component acts by $u \otimes v \mapsto v \otimes u$. Thus, both models are over $\mathbb{Q}$. Conjugation by n_0 corrects inversion of θ and fixes y and these Weyl matrices. Induction yields $\psi_\pm$ with rational values and $\psi_\pm^{F_r} = \psi_\pm$. Moreover, the regular weight on y differs by the quadratic character of D/S. Thus, the outer diagonal acts by $\psi_+ \mapsto \psi_-$.

By [208, Lemma 11.14 and proofs of Propositions 11.20 and 11.22], the global characters χ_+, χ_- are F_r-fixed and $\chi_+ + \chi_-$ is rational. Take an outer diagonal d over $\mathbb{F}_r$. Then, $d^2 \in S$ and $F_r(d) = d$. The image of d in D/S is nontrivial since f is odd. For one member χ, set $\tau = c$ if $\chi^c = \chi$, and $\tau = dc$ otherwise, with c complex conjugation. Then, $\chi^\tau = \chi$ and $\nu^\tau = \nu^{-1}$ for each character ν of the odd field quotient $\langle F_r\rangle$. We normalize by the unique extension $\widehat{\chi}$ with $\widehat{\chi}^\tau = \widehat{\chi}$. Moreover, on characters, τ satisfies $(\widehat{\chi}^\sigma)^\tau = (\widehat{\chi}^\tau)^\sigma$ for each σ in the coefficient stabilizer. Locally, the unique real extension $\widehat{\psi}_+$ has d-conjugate $\widehat{\psi}_-$. By Sylow conjugacy, d has a representative in $N_D(P)$. If $\chi_\pm^\sigma = \chi_\pm$, the normalized extensions satisfy $\widehat{\chi}_\pm^\sigma = \widehat{\chi}_\pm$ and $\widehat{\psi}_\pm^\sigma = \widehat{\psi}_\pm$. Inflation from S to G gives central value 1 at z on both sides. □

Proposition 7.142. *Moreover, the* 53 *characters in the $D_4A_1^2$ case of* Lemma 7.137 *satisfy $\succeq_c$ at M. For each of the* 18 *pairs, $\sigma(\sqrt{-r}) = \sqrt{-r}$ implies $\chi_\pm^\sigma = \chi_\pm$.*

Proof. By Lemma 7.141, consider only χ_+, χ_-. In [208, Proposition 11.20, Step 1], we put $v = \omega_1^\vee + \omega_6^\vee$, $t = \omega_1^\vee - \omega_6^\vee$, $s = v(i)$ and $a = \alpha_0 + \alpha_7$. In E_7, choose $\alpha^\vee = \alpha$. Moreover, the roots α_0, α_7 are orthogonal. The component fixes a and acts by $t \mapsto -t$. Moreover, the vector t has squared length 2 and is perpendicular to a and D_4. The vector a

is conjugate to $\omega_6^\vee$. Indeed, $\omega_1^\vee = -\alpha_0$, and $\omega_6^\vee = \omega_1^\vee - t$ is an orthogonal root sum. In $\alpha_0^\perp$, the roots form D_6. The group $W(D_6)$ is transitive on them. Consequently, $\mathbf{L} = C_{\mathbf{G}}(a(\mathbf{G}_m))$ has simply connected derived factors D_5, A_1. In D_5, no root is perpendicular to D_4. Thus, the separate A_1 is the root system $\{\pm t\}$.

The cocharacter a defines the parabolic. The element $s_{\alpha_0}s_{\alpha_7}$ belongs to the connected centralizer of s and acts by $a \mapsto -a$. Thus, the negative mark corrected by $s_{\alpha_0}s_{\alpha_7}$ fixes a. By Lang's theorem in $C(s)^\circ$, the modified mark gives $\mathcal{E}(G, \bar{s})$. The dual Levi intersection has D_4 and split tori on A_1^2. For graph-fixed $\tau \in \mathrm{Irr}(W(D_4))$, take the corresponding regular Levi character $\widetilde{\psi}$. The component fixes a, so its representatives lie in $\mathbf{L}$. Since the nontrivial quotient twist fixes τ, $\widetilde{\psi}|_L = \psi_+ + \psi_-$. By [9, §6], Harish–Chandra induction of $\widetilde{\psi}$ has label τ on D_4 and $1 + \mathrm{St}$ on each rational A_1. Thus, each invariant target χ with entry τ has multiplicity 1, independently of $\chi(1)$. The rational regular parabolic $\widetilde{P}_a$ surjects onto $\widetilde{G}/G$. Hence, restriction to G of induction from $\widetilde{L}$ is induction from L of $\widetilde{\psi}|_L$. The matrix B between (ψ_+, ψ_-) and (χ_+, χ_-) has column sums 1. The outer diagonal d exchanges both pairs, so B is a permutation matrix.

The pair ψ_+, ψ_- is stable under coefficient operations. Indeed, $(v, t) = -2$ and $s_t(v) = v + 2t = -v + 4\omega_1^\vee$, so $s_t(s) = s^{-1}$. The rational reflection s_t acts only on the isolated A_1 factor and fixes the label τ. After s_t, the lift differs by a $\widetilde{L}/L$ twist. Next, we put $H = (\mathbf{L}_{\mathrm{der}})^F$. Restriction of $\widetilde{\psi}$ to H is multiplicity free. Moreover, each of ψ_+, ψ_- restricts to one L-orbit $O_\pm$ on $\mathrm{Irr}(H)$, with $O_+ \cap O_- = \varnothing$. The factorization is $H = H_D \times \mathrm{SL}_2(q)$, with spin factor H_D. On the dual of $\mathrm{SL}_2(q)$, the image of s is a nontrivial projective involution, since $t(s) = -1$. By [165, Theorem B], its two constituents $\rho_\pm$ have values in $\mathbb{Q}(\sqrt{-r})$.

On H_D, choose the dual lift s_0 in SO_{10}. Next, we write δ for the rational form of D_5. The four torus coordinates on D_4 can be $+1$. The fifth coordinate is -1, since s_t fixes this factor and inverts s, leaving precisely D_4. The eigenspaces of s_0 for $1, -1$ have dimensions $8, 2$. Thus, $F(s_0) \neq -s_0$, and isotypy from s_0 gives the series on H_D. By [165, Theorem A], the characters in $\mathcal{E}(\mathrm{SO}_{10}^\delta(q), s_0)$ are rational. The sign of $\mathrm{SO}_{10}^\delta(q)/\Omega_{10}^\delta(q)$ corresponds to $s_0 \mapsto -s_0$, in a distinct class from s_0. Thus, restriction to $\Omega_{10}^\delta(q)$ is irreducible, and inflation to H_D is rational. Thus, $\sigma(\sqrt{-r}) = \sqrt{-r}$ gives $\xi^\sigma = \xi$ for $\xi \in O_+ \cup O_-$. The disjoint orbits O_+, O_- distinguish the two characters ψ_+, ψ_-. Since B is a permutation matrix, $\chi_\pm^\sigma = \chi_\pm$. By [156, Lemma 2.3(c)], $\mathcal{H}_\ell$ fixes $\sqrt{-q}$, since $d = 2$. Thus, it fixes $\sqrt{-r}$ as f is odd. By Lemma 7.141, we obtain $\succeq_c$. □

Proposition 7.143. *The* 20 *singleton characters and* 10 *diagonal pairs in the* $A_3^2A_1$ *case of* Lemma 7.137 *satisfy* $\succeq_c$ *at* M.

Proof. Take a pair above $(\lambda, \lambda, \epsilon)$, with $\lambda \vdash 4$ and outside A_1 label ϵ. On the first $A_3 = D_3$, we choose the unit axes $a = (1, -1, -1, 1)/2$, $b = (1, 1, -1, -1)/2$, $c = (1, -1, 1, -1)/2$. The second factor has axes a', b', c'. Based opposition is $(x_1, x_2, x_3, x_4) \mapsto (-x_4, -x_3, -x_2, -x_1)$. Thus, it acts by $a \mapsto -a$ and fixes b, c. Choose a generic integral cocharacter u_0 in the plane of $b + c + b' + c'$ and the outside root of A_1. By [208, proof of Proposition 11.22, Step 1], the component and mark fix u_0. Hence, u_0 gives a rational Levi $\mathbf{J}$ and rational parabolic $\mathbf{P}_{u_0}$. The component has a representative in $\mathbf{J}$. Since $b + c = (1, 0, 0, -1)$, the isotropic-line Levi roots are $b - c$ and $b' - c'$. On the outside A_1, the intersection is the split torus T_1.

Every partition of 4 has empty 2-core. Also, a relative character $\eta \in \mathrm{Irr}(W(B_2))$ on either unitary factor has constituent $\tau \in \{1, \mathrm{St}\}$ of multiplicity 1 on the rank-one subgroup $W(A_1)$. If $\eta(1) = 1$, its restriction is linear. If $\eta(1) = 2$, the reflection has eigenvalues $1, -1$. For the target $(\lambda, \lambda, \epsilon)$, choose (τ, τ) on the middle A_1 factors. The component in $W(\mathbf{J})$ fixes (τ, τ). Hence, the regular Levi character $\widetilde{\psi}$ restricts to $\psi_+ + \psi_-$. To check the pair under σ, put $\beta_\pm = a \pm a'$. The vectors $\beta_\pm$ are orthogonal to u_0. The vectors $\beta_\pm$ have weight class $(2, 2, 0)$, in the root lattice by [208, (11.13)]. Since $(\beta_\pm, \beta_\pm) = 2$, they are roots of E_7. On the lift $s = (iI_4, I_4, I_2)$, both roots have value -1. Thus, $s_{\beta_+}s_{\beta_-}$ multiplies s by $(2a)(-1) = s^2$, acts by $s \mapsto s^{-1}$ and fixes the labels (τ, τ). By Lang's theorem in $C_{\widetilde{\mathbf{J}^*}}(s)^\circ$, the series of s^{-1} differs by a $\widetilde{J}/J$ twist.

In the regular groups, $(\lambda, \lambda, \epsilon)$ has induction coefficient 1 for either ϵ. If $\tau = 1$, the same source has coefficient 1 at the semisimple pair with partitions $(4), (4)$, whereas for $\tau = \mathrm{St}$, it has coefficient 1 at the regular pair with partitions $(1^4), (1^4)$. Restriction and the common diagonal d give permutation matrices B_0, B_λ by Proposition 7.142. For exponent u on μ_r, the split Whittaker character λ_{reg} changes by conjugation with $(2\rho^\vee)(\sqrt{u})$. The pair permutation has sign $+1$ exactly when u is a square in $\mathbb{F}_q^\times$. Rational Alvis–Curtis duality gives sign $+1$ under the same condition on u for the semisimple pair. By [156, Lemma 2.3(c)], $\mathcal{H}_\ell$ fixes $\sqrt{-q}$. Thus, the sign is $+1$ on both pairs since $r \equiv 3 \bmod 4$ and f is odd. The matrix B_0 gives $\psi_\pm^\sigma = \psi_\pm$. The matrix B_λ gives $\chi_\pm^\sigma = \chi_\pm$ at $(\lambda, \lambda, \epsilon)$. The field normalization in

Lemma 7.141 gives $\succeq_c$. □

Lemma 7.144. *Let $G = E_{7,\mathrm{sc}}(q)$, where $q = r^f$ with r, f odd. Next, fix an odd prime $\ell \mid q+1$ and a torus $\mathbf{T}$ with Frobenius $-q$ on $X(\mathbf{T})$. Let $T = \mathbf{T}^F$, $W = W(E_7)$, and take $P \in \mathrm{Syl}_\ell(N_G(\mathbf{T}))$. By* Lemma 7.119*, $P \in \mathrm{Syl}_\ell(G)$ has image $Q \in \mathrm{Syl}_\ell(W)$ and $M = \mathrm{N}_G(P)$ satisfies $T \cap M = T_* = T_\ell C_{T_{\ell'}}(Q)$. The local weights in M are restrictions to T_*. Let $s \in T^*$ have weight $\theta \in \mathrm{Irr}(T)$, and let R consist of the roots with value 1 on s. Next, we let $\mathbf{K}_R$ be generated by $\mathbf{T}$ and the primal subgroups for R. There is a unique linear character κ_s of $\mathbf{K}_R^F$ with $\kappa_s|_T = \theta$ and $H_0 \le \ker \kappa_s$, where H_0 is the image of $\mathbf{H}^F$ for the simply connected derived cover $\mathbf{H}$. For root or coefficient actions d, $\kappa_{s^d} = \kappa_s^d$. Moreover, for a regular lift $\widetilde{s}$, the character $\kappa_{\widetilde{s}}$ restricts to κ_s.*

Proof. If $\alpha, \beta \in R$ and $\alpha + \beta$ is a root, $(\alpha+\beta)(s) = 1$. Thus, R is closed, and scalar F fixes the subsystem R and the group $\mathbf{K}_R$. Let $\mathbf{H}$ be the simply connected cover of $[\mathbf{K}_R, \mathbf{K}_R]$. Next, we write H_0 for the image of $\mathbf{H}^F$. The full preimage of $\mathbf{T} \cap [\mathbf{K}_R, \mathbf{K}_R]$ in $\mathbf{H}$ is its maximal torus $\mathbf{T}_H$. Moreover, $T \cap H_0$ is the image of $\mathbf{T}_H^F$. Its cocharacter lattice has basis $\{\beta^\vee : \beta \in \Delta(R)\}$. Thus, $\mathbf{T}_H^F$ is generated by $\beta^\vee(\mu_{q+1})$ for $\beta \in \Delta(R)$. By torus duality, θ is trivial on these subgroups.

If $x \in \mathbf{K}_R^F$, choose a lift $\widehat{x}$ in $Z(\mathbf{K}_R)^\circ \times \mathbf{H}$. Its Frobenius discrepancy $\widehat{x}^{-1}F(\widehat{x})$ lies in the central kernel over $\mathbf{T}$. By [114, corollary to Theorem 1], we choose t_0 in the covering torus with $t_0^{-1}F(t_0) = \widehat{x}^{-1}F(\widehat{x})$. Therefore, $\widehat{x}t_0^{-1}$ is rational, and t_0 maps into T. Thus, $\mathbf{K}_R^F = TH_0$. Since $H_0 \lhd \mathbf{K}_R^F$ and $\theta(T \cap H_0) = 1$, the formula $\kappa_s(th) = \theta(t)$ is unique. Also, its restrictions to T, H_0 give $\kappa_{s^d} = \kappa_s^d$. Given a lift $\widetilde{s}$, the coroot groups $\beta^\vee(\mu_{q+1})$ agree in the two images of H_0. Moreover, a central twist changes $\kappa_{\widetilde{s}}|_{\widetilde{T}}$ and is trivial on H_0. The argument covers any subsystem R and includes non-Levi subgroups $\mathbf{K}_R$. □

Proposition 7.145. *Suppose that $s \ne 1$, $C_{\mathbf{G}^*}(s)$ is connected and $\mathcal{E}(G,s) \cap \mathrm{Irr}_{\ell'}(G) \ne \varnothing$. If $2 \nmid f$, we have the mixed comparison $\succeq_c$ on M. Given $\ell \in \{3,5,7\}$, the local labels belong to $\mathrm{Irr}_{\ell'}(Y)$ for $Y = \mathrm{N}_{W(R)}(Q)$ after rational conjugation to $Q \le W(R)$.*

Proof. By Lemma 7.119, we may take $Q \le W_s = W(R)$. Given any neutral root α, the reflection formula gives $s_\alpha(\widetilde{s}) = \widetilde{s}$ for every regular lift $\widetilde{s}$. Thus, in W, we find stabilizer $W(R)$ for $\widetilde{s}$. A diagonal self-twist of $\widetilde{\chi}$ would require $\widetilde{s}z_0$ in the Weyl orbit of $\widetilde{s}$ for some central $z_0 \ne 1$. But $w\widetilde{s} = \widetilde{s}z_0$ implies $w \in W_s = W(R)$ and so $z_0 = 1$. Thus, the quotient formula gives $\langle \widetilde{\chi}|_G, \widetilde{\chi}|_G \rangle_G = 1$. $\widetilde{\psi}$ has only the self-twist 1. The weight argument gives that $\widetilde{\psi}|_M$ is irreducible. Indeed, $T_{\ell'} = C_{T_{\ell'}}(Q)[T_{\ell'}, Q]$ by [97, Theorem 4.34]. Given a Q-fixed weight θ, the subgroup $[T_{\ell'}, Q]$ lies in $\ker\theta$. Thus, θ is determined by $\theta|_{T_*}$.

Write $C = C_{\mathbf{G}^*}(s)^F$. Moreover, the global labels are unipotent ℓ'-degree characters of C. The minimal 2-split Levi is the torus T^*. Moreover, the relative labels lie in $\mathrm{Irr}_{\ell'}(W(R))$. The eligible classical unipotent characters η are rational. For a proper E_6 component, irrational labels are outside $\mathrm{Irr}_{\ell'}(C)$ on this mark. Also, the scalar F fixes each component of R. The ambient $-I$ acts by $s^{-1} \mapsto s$ and induces opposition up to $W(R)$-conjugation. Its action on $W(R)$ is inner. By [156, Theorem B, Proposition 4.6, Corollary 4.7, Table 4.1 and §4.4], $\overline{\widetilde{\chi}} = \widetilde{\chi}\nu$ for a central quotient character ν. Thus, $\chi = \widetilde{\chi}|_G$ is real.

On the inertia $N_G(\mathbf{T})_\theta$, we choose κ_s from Lemma 7.144. Induce the product $\kappa_s\eta$ to $N_G(\mathbf{T})$. If $\ell > 7$, $Q = 1$ and $M = N_G(\mathbf{T})$. Given a field and coefficient operation d in the joint stabilizer, the corrected lift $\widetilde{s}^d$ gives the same central character ν_d on G, M. Moreover, the corrections differ by $w \in W(R)$, with $w\widetilde{s} = \widetilde{s}$ and inner action on $W(R)$. The pure field quotient F_0 has odd order dividing f. By [143, Lemma 2.1], each real F_0-fixed character χ has a unique real extension $\widehat{\chi}$ over F_0. The regular diagonal restriction and $\widehat{\chi}$ determine the mixed stabilizer operation d by Proposition 7.124.

Suppose that $Q \ne 1$, and put $I = M_{\theta|_{T_*}}$ for a Q-fixed weight θ. The Weyl image I/T_* is Y. Indeed, the restriction of Q-fixed weights to T_* is injective. Given $y \in Y$, choose a Weyl lift n_y in the preimage of $\mathrm{N}_W(Q)$. Sylow conjugacy in the preimage of Q corrects n_y to an element of M. Since $Y \le W(R)$, $I \le \mathbf{K}_R^F$. Put $\mu_0 = \mathrm{Ind}_I^M(\kappa_s|_I)$. Its T_*-weights are distinct conjugates of $\theta|_{T_*}$, so μ_0 is irreducible. The rational correction $-I$ acts by $\theta^{-1} \mapsto \theta$, so $\overline{\mu_0} = \mu_0$. On the inertia A_η of the final label η, take the real field extension of μ_0 and its matched diagonal restriction on A_η. The weight space L_θ for $\theta|_{T_*}$ has dimension 1, since only the coset I affords $\theta|_{T_*}$. Each mixed stabilizer operator A_d restricts to L_θ. Thus, the products A_dA_e and coefficient actions on A_d give the same scalars on L_θ as on μ_0. This step depends on reality of μ_0 alone.

We extend η to its stabilizer. If $\ell \in \{5,7\}$, only one component can carry ℓ, since $v_\ell(|W|) = 1$. At $\ell = 5$, we find types A_{m-1} with $5 \le m \le 8$, D_5, D_6 and E_6. If $\ell = 7$, the types are A_6, A_7. Given a full-rank D_7, the determinant ratio is $4/2$, which is not a square. If $\ell = 3$, $v_3(|W|) = 4$. The bounds $v_3((a+1)!) \le a/2$ and $v_3(a!) \le a/2$ for $1 \le a \le 7$ give $v_3(|W(R)|) \le 3$ for classical R of rank at most 7. Thus, the remaining proper component is E_6. In type E_6A_1, the determinant ratio $6/2 = 3$ is not a square.

Given A_{m-1} with $\ell \le m < 2\ell$, the normalizer of an ℓ-cycle is $\mathrm{AGL}_1(\ell) \times S_{m-\ell}$. By Proposition 7.124, the core κ gives the residual Specht model V_κ. The occupied runner gives the $\mathrm{AGL}_1(\ell)$ label. These characters have values in $\mathbb{Q}(\mu_{\ell-1})$. Moreover, root actions on $S_{m-\ell}$ are inner, and equal inactive factors V_i have tensor permutations. Moreover, for D_5, D_6 at $\ell = 5$, sign vectors in the normalizer are $(\epsilon, \dots, \epsilon)$ on the 5 moving coordinates. Thus, parity gives $\mathrm{AGL}_1(5)$ for D_5 and its product with $\langle -I \rangle$ for D_6. The root automorphism normalizer adds C_2 in ranks $5, 6$. Moreover, the graph element γ exchanges only the global constituents of (λ, λ). There is no such bipartition for D_5, and for D_6 the split degree $\binom{6}{3}(f^\lambda)^2/2 = 2 \cdot 5(f^\lambda)^2$ is divisible by 5, where f^λ is the Specht degree. Next, for E_6 at $\ell = 5$, take the cycle in the A_5 perpendicular to a root α. Moreover, the three-subset weights of A_5 have orbits of size 5. Thus, the fixed roots of Q are $\pm\alpha$. Their line stabilizer $S_6 \times \langle s_\alpha \rangle$ gives the normalizer $\mathrm{AGL}_1(5) \times C_2$. The full root automorphism normalizer adds $\langle -I \rangle$. If $\ell = 3$, the calculation on A_2^3 gives $Y = (C_3^3 \rtimes C_3) \rtimes (C_2 \times C_2)$ in $W(E_6)$. Let $U = C_3^3 \rtimes C_3$, with $U/[U,U] \simeq C_3^2$. Choose the complementary involutions a, b on $\mathbb{F}_3^2$, with $a(x,y) = (-x, y)$ and $b(x,y) = (x, -y)$. Thus, $Y/[U,U] \simeq \mathfrak{S}_3 \times \mathfrak{S}_3$. $\mathbb{Q}(\eta) = \mathbb{Q}$ for $\eta \in \mathrm{Irr}_{3'}(Y)$, and take value 1 on the extra $-I$. For non-A factors, the graph and coefficient actions d fix the eligible labels η, with $\eta^d = \eta$. By [35], pair the η-labels.

For the extended Weyl model ρ, write $\rho(-I) = h\,\mathrm{Id}$, with $h = \pm 1$. A root stabilizer operation v on $\mathbf{T}$ with field exponent j has operator $\rho(v)h^j$ before restriction to R. Moreover, the field F_r fixes the odd complement $\widehat{Q}$ in the Tits group V. A diagonal representative can normalize $\widehat{Q}$ after correction. Changing j by f changes v by $-I$. Since f is odd, $h^{f+1} = 1$ and the two operators $\rho(v)h^j, \rho(-v)h^{j+f}$ are equal. Each ρ is $\mathcal{H}_\ell$-fixed. Next, induce the product of ρ with L_θ. The factor set α and the scalars for the stabilizer action come from L_θ. The global restriction χ has matching diagonal twist ν_d and unique real extension over F_0. By Clifford theory on T_*, the characters exhaust the fibre of $\theta|_{T_*}$. □

Theorem 7.146. *Let $G = E_{7,\mathrm{sc}}(r^f)$, with r, f odd. The marked classical comparisons for R, signed Levi comparisons for G and finite-action comparisons for M, on their stated domains, give $\succeq_c$ on $M = \mathrm{N}_G(P)$ for every odd prime $\ell \mid r^f + 1$.*

Proof. If $s = 1$, the weight is trivial. If $\ell = 3$, the fixed line of Q gives $\mathrm{N}_W(Q) = \mathrm{N}_{W(E_6)}(Q) \times \langle -I \rangle$. If $\ell = 7$, we set $Q = \langle (1234567) \rangle$ in $W(A_7)$. Its moving space has roots A_6 and no perpendicular root of E_7. Indeed, neither $e_i - e_j$ nor a half-root with four $+$ signs is constant on coordinates $1, \dots, 7$. Thus, the pointwise stabilizer in W is 1. The affine group $\mathrm{AGL}_1(7)$ gives the permutations and $-I$ gives the other sign. If $\ell = 5$, the perpendicular roots are the A_2 on coordinates $6, 7, 8$. Their group S_3 centralizes $\mathrm{AGL}_1(5)$ and $-I$. Thus, $\mathrm{N}_W(Q) = \mathrm{AGL}_1(5) \times C_2 \times S_3$ for $\ell = 5$, and $\mathrm{N}_W(Q) = \mathrm{AGL}_1(7) \times C_2$ for $\ell = 7$. The eligible characters are rational at $\ell = 3$ and have values in $\mathbb{Q}(\mu_{\ell-1})$ at $\ell = 5, 7$. Pair the Weyl labels η by the count for W. Moreover, the local lift U acts by $h = \rho(-I) \in \{\pm 1\}$. Since f is odd, $h^f = h$. By [156, proof of Theorem 8.5], the adjoint unipotent extensions $\widehat{\chi}$ are $\mathcal{H}_\ell$-fixed. Both inflations have value 1 on z and trivial additional quotient twists $\nu_d = 1$. If $\ell > 7$, the unipotent comparison is already on $M = N_G(\mathbf{T})$.

By Proposition 7.145, the comparison holds for $s \ne 1$ with connected $C_{\mathbf{G}^*}(s)$. For the involution with connected roots A_1D_6, by Proposition 7.136, we obtain $\succeq_c$ on both scalar marks. For disconnected $C_{\mathbf{G}^*}(s)$, the map $w \mapsto w(\widetilde{s})\widetilde{s}^{-1}$ embeds $W_s/W(R)$ in $\langle z \rangle$. The centralizer of $\widetilde{s}$ is connected, with Weyl group $W(R)$. Thus, the odd Sylow subgroup Q lies in $W(R)$. If $\ell = 3$, a proper qualifying E_6 has parabolic closure of rank 6. It has 72 roots, more than any classical R of rank 6. Thus, E_6 is a Levi. If $\ell = 7$, the closure of the rank-six A_6 is A_6. Indeed, the rank-six alternatives R that are irreducible have $|W(R)|$ prime to 7. The nonsquare ratios $7, 2$ exclude A_6A_1, D_7, and full-rank A_7 is isolated. In the proper Levi, $\mathbb{Z}R$ is a summand of $X(\mathbf{T}^*)$. Thus, the central line Z° is connected with pointwise stabilizer $W(R)$. Any additional stabilizer inverts Z°, so its fixed points satisfy $s^2 = 1$. For the involution in A_7 see Proposition 7.154. For the other s of order 2, Propositions 7.156 and 7.157 give $\succeq_c$ on the domain of Lemma 7.155.

If $\ell = 5$, by Corollary 7.133, we obtain $\succeq_c$ for the other s with disconnected $C_{\mathbf{G}^*}(s)$. By Lemma 7.137, for disconnected quasi-isolated s with $s^2 \ne 1$, no character is eligible at $\ell = 3, 5, 7$. If $\ell > 7$, Proposition 7.140 covers order 6 and Propositions 7.142 and 7.143 cover order 4. For the connected cubic s, see Proposition 7.138. For non-quasi-isolated s, Lemma 7.126 gives the minimal Levi $\mathbf{L}$. Moreover, the orthogonal factors L_D have $\succeq_c$ by Propositions 7.127

and 7.128. The unitary factors L_A have $\succeq_c$ by Propositions 7.129 and 7.130. Moreover, the orbit $W\cdot s$ gives the torus weight orbit $W\cdot\theta$. Signed induction is indexed by $W\cdot s$. Given T_*, conjugate the two Sylow subgroups within W_s to Q. Then, $\theta\mapsto\theta|_{T_*}$ is injective on $\mathrm{Irr}(T)^Q$. Thus, disjoint local sets exhaust $\mathrm{Irr}_{\ell'}(M)$. □

Let $q=r^f$ be odd with even f, with the same torus mark $-q$. Write $s=ty$, where t is the 2-part and $y\neq 1$ the odd part. Assume that $C_{\mathbf{G}^*}(s)$ is disconnected. By Proposition 7.125, $t^2=1\neq t$ and $\mathbf{K}^*=C_{\mathbf{G}^*}(y)$ is a proper Levi. Write R_K for its roots and R for the roots with value 1 on s. The dual cover has kernel $\langle z\rangle$. Next, we let y_0 denote the odd-order lift of y. Given $w\in W_t$, we put $\epsilon_t(w)=(-1)^e$ if $w(t_0)=z^e t_0$. Moreover, the two choices t_0, zt_0 give the same character ϵ_t. On W_s, its kernel is $W(R)$ by connectedness of the centralizer of t_0y_0. Put $f_2=2^{v_2(f)}$. Since $r^f\equiv -1 \pmod{\ell}$, we have that $2f_2\mid \ell-1$.

Proposition 7.147. *Let $\ell>7$. For regular or semisimple $\chi\in\mathcal{E}(G,s)$, we have that the extension $\widehat{\chi}$ matches the field extension $\widehat{\psi}$ of its partner $\psi\in\mathrm{Irr}(M)$. If σ has exponent a on μ_r, we have that the correction at ϕ^k for the distinguished member χ_+ is $\left(\frac{a}{r}\right)^k$. If $R=\varnothing$, these characters exhaust $\mathcal{E}(G,s)$.*

Proof. First, let L be the E_7 coroot lattice and $c=2\omega_7^\vee$, with $z=c(-1)$. The norm form on $L/2L$ is $q_L(x)=(x,x)/2 \bmod 2$. A direct computational verification gives that $(c,L)\subseteq 2\mathbb{Z}$ and $q_L(c)=1$. Thus, $q_L(x+c)=q_L(x)+1$. If $t_0=x(-1)$, the two lifts would have distinct q_L-values. Since W_s exchanges the lifts, $t_0^2=z$. Consequently, $\epsilon_t(-I)=-1$. Since $(q+1)_2=2$ and $y_0^{q+1}=1$, $(t_0y_0)^{q+1}=z$. Moreover, the dual isogeny pairing gives $\chi(z)=-\chi(1)$ for $\chi\in\mathcal{E}(G,s)$.

In the regular overgroup $\widetilde{G}$, take the Steinberg Jordan label St. It is W_s-fixed, and the lift orbit above s has size 2. Thus, the regular parent $\widetilde{\chi}_{\mathrm{reg}}$ has two diagonal self-twists $1,\varepsilon$. The diagonal quotient is cyclic, so $\widetilde{\chi}_{\mathrm{reg}}|_G=\chi_{\mathrm{reg},+}+\chi_{\mathrm{reg},-}$. The outer diagonal d exchanges $\chi_{\mathrm{reg},+},\chi_{\mathrm{reg},-}$. By Alvis–Curtis duality D_G, the semisimple pair $\chi_{\mathrm{ss},\pm}$ has the same diagonal action d. The local lines on W_s are $\det_R, \epsilon_t\det_R, 1, \epsilon_t$. Next, put $\det_R(w)=\det(w|_{\mathbb{R}R})$ for the real span $\mathbb{R}R$ of R. Each local pair differs on $W_s/W(R)$. If $\alpha\in R$, the values at s_α distinguish the two pairs. If $R=\varnothing$, both parents coincide, with local lines $1,\epsilon_t$. Since $\ell\nmid|W|$, the Weyl indices are prime to ℓ. The unipotent degrees $1, r^b$ then give characters of ℓ'-degree.

On the split radical U_G, we choose the Whittaker character λ from a nontrivial additive character ν_r of $\mathbb{F}_r$. On each simple root coordinate $u_\beta(t)$, its value is $\nu_r(\mathrm{Tr}_{q/r}(t))$. Moreover, the Gelfand–Graev character Γ_λ induces to the Gelfand–Graev character $\widetilde{\Gamma}_\lambda$ on $\widetilde{G}$. There $\widetilde{\chi}_{\mathrm{reg}}$ has multiplicity 1. Thus, Γ_λ has constituent $\chi_{\mathrm{reg},+}$ with multiplicity 1. The split field generator ϕ fixes λ. If σ has exponent a on μ_r, set $x=(2\rho^\vee)(\sqrt{a})$ to correct λ^σ, up to inversion convention. Since f is even, $\sqrt{a}\in\mathbb{F}_q$. By [208, proof of Proposition 11.19, Step 1], $(2\rho^\vee)(-1)=z$ in G. Thus, $\phi^k(x)x^{-1}=z^{ek}$, where $(-1)^e=\left(\frac{a}{r}\right)$. Extend λ with value 1 on the ordinary generator ϕ^k and central value -1 at z. Its induction has a unique extension $\widehat{\chi}_{\mathrm{reg},+}$ of $\chi_{\mathrm{reg},+}$, since the base multiplicity is 1. Conjugation by x multiplies the value at ϕ^k by $\left(\frac{a}{r}\right)^k$. Since $\lambda^{\phi^j}=\lambda$, the same sign holds for (ϕ^j,σ). Next, apply Alvis–Curtis duality D_G to the pinned extension $\widehat{\chi}_{\mathrm{reg},+}$. The parabolics P_J are fixed by ϕ, and the radical projectors e_{U_J} are rational. Moreover, each summand satisfies $D_{G,J}(\widehat{\chi}_{\mathrm{reg},+}\nu)=D_{G,J}(\widehat{\chi}_{\mathrm{reg},+})\nu$ for a field quotient character ν. The signed base multiplicity is ±1. Extraction on the cyclic quotient $\langle\phi^k\rangle$ gives the same twist character for $\widehat{\chi}_{\mathrm{ss},+}$. Moreover, the diagonal d gives the extensions $\widehat{\chi}_{\mathrm{reg},-},\widehat{\chi}_{\mathrm{ss},-}$.

By Lemma 7.119, $M=N_G(\mathbf{T})$ for $\ell>7$. Let n_0 lift $-I$. By Lang's theorem, $U^f=n_0^{-1}$ for the marked field lift U. Moreover, $n_0^2=z$ acts by -1 on the weight space L_θ. By Proposition 7.124, inversion has normalized representative $n_0/\delta(\theta)$. For a longest word, $\delta(\theta)=i^{|\{\beta>0:\theta(\beta^\vee(-1))=-1\}|}$. The tests on β and $-\beta$ agree, so the orbit of θ determines $\delta(\theta)$. Since $(n_0/\delta(\theta))^2=1$ on L_θ, $\delta(\theta)^2=-1$. Given a local line η, we let $h=\eta(-I)$. Give field exponent j the scalar α^j, with $\alpha^f=\delta(\theta)^{-1}h^{-1}$. Choose α of order $4f_2$, since the odd power f/f_2 permutes the primitive fourth roots $\pm i$. Thus, $U^f n_0=1$ holds on the corrected inertia model $L_\theta\otimes\eta$. Induction gives $\widehat{\psi}$, with $\widehat{\psi}|_M$ irreducible. Returning Weyl conjugation fixes η. Moreover, the outer diagonal acts by $\eta\mapsto\eta\epsilon_t$. Given $\sigma\in\mathcal{H}_\ell$ of exponent a on μ_r, we have $\sigma(i)=i$ since $4\mid\ell-1$. By Section 2, the correction at U^k is $(\sigma(\alpha)/\alpha)^k=(a/r)^k$. Thus, the corrections at ϕ^k and the central values at z agree. The outer diagonal d exchanges both pairs. □

Proposition 7.148. *Suppose that $\ell>7$ and all irreducible components of R_K have type A. For each eligible partition tuple Λ on R, we have that the paired characters χ,ψ have equal stabilizers and satisfy $\succeq_c$ at M. A fixed tuple Λ gives two constituents $\chi_\pm$ with distinct determinant parity on K.*

Proof. In a standard Levi, z has coroot signs at nodes $2,5,7$ by Proposition 7.125. Write z_j for the projection onto factor j of the simply connected cover. Take $\mathcal{A} = \{j : z_j \neq 1\}$. If $j \in \mathcal{A}$, the factor is SL_{2a_j}, and z_j is the product of coroot signs $\alpha_i^\vee(-1)$ for odd i. Thus, $\sum_{j\in\mathcal{A}} a_j = 3$. If $w(t_0) = zt_0$, the element w exchanges the opposite eigenvalues of multiplicity a_j on each $j \in \mathcal{A}$. Given $j \notin \mathcal{A}$, it fixes the lifted element $t_{0,j}$. Thus, the component $W_s/W(R)$ exchanges the partition pairs for every $j \in \mathcal{A}$ simultaneously. By [156, §4.4], the labels are Λ, the unitary partition tuples.

If $W_s/W(R)$ moves Λ, the diagonal self-twist group of $\widetilde{\chi}$ is 1. Thus, $\widetilde{\chi}|_G$ is irreducible. Locally, $\mathrm{Ind}_{W(R)}^{W_s} \eta_\Lambda$ is irreducible and depends on $W_s \cdot \Lambda$. Moreover, any two Weyl corrections restoring Λ differ by $w \in W(R)$, with $w(\widetilde{s}) = \widetilde{s}$. The lift gives the diagonal character ν on both sides. For complex conjugation, $-I$ acts by $s^{-1} \mapsto s$. Moreover, on R, the opposition is inner modulo $W_s/W(R)$. Hence, the singleton characters χ, ψ are real, and the twist of ψ by ϵ_t fixes ψ.

Let ϕ^k, with $k \mid f$, generate the field inertia of χ. The field action fixes the orbit $W \cdot y$. Next, choose its Weyl correction w based on R_K. Moreover, both $w^{f/k}$ and $(-I)w_{0,R_K}$ act by $y \mapsto y^{-1}$ and fix the chamber C_K. Their quotient belongs to $W_y = W(R_K)$, with free action on chambers. Thus, $w^{f/k} = (-I)w_{0,R_K}$. If $a_j \geq 2$, the rank $2a_j - 1$ has odd multiplicity in R_K. Indeed, two components of rank 5 exceed rank 7, and two of rank 3 leave no room for the additional A_1. The diagram sign product of $(-I)w_{0,R_K}$ on this rank is -1. Thus, f/k is odd. If every $a_j = 1$, the nodes $2,5,7$ are isolated in R_K, and extra nodes lie in $\{1,3\}$. With one extra node, $\mathrm{Fix}(W(R_K))$ has odd dimension, and $w^{f/k}$ has determinant -1 there. With both nodes $1,3$, the unique A_2 has nontrivial opposition. Thus, f/k is odd in either case. If $R_K = A_1^3$, then $R = \varnothing$, so Proposition 7.147 gives $\succeq_c$. For moved Λ, Proposition 7.124 normalizes the extension over $\langle \phi^k \rangle$ by reality.

With fixed Λ, the two blocks of $j \in \mathcal{A}$ have the same partition λ_j. Moreover, the parent $\widetilde{\chi}_j$ on $\mathrm{GU}_{2a_j}(q)$ has two constituents on $\mathrm{SU}_{2a_j}(q)$. Indeed, the only nontrivial self-twist ε_j exchanges the two eigenvalues and fixes λ_j. Choose a unipotent class $\mathcal{O}_j$ with generalized Gelfand–Graev character $\widetilde{\Gamma}_j$ satisfying $\langle \widetilde{\Gamma}_j, \widetilde{\chi}_j \rangle = 1$. By [163, Proposition 6.4], an odd block of $\mathcal{O}_j$ would give odd orbit size under $\mathrm{GU}_{2a_j}(q)/\mathrm{SU}_{2a_j}(q)$. The orbit has size 2, so the block sizes are $2b$. On a block of size $2b$, take the regular nilpotent shift N_b and an alternating antidiagonal form J_b with $N_b^{\mathsf{T}} J_b + J_b N_b = 0$. Also, the Cayley transform $u_b = (1 + N_b)(1 - N_b)^{-1}$ is symplectic over $\mathbb{F}_r$, with block size $2b$. The orthogonal sum u_j is fixed by the symplectic graph γ and F_r. The full algebraic centralizer $C_{\mathrm{GL}_{2a_j}}(u_j)$ is connected. Hence, $\mathcal{O}_j$ has one rational orbit $\mathcal{O}_j^F$. By [163, Lemma 5.3] and [188, §5], the character Γ_{u_j} of $\mathrm{SU}_{2a_j}(q)$ induces to $\widetilde{\Gamma}_j$. Since q is square, Γ_{u_j} is rational by [163, Proposition 2.6]. Thus, $\langle \Gamma_{u_j}, \chi_{j,+} \rangle = 1$ distinguishes the constituent $\chi_{j,+}$, compatibly with pinned actions on $\widetilde{\chi}_j$. Any determinant twist of $\widetilde{\chi}_j$ restricts to $\chi_{j,+} + \chi_{j,-}$. On equal factors the choices u_j agree under permutation. An inner-diagonal g_j acts by $\chi_{j,+} \mapsto \chi_{j,-}$ exactly when the quadratic determinant $\varepsilon_j(g_j)$ is -1.

Let H_j be the simply connected factor and D_j its rational adjoint group. With $H = \prod H_j$, we put $D = \prod D_j$ and $K = \mathbf{K}^F$. Next, write $\varepsilon = \prod_{j\in\mathcal{A}} \varepsilon_j$ on D. The image of K is $\ker \varepsilon$. The dual derived cover has kernel $\langle z \rangle$, so $|\pi_0 Z(\mathbf{K})| = 2$. Lang's theorem gives index 2 in D, and the character for z is ε. The regular Levi $\widetilde{K}$ surjects onto D. Thus, its constituents on H form the product $\prod O_j$ of the factor orbits. The two K-orbits O_+, O_- are fibres of $\prod_{j\in\mathcal{A}} s_j$, where $s_j = \pm 1$ indexes $\chi_{j,\pm}$. Moreover, restriction to H distinguishes the two K-characters $\chi_{K,\pm}$ by $O_\pm$. The factors $j \notin \mathcal{A}$ do not affect the product $\prod_{j\in\mathcal{A}} s_j$.

Given a mixed stabilizer element d, its pinned part p_d has field, graph and permutations of j. Moreover, the inner-diagonal part g_j acts on factor j. The determinant sign $\varepsilon((g_j)_j)$ is independent of Λ. Thus, it equals the sign for $\chi_{\mathrm{reg},\pm}$. Signed induction $R_{\mathbf{K}}^{\mathbf{G}}$ gives the same pair action on G. For the correction at ϕ^k, see Proposition 7.147. Moreover, the local model $\rho = \bigotimes_i \rho_i$ has rational Specht factors ρ_i and operators P_w for permutation and opposition. On W_s, the two extensions are $w \mapsto \rho(w|_R)$ and $w \mapsto \epsilon_t(w)\rho(w|_R)$. Their restriction to $W(R)$ is irreducible, with distinct values on $W_s \setminus W(R)$. Hence, they exhaust the extensions of $\rho|_{W(R)}$. Also, for w fixing Λ, its action is conjugation by $P_w = \rho(w|_R)$. The field and $\mathcal{H}_\ell$ fix i. Moreover, the diagonal d multiplies the local extension by ϵ_t, giving the global pair action.

Let σ_0 fix μ_{2^∞}. On odd-order roots ζ, set $\sigma_0(\zeta) = \zeta^{-1}$. Moreover, the correction $-I$ acts by $y^{-1} \mapsto y$, and σ_0 fixes both signs. Thus, $\chi_\pm, \psi_\pm$ are σ_0-fixed. If $R \neq \varnothing$, the field inertia quotient $\langle \phi^k \rangle$ has odd order. Thus, take extensions $\widehat{\chi}, \widehat{\psi}$. On quotient characters, $\nu^{\sigma_0} = \nu^{-1}$, and the square map $\nu \mapsto \nu^2$ is bijective. Thus, $\widehat{\chi}, \widehat{\psi}$ are unique with $\widehat{\chi}^{\sigma_0} = \widehat{\chi}$ and $\widehat{\psi}^{\sigma_0} = \widehat{\psi}$. Moreover, the mixed stabilizer operations fix $\widehat{\chi}, \widehat{\psi}$ under σ_0-normalization. The central subgroup C has character ω. The diagonal d gives companions $\widehat{\chi}^d, \widehat{\psi}^d$. Thus, both pairs satisfy $\succeq_c$. □

Lemma 7.149. *Let $q = r^f$ with r odd and f even, and let $\ell \mid q+1$ be odd. Since -1 is a square modulo ℓ, the only possible common prime of $q+1$ and $|W(E_7)|$ among odd primes is 5. Put $M = N_G(P)$. If $\ell = 5$,* Lemma 7.119 *gives the torus kernel T_* and quotient $N_W(Q)$. Let $\ell = 5$ and $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ be a proper Levi with some R_j of non-A type. Suppose that $Q \in \mathrm{Syl}_5(W)$ lies in $W(\mathbf{K}^*)$. The unipotent partners χ, ψ on $K, N_K(P)$ admit $\mathcal{H}_5$-fixed extensions $\widehat{\chi}, \widehat{\psi}$ with matching invariants. The common twist κ_s is allowed. If $s = 1$, the same extensions exist on G, M.*

Proof. The Levi $\mathbf{K}^*$ has rank at most 6. A non-A factor has rank at least 4, so no separate A factor contains Q. The factor containing Q is D_5, D_6 or E_6, with at most one additional A_1. By central isotypy, an adjoint unipotent χ_{ad} gives an irreducible character χ before the twist by κ_s. The cyclotomic Levi is T, with source 1_T. The unipotent $5'$-degrees correspond to $\eta \in \mathrm{Irr}_{5'}(W(R))$. By [156, proofs of Theorems 8.4 and 8.5], the graph and field extensions $\widehat{\chi}$ are $\mathcal{H}_5$-fixed for the trivial extension of 1_T.

Locally take the trivial weight on T_*, with quotient $Y = N_{W(R)}(Q)$. By Proposition 7.145, Y has factor $\mathrm{AGL}_1(5)$ for D_5 and $\mathrm{AGL}_1(5) \times C_2$ for D_6, E_6. These Weyl calculations are independent of f. Moreover, the root normalizer adds C_2 centrally. The extended models are $\mathbb{Q}(i)$-valued. By Proposition 7.145, the split D_6 labels have degrees divisible by 5. Keep the separate A_1 label η_1 when present.

The semilinear action on $\mathbf{T}$ has root isometry v and power r^j. The ambiguity on (v, j) is generated by $(-I, f)$. To see this, reduce to $0 \le j < f$ and suppose that $v \equiv r^j I \pmod{q+1}$. For adjacent simple roots α, β, the integer $\langle v\alpha, \beta^\vee \rangle$ lies in $\{0, \pm 1, \pm 2\}$ and is congruent to $-r^j$ modulo $q+1$. It is odd, so $r^j \equiv \pm 1 \pmod{q+1}$. Thus, $j = 0$ since $r^j < q$. At $j = 0$, $\langle v\alpha, \beta^\vee \rangle = \langle \alpha, \beta^\vee \rangle$ since $q + 1 \ge 10$. Thus, $v = I$. In type A_1, the root isometries $\pm I$ give kernel $\langle (-I, f) \rangle$. Moreover, the root normalizers act on $\mathbf{T}$. Choose the odd Tits lift $\widehat{Q}$ of the 5-cycle subgroup Q. Moreover, the corresponding root actions normalize Q.

Write $\rho(-I) = h\,\mathrm{Id}$ for $h \in \{1, -1\}$ in the extended Weyl model ρ. The order of r mod 5 is 4 and $f \equiv 2 \pmod 4$, since $r^f \equiv -1 \pmod 5$ and f is even. In μ_4, we choose a with $a^f = h^{-1}$. Thus, $(v, j) \mapsto \rho(v)a^j$ has value 1 at $(-I, f)$ and inflates the Weyl model for $j = 0$. Its characters are $\mathcal{H}_5$-fixed. Both mixed twists are 1. By Lemma 7.7, κ_s restores the weight. Its contribution at x is common to both models. If $s = 1$ in G, Theorem 7.146 computes $N_W(Q) = \mathrm{AGL}_1(5) \times C_2 \times S_3$. The ordinary McKay count pairs the $5'$-labels η, all $\mathcal{H}_5$-fixed. Choose the scalar a locally and the adjoint extensions $\widehat{\chi}_{\mathrm{ad}}$ globally. Moreover, on $Z(G)$, both models have scalar 1. □

Proposition 7.150. *Suppose that $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ is a proper Levi with R_j of non-A type. Given any $\chi \in \mathcal{E}(G, s) \cap \mathrm{Irr}_{\ell'}(G)$, the comparison $\succeq_c$ holds on $M = N_G(P)$ over the torus weight orbit of s.*

Proof. Let $\Gamma = \widetilde{G} \rtimes \langle \phi \rangle$ for split ϕ. Lang conjugation gives $\mathbf{T}, \mathbf{K}$ there. With $\chi = \epsilon_{\mathbf{G}}\epsilon_{\mathbf{K}} R_{\mathbf{K}}^{\mathbf{G}}(\beta)$, we write $B = N_\Gamma(\mathbf{K})_\beta$. The Weyl stabilizer of s is $W_K = W(\mathbf{K}^*)$. On both marked tori $\mathbf{T}, \mathbf{T}^*$, all Weyl elements lift rationally since F has Weyl part $-I$. Moreover, an ordinary stabilizer element of χ admits an inner correction that stabilizes $\mathbf{T}$. The diagonal representative lies in $\widetilde{\mathbf{T}}$. Each mixed stabilizer element acts by $s \mapsto s'$, including the coefficient power a. Also, the elements $s, s' \in T^*$ are conjugate, with corrections in one W_K-coset. Since $\gcd(a, |s|) = 1$, $C_{\mathbf{G}^*}(s^a) = C_{\mathbf{G}^*}(s)$. A rational correction acts by $C_{\mathbf{G}^*}(s') \mapsto \mathbf{K}^*$, and restores $\mathbf{K}$ in $\mathbf{G}$. After the series on K is restored, injectivity of $R_{\mathbf{K}}^{\mathbf{G}}$ restores β for the corresponding coefficient action. Thus, $\Gamma_\chi = GB$ as in Proposition 7.2. On $\mathbf{T}$, take the corrected maps d of Corollary 7.3. The stabilizers act within $\widetilde{K}$, including diagonal stabilizer elements d.

If $\ell = 5$, the eligible series are those with Q conjugate into W_K. Choose $P \le K$ above $Q \le W_K$. Next, put $L = N_K(P)$ and $B_P = N_B(P)$. The kernel on L and M is T_*. The quotients are $L/T_* = \mathrm{N}_{W_K}(Q)$ and $M/T_* = \mathrm{N}_W(Q)$. By [97, Theorem 4.34] for $T_{5'}$, we distinguish Q-fixed weights by $\theta \mapsto \theta|_{T_*}$. The toral 5-part is contained in T_*. Thus, $L = M_{\theta|_{T_*}}$. Clifford theory gives an irreducible $\mathrm{Ind}_L^M \psi_0$ for each source ψ_0. Its index $[M : L]$ is prime to 5, since both quotients contain Q. Sylow conjugacy in K gives $B = KB_P$, with $K \cap B_P = L$. Consequently, $N_{GB}(P) = MB_P$ and $(GB)/G \simeq B/K \simeq B_P/L$. Each joint stabilizer element admits K-correction to fix P. The local character $\mathrm{Ind}_L^M \psi_0$ determines the orbit of $\theta|_{T_*}$. After restoring $\theta|_{T_*}$, its Clifford correspondent is ψ_0. Thus, the corrections recover inertias GB, MB_P. Choose the twist of Lemma 7.149 by κ_s. By Proposition 7.2, transfer D_G with factor set c. Locally, induce the c-twisted model D_L. With coset representatives in M, the effect of a multiplier $u : B_P/L \to \mathbb{C}^\times$ agrees on D_L and its induction. Consequently, $\succeq_c$ holds on M.

If $\ell > 7$, Lemma 7.119 gives $M = N_G(\mathbf{T})$. The unit source 1_T gives the labels $\mathrm{Irr}(W(R_K))$, all of ℓ'-degree. On M, we choose the weight line κ_s from Lemma 7.144. Moreover, each neutral reflection s_α fixes the lift $\widetilde{s}$. Thus, $\widetilde{\chi}|_G, \widetilde{\psi}|_M$ are irreducible and share diagonal twist ν. By [156, Proposition 4.6 and Table 4.1], the labels η are real. By

Proposition 7.145, inversion by $-I$ and opposition give $\overline{\chi} = \chi$. If the field inertia F_χ has odd order, the real extension $\widehat{\chi}$ determines the mixed functions.

Suppose that $|F_\chi| = m$ is even. By Proposition 7.124, $\operatorname{codim}(R_K)$ is even, so $\operatorname{rank}(R_K) \leq 5$ is odd. A unique D_5 factor would have nontrivial opposition. Thus, $R_K = D_4A_1$. The opposition is inner on D_4, A_1, with split adjoint groups $D_4(q)_{\mathrm{ad}}, A_1(q)_{\mathrm{ad}}$. The quadratic invariant of Proposition 7.124 selects s_0. Next, we choose the adjoint parent $\widetilde{\chi}_0$ in the series of s_0.

On the longest torus $\mathbf{T}_4$ of the split adjoint D_4, pinned triality fixes n_0. Thus, $W(D_4) \rtimes \Gamma_{\eta_4}$ embeds in the extended normalizer modulo T_4. By [156, Lemma 8.6], its extension $\widehat{\eta}_4$ is $\mathcal{H}_\ell$-fixed. Tensor with the A_1 model η_1. As in Proposition 7.124, the scalar a_0 satisfies $a_0^f = \delta(\theta)^{-1}h^{-1}$. Since $\delta(\theta)^2 = 1$ and $h = \pm 1$, choose $a_0 \in \mu_{2f_2}$. The corrected mixed stabilizer operations fix a_0. Thus, the local extension $\widehat{\psi}$ satisfies $U^f = n_0^{-1}$.

Write the adjoint Levi source as $\kappa\tau$, with central line κ and adjoint unipotent product τ. Next, on the corrected field stabilizer A_0, choose the normalized Tits operators for κ with Weyl label 1. Their restrictions to $A_0 \cap K$ equal $\kappa|_{A_0\cap K}$. Indeed, the neutral Tits elements n_α in the rational derived image satisfy $\kappa(n_\alpha) = 1$. Thus, $\widehat{\kappa}$ on KA_0 is fixed by mixed stabilizer operations d. By [156, §8.1 and proof of Theorem D], the unipotent factors on D_4, A_1 admit $\mathcal{H}_\ell$-fixed graph and field extensions $\widehat{\tau}_4, \widehat{\tau}_1$. Moreover, the full source extension is $\widehat{\kappa} \otimes \widehat{\tau}_4 \otimes \widehat{\tau}_1$. In Proposition 7.2, take the adjoint overgroup $D \rtimes \langle\phi\rangle$. The stabilizer of s_0 projects into W_K, whose neutral reflections fix s_0. Moreover, the source extension induces to $\widetilde{\widetilde{\chi}}_0$ with the same mixed functions $\mu_{a,\sigma}$. Each corrected mixed stabilizer operation fixes s_0. Restriction and inflation to G then give $\succeq_c$, as in Proposition 7.124. □

For example, let $q = 3^6$, $\ell = 5$ and $s = \alpha^\vee(\zeta)$, with root α and $|\zeta| = 5$. The element s is rational because $5 \mid q+1$. The pairings with $\alpha^\vee$ lie in $[-2, 2]$. Thus, the neutral roots are D_6 in $\alpha^\perp$. Each lift from $C_{\mathbf{G}^*}(s)$ fixes the unique odd-order lift s_0. Moreover, its connected centralizer gives that $C_{\mathbf{G}^*}(s) = C_{\mathbf{G}^*}(\alpha^\vee(\mathbf{G}_m))$ has Weyl group $W(D_6)$. A direct computational verification gives that $q + 1 = 2 \cdot 5 \cdot 73$, $T_5 = C_5^7$ and $|W(D_6)|_5 = 5$. The 5-cycle acts nontrivially on T_5, as seen on coordinate differences in D_6. Hence, we obtain that $|P| = 5^8$ and P is nonabelian.

Choose κ_s with unipotent label 1. The field square ϕ^2 acts by $s \mapsto s^{-1}$, since $3^2 \equiv -1 \pmod 5$. The reflection s_α corrects ϕ^2. Each ordinary field exponent j satisfies $3^j \equiv \pm 1 \pmod 5$, since the correction on $\alpha^\vee(\mathbf{G}_m)$ is a sign. Thus, $\langle\phi^2\rangle$ is the field image, of order 3. A rational lift n_α exchanges the parabolics for $\alpha^\vee, -\alpha^\vee$. For $\sigma \in \mathcal{H}_5$ inverting the 5-power roots and fixing the $5'$-roots, we have that $\kappa_s^{(n_\alpha,\sigma)} = \kappa_s$. Moreover, the action of n_α exchanges the radicals U, U^- at each position. Proposition 7.2 compares the $\kappa_s^\vee$-summands on the ordinary field image C_3 modulo inner and diagonal automorphisms.

Proposition 7.151. *Let $s = ty$, with $y \neq 1$ of odd order and $C_{\mathbf{G}^*}(s)$ connected and non-Levi. If $\ell > 7$, we have that the sources are D_aD_b with optional A_1, or A_5A_1 in ${}^2E_6$. If $\ell = 5$, only the A_5A_1 source occurs. The marked comparisons give $\succeq_c$ on $M = N_G(P)$.*

Proof. We find $\gcd(|y|, 6) = 1$. By the argument of Proposition 7.125, $\mathbf{K}^* = C_{\mathbf{G}^*}(y)$ is a proper Levi. Since $(q+1)_2 = 2$, $t^2 = 1$. Also, $C_{\mathbf{G}^*}(s) = C_{\mathbf{K}^*}(t)$ and $t \neq 1$. Next, on a simple factor of $\mathbf{K}^*$, let t_j be the adjoint image of t. The order of t_j divides 2. In type A, the eigenspaces V_+, V_- of t_j give the block Levi. Next, take λ with centralizer $\mathbf{K}^*$. Choose μ with the desired centralizer in $\mathbf{K}^*$. Then, $n\lambda + \mu$ has the same centralizer in $\mathbf{G}^*$ for large n. Indeed, on the roots of $\mathbf{K}^*$ its pairings are those of μ. For the other roots β, the value $\langle\beta, n\lambda + \mu\rangle$ is nonzero. Hence, some R_j of non-A type has a non-Levi centralizer C_j.

In type D, lift t_j to $g \in \mathrm{SO}_{2n}$ with scalar g^2. If $g^2 = -I$, its isotropic eigenspaces V_i, V_{-i} give a Levi GL_n. If $g^2 = I$ and g is nonscalar, its eigenspaces have dimensions $2a, 2b$. When $\min(a, b) = 1$, the root system is the Levi D_{n-1} with its torus. Thus, $a, b \geq 2$ in a non-Levi case. A D-component has the trivalent node α_4 and its 3 neighbors. The arms in E_7 give $4 \leq a + b \leq 6$, with at most one additional A_1. Moreover, the neutral Weyl group on D_{a+b} is $W(D_a) \times W(D_b)$, with $a, b \leq 4$. The optional A_1 contributes a factor of order 2. Therefore, $5 \nmid |W(R)|$, so $\mathcal{E}(G, s)$ has no character of $5'$-degree.

If $\ell > 7$, take the orthogonal sources for D_4, D_5, D_6. Moreover, for D_4, each split eigenspace V_+ or V_- is the sum of 2 negative planes and has dimension 4. Triality on D_4 gives the other vector coordinates. Given D_5, D_6, a rational exchange of V_+, V_- requires $a = b$. In particular, $(a, b) = (2, 4)$ has no exchange. In type D_{a+b}, see Propositions 7.127 and 7.128. In type A_1, the weight is 1 or the quadratic character θ_2. Next, for 1, we choose the labels $1, \mathrm{St}$. For θ_2, we choose the extensions of [186, Theorem 4.2], denoted $\widehat{\theta}_{2,\pm}$. The tensor comparison is $\succeq_c$ by Proposition 7.6. With fixed adjoint t, triviality on Z_{odd} selects the involution lifts t_0 by Section 7.5. For central $y \in \mathbf{K}^*$, $C_{\mathbf{K}^*}(ty) = C_{\mathbf{K}^*}(t)$.

Thus, connectedness holds for all translated ty.

On E_6, the non-Levi type is A_5A_1 by [208, §§10.5–10.7]. The negative mark gives $H = {}^2E_6(q)$. The central product with $\mathbf{Z}$ has kernel C of order dividing 3. Moreover, on $\mathbf{Z}$, the action of F is $-q$. Since $3 \nmid q+1$, $C^F = 1$ and $H^1(F, C) = 1$. In characteristic 3, the geometric kernel is trivial. Thus, $K = H \times Z$, with $|Z| = q+1$ and $N_K(P) = N_H(P \cap H) \times Z$. Keep the character ω_Z of Z.

By [208, Proposition 10.7, Steps 3–4], the adjoint principal-series extensions $\widehat{\chi}$ are rational. If $\lambda = (3,2,1)$, the parabolic source is $\theta_2\tau_\lambda$, with unitary cuspidal character τ_λ and quadratic line θ_2. At $-q \equiv 1 \pmod{\ell}$, the generic degree is congruent to $\pm 2^4$, the signed Specht degree modulo ℓ. Extend θ_2 by 1 on the pinned field ϕ. By [156, §8], $\widehat{\tau}_\lambda$ is $\mathcal{H}_\ell$-fixed. On the parabolic overgroup inflate $\widehat{\theta}_2\widehat{\tau}_\lambda$. Each rational constituent χ_i has multiplicity 1. By [208, Lemma 3.13], induction gives the unique extension $\widehat{\chi}_i$ above χ_i, fixed by $\mathcal{H}_\ell$.

Let α be perpendicular to A_5, with torus coordinate $T_\alpha \simeq \mu_{q+1}$. Moreover, its weight inertia acts on H_α by conjugation. The induced map to $\mathrm{PGL}_2(q)$ gives the determinant square class ε_α, with value 1 on ϕ. A generator b of $\mathbb{F}_{q^2}^\times$ has nonsquare norm, and b/b^q generates μ_{q+1}. Thus, the rational extension $\widehat{\theta}_2$ affords the given quadratic weight on T_α. If $\ell > 7$, the relative group on $N_H(P \cap H)$ is $S_6 \times C_2$. If $\ell = 5$, we let Q be the 5-cycle subgroup of S_6. Choose an odd Tits lift $\widehat{Q}$ with the same toral Sylow T_5. By Proposition 7.124, the quotient over T_* is $\mathrm{AGL}_1(5) \times C_2$. Next, we restrict $\widehat{\theta}_2$ to this inertia. The 5 affine labels over $\mathbb{Q}(i)$ and the 2 signs match the labels of S_6 of weight one. The marked lift U fixes the Tits lifts $\widehat{Q}$ and the Weyl quotient Y. Moreover, on H, its outer order is $2f$. There $U^f = \mathrm{Int}(n_0^{-1})\gamma$, where γ is the diagram involution, and $U^{2f} = 1$. Next, extend the Weyl model D_W by 1 on U, tensor with $\widehat{\theta}_2$, and induce. If $\ell > 7$, the model D_W has group S_6 in place of $\mathrm{AGL}_1(5)$. By Lemma 7.7, restore ω_Z on the H models. Given an ambient $U^{2f} \in Z$, the scalar is $\omega_Z(U^{2f})$. Since $C_{\mathbf{G}^*}(s) \le \mathbf{K}^*$, signed induction to G and local induction to M give the comparison over y. □

Lemma 7.152. *Let $s \in G^*$ be the involution with connected roots A_7. By* Zhang [208, Proposition 11.23]*, we find two rational forms $s_{\mathrm{sp}}, s_{\mathrm{u}}$ and diagonal pairs. Let $q = r^f$ be odd, $\ell \mid q+1$ odd, $P \in \mathrm{Syl}_\ell(G)$ and $M = N_G(P)$. The unitary form of s contributes precisely for $\ell \neq 3$, and the split form contributes no ℓ'-degree characters. Moreover, each eligible partition $\lambda \vdash 8$ gives local characters $\psi_\pm$ with line values $j, -j$, where $j^2 = k_z = (-1)^{(q+1)/2}$. The numbers are $44, 30, 14$ for $\ell > 7, \ell = 5, \ell = 7$, respectively.*

Proof. For form $\varepsilon = \pm 1$, put $X = \varepsilon q$. The ambient degrees are $2, 6, 8, 10, 12, 14, 18$, and the denominator has factors $X^d - 1$ for $2 \le d \le 8$. All remaining factors are powers of 2 from isogeny, components and restriction. If $\varepsilon = 1$, the denominator has 4 even degrees against 7 in the numerator. $\Phi_2(q)$ divides the index, so its ℓ-valuation is positive. If $\varepsilon = -1$, $X \equiv 1 \pmod{\ell}$. Moreover, the valuation formula is $v_\ell(X^d - 1) = v_\ell(q+1) + v_\ell(d)$. The seven torus contributions cancel from $v_\ell([G^* : C_{G^*}(s)]_{r'})$. A direct computational verification gives that $(2 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 18)/(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8) = 2^3 \cdot 3^2$. Thus, the index excludes $\ell = 3$.

Given an eligible unipotent label λ, the unitary hook formula gives the ℓ-valuation of the Specht degree f^λ. Division by 2 has no effect on $v_\ell(f^\lambda)$. If $\ell > 7$, all 22 partitions in [208, proof of Proposition 11.23, Step 2] are eligible. If $\ell \in \{5, 7\}$, their weight is 1 and the cores have sizes $3, 1$, respectively. For each core, the ℓ runners label the partitions. By [208, Proposition 11.23(iii)–(iv)], $\chi_\pm(z)/\chi_\pm(1) = k_z$.

Let n_0 be the Tits lift of $-I$. By [208, proof of Proposition 11.23, Step 1], $s_0^2 = z$ and n_0 exchanges s_0, zs_0. Thus, $W_s = W(A_7) \times \langle -I \rangle$. Whenever $\alpha \in A_7$, $\theta(\alpha^\vee(\mu_{q+1})) = 1$. Thus, the neutral line θ_0 extends θ with value 1 on n_α. Since $n_0 n_\alpha = n_\alpha n_0$ and $n_0^2 = z$, its extensions have values $j, -j$ on n_0. If $\ell > 7$, tensor with the Specht model V_λ. If $\ell \in \{5, 7\}$, we let $Q = \langle (1\,2\,\ldots\,\ell) \rangle$. By Proposition 7.136, choose an odd complement $\widehat{Q}$ in the Tits preimage and keep $T_* = T_\ell C_{T_{\ell'}}(Q)$. The inertia quotient over T_* is $\mathrm{AGL}_1(\ell) \times S_{8-\ell} \times \langle -I \rangle$. Next, take the core λ_0 for $S_{8-\ell}$ and runners for the affine labels η, all $\mathcal{H}_\ell$-fixed. The restricted lines still take $j, -j$ on n_0. Moreover, induction exhausts the fibre of $\theta|_{T_*}$. The lift discrepancy $s_0^{-1} = zs_0$ makes the diagonal act by $j \mapsto -j$ and fix η. □

Lemma 7.153. *Assume $q = r^f$ is odd. Next, we choose s in either half-spin class and $\lambda = (3,2,1)$ on its unitary form. Assume that $\ell \neq r$ is odd and $v_2(\mathrm{ord}_\ell(q)) = 1$. The pinned field group fixes $\chi_\pm$. If $\sigma \in \mathcal{H}_\ell$ with exponent a on μ_r, their permutation has sign $\left(\frac{a}{r}\right)^f$. If $2 \mid f$, the quotient twist of $\widehat{\chi}_\pm$ at ϕ is $\left(\frac{a}{r}\right)$. If $2 \nmid f$, the permutation on the chosen $\widehat{\chi}_\pm$ is that on $\chi_\pm$.*

Proof. Let $\mathbf{H} = \mathrm{Spin}_{12}$. By [208, Proposition 7.36, Steps 1, 3–5], the Jordan type is $(5,5,1,1)$ with nonsplit planes V_5, V_1. If $q \equiv 1 \pmod 4$, Step 4 gives nonsquare determinants on both V_d. If $q \equiv 3 \pmod 4$, Step 5 gives nonsplit

V_d. In the character Γ_θ, the multiplicity is 2 for $\widetilde{\chi}$ and 1 for each $\chi_\pm$. Moreover, the radical U has common additive character θ.

If $d \in \{5, 1\}$, we choose a block with basis $v_0, \dots, v_{d-1}$, shift $Xv_j = v_{j+1}$ for $j < d-1$, and $Xv_{d-1} = 0$. Moreover, the form B_d has entries $(v_j, v_{d-1-j}) = (-1)^j$ and other entries 0. Tensor with the split plane $V_{d,0}$ over $\mathbb{F}_r$. Next, choose $e_d \in V_{d,0}$ with norm 1, and let s_{e_d} be its reflection. The simultaneous isometry g has minus eigenspace of dimension 6. This space has square discriminant since $\det(B_d) = 1$. The volume element gives w in the spin cover over $\mathbb{F}_r$. A direct computational verification gives that $w^2 = (-1)^{6\cdot 5/2} = z_0$, the spin kernel involution. If $F = \mathrm{Int}(w)F_q$, the planes become nonsplit and $H = \mathbf{H}^F$ is split. On U, the Cayley trace gives $\theta^w = \theta^{F_r} = \theta$. Indeed, X, B_d are defined over $\mathbb{F}_r$, and the absolute trace is F_r-invariant.

On the two θ-lines, the eigenvalues of w are $\kappa, -\kappa$. Moreover, for $q \equiv 1 \pmod 4$, Step 4 gives the two characters on $\langle w, Z(H)\rangle$ with distinct values on w and equal central restriction. Given $q \equiv 3 \pmod 4$, Step 5 takes simultaneous inversion w. Changing w multiplies both by the weight ξ of the rotations in rank 6. In both cases $\kappa^2 = \omega(z_0) = (-1)^{(q+1)/2}$. Next, on $K = U\langle w\rangle$, we let $\nu_\kappa|_U = \theta$ and $\nu_\kappa(w) = \kappa$. This character has multiplicity 1 in one restriction and 0 in the other.

Let u act by F_r on H, with $u^f = w^{-1}$. By [133, Theorem 21.7], choose $h \in \mathbf{H}$ with $h^{-1}F_q(h) = w$. After conjugation by h, $H \cong \mathbf{H}^{F_q}$ and acts by $u \mapsto hF_r(h)^{-1}\phi$, where ϕ is the pinned generator. The multiplier $hF_r(h)^{-1}$ is F_q-fixed and $(hF_r(h)^{-1}\phi)^f = hw^{-1}h^{-1}$. By Step 1, ϕ fixes the rational series and λ. Since u fixes each ν_κ, the pinned generator ϕ fixes $\chi_\pm$. Extend ν_κ to $K\langle u\rangle$ with scalar α_κ on u, where $\alpha_\kappa^f = \kappa^{-1}$. Moreover, its induction to $H\langle u\rangle$ restricts to $\mathrm{Ind}_K^H \nu_\kappa$. The base multiplicity 1 selects an extension $\widehat{\chi}_\kappa$ uniquely. On each size-5 block take weights $2, 1, 0, -1, -2$, and weight 0 on each size-1 block. The positive weights sum to $2(2+1) = 6$, so the cocharacter lifts to d_0 in the spin coroot lattice. Over $\mathbb{F}_r$, $w(d_0) = d_0$. Since X has degree -1, the corrector for θ^σ is $d_0(a^{\pm 1})$ with $d_0(a^{\pm 1}) \in H$ centralizing u. Thus, σ replaces (κ, α_κ) by $(\sigma(\kappa), \sigma(\alpha_\kappa))$. If f is even, then $\kappa = \pm i$ and we choose α_κ of order $4f_2$, where $f_2 = 2^{v_2(f)}$. By Section 2, $\sigma(\alpha_\kappa)/\alpha_\kappa = (a/r)$. If $2 \nmid f$, take the unique $\alpha_\kappa \in \langle\kappa\rangle$ with $\alpha_\kappa^f = \kappa^{-1}$. Then, $\sigma(\alpha_\kappa) = \alpha_{\sigma(\kappa)}$. By Section 2, $\sigma(\kappa)/\kappa = (a/r)$. Moreover, each quotient character ν satisfies $\nu(u) = \nu(\phi)$. □

Proposition 7.154. *The characters of* Lemma 7.152 *satisfy* $\succeq_c$ *at* M *for each eligible* $\lambda \vdash 8$*, including* $\lambda = (5,2,1), (5,2,1)^{\mathrm{t}}$.

Proof. The marked lift U fixes the Tits lifts n_α and the weight θ. Since $U^f = n_0^{-1}$, take scalar α on U with $\alpha^f = j^{-1}$. If f is even, $k_z = -1$, take α of order $4f_2$ with $f_2 = 2^{v_2(f)}$. The base characters are $\mathcal{H}_\ell$-fixed. Given σ of exponent a on μ_r, the extra sign on U is (a/r) by Section 2. If f is odd, then $\sigma(j)/j = (a/r)$ by Section 2. Thus, $(a/r)^f$ is the base sign.

If $2 \nmid f$, we set $u_0 = \sum_{x\in Q} x(2\omega_7^\vee)$, with $Q = 1$ for $\ell > 7$. Then, u_0 is Q-fixed and $u_0(-1) = z$ since $|Q|$ is odd. Put $V = n_0U$ and $h = [u_0(b), c]$ in the regular overgroup. Next, choose $b^{-r-1} = -1$ and $c^{r-1} = -1$, with c of order $2(r-1)_2$. The discrepancy at V is $[z, -1] = 1$. Thus, h is rational for the mark with f odd, and c^2 is a nonsquare in $\mathbb{F}_q^\times$. Since Q fixes u_0, the element h normalizes P. Also, h^2 is inner times the common scalar c^2. The local lift has $V^f = z^{(f-1)/2}$, so take scalar $k_z^{(f-1)/2}$ on V. Since f is odd, the f-th power is $k_z^{(f-1)/2}$, independent of j. The choice is invariant under $h, \mathcal{H}_\ell$ for every affine label η.

By [208, proof of Proposition 11.23, Step 3], $\chi_+ + \chi_-$ is rational. Let c_0 be complex conjugation. For χ in this pair, we set $\tau = c_0$ if $\chi^{c_0} = \chi$, and $\tau = hc_0$ otherwise. Then, $\chi^\tau = \chi$ and $\nu^\tau = \nu^{-1}$ on the odd field quotient $\langle V\rangle$. By [208, Lemma 3.1 and proof of Lemma 11.14], χ has a unique extension $\widehat{\chi}$ with $\widehat{\chi}^\tau = \widehat{\chi}$. Given a diagonal or coefficient action d, $(\widehat{\chi}^d)^\tau = (\widehat{\chi}^\tau)^d$. Thus, the base permutations determine $\mu_d(V)$ for these extensions. On $C = \langle z, c^2\rangle$, take the common central character.

If $0 \le d \le 4$, let $\mathbf{J}_d$ centralize the directions $a_i = e_i - e_{9-i}$ for $1 \le i \le d$. If $c_1 \gg c_2 \gg c_3 \gg c_4 > 0$, we have that the parabolics for $\sum_{i=1}^d c_i a_i$ are nested. The element $w = -w_{0,A_7}$ fixes each a_i and exchanges the two lifts of s. Next, we choose w as the unitary mark, with $(-I)w \in W(A_7)$. The chain is rational and $w \in W(\mathbf{J}_d)$. Moreover, every middle partition λ_d is opposition-fixed. The regular parent $\widetilde{\chi}_d$ has two constituents because w exchanges its lifts and fixes λ_d. Distinct labels λ_d give disjoint pairs. Moreover, correction of s_0^{-1} gives rational sums $\chi_+^{(d)} + \chi_-^{(d)}$.

At each step exclude L_i, L_i', paired and isotropic. Next, for fixed 2-core κ and 2-weight k, we apply the unitary branching $W(B_k) \downarrow W(B_{k-1})$. A rim domino δ gives one quotient box of multiplicity 1. The induction from a source pair $\chi_\pm^{(d)}$ to a target pair $\chi_\pm^{(d-1)}$ has matrix B_d with column sums 1. Indeed, the sum $\chi_+^{(d)} + \chi_-^{(d)}$ induces with coefficient 1 on each target. The diagonal h exchanges both pairs, so B_d is a permutation matrix. Thus, B_d intertwines the pair

permutations.

The mark w lies in $W(\mathbf{J}_4)$. By [114, corollary to Theorem 1] on $\mathbf{J}_4$, choose the split prime-field chain $(\mathbf{P}_d)_d$. Induction and radical invariants for $\mathbf{P}_d$ are equivariant for σ and quotient twists ν. A base multiplicity of 1 gives one extension $\widehat{\chi}_{\pm}^{(d-1)}$ in the upward direction and one $\widehat{\chi}_{\pm}^{(d)}$ by reciprocity. Thus, for $\kappa = \varnothing$ the extensions are those for (8) through $d = 4$. If $\lambda = (8)$, we choose the dual $D_G(\chi_{\mathrm{reg},\pm})$. The Whittaker calculation on U_G is Proposition 7.147, with sign k_z at z. Its corrector $x = (2\rho^\vee)(\sqrt{a})$ belongs to G when a is a square in $\mathbb{F}_q^\times$. Otherwise, x induces the outer diagonal d. Thus, the base sign is $(a/r)^f$. If $2 \mid f$, $x \in G$ and $\phi(x)x^{-1} = z^e$, with $(-1)^e = (a/r)$. The diagonal d gives the companion orbit. Next, let ϕ_d lift the field map corrected by $d\phi(d)^{-1}$. In the covering group, ϕ_d^f can be z. The chosen scalar $\alpha_d \in \mu_{2f_2}$ is $\mathcal{H}_\ell$-fixed, so the extra twist is 1. The extended duality D_G is equivariant under the quotient twist ν. The parabolic sum has signed multiplicity ± 1 on $\chi_{(8)}$. Moreover, the corresponding quotient character determines $\widehat{\chi}_{(8)}$. If f is odd, apply the normalization by V, h. The chain covers 20 of the 22 uncut pairs.

For size 8, the other 2-core is $(3, 2, 1)$. Moreover, its weight is 1, giving $\lambda = (5, 2, 1), (5, 2, 1)^{\mathrm{t}}$. Both have hooks of lengths $5, 7$, so both survive at $\ell = 5, 7$. The direction $a_1 = e_1 - e_8$ is an E_7 root with perpendicular system D_6. Moreover, the middle differences $e_i - e_j$, for $2 \le i, j \le 7$, form A_5 with value 1 on s. For the other D_6 roots β, the end signs agree and $\beta(s) = -1$. Also, in orthogonal coordinates, all 6 entries of the lift equal b. The pair sums have value $b^2 = -1$. Thus, $H = \mathrm{Spin}_{12}^+(q)$ has unitary source with label $(3, 2, 1)$. The mark w fixes a_1 and lies in $W(D_6)$. After the inner untwist, ϕ acts on H with the same field quotient F_0.

Moreover, the regular Levi $\widetilde{\mathbf{J}}_1$ has connected centre and derived group $\mathbf{H}$ with roots D_6. The Levi root lattice $\mathbb{Z}D_6$ is a summand in the saturation L_{sat} of the ambient roots in $X(\widetilde{\mathbf{T}})$. By [208, Proposition 7.36, Step 1], the parent $\widetilde{\chi}_1$ with label $(3, 2, 1)$ restricts on H to $\chi_{H,+} + \chi_{H,-}$. Its restriction to J_1 has two constituents $\chi_{1,+}, \chi_{1,-}$. Hence, each restricts to one distinct $\chi_{H,\pm}$. Given any field extension $\widehat{\chi}_{1,\pm}$ on J_1, a quotient twist gives restriction $\widehat{\chi}_{H,\pm}$. The common generator ϕ detects each twist ν on restriction. By Lemma 7.153, at odd f the base sign is (a/r). At even f, the characters $\chi_{H,\pm}$ are fixed and the sign at ϕ is (a/r), with scalar order $4f_2$. On the first edge, $W(B_1)$ branching has multiplicity 1 on both target labels λ. Thus, the field signs agree for $\widehat{\chi}_{\pm}, \widehat{\psi}_{\pm}$. □

Let $G = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ odd and odd $\ell \mid q + 1$. By Zhang [208, proof of Proposition 11.19, Steps 1–2], the dual involution s has connected centralizer of type E_6T_1 and component group C_2. The rational forms have signs $\varepsilon = \pm 1$, each with 30 diagonal pairs $\chi_\pm$ in the full series. Next, choose s on the $-q$ mark.

Lemma 7.155. *The twisted form contributes for $\ell \neq 7$, and the split form contributes no ℓ'-degree characters. If $Q \in \mathrm{Syl}_\ell(W(E_7))$ in $W(E_6)$, the number of eligible pairs is $|\mathrm{Irr}_{\ell'}(N_{W(E_6)}(Q))|$. On the local lines in $M = N_G(P)$, the values at n_0 are $j, -j$, with $j^2 = (-1)^{(q+1)/2}$. Moreover, for each eligible η, the two base and field signs ϵ_η, μ_η are those in* Proposition 7.154.

Proof. First, put $x = \varepsilon q$. The fundamental degrees give the factor $I_x = |(x^5 + 1)(x^9 + 1)(x^{14} - 1)/(x - 1)|$ in the centralizer index I. Central maps and restriction contribute 2-powers. If $x = q$, the factor $x^5 + 1$ is divisible by ℓ. If $x = -q$, the unsigned factor is congruent to $2^3 \cdot 7$ modulo ℓ. Thus, $\ell \mid I_x$ exactly for $\ell = 7$. By [40, pp. 461–481], the cubic cuspidal degree is $\frac{q^7}{3}|\Phi_1(x)^6\Phi_2(x)^4\Phi_4(x)^2\Phi_5(x)\Phi_8(x)|$. Since $x \equiv 1 \pmod{\ell}$, they are excluded even at $\ell = 3$, where $v_3(\Phi_1(x)^6) \ge 6$. The rational cuspidal label ${}^2E_6[1]$ has degree $\frac{q^7}{6}(q-1)^4\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{18}(q)$. At $\ell = 5$, the factor $\Phi_{10}(q)$ excludes that label. At $\ell = 3$, $v_3(\Phi_{18}(q)) = 1$ since $q^3 \equiv -1 \pmod 9$. Thus, the factor cancels the 3-part of the denominator. If $\ell > 7$, the degree of ${}^2E_6[1]$ is a unit at $q \equiv -1 \pmod{\ell}$.

By Ennola substitution, we choose $\pm d_\lambda(-q)$ for split degree polynomial d_λ. Next, let m exceed all ℓ-valuations of denominators of d_λ. By Dirichlet, take an odd prime $q' \equiv -q \pmod{\ell^{m+1}}$. After clearing denominators, $\ell \mid d_\lambda(q')$ exactly when $\ell \mid d_\lambda(-q)$. By [166, Proposition 6.1 and Theorem 6.2], the split count is $|\mathrm{Irr}_{\ell'}(W(E_6))|$, including at $\ell = 3$. By Weyl–McKay, the count is $|\mathrm{Irr}_{\ell'}(N_{W(E_6)}(Q))|$. The count is over all labels λ.

The marked quadratic weight θ has Weyl inertia $W(E_6) \times \langle -I \rangle$. By Lemma 7.144, take the neutral extension κ_s. It is inversion-invariant since $\theta^2 = 1$. By [208, proof of Proposition 11.19, Steps 1–2], $n_0^2 = z$ and its central sign is $(-1)^{(q+1)/2}$. Thus, the line value j satisfies $j^2 = (-1)^{(q+1)/2}$. Next, for $\ell = 3, 5$, we choose an odd Tits complement $\widehat{Q}$ over Q, with 2-kernel below it. The corrections of $\widehat{Q}$ lie in the subsystem. Moreover, on T_*, restriction is injective on Q-fixed weights. The inertia quotient is $N_{W(E_6)}(Q) \times \langle -I \rangle$. If $\ell = 3$, the calculation on A_2^3 gives rational labels η. At $\ell = 5$, the factor $\mathrm{AGL}_1(5) \times C_2$ has values in $\mathbb{Q}(i)$. If $\ell > 7$, take the rational characters of $W(E_6)$. The field fixes the Tits lifts n_α. Moreover, the outer diagonal exchanges $j, -j$ and fixes the labels η. Choose v on the central line with $v(-1) = z$. The group $W(E_6)$ fixes v. Thus, the diagonal h of Proposition 7.154 centralizes E_6 and gives

the normalization on $V = n_0 U$. If $2 \mid f$, we choose the scalar α on U with $\alpha^f = j^{-1}$. The local sign at U is (a/r) for exponent a on μ_r by Proposition 7.154 at U. □

Proposition 7.156. *For each eligible noncuspidal pair $\chi_\pm$ on ${}^2E_6$, the base sign is $(a/r)^f$. If $\sigma \in \mathcal{H}_\ell$ and even f, the extensions have twist (a/r) at ϕ. If $2 \nmid f$, they admit the normalization on V. The nonprincipal source $\lambda = (3,2,1)$ is in* Lemma 7.153.

Proof. First, take the 25 ordinary principal labels η before the degree cut. If $w = -w_{0,E_6}$, we choose the fixed-space torus $\mathbf{T}_+$. Next, we let $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}_+)$ with mark w. The fixed space contains x in the E_6 chamber. Thus, its pointwise stabilizer in $W(E_6)$ is trivial. The other component contains w, and $C_{\widetilde{\mathbf{L}}^*}(\widetilde{s})$ is a torus. The root lines $\gamma_1, \gamma_2, \gamma_3$ have type A_1. Moreover, their negative mark gives rational $\mathrm{SL}_2(q)$ factors with elliptic tori T_i. The regular character $\widetilde{\psi}$ restricts to $\psi_+ + \psi_-$. Also, for a nondegenerate Whittaker λ, regular inclusion and reciprocity give $\langle \Gamma_\lambda, \psi_+ + \psi_- \rangle_L = 1$.

Given any additive coordinate λ_i, take its square class relative to B_i. Specifically, the square classes form an $\mathbb{F}_2^3$-torsor. The regular Levi $\widetilde{L}$ surjects onto the three adjoint factors D_i, giving all translations of $\mathbb{F}_2^3$. The image of L has index at most 2 since the regular diagonal quotient modulo $Z(\widetilde{L})$ has order 2. Moreover, the two Whittaker detectors for $\psi_\pm$ make this image proper. The rational 3-cycle normalizes $\mathbf{L}$ and centralizes w. It permutes the three coordinates of $\mathbb{F}_2^3$. Specifically, the invariant hyperplane is $H_0 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$. Hence, scaling all additive coordinates by a changes detector exactly when a is a nonsquare in $\mathbb{F}_q^\times$. Thus, the permutation of $\psi_\pm$ has sign $(a/r)^f$.

If $2 \mid f$, Lang conjugation in $\mathbf{L}$ gives a prime-field parabolic $\mathbf{P}$. Put $c_i = \gamma_i^\vee(-1)$. The quadratic weight on T_i gives value -1 on c_i, since $(q+1)/2$ is odd. For detector λ_+, we put $y = 1$. For detector λ_-, put $y = \gamma_1^\vee(\sqrt{b})$ with b nonsquare in $\mathbb{F}_q^\times$. Then, $t = yF_r(y)^{-1} \in L$ and $u = t\phi$ fixes λ_+ or λ_-, respectively. Its powers u^f are $1, c_1$, respectively. Next, choose scalar $\beta \in \mu_{2f_2}$ on u with the given β^f, and the central values on c_i. The coefficient corrector $x = \prod_i \gamma_i^\vee(\sqrt{a})$ changes u by $(c_1c_2c_3)^e$, with $(-1)^e = (a/r)$. Since $\sigma(\beta) = \beta$, multiplicity 1 in the induced module gives twist (a/r) on both extensions.

The characters $\psi_\pm$ are Harish–Chandra cuspidal. Otherwise, $\widetilde{\psi}^{U_0} \neq 0$ for a proper rational parabolic $P_0 = L_0U_0$. Then, L_0^* contains $\widetilde{s}$ and its unique centralizing torus $\widetilde{T}_s$, contrary to ellipticity of $\widetilde{T}_s$. Thus, the supports of $I_\pm$ are equal or disjoint. If $\eta = 1$, the multiplicities of a fixed χ_1 in I_+, I_- sum to 1, by the Borel coefficient in $C_{G^*}(s)^\circ$. Therefore, I_+, I_- have disjoint supports. The outer diagonal d exchanges I_+, I_-. Given any principal label η, the Borel coefficient m_η is positive. Thus, $I_\pm$ contains one of $\chi_{\eta,+}, \chi_{\eta,-}$ with multiplicity m_η. By [208, proof of Proposition 11.19, Step 3], the sum $\chi_{\eta,+} + \chi_{\eta,-}$ is rational for each principal label η. The permutation of $I_\pm$ determines that of $\chi_{\eta,\pm}$.

Next, induce $\widehat{\psi}_\pm$ over the standard parabolic field group $P\langle\phi\rangle$. The field fixes each supporting coset PvP. Moreover, its intertwining line is $\mathbb{C}T_v$ by cuspidal Mackey and Schur. Thus, ϕ acts on Howlett–Lehrer generators by $T_v \mapsto a_v T_v$. Given a reflection v, the quadratic relation has constant coefficient $b_v \neq 0$. Given a complement element v, the inverse coset $Pv^{-1}P$ gives T_v^{-1} up to scalar. Thus, T_v acts on χ_1 by a nonzero scalar, with multiplicity 1. By [208, proof of Proposition 11.19, Step 3], $\chi_1^\phi = \chi_1$. Thus, the scalar is ϕ-fixed and $a_v = 1$. Hence, the field centralizes $\mathrm{End}(I_\pm)$. On each isotypic summand $\chi_\eta^{\oplus m_\eta}$, there is a unique extension $\widehat{\chi}_\eta$ repeated m_η times. The induced extension of $I_\pm$ has twist (a/r), so each $\widehat{\chi}_\eta$ has that twist for every m_η.

For the nonprincipal labels, we let γ be the highest root of E_6. Its perpendicular systems are A_5 in E_6 and D_6 in E_7. Since $w(\gamma) = \gamma$, its parabolic $\mathbf{P}_\gamma$ is rational. The inner untwist gives $H = \mathrm{Spin}_{12}^+(q)$. The minuscule weight restricts to ω of half-spin type with stabilizer $W(A_5)$. Moreover, the mark w gives unitary type and label $\lambda = (3,2,1)$. Inverting the 6 coordinates exchanges the lifts and fixes λ. Thus, the regular parent $\widetilde{\chi}_L$ splits into $\chi_{L,\pm}$. By [208, Proposition 7.36, Step 1], these restrict irreducibly to distinct $\chi_{H,\pm}$. Regular induction of λ has coefficient 1 on each target $\chi_\pm$. The diagonal gives one constituent from $\chi_{L,+}$ and one from $\chi_{L,-}$ in each pair. Moreover, on H, L, the common field quotient is F_0, as in Proposition 7.154. Lemma 7.153 gives sign $(a/r)^f$. If $2 \mid f$, the twist at ϕ is (a/r). If $2 \nmid f$, choose the normalization by V, h. □

Proposition 7.157. *If $\ell = 3$ or $\ell > 7$, the pair $\chi_{\mathrm{cusp},\pm}$ for ${}^2E_6[1]$ has the same signs as $\chi_{\mathrm{ss},\pm}$. Both base signs are $(a/r)^f$. Together with* Proposition 7.156*, we obtain that $\succeq_c$ holds on M for the eligible series of s.*

Proof. Take $\mathbf{L}$ of type $A_2(q)A_1(q^3)\Phi_3$ from the first E_7 entry of [129, Table 3]. The semisimple character $\widetilde{\psi}$ has coefficient ± 1 at $\widetilde{\chi}_{\mathrm{ss}}, \widetilde{\chi}_{\mathrm{cusp}}$ by [129, proof of Theorem 3.2]. The latter has label ${}^2E_6[1]$. In $\mathbf{L}^*$, the connected centralizer has type A_2. Moreover, its component exchanges the lifts s_0, zs_0 and fixes the label 1. Thus, $\widetilde{\psi}|_L = \psi_+ + \psi_-$. The

diagonal d exchanges both $\psi_\pm$ and each target pair $\chi_\pm$. The signed induction matrix B_χ has rows $(A,B),(B,A)$, with $A+B=\pm1$. Indeed, Lusztig induction $R_{\mathbf{L}}^{\mathbf{G}}$ intertwines the restrictions from $\widetilde{L},\widetilde{G}$. In the lifted series, only $\widetilde{\chi}$ lies above the pair. Thus, $A\neq B$ and $A\not\equiv B \pmod 2$. By [53, Proposition 3.3], the alternating trace Ξ is integral. If $2\nmid r$, parabolic independence gives d-equivariance. Both target sums $\chi_+ + \chi_-$ are rational. Since $A\neq B$, the permutation of $\psi_\pm$ determines that of $\chi_\pm$. Thus, the cuspidal base sign is $(a/r)^f$. If $2\nmid f$, take the normalization by V,h.

Suppose that f is even, and put $n=f_2$ and $q_0=r^{f/n}$. In a split model of $\mathbf{L}$, we choose a prime-field representative b of the mark, with $F=\mathrm{Int}(b)F_q$. Its image in $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ has order 3. Moreover, it fixes the A_2 diagram, cycles the three A_1 factors and acts by Φ_3 on the perpendicular plane. Choose $kn\equiv 1 \pmod 3$ and put $D=\mathrm{Int}(b^k)F_{q_0}$. Then, $b^{1-kn}\in\mathbf{L}$. By [133, Theorem 21.7] on $\mathbf{L}$, choose x with $x^{-1}D^n(x)=b^{1-kn}$. With $g=x^{-1}D(x)b^k$, we put $F_0=\mathrm{Int}(g)F_{q_0}$. The product $gF_{q_0}(g)\cdots F_{q_0}^{n-1}(g)$ is $x^{-1}D^n(x)b^{kn}=b$. Thus, $F_0^n=F$ with equal multipliers. Since $F_{q_0}(b)=b$, cycling the product gives $g=bF_q(g)b^{-1}$. Consequently, $gb^{-k}\in\mathbf{L}^F$. In $\mathbf{G}$, we choose h with $g=h^{-1}F_{q_0}(h)$. Then, $b=h^{-1}F_q(h)$, so h transports F_0 to the pinned field of order n.

Represent the involution s_L in the split torus of $\mathbf{L}^*$. Moreover, the split fields fix s_L. The Weyl mark b fixes its geometric class $\mathcal{C}$, as does b^k. By [133, Theorem 21.11(a)], $\mathcal{C}$ contains $s'_L\in(\mathbf{L}^*)^{F_0}$. Write $C=C_{\mathbf{L}^*}(s'_L)$. By [133, Theorem 21.11(b)], the rational classes for F_0^n correspond to F_0^n-conjugacy classes in C/C°. Since $C/C^\circ\cong C_2$, F_0 acts trivially on it. Hence, $\mathcal{C}$ has two rational classes for F_0^n, both F_0-stable. The label 1 gives the stable pair $\psi_\pm$. Each member is fixed since $\chi_{\mathrm{ss},\pm}^{F_0}=\chi_{\mathrm{ss},\pm}$ and $A\neq B$. By [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], take the rotation ρ and $\mathcal{F}=\rho^{-1}\,\mathrm{diag}(F_0)$ on $\mathbf{G}^n,\mathbf{L}^n$. Projection gives the fixed groups G,L, and ρ induces F_0. Moreover, given a parabolic $\mathbf{P}$ with Levi $\mathbf{L}$, the corresponding variety Y for $\mathbf{P}^n$ has the algebraic action of ρ, with $\rho^n=1$.

On $G\times L$, we compute the ordinary coefficients. Since $q_0>2$, Lusztig induction from $\mathbf{L}^n$ to $\mathbf{G}^n$ is independent of $\mathbf{P}^n$ by [72, Theorems 3.3.7 and 3.3.8]. Compute with $U_i=F_0^i(U_0)$ for $0\le i<n$, with U_0 from Malle's induction. By [208, proof of Lemma 4.19(b)], projection to g_0U_0 gives the $G\times L$-equivariant isomorphism with $Y_{U_0}^{F_0^n}$. Thus, induction for $\mathbf{P}^n$ has coefficients A,B. By [53, Proposition 3.3], the alternating trace Ξ is integral also on cosets of ρ. The tensor product $\Xi\otimes_L\widehat{\psi}$ is equivariant for coefficient operations σ and quotient twists ν. On the virtual χ-part, the signed multiplicity is m. If m is odd, then $\gcd(m,n)=1$. Thus, [208, Lemma 9.10] gives an extension with twist ν. This normalization transports with χ.

Choose an extension $\widehat{\psi}_\pm$. Moreover, the odd matrix entry m gives a semisimple extension with twist ν equal to that of $\widehat{\psi}_\pm$. The order of r mod ℓ has 2-part $2n$, since $r^f\equiv -1 \pmod\ell$. Hence, $\ell\equiv 1 \pmod n$, so $\mathcal{H}_\ell$ fixes $\mathrm{Irr}(C_n)$. Each extension over C_n of $\chi_{\mathrm{ss},\pm}$ has twist (a/r) at $\phi^{f/n}$, whose exponent f/n is odd. Thus, both rows for $\psi_\pm$ have this twist. Moreover, each cuspidal $\chi_{\mathrm{cusp},\pm}$ has an odd coefficient m in one row. Extraction gives $\widehat{\chi}_{\mathrm{cusp},\pm}$ over C_n with twist ν.

On $C_{f/n}$, [208, Lemma 3.1] gives the rational extension $\widehat{\chi}_{\mathrm{odd}}$. Since $C_f=C_n\times C_{f/n}$, [156, Lemma 7.1] combines the extensions on $C_n,C_{f/n}$. If f is even, the bases $\chi_\pm,\psi_\pm$ are $\mathcal{H}_\ell$-fixed. The ordinary stabilizer is $G\langle\phi\rangle$ modulo $Z(G)$, since the outer diagonal d exchanges the branches. Next, choose $G\rtimes\langle\phi\rangle$ and $N_{G\rtimes\langle\phi\rangle}(P)$ as preimages, with kernel $Z(G)$ on G. Both central characters take z to k_z. A common section gives factors $c(a,b)$ and equal conjugation scalars on $Z(G)$. By Lemma 7.155, the twist at ϕ is (a/r). The cubic degrees are divisible by ℓ. The other labels satisfy $\succeq_c$ by Proposition 7.156. □

Lemma 7.158. *Let $q=r^f$ be odd and $H=\mathrm{Spin}_{12}^+(q)$. Next, in either half-spin class s with unitary connected centralizer, we write $\chi_{\lambda,\pm}$ for the pair over $\lambda\vdash 6$. The pinned field group $\langle\phi\rangle$ fixes each $\chi_{\lambda,\pm}$. For even f and odd $\ell\mid q+1$, we have that both $\mathcal{H}_\ell$ and σ_0 fix $\chi_{\lambda,\pm}$, where σ_0 fixes μ_{2^∞} and inverts each odd-order root ζ. Moreover, for odd f and odd $\ell\mid q+1$, the pair sign of $\sigma\in\mathcal{H}_\ell$ is (a/r), where $\sigma(\zeta_r)=\zeta_r^a$. In particular, we have that $\mathcal{H}_5$ fixes both characters when $5\mid q+1$.*

Proof. If $\lambda=(3,2,1)$, apply Lemma 7.153. If $2\mid f$, the sign for σ_0 is 1 since $\sigma_0(i)=i$. If f is odd and $5\mid q+1$, $r^f=-1$ in $\mathbb{F}_5^\times$ gives $r\equiv -1 \pmod 5$. Thus, quadratic reciprocity gives that $(5/r)=(r/5)=1$. An exponent a of $\mathcal{H}_5$ on μ_r is a power of 5 mod r, so $(a/r)=1$. The other partitions $\lambda\vdash 6$ have 2-core $\varnothing$. Moreover, parabolic induction compares them with $\chi_{(6),\pm}$.

In half-spin coordinates, we choose $u_j=e_{2j-1}-e_{2j}$ for $1\le j\le 3$. Moreover, the Weyl mark w exchanges the isotropic halves V_+,V_- and swaps the letters $(2j-1,2j)$ in each pair. Thus, $w(u_j)=u_j$ and $w(s_0)=-s_0$ for the natural lift s_0. Moreover, $w\in W(\mathbf{J}_3)$, generated by reflections in $e_{2j-1}+e_{2j}$. Lang conjugation in $\mathbf{J}_3$ gives the rational chain $(\mathbf{J}_j)_j$ fixed by the pinned field ϕ. Moreover, at level j, the nonidentity component has representative w. On the

unitary factor U_j^* of the regular centralizer, opposition fixes λ_j. By [208, Propositions 7.21 and 7.36, Step 1], we have $\widetilde{\chi}_j|_{J_j} = \chi_{j,+} + \chi_{j,-}$ with $\chi_{j,+} \neq \chi_{j,-}$.

Given a rim domino $\delta \subset \lambda_j$, the partition $\lambda_{j+1} = \lambda_j \setminus \delta$ has one fewer 2-quotient corner. Hence, in R_j, its coefficient is 1. By [208, Proposition 7.21], the separated plane V_j has the same sign ϵ at each step. At $\mathbf{J}_3$, the roots $e_{2j-1} + e_{2j}$ take value -1 on s_0. Thus, the lift s_0 gives one label on J_3. For the two pairs $\{\chi_{j,\pm}\}$ and $\{\chi_{j+1,\pm}\}$, restriction of R_j gives column sums 1 in B_j. By Lang's theorem, $\widetilde{J}_3$ surjects onto $D = \widetilde{H}/H$ and D acts transitively on each pair $\{\chi_{j,\pm}\}$. Thus, both rows of B_j are nonzero, so B_j permutes $\{+,-\}$. The actions of ϕ and σ fix each pair sum $\chi_+ + \chi_-$, since on the marked torus $s^\phi = s^a = s$ for odd a. A central discrepancy at s_0 gives a twist ν with $\nu|_{J_j} = 1$. The labels λ_j and plane signs ϵ are fixed. Thus, B_j transfers the action to core $\varnothing$, including $\lambda = (6)$.

If $\lambda = (6)$, we compare the regular pair $\chi_{\mathrm{reg},\pm}$ and its Alvis–Curtis dual $D_H(\chi_{\mathrm{reg},\pm})$. In a rational split Borel B, we choose λ_{Wh} with simple-root coefficients 1. Moreover, the regular parent $\widetilde{\chi}_{\mathrm{reg}}$ has multiplicity 1 in the Gelfand–Graev character $\widetilde{\Gamma}_{\mathrm{Wh}}$. Restriction to the common radical gives $\langle \Gamma_{\mathrm{Wh}}, \chi_{\mathrm{reg},+} + \chi_{\mathrm{reg},-} \rangle_H = 1$, so exactly one constituent occurs. $\lambda_{\mathrm{Wh}}^\phi = \lambda_{\mathrm{Wh}}$. If f is even, each $a \in \mathbb{F}_r^\times$ is a square in $\mathbb{F}_q^\times$. If f is odd and $\ell = 5$, we found $(a/r) = 1$, so $a \in (\mathbb{F}_r^\times)^2$. The element $x = (2\rho^\vee)(\sqrt{a}) \in H$ then corrects the coefficient action on λ_{Wh}. Thus, σ fixes $\chi_{\mathrm{reg},\pm}$ and its duals $D_H(\chi_{\mathrm{reg},\pm})$.

For odd f and nonsquare a, the adjoint image of x is rational. The Lang discrepancy of x is $(2\rho_D^\vee)(-1)$ in the spin cover. Moreover, in D_6 coordinates, $\rho_D^\vee = (5,4,3,2,1,0)$ is congruent to e_1 modulo $\mathbb{Z}D_6$. The pairing of $\rho_D^\vee$ with $\omega_\pm = (e_1 + \cdots + e_5 \pm e_6)/2$ is $1/2$ modulo $\mathbb{Z}$. Thus, the Lang discrepancy $(2\rho_D^\vee)(-1)$ pairs to -1 with the lift commutator $w(s_0)s_0^{-1}$. By [208, §3.1 and Proposition 7.36], the diagonal action exchanges $\chi_{\mathrm{reg},+}$ and $\chi_{\mathrm{reg},-}$. The matrices B_j transfer the exchange to the 10 empty-core partitions. If $\lambda = (3,2,1)$, by Lemma 7.153, we obtain the sign (a/r) also for $\ell \equiv 3 \pmod 4$. □

Proposition 7.159. *Let $q = r^f$ be odd with f even. Suppose that $s = ty$ has $y \neq 1$ of odd order and disconnected $C_{\mathbf{G}^*}(s)$. Assume that y lies on a root line of $C_{\mathbf{G}^*}(t)^\circ$ and that $\ell \nmid [G^* : C_{G^*}(s)]$. Each eligible λ gives two pairs with comparison $\succeq_c$ on $M = N_G(P)$. If $\ell = 5$, every eligible disconnected series with $y \neq 1$ is of this form and has 5 eligible partitions.*

Proof. Write $y = \alpha^\vee(u)$, where $|u| > 1$ is odd. Every root β has $|\langle \beta, \alpha^\vee \rangle| \leq 2$. Thus, the roots of $C_{\mathbf{G}^*}(y)$ are D_6 in $\alpha^\perp$. The roots of $C_{\mathbf{G}^*}(t)^\circ$ have type A_7 or E_6. In both cases, the roots in $\alpha^\perp$ have type A_5. In half-spin coordinates for D_6, the A_5 roots are $e_i - e_j$. The natural lift t_0 has 6 equal coordinates b with pair-sum value $b^2 = -1$. Thus, $t_0^2 = -I$ on the natural module V. The unitary series on $X = \mathrm{Spin}_{12}^+(q)$ is in [208, Lemma 7.15 and Propositions 7.21, 7.36]. The lift n_α centralizes X. Thus, for the pinned longest lift $n_D \in X$, we mark the negative torus by $n_D n_\alpha$.

The component $-I_D$ exchanges the lifts of t and fixes the partition λ. Thus, the regular parent $\widetilde{\chi}$ has two constituents $\chi_{K,\pm}$ on the finite Levi K. By Lemma 7.158, the restriction is $\widetilde{\chi}|_X = \chi_{X,+} + \chi_{X,-}$, with distinct constituents. Thus, $\chi_{K,\pm}|_X = \chi_{X,\pm}$ after ordering. A correction returning y on the root line can be 1 or n_α, both acting trivially on X. Lang conjugation for n_D changes the field on X by an inner automorphism ι_x with $x \in X$. By Lemma 7.158, the field, $\mathcal{H}_\ell$ and σ_0 fix $\chi_{X,\pm}$. The outer diagonal d exchanges $\chi_{X,+}$ and $\chi_{X,-}$. Thus, signed induction $R_{\mathbf{K}}^{\mathbf{G}}$ gives these permutations on $\chi_\pm$.

Locally extend the neutral A_5 weight θ. The values j at n_D satisfy $j^2 = \theta(n_D^2) = -1$. By Proposition 7.147, $(n_D n_\alpha)^2 = z$ and $\theta(z) = -1$. Since $\alpha(t) = 1$ and y has odd order, $\theta(n_\alpha^2) = \theta(\alpha^\vee(-1)) = 1$. Thus, $j = \pm i$, exchanged by the outer diagonal d. If $\ell > 7$, take the S_6-label λ on each line. At $\ell = 5$, Proposition 7.132 gives the local affine label η on $N_X(P \cap X)$. Moreover, the 4 linears have values in μ_4, and the rational augmentation has degree 4. At n_D the two extensions have values j and $-j$. Their quotient character ϵ_t takes values in $\{1,-1\}$. Thus, the permutation on $j = \pm i$ equals that on $\chi_{X,\pm}$.

We write $\langle \phi^k \rangle$ for the field inertia, with $k \mid f$. The Weyl element w_k normalizes $\alpha^\vee(\mathbf{G}_m)$, so $r^k \equiv \pm 1 \pmod{|u|}$. Since $r^f \equiv -1 \pmod{|u|}$ and $|u| > 2$, $r^k \equiv -1 \pmod{|u|}$ and f/k is odd. We choose n_α with $(y^{\sigma_0})^{n_\alpha} = y$. Put $C = \langle \phi^k \rangle$. Since σ_0 fixes both characters and inverts $\mu_{|C|}$, $\mu_a \notin K$ for $K = \mathbb{Q}(\chi), \mathbb{Q}(\psi)$ and primes $a \mid |C|$. By [208, Lemma 3.2], the unique extensions $\widehat{\chi}, \widehat{\psi}$ over $\mathbb{Q}(\chi), \mathbb{Q}(\psi)$ are σ_0-fixed. Both $\phi^k, \mathcal{H}_\ell$ fix each sign j, whereas d exchanges j and $-j$. Thus, d is outside the joint stabilizer J. Since C has odd order and D_2 is cyclic of 2-power order, we have $[C, D_2] = 1$. With identical scalars on $\mathrm{Z}(G)$, we obtain $\succeq_c$ for both j. Finally, at $\ell = 5$, Lemma 7.131 puts each eligible disconnected s with $y \neq 1$ on $b\alpha^\vee(\mathbf{G}_m)$. If $s = b\alpha^\vee(v)$, the two coordinates of b at α coincide, and $v_{(2)} = \pm 1$. Thus, $\alpha(t) = \alpha(b)v_{(2)}^2 = 1$, so α is a root of $C_{\mathbf{G}^*}(t)^\circ$. □

Corollary 7.160. *Let $q = r^f$ be odd and $\ell \in \{3,5,7\}$ divide $q+1$. Put $G = E_{7,\mathrm{sc}}(q)$ and $P \in \mathrm{Syl}_\ell(G)$. Then,* $\mathrm{iMN}(E_7(q),\ell)$ *holds with local group* $\mathrm{N}_G(P)$.

Proof. By Lemma 7.119, choose the negative torus T with $P \le \mathrm{N}_G(T)$. Put $M = \mathrm{N}_G(P)$, $T_* = M \cap T$, and let Q be the image of P in $W = W(E_7)$. Given $\psi \in \mathrm{Irr}_{\ell'}(M)$, we let $\mathcal{O} \subseteq \mathrm{Irr}(T_*)$ be its weight orbit. Since $|\mathcal{O}| \mid \psi(1)$ and $Q \lhd N_W(Q)$, we have $\mathcal{O} \subseteq \mathrm{Irr}(T_*)^Q$. By Lemma 7.119, we have the bijection $\mathrm{Irr}(T)^Q \to \mathrm{Irr}(T_*)^Q$ given by $\theta \mapsto \theta|_{T_*}$. Moreover, for $\theta, \theta' \in \mathrm{Irr}(T)^Q$, we have $\theta' \in W \cdot \theta$ exactly when $\theta' \in N_W(Q) \cdot \theta$.

If $\ell \in \{3,7\}$, the field degree f is odd because -1 is a nonsquare modulo ℓ. If $s = 1$, by Theorem 7.146, we obtain $\succeq_c$ with labels on $N_W(Q)$. Given $s \ne 1$ with connected centralizer, apply Proposition 7.145. If $C_{\mathbf{G}^*}(s)$ is disconnected, its component order divides 2, so $Q \le W(R)$. If $\ell = 3$, by Theorem 7.146, $R = E_6$, since $v_3(|W(R)|) \le 3$ for classical R and the ratio for E_6A_1 is nonsquare. If $\ell = 7$, by Theorem 7.146, $R = A_6$ or A_7. The maximal Levis of types E_6, A_6 have connected central line $\mathbf{Z}$. An extra Weyl stabilizer w inverts $\mathbf{Z}$, so $s = s^w = s^{-1}$. In type A_7, the root lattice has index 2, which also gives $s^2 = 1$. Corollary 7.214 gives the comparisons on M.

For $\ell = 5$ and odd f, the unipotent comparison covers $s = 1$. If $C_{\mathbf{G}^*}(s)$ is connected and $s \ne 1$, by Proposition 7.145, we obtain $\succeq_c$. By Lemma 7.131, the other s lie on $s(v) = b\alpha^\vee(v)$. If $v = \pm 1$, apply Corollary 7.214 on M. If $v^2 \ne 1$, we apply Corollary 7.133 on M. For $\ell = 5$ and even f, $v_2(f) = 1$, $r \equiv 2,3 \pmod 5$ and $3 \nmid q+1$. Write $s = ty$ with t its 2-part. Since $(q+1)_2 = 2$, $y = 1$ implies $s^2 = 1$. Lemma 7.149 covers $s = 1$, and Corollary 7.214 covers $s \ne 1$. If $y \ne 1$ and $C_{\mathbf{G}^*}(s)$ is a Levi $\mathbf{K}^*$, Propositions 7.124 and 7.150 give $\succeq_c$ on M. For connected non-Levi $C_{\mathbf{G}^*}(s)$, Proposition 7.151 excludes orthogonal roots R since $5 \nmid |W(R)|$. Only A_5A_1 in ${}^2E_6$ remains. By Proposition 7.151, we obtain 20 rational principal-series extensions and 2 parabolically induced extensions from $\lambda = (3,2,1)$. Moreover, for the cuspidal $\widehat{\chi}$, $\mathrm{ord}_5(-q) = 1$ meets the hypothesis of [156, Lemma 8.1 and proof of Proposition 8.5]. On each A_1-sign, the local $\mathrm{AGL}_1(5)$-labels are 4 linears and the augmentation η_{aug}. For disconnected $C_{\mathbf{G}^*}(s)$, by Lemma 7.131, we obtain the D_6 root line. Thus, for $y \in \alpha^\vee(\mathbf{G}_m)$, we obtain $\succeq_c$ from Proposition 7.159. If $2 \mid f$, $\gcd(|y|,6) = 1$. By Proposition 7.151, $C_{\mathbf{G}^*}(y)$ is the Levi $\mathbf{L}^*$. Thus, $R_{\mathbf{L}}^{\mathbf{G}}$ extends to Γ_χ by Corollary 7.3. By Clifford theory, ψ determines its T_*-orbit and ψ_0 on the weight inertia M_θ. Thus, disjoint local sets exhaust $\mathrm{Irr}_{\ell'}(M)$. Apply Lemma 9.4 for each prime ℓ. □

Lemma 7.161. *Let $q = r^f$ be odd and $\ell > 7$ divide $q+1$. Suppose that $C_{\mathbf{G}^*}(s)$ is disconnected and the proper Levi $\mathbf{K}$ has type D_4A_1 as in* Proposition 7.128*, also for even f. Then, we have $\succeq_c$ on $M = N_G(P)$. Moreover, we have $\succeq_c$ if $C_{\mathbf{G}^*}(s)$ is connected and non-Levi and $C_{\mathbf{G}^*}(y)$ has type D_4A_1 as in* Proposition 7.151.

Proof. The inversion w on the negative planes $V_1, \dots, V_4$ lies in $W(D_4)$. Thus, we have the split factor $H_4 = \mathrm{Spin}_8^+(q)$. In the vector marking, write (a,b) for the sign-space ranks. If $(a,b) = (1,3)$, Proposition 6.120 gives both orientations on $W(D_4) \cdot \theta$, including the diagram images θ^γ. If $(a,b) = (2,2)$, each four-space $V_\pm$ is the orthogonal sum $V_i \perp V_j$. Since both planes are negative, $V_\pm$ is split. Moreover, inversion of V_i, V_j lies in $W(D_2)$. Hence, the cocycle c is trivial on the four A_1-factors. By Proposition 6.114, we have all 16 branches with their triality action γ. For connected non-Levi $C_{\mathbf{G}^*}(s)$, only $(a,b) = (2,2)$ occurs. Also, on the A_1-factor H_1, the extensions $\widehat{\theta}_{1,\pm}$ of the quadratic weight θ_1 give the nonsplit pair $\chi_{1,\pm}$ of [186, Theorem 4.2 and its proof]. The exception $q = 9$ has no $\ell > 7$ dividing $q+1$. For central projection s_{H_1}, we have $\theta_1 = 1$ and the two unipotent labels on H_1.

Next, we write $H = [\mathbf{K},\mathbf{K}]^F$, $K = \mathbf{K}^F$ and $Z_\ell = (Z(\mathbf{K})^{\circ F})_\ell$. Since the central isogeny $[\mathbf{K},\mathbf{K}] \times Z(\mathbf{K})^\circ \to \mathbf{K}$ has kernel of 2-power order, we have $P = (P \cap H)Z_\ell$. On the two derived factors H_i for $i = 4,1$, we have $\ell \nmid |W(H_i)|$, so their local groups are $N_{H_i}(T_i)$ for the marked tori T_i. Since $H_4 \not\cong H_1$, we form $\rho_4 \otimes \rho_1$ for operations returning both factors simultaneously. Write E_T for the external torus normalizer. On H, we combine its image with $\mathrm{Inn}(H)$. If an inner correction $x \in H$ normalizes $P \cap H$, $x \in N_H(P \cap H)$. Moreover, an element g in the external normalizer of $\mathbf{K}$ that centralizes H fixes $P \cap H$ and normalizes Z_ℓ. Hence, $P^g = P$, so the two images on H coincide for the stabilizer of $P \cap H$. By [141, Theorem 2.9], we lift $\succeq_c$ through the common image on H. Then, Proposition 7.6 gives the characters of $K, N_K(P)$ common quotient multiplicity U_0. Thus, the two central scalars on Z coincide.

If f is odd, take all lifts of (s_H, κ) as in Proposition 7.128. If f is even, we compare the involution t over $1_{Z_{2'}}$ on the odd central torus $Z_{2'}$. By Section 7.5, we twist by ν_y on K and $N_K(P)$. Thus, the unions $\mathcal{U}, \mathcal{V}$ are invariant and $C_{\mathbf{G}^*}(s) \le \mathbf{K}^*$. The assignments from $\mathbf{A}$ and $C_{\mathbf{G}^*}(y)$ are equivariant for conjugacy and F_r. Moreover, $s \mapsto s^a$ fixes both choices for $\gcd(a,|s|) = 1$. Consequently, under $R_{\mathbf{K}}^{\mathbf{G}}$ the comparison extends to Γ_χ by Corollary 7.3. Locally, $W_\theta \le W(\mathbf{K})$, so Clifford induction to M gives the corresponding orbits. □

Proposition 7.162. *Let $q = r^f$ be odd and $\ell > 7$ divide $q+1$. If $C_{\mathbf{G}^*}(s)$ is connected, then each eligible $\chi \in \mathcal{E}(G,s)$ has the comparison $\succeq_c$ on $M = N_G(P)$.*

Proof. If $s = 1$, we have the correspondence of [156, Theorem D] with $C_G(T) = T$ and $N_G(T) = M$. If f is odd and $s \neq 1$, we apply Proposition 7.145. If $2 \mid f$, Propositions 7.124 and 7.150 give $\succeq_c$ for genuine proper Levi $C_{\mathbf{G}^*}(s)$. Moreover, if $s^2 = 1$, by Corollary 7.214, we obtain $\succeq_c$. For the remaining $s = ty$ with odd part $y \neq 1$, by Proposition 7.151, we obtain the Levi $\mathbf{K}^* = C_{\mathbf{G}^*}(y)$ and the possible non-Levi $C_{\mathbf{K}^*}(t)$.

Suppose that the field inertia of χ has odd order m. The eligible Jordan labels lie in $\mathrm{Irr}(W(R))$ for the negative scalar torus of $C_{\mathbf{G}^*}(s)$. In the orthogonal cases R has factors D_a, D_b and possibly A_1. Alternatively, in the exceptional case R has factors A_5, A_1. Moreover, scalar Frobenius F fixes each component of R. Since $W_s = W(R)$ fixes $\widetilde{s}$, we have the neutral-root character κ_s even for a non-Levi $C_{\mathbf{G}^*}(s)$. Thus, by Proposition 7.145, the restrictions $\widetilde{\chi}|_G, \widetilde{\psi}|_M$ are irreducible. Moreover, each corrected mixed stabilizer operation d gives a common central quotient twist ν_d. Since $(-I)(s^{-1}) = s$ and opposition is inner on $W(R)$, both base characters χ, ψ are real.

Write D for the cyclic 2-part of the regular quotient and C_m for the field inertia quotient. Since $|\mathrm{Aut}(D)|$ is a power of 2, we have $[D, C_m] = 1$ and inertia quotient $D \times C_m$ over the preimage of D. Since $\gcd(m, |D|) = 1$, translations of C_m on the $\mathrm{Irr}(D)$-torsor are trivial. Extend $\widetilde{\chi}, \widetilde{\psi}$ and normalize over C_m by reality. Moreover, each mixed stabilizer operation d normalizes the preimage of C_m, since field powers centralize C_m and $[D, C_m] = 1$. Thus, the twists coincide on C_m by uniqueness and on D by the comparison of $\widetilde{s}$. Thus, odd m suffices even when $2 \mid f$.

Finally, let $m = f/k$ be even, with field generator ϕ^k, and let R_K be the roots of $\mathbf{K}^*$. Since ϕ^k returns the series of s, we choose w with $w(y) = y^{r^k}$ and w fixing a chamber of R_K. Then, $w^m = (-I)w_{0,R_K}$. Indeed, both elements take y to y^{-1} and their quotient in $W_y = W(R_K)$ fixes a chamber. On $V_0 = \mathrm{Fix}(W(R_K))$, the right side acts by $-I$. Since m is even, $\det(w^m|_{V_0}) = 1$, so $7 - \mathrm{rk}(R_K)$ is even. Thus, R_K cannot be E_6, D_6 or D_5A_1. If $R_K = D_5$, we have that the image of $(-I)w_{0,R_K}$ in $C_2 = \mathrm{Aut}(\Delta_{D_5})$ is nontrivial, whereas w^m maps to 1. Consequently, Proposition 7.151 leaves only $R_K = D_4A_1$. By Lemma 7.161, we obtain $\succeq_c$ on M. □

Lemma 7.163. *Let $\ell > 7$ divide $q+1$. Moreover, on the negative scalar torus $\mathbf{T}$, we have $\succeq_c$ on the rational weight orbits for $\mathrm{I}(1,4), \mathrm{I}(2,3)$ in $\mathrm{Spin}^-_{10}(q)$ and for $\mathrm{I}(3,3)$ in $\mathrm{Spin}^+_{12}(q)$.*

Proof. Each j-plane sign block Y_j has type $(-1)^j$. Moreover, the Weyl labels ρ have $\ell \nmid \rho(1)$. Given $\mathrm{I}(3,3)$, both six-spaces $Y_\pm$ have type $^-$. Also, their labels $\lambda \vdash 4$ on ${}^2A_3$ correspond to bipartitions of 2 on rational flags $E_1 < E_2$. Each symbol Λ has absolute defect 2: the defect is 2 mod 4, and absolute defect $2s$ has rank at least s^2. Thus, two ordinary 1-hooks give the rank-one symbol Λ_0 through nondegenerate Λ_j. If $\Lambda_+ = \Lambda_-$, choose one hook h in both. Otherwise, we choose h in one factor until $\Lambda'_+ = \Lambda'_-$. By [208, Proposition 7.18(i), Steps 1–2], we have the columns at E_i and the rational component actions γ_i. The complements R_+, R_- each contain a plane, and Λ_0 has empty additional partition $\varnothing$. By Lemma 6.57 we obtain Φ, the nonsplit pair.

From Φ, we apply Propositions 6.58 and 6.74 along $(\Lambda_j)_j$. For paired h, Lemma 6.56 gives the determinant intertwiner on $E_i \oplus E'_i$. At a single hook on E_i, take 1 or $\eta_q^{\mathrm{fld}} \circ \det_{E_i}$. Thus, on the multiplicity-one spaces L_i we get the field powers A_i^f and coefficient signs g_σ as above. By Proposition 6.220, there are 2 branches for each of the $\binom{5}{2} = 10$ unequal pairs and 4 for each equal pair (Λ, Λ). Given a mixed stabilizer operation (g, σ), its operator H gives the sign at the similitude g.

Given $\mathrm{I}(2,3)$, the four-space Y_2 is split and the six-space Y_3 is minus. Moreover, the equal A_1 entries on Y_2 give $1, \mathrm{St}$, with two orientations each. Choose E_- in Y_2 and a two-line flag $E_{+,1} < E_{+,1} + E_{+,2}$ in Y_3. The split residue R_- and nonsplit residue R_+ have the models $\Phi_\pm$ of Lemma 6.88. By [208, Proposition 7.18(i)], the columns at E_i give the ordinary extensions $\widehat{\chi}_\pm$. The distinct A_1 entries form one orbit $\{(1,\mathrm{St}), (\mathrm{St}, 1)\}$. Specifically, on the rulings E, E', the coefficients are $1, 0$, respectively. The degree fraction $q/(q+1)$ has the Y_3-degree cancelled. Moreover, each partition $\lambda \vdash 4$ has $\mathrm{core}_2(\lambda) = \varnothing$. Thus, Proposition 6.187 gives the extension $\widehat{H}_\lambda$ on Y_3 in both cases. Thus, $W(B_2) \times W(D_3)$ give $5 \cdot 5 = 25$ local characters.

Given $\mathrm{I}(1,4)$, the negative eigenspace Y_4 is split, and the positive plane Y_1 is anisotropic. A nondegenerate Λ_4 of defect 0 gives a split plane by three 1-hooks, with unequal rows at each Λ_j as in Proposition 6.58. Pair R_- with Y_1 by Lemma 6.88. On E_1, E_2, E_3, the marked induction of [208, Proposition 7.18(i)] has weights $\eta_q^{\mathrm{fld}} \circ \det_{E_i}$. For degenerate (α, α), we have $\alpha \in \{(2), (1,1)\}$. On the first isotropic plane E_1, the marked constituent χ_1 has degree fraction $q/(q+1)$. By [208, Proposition 7.20(1), Step 1], the box on E_2 gives opposite orientation coefficients $1, 0$. Both flags leave Y_1 fixed. By Proposition 6.168, we obtain $\widehat{\chi}_\alpha$ for $\alpha = (2), (1,1)$.

The remaining symbol $\Lambda_C = (\{0,1,2,3\}, \varnothing)$ has absolute defect 4 and gives the cusp ψ_C. On $H = \mathrm{SO}(Y_4 \perp Y_1)$, take $t = (-I_{Y_4}, I_{Y_1})$. Moreover, the full centralizer $C_{\mathbf{H}}(t)$ is quasi-isolated. Its reflection group $W(D_4)$ has zero fixed space on $X(T_4) \otimes \mathbb{Q}$ for a maximal torus T_4 on Y_4, and the improper component inverts T_1 on Y_1. Next, choose $\psi_C \boxtimes 1$ with cusp ψ_C on Y_4. A proper Harish–Chandra restriction needs $\lambda : \mathbf{G}_m \to C_{\mathbf{H}}(t)$ split over $\mathbb{F}_q$. Since T_1 is anisotropic, λ gives a proper Levi L_4 in the Y_4-factor. Thus, cuspidality of ψ_C gives zero restriction. For $\chi \in \mathrm{Irr}(H)$ above, Theorem 10.8 of [165] gives an extension $\widehat{\chi}$ to $\mathrm{O}(Y_4 \perp Y_1)$ fixed by $F_r, \mathcal{H}_\ell$. The character $\nu_H \neq 1$ of H/Ω gives $t \mapsto -t$. Since $\dim \ker(-t + I) = 2$, the classes of $t, -t$ are distinct and $\chi\nu_H \neq \chi$. Thus, $\chi|_\Omega$ is irreducible.

By [208, Proposition 7.25(ii), Steps 2–4], an oriented constituent ξ on Ω has a one-dimensional source space L_ξ for the generalized Gelfand–Graev character Γ. On each odd block J_d, we choose the shift X_d and form B_d of Lemma 6.238. Choose w over $\mathbb{F}_r$ with $w^2 = 1$ giving the discriminants of the spaces M_d. Then, the field operator B fixes L_ξ and has $B^f = w^{-1}$. Normalize its eigenvalue β on L_ξ in μ_h for $h = \mathrm{ord}_\ell(r)$. Since $\mathrm{ord}_\ell(q) = 2$, the map $\beta \mapsto \beta^f$ on μ_h has image μ_2, so β^f can equal the scalar of w^{-1}. Given $\sigma \in \mathcal{H}_\ell$, the half-grading isometry b corrects the additive source and satisfies $[b, w] = [b, B] = 1$. Since $h \mid \ell - 1$, we have $\beta^\sigma = \beta$ for $\sigma \in \mathcal{H}_\ell$. Hence, evaluation on L_ξ gives quotient twist 1. The other orientation is ξ^g for a common nonsquare similitude g. The ordered volume scalar at $t = 2$ is $+1$ by the cited Proposition 7.25. Thus, on $W(B_4)$, Proposition 6.185 gives the comparison $\succeq_c$. For equal rows (α, α), the two similitude extensions finish the 20 comparisons, with identical central scalars. □

Lemma 7.164. *Let $q = r^f \equiv 3 \pmod 4$ and $\ell > 7$ divide $q + 1$. If $X = \mathrm{Spin}^+_{12}(q)$, the homogeneous $\mathrm{Q}(2,2)$ orbit has comparison $\succeq_c$ on $N_X(P)$, where $P \in \mathrm{Syl}_\ell(X)$. Each unitary partition $\lambda = (2), (1,1)$ gives 16 characters.*

Proof. On the six negative planes V_i, the lift s_0 is rational. Moreover, the sign spaces $Y_\pm$ are split of dimension 4. The quartic planes give $\mathrm{GU}_2(q)$ with $\lambda = (2)$ or $(1,1)$. In both cases, $\mathrm{core}_2(\lambda) = \varnothing$. On a split four-space, put $A = (1,1)$ and $B = (\mathrm{St}, \mathrm{St})$. The two orientations $C = (1, \mathrm{St}), (\mathrm{St}, 1)$ give the degenerate symbol with partition (1). Thus, the nondegenerate symbols are A, B.

An ordinary 1-hook gives the split sign torus T_1. For equal symbols, take paired h. Given A, B, choose a hook from A and then from B. On $Y_\pm$, we choose marked isotropic lines $E_\pm$ and their opposite lines $E'_\pm$. By [208, Proposition 7.18(i), Steps 1–2], the columns have multiplicity 1 and component actions γ_i. The split sign residues $R_\pm$ have the same λ. By Lemma 6.53, its unique domino d gives a flag. Next, choose $E = \langle e_1, f_2 \rangle$ and its opposite E' give the flag. On $R_\pm$, the simultaneous reflection $r_+ r_-$ and the swap w generate the sign-space component A_s. The exchange u fixes E, E' after transposing indices $1, 2$. Thus, the flag component map π_E is onto. The connected stabilizer L_E is a Levi, so $\ker \pi_E = 1$. By [208, Proposition 7.21, Step 1], each λ occurs with multiplicity 1 in the flag character $1 + \mathrm{St}$. By Lemma 6.57, take the residual pair Φ. By Lemma 6.44, the parent $\widetilde{\rho}$ on $\mathrm{PGL}_2(q)$ has its determinant intertwiner D_0. Thus, on $\Phi, \widetilde{\rho}$ we choose the operators A, J of Propositions 6.58 and 6.74. For paired h, by Lemma 6.56, we obtain the operator D_h. At equal rank 2, the sign is $\eta_q(a)^3$ for $\sigma(\zeta_r) = \zeta_r^a$. By [156, Lemma 2.3(c)], $\sigma(\sqrt{-q}) = \sqrt{-q}$ for $\sigma \in \mathcal{H}_\ell$. Since $r \equiv 3 \pmod 4$ and f is odd, $\eta_q(a)^3 = \sigma(\sqrt{-q})/\sqrt{-q} = 1$.

Given C, C, the paired box is $\square$. Next, we choose maximal planes $E_\pm$ in the selected rulings. Simultaneous reflection $r_+ r_-$ exchanges both rulings. At quartic size 2, the aligned and crossed orbits $\mathcal{O}_+, \mathcal{O}_-$ are distinct. By [208, Proposition 7.20(5), Step 3], each stabilizer S_E has component C_2 and column multiplicities $1, 1$. The split residual group H_0 acts on a four-space V_0. Its parents on $\mathrm{PGL}_2(q)^2$ are $\rho_{\mathrm{cusp}} \boxtimes 1$ and $\rho_{\mathrm{cusp}} \boxtimes \mathrm{St}$. Across the swap w_0, we induce by Lemma 6.54. On $H_0 w_0$, we choose the rank-two extension $\widehat{\rho}$ of Proposition 6.69. Before extraction, tensor with $U_\square$ of Lemma 6.65. Thus, the source on H_0 and the maps for $E_\pm$ give Proposition 6.66. The operators D, J satisfy $DJ = JD$ for quartic size 2, giving four simultaneous eigenspaces.

Next, for A, C or B, C, we put C on the negative sign space Y_-. On Y_+, the trivial and Steinberg modules are $U_1 \otimes U_1$ and $U_{\mathrm{aug}} \otimes U_{\mathrm{aug}}$ on the rulings $\mathcal{R}_1, \mathcal{R}_2$. The ruling swap w_R acts by the tensor flip $\mathcal{T}$. Thus, the extensions $\widehat{A}, \widehat{B}$ have central scalar 1. In the involution group, adjoin R_- with weight $\eta_q \circ \det_{R_-}$. Moreover, the projective-line induction I_R contains χ_{mix} with multiplicity 1 with relative degree $q/(q+1) > 1/2$. On restriction of $\widetilde{\chi}$, the rulings satisfy $\mathcal{R}_1^{w_R} = \mathcal{R}_2$. By Lemma 6.79 and Proposition 6.92, we obtain $\widehat{\chi}_{\mathrm{mix}}$ over $\mathbb{Q}$ with central scalar 1. The unitary two-space contributes 1 or St. Moreover, Frobenius exchanges the opposite parabolics P_1, P_2. The transition on the cover is $(\alpha_\sigma \circ \mu)^2 = 1$ and the scalar character is λ_b^2. The torus parent ρ_T has transition $(\alpha_\sigma \circ \mu)^2 = 1$ by Lemma 6.91. Moreover, the signed period $e = 1$, weights $2, 2$, and residual dimension 0 give Proposition 6.94 on χ_{mix}.

On the six planes V_i, choose the anisotropic norm $N_{\mathbb{F}_{r^2}/\mathbb{F}_r}$. Their orthogonal sum V is split. Since f is odd, V_i remains anisotropic over $\mathbb{F}_q$. On each $\mathbb{F}_{q^2}$, the operator is $F_r : x \mapsto x^{rq}$. Indeed, it acts as $x \mapsto x^r$ on $\mathbb{F}_q$

and fixes $\mathbb{F}_{r^2}$. Thus, $F_r^f = 1$, and $rq \equiv 1 \pmod 8$ fixes the order-eight weight θ_8. Take the coordinatewise norm multipliers m_i, plane permutations w and reflections r_i. For the spin map $\pi : X \to \Omega(V)$, by Theorem 6.63, $\pi(N_X(P)) = N_{\mathrm{O}(V)}(T) \cap \ker\det \cap \ker \mathrm{sn}$, where sn is the spinor norm. The ℓ-weights θ_i recover the planes V_i up to permutation. By Corollary 2.10, $N_X(P) = \pi^{-1}(N_{\Omega(V)}(T_\ell))$. Since the dual lift s_0 is rational, $\ker\pi \le \ker\chi$ on this orbit. Conjugation by g carries m_i, w, r_i, F_r together. Alternatively, as in Lemma 7.161, [141, Theorem 2.9] lifts $\succeq_c$ from the induced automorphisms. Given C, C, the idempotents are $(1 \pm D)(1 \pm J)/4$, with the multiplier of Proposition 6.69. For A, A or B, B, the self-twist group is $\langle\nu\rangle$, with norm character ν. Thus, Theorems 6.70 and 6.224 give $\succeq_c$ for all pairs.

On $\{1,2,3,4\}$, the two permutations are $(12)(34)$ for reflection and $(13)(24)$ for block exchange. Next, let $K_4 = \langle (12)(34), (13)(24)\rangle$. For subsets of size $0, 4$, both stabilizers are K_4, giving 8 extensions. Moreover, at sizes $1, 3$, the stabilizers are trivial, giving 2 extensions. At size 2, the three stabilizers are C_2, giving 6 extensions. The quartic transposition τ carries the independent Specht label λ. Thus, at both ends there are 16 branches for each λ. By Proposition 6.223, distinct labels ρ exhaust the characters over $W \cdot \theta$. □

Theorem 7.165. *Let q be odd and $\ell > 7$ divide $q+1$. For the universal cover G of $E_7(q)$ and $P \in \mathrm{Syl}_\ell(G)$, the comparison* $\mathrm{iMN}(E_7(q), \ell)$ *holds on* $\mathrm{N}_G(P)$.

Proof. If $C_{\mathbf{G}^*}(s)$ is connected, Proposition 7.162 gives $\succeq_c$. If s is quasi-isolated, Corollary 7.229 gives $\succeq_c$. For the other s and odd f, we choose the minimal Levi $\mathbf{K}^*$ of Lemma 7.126. If $2 \mid f$, we write $s = ty$ with odd part $y \ne 1$, and take $\mathbf{K}^* = C_{\mathbf{G}^*}(y)$ by Proposition 7.125. For a type-A product and odd f, Propositions 7.129 and 7.130 give $\succeq_c$. If f is even, Propositions 7.147 and 7.148 give $\succeq_c$. In type D_4A_1, we apply Lemma 7.161.

Next, for D_5A_1, choose the D_5 comparisons of Lemma 7.163 and the nonsplit A_1 comparison of [186, Theorem 4.2]. The central isogeny π has 2-power kernel and the factors D_5, A_1 are nonisomorphic. By Lemma 7.161, models $\rho_5 \otimes \rho_1$ give common multiplicity U_0. Signed induction $R_{\mathbf{K}}^{\mathbf{G}}$ gives $\succeq_c$ on the lifts of (s_H, κ) in Proposition 7.128. In type D_6, the single factor gives $\succeq_c$ on $\mathrm{I}(3,3)$ by Lemma 7.163. If $2 \nmid f$, the derived projection s_H and central character κ determine the $\mathbf{K}^{*F^*}$-orbit of s by Proposition 7.127. If $2 \mid f$, the central twist ν_y selects the lifts of t. The pure unitary case has $\succeq_c$ by Proposition 7.134.

Finally, consider $\mathrm{Q}(2,2)$. On the natural torus T_0, we write s_0 with coordinates $1, 1, -1, -1, u, v$ with $|u| = |v| = 4$, modulo negation and Weyl conjugacy. Then, $F : x \mapsto x^{-q}$ gives $F(s_0) \in \{s_0, -s_0\}$. Since the sign coordinates are fixed, $F(s_0) = s_0$. Hence, $u^{-q} = u$ and $v^{-q} = v$, so $4 \mid q+1$. By Lemma 7.164, we obtain $\succeq_c$ on $\mathrm{Q}(2,2)$. By Proposition 7.127, the commutator $[x, s_0]$ gives the Levi class from (s_H, κ). As for $\mathrm{I}(3,3)$, signed induction $R_{\mathbf{K}}^{\mathbf{G}}$ gives $\succeq_c$ on $M = \mathrm{N}_G(P)$. Since $\ell \nmid |W(E_7)|$, we have $M = \mathrm{N}_G(T)$. The torus weight orbits $W \cdot \theta$ partition $\mathrm{Irr}_{\ell'}(M)$. Thus, the invariant unions give $\mathrm{iMN}(E_7(q), \ell)$ by Lemma 9.4. □

7.6 Primitive tori in E_7

Proposition 7.166. *Let $G = \mathbf{G}^F = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ odd. Next, for an odd prime $\ell \ne r$, we put $d = \mathrm{ord}_\ell(q)$ and choose a Sylow d-torus $\mathbf{D}$. With $P = (\mathbf{D}^F)_\ell$, we write $\mathbf{L} = \mathrm{C}_{\mathbf{G}}(\mathbf{D})$ and $L = \mathbf{L}^F$ and $M = \mathrm{N}_G(P)$. Given a rational dual element s, the neutral roots are α with $\alpha(s) = 1$. Next, for a torus $\mathbf{S}$, we write $\mathrm{N}_G(\mathbf{S}) = G \cap \mathrm{N}_{\mathbf{G}}(\mathbf{S})$, and define $\mathrm{N}_{G^*}(\mathbf{S})$ analogously. Let $m \in \{7, 9\}$. When $d = m$ or $d = 2m$, we set $\delta = 1, -1$, respectively, and put $u = \delta q$. There is a marked maximal torus T of order $(q-\delta)\Phi_m(u)$ with $\mathrm{C}_{\mathbf{G}}(P[\ell]) = \mathbf{T}$ and $M = \mathrm{N}_G(P[\ell]) = \mathrm{N}_G(\mathbf{T})$. Then, we obtain that $M/T = \langle a\rangle \times \langle -I\rangle$ with $|a| = m$. Moreover, the corresponding formulas hold on G^*. Every eligible s is conjugate into T^*, where M^*/T^* controls fusion. If $s^2 \ne 1$, we have that $C_{\mathbf{G}^*}(s)$ is connected and lies in $\mathbf{K}^*$ of type A_6 for $m = 7$ and E_6 for $m = 9$.*

Proof. If $m = 7$, choose a Coxeter seven-cycle a in the A_6 Levi with node 2 deleted. If $m = 9$, we set $a = c^{10} = -c$ for a Coxeter element c by Lemma 7.9. The Weyl mark is δa. The primitive space U is irreducible over $\mathbb{Q}$ of dimension 6, and a fixes $V = U^\perp$. If $m = 7$, a root in V would give the sublattice $A_6 \perp A_1$ with squared index $(7 \cdot 2)/2 = 7$, impossible. If $m = 9$, Lemma 7.9 gives the corresponding exclusion of $E_6 \perp A_1$. If a root β annihilates the eigenline E, then it annihilates its conjugates spanning U. Since V is rootless, E is regular. Thus, by [178], $C_W(\delta a)$ has order $2m$, so it equals $\langle a\rangle \times \langle -I\rangle$. Only fundamental degree $2m$ contributes to $|G|_\ell$. Since the eigenline modulo ℓ has multiplicity 1, P is cyclic and Sylow.

On $P[\ell]$, apply the test of Lemma 7.9. Let R be the roots trivial on $P[\ell]$. If $R \neq \varnothing$, its rational span is δa-stable and contains U since V has no roots. Choose independent roots β_i spanning $\langle R\rangle_{\mathbb{Q}}$, with Gram determinant $D_R \leq 2^7$. Thus, the saturation index j_R satisfies $j_R^2 \leq D_R$. Since $\ell \geq 29$ for $m = 7$ and $\ell \geq 19$ for $m = 9$, we have $\ell \nmid j_R$, so the span modulo ℓ is unaffected. Averaging over $\langle a\rangle$ gives the orthogonal splitting $U \perp V$ over $\mathbf{Z}_{(\ell)}$. Consequently, U remains nondegenerate modulo ℓ. Thus, R cannot annihilate the Sylow line $E_\ell \subset U$.

On the adjoint form, we choose $\bar{x}$ generating $P[\ell]$. It has unique odd-order lift x across the double cover π. Thus, $\pi^{-1}(C_{\mathbf{G}}(\bar{x})) = C_{\mathbf{G}_{\mathrm{sc}}}(x)$, connected by Steinberg's theorem. Thus, $\mathrm{C}_{\mathbf{G}}(P[\ell]) = \mathbf{T}$ in both forms. Since P is the unique Sylow ℓ-subgroup of T, $M = \mathrm{N}_G(P[\ell]) = \mathrm{N}_G(\mathbf{T})$. Moreover, [133, Proposition 25.3(a)] gives rational representatives of $C_W(\delta a)$.

If s is eligible, the Jordan index has $\ell \nmid [G^* : C_{G^*}(s)]$, since the central-form indices divide 2. Hence, conjugate a Sylow subgroup of $C_{G^*}(s)$ to P^*, giving $s \in T^*$. If $s, t \in T^*$ are conjugate, a correction in $C_{G^*}(t)$ returns the conjugated P^* to P^* by Sylow conjugacy. Thus, s, t are conjugate under $M^* = \mathrm{N}_{G^*}(P^*)$. The span of neutral roots R_s is $0, U$, or $U \oplus V$. If $\mathrm{rk}(R_s) = 7$ and $R_s \neq E_7$, there is no action of order 9. Indeed, the A-factors act on at most 8 letters and their exchanges have order ≤ 2, while D_6 has 6 signed letters. If $m = 7$, only A_7 admits the action of order 7 [208, §11.5]. Since $[Q(E_7) : Q(A_7)] = 2$, an adjoint s neutral on A_7 satisfies $s^2 = 1$.

Suppose that $s^2 \neq 1$. Then, $C_{\mathbf{G}^*}(s)^\circ \leq \mathbf{K}^*$. The component group A_s has order ≤ 2, so Lang gives a rational representative g in each component. Since ℓ is odd, P^* is Sylow in $C_{\mathbf{G}^*}(s)^{\circ F^*}$. Thus, correct g by $h \in C_{\mathbf{G}^*}(s)^{\circ F^*}$ with $gh \in M^*$. If its Weyl image w lies in $-\langle a\rangle$, $w^m = -I$, giving $s = s^{-1}$, a contradiction. The preimage of $\langle a\rangle$ lies in $\mathbf{K}^*$. The simply connected derived centre Z_K has order 7 or 3. Moreover, the ambient kernel has order 2. Thus, $(\mathbf{K}^*)_{\mathrm{der}}$ is simply connected, so $C_{\mathbf{K}^*}(s)$ is connected. Consequently, $C_{\mathbf{G}^*}(s)$ is connected. □

Proposition 7.167. *Let $d \in \{5, 10, 8, 12\}$. Then, $P \in \mathrm{Syl}_\ell(G)$ is cyclic. On both forms, $\mathrm{C}_{\mathbf{G}}(P[\ell]) = \mathbf{L}$ and $\mathrm{N}_G(P[\ell]) = \mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{D})$. If $d \in \{5, 10\}$, the derived type of $\mathbf{L}$ is A_2 and $M/L \cong C_{10}$, whereas for $d = 8, 12$, the type is A_1^3 and $M/L \cong C_d$. Moreover, the unique involution of M/L acts by $-I$ on the primitive four-space.*

Proof. If $d \in \{5, 10\}$, set $\delta = 1, -1$, respectively, and take the mark δc with $c = (12345)$ in the A_7 model of E_7. The roots are $e_i - e_j$ and the half-roots $\frac{1}{2}\sum_i \epsilon_i e_i$ with four positive ϵ_i. Moreover, the active span U has type A_4, and the roots in $U^\perp$ are $e_i - e_j$ for $i, j \in \{6, 7, 8\}$. We have $|\mathbf{D}^F| = \Phi_5(\delta q)$. Moreover, only degree 10 contributes to $|G|_\ell$, and the eigenline modulo ℓ has multiplicity 1.

Let x have order ℓ in $\mathbf{D}^F$. The five coordinates x_i cannot all agree. Indeed, $\ell \geq 11$ and the map to the adjoint A_4-torus has kernel of order dividing 5. The relation $x_i = x_j$ is c-invariant, so the x_i are pairwise distinct. If a root $e_i - e_j$ with $i \leq 5 < j$ is neutral, cycling gives $x_1 = \cdots = x_5$. For a half-root, we let I be the positive indices in $\{1, \ldots, 5\}$, with $1 \leq |I| \leq 4$. After taking the complement if needed, I has one or two letters. If $I = \{i_1, i_2\}$, translate by $i_2 - i_1$. If $|I| = 1$, take cI. Thus, some $c^k I$ replaces index i by j. The root-value ratio x_i/x_j is independent of signs in SL_8. Consequently, the neutral subsystem is A_2. By the pointwise stabilizer theorem, $W_U = \mathrm{Stab}_W(U)$ has quotient $W_U/W(A_2) \simeq S_5 \times \langle -I\rangle$. By Lang on $\mathbf{L}$, $M/L \cong C_{S_5 \times \langle -I\rangle}(c) \cong C_{10}$.

If $d \in \{8, 12\}$, we set $U = D_4$ on the first four coordinates in $E_7 = (e_7 + e_8)^\perp$. The perpendicular simple roots are $e_5 - e_6, e_5 + e_6, e_7 - e_8$. The index-four gluing of $D_4 \perp A_1^3$ pairs the nonzero classes in $D_4^\vee/D_4$ with the even nonzero classes of $(A_1^\vee/A_1)^3$. Indeed, the D_4 classes have norm 1 mod 2, and a leg vector of weight j has norm $j/2$ mod 2. Thus, $g \in \mathrm{Aut}(D_4)$ lifts by permuting $\beta_1, \beta_2, \beta_3$, the perpendicular positive roots. On U, the kernel is $W(A_1^3)$. Next, in $\mathrm{Aut}(D_4) = W(F_4)$, take a negative four-cycle for $d = 8$ and a Coxeter element for $d = 12$. The mark w has characteristic polynomial Φ_d on U. Moreover, its diagram image $\bar{w}$ is a transposition for $d = 8$ and a three-cycle for $d = 12$. Indeed, the negative cycle w lies outside $W(D_4)$ at $d = 8$. The preimage of $C_2 \leq S_3$ is a signed-permutation group $W(B_4)$ with no eigenvalue of order 12. By regularity and degrees $2, 6, 8, 12$, $C_{W(F_4)}(w) \cong C_d$ [178].

$|\mathbf{D}^F| = \Phi_d(q)$, with the selected eigenvalue of multiplicity 1. The sole divisible fundamental degree is d, so $P \in \mathrm{Syl}_\ell(G)$ is cyclic. For odd $t \mid \Phi_d(q)$, we have $\mathrm{ord}_t(q) = d$, so $t \geq 17, 13$ at $d = 8, 12$, respectively. Moreover, $\gcd(\Phi_{12}(q), 6) = 1$. If $\alpha(x) = 1$ for $\alpha \notin U^\perp$ and $|x| = t$, the neutral span contains the irreducible U, which has multiplicity 1. The saturation index j_R satisfies $j_R \leq \sqrt{2^7} < 13$. The weight lattice adds index 2, and U is nondegenerate modulo t since $t \nmid d$ and perpendicular eigenvalues have orders dividing 2 or 3. Thus, the neutral span cannot annihilate $E_t \subset U$, the nonzero order-t line.

For all four d, the generator x of $P[\ell]$ has unique odd-order lift $\widetilde{x}$. Thus, $C_{\mathbf{G}}(x)$ equals $\mathbf{L}$ also on the adjoint form, with $\mathbf{D}$ the unique primitive d-subtorus of $Z(\mathbf{L})^\circ$. Since $P = (\mathbf{D}^F)_\ell$ is characteristic in $\mathbf{D}^F$, we get $\mathrm{N}_G(\mathbf{D}) \leq \mathrm{N}_G(P) \leq$

$\mathrm{N}_G(P[\ell])$. Conversely, each normalizer of $P[\ell]$ stabilizes $\mathbf{L}$ and $\mathbf{D}$. By [133, Proposition 23.2], the rational quotients are C_{10}, C_d as above. Their involution acts by $-I$ on U. □

Lemma 7.168. *In* Section 7.1*, let a finite group B act algebraically on $\mathbf{G}_0$ and centralize $\mathcal{F}$. Suppose that B fixes $\mathbf{K}_0$, both parabolics with radicals U, V, and the rational series idempotent e_s. If $C_{\mathbf{G}_0^*}(s)^\circ \le \mathbf{K}_0^*$, then the top cohomologies satisfy $H_U e_s \cong H_V e_s$ as bimodules with diagonal B-action. Suppose also that $C_{\mathbf{G}_0^*}(s) \le \mathbf{K}_0^*$ and Δ acts as in* Proposition 7.2*. Let (t,σ) have algebraic action $d(t)$ on $\mathbf{G}_0$ that normalizes the Δ-action with $d(t)U = V$ and $(\beta^t)^\sigma = \beta$. Then, the extension of $\chi \boxtimes \beta^\vee$ on Δ is fixed by (t,σ).*

Proof. On $\mathbf{G}_0 \rtimes B$, we extend $\mathcal{F}$ by the identity on B. Next, for $W = U, V$, we choose a cocharacter λ_W in the cone for W and set $\lambda'_W = \sum_{b\in B} b(\lambda_W)$. Each $b(\lambda_W)$ has the root signs of λ_W and pairs to zero on $\Phi(\mathbf{K}_0)$. Thus, λ'_W is B-fixed and defines $\mathbf{P}_W \rtimes B$ with Levi $\mathbf{K}_0 \rtimes B$. The relative trace of e_s to $(\mathbf{K}_0 \rtimes B)^{\mathcal{F}}$ is e_s. In each dual radical W^*, the fixed subgroup $C_{W^*}(s)$ is connected and is trivial by $C_{\mathbf{G}_0^*}(s)^\circ \le \mathbf{K}_0^*$. By [19, Corollary 6.5 and Remark 6.7], the conditions hold for (U,V) and (V,U). Thus, [160, Lemma 3.1] gives $H_U e_s \cong H_V e_s$ on $(\mathbf{G}_0^{\mathcal{F}} \times (\mathbf{K}_0^{\mathcal{F}})^{\mathrm{opp}})\delta((\mathbf{K}_0 \rtimes B)^{\mathcal{F}})$.

The finite action on Y_W is defined over $\mathbb{F}_{q_0}$ for a Frobenius power $F_{q_0} = \mathcal{F}^m$. Thus, [53, Proposition 3.3] gives integral alternating traces $E_W(h)$ on the unprojected cohomology, for $W = U, V$. Since $V = d(t)U$, we have $E_U^t = E_V$ and $n_U = n_V$, with $n_W = \dim Y_W$. Under $C_{\mathbf{G}_0^*}(s) \le \mathbf{K}_0^*$, [18, Theorem B$\prime$] gives concentration in degree n_W on e_s. Let X_W denote the right $\beta^\vee$-part. Its projector is $e_{\beta^\vee} = \frac{\beta(1)}{|K|}\sum_{k\in K}\beta(k)(1,k)$, because $\beta^\vee(k^{-1}) = \beta(k)$. Thus, $\mathrm{tr}_{X_W}(h) = (-1)^{n_W}\frac{\beta(1)}{|K|}\sum_{k\in K}\beta(k)E_W(h(1,k))$. At (t,σ) the projector is carried to $e_{(\beta^{(t,\sigma)})^\vee} = e_{\beta^\vee}$. Since $E_U^{(t,\sigma)} = E_V$, we obtain $\mathrm{tr}_{X_U}^{(t,\sigma)} = \mathrm{tr}_{X_V}$. Moreover, the D-isomorphism gives $X_U \simeq X_V$ by right K-linearity. Thus, (t,σ) fixes the extension. □

Corollary 7.169. *With $X = E_{7,\mathrm{sc}}(q)$, we let ℓ be odd, nondefining, and divide $|X/\mathrm{Z}(X)|$. Next, we put $d = \mathrm{ord}_\ell(q)$ and $P \in \mathrm{Syl}_\ell(X)$. Given $d \in \{5,7,8,9,10,12,14,18\}$, we find an $\mathrm{Aut}(X)_P \times \mathcal{H}_\ell$-equivariant bijection $\mathrm{Irr}_{\ell'}(X) \to \mathrm{Irr}_{\ell'}(\mathrm{N}_X(P))$ satisfying $\succeq_c$.*

Proof. Among $2,6,8,10,12,14,18$, only $2d$ is divisible by d at $d = 5,7,9$, and only d otherwise. Thus, Φ_d has multiplicity 1 in the order polynomial. Since $d \mid \ell - 1$, the listed d give $\ell \notin \{2,3,5,7\}$, so $\ell \nmid |W(E_7)|$. By Lemma 2.11, P is cyclic and $\mathrm{N}_X(P[\ell]) = \mathrm{N}_X(P)$. By Proposition 2.9, the comparison holds on $\mathrm{N}_X(P)$. □

Lemma 7.170. *Let q be odd. If $d \in \{3,6\}$, we have that the Sylow subgroup P is toral of rank 3, with $\mathrm{C}_G(P) = T$ a maximal torus and $\mathrm{N}_G(P) = \mathrm{N}_G(T)$. If $d = 4$, P is toral of rank 2 in $\mathbf{D}$, with $\mathrm{C}_G(P) = L = \mathrm{C}_{\mathbf{G}}(\mathbf{D})^F$ and $\mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{D})$. The residual root system at $d = 4$ is A_1^3. These identities also hold for G^*.*

Proof. For $d = 3,6$, put $u = \delta q$ with $\mathrm{ord}_\ell(u) = 3$ and $\delta \in \{1,-1\}$. Then, take the torus of type δb, with b the fourth Coxeter power on U_6 of type E_6. On $U_6^\perp$, $b|_{U_6^\perp} = I$. The torus has order polynomial $(q-\delta)\Phi_3(u)^3$ by [27, Table 3]. By [130, Table 2], at $d = 3,6$, its Sylow d-torus $\mathbf{S}_d$ has maximal torus centralizer. At $d = 4$, the two negative two-cycles on D_4 give $\mathbf{S}_4 = \mathbf{D}$. The same table gives its polynomial Φ_4^2 and residual root system A_1^3. By [132, Proposition 5.6], $P = (\mathbf{S}_d^F)_\ell$ has rank 3 at $d = 3,6$ and rank 2 at $d = 4$. Since $\ell d > 18$, only $\Phi_d(q)$ contributes to the ℓ-part of $|G|$. By [63, after the proof of Theorem 3.2], $\mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{S}_d)$ and $\mathrm{C}_G(P) = \mathrm{C}_G(\mathbf{S}_d)$. Thus, at $d = 3,6$, $\mathrm{N}_G(P) = \mathrm{N}_G(T)$. The polynomials and roots agree on $\mathbf{G}^*$. Thus, both formulas hold for P^*. Since P is abelian and normal in $\mathrm{N}_G(P)$, Itô's theorem in [95, Theorem 6.15] gives $\chi(1) \mid [\mathrm{N}_G(P) : P]$ for $\chi \in \mathrm{Irr}(\mathrm{N}_G(P))$. By the Jordan degree formula [58, Proposition 11.5.6], ℓ'-degree in $\mathcal{E}(G,s)$ implies $\ell \nmid [G^* : \mathrm{C}_{G^*}(s)]$. Thus, by Sylow conjugacy, the G^*-class of s meets $\mathrm{C}_{G^*}(P^*)$. □

For induction from $\mathbf{K}$, we apply Section 7.1 to $\mathcal{E}(K,s)$ and the finite Levi characters β. If $C_{\mathbf{G}^*}(s) \le \mathbf{K}^*$, the mixed stabilizer operations (a,σ) match for signed induction $R_{\mathbf{K}}^{\mathbf{G}}$ and Clifford induction. The Sylow correction is unique modulo $K = \mathbf{K}^F$. From K_{der} to K, we have that the two sections s, s' yield common factor c on the inertia quotient. Tensor by U_0 of factor c^{-1}, then take Clifford induction Ind. By Proposition 7.6, they have equal scalars ω and mixed functions $\mu_{t,\sigma}$ whenever they coincide on each $\mathcal{E}(K,s)$.

Proposition 7.171. *Let $d = 3,6$, $u = \delta q$ as above, and $h = \Phi_3(u)$. Under the comparisons below, each ℓ'-series has $\succeq_c$ on $\mathrm{N}_G(P)$ if $s^{2h} \ne 1$, its neutral system is empty, or $|s|$ is odd. If $s^{2h} = 1$ and $R_s = \varnothing$, the toral case or* Tapp [186, Theorems 4.1 and 4.2] *gives $\succeq_c$.*

Proof. Let $\mathbf{K}_0^*$ be the minuscule Levi with centre $\mathbf{Z}^*$, and put $x = s^h$. On U_6, we have $(1-ub)(1-ub^2) = hI$. Thus, the two marked tori $T_{\mathrm{ad}}, T_{\mathrm{der}}$ have exponent dividing h. Consequently, $x \in \mathbf{Z}^*$. Equal scalars on $\mathrm{Z}(\mathbf{K}_0)^{\circ F}$ give dual elements differing by $T_{\mathrm{der}} \subseteq \ker h$. This determines x also at 3. If $x^2 \neq 1$, $\mathrm{C}_{\mathbf{G}^*}(x) = \mathbf{K}_0^*$. Indeed, outside roots have exponents ± 1 on $\mathbf{Z}^*$. Its pointwise stabilizer is $W(E_6)$, and the only further action on $\mathbf{Z}^*$ is inversion. Thus, $\mathrm{C}(s) \leq \mathbf{K}_0^*$. If $H = (\mathbf{K}_{0,\mathrm{der}})^F$, we have $P \leq H$, since the complementary quotient has order $q - \delta$. By Theorem 7.107, we have $\succeq_c$ on $H, \mathrm{N}_H(P)$. Since $\mathrm{ord}_\ell(\delta q) = 3$, the split and twisted orders are $3, 6$, respectively. Lift to $K_0, \mathrm{N}_{K_0}(P)$ with equal central scalar, giving equal x. Ambient fusion on $\mathbf{Z}^*$ is inversion by $-I$, with no exterior ordinary stabilizer of x. The inductions compare the unions modulo $x \mapsto x^{-1}$, with $\succeq_c$ by Lemma 7.168. Corrected powers stabilize $\mathrm{C}(x)$. It remains to take $x^2 = 1$, so $|s| \mid 2h$.

Write $s = ty$ with t its 2-part and $|y|$ odd. The quotient by $\mathbf{Z}^*$ has odd order, so the 2-part is cyclic on $\mathbf{Z}^*$. If $s = y$, [20, Corollary 2.9] gives $\mathrm{C}(s) = \mathrm{C}(s)^\circ$, since the fundamental group has order 2. For toral $\mathrm{C}(s)$, choose the torus character θ_s for both inductions. For rootless s with $t \neq 1$, this suffices when y is rootless. Otherwise, each neutral root α of y lies outside $\mathbf{K}_0^*$ and has $\alpha(t) = -1$. If $\alpha \neq \pm\beta$ and $(\alpha, \beta) \neq 0$, one root $\alpha \pm \beta$ has value 1 on s, impossible. Thus, the neutral system of y has type A_1^j. No root line is fixed by b. Each triple contributes dimension 1, so $\dim \mathrm{Fix}(b) = 1$ gives $j = 3$. For another root γ in the span, $\|\gamma\|^2 = 2$ needs four half-integer coordinates. Thus, $\mathrm{C}(y)$ is a proper Levi of type A_1^3 containing $\mathrm{C}(s)$.

Its rational derived group is $H = \mathrm{SL}_2(q^3)$. The outside roots have exponents ± 1 on $\mathbf{Z}^*$, and $v_2(q^3 - \delta) = v_2(q - \delta)$. Thus, the derived quadratic class determines t. The isogeny has 2-power kernel, so the scalar κ_y determines y. The central ℓ-part splits off, giving $\mathrm{N}_K(P) = \mathrm{N}_K(P \cap H)$. Since $q^3 \geq 27$ and $\ell \mid q^3 - \delta$, the quadratic label $\theta_s|_{H \cap T}$ has two extensions to $\mathrm{N}_H(P \cap H)$. At $\delta = 1$ apply [186, Theorem 4.1], and at $\delta = -1$ apply [186, Theorem 4.2]. Their bijections pair $\chi_\pm$ with these extensions and give $\succeq_c$ for $\mathrm{Aut}(H)_{P \cap H} \times \mathcal{H}_\ell$. Up to inner-diagonal g, rotation gives F_q on tuples $(v, F_q(v), F_q^2(v))$. Thus, a correction n includes its field action on H, with equal inner scalars ω. By Proposition 7.6, take common U_0 on the finite Levi. Since y is the odd part of s, by Corollary 7.3, we obtain $\succeq_c$ for $\mathrm{C}(y)$ on full inertia. Sylow conjugacy in $\mathrm{C}(y)^F$ restores the orbit of θ_s.

Next, suppose $|s|$ is odd. For quasi-isolated s, apply its comparison. Otherwise, take $\mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(\mathrm{Z}(\mathrm{C}(s))^\circ)$. In each adjoint factor of $\mathbf{K}^*$, the projection s_i is isolated. By [20, Theorem 5.1 and Tables II–III], $s_i = 1$ in types A, D. Thus, $\mathrm{C}(s) = \mathbf{K}^*$, or an E_6 factor has the cubic A_2^3-class. If $\mathrm{C}(s) = \mathbf{K}^*$, apply [156, Theorem D] to the unipotent characters of K_i. By [18, §11.5], $\mathcal{E}(K, s) = \kappa_s \mathcal{E}(K, 1)$. On $\mathrm{N}_K(P)$, twist by $\kappa_s|_{\mathrm{N}_K(P)}$. The Sylow torus has toral centralizer T^* in $\mathbf{K}^*$. Next, on adjoint K_i, we choose the minimal-torus models ρ_i. Permuted factors have identical ρ_i with tensor flips P_w. Lang gives the relative quotients W_i. After the twist by κ_s, both section factors coincide.

For cubic s, the span U has even fixed dimension ≤ 1, so $U = U_6$. The rank-six class s_H is rational at signed order 3. Moreover, for fixed (s_H, κ_s), the lifts differ by Z_3, the derived centre of order 3. The component C_3 rotates the A_2 factors and acts by translations from Z_3. The lift $\widetilde{s}$ has connected centralizer. Thus, by [20, Corollary 2.8(a)], $C_3 \hookrightarrow Z_3$. By Lang, rational translates $z_0 s$ are conjugate to s in $\mathbf{K}_0^{*F^*}$. Thus, (s_H, κ_s) determines the Levi series $\mathcal{E}(K_0, s)$. □

Proposition 7.172. *With $d = 4$, we let π be the odd prime divisors of $q^2 + 1$. For aligned $s \in \mathbf{L}^{*F}$, we have $y = s_\pi \in \mathbf{D}^{*F}$. If $y \neq 1$, we have that $\mathrm{C}(y)$ is $\mathbf{L}^*$ or a proper Levi of type D_4A_1. If $y = 1$, we have that a projective ratio $u_i^2 \neq 1$ gives a proper Levi $\mathbf{K}^*$ containing $\mathrm{C}(s)$. If $k \geq 2$ legs move with distinct $[u_i]$, we have that the marked D_4 comparison gives $\succeq_c$.*

Proof. The three residual adjoint factors have π'-order since $\gcd(q^2 + 1, q^2 - 1) = 2$. Thus, y lies in the projection kernel $\mathbf{D}^*$. Take odd $m > 1$ dividing $q^2 + 1$. On m-torsion the coordinates are equivalent. Then, $qJy = y$ gives $y = (v, qv, w, qw) \bmod m$. The projections $e_i, e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ have zeros on six lines, namely $v = 0$, $w = 0$, $w = \pm v$ and $w = \pm qv$. Indeed, $q^2 = -1$ modulo m, the elements $2, q, 1 + q, 1 - q$ are units, and $(1 - q)/(1 + q) = -q$. Thus, distinct lines intersect in 0, including composite m. Otherwise, the neutral projection is $\varnothing$. On $v = 0$, $w \neq 0$, the neutral projections are $\pm e_1, \pm e_2, \pm e_1 \pm e_2$.

The six lines form one $\mathrm{C}_{W(F_4)}(J)$-orbit. With $J = i$ on (z_1, z_2), take plane exchange, quarter-turns and $g : (z_1, z_2) \longmapsto \frac{1+i}{2}(z_1 + z_2, z_1 - z_2)$. The columns of g are orthogonal half-sums $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$. Thus, g permutes both root sets $\Phi_{\mathrm{short}}, \Phi_{\mathrm{long}}$, and $gJ = Jg$. Moreover, g takes $w = 0$ to $w = v$, while quarter-turns act by $\pm q$. Thus, singular nonzero y has projected neutral system B_2. By Proposition 7.117, the lattice D_4 and its cosets give D_4A_1. Since $|y|$ is odd, the gluing of index 2^a gives no extra roots. On D_4, the mark is $J_2 \oplus I_2$ with negative two-cycle J_2,

giving ${}^2D_4$. By [20, Corollary 2.9], $\mathrm{C}(y)$ is connected since $(|y|,2)=1$. If the neutral projection is $\varnothing$, $\mathrm{C}(y)=\mathbf{L}^*$, with common source β on $\mathbf{L}^F$. By Lemma 7.170 and [130, Table 2], $M/L\cong G_8$. Thus, $|M/L|=96$ by [171, Table II]. For regular y, the stabilizer is 1. On an axis, $\mathrm{C}(y)$ fixes $Z(\mathrm{C}(y))^\circ$, so the stabilizer fixes V_1 pointwise. On the neutral plane V_2, it lies in $C_{W(B_2)}(J|_{V_2})$ and contains $\langle J|_{V_2}\rangle$. Thus, its order is 4. The $6(m-1)$ singular points give $(m-1)/4$ orbits, while the $(m-1)(m-5)$ regular points give $(m-1)(m-5)/96$.

Suppose $y=1$. Since $J^2=-I$, the primitive quotient has exponent dividing q^2+1, so s projects to order at most 2. The character lattice of $\mathbf{L}^*/\mathbf{L}^*_{\mathrm{der}}$ is $Q(D_4)$, since the coroot projection gives $P(D_4)$ via gluing. The nontrivial J-fixed signs on $Q(D_4)$ are $x\mapsto(-1)^{x_i+x_j}$ for the $2+2$ partitions. On $Q(D_4)$, complementing $\{i,j\}$ fixes its sign θ_{ij}. Moreover, the double transposition fixes θ_{ij}, and $\det(J-I)=4$. Choose a geometric lift $a=(e_i+e_j)(-1)$, or $a=1$. We do not assume a rational. Write $s=a(t_1,t_2,t_3)$ modulo simultaneous negation. Next, we put $u_i=t_i^2$, $[u]=\{u,u^{-1}\}$.

For $[u]$ with $u^2\neq1$, put $b_{[u]}=\sum_{u_i=u}\alpha_i^\vee-\sum_{u_i=u^{-1}}\alpha_i^\vee$. Every neutral root α has $\langle\alpha,b_{[u]}\rangle=0$. Moreover, this is direct for primitive α and individual leg roots α_i. Given a gluing root on j,k, squaring its value gives $u_j^e u_k^{e'}=1$, so its two contributions to $b_{[u]}$ cancel. By [20, Corollary 2.9], $|\pi_0\mathrm{C}(s)|\mid 2$, so F fixes each component. Choose $g\in\mathrm{C}(s)^F$ in each component by [133, Theorem 21.7]. Since $P^*\le\mathbf{D}^{*F}\le\mathrm{C}(s)^{\circ F}$, Sylow conjugacy gives a correction of g in $\mathrm{C}(s)^{\circ F}$ to normalize P^*. By Lemma 7.170, g now normalizes $\mathbf{D}^*$. In $\mathrm{C}_{\mathbf{L}^*}(s)^\circ$, the tori $\mathbf{T}^*$ and ${}^g\mathbf{T}^*$ are maximal. By [133, Chapter 6], choose $h\in\mathrm{C}_{\mathbf{L}^*}(s)^\circ$ with ${}^{hg}\mathbf{T}^*=\mathbf{T}^*$. Then, the signed permutation of hg fixes each $b_{[u]}$ by the ratios u_i. Changing a changes only leg central values. Thus, $\mathrm{C}(s)$ lies in the centralizer of $\langle b_{[u]}\rangle$. This span is F-stable and nonzero if some $u_i^2\neq1$. Otherwise, $|s|\mid 4$.

With 3 distinct moving classes the Levi is D_4, and with 2 it is D_4A_1. No gluing root annihilates all $b_{[u]}$. Since $J\in W(D_4)$, the factor D_4 is split. Its sign a is trivial or represented by $(1,1,-1,-1)$. Triality gives the packets for θ_{ij}. If $a=1$, we apply [156, Theorem D] for $\succeq_c$. For quadratic a, by Lemma 6.33, we obtain 10 branches. The involution has minus spaces V_+,V_- of dimension 4. Triality carries the kernel Z_0 and orbit $W\cdot\theta$ together. By Proposition 6.123, each $W\cdot\theta$ is on $\mathrm{N}_K(\mathbf{D})$. The remaining A_1 has order prime to ℓ and comparison id. The translates fix u_i and the primitive root values $\alpha(a)$.

Next, we let $\mathbf{K}^*$ centralize the cocharacter span, set $\mathbf{L}_0^*=\mathbf{K}^*\cap\mathbf{L}^*$, and put $J_0=\mathrm{N}_K(\mathbf{D})$. Moreover, on moving legs, L_0 contains only the tori. Signed induction $R_{\mathbf{L}_0}^{\mathbf{L}}$ is irreducible on the source since $\mathrm{C}_{\mathbf{L}^*}(s)\le\mathbf{L}_0^*$. Correct each stabilizer element to the torus $\mathbf{T}^*$. Its Weyl element w fixes $\langle b_{[u]}\rangle$. Hence, w corrects to J_0. Thus, the two quotient stabilizers A,A' coincide. At a fixed moving position i, a transporter cannot invert u_i. Choose B_i on each position orbit $\mathcal{O}$ and propagate by $A_{\mathcal{O}}$. By Lemma 7.168, the radicals U,V admit mixed leg permutations w. Clifford induction to $\mathrm{N}_G(P)$ from the common lift gives c, scalars ω and functions $\mu_{a,\sigma}$ of $R_{\mathbf{K}}^{\mathbf{G}}$. □

Proposition 7.173. *Assume the comparison of* Proposition 7.171 *for $d\in\{3,6\}$ on its stated domains. On balanced $\mathrm{GU}_{2b}(q)$ with $z_i^{-q}=z_i$, take the operators K_χ,A_{u_J} of* Proposition 6.109 *for $\lambda\neq\mu$, with their source identities. If $\lambda=\mu$, take its normalized operators. At $d=3,6$, the remaining series have $\succeq_c$ on $\mathrm{N}_G(P)$.*

Proof. In the remaining case, $t=\omega_7^\vee(-1)\neq1$. Suppose that $\mathrm{C}(y)$ is a b-stable proper Levi. Each factor orbit $\mathcal{O}$ of fixed dimension 0 has its roots in U_6, where t is central. Moreover, at most one $\mathcal{O}$ has noncentral t. An orbit of length 3 consists of three A_1's with quadratic projection of t. On a stable A-factor, $\dim\mathrm{Fix}(b)=n_c-1$ for the cycle count n_c. Specifically, $n_c=2$. Singleton cycles $(i),(j)$ give the fixed root e_i-e_j. For lengths $3,1$, the generator $(1,1,1,-3)$ has norm squared 12, whereas $\|2\omega_7^\vee\|^2=6$. The ratio $2\notin\mathbb{Q}^{\times2}$. For lengths $3,3$, the generator $(1,1,1,-1,-1,-1)=\pm2\omega_7^\vee$ gives the balanced A_5 case. Moreover, a further factor A_1 would give a fixed root. On D_5,D_6, a signed permutation b of order 3 has $\dim\mathrm{Fix}(b)\ge2$. Moreover, on D_4,E_6, parity gives fixed dimension 0.

We may take $x=t$ and $y^h=1$. Write $y=jy_3$, where $j=y_{6'}$ and $h_3\le3$. If $j\neq1$, the lift $\widetilde{j}$ and highest-root marks give $\mathbf{K}^*=\mathrm{C}(j)$ containing $\mathrm{C}(s)$. If t is central there, $\mathrm{C}(s)=\mathrm{C}(y)$. At a cubic vertex, $y^3\in\mathbf{Z}^{*F^*}$, where $|\mathbf{Z}^{*F^*}|=q-\delta$ and $|y^3|\mid h/3$. If $3\mid h$, we have $\gcd(h/3,q-\delta)=1$, so $j=1$. At a full-rank odd vertex, $j=1$ as well. Thus, for central t and $j\neq1$, we apply the unipotent twist in $\mathrm{C}(y)$.

Suppose t is noncentral. For the three cycled A_1's, only A_2 can occur as another factor. Given A_5, the remaining root system is $\varnothing$. The central isogeny indices divide 2^a3^b, so $P=(P\cap H)P_Z$ for central P_Z and $\mathrm{N}_K(P)=\mathrm{N}_K(P\cap H)$. In A_5, order the triples T_1,T_2. Given a lift $\widetilde{y}_3$, all ratios lie in μ_3. If $y_3\neq1$, then $3\mid h$ and $u\equiv1\pmod3$. As the ratios are fixed under u-th powers, rotation scales $\widetilde{y}_3$ by $\xi_0\in\mu_3$. For $\xi_0\neq1$, each triple has distinct entries z_i. Cross ratios lie in $-\mu_3$ because of t, so $R_s=\varnothing$ and Proposition 7.171 applies. If $\xi_0=1$, each triple is constant, with ratio

$-\xi$ on s and $\xi^3 = 1$. This includes $y_3 = 1$. The kernel $\mathbf{G}_m$ is connected, so s lifts to s_0 in the rational torus T of linear or unitary type. On T, signed Frobenius fixes z_1, z_2 separately.

If $\xi \neq 1$, the ratio $-\xi$ has order 6, so the block permutation moves $[s_0]$. The neutral roots have type A_2^2. Since $\mathrm{C}(s) \leq C(j)$ and the Weyl image is S_6, the full centralizer is Levi $\mathbf{L}_s^*$. Apply the unipotent twist κ_s and Corollary 7.3. On each A_2, the cycle torus gives $\mathrm{Irr}(C_3)$. The adjoint extensions $\widehat{\chi}$ are $\mathbb{Q}_\ell$-valued, also for unitary $(2,1)$ with $\mathrm{ord}_\ell(-q) = 3$ [156, Lemma 8.1 and proof of Proposition 8.5]. Given a rotation scalar ω, the local lift has an f-th root of ω^{-1} in μ_{3f_3}, where $f_3 = 3^{v_3(f)}$. Indeed, $\mathrm{ord}_\ell(r^f) \in \{3, 6\}$ gives $3f_3 \mid \ell - 1$. The opposition γ has value 1 by Proposition 7.100. Choose identical $\widehat{\chi}$ on isomorphic factors with tensor permutation P_w. By Lemma 7.7, the twist κ_s has equal section values.

If $\xi = 1$, the twist by z_1^{-1} gives colours $1, \eta$ with $\eta^2 = 1$. Both slots have $b = d' = 3$, with $d' = \mathrm{ord}_\ell(\delta q)$, and residual dimension 0. All pairs $\lambda, \mu \vdash 3$ occur since both cycles lie in $\mathrm{C}(s)$ and $\ell \geq 7$. Only $\lambda = \mu$ gives a quadratic self-twist $\eta \circ \det$. Given $\mathrm{GL}_3(q)$, the sources χ_i are those of Proposition 6.106 for $b = 3$. Moreover, the radicals have orders $q^9, 1$. Since $q \bmod \ell$ has odd order, $\sqrt{q} \in \mathbb{Q}_\ell$ by [156, Lemma 2.3(d)]. The middle label $(2,1)$ is on the augmentation of $\mathbb{P}^2(\mathbb{F}_q)$. For the standard dot product, put $D_{L,M} = 1$ if $L \perp M$, and 0 otherwise. Given $L \neq M$, there is one common perpendicular. If $L = M$, there are $q + 1$. Hence, $D^2 = qI + J$ with J the all-ones matrix. On augmentation, $D/\sqrt{q}$ gives graph action through $g \mapsto (g^{-1})^{\mathrm{t}}$. The field identity is $F_r(D/\sqrt{q}) = (D/\sqrt{q})F_r$. The modules $1, \mathrm{St}$ also have rational actions, with St on the building. For unitary colours $1, \eta$, Proposition 6.109 gives the field power F_r^{2f}. The determinant twist restores θ_1, θ_2. By Lemma 6.193 and Proposition 6.194, the parents $\widetilde{\chi}$ descend to two constituents if $\lambda = \mu$ and one if $\lambda \neq \mu$.

For a triple of A_1's, the rational group is $\mathrm{SL}_2(q^3)$. Moreover, Tapp matches a nonquadratic index i with induction from $\{i, -i\}$, and the quadratic weight θ with its two extensions. This includes $|s_H| = 6$ [186, Theorems 4.1–4.2]. On an additional A_2, the y_3-ratios lie in μ_3, so its entries z_i are constant or distinct. For distinct z_i, the series is rootless since t is noncentral on each A_1. Otherwise, take the unipotent comparison on A_2. For nonisomorphic H_1, H_2, the product model $\rho_1 \otimes \rho_2$ has factor $c_1 c_2$ after pullback. By Proposition 7.6, pass to K.

The derived projection has order dividing 6, so the connected central scalar determines j. An active root has exponent ± 1 on the minuscule line $\mathbf{Z}^*$, so the projection on $\mathbf{Z}^*$ is injective. Writing $\overline{s}$ for the derived projection, we have $|\overline{s}^h| = 2$ precisely for $x = t$. The admitted translates fix j, x, with centralizer in $\mathbf{K}^*$. Fusion fixing j is in $\mathbf{K}^*$, so both inductions compare the union $\mathcal{U}$. Since $C(j)$ depends on $\langle s \rangle$, Corollary 7.3 gives signed transfer on Γ_χ and on ordinary inertia.

If $j = 1$, either $y = 1$ or $|y| = 3$. If $y = 1$, choose the involution series. If $|y| = 3$, the affine numerators satisfy $\sum_i m_i a_i = 3$, with only two marks $m_i = 1$. Thus, at most two $a_i > 0$, and the neutral rank is 6 or 7. At rank 6, parity gives span U_6, so t centralizes $\mathrm{C}(y)$. Moreover, for A_2^3, the branch edge $2, 4$ must remain, deleting $3, 5$. Deleting 4 leaves an isolated leaf or two A_3's. Deleting 2 and a path vertex i gives at most 2 components. But $m_3 = m_5 = 3$ excludes $a_3, a_5 > 0$. Thus, $\mathrm{C}(y)$ is a Levi.

At rank 7, the cubic system is $A_5 A_2$. The fixed dimension 1 gives two 3-cycles on A_5 and one on A_2. Moreover, the triple signs give $R_s = A_2^3$ and $|s| = 6$. The coordinates $a_3 = a_5 = 1/6$ give the quasi-isolated $[s]$ [208, Proposition 11.18, Step 1]. By Lemma 7.218, the index for $[s]$ excludes $[s']$ with exchanged blocks. Proposition 7.220 gives $\succeq_c$ for the other form. The sets $x^2 \neq 1$, $x^2 = 1$ exhaust the series with local group $\mathrm{N}_G(P)$. □

Lemma 7.174. *Let $q = r^f$ be odd with $\mathrm{ord}_\ell(q) = 6$. Next, take separately rational colours $1, \eta$ of multiplicity 3 in $\mathrm{GU}_6(q)$, where η is quadratic on μ_{q+1}. The operators of* Proposition 6.109 *give $\succeq_c$ on the marked normalizer for every (λ, μ). If $\lambda = \mu$, both special-group constituents occur.*

Proof. Each $\lambda \vdash 3$ has $\mathrm{core}_3(\lambda) = \varnothing$ and one relative letter. The degree $\chi_\lambda(1)$ is $1, q(q-1)$, or q^3, prime to ℓ. Moreover, the blocks $1, \eta$ give both factors $\Phi_6(q)_\ell$ in $|\mathrm{GU}_6(q)|$. Thus, every (λ, μ) is eligible. The scalar character $\eta^{h_{\lambda\mu}}$ on even strings has $h_{\lambda\mu} \in \{0, 1\}$. Specifically, the string lengths are $D_i = \lambda_i' + \mu_i'$. With rows and columns ordered $3, 21, 1^3$, we obtain $(h_{\lambda\mu}) = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$. The column contribution on even D_i is $\eta^{\mu_i'}$. For opposite parities of D_i, D_j, the Heisenberg contribution is $\eta^{\min(D_i, D_j)}$ by Proposition 3.28. If $\lambda = \mu$, all lengths D_i are even and $h_{\lambda\lambda} = 3 \bmod 2$. For $(3, 21), (3, 1^3), (21, 1^3)$, the lengths are $(3, 2, 1), (4, 1, 1), (5, 1)$. The moving dimensions are $3, 2, 0$, and the even column exponents are $1, 3, 0$. We have $h_{\lambda\mu} = h_{\mu\lambda}$ since $\lambda_i' \equiv \mu_i' \pmod 2$ on even D_i.

If $\sigma(\zeta_r) = \zeta_r^a$, we put $h = h_{\lambda\mu}$. For the field operator P_0, field conjugation gives commutator $(a/r)^h$. At power $2f$, the field relation gives $P_0^{2f} = \eta(-1)^h \rho(g_0)$ for the chosen inner element g_0. The parity dimensions differ by 1

and the determinant quotient is 1, so Proposition 3.24 gives multiplicative normalization of P_0. Scale P_0 by β_*^h from Proposition 6.109. We have $(\beta_*^h)^{2f} = \eta(-1)^h$ and $\sigma(\beta_*^h)/\beta_*^h = (a/r)^h$. Thus, the normalized commutator is 1 and power $2f$ gives the given inner operator.

Choose $\mathbb{F}_{q^6}$ over $\mathbb{F}_{q^2}$ with Hermitian form $\mathrm{Tr}_{q^6/q^2}(xy^{q^3})$ on each block. The rotation $v : z \mapsto z^{q^4}$ acts by $-q$ on μ_{q^3+1}. Moreover, the field map $U : z \mapsto z^r$ satisfies $U\,v = v\,U$ and has $U^{2f} = v^2$. If $z \in \mu_{q^3+1}$, the determinant z^{q^2-q+1} ranges over μ_{q+1}. For $\omega_1, \omega_2 \in \mu_3$, the local field power P_{loc}^{2f} has scalar $(\omega_1\omega_2)^{-2}$ relative to the inner operator. In μ_{3f_3}, choose α with $\alpha^{2f} = (\omega_1\omega_2)^2$, where $f_3 = 3^{v_3(f)}$. Since $v_3(\mathrm{ord}_\ell(r)) = 1 + v_3(f)$, such α exists in $\mathbb{Q}_\ell$. Then, αP_{loc} has the given $2f$-th power and commutator 1. Given $\lambda \neq \mu$, only 1 fixes χ among determinant twists. If $\lambda = \mu$, we have that the block swap s_0 has determinant -1. The raw quadratic intertwiner J_B has $J_B^2 = \eta(-1)$. The normalized J_B has coefficient sign $\sigma(i)/i$ if $q \equiv 1 \pmod 4$, and 1 for $q \equiv 3 \pmod 4$. By Proposition 6.109, the signs coincide for J_η, J_B.

The simultaneous lift is $(A_1 \oplus A_1)F_r$ in orthonormal coordinates with linear determinant $(\det A_1)^2$. Thus, $\eta((\det A_1)^2) = 1$. Write P_B, P_η for the field operators. Then $P_B J_B = J_B P_B$ and $P_\eta J_\eta = J_\eta P_\eta$, so eigenspaces are invariant. Next, put $G^1 = \mathrm{SU}_6(q)$ and $M^1 = M \cap G^1$, where M is the marked normalizer. Restriction to G^1, M^1 matches both constituents. Also, the twist $\theta \circ \det$ restores the original colours z_1, z_2. For matched inner corrections g, g', equal determinants give $\theta(\det g) = \theta(\det g')$. □

Theorem 7.175. *Let q be odd with $\mathrm{ord}_\ell(q) \in \{3, 6\}$. Then, we have that $G = E_{7,\mathrm{sc}}(q)$ has the inductive comparison on $M = \mathrm{N}_G(P)$.*

Proof. First, take $\delta = 1$ for $d = 3$ and $\delta = -1$ for $d = 6$. With $h = \Phi_3(\delta q)$, we put $x = s^h$. Given $x^2 \neq 1$, the Levi for $C(x)$ has derived group $H = E_{6,\mathrm{sc}}^\delta(q)$, marked as in Theorem 7.107. Next, put $Z_0 = (\mathrm{Z}(\mathbf{K})^\circ)^F$. By [114, corollary to Theorem 1] on $\mathbf{H}$, $K/H = (\mathbf{K}/\mathbf{H})^F$, of order $q - \delta$. Since $\ell \nmid q - \delta$, $P \leq H$. The Frattini argument [97, Theorem 1.12] gives $K = H\mathrm{N}_K(P)$, also when $[K : HZ_0] = 3$. Apply the rank-six comparison on $H, \mathrm{N}_H(P)$. Let $\mathcal{N}$ be the rational normalizer of $\mathbf{K}$, acting on H. The stabilizer of P induces the marked image on H since $P \leq H$. The kernel on H centralizes P, so [141, Theorem 2.9, as in the proof of Theorem 3.3] transports $\succeq_c$ to $\mathcal{N}$. Proposition 7.6 gives the comparison on $K, \mathrm{N}_K(P)$ with common multiplicity operator D. The action restricts to the ℓ'-degree characters. Moreover, each $z \in Z_0$ centralizes H and has equal scalars on χ, ψ.

Restriction to Z_0 is dual to $\mathbf{T}^*/\mathbf{T}_H^*$, with derived torus $\mathbf{T}_H^*$. The kernel $\mathbf{T}_H^*$ is connected, so [114, corollary to Theorem 1] gives exactness on F-points. Take $L = (2, 3, 4, 6, 5, 4, 3)$ in fundamental coweight coordinates and $e_7 = \omega_7^\vee$. The primitive row L is zero on $\alpha_1^\vee, \dots, \alpha_6^\vee$ and has $L(e_7) = 3$. We have $1 + b + b^2 = e_7 L$, since it is zero on U_6 and is $3I$ on $\mathbb{Q}e_7$. On the rational torus, b acts as u^{-1} for $u = \delta q$, with u^3 acting trivially. Thus, $s^h = s\,b(s)\,b^2(s) = \omega_7^\vee(L(s))$. The character L has kernel $\mathbf{T}_H^*$.

The scalar on Z_0 determines x. Next, take the full fibre in $K, \mathrm{N}_K(P)$. If $[K : HZ_0] = 3$, a fixed $\chi_0 \in \mathrm{Irr}(HZ_0)$ has 3 extensions. Moreover, an orbit of length 3 gives $\mathrm{Ind}_{HZ_0}^K \chi_0$. The quotient K/HZ_0 matches both cases with $\mathrm{N}_K(P)$. The rational Weyl group acts on $\mathbb{Q}e_7 = \mathrm{Fix}(b)$, with pointwise stabilizer $W(E_6)$. Thus, the exterior action on $\mathbb{Q}e_7$ is generated by $-I$. Since $x \neq x^{-1}$, ordinary inertia fixes x on this line. Moreover, over $\{x, x^{-1}\}$, signed induction to G and Clifford induction to M give one pair (χ, ψ) per conjugacy orbit. We have $C(s) \leq C(x)$, and the joint stabilizer fixes $C(x) = C(s^h)$. Moreover, on full inertia, Corollary 7.3 transfers $\succeq_c$ for the class of s in $\mathbf{T}^*/\mathbf{T}_H^*$.

Given a proper Levi $\mathbf{K}^* = C(s)$, put $\mathbf{J} = \mathbf{K}/\mathrm{Z}(\mathbf{K})$. Next, we write $\overline{\mathbf{T}}$ for the image of the marked $\mathbf{T}$ in $\mathbf{J}$. The roots of $\mathbf{K}$ span a direct summand of the E_7 root lattice, of index 2 in the weight lattice. Thus, $\mathrm{Z}(\mathbf{K})/\mathrm{Z}(\mathbf{K})^\circ$ has exponent dividing 2. By Lang's theorem for $\mathrm{Z}(\mathbf{K})^\circ$, the image of K has ℓ'-index in $\mathbf{J}^F$. Therefore, the image $\overline{P}$ is Sylow in $\mathbf{J}^F$. Both torus normalizers have quotient $W(\mathbf{K})^F$ on rational points by [114, corollary to Theorem 1] on $\mathbf{T}, \overline{\mathbf{T}}$.

In each F-orbit of length m, one factor has field size q^m and torus action $(\delta b)^m$. Moreover, its rational ℓ-torsion lies in the Φ_{d_m}-subtorus $\mathbf{S}_m$, with $d_m = \mathrm{ord}_\ell(q^m)$. If $m = 1$, the eigenvalues are $\delta\mu_3$, separated modulo ℓ by $X^3 - \delta$. If $m = 3$, the mark is δI and $q^3 \equiv \delta \pmod \ell$. Each root component is δI-stable, so $m \in \{1, 3\}$. For multiplicity a of Φ_{d_m}, the torus valuation is $a\,v_\ell(\Phi_{d_m}(q^m))$. Moreover, the other eigenvalues contribute 0. A larger multiplicity would exceed $|\overline{P}_m|$, the projected Sylow order. Thus, $\mathbf{S}_m$ is a Sylow d_m-torus. By Lemma 7.170, a projected root α is nontrivial on the image of $P[\ell]$. Write $\overline{\mathbf{T}}_m$ for the projected maximal torus. Then $C(\mathbf{S}_m) = \overline{\mathbf{T}}_m$. Given any adjoint factor J_m, we apply [156, §8 and Theorem D] with torus source 1. This matches unipotents χ_i with Weyl labels ψ_i and their extensions $\widehat{\chi}_i, \widehat{\psi}_i$. Next, on a factor orbit $\mathcal{O}$, take identical extension pairs $(\widehat{\chi}_i, \widehat{\psi}_i)$. Next, choose tensor

permutation P_w as in Proposition 7.279. Regularity of $\mathbf{S}_m$ allows fixed directions in $\overline{\mathbf{T}}$. Let π_i be the factor map from K for χ_i. By central isotypy, $\chi_i \circ \pi_i$ is irreducible on K. Moreover, the local label is $\psi_i \circ \varpi_i$, with Weyl factor map ϖ_i from $W(\mathbf{K})^F$. Apply κ_s by Lemma 7.7 and transfer on $C(s)$ by Corollary 7.3. Given $x^2 = 1$ with s of odd order, we have $s^h = 1$. At a cubic vertex in $\mathbf{K}^*$, the E_6-span is U_6 by Proposition 7.171. Then, s^3 lies on $\mathbf{Z}^{*F^*}$ of order $q - \delta$. Since $\gcd(q - \delta, h) \mid 3$ and $v_3(h) \leq 1$, we have $s^3 = 1$. We compute $R(y), R(ty)$ for marked y of order 3 and $t = \omega_7^\vee(-1)$.

These points require $u \equiv 1 \pmod 3$. Next, we choose the Coxeter product $s_1 \cdots s_6$ and its fourth power b. Next, we write $\alpha_i(y) = \zeta_3^{k_i}$. Row reduction of $(b-I)k = 0$ over $\mathbb{F}_3$ gives $k = (a, c, -a-c, -a-c, -a, a+c, e), \qquad a, c, e \in \mathbb{F}_3$. If $(a, c) = (0, 0)$, the two nonzero choices of k give $R(y) = R(ty) = E_6$. Otherwise, the 24 choices give $(R(y), R(ty)) = (A_5A_2, A_2^3)$. For the Cartan matrix A, take edges $13, 34, 24, 45, 56, 67$. Reflection s_i acts on k by $k \mapsto k - k_i A_{\bullet i}$. Thus, the orbits of e_i under $v \mapsto v - (Av)_i e_i$ give the roots v. Next, keep $v \geq 0$ with $v \cdot k \equiv 0 \pmod 3$, adding $v_7 \equiv 0 \pmod 2$ for ty. The positive-root counts for y, ty are $36, 36$ if $(a, c) = (0, 0)$ and $18, 9$ otherwise. Next, discard $v = v_1 + v_2$ with both v_i chosen. The simple components have ranks $6, 6$ and $(5, 2), (2, 2, 2)$ in the two cases.

For odd s with Levi $C(s)$, apply the twist κ_s. Moreover, the nonidentity isolated type is A_5A_2. $r \geq 5$, and Proposition 7.217 with Lemma 7.216 gives $\succeq_c$ on M. If $s = 1$, apply [156, Theorem D]. For even $|s|$, its two-part is t. The series for $s = t$ is covered by Corollary 7.214. Otherwise, $s = tjy_3$, with $j = s_{6'}$ and $y_3^3 = 1$. If $j = 1$, Proposition 7.173 gives either a Levi $C(s)$ or isolated $[s]$ of order 6. In the Levi case apply κ_s. By Proposition 7.220, we obtain $\succeq_c$ on isolated $[s]$, with connected index excluding $[s']$. If $j \neq 1$, $C(j)$ is a proper Levi. If $t \in \mathrm{Z}(C(j))$, the argument for s gives $C(s) = C(y)$, a Levi.

For noncentral t, we apply [186, Theorems 4.1–4.2] to the cycle of A_1's at $\ell \mid q^3 - \delta$, including $|s_H| = 6$. On an additional A_2, distinct colours give $R(s) = \varnothing$. Next, for constant colours on A_2, take its unipotent comparison. In balanced A_5, nonconstant triples give $R(s) = \varnothing$. By Proposition 7.173, the ratio $|z_1/z_2| = 6$ gives a Levi $C(s)$. If $z_1/z_2 = -1$, the diagram action of b is trivial and $-I$ induces opposition. The rational forms are $\mathrm{GL}_6(q)$ for $\delta = 1$ and $\mathrm{GU}_6(q)$ for $\delta = -1$.

Apply Proposition 6.106 for $\delta = 1$ and Lemma 7.174 for $\delta = -1$, with $b = 3, c = 0$. Next, choose the twist $\theta \circ \det$ and the transfer through $C(j)$ of Proposition 7.173. If $R(s) = \varnothing$, apply Proposition 7.171 on M with the A_1 or torus models T. The local weight orbit on T recovers s by Sylow conjugacy in $C(s)^F$. Lemma 9.4 combines the comparisons on M. □

Lemma 7.176. *Assume $q = r^f$ is odd and $\mathrm{ord}_\ell(q) = 4$. In* Proposition 7.142*, take the rational lift s of order 4. Then, we have that $\mathcal{E}(G, s)$ is trivial on $\mathrm{Z}(G)$ for $G = E_{7,\mathrm{sc}}(q)$. Put $D = E_{7,\mathrm{ad}}(q)$ and $S = G/\mathrm{Z}(G) \leq D$. Given $P \in \mathrm{Syl}_\ell(S)$, we write $M_S = \mathrm{N}_S(P)$ and $M_D = \mathrm{N}_D(P)$. Therefore, the series on G is inflated from S. Let $\mathbf{T}_0$ be the primitive torus on the D_4-space and put $T_0 = \mathbf{T}_0^F$. Then $M_D/T_0 = \mathrm{PGL}_2(q)^3 \rtimes W_c$, $M_S/T_0 = K_0 \rtimes W_c$. Let $W_c = C_{W(F_4)}(c)$ with $c = \pm c_1c_2$ for negative two-cycles c_i of disjoint support. Moreover, $W_c \to S_3$ is onto with kernel $B = G(4, 2, 2)$ and third-factor stabilizer $A = C_4 \wr S_2$. On $\mathrm{PGL}_2(q)$, we write e for the determinant square class. Then $K_0 = \ker(e \otimes e \otimes e)$. The field lift U acts entrywise on the factors, fixes W_c pointwise, and satisfies $U^f = c^{-1}$ modulo T_0.*

Proof. Take D_4 on nodes $2, 3, 4, 5$, with perpendicular roots α_0, α_7, t as in Proposition 7.142. Since $c^2 = -I$ there, the Φ_4-eigenspaces span the D_4-space. Pin the root groups on α_0, α_7, t. The E_7 degrees $8, 12$ give both $\Phi_4(q)$-factors. No degree is divisible by 4ℓ, so T_0 contains a Sylow subgroup. By Proposition 7.172, $M_S = \mathrm{N}_S(\mathbf{T}_0)$. Since $\mathbf{T}_0 \cap \mathrm{Z}(G) = 1$, the normalizers modulo $\mathrm{Z}(G)$ are isomorphic via the isogeny. Since $S \lhd D$, Sylow conjugacy gives $D = SM_D$ and $M_D = \mathrm{N}_D(\mathbf{T}_0)$.

Orient the perpendicular roots β_i with sum $2\omega_7$. For any other orientation, s_{β_i} transports the construction whenever β_i changes sign. Moreover, the based $W(F_4)$-action lies in $W(E_6)$ perpendicular to ω_7. A derived E_6-lift permutes the pinned maps x_{β_i} with scalars a_i. Since their total weight is $2\omega_7$, the torus has $\prod_i a_i = 1$. The D_4 simple lifts centralize x_{β_i}, so the product factors through S_3. In D_5, we have that the transposition lift w is the product for $e_4 + e_5, e_4 - e_5$. Also, w fixes the third map x_{β_i}, and $w^2 = (2e_4)(-1)$ acts trivially on both exchanged x_{β_j}. Thus, $\prod_i a_i = 1$ on the transpositions generating S_3.

The scaling map on the E_6-torus has kernel dimension 4 and image $\{(a_i) : \prod_i a_i = 1\}$. Indeed, the Levi roots β_i span a primitive root sublattice Λ. The weight-lattice saturation is $\Lambda + \mathrm{Z}\omega_7$, of index 2 over Λ. Thus, the residual centre is $\mathbf{T}_0 \times \mathrm{Z}(G)$, and ω_7 is nontrivial on $\mathrm{Z}(G)$. Thus, the E_6-torus kernel is $\mathbf{T}_0$. Correct the scalars a_i to obtain

pure permutations modulo $\mathbf{T}_0$.

By [133, Proposition 23.2] for $\mathbf{T}_0$, we choose a prime-field lift v of c. If $q \equiv 3 \pmod 4$, the mark includes w_0, the simultaneous longest lift. Since diagonal SL_2 centralizes W_c, Lang conjugation there gives Frobenius $\mathrm{Int}(v)F_q$ on the series, also for $q \equiv 1 \pmod 4$. In $W(B_4)$, we have that rotating and exchanging c_1, c_2 gives $C_{W(B_4)}(c) = C_4 \wr S_2 = A$. Its even-sign subgroup $A \cap W(D_4) = B$ has order 16. The $W(B_4)$-class stays one $W(D_4)$-class because $A \not\leq W(D_4)$. The equation $c^2 = -I$ determines that class within $W(D_4)$. The class of c is S_3-stable, so $W_c \to S_3$ is onto. In $W(B_4)$, the transposition τ induces the graph action on D_4. Hence, we obtain the stabilizer A and $|W_c| = 96$. On $\mathbf{T}_0$, [114, corollary to Theorem 1] gives rational pure representatives modulo T_0.

Write z_1, z_2, z_3 for the central involutions of SL_2^3. Indeed, we have $(\omega_7)|_{\mathbf{T}_0} = 1$ and takes value -1 on each z_i. Thus, the kernel modulo $\mathbf{T}_0$ consists of the even products $z_1^{a_1} z_2^{a_2} z_3^{a_3}$. Quotient by the remaining involution gives PGL_2^3. Then (g_1, g_2, g_3) lifts rationally if and only if $\prod_i e(g_i) = 1$. The pure representatives from G yield M_S/T_0 and M_D/T_0. The marked lift U acts as F_r. Thus, U^f cancels the marking by v. Pure rational representatives of c^{-1} differ by T_0, giving $U^f \equiv c^{-1} \pmod{T_0}$. □

Lemma 7.177. *Moreover, on M_S, there are 14 singletons and 12 diagonal pairs. The lift U fixes every label. If $\sigma \in \mathcal{H}_\ell$ acts on μ_r by ℓ^j, each pair has sign $(\ell/r)^{fj}$, with Legendre symbol $(\,\cdot\,/r)$.*

Proof. Take $x, y \in \{1, \mathrm{St}\}$ on $\mathrm{PGL}_2(q)$. The lift $s = (\omega_1^\vee + \omega_6^\vee)(i)$ has coordinates $(-1, 1, t^\vee(-i))$ on the three dual factors since $\omega_1^\vee + \omega_6^\vee = -2\alpha_0^\vee - t^\vee$. Thus, T_0 acts trivially with ordered label (ex, y, π). The dual element s_π of order 4 is split if $q \equiv 1 \pmod 4$, nonsplit if $q \equiv 3 \pmod 4$. The Weyl group conjugates s_π to s_π^{-1}, so π is rational with $\pi^U = \pi$. The twist e also inverts s_π, so $e\pi = \pi$.

Consequently, $\pi|_{\mathrm{PSL}_2(q)} = \xi_+ + \xi_-$. With $\mathrm{Tr}_{\mathbb{F}_{q^f}/\mathbb{F}_r}$, the Gelfand–Graev character Γ contains exactly one $\xi_\pm$. Thus, U fixes both $\xi_\pm$. The determinant separates ex from y, while only π splits. The inertia quotient B acts trivially on the tensor model ρ. Thus, extend ρ by 1_B, tensor by $\eta \in \mathrm{Irr}(B)$, and induce to M_D/T_0. Clifford theory gives distinct irreducibles $\chi_{\eta,x,y}$. Their degrees are prime to ℓ, since the index is 6, $\eta(1)$ is a power of 2, and $r, q-1, q+1$ are prime to ℓ.

After reordering, the twist by D/S induces $(\eta, x, y) \mapsto (\eta^w, y, x)$ for $w \in A \setminus B$. The action of w is the component action in [208, proof of Proposition 11.20, Step 1] for D_4. Indeed, the eligible unipotents form the $d = 4$ series with relative group B. The two negative cycles c_1, c_2 exhaust the primitive space of D_4. The torus source is 1_{T_0} and $|B| = 16$ is prime to ℓ, including $\ell = 5$. Thus, the equivariant parametrization gives precisely $\mathrm{Irr}(B)$ [156, §4.3]. If $A = C_4^2 \rtimes S_2$, equal colours give 8 extensions and unequal colours give 6 inductions. The twist by A/B adds 2 to both colours modulo 4, fixing $\{0, 2\}$ and $\{1, 3\}$. Thus, $\mathrm{Irr}(B)$ has 6 fixed and 4 moved members under w. Among the 40 parents, 12 split on restriction to M_S/T_0, and 28 form 14 orbits.

If (η, x, y) is fixed, we put $K_1 = \{g \in K_0 : e(g_3) = 1\}$. The ordered tuple (x, x, ξ_+) has inertia $K_1 \rtimes A$. Given $g \in K_1$, we choose $\rho_+(g) = e(g_1)\rho_x(g_1) \otimes \rho_x(g_2) \otimes \rho_{\xi_+}(g_3)$, with ρ_x, ρ_{ξ_+} affording x, ξ_+. Since $e(g_1) = e(g_2)$ on K_1, the tensor flip extends ρ_+ to $K_1 \rtimes A$. Tensor by either extension $\widehat{\eta}$ of η to A and induce to M_S/T_0. Therefore, Clifford induction gives distinct characters ψ_+, ψ_-. The multiplicity space of ρ_+ in the restricted parent is $\mathrm{Ind}_B^A \eta$. Thus, $\chi_{\eta,x,x}$ restricts to $\psi_+ + \psi_-$.

Take $j_0 = (l, 1, l)$ with $e(l) = -1$. If ξ_+ is replaced by $\xi_- = \xi_+^{j_0}$, the swap lift changes by $(l, l^{-1}, 1)^{\pm 1}$. Conjugation absorbs both $\rho_x(l^{\pm 1})$ factors, leaving $e(l) = -1$. The correction on B is 1. Thus, changing ξ_+ to ξ_- twists $\widehat{\eta}$ by the sign of A/B. By the two-colour description, $\widehat{\eta}$ has values in $\mathbb{Q}(i)$. The B-characters have values there too since B has exponent 4. Since $\ell \equiv 1 \pmod 4$, the action on $\xi_\pm$ determines the pair action of σ. Their individual fields are $\mathbb{Q}$ if f is even and $\mathbb{Q}(\sqrt{r^*})$ if f is odd, with $r^* = (-1)^{(r-1)/2} r$. Thus, the pair sign is $(\ell/r)^{fj}$. The lift U acts on the pinned models $\rho_\pm$ and the Weyl lifts [165, Theorem B]. □

Proposition 7.178. *Let $\mathrm{ord}_\ell(q) = 4$ and take the rational $D_4A_1^2$-series $\mathcal{E}(G, s)$ with trivial character on $\mathrm{Z}(G)$. Then $\succeq_c$ holds on $\mathrm{N}_G(P)$ for every f, with diagonal and field extensions.*

Proof. Match χ, ψ by Lemma 7.177. If $a = \alpha_0 + \alpha_7$, choose the rational parabolic $\mathbf{P}_a$ of Proposition 7.142. If opposition is negative, $s_{\alpha_0} s_{\alpha_7} \in C(s)^\circ$ corrects the mark to fix a. The split mark fixes a already. The connected intersection includes D_4 and both split flag tori $\mathbf{T}_1, \mathbf{T}_2$. Thus, source exchange induces permutation on $\mathcal{E}(G, s)$. The Levi reflection s_t acts by $s^{-1} \mapsto s$ and fixes the D_4-label. On $H = \mathrm{Spin}_{10}(q) \times \mathrm{SL}_2(q)$, the spin labels θ are rational.

Indeed, the lift s_{sp} has eigenspace dimensions $8, 2$, so the spinor twist moves $[s_{\mathrm{sp}}]$. For $\xi_\pm$, apply Lemma 7.177 for their fields.

Suppose f is odd and σ exchanges ξ_+, ξ_-. The central torus $a(\mathbf{G}_m)$ meets the derived group in μ_4. For a conjugate to $\omega_6^\vee$, the primitive quotient character has degree 4 on $a(\mathbf{G}_m)$. Since $|\mathrm{Z}(\mathrm{SL}_2)| = 2$, projection gives $\mu_4 \cong \mathrm{Z}(\mathrm{Spin}_{10})$. Apply Lang to $a(\mathbf{G}_m)$ and $\mathbf{H}$. Thus, L induces full inner-diagonal action on $\mathrm{Spin}_{10}(q)$. The quotient L/H is cyclic, and the spin orbit $L \cdot \theta$ has 2-power size. If $|L \cdot \theta| > 1$, its stabilizer in L/H consists of even powers, fixing each $\xi_\pm$. If $|L \cdot \theta| = 1$, the regular parent has at most 2 constituents on H by the full spin action of L. By multiplicity freeness, $\chi_\pm|_H$ are disjoint and nonempty. The bound 2 makes each $\chi_\pm|_H$ irreducible. Thus, exchanging $\xi_\pm$ exchanges $\chi_\pm$, as locally.

If $2 \nmid f$, we let ι fix μ_{2^∞} and invert every odd-order root ζ. Since ψ is $\mathbb{Q}(i)$-valued and f is odd, [208, Lemma 3.2] gives its unique extension $\widehat{\psi}$ over $\mathbb{Q}(i)$. Thus, ι and $\mathcal{H}_\ell$ fix $\widehat{\psi}$. Given a rational global singleton χ, choose its unique real extension $\widehat{\chi}$ by [143, Lemma 2.1 and Corollary 2.2]. On each split pair $\{\chi_+, \chi_-\}$, we set $\kappa = \iota$ or $d\iota$ according as $\chi_+^\iota = \chi_+$ or χ_-. For nonsquare $u \in \mathbb{F}_r^\times$, put $d = \prod_i \beta_i^\vee(\sqrt{u})$ in the adjoint group. Then d satisfies $d\,v = v\,d$, and $F_r(d)d^{-1}$ is central. With f odd, d is outer and d^2 is inner. On odd field twists, κ acts by inversion. By [208, proof of Lemma 3.1], κ fixes a unique extension $\widehat{\chi}_\pm$ of each constituent. On characters, $\kappa d = d\kappa$ and $\kappa\sigma = \sigma\kappa$ for $\sigma \in \mathcal{H}_\ell$. Hence, the mixed stabilizer elements (d^i, σ) fix the choices.

If f is even, the torus for π is split. On $\mathrm{PGL}_2(q)$, take the field permutation P_r if $r \equiv 1 \pmod 4$. If $r \equiv 3 \pmod 4$, choose $JP_r/\sqrt{q}$ with longest intertwiner J. The normalization of [208, Proposition 4.16] is over $\mathbb{Q}(i)$. For coefficient conjugation γ, the invertible antilinear map $J\gamma$ satisfies $J\gamma\,U = U\,J\gamma$. Indeed, J has rational coefficients and P_r permutes its summands, fixing the representative w_0. Also, $\sqrt{q} \in \mathbb{Q}$. Thus, $\widehat{\pi}$ is real and $\mathbb{Q}(i)$-valued. Hence, $\widehat{\pi}$ is rational.

On ex, y, take the projective-line action on $1, \mathrm{St}$ and field value 1 on e. On the Weyl multiplicity space take $\alpha^f = \eta(c^{-1})/\eta(1)$ with $\alpha^{4f_2} = 1$, where $f_2 = 2^{v_2(f)}$. Since c is central in W_c, this satisfies the relation for U in Lemma 7.176. Such α exists because f/f_2 is odd. The order of r mod ℓ has two-part $4f_2$, so $\mathcal{H}_\ell$ fixes α. Induction yields an extension $\widehat{\psi}$ fixed by $\mathcal{H}_\ell$ on M_D. The constituents on M_S are U-stable of multiplicity 1. Since each constituent ψ_i is $\mathcal{H}_\ell$-fixed for even f, so is $\widehat{\psi}_i$.

On L, take split coordinates for $\mathbf{P}_a$ by Lang conjugation. Then F_r acts separately on $\mathrm{Spin}_{10}(q)$ and $\mathrm{SL}_2(q)$. Since s_t corrects inversion of s and fixes the split D_4-label τ, F_r stabilizes the Levi series at τ, including degenerate labels. On $\mathrm{SL}_2(q)$, $\xi_\pm(-I) = \xi_\pm(1)$. In the Gelfand–Graev model Γ, a coefficient exponent u on μ_r is compensated by the coroot at $\sqrt{u}$, since f is even. The commutator with F_r is in $\{I, -I\}$, acting trivially. Let h be the coroot at $\sqrt{b}$ for a nonsquare $b \in \mathbb{F}_q^\times$. Then Γ^h detects the other trace orbit. Take $hF_r(h)^{-1}F_r$ on Γ^h. Its f-th power is $-I$, acting as $+1$. Thus, multiplicity 1 gives $\mathcal{H}_\ell$-fixed extensions $\widehat{\xi}_\pm$.

In split $\mathrm{SO}_{10}(q)$, both eigenspaces $V_\pm$ are split. Take the Levi at $\mathbf{P}_-$, the rational line parabolic on V_-. Then the source on $\mathrm{GL}_1(q) \times \mathrm{SO}_8^+(q)$ is $\eta_{\mathrm{line}} \otimes \chi_\tau$, with η_{line} quadratic and χ_τ unipotent. The regular induction has multiplicity 1 because it acts identically on D_4. On the orthogonal Levi L_-, the source restricts irreducibly since s_0 is central. If the target splits, θ_+, θ_- each occur with multiplicity 1. For degenerate τ, the translate $\tau' \neq \tau$ gives a distinct source $\chi_{\tau'}$. Thus, their common singleton θ occurs with multiplicity 1. The adjoint character gives $\chi_\tau = \overline{\chi}_\tau \circ \pi_{\mathrm{ad}}$. Since $\overline{\chi}_\tau$ is rational with Frobenius eigenvalue ± 1, take an f-th root in μ_{2f_2}, fixed by $\mathcal{H}_\ell$. Thus, $\widehat{\chi}_\tau \otimes \widehat{\eta}_{\mathrm{line}}$ is fixed by $\mathcal{H}_\ell$ [156, Theorem 7.5].

For a stable θ of multiplicity 1 in a restriction $\chi|_Y$, the θ-space of $\widehat{\chi}$ affords an extension $\widehat{\theta}$. If λ is a field-quotient character, then $\lambda\widehat{\chi}$ affords $\lambda\widehat{\theta}$ on that space. Thus, a fixed $\widehat{\theta}$ determines a fixed $\widehat{\chi}$ by uniqueness. For the unipotent radical R, the rational idempotent $|R|^{-1}\sum_{u \in R}\rho(u)$ projects onto V^R. Thus, V^R affords the Harish–Chandra restriction of $\widehat{\chi}$. For V_- and the spin pullback θ, multiplicity is 1. Tensor with $\widehat{\xi}_\pm$, then restrict to H with multiplicity 1 per summand. Finally, induction from $\mathbf{P}_a^F$ gives the ambient target with multiplicity 1. Keep the regular label on D at each singleton χ. Thus, $\mathcal{H}_\ell$ fixes $\widehat{\chi}$ at singletons and $\widehat{\chi}_\pm$ at pairs. Restrict the singleton extensions $\widehat{\chi}, \widehat{\psi}$ to S, M_S. Take D/S to transport paired extensions. Hence, on G both inflated models have scalar $+1$ at the central involution, giving $\succeq_c$. □

For odd $q = r^f$ and $\mathrm{ord}_\ell(q) = 4$, we put $G = E_{7,\mathrm{sc}}(q)$. Take the second rational $D_4A_1^2$-form of Zhang [208, Proposition 11.20], with component swapping $A_1^{(1)}, A_1^{(2)}$. Hence, the derived factor types are ${}^2D_4(q), A_1(q^2)$. The series is faithful on $\mathrm{Z}(G) = \langle z \rangle$ with $|z| = 2$. Let $\mathbf{T}$ be the primitive torus of dimension 4, and put $P = (\mathbf{T}^F)_\ell$, $L = C_{\mathbf{G}}(\mathbf{T})^F$ and $M = \mathrm{N}_G(P)$.

Lemma 7.179. *Then the weight κ on $T = \mathbf{T}^F$ has order 2. $[M : I_M(\kappa)] = 3$. Moreover, for $x, y \in \{1, \mathrm{St}\}$ the four sources ψ_{xy} are irreducible on L, rational and field-fixed. The turns ρ, υ have order 4 and give $B_{xy} = I_M(\psi_{xy})/L$. If $x = y$, we find $B_{xy} = \langle \rho \rangle \times \langle \upsilon \rangle$. Otherwise, we have that $B_{xy} = \{\rho^i \upsilon^j : i + j \equiv 0 \bmod 2\}$. Also, the two labels with $x \neq y$ form one M-orbit.*

Proof. On signed D_4-coordinates $f_1, \dots, f_4$, the graph action is odd. With $\alpha = \alpha_0, \gamma = \alpha_7$, choose t from Proposition 7.142. Then, choose c with negative two-cycles on $(f_1, f_2), (\alpha, \gamma)$, fixing f_3, f_4. Indeed, $W(D_4)$ adjusts the odd action on $f_1, \dots, f_4$. The outer signs come from s_α, s_γ. If $q \equiv 3 \pmod 4$, opposition $-I$ changes both plane orientations. Lang in $C(s)^\circ$ transports the rational form.

The frame $\mathcal{F} = \{\alpha, \gamma, \delta, \delta'\}$ has $\delta = f_1 - f_2$ and $\delta' = f_1 + f_2$. The planes give both $\Phi_4(q)$-factors in $|G|$, so P is Sylow also at $\ell = 5$. Conjugacy of Sylow 4-tori gives the cocharacter lattice $Q(D_4)$ for the primitive D_4. The perpendicular lines are $f_3 - f_4, f_3 + f_4, t$. Thus, Lemma 7.176 gives relative group W_c of order 96. In simply laced coordinates $s = \alpha(-1)t(-i)$, so its primitive projection is $\alpha(-1)$. A root β of D_4 with $(\beta, \alpha) = 1$ detects this projection. For distinct $u, v \in \mathcal{F}$, $(u + v)/2 \in P(D_4)$, so all axes agree modulo $2P(D_4)$. Exactly the frame roots $\pm\mathcal{F}$ take value 1 at $\alpha(-1)$. The other roots have form $\frac{1}{2}\sum_{u \in \mathcal{F}} \epsilon_u u$ for $\epsilon_u = \pm 1$, with odd pairing against α. Thus, $W_\mathcal{F}$, the signed permutation group of $\mathcal{F}$, is the isometry stabilizer. Intersect $W_\mathcal{F}$ with $C(c)$. Rotating or exchanging the cycles gives $C_{W_\mathcal{F}}(c) \simeq C_4 \wr S_2$, of order 32. Hence, rational torus duality gives $[M : I_M(\kappa)] = 96/32 = 3$.

Put $\mathbf{H} = [C_\mathbf{G}(\mathbf{T}), C_\mathbf{G}(\mathbf{T})]$. Let z_i be the involution of factor i. Thus, the kernel $K = \mathbf{T} \cap \mathbf{H}$ consists of even products of z_1, z_2, z_3 by Lemma 7.176. Since $f_3 - f_4, f_3 + f_4, t$ are rational lines, $K = K^F$. Since $|T| = (q^2 + 1)^2$ has 2-part 4, K is a Sylow 2-subgroup of T. Thus, $\kappa|_K \neq 1$. Each z_i lies in $\mathrm{SL}_2(q)'$, including $q = 3$, where $\mathrm{SL}_2(3)' = Q_8$. Thus, κ has no linear extension to L.

On the dual Levi, s has root values $1, 1, -1$. The two neutral reflections fix s, and s_t acts by $s \mapsto s^{-1}$. Since $|\bar{s}| = 4$, s_t moves $\bar{s}$. Also, the Levi Weyl orbit $W_L \cdot s$ excludes zs. Hence, in the dual Levi $C(\bar{s})$ is connected. Thus, the parent $\widetilde{\psi}_{xy}$ has trivial quotient self-twist. Put $\psi_{xy} = \widetilde{\psi}_{xy}|_L$, which is irreducible. On $\mathbf{H}^F = \mathrm{SL}_2(q)^3$, the labels are $x, y, \xi_\pm$, with quadratic pair $\xi_\pm$ on t. The old torus T_t is nonsplit if $q \equiv 1 \pmod 4$, split if $q \equiv 3 \pmod 4$. Thus, both cases give $\eta_t(-1) = -1$ for the quadratic character η_t. Each coefficient or field action σ gives $s^\sigma = s^{\pm 1}$. Since s_t corrects inversion and fixes x, y, the remaining quotient twist restricts to 1_L. Thus, ψ_{xy} is rational and field-fixed. Since T contains both primitive factors, $\ell \nmid \psi_{xy}(1)$.

The turns ρ, υ on (α, γ), (δ, δ') have product c on the primitive space. Next, for ρ, choose the graph element $f_4 \mapsto -f_4$ with an outer sign correction. υ compensates the negative two-cycle on (f_1, f_2) by $f_4 \mapsto -f_4$. Thus, both turns exchange $f_3 - f_4, f_3 + f_4$ and fix t. Indeed, the image of $I_M(\kappa)/L$ in S_3 contains this transposition and is a 2-group, so fixes t.

Exchanging the primitive planes acts by $\alpha(-1) \mapsto \delta(-1)$ or $\delta'(-1)$. In $Q(D_4)/2Q(D_4)$, the axes have sole relation $\sum_{u \in \mathcal{F}} u = 0$. Indeed, $\mathbb{Z}\mathcal{F}$ has index 2, and the half-sum generates $Q(D_4)/\mathbb{Z}\mathcal{F}$. Thus, neither change from $\alpha(-1)$ lies in $\{1, \alpha(-1)\gamma(-1)\} = \{1, zt(-1)\}$. Since $\langle z, t(-1)\rangle \cap \mathbf{T} = \{1, zt(-1)\}$, the exchange survives both s_t and the regular scalar corrections. With t positively oriented, ρ acts by $s \mapsto zs^{-1}$, which the reflection on t corrects to zs. The turn υ fixes s. Thus, ρ, υ act on the series by $(x, y) \mapsto (y, x)$, giving B_{xy}. □

Lemma 7.180. *Fix x, y and put $B = B_{xy}$, $I = I_M(\psi_{xy})$. There is $D_B \leq I$ with $I = LD_B$ and $L \cap D_B = T$. Thus, D_B permutes the pinning without scalars. Moreover, D_B has a field-invariant character $\lambda_0 : D_B \to \{1, -1\}$ with $\lambda_0|_T = \kappa$. If $c = \rho\upsilon$, the prime-field lift v may be chosen with $\lambda_0(v) = 1$.*

Proof. Consider the transported permutation group $\mathbf{D}$ of Lemma 7.176. Since $\mathbf{T}$ is connected, Lang gives $v \in \mathbf{D}$, a prime-field lift of c. Its permutation on $\mathbf{H}$ is trivial, so v centralizes $\mathbf{H}$. Thus, Lang conjugation in $\mathbf{H}$ gives trivial residual Weyl action. Then $F = \mathrm{Int}(v)F_q$, with the transported Levi class. Thus, D_B is the preimage of B in $\mathbf{D}^F$. By [133, Proposition 23.2] for $\mathbf{T}$, $D_B/T \simeq B$. Moreover, D_B centralizes $\mathrm{SL}_{2,t}$. The coroot torus $t^\vee(\mathbf{G}_m)$ has centralizer $\mathrm{Spin}_{12} \mathbf{T}_{\max}$, with maximal torus $\mathbf{T}_{\max}$. Since D_B centralizes the root groups on t, its remaining torus multiplier lies in $\ker t$. This kernel is connected because t pairs to 1 with a coroot. Thus, $\ker t$ is the spin maximal torus, and $D_B \leq \mathrm{Spin}_{12}$.

Take the prime-field representation $\varpi : \mathrm{Spin}_{12} \to \mathrm{SO}_{12}$ with coordinates $e_1, \dots, e_6$. Choose $\alpha = e_5 - e_6$, $\gamma = e_5 + e_6$. Indeed, disjoint support gives A_1^4 perpendicular to α, γ, whereas $\{\alpha, \gamma\}^\perp$ contains the old D_4. The second plane is on e_1, e_2 of the old D_4. Thus, the four-spaces V_1, V_2 are $\varpi(D_B)$-stable of these planes. On V_1, we put $v_1 = \varpi(v)|_{V_1}$ and let $\mathbf{T}_1$ be its coordinate torus. The square v_1^2 exchanges each pair of weight lines $V_\nu, V_{-\nu}$. Since v_1^2 fixes $\langle V_\nu, V_{-\nu}\rangle$

setwise, its scalars are a, a^{-1}. Thus, $v_1^4 = 1$ on V_1. For $g \in D_B$ above $\rho^i \upsilon^j$, the first restriction has unique form $d_1 v_1^i$ with i mod 4 and $d_1 \in \mathbf{T}_1^{F_1}$, where $F_1 = \mathrm{Int}(v_1)F_q$. The torus $\mathbf{T}_1^{F_1}$ is cyclic of order q^2+1. Put $\lambda_0(g) = \eta_1(d_1)$ for its quadratic character η_1. We have $\eta_1^{v_1} = \eta_1$, so λ_0 is multiplicative. Moreover, F_r fixes v_1 and gives $\eta_1^{F_r} = \eta_1$. If $g = v$, the torus factor is $d_1 = 1$, so $\lambda_0(v) = 1$. On the orthogonal torus, $\eta_1 \otimes 1$ is dual to $(-1,-1,1,1)$. The dual isogeny ϖ^* gives the image $\alpha(-1)$ in the adjoint D_4-torus. Thus, $\lambda_0|_T = \kappa$ by the pullback in Lemma 7.179. □

Proposition 7.181. *At $\mathrm{ord}_\ell(q) = 4$, the faithful rational form of $D_4A_1^2$ has $\succeq_c$ on $M = \mathrm{N}_G(P)$ for every f. Its 40 members form 20 diagonal pairs. If $\sigma \in \mathcal{H}_\ell$ acts on μ_r by a, each pair has permutation sign $(a/r)^f$. If f is even, take extensions with twist (a/r) at U on both sides.*

Proof. Let V afford ψ_{xy}, with flag Borels B_x, B_y and trace character λ_t on the pinned t-root group. The regular parent $\widetilde{\psi}$ restricts multiplicity freely to $\mathbf{H}^F$. Both constituents $\xi_\pm$ occur, since K allows nonsquares on t and one flag factor x. Indeed, by Lang on $\mathbf{T}, \mathbf{H}$, the even cocycle is represented in K. The flag spaces $V_x^{B_x}, V_y^{B_y}$ have dimension 1, and λ_t detects exactly one $\xi_\pm$. Thus, the detector E has dimension 1, with conditions stable under D_B, F_r.

Given $g \in D_B$, normalize $X(g)$ on V by $X(g)|_E = \lambda_0(g)$. By [95, Corollary 1.6], this determines $X(g)$. Since $\lambda_0(gh) = \lambda_0(g)\lambda_0(h)$, we have $X(gh) = X(g)X(h)$. On $L \cap D_B = T$, this is the torus action on V. Thus, V extends to X on I. Given any representative (x,y), Clifford induction gives distinct $\mathrm{Ind}_I^M(X\lambda)$ for $\lambda \in \mathrm{Irr}(B)$. If $x = y$, each label gives 16 characters. The $x \neq y$ orbit gives 8. The indices and source degrees are ℓ', so all induced degrees are ℓ'.

The regular parent satisfies $\widetilde{\psi}|_L = \psi_{xy}$. For the quadratic quotient e, a corrected $b = \rho^i \upsilon^j \in B$ acts by $\widetilde{\psi} \mapsto \widetilde{\psi} e^i$. Indeed, s_t fixes the scalar coordinate. Multiplication by z on the first coordinate gives twist $[1,-1]$. Also, the flags return to (x,y). Choose an outer diagonal torus element h with $e(h) = -1$ and $[h,b] \in L$. Given a lift $\dot{b} \in I$, the regular intertwiner on L equals $X(\dot{b})$ up to scalar. At h the relation gives $e(h)^i$, since the commutator is in L. Hence, the diagonal gives $\lambda \mapsto \lambda\epsilon$, with $\epsilon(\rho^i\upsilon^j) = (-1)^i$. Since $c = \rho\upsilon$ has $\epsilon(c) = -1$, the pairs have labels $\lambda|_{\ker\epsilon}$. If $x = y$, $\ker\epsilon = \langle\rho^2\rangle \times \langle\upsilon\rangle$. If $x \neq y$, $\ker\epsilon = \langle\rho^2\rangle \times \langle\upsilon^2\rangle$. The nonsplit torus over q^2 gives the two $\langle\rho^2\rangle$-labels globally. In twisted D_4, the negative cycle stabilizes the split D_2 source and C_4 exchanges its flags. Thus, $x = y$ gives stabilizer C_4, and $x \neq y$ gives C_2. This gives $4+4+2 = 10$ labels in twisted D_4. Both the indices and degrees are ℓ'. Thus, by [208, Proposition 11.20, Step 2], each of the 20 global parents splits. Thus, match the 20 pairs by $\lambda|_{\ker\epsilon}$.

The field lift U acts by F_r with $U^f = v^{-1}$. Normalize $X(U)$ to act as 1 on E. Since λ_0 is field-fixed, $X(U)$ conjugates the D_B-operators by F_r. Moreover, $X(U)^f = X(v^{-1})$ by the action on L and scalar 1 on E. For the relative label take $\beta^f = \lambda(c^{-1})$ in μ_{4f_2}, where $f_2 = 2^{v_2(f)}$. This is possible because $\lambda(c^{-1}) \in \mu_4$ and the odd part of f acts invertibly on μ_{4f_2}. Since $\mathrm{ord}_\ell(r^f) = 4$, the two-part of $\mathrm{ord}_\ell(r)$ is $4f_2$. Thus, we have that $\mathcal{H}_\ell$ fixes β and the relative label. Induction to $M\langle U\rangle$ extends the character of M. Put $\mu = \sum_i \beta_i^\vee$ for the oriented residual roots β_i, so $\mu(-1) = z$. If $\sigma(\zeta_r) = \zeta_r^a$, the corrector is $g_a = \mu(\sqrt{a})^{\pm 1}$ with the conjugation sign. The corrector g_a fixes the flags and centralizes D_B. If f is even, $g_a \in L$. Moreover, $F_r(g_a)g_a^{-1} = z^e$, with $(-1)^e = (a/r)$. Thus, the scalar ratios on D_B, U are $1, (a/r)$ because z acts by -1. The corrected models agree on L and intertwine the actions on E, E^σ. Hence, $I\langle U\rangle$ has the given twist. If f is odd and a is square, g_a is inner over $\mathbb{F}_r$. For nonsquare $b \in \mathbb{F}_r^\times$, combine $\mu(\sqrt{b})$ with a regular scalar c_0 satisfying $c_0^r = -c_0$. Then the outer diagonal h satisfies $hU = Uh$, $hD_B = D_B h$. Thus, correction fixes $X|_E$, with $\lambda \mapsto \lambda\epsilon^e$ before correction.

Choose $\mathbf{P}_{a_0}$ from Proposition 7.142, with $a_0 = \alpha + \gamma$. Since the corrected mark exchanges α, γ positively, their Borel is over q^2. Induction gives $1 + \mathrm{St}$ with the D_4-label fixed. Thus, each coefficient is 1, giving a permutation matrix P_0. The spin constituents θ are rational by eigenspace dimensions $8, 2$. The central torus $a_0(\mathbf{G}_m)$ meets the derived group in $\mu_4 \simeq \mathrm{Z}(\mathrm{Spin}_{10})$. Thus, Proposition 7.178 transfers $\xi_+ \leftrightarrow \xi_-$ to the Levi pair χ_+, χ_-. Thus, the global sign is $(a/r)^f$, with $\chi^{F_r} = \chi$ for the ambient members [208, Lemma 11.14 and proof of Proposition 11.20, Step 3].

If f is even, the mark fixes a_0. Thus, Lang gives split coordinates on $H_a = \mathrm{Spin}_{10}(q) \times \mathrm{SL}_2(q)$. In $\mathrm{SO}_{10}(q)$, the sign spaces V_+, V_- have dimensions $8, 2$ and minus type. On V_+, the symbol $\Lambda = (A,B)$ has $\big||A| - |B|\big| = 2$, since difference 6 requires rank at least 9. Each valid hook (C,x) has $C \in \{A,B\}$, $x \in C$, $x > 0$, $x-1 \notin C$. Then C becomes $(C \setminus \{x\}) \cup \{x-1\}$, with row-size difference 2. Thus, at $A = \{0,\dots,u-1\}$, $B = \{0,\dots,v-1\}$, we get $|u-v| = 2$ and $\mathrm{rk}(\Lambda) = (u-v)^2/4 = 1$.

At (C,x), we choose an isotropic-line parabolic $\mathbf{P}_J$ for a hyperbolic plane $J \leq V_+$. Then, its residual space on V_+ is $V_+ \cap J^\perp$. Take the line character 1. Thus, connected multiplicity is 1 by [208, Lemmas 7.16–7.17] and [9, §6]. On V_+, V_-, paired odd isometries fix both symbols Λ. Thus, each regular parent $\widetilde{\chi}$ splits, with one above each $\{\chi_+, \chi_-\}$. The matrix $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ has $a + b = 1$, so each child has multiplicity 1. On the residual split $\mathrm{SO}_{2n}^+(q)$, the rational labels χ

satisfy $\chi^{F_r} = \chi$ [165, Theorem A].

At $\mathrm{SO}_4^+(q)$, the planes $V_\pm$ are anisotropic. Hence, on the cover $\mathrm{SL}_2(q)^2$, the coordinates are $1 \cdot (-1) = -1$ and $1/(-1) = -1$. At $-I$, each nonsplit quadratic sign is $(-1)^{(q+1)/2} = -1$. Thus, Proposition 7.178 gives twist (a/r) on each trace constituent ξ. Indeed, if a is nonsquare, the corrector changes U by $-I$. On the translated detector, we choose $\beta \in \mu_{2f_2}$ with the given f-th power ± 1, fixed by $\mathcal{H}_\ell$.

The tensor $\xi_1 \otimes \xi_2$ has twist $(a/r)^2 = 1$ and is trivial on Z_4. Thus, it descends to $H_4 \leq \mathrm{SO}_4^+(q)$ of index 2. Restriction to H_4 is multiplicity free. Thus, $\widehat{\theta}$ transfers at (C, x) by Proposition 7.178. The other lift gives $V_+ \leftrightarrow V_-$ with $\dim V_+ \neq \dim V_-$, so pullback to $\mathrm{Spin}_{10}(q)$ is irreducible. On the t-factor, the nonsplit torus has sign -1, giving twist (a/r). On H_a, tensor and transfer through $\mathbf{P}_{a_0}$ by restriction and induction. Thus, U has global and local twist (a/r).

If f is odd, let ι fix μ_{2^∞} and invert odd-order roots ζ. The permutation on $\xi_\pm$ and the labels λ_0, λ show that ι fixes each pair. If it exchanges $\chi_\pm$, compose with h. The involution on $\mathcal{X}_\chi$ inverts the twists in $\mathrm{Irr}(C_f)$. By [208, proof of Lemma 3.1], the unique fixed extension is $\widehat{\chi}$. Hence, both transports fix $\widehat{\chi}$. Extend the common central sign to $\langle z, c_0^2 \rangle$. If f is even, $\chi_\pm^\sigma = \chi_\pm$. Thus, ordinary diagonal inertia lies in $S = G/\mathrm{Z}(G)$. On the scalar torus $\mathbf{Z}_a$, we choose common κ_0 with $\kappa_0(z) = -1$. Then, nbU^i acts by $\kappa_0(b)X(nU^i)$. Both formulas have κ_0 on the overlap. Thus, for F_r, σ the scalar ratios match, yielding common $c, \mu_{a,\sigma}$ on the preimages. □

Lemma 7.182. *Take odd $q = r^f$ with $\mathrm{ord}_\ell(q) = 4$. Moreover, for $\bar{s}$ of type $A_3A_3A_1$, we write C for the outer A_1. The component swaps the A_3's up to common opposition. If $k = 1$, we find a rational lift s. If $k = -1$, Frobenius swaps the A_3's, giving derived types $\mathrm{SL}_4(q^2), \mathrm{SL}_2(q)$.* (Zhang [208, proof of Proposition 11.22, Step 1]) *For the primitive torus $T = \mathbf{T}^F$, we put $P = T_\ell$, $L = C_{\mathbf{G}}(\mathbf{T})^F$, and $M = \mathrm{N}_G(P)$. In both forms, the sources ψ_x on L are distinct for $x = 1, \mathrm{St}$ on C. If $k = 1$, $B = I_M(\psi_x)/L$ is $C_4 \wr S_2$. If $k = -1$, $B = D_8 * C_4$, with common center $\mathrm{Z}(D_8) = \langle c^2 \rangle$ for $C_4 = \langle c \rangle$. The diagonal acts by $\lambda \mapsto \lambda\epsilon$, where ϵ is the sign on the A_3 planes. For both x together, $k = 1$ gives 12 singletons and 8 pairs, and $k = -1$ gives 10 pairs.*

Proof. If $k = 1$, mark each A_3 by $(1, 2, 3, 4)$ with the common sign. Thus, $e_1 - e_3, e_2 - e_4$ span the plane. Its perpendicular in A_3 is spanned by $(1, -1, 1, -1)/2$. Write a, b for these unit vectors. If $k = -1$, the block swap composed with $(1, w)$ squares to (w, w) for a double transposition w. Thus, the opposition sign disappears in c^2. Matched differences β_1, β_2 span each plane. Thus, the perpendiculars a, b have coordinates $(1, 1, -1, -1)/2$ in pair order. In both cases, $\dim V_4 = 4$ and the primitive space gives both $\Phi_4(q)$-factors in $|G|$. By Lemma 7.176, $M = \mathrm{N}_G(\mathbf{T})$, including $\ell = 5$.

Let α_C be the root on C. Coroot pairings give $ma + nb + h\alpha_C/2$ for perpendicular roots, with $m, n, h \in \mathrm{Z}$. Thus, $m^2 + n^2 + h^2/2 = 2$ gives lines on $\alpha_C, \gamma_+ = a + b, \gamma_- = a - b$. By [208, (11.13)], $\gamma_\pm$ have lattice class $(2, 2, 0)$ of $2\alpha_4$. Thus, $\gamma_\pm$ are roots of squared norm 2. If $s = (iI_4, I_4, I_2)$, both $\gamma_\pm(s) = -1$ and $\alpha_C(s) = 1$. Moreover, both $\gamma_\pm$ lines are rational.

The reflection s_C fixes s. Each $s_{\gamma_\pm}$ multiplies s by $\gamma_\pm(-1)$. This is noncentral since $(\gamma_\pm, \beta)$ is odd for some ambient root β. Their product multiplies by $(2a)(-1) = s^2 \neq z$. Hence, the product corrects inversion of s. The residual Weyl orbit excludes zs. Thus, $C_{\mathbf{L}^*}(\bar{s})$ is connected with neutral root α_C. The parent $\widetilde{\psi}_x$ has trivial quotient self-twist. Hence, $\psi_x = \widetilde{\psi}_x|_L$ is irreducible. By Lang in $C_{\mathbf{L}^*}(\bar{s})$, we obtain a rational inversion correction g. The element g fixes x, and the scalar twist restricts to 1_L. Thus, ψ_x is rational and satisfies $\psi_x^{F_r} = \psi_x$.

In S_4, the matching stabilizer induces D_8 on the oriented differences β_1, β_2. Thus, the two-plane group is $W = D_8 \wr S_2$. Since the residual kernel $C_{\mathbf{L}^*}(\bar{s})$ is connected, Lang gives $B = C_W(c)$. The C-labels remain invariant since $1, \mathrm{St}$ have distinct degrees $1, q$. If $k = 1$, the identical quarter-turns give $B = C_4 \wr S_2$. If $k = -1$, c swaps the planes with $c^2 = -I$ on each. The subgroup of B with both planes invariant is diagonal D_8 under transport by c. The other coset is D_8c, with c^2 the common central involution. Hence, $B = D_8 * C_4$. If $\epsilon(g) = 1$, then $s^g = s$ up to residual correction. If $\epsilon(g) = -1$, then $s^g = zs$ up to that correction. Indeed, $\ker \epsilon \leq C(s)^\circ$. Thus, a swap gives quadratic twist e. At a regular diagonal h, the intertwiner of Proposition 7.181 gives $\lambda \mapsto \lambda\epsilon$ for the extension below.

If $k = 1$, a partition $\nu \vdash 4$ needs a 4-hook to cancel $\Phi_4(q)$ in the degree, since $4\ell > 18$. The label x is unrestricted. Take runners $i \in \mathrm{Irr}(C_4)$. If $i \neq j$, induction to $C_4 \wr S_2$ gives one ϵ-fixed label over $\{i, j\}$. If $i = j$, the two extensions are exchanged by ϵ. Thus, both x give 12 singletons and 16 paired constituents. If $k = -1$, $\mathrm{ord}_\ell(q^2) = 2$. Since 0 is the only even staircase size ≤ 4, each partition of 4 has empty 2-core. The two-quotients give $\mathrm{Irr}(D_8)$ for $D_8 = C_2 \wr S_2$. The choices $\pm\omega$ for c have square ω^2 and are exchanged by ϵ. Thus, the Jordan series gives 20 restrictions for the two

x. The relative indices and degrees are $2,3$-numbers. Since $P \leq \mathrm{Z}(L)$, the local degrees are ℓ'. □

Lemma 7.183. *In either form, D_B permutes the pinning, with $D_B/T \simeq B$. Moreover, the weight of ψ_x on T extends to an F_r-invariant $\lambda_0 : D_B \to \mu_4$. For the mark c, choose a prime-field lift v with $\lambda_0(v) \in \{1, -1\}$.*

Proof. Let Λ be generated by the matching axes $r_1, \ldots, r_4$. The Gram determinants of $\Lambda, Q(D_4)$ are $16, 4$, so the index is 2. The vector $\omega = \frac{1}{2}\sum_i r_i$ represents the other coset. On each A_3, its class is 2, so $s(\omega) = -1$ and $s|_\Lambda = 1$. Thus, $\kappa \neq 1_T$ and $\kappa^2 = 1_T$ in both forms. Transport ψ_x, κ under Lang conjugation. If $k = -1$, diagonal D_8 acts with the same sign on a, b. Indeed, swapping matched pairs gives -1. Both D_8 and c stabilize the lines $\gamma_\pm, \alpha_C$. Thus, pure lifts centralize the residual SL_2-factors. Since S_3 acts faithfully on the residual lines $\{\gamma_\pm, \alpha_C\}$, the primitive image lies in $W(D_4)$. Take the pairs $e_1 - e_2, e_1 + e_2$ and $e_3 - e_4, e_3 + e_4$, each exchanged by c. The frame and half-sums give D_4. Orient the cycles so that $c = n_1 n_2$. By [130, proof of Proposition 4.19], the quartic relative group in $W(D_4)$ is $G(4, 2, 2)$ of order 16, hence equals B for $k = -1$. Write its elements as $n_1^i n_2^j s_0^e$ with $i + j$ even and s_0 exchanging the vector blocks. Moreover, ϵ records the original permutation of the A_3 factors.

In block V_k, n_k cycles the isotropic basis e_1, e_2, f_1, f_2 with $(e_i, f_j) = \delta_{ij}$. Thus, s_0 exchanges equally based blocks. Both n_k have determinant -1 and equal spinor norm. Thus, $n_1 n_2$ has a prime-field lift $v \in \mathrm{Spin}_8$ centralizing the perpendicular factors $\mathbf{H}$. By Lang in $\mathbf{H}$, take their inner markings to be 1. Then, $F = \mathrm{Int}(v)F_q$, and $D_B = \mathrm{N}_{\mathrm{Spin}_8}(\mathbf{T})^F$ has $D_B/T \simeq B$ by Lang. Above $b = n_1^i n_2^j s_0^e$, the orthogonal image is $\mathrm{diag}(d_1, d_2)b$. Since b is prime-field and satisfies $b\, n_1 n_2 = n_1 n_2\, b$, d_k lies in the cyclic marked torus T_k of order $q^2 + 1$.

By [189, p. 165], $-I_4$ has nonsquare spinor norm on V_k of minus type. On the cyclic T_k, this sign is its quadratic character θ. Given s_0, write Δ for its rational spinor sign. Since $i + j$ is even, the signs multiply to 1. Thus, a rational spin lift gives $\theta(d_1)\theta(d_2) = \Delta^e$. Choose $b_0^2 = \Delta$ and set $\lambda_0(g) = \theta(d_1)b_0^e$. If g' has exponent e', a swap by g replaces $\theta(d_1')$ by $\theta(d_2')$, producing $\Delta^{ee'}$. The carry $e + e' - (e + e' \bmod 2) = 2ee'$ gives the same sign from b_0. Thus, λ_0 is multiplicative. The character $\theta \otimes 1$ is dual to $(-1, -1, 1, 1)$. Its image annihilates Λ and has value -1 at ω in the adjoint D_4-torus. Thus, the spin isogeny gives $\lambda_0|_T = \kappa$. Since F_r stabilizes the blocks and fixes θ, it fixes λ_0. On v, the formula gives 1.

If $k = 1$, the system perpendicular to C is D_6. The two $A_3 = D_3$ factors have disjoint triples I_1, I_2. Given $i \in I_1, j \in I_2$, the roots $\pm e_i \pm e_j$ have class $(2, 2, 0)$. Thus, in SO_{12} the active spaces are V_1, V_2. The remaining space V_0 has roots $\gamma_\pm$. Since n_k acts by -1 on the unused coordinate, it exchanges γ_+, γ_- as lines. Both lines are s_0-stable. On $V_0 = (M_2, \det)$, adjugation J is a linear isometry with $J^2 = 1$ and $\det J = -1$. Moreover, $J(gXh^{-1}) = hJ(X)g^{-1}$ $(g, h \in \mathrm{SL}_2)$. Thus, J gives the pinned block swap. If $b = n_1^i n_2^j s_0^e \in C_4 \wr S_2$, we define $\rho(b)$ by these rotations and swap on V_1, V_2 and by $J^{i+j \bmod 2}$ on V_0. Since parity is additive under swaps, ρ is a prime-field homomorphism into SO_{12}. Since $c = n_1 n_2$ is trivial on V_0, reuse v. The active torus $\mathbf{T}_{\mathrm{act}}$ has connected spin preimage $\mathbf{T} \leq \mathrm{Spin}_8$, containing $\ker(\mathrm{Spin}_8 \to \mathrm{SO}_8)$. Thus, the spin preimage of $\mathbf{T}_{\mathrm{act}}\rho(B)$ has kernel $\mathbf{T}$ over B. Lang for F gives $D_B/T \simeq B$. Moreover, D_B permutes the pinning and centralizes C.

The image of g is $(d_1, d_2, 1)\rho(b)$. For rational spinor sign sn, we put $\tau = \mathrm{sn} \circ \rho$. The rotations are conjugate, so write $\tau(n_1) = \tau(n_2) = (-1)^{k_0}$ and $\tau(s_0) = (-1)^{t_0}$ with $k_0, t_0 \in \{0, 1\}$. If $\iota^2 = -1$, take $\lambda_0(g) = \theta(d_1)(-1)^{k_0 j}\iota^{t_0 e}$. Since g has a rational spin lift, $\theta(d_1)\theta(d_2) = \tau(b)$. In a product with g', the swap introduces $\tau(b')^e$ from the torus and $(-1)^{k_0 e(i'+j')}$ from exchanging exponents. The carry in $e + e'$ contributes $(-1)^{t_0 ee'}$. Thus, their product 1 makes λ_0 multiplicative. The dual block sign gives $\lambda_0|_T = \kappa$, and $\lambda_0(v) = (-1)^{k_0}$. The block labels are F_r-stable, so λ_0 is F_r-invariant. □

Lemma 7.184. *We write $I = I_M(\psi_x)$.* Lemmas 7.182 and 7.183 *give M, L, D_B, λ_0 and v over c. Next, we write $\widetilde{M}_D$ for the adjoint normalizer with field. χ_η, indexed by $\eta \in \mathrm{Irr}(B)$, extends over its marked preimage. If $\sigma \in \mathcal{H}_\ell$ acts on μ_r by a, its diagonal pair sign is $(a/r)^f$. If $2 \mid f$, the quotient twist at U is k^e, where $(-1)^e = (a/r)$. When $k = 1$, each singleton extends to $\widetilde{M}_D$ with $\mathcal{H}_\ell$-fixed values.*

Proof. The pinned groups $U_{\gamma_\pm}$ carry a common nondegenerate trace ϑ. On C, we fix the Borel line in x. Also, the joint space E in ψ_x has dimension 1. Indeed, its regular parent on $\mathbf{H}^F = \mathrm{SL}_2(q)^3$ has distinct constituents there. On each $\gamma_\pm$-factor the quadratic parent has 2 constituents, with ϑ of multiplicity 1 in one constituent. The signs from L include all triples (d_+, d_-, d_C) with $d_+ d_- d_C = 1$. Moreover, the even-sign central kernel Z_0 is rational. By Lang on $\mathbf{T}, \mathbf{H}$, we obtain each cocycle. Since x extends to the projective group of C, the sign d_C compensates arbitrary d_+, d_-.

Thus, all 4 pairs occur. Exactly one satisfies both ϑ-conditions, and the flag line of x has dimension 1, including at $q = 3$.

The line E is invariant under the pure D_B, F_r-actions. If $g \in D_B$, its intertwiner acts on E by $\lambda_0(g)$. Since $\lambda_0|_T = \kappa$, the line argument of Proposition 7.181 gives X on I. Choose the scalar β of the prime-field operator U by $\beta^f = \lambda_0(v)^{-1}$. Then, $U^f = X(v^{-1})$ and the action on D_B follows from $\lambda_0^{F_r} = \lambda_0$. Let ρ_η afford η on B, and let $\pi_B : I \to B$ be the quotient. On $X_\eta = X \otimes (\rho_\eta \circ \pi_B)$, the relative field scalar α satisfies $\alpha^f = \eta(c^{-1})/\eta(1)$, since $c \in \mathrm{Z}(B)$. As in the preceding proposition, $\alpha, \beta \in \mu_{4f_2}$ exist and are $\mathcal{H}_\ell$-fixed. Let $\widehat{X}_\eta$ be the extension to $I\langle U\rangle$. By [95, Problem 5.6], $\mathrm{Ind}_{I\langle U\rangle}^{M\langle U\rangle} \widehat{X}_\eta$ restricts to $\mathrm{Ind}_I^M X_\eta$, as desired.

Let μ be the sum of the 3 oriented residual coroots. The element $t = \mu(\sqrt{a})$ returns ϑ^σ to ϑ on $U_{\gamma_+}, U_{\gamma_-}$. Moreover, it fixes E on the flag factor and satisfies $tg = gt$ for the pure representatives g. The characters η have values in $\mathbb{Q}(i)$. Specifically, for $C_4 \wr S_2$, the two-colour model suffices. Given $D_8 * C_4$, the dihedral models are real and c acts by μ_4-scalars. If f is even, $t \in L$ and $F_r(t) = tz^e$. Thus, $\chi_\eta^\sigma = \chi_\eta$ and the twist at U is k^e, including the relative scalar of η. If f is odd and $(a/r) = -1$, the correction is the diagonal δ of Proposition 7.181. Both actions on E give equal scalars. Thus, σ acts by $\eta \mapsto \eta\epsilon$, where ϵ is -1 on plane swaps. If $(a/r) = 1$, $\chi_\eta^\sigma = \chi_\eta$. Lemma 7.182 gives the orbits under ϵ.

Given a singleton with $k = 1$, the ordered planes give $B_0 = C_4^2$. Next, on the adjoint Levi L_D, let ψ_D lie over s. Inversion via both $\gamma_\pm$-reflections gives $\mathbb{Q}(\psi_D) = \mathbb{Q}$ and $\psi_D^{F_r} = \psi_D$. Next, we let D_0 be the pure lift image over B_0. The inertia of ψ_D is $L_D D_0$, since a plane swap takes s to zs in another L_D-orbit. By primitive Weyl projection, $L_D \cap D_0 = \bar{T}$, the image of T. With the central map $\pi_D : L \to L_D$, we have $\psi_D \circ \pi_D = \psi_x$. Thus, E gives line scalars for D_0, U. In the adjoint group $t = \mu(\sqrt{a})$ is rational over $\mathbb{F}_r$, since the image of $\mu(-1) = z$ is 1. Moreover, we have that $tg = gt$ for each pure lift g, and $F_r(t) = t$. Thus, the ordered B_0-extensions and their inductions to $\widetilde{M}_D$ are $\mathcal{H}_\ell$-fixed. The rational torus $T_D \leq L_D$ surjects onto the diagonal quotient Δ_D. The derived restriction gives $\mathrm{Ind}_{B_0}^B(\eta_1 \boxtimes \eta_2)$, with η_i on plane i. If $\eta_1 \neq \eta_2$, it is irreducible and gives 12 singletons. Thus, each singleton extends over $\widetilde{M}_D$. □

Proposition 7.185. *Suppose that* $\mathrm{ord}_\ell(q) = 4$. *Given any* f, *both rational forms of* $A_3^2A_1$ *give* $\succeq_c$ *at* $\mathrm{N}_G(P)$. *Moreover, the domains have sizes* 28 *at* $k = 1$ *and* 20 *at* $k = -1$.

Proof. Let σ act on μ_r by a, and set $t = (2\rho^\vee)(\sqrt{a})$. Moreover, a pinned trace ϑ detects χ_+ in the paired restriction [208, Proposition 11.22]. The identities $(\vartheta^\sigma)^t = \vartheta$ and $(2\rho^\vee)(-1) = z$ give pair sign $(a/r)^f$. By [208, Proposition 11.22(ii)–(iii)], both members are F_r-fixed. If $2 \mid f$, we have $F_q(t) = t$ and $F_r(t) = tz^e$, where $(-1)^e = (a/r)$. On the detected constituent, the multiplicity 1 in extended induction gives twist k^e at U. For the other detector choose y with $F_q(y) = yz$. Then $(yF_r(y)^{-1}F_r)^f = z$. A scalar $\beta \in \mu_{2f_2}$ with $\beta^f = k$ is $\mathcal{H}_\ell$-fixed. Since $ty = yt$, the second twist is also k^e. On the field group, D is Alvis–Curtis duality for the F_r-stable parabolics. Given a quotient character δ, $D(\chi\delta) = D(\chi)\delta$, and $D(\chi^\sigma) = D(\chi)^\sigma$. Thus, D gives both semisimple actions on $\psi_\pm$ as in Proposition 7.147.

On the exterior $\mathrm{SL}_2(q)$, the flag label is $\nu \in \{1, \mathrm{St}\}$. Moreover, regular and semisimple anchors are $((4), 1)$ and $((1^4), \mathrm{St})$, respectively. In the parabolics we keep $\bar{s}$ rational. For exchanged A_3's choose $v \in W^\circ$ so the mark $w' = vw$ gives the signed swap for $q \bmod 4$. On the root spaces, $(w')^2 = 1$. Next, set $h = (u, w, -w, -u)$ with four distinct entries and $y = (h, \pm h, 0)$ with the sign of w'. Scale y to lie in $X_*(\mathbf{T})$. The component and inversion operations fix y after simultaneous $(1\,4)(2\,3)$ if needed. Moreover, both lift to $\mathbf{J}$ of y. In $\mathrm{SL}_4(q^2)$ the intersection is a principal torus T_1, split over $\mathbb{F}_{q^2}$. It meets the exterior factor in C. Moreover, the regular source $\widetilde{\psi}_\nu$ restricts to $\psi_1 + \psi_2$. Its only nontrivial scalar self-twist δ comes from the component. Inversion in $\mathbf{J}$ fixes ν. For the connected regular centralizer $\mathbf{C}_0$, Lang gives conjugate rational lifts. Hence, $\{\psi_1, \psi_2\}$ is stable under σ, F_r. Since $w'(y) = y$, we have $w' \in W_J$. Lang on $\mathbf{J}$ gives split coordinates on the parabolic $\mathbf{P}_y$ and its field induction.

At flag ν, branching and [9, §6] give regular coefficient $m_\lambda = \dim S^\lambda$ for $\lambda \vdash 4$. A hook of leg j has odd dimension $\binom{3}{j}$, with $0 \leq j \leq 3$. The coefficient at either anchor is 1. Moreover, for the cover pair $\chi_\lambda^\pm$ the coefficient matrix is $H_\lambda = \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$. Thus, $A + B = m_\lambda$. The diagonal δ_{diag} swaps both pairs, and $\widetilde{\psi}_\nu\delta = \widetilde{\psi}_\nu$. The anchor matrix H_λ is a permutation matrix. If $A \neq B$, H_λ gives equivalent permutation actions on $\{\psi_1, \psi_2\}$ and $\{\chi_\lambda^+, \chi_\lambda^-\}$.

If $2 \mid f$, the matrices give $\mathbb{Q}(\chi_\lambda^\pm) = \mathbb{Q}$ and $(\chi_\lambda^\pm)^{F_r} = \chi_\lambda^\pm$ on the hooks. The anchor entry 1 gives the source extension $\widehat{\psi}_i$ on radical invariants with twist k^e. Moreover, at any hook λ, some entry m per target is odd. Let Γ be the field quotient and Γ_2 its Sylow 2-subgroup. The induced target fibre over Γ_2 is $\sum_\delta m_\delta[Y_0\delta]$, with extensions $Y_0\delta$ for $\delta \in \mathrm{Irr}(\Gamma_2)$. Next, we set $m = \sum_\delta m_\delta$, which is odd, and choose b with $bm \equiv 1 \bmod |\Gamma_2|$. Define $Y_1 =$

$Y_0\left(\prod_{\delta\in\mathrm{Irr}(\Gamma_2)}\delta^{m_\delta}\right)^b$. If Y_0 changes to $Y_0\delta_0$, the product changes by δ_0^{-m}, and Y_1 changes by $\delta_0^{1-bm}=1$. Under a coefficient map σ, the formula gives Y_1^σ. A twist δ_1 of the fibre multiplies it by $\delta_1^{bm}=\delta_1$. Hence, Y_1 has the source twist over Γ_2. Put $O=O_{2'}(\Gamma)$. Over O, the real extension ψ_{o} is unique as in Lemma 7.184. Since $\Gamma=\Gamma_2\times O$, one extension has both restrictions. At U, its twist is (a/r) for $k=-1$.

If $\lambda=(2,2)$, put $h=(1,1,-1,-1)$. Both transporters still fix y. Moreover, the rational intersection J_2 is the two-block Levi. Choose $1\boxtimes 1$ on its blocks and flag ν outside. The component and inversion fix $1\boxtimes 1$. By the horizontal-strip rule, both $(4),(2,2)$ have regular coefficient 1 from $1\boxtimes 1$. Thus, both cover matrices $H_{(4)},H_{(2,2)}$ are permutation matrices. Moreover, the (4)-pair determines both on $\{\psi_1,\psi_2\}$; the $(2,2)$-pair inherits both. This accounts for 20 faithful members.

For equal hooks choose $\mathbf{P}_a$ of Proposition 7.143 with cocharacter y_a. Also, opposition and component fix y_a. Thus, we have that the Levi $\mathbf{J}_a$ is rational in both forms. In A_3 the intersection has the middle A_1, with a flag torus on C. The unitary $\mathcal{H}(B_2)$-module gives a rank-one label $\tau\in\{1,\mathrm{St}\}$ of multiplicity 1 there. Moreover, in split type $\bigwedge^j V$ gives the hooks, where V is the reflection module of S_4. On S_2, $V=1\oplus 1\oplus\mathrm{sgn}$. For each $j=0,1,2,3$ at least one of $1,\mathrm{sgn}$ has multiplicity 1. Moreover, with $\tau\boxtimes\tau$, the source restricts to $\theta_1+\theta_2$. The perpendicular reflections in $\mathbf{J}_a$ correct inversion. Thus, $\{\theta_1,\theta_2\}$ is stable under coefficients. For either ν, the equal-hook target (λ,λ,ν) and its anchor have coefficient 1. Hence, both cover matrices give sign $(a/r)^f$ on the target pair.

At even f, Lemma 7.224 gives rational extensions $\widehat{\chi}_D$ for the split principal parents χ_D. By [208, Proposition 11.22(ii)], each summand χ_i of a pair is F_r-fixed and occurs with multiplicity 1. Thus, $\widehat{\chi}_D$ gives one extension of each χ_i. Each χ_i is $\mathcal{H}_\ell$-fixed by the pair calculation, so $\widehat{\chi}_i$ has twist 1. At odd f, the 12 singleton parents are rational and F_r-fixed. Their unique real extensions $\widehat{\chi}_D$ are rational by [143, Lemma 2.1 and Corollary 2.2].

Finally, we put $\zeta^{\sigma_0}=\zeta$ for 2-power roots and $\zeta^{\sigma_0}=\zeta^{-1}$ for odd-order roots. If $2\nmid f$, σ_0 fixes the local μ_4-scalars. On the global pair, σ_0 acts as on the anchors. If σ_0 swaps the pair, the noninner diagonal δ_{diag} over $\mathbb{F}_r$ corrects it. The odd quotient Γ gives the stabilizer operations of Proposition 7.181. If $2\mid f$, that calculation has scalar k. A singleton extends to $\widetilde{M}_D$ by Lemma 7.184. Thus, we have $\succeq_c$ at $M=\mathrm{N}_G(P)$. □

Lemma 7.186. *Let $q=r^f$ be odd and $G=E_{7,\mathrm{sc}}(q)$. If $\mathrm{ord}_\ell(q)=8$, then $\ell\mid q^4+1$ and $\ell\nmid q^2+1$. The negative 4-cycle on D_4 swaps two perpendicular root lines. Moreover, F_r acts on the corresponding torus T and its root factors. Let w be the negative 4-cycle on primitive D_4, swapping 2 perpendicular root lines. Moreover, the lift v over $\mathbb{F}_r$ has order 8, with pinned action $(g_1,g_2,g_3)\mapsto(g_2,g_1,g_3)$. Next, we set $F=\mathrm{Int}(v)F_q$, $q=r^f$, $T=\mathbf{T}^F$ and $L=C_{\mathbf{G}}(\mathbf{T})^F$. Moreover, its rational derived factor is $H=\mathrm{SL}_2(q^2)\times\mathrm{SL}_2(q)$. Then, we obtain that $|T|=q^4+1$, $|T\cap H|=2$ and $|L:TH|=2$. The coset outside TH acts by nonsquare diagonals on both H-factors. The field lift satisfies $U^f=v^{-1}$.*

Proof. First, we choose the pure lift v over $\mathbb{F}_r$ as in Lemma 7.176. On $\mathbf{H}=\mathrm{SL}_2^3$, let $\pi=(1\,2)$ act on pinned factors. If $F_0=\pi F_q$, an inner correction has form $\mathrm{Int}(b)F_0$. By [133, Corollary 21.8], we choose $h\in\mathbf{H}$ with $b=hF_0(h)^{-1}$. Thus, $\mathrm{Int}(b)F_0=\mathrm{Int}(h)F_0\,\mathrm{Int}(h^{-1})$. Since v centralizes the third SL_2, it lies in Spin_{12} on its perpendicular by Lemma 7.180. Let $\bar{v}$ be the orthogonal image on 12 dimensions. On the active 8-space it cycles the 8 weight lines. Then $\bar{v}^4$ swaps $L_i,L_i^\vee$ with scalars α_i,α_i^{-1} by the pairing. Thus, $\bar{v}^8=1$ there and its polynomial is X^8-1. On the remaining 4-space M_2, $\bar{v}(B)=\pm\,\mathrm{adj}(B)$. Thus, $\bar{v}^2=1$ there. In a torus of SO_{12} containing $\bar{v}$, we choose $\zeta_8^{a_i}$ in reciprocal pair i. The nonsign exponents are $\pm1,\pm2,\pm3$. Moreover, the other exponents are $0,4$. Consequently, $\sum_i a_i\equiv 0 \bmod 2$ for every orientation. For the weight ω given by the spin half-sum, $\omega(v)^2=\prod_{i=1}^6\zeta_8^{a_i}$. Hence, $\omega(v^8)=(-1)^{\sum_i a_i}=1$. On the spin kernel $\omega(z_{\mathrm{sp}})=-1$, so $v^8=1$. The Weyl image w has order 8, so $|v|=8$.

The central intersection is $K=\{(\epsilon_i)\in\{\pm1\}^3:\epsilon_1\epsilon_2\epsilon_3=1\}$. Since F swaps the first two coordinates, $K^F=\langle(-1,-1,1)\rangle$ and $|H^1(F,K)|=2$. The negative 4-cycle gives $T=C_{q^4+1}$, with Sylow 2-subgroup K^F. By [114, corollary to Theorem 1] on $\mathbf{T}\times\mathbf{H}$, the connecting map gives $L/(TH)\simeq H^1(F,K)$. The nontrivial class has representative $(-1,1,-1)$. Next, on $\mathbf{H}$ choose $(D,F_q(D),E)$ with $F_{q^2}(D)=-D$ and $F_q(E)=-E$. The torus correction follows from [114, corollary to Theorem 1] on $\mathbf{T}$. Choose γ with $\gamma^{q^2-1}=-1$. Then, γD is rational over $\mathbb{F}_{q^2}$, with determinant γ^2 satisfying $(\gamma^2)^{(q^2-1)/2}=-1$. Similarly, a scalar β with $\beta^{q-1}=-1$ gives βE with nonsquare determinant over $\mathbb{F}_q$. Finally, $F_r^f=F_q$ acts on the marked points by $\mathrm{Int}(v^{-1})$, giving $U^f=v^{-1}$ for the marked lift. □

Proposition 7.187. *At $\mathrm{ord}_\ell(q)=8$, the faithful rational form of $A_3^2A_1$ gives 8 diagonal pairs and $\succeq_c$ at $\mathrm{N}_G(P)$ for every f.*

Proof. For the mark c of Lemma 7.182, now c^2 is a 4-cycle on each A_3. On the complementary axes a, b, $c^2 = -I$. Thus, c swaps the lines of $a+b, a-b$. The corresponding roots give $\mathrm{SL}_2(q^2)$, with nonsplit torus since c^2 acts as -1 on a root. On C the flag label is $x \in \{1, \mathrm{St}\}$. Moreover, on the primitive space, c has polynomial X^4+1. The centralizer fixing both planes is $B_{\|} = C_4$, diagonal on quarter-turns. The coset $cB_{\|}$ swaps the planes. Thus, $I_M(\psi_x)/L = M/L = \langle c \rangle = C_8$ by Proposition 7.167.

On $\mathrm{SL}_2(q^2)$ the quadratic torus character gives ξ_+, ξ_-, detected by distinct traces. By [165, Theorem B], $\mathbb{Q}(\xi_\pm) = \mathbb{Q}$ and the central sign is $(-1)^{(q^2+1)/2} = -1$. For the involution z_T of T, by Lemma 7.183, $\kappa(z_T) = -1$ for the quadratic weight κ. The two signs cancel on $T \cap H$, so $\tau_x(th) = \kappa(t)(\xi_+ \boxtimes x)(h) \qquad (t \in T,\ h \in H)$ defines a character of TH. By Lemma 7.186, the outer $L/(TH)$-coset swaps ξ_+, ξ_- and fixes the flag label x through its projective-line model. Hence, $\mathrm{Ind}_{TH}^{L} \tau_x = \psi_x$ is irreducible. As at $d = 4$, the reflections $s_{\gamma_+}, s_{\gamma_-}, s_{\alpha_C}$ give inversion on s, with scalar self-twist group $\{1\}$. Thus, $\psi_x|_H = (\xi_+ \boxtimes x) + (\xi_- \boxtimes x)$.

On ξ_+, the prime-field operator A_0 from trace induction satisfies $A_0^{2f} = 1$. On x, the counterpart B_0 satisfies $B_0^f = 1$. If $\lambda \in \mu_8$, we choose $\alpha \in \mu_{8f_2}$ with $\alpha^f = \lambda^{-1}$. Moreover, the operators at v, U are $V = \lambda A_0^f \otimes I, \qquad W = \alpha A_0 \otimes B_0$. They satisfy $VW = WV$, $V^8 = I$ and $W^f = V^{-1}$. They give the marked actions on TH, including $T \cap H$. The inertia $I_0 = TH\langle v, U \rangle$ has $I_0/(TH) \simeq C_{8f}$, so the relations give an extension $\widehat{\tau}_x$. The representation $\mathrm{Ind}_{I_0}^{M\langle U \rangle} \widehat{\tau}_x$ restricts irreducibly to M, since the two TH-constituents are distinct. Each x gives 8 labels λ. The 4 restrictions to $\langle c^2 \rangle$ correspond to the hooks of 4, since $\mathrm{ord}_\ell(q^2) = 4$ and $8\ell > 18$. The lifts are $\lambda, -\lambda$ over each hook.

If σ acts on μ_r by a, its trace correction in $\mathrm{SL}_2(q^2)$ is the coroot at $\sqrt{a}$. Write t_a for that element. It lies in $\mathrm{SL}_2(q^2)$ with $F_r(t_a) = (a/r)t_a$. Hence, the twists at A_0, A_0^f are $(a/r), (a/r)^f$, respectively. Since $\mathrm{ord}_\ell(r)/\gcd(\mathrm{ord}_\ell(r), f) = 8$, we have $8f_2 \mid \ell - 1$, so $\mathcal{H}_\ell$ fixes α, λ. Let μ be the sum of the 3 residual coroots. Given a nonsquare $b \in \mathbb{F}_q$, the diagonal $\mu(\sqrt{b})$ acts internally on $\mathrm{SL}_2(q^2)$. Also, its corrector t satisfies $F_q(t) = -t$, giving $\lambda \mapsto -\lambda$ at v. The flag model on x contributes scalar 1.

The principal torus on 4 letters has hook dimensions $\binom{3}{j}$, all odd. Since $2f_2 \mid \ell - 1$, by Proposition 7.185, we obtain the U-twist at the anchor. Given any x, the 4 hook pairs give signs $(a/r)^f$ and even-field twist (a/r) at U. If $2 \nmid f$, the diagonal over $\mathbb{F}_r$ gives the odd cyclic normalization. If $2 \mid f$, the central scalar is $k = -1$. Thus, the field preimages give $\succeq_c$ at $M = \mathrm{N}_G(P)$. □

Proposition 7.188. *Let the projective involution $\bar{s}$ have connected centralizer of type A_7 and sign ε. The central sign at z is $k = (-1)^{(q-\varepsilon)/2}$. At $d = \mathrm{ord}_\ell(q) = 4, 8$, we have that the domains of $\succeq_c$ at $\mathrm{N}_G(P)$ have sizes $28, 16$, respectively, in each rational form. If $2 \mid f$ and the coefficient power on μ_r is a, we have that the twist at U is k^e, with $(-1)^e = (a/r)$.*

Proof. Since $d = 4, 8$, $\mathrm{ord}_\ell(\varepsilon q) = d$. The primitive factors give $\ell \nmid [G^* : \mathrm{C}_{G^*}(\bar{s})]$. Since $d\ell > 18$, the type A hook formula leaves precisely $\lambda \vdash 8$ of empty d-core. At $d = 4$, their weight-2 quotients give $2 \cdot 4 + \binom{4}{2} = 14$ labels. At $d = 8$, the quotient has one box on 8 possible runners. Since $s^2 = z$, the regular restriction is $\chi_1 + \chi_2$ with sign k by [208, Proposition 11.23].

Moves by d keep runner counts n_0, n_1 on the two runners. Thus, the 2-core is empty. Proposition 7.154 gives $\mathbf{J}_j$ centralizing $e_i - e_{9-i}$ for $1 \le i \le j \le 4$, with $\mathbf{J}_0 = \mathbf{G}$. The mark is 1 if $\varepsilon = 1$, and $-w_{0,A_7}$ otherwise. It fixes $e_i - e_{9-i}$ and gives the component on $\mathbf{J}_j$. Each stage has a restriction pair $\{\psi_{j,1}, \psi_{j,2}\}$. If $\varepsilon = -1$, the rim domino gives a 2-quotient edge of multiplicity 1. If $\varepsilon = 1$, the 2 boxes have unique order $\lambda_2 \supset \lambda_1 \supset \lambda_0$. Each step has multiplicity 1. Thus, every λ reaches the bottom pair for (8). For the edge matrix H_j, the common swap gives $H_j S = S H_j$ for the swap matrix S. Both column sums are 1, so $H_j \in \{I, S\}$. The bottom Levi $\mathbf{J}_{\mathrm{bot}}$ is untwisted. Thus, the chain is F_r-stable, with maps $R_{\mathbf{J}_j}^{\mathbf{J}_{j-1}}$ and ${}^*R_{\mathbf{J}_j}^{\mathbf{J}_{j-1}}$ on field extensions. At anchor (8), Proposition 7.185 gives both actions with k. Since $2f_2 \mid \ell - 1$, the alternative detector has scalar $\beta \in \mu_{2f_2}$ with $\beta^f = k$. Thus, the pair sign is $(a/r)^f$, with twist k^e at U.

Locally mark the eight letters by $w = \varepsilon w_0$, where w_0 has cycle type 4^2 or 8. Specifically, for opposite letters i, i', the vectors $b_i = e_i - e_{i'}$ form an orthogonal frame in V_d. A perpendicular root is constant on i, i'. No root $e_u - e_v$ qualifies. Moreover, the half-roots have 2 positive and 2 negative pairs. They give 3 perpendicular lines γ_j. On each, $\gamma_j(s) = -1$. Moreover, the frame lattice $Q_0 = \sum_i \mathbb{Z} b_i$ has $\kappa|_{Q_0} = 1$ and $\kappa = -1$ on $h + Q_0$, where $h = \frac{1}{2}\sum_i b_i$. The letter permutations give $W(B_4)$ on the frame. Put $b = (-I)\prod_i (i\, i')$. Then, $b(b_i) = b_i$, $b(\gamma_j) = -\gamma_j$ and $b(s) = zs$ in the residual $\mathbf{L}$. Thus, the regular source restricts to $\psi + \psi^\delta$. The relative stabilizer lies in $\mathrm{C}_{W(B_4)}(w)$. Specifically, for the negative 2-cycles at 4 or the negative 4-cycle at 8, this is $C_4 \wr S_2$ or C_8, respectively. Pure lifts attain them.

If $d = 4$, each factor $\mathrm{SL}_2(q)$ has 2 quadratic constituents. The signs from L are all triples with $\epsilon_1\epsilon_2\epsilon_3 = 1$. Thus, there are 2 orbits of size 4. If $d = 8$, the factors are $\mathrm{SL}_2(q^2)$ and $\mathrm{SL}_2(q)$. The diagonal of Lemma 7.186 acts by $(i, j) \mapsto (-i, -j)$ on signs, giving 2 orbits of size 2. Moreover, the multiplicity-free restriction to H gives one orbit per source. Choose ϑ_j on the pinned group U_{γ_j}, with the $\mathbb{F}_{q^2}$-trace on the exchanged pair. The ϑ_j-conditions select one tensor with multiplicity 1 in ψ. The pure permutations fix its line E, so they fix ψ. The other source ψ^δ lacks the selected tensor, with δ a noninner diagonal.

If $d = 4$, the separating line γ_0 is the exterior line of Lemma 7.183. u_i swaps the other lines, while u_1u_2 fixes them. Thus, the section gives λ_0 on pure lifts g, which satisfy $\vartheta_0^g = \vartheta_0$. At $d = 8$, the lift v is that of Lemma 7.186. Extend κ to $T\langle v\rangle$ by $\lambda_0(v) = 1$. This is valid since $\kappa^v = \kappa$, $v^8 = 1$ and the Weyl image has order 8. Let D_B denote the pure preimage at either d. Given $g \in D_B$, the intertwiner has scalar $\lambda_0(g)$ on E. Thus, ψ extends to LD_B as in Lemma 7.184. The field scalar β satisfies $\beta^f = \lambda_0(v)^{-1}$. Next, for the relative η take $\alpha^f = \eta(c^{-1})/\eta(1)$. Both α, β can be chosen in μ_{df_2}, fixed by $\mathcal{H}_\ell$. The relative values lie in $\mathbb{Q}(i)$ at 4 or $\mathbb{Q}(\zeta_8)$ at 8. By Clifford induction, we obtain 14 or 8 labels per source, in bijection with d-quotients.

The sum μ of oriented coroots satisfies $\mu(-1) = z$. If $2 \nmid f$, the nonsquare diagonal over $\mathbb{F}_r$ swaps the determinant orbits. At 4 its three positions are nonsquare. At 8 it is square over $\mathbb{F}_{q^2}$ and nonsquare over $\mathbb{F}_q$. Moreover, in the regular overgroup, $\delta g = g\delta$ for pure lifts g, and $\delta U = U\delta$. The coefficient correction $\mu(\sqrt{a})$ from Lemma 7.184 gives permutation sign $(a/r)^f$. If $2 \nmid f$, the odd normalization applies.

If $2 \mid f$, the corrector $t = \mu(\sqrt{a})$ lies in L, and $F_r(t) = tz^e$. Thus, the twist is k^e. Given a nonsquare $b \in \mathbb{F}_q$, put $y = \mu(\sqrt{b})$. The lift $V = yF_r(y)^{-1}U$ fixes the translated trace ϑ_j^y. Its multiplier is rational since $F_q(y) = yz$. Since $vy = yv$, multiplication of the field translates gives $V^f = zv^{-1}$. Multiply the line scalar at V by $\beta_1 \in \mu_{2f_2}$ with $\beta_1^f = k$. If $u = yF_r(y)^{-1}$, $tu = ut$. Thus, the twist at V is also k^e. Moreover, the global twists at U, V are both k^e. On the regular scalar torus, extend k as in Proposition 7.181. □

Lemma 7.189. *Let $G = E_{7,\mathrm{sc}}(q)$, $q = r^f$ odd, $S = G/\mathrm{Z}(G)$, and $D = E_{7,\mathrm{ad}}(q)$. If $2 \nmid \ell$ and $\ell \mid |G|$ with $\ell \neq r$, we write $d = \mathrm{ord}_\ell(q)$. The connected involution has split roots A_1D_6, with lifts s, zs from* Lemma 7.135. *Next, on D, we choose s, and put $\phi = F_r$. By* Zhang [208, Proposition 11.9, Steps 1–2]*, s, zs are geometrically distinct in $\mathbf{D}^*$, and the series $\mathcal{E}(D, s), \mathcal{E}(D, zs)$ have equal restrictions to S. The nonscalar orders are $d = 4, 5, 8, 10$. Each eligible S-constituent has an $\mathcal{H}_\ell$-fixed extension to $D \rtimes \langle\phi\rangle$.*

Proof. By Lemma 7.135, the Jordan index $J(s)$ has cyclotomic factors at $3, 6, 12, 7, 14, 9, 18$. If $2 \nmid \ell$, $\ell \mid \Phi_j(q)$ if and only if $j = d\ell^a$ for $a \geq 0$. Among divisors of the E_7 degrees, only $d = 4, 5, 8, 10$ survive outside $1, 2$. Indices $d\ell^a$ with $a > 0$ are absent from $J(s)$. Thus, the label 1 survives. Each other label λ survives exactly when $\ell \nmid \lambda(1)$. For even d, Lemma 7.135 gives invariant extensions of χ in both Harish-Chandra series. At $d = 5$, the toral $\mathbf{J}_5$ in D_6 has unipotent label 1. Thus, in D_6, the ordinary cuspidal Levi $\mathbf{L}_0$ is split toral by [156, Lemma 8.2]. Moreover, the A_1-labels $1, \mathrm{St}$ are principal. Thus, $\widehat{\chi}$ is rational at $d = 5$. □

Proposition 7.190. *At $d = 4, 8$ the connected involution gives $\succeq_c$ at $\mathrm{N}_G(P)$, for $P \in \mathrm{Syl}_\ell(G)$. The adjoint field extensions $\widehat{\chi}, \widehat{\psi}$ are $\mathcal{H}_\ell$-fixed. The domains have sizes $76, 32$, respectively.*

Proof. First, we put E_7 in $(e_7+e_8)^\perp$, with exterior root $b = e_7-e_8$. The perpendicular subsystem is D_6 on $e_1, \ldots, e_6$. In the adjoint algebraic group $\mathbf{D}$, set $\mathbf{K} = \langle \mathbf{T}, U_\alpha : \alpha \in \{\pm b\} \cup D_6\rangle$. Moreover, with the full $\mathbf{T}$, we have that the dual is $\mathbf{K}^* \simeq \mathrm{C}_{\mathbf{D}^*}(s)^\circ$ for $s = b^\vee(-1)$. Since $s \in \mathrm{Z}(\mathbf{K}^*)$, it defines a quadratic $\kappa \in \mathrm{Irr}(\mathbf{K}^F)$. On a rational maximal torus T_1, $\kappa|_{T_1}$ is dual to s. This determines $\kappa|_L$ on a common Levi $L \geq T_1$ [58]. Also, the pinned ϕ fixes κ. Thus, $\widehat{\kappa}(a\phi^j) = \kappa(a)$ gives a rational extension.

Put $V = \langle e_1, \ldots, e_4\rangle$. If $d = 4$, w has two negative 2-cycles on V, fixing the other coordinates. If $d = 8$, $w|_V$ is a negative 4-cycle and $w(e_6) = -e_6$, giving $w \in W(D_6)$. Choose its Tits lift n over $\mathbb{F}_r$ in $\mathbf{K}$. With $F = \mathrm{Int}(n)F_q$, the primitive torus $\mathbf{A}$ has $|\mathbf{A}^F| = \Phi_4(q)^2$ or $\Phi_8(q)$. The ambient degrees give $P_D \in \mathrm{Syl}_\ell(D)$ in $\mathbf{A}^F$, including $(\ell, d) = (5, 4)$. Next, we set $L = C_{\mathbf{D}}(\mathbf{A})^F$ and $M = N_D(\mathbf{A})$. On S, Propositions 7.172 and 7.167 give $\mathrm{N}_S(\mathbf{A}) = \mathrm{N}_S(P_D)$ at $4, 8$, respectively. Choose a rational maximal torus $T_0 \leq L$. By [133, Theorem 21.7] on its covering torus, $T_0 \to D/S$ is onto. T_0 centralizes P_D, so Sylow conjugacy gives $M = \mathrm{N}_D(P_D)$. The quotient $G \to S$ gives $P \simeq P_D$.

Moreover, the perpendicular roots are $e_5 - e_6, e_5 + e_6, b$. Every half-root has nonzero V-projection. Hence, $C_{\mathbf{D}}(\mathbf{A}) = C_{\mathbf{K}}(\mathbf{A}) = \mathbf{L}$. The sources are $\kappa|_L\lambda$, with λ given by $1, \mathrm{St}$ on factors over q, q, q at 4 or q^2, q at 8. Because

$\mathrm{Z}(\mathbf{L})$ is connected, L maps onto $(\mathbf{L}/\mathrm{Z}(\mathbf{L}))^F$ by [114, corollary to Theorem 1]. Since $\mathbf{A} \le \mathrm{Z}(\mathbf{L})$, these exhaust Sylow-torus labels in $\mathbf{K}^*$. Central isotypy gives λ across the isogenies [156, §§4.3–4.4].

Let $M_0 = N_{\mathbf{K}}(\mathbf{A})^F$. Moreover, the common Weyl kernel W_L fixes s, and a transporter of its L^*-class has a representative in $W(\mathrm{C}(s)^\circ) = W(\mathbf{K})$. By [114, corollary to Theorem 1] on $\mathbf{L}$, choose rational representatives in M_0. Thus, M_0 stabilizes the entire L-series. Geometrically the action on V is $W(B_4)$, since an odd sign on V is compensated on e_5 or e_6. Also, we obtain the rational quotient $C_{W(B_4)}(w|_V)$. At 4 it is $C_4 \wr S_2$. On the two D_6-lines the sign is $(-1)^{j_1+j_2}$ for rotation exponents j_1, j_2. Moreover, at 8 it is C_8. The lines form one F-orbit, with fixed flag label λ_1.

At $d = 4$, the inertia $B_\lambda = C_4 \wr S_2$ if the D_6-labels are equal. Otherwise, $B_\lambda = A_{\mathrm{ev}} \rtimes \langle \pi \rangle$, with even-sum rotations A_{ev} and swap π. Let A_0 be the rotation subgroup. If $\theta^\pi = \theta$ for $\theta \in \mathrm{Irr}(A_0)$, its extension has value 1 at π. Otherwise, $\mathrm{Ind}_{A_0}^{B_\lambda} \theta$ is irreducible. Thus, $\mathbb{Q}(\eta) \subseteq \mathbb{Q}(i)$ for $\eta \in \mathrm{Irr}(B_\lambda)$. At 8, $B_\lambda = C_8$ and $\mathbb{Q}(\eta) \subseteq \mathbb{Q}(\zeta_8)$. The full wreath product has $2 \cdot 4 + \binom{4}{2} = 14$ labels. Since $(\pi - 1)A_{\mathrm{ev}} = \langle (2,2) \rangle$, we have $|\mathrm{Irr}(A_{\mathrm{ev}})^\pi| = 4$. The unequal orbit gives $2 \cdot 4 + (8-4)/2 = 10$. Moreover, there are 2 exterior choices, giving $2(2 \cdot 14 + 10) = 76$. At 8, we have $4 \cdot 8 = 32$.

Choose a Lang transporter in $\mathbf{K}$ for the mark. Then U acts by F_r with $U^f = n^{-1}$ on the transported $\widehat{\kappa}$. Given a factor over $\mathbb{F}_{q_i}$, the flag module $\mathbb{Q}[\mathbb{P}^1(\mathbb{F}_{q_i})]$ has constants $V_{i,1}$ and augmentation $V_{i,\mathrm{St}}$. If $\lambda = (\lambda_i)$, choose $\bigotimes_i V_{i,\lambda_i}$. The flag action permutes the V_{i,λ_i} and gives $U^f = n^{-1}$ on their tensor product. Since U fixes B_λ and $w^{-1} \in \mathrm{Z}(B_\lambda)$, we extend the inflated η by scalar α with $\alpha^f = \eta(w^{-1})/\eta(1)$. We can take $\alpha \in \mu_{df_2}$ since d is a power of 2 and $df_2 \mid \ell - 1$. All three factors are $\mathcal{H}_\ell$-fixed. Let $I_\lambda = I_M(\kappa|_L \lambda)$ and $\widehat{\tau}_{\lambda,\eta}$ be the tensor model on $I_\lambda \langle U \rangle$. Then, $\mathrm{Ind}_{I_\lambda \langle U \rangle}^{M\langle U \rangle} \widehat{\tau}_{\lambda,\eta}$ restricts to $\chi_{\lambda,\eta} = \mathrm{Ind}_{I_\lambda}^{M} \tau_{\lambda,\eta}$, irreducible by Clifford theory. Since $P_D \le L$, the indices are prime to ℓ.

The dual $A_1 D_6$ gives eligible global labels (λ, η) on its Sylow torus $\mathbf{A}^*$, with the computed line action. Moreover, $s \mapsto zs$ on L-series is the twist by $1 \ne \epsilon \in \mathrm{Irr}(D/S)$. By [208, Proposition 11.9, Steps 1–2], the restrictions $\chi|_S$ are distinct irreducibles. By [95, Lemma 5.2 and Problem 5.3], $\chi_{\lambda,\eta}|_{M \cap S}$ are distinct irreducibles as well. Since T_0 maps onto D/S, $M\langle U \rangle$ covers the outer stabilizer. The map $G \to S$ gives central sign $+1$. The local argument and Lemma 7.189 give $\mu_h^{\mathrm{glob}} = \mu_h^{\mathrm{loc}} = 1$. □

Proposition 7.191. *At $d = 5, 10$, the connected involution has 20 labels with $\succeq_c$ at $\mathrm{N}_G(P)$ for each f. With $d = 1, 2$ and* Proposition 7.190*, this covers odd $\ell \ne r$.*

Proof. Let $c = (1\,2\,3\,4\,5)$ and $V = \{x \in \langle e_1, \dots, e_5 \rangle : \sum_i x_i = 0\}$. The marks are $w = c, -c$ for $d = 5, 10$, with $-1 \in W(A_1 D_6)$. The torus $\mathbf{A}$ has $|\mathbf{A}^F| = \Phi_d(q)$ and contains $P_D \in \mathrm{Syl}_\ell(D)$, since only degree 10 contributes. By Propositions 7.167 and 7.190, $M = \mathrm{N}_D(\mathbf{A}) = \mathrm{N}_D(P_D)$. Set $L = C_{\mathbf{D}}(\mathbf{A})^F$. A root orthogonal to V has equal first 5 coefficients, so no D_6-root is orthogonal to V. Therefore, the residual roots in $C(s)$ are $\pm b$. A signed permutation stabilizing V has sign ϵ on all 5 coordinates. The sign on e_6 is then ϵ by parity. The relative centralizer of c is $\langle c \rangle \times \langle -1 \rangle = C_5 \times C_2$. It fixes the exterior labels $1, \mathrm{St}$. Thus, there are $2 \cdot 10 = 20$ eligible pairs.

In $\mathbf{D}$, the half-roots orthogonal to V have equal signs on $e_1, \dots, e_5$, the opposite sign on e_6, and opposite signs at e_7, e_8. Together with $\pm b$ they give A_2. In this dual Levi, $s = b^\vee(-1)$ has eigenvalues $(-1, -1, 1)$. The connected centre of $\mathbf{L}$ gives a quotient $\pi : L \to H = (\mathbf{L}/\mathrm{Z}(\mathbf{L}))^F$. If $\chi \in \mathcal{E}(H, s)$, the characters $\chi \circ \pi$ exhaust $\mathcal{E}(L, s)$. The dual A_1 gives 2 unipotent labels. If $d = 5$, $H = \mathrm{PGL}_3(q)$. At $d = 10$, inversion gives the pinned graph γ up to inner action. Hence, $H = \mathrm{PGU}_3(q)$.

Choose the pinned $B_H = T_H U_H$ with maximally split T_H. In dual coordinates $s = \mathrm{diag}(-1, 1, -1)$ satisfies $\gamma(s) = F_r(s) = s$. Also, its quadratic torus character is θ. Write $\pi_T : B_H \to T_H$ for the projection. On $\mathcal{V} = \mathrm{Ind}_{B_H}^{H}(\theta \circ \pi_T)$, the graph and field action is $(\alpha u)(h) = u(\alpha^{-1}(h))$. By the Mackey formula in [95, Problem 5.6], $\dim \mathrm{End}_H(\mathcal{V}) = |(W_H)_\theta| = 2$. In split type $W_H = \mathfrak{S}_3$, and in unitary type $W_H = C_2$ fixes θ. Thus, $\mathcal{V} = V_1 \oplus V_2$ with two constituents in $\mathcal{E}(H, s)$. Their degrees are $q^2 + \varepsilon q + 1$ and $q(q^2 + \varepsilon q + 1)$, with $\varepsilon = 1, -1$ respectively. The distinct degrees make the V_i rational and $\gamma V_i = F_r V_i = V_i$. Thus, each affords an extension ρ_i to $\mathcal{A}_H = \langle \mathrm{Inn}(H), \gamma, F_r \rangle$, with $\gamma = F_q$ on unitary H. The characteristic quotient H gives the conjugation map $\pi_{\mathrm{fld}} : M\langle U \rangle \to \mathcal{A}_H$. Moreover, the rank-two models are $\rho_i \circ \pi_{\mathrm{fld}}$ before the twist.

The roots in V give $\Phi(A_4)$, with pointwise stabilizer $W(A_2)$ of V in $W(E_7)$. Moreover, permutations and inversion give $\mathrm{Aut}(\Phi(A_4))$ on V. Rational descent gives $M/L = C_5 \rtimes C_2$, as in the dual centralizer. If $\eta \in \mathrm{Irr}(M/L)$, the marked field U fixes M/L, with $U^f L = w^{-1}$ of order d. Put $f_{[d]} = \prod_{p \mid d} p^{v_p(f)}$. Next, choose $\alpha \in \mu_{df_{[d]}}$ such that $\alpha^f = \eta(w^{-1})$. Given $p \mid d$, $v_p(\mathrm{ord}_\ell(r)) = v_p(d) + v_p(f)$, so $df_{[d]} \mid \ell - 1$. Thus, $\mathcal{H}_\ell$ fixes α and the μ_{10}-values of η, since ℓ is odd. The tensor models give 20 fixed extensions, with 2 sources and 10 relative labels. By [208, Proposition 11.9, Step 1], s, zs are geometrically distinct in $\mathbf{D}^*$. On the L-series, $s \mapsto zs$ is the twist by $1 \ne \epsilon \in \mathrm{Irr}(D/S)$. By [95, Lemma 5.2

and Problem 5.3], restriction to $M \cap S$ gives distinct irreducibles. Compose with $G \to S$ as at 4 for $\succeq_c$ at $\mathrm{N}_G(P)$. Lemma 7.189 gives the global extensions $\widehat{\chi}$. Its index cut and Proposition 7.136 exhaust d. □

Lemma 7.192. *Let $G = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ odd. By* Zhang [208, proof of Proposition 11.19, Step 1]*, the involution $\bar{s}$ has connected type E_6T_1. For $v = 2\omega_7^\vee$, we have $z = v(-1)$ and $n_0^2 = z$ for the longest Tits representative n_0. Next, in the split E_6-form, we put $\phi = F_r$, $k = (-1)^{(q-1)/2}$ and $j^2 = k$. The choice $-j$ swaps the pair labels. If $2 \nmid \ell$ and $\ell \neq r$ and eligible split τ, the pair $\{\chi_+, \chi_-\}$ extends over $G \rtimes \langle\phi\rangle$. If $\sigma \in \mathcal{H}_\ell$, the permutation sign on extensions is $\frac{\sigma(j\sqrt{q})}{j\sqrt{q}}$. The diagonal swaps χ_+, χ_-. If $2 \nmid f$, we have that a diagonal d acts on both extensions. If $2 \mid f$, we have that both extensions are $\mathcal{H}_\ell$-fixed.*

Proof. Let $\mathbf{K}$ be the split Levi of type E_6T_1, with parabolic KR and radical R. The element $\bar{s} \in \mathrm{Z}(\mathbf{K}^*)$ gives the quadratic character κ of K. Next, we write $\pi : K \to K_{\mathrm{ad}}$ for the map onto adjoint E_6, and set $\lambda = \kappa(\tau \circ \pi)$. In a regular inclusion $G \leq \widetilde{G}$, we have that the lift s has $\mathrm{C}_{\widetilde{G}^*}(s) = \widetilde{K}^*$. Thus, its Jordan parent is $\mathrm{Ind}_{KR}^{\widetilde{G}} \widetilde{\lambda}$, irreducible. The Levi $\widetilde{K}$ surjects onto $\widetilde{G}/G$. Central twists and unipotent isotypy give $\widetilde{\lambda}|_K = \lambda$. By [95, Problem 5.2], the parent restricts to $V = \mathrm{Ind}_{KR}^G \lambda$. By [208, proof of Proposition 11.19, Step 2], its character is $\chi_+ + \chi_-$. Moreover, the diagonal swaps χ_+, χ_-. Thus, $\chi_+(1) = \chi_-(1)$.

On K_{ad}, n_0 induces the inner-graph γ with $\gamma^2 = 1$ and $\gamma\phi = \phi\gamma$. Next, on $\langle K_{\mathrm{ad}}, \gamma, \phi\rangle$, we choose $\widehat{\tau}$ fixed by $\mathcal{H}_\ell$ as in [156, Theorems 7.4 and 7.5, proof of Theorem 8.5]. Opposition fixes κ. Moreover, its extension has values j at n_0 and 1 at ϕ. The operators $X(n_0), X(\phi)$ give an extension of λ, since $X(n_0)X(\phi) = X(\phi)X(n_0)$ and $X(n_0)^2 = kI$. On $\langle K, \phi\rangle$, the coefficient transform is equivalent to X. Also, on the n_0-coset the scalar is $\sigma(j)/j$.

By [208, proof of Proposition 4.16], the invariant form on X gives the coset-sum form on V. Let A have kernel supported on $(KR)n_0(KR)$, with value $X(n_0)$ at n_0, or $X(n_0)^{-1}$ for the inverse-kernel convention. Since $KR \cap n_0(KR)n_0^{-1} = K$, the kernel covariance holds. Since $n_0^2 = z$, unitarity gives $A = A^*$. Its blocks satisfy $A_{xx} = 0$, with $|R|$ unitary blocks per row. Hence, $\mathrm{tr}\, A = 0$, $\mathrm{tr}(A^*A) = |R| \dim V$. The eigenvalues $a_\pm$ are real and $a_+ + a_- = 0$. The root counts $63, 36$ give $|R| = q^{27}$, so $a_\pm = \pm q^{27/2}$. The field action satisfies $\phi A = A\phi$. Thus, its eigenspaces $V_\pm$ give $\widehat{\chi}_\pm$. An equivalence on $\langle K, \phi\rangle$ carries A^σ to $(\sigma(j)/j)A$, by the kernel value at n_0. Thus, the permutation sign is $\sigma(j\sqrt{q})/(j\sqrt{q})$ on the extensions.

If $2 \nmid f$, the one-torus inclusion gives $d = [v(t), b]$ with $t^{r-1} = b^{r-1} = -1$. Then, d is rational and noninner over $\mathbb{F}_q$. It centralizes K', with $d\phi = \phi d$ and $d(KR)d^{-1} = KR$. The commutator $[d, n_0]$ is $v(t^2)$ up to inversion, of κ-value -1. Indeed, inversion takes s to zs. The quotient twist ϵ has $\epsilon(d) = -1$. Therefore, $dAd^{-1} = -A$ and d swaps the extensions. If $2 \mid f$, $j, \sqrt{q} \in \mathbb{Q}$, so $\widehat{\chi}_\pm^\sigma = \widehat{\chi}_\pm$. □

Proposition 7.193. *At $d = \mathrm{ord}_\ell(q) \in \{4, 8, 12\}$, we have that each eligible split E_6T_1-pair gives $\succeq_c$ at $\mathrm{N}_G(P)$. The relative labels are $\mathrm{Irr}(C_{W(F_4)}(b))$ for the primitive mark b. The extensions $\widehat{\chi}_\pm$ are from* Lemma 7.192*.*

Proof. First, let $V_4 = \langle e_1, \ldots, e_4\rangle$ in the signed $D_5 \leq E_6$. No E_6-root is perpendicular to V_4. Moreover, a D_5-root needs 2 coordinates, and half-spin roots meet V_4. Orient the three perpendicular roots β_i by $\langle\beta_i, v\rangle > 0$. Since $\omega_7^\vee$ is minuscule, $\langle\beta_i, v\rangle = 2$ and orthogonal projection gives $v = \sum_{i=1}^3 \beta_i^\vee$. Next, for the primitive D_4, put $\mathcal{D} = Q(D_4)^\vee/Q(D_4)$. The isometries act on $\mathcal{D} \setminus \{0\}$ as on the lines $\langle\beta_i\rangle$. The based lifts fix v, and give the stabilizer of V_4 in $W(E_6)$. Moreover, in $W(E_7)$ the kernel on V_4 is generated by $s_{\beta_1}, s_{\beta_2}, s_{\beta_3}$. Let b have two negative 2-cycles at 4, a negative 4-cycle at 8, and type F_4 Coxeter at 12. The Coxeter exponents are $1, 5, 7, 11$. The residual permutation has order $1, 2$ or 3, so it has no primitive d-eigenvalue. Take the Tits lift n in the derived E_6 over $\mathbb{F}_r$ and put $F = \mathrm{Int}(n)F_q$. The inner twist is split on E_6. Moreover, the primitive torus $\mathbf{A}$ contains $P \in \mathrm{Syl}_\ell(G)$ by the degrees, including $(\ell, d) = (5, 4)$. Propositions 7.167 and 7.172 give $M = \mathrm{N}_G(\mathbf{A}) = \mathrm{N}_G(P)$.

Put $\mathbf{T} = C_{\mathbf{K}}(\mathbf{A})$, a torus since no E_6-root is perpendicular to V_4. With $T = \mathbf{T}^F$, set $C = N_{\mathbf{K}}(\mathbf{A})^F$ and $L = C_{\mathbf{G}}(\mathbf{A})^F$. Lang on $\mathbf{T}$ and the connected ambient Levi gives $M = LC$, $C \cap L = T$, $M/L \simeq C/T = B = C_{W(F_4)}(b)$. Thus, B is the relative group in E_6 on $\mathbf{A}^*$. The labels are $\eta \in \mathrm{Irr}(B)$ by [156, §4.4]. The index $J(\bar{s})$ of Lemma 7.155 is $\Phi_2^3\Phi_6\Phi_{10}\Phi_{18}\Phi_7\Phi_{14}$ up to sign. It excludes no $d \in \{4, 8, 12\}$. Since $P \leq T$, we have $\ell \nmid |B|$.

Set $R_\pm = (\prod_{i=1}^3 U_{\pm\beta_i})^F \leq L$. Their orders are q^3 even when F permutes roots. The group C normalizes both, so $H = R_+C \leq M$ and $H \cap L = R_+T$. Also $C^{n_0} = C$, $R_+^{n_0} = R_-$, and $n_0 n = n n_0$. Hence, $H \cap H^{n_0} = C$. Write $\vartheta(uc) = \kappa(c)$ for $u \in R_+$ and $c \in C$. With $V_0 = \mathrm{Ind}_H^M \vartheta$, [95, Problem 5.2] gives $V_0|_L = \mathrm{Ind}_{R_+T}^L(\vartheta|_{R_+T})$, the principal induction at $\theta = \kappa|_T$. Moreover, its endomorphism algebra has dimension 2. Indeed, W_L fixes the line of v through

$\{1, \omega_0\}$, where $\omega_0 = \prod_i s_{\beta_i}$. Also, both fix $\bar{s}$. The element ω_0 is rational and takes s to zs. The toral parent has Weyl stabilizer 1, and $V_0|_L$ has constituents ξ_+, ξ_-, each of multiplicity 1.

Next, we extend $\kappa|_C$ to $C\langle n_0\rangle$ by value j at n_0. The operator A_L with kernel on Hn_0H and value j at n_0 satisfies $A_L = A_L^*$, and $(A_L)_{xx} = 0$ by Lemma 7.192. The eigenspaces $W_\pm$ extend $\xi_\pm$ to M. The rational diagonal $\delta = [v(t), b_0]$ with $t^{q-1} = b_0^{q-1} = -1$ centralizes C and normalizes H. The regular quotient calculation gives $\delta A_L \delta^{-1} = -A_L$, so $\dim W_+ = \dim W_-$. Each row has $[H : C] = q^3$ unitary entries, giving eigenvalues $\pm q^{3/2}$ by the trace norm. Since $q^{27/2}/q^{3/2} = q^{12}$, the signs coincide.

With the marked U we have $U^f = n^{-1}$, with n rational in derived E_6 and $\kappa(n) = 1$. Moreover, the source extension has $\widehat{\vartheta}(U) = 1$. On the field module $UA_L = A_L U$, giving eigenspace extensions $\widehat{\xi}_\pm$. On M, we obtain $\widehat{\xi}_\pm|_M(\eta \circ \pi_M)$ by [95, Corollary 6.17] with $\pi_M : M \to B$. The relative η's are $\mathcal{H}_\ell$-fixed by [156, Theorem 4.1 and Lemma 4.3]. If $r \mid d$, take split E_6 and mark b in characteristic prime to d. By Lang, the quotient remains $C_{W(F_4)}(b)$, so the cited bound holds for η. Since U fixes B, we choose $\alpha \in \mu_{df_{[d]}}$ with $\alpha^f = \eta(b^{-1})/\eta(1)$. By Proposition 7.191, $df_{[d]} \mid \ell - 1$. Hence, $\alpha^\sigma = \alpha$ and the extension is fixed. If $2 \nmid f$, take the diagonal δ_r of Lemma 7.192, with Lang transporter in $\mathbf{K}'$. Then $\delta_r c = c\delta_r$ for $c \in C$, and $\delta_r U = U\delta_r$. If $2 \mid f$, $\xi_\pm^\sigma = \xi_\pm$, so no diagonal coset returns $\xi_\pm$. On the regular scalar torus, choose ν with $\nu(z) = k$ on both sides by Proposition 7.181. Thus, $\succeq_c$ holds at $M = \mathrm{N}_G(P)$. □

Lemma 7.194. *Let $G = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ odd. For the other rational E_6T_1-form of $\bar{s}$, the derived type is ${}^2E_6$. Moreover, we obtain the central sign $k = (-1)^{(q+1)/2}$. If $d = \mathrm{ord}_\ell(q) \in \{4, 8, 12\}$, we find $\mathrm{ord}_\ell(-q) = d$. The index $J(\bar{s})$ of* Lemma 7.155 *is prime to ℓ. At $d = 4, 8, 12$ the residual Levi has two characters $\psi_\pm$ over $\bar{s}$, fixed by $M = \mathrm{N}_G(P)$. Each gives labels $\eta \in \mathrm{Irr}(B)$, with B from* Proposition 7.193. *If $\sigma \in \mathcal{H}_\ell$ acts on μ_r by a, the permutation sign is $(a/r)^f$ at each η. If $2 \mid f$, the twist at U is (a/r).*

Proof. The twisted mark is $-b$ for split mark b. The primitive space V_4 has the same perpendicular roots β_i. Indeed, $\Phi_d(-X) = \Phi_d(X)$ for $d = 4, 8, 12$. The extra eigenvalues have orders dividing 2 or 6. Thus, $\mathbf{A}, L, M$ are taken with the twisted Frobenius F. If $v = 2\omega_7^\vee$, we have that the stabilizer of $\langle v\rangle$ in W_L is generated by simultaneous opposition ω_0. Moreover, it swaps s, zs and satisfies $\omega_0(-b) = (-b)\omega_0$. In $\widetilde{L}^*$, the centralizer of s is a torus. The parent $\widetilde{\psi}$ has only the quadratic self-twist ϵ. Thus, $\widetilde{\psi}|_L = \psi_+ + \psi_-$, with the summands swapped by δ. The sum is rational since $\bar{s}^2 = 1$ and the unipotent label is 1.

In derived E_6, Lemma 7.176 gives pure lifts on β_i. Moreover, their kernel is $\mathbf{A}$, and they centralize the diagonal SL_2-group Δ. Put $c = -b$ on V_4, with the pinned action on β_i. Since $-I \in W(D_4)$, Lang on $\mathbf{A}$ gives the lift n_c over $\mathbb{F}_r$. Also, the extra lift $n_\Delta \in \Delta$ is longest on residual SL_2-factors. By Lang in Δ, we obtain the model $F = \mathrm{Int}(n_c)F_q$ and the corresponding Levi series. With D_0 the rational pure lifts, we have $M = LD_0, \quad L \cap D_0 = \mathbf{A}^F, \quad D_0/\mathbf{A}^F = B = C_{W(F_4)}(b)$. Moreover, the old torus weight is $\kappa = \kappa_6|_T$, where κ_6 is linear on K. It is trivial on $(\mathbf{K}')^F$, with $\mathbf{K}'$ the simply connected derived group. Since $\mathbf{A} \le \mathbf{K}'$, $\kappa|_{\mathbf{A}^F} = 1$. The Lang change in Δ centralizes $\mathbf{A}$. Thus, the pure lifts fix κ and the transported L-series.

Set $R_+ = (\prod_i U_{\beta_i})^F$. An orbit $\mathcal{O}$ gives coordinate $u_\mathcal{O} \in \mathbb{F}_{q^{|\mathcal{O}|}}$. Next, we define $\lambda(u) = \zeta_r^{\sum_\mathcal{O} \mathrm{Tr}_{\mathbb{F}_{q^{|\mathcal{O}|}}/\mathbb{F}_r}(u_\mathcal{O})}$. The regular induction detects $\widetilde{\psi}$ with multiplicity 1. Thus, λ detects one $\psi_\pm$ with multiplicity 1, and $\lambda^g = \lambda$ for $g \in D_0$ by the orbit trace. Put $\mu = \sum_i \beta_i^\vee$. Then $\mu(\sqrt{t})$ scales all three coordinates by $t \in \mathbb{F}_q^\times$. The value $\mu(-1) = z$ in either pinning. If $t \in (\mathbb{F}_q^\times)^2$, $\mu(\sqrt{t}) \in L$. Otherwise, $F_q(\mu(\sqrt{t})) = \mu(\sqrt{t})z$. Then, $b_q^q = -b_q$ gives the rational diagonal $[\mu(\sqrt{t}), b_q]$, selecting the other $\psi_\pm$. Thus, both characters are M-fixed because D_0 fixes both trace conditions.

On $J = R_+\langle z\rangle D_0$, we extend λ by $\lambda_{\mathrm{ext}}|_{D_0} = 1$ and $\lambda_{\mathrm{ext}}(z) = k$. Indeed, $z \notin D_0$ and the weight on $\mathbf{A}^F$ is trivial. We have $M = LJ$ and $L \cap J = R_+\mathbf{A}^F\langle z\rangle$. Thus, $(\mathrm{Ind}_J^M \lambda_{\mathrm{ext}})|_L = \mathrm{Ind}_{L\cap J}^L(\lambda_{\mathrm{ext}}|_{L\cap J})$, the central summand of $\Gamma_\lambda = \mathrm{Ind}_{R_+}^L \lambda$ at the chosen weight. Select ψ_+ by this trace. Multiplicity 1 gives the unique constituent $\widehat{\psi}_+$ over ψ_+. Moreover, the translated trace gives $\widehat{\psi}_-$. By [95, Corollary 6.17], we obtain precisely the labels $\widehat{\psi}_\pm(\eta \circ \pi)$ for $\eta \in \mathrm{Irr}(B)$ and $\pi : M \to B$. Globally the toral centralizer in twisted ${}^2E_6$ gives the labels $\mathrm{Irr}(B)$ by [156, §4.4], since $C(b) = C(-b)$ on V_4.

Let U act as F_r, with $U^f = n_c^{-1}$. The Lang transport gives lift $l_0 U$ for some $l_0 \in L$. On the first line set $U = 1$ as operator, since $n_c \in D_0$. Given η, choose $\alpha \in \mu_{df_{[d]}}$ with $\alpha^f = \eta(c^{-1})/\eta(1)$. The split conductor bounds give $\eta^\sigma = \eta$, $\alpha^\sigma = \alpha$. If $2 \mid f$, we choose nonsquare $t \in \mathbb{F}_q$ and put $y = \mu(\sqrt{t})$. The field lift $U' = yF_r(y)^{-1}U$ fixes the trace λ^y. Moreover, the rational multiplier $u = yF_r(y)^{-1}$ gives $(U')^f = zn_c^{-1}$ by telescoping. On the second line, $\beta \in \mu_{2f_2}$ with $\beta^f = k$. For exponent a on μ_r, the correction $t_a = \mu(\sqrt{a})$ is rational over $\mathbb{F}_q$ and centralizes D_0, y.

Since $F_r(t_a) = t_a z^e$, $(-1)^e = (a/r)$, both line scalars acquire $k^e = (a/r)$ at their field lifts. On M the twist is trivial. Multiplicity 1 gives those twists on $\widehat{\psi}_\pm$.

If $2 \nmid f$, we choose nonsquare $t \in \mathbb{F}_r$. The traces satisfy $\lambda^{F_r} = \lambda$ and $(\lambda^y)^{F_r} = \lambda^y$. The diagonal $\delta = [\mu(\sqrt{t}), b_0]$ with $b_0^r = -b_0$ is rational, centralizes D_0 and satisfies $\delta U = U\delta$. Moreover, it swaps the classes and $\delta^2 = [\mu(t), b_0^2]$, an inner factor times an $\mathbb{F}_r$-scalar. The corrector for a is prime-field inner if $(a/r) = 1$, or δ times inner if $(a/r) = -1$. Thus, the extension permutation has sign $(a/r)^f$. □

Lemma 7.195. *At $d = 4, 8$, global pairs match $\psi_\pm$ in* Lemma 7.194. *If $d = 4$, the rational cuspidal source ρ_{cusp} is from* Proposition 7.157. *At $d = 12$ the exceptions are the cubic labels $\tau, \bar{\tau}$.*

Proof. For the 25 principal labels, Proposition 7.156 takes parabolic Q independent of d. Its cuspidal sources $\rho_\pm$ give disjoint induced supports $S_\pm$. Moreover, each pair $\{\chi_+, \chi_-\}$ meets each $S_\pm$ at Borel multiplicity m_χ. The Bruhat intertwining lines $\langle T_w \rangle$ are F_r-stable. The generator T_w has nonzero scalar on fixed χ_w of multiplicity 1, so $F_r(T_w) = T_w$. On $V_\chi \otimes E_\chi$ the normalized field action gives $F_r|_{E_\chi} = 1$. Thus, $\mu_{\mathrm{tar}}(U) = \mu_{\mathrm{src}}(U)$ for every m_χ. The normalization needs $2f_2 \mid \ell - 1$, which holds for $d = 4, 8, 12$. Consequently, the global sign is $(a/r)^f$ with twist (a/r) at U when $2 \mid f$.

In ${}^2E_6$, the nonprincipal noncuspidal degrees are D and $q^9 D$, with $D = q^4\Phi_1(q)^3\Phi_3(q)^2\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)$. The unitary staircase 321 has hooks $5, 3, 3, 1, 1, 1$. By the proof of [61, Lemma 4.19], $D = Id_{321}/(1 + q^9)$, with index I and unitary hook degree d_{321}. The ${}^2A_5$-factors $q^i - (-1)^i$ cancel for $2 \le i \le 6$. Hence, $\ell \mid D$ and $\ell \mid q^9 D$ at $d = 4, 8, 12$. The cubic degree D_{cub} is from Lemma 7.155 at $x = -q$. The factors $\Phi_4(x), \Phi_8(x)$ exclude them at $d = 4, 8$. At $d = 12$ those degrees are ℓ', and $\ell \equiv 1 \bmod 3$, so $\tau^\sigma = \tau$ and $\bar{\tau}^\sigma = \bar{\tau}$ for $\sigma \in \mathcal{H}_p$. The rational cuspidal degree is $D_{\mathrm{rat}} = q^7(q-1)^4\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{18}(q)/6$. If $d = 4$, no index $4\ell^a$ occurs, so this degree is ℓ'. The factors at $d = 8, 12$ exclude it there.

If $d = 4$, Proposition 7.157 gives matrices A_s, A_c for $A_2(q)A_1(q^3)\Phi_3$. Their column sums are ± 1, each with an odd entry. Thus, A_s, A_c transfer the semisimple permutation via ρ_{cusp} to the cuspidal pair. If $2 \mid f$, the relative mark c has order 3 modulo the Levi. Since $(3, f_2) = 1$, we choose $c_0 \in \langle c \rangle$ with $c_0^{f_2} = c$. The bimodule action of C_{f_2} transfers twists along odd entries of A_s, A_c. The characters of C_{f_2} are $\mathcal{H}_p$-fixed since $f_2 \mid \ell - 1$. The odd quotient gives $\widetilde{\chi}_{\mathrm{odd}}$, the unique real extension. If $k = -1$, the Whittaker twist at U is (a/r) with the $2f_2$-root normalization. If $2 \nmid f$, the prime-field diagonal gives h_0 on the odd extension torsor Y, with $|Y^{h_0}| = 1$. Hence, ρ_{cusp} gives both actions at $d = 4$. □

Proposition 7.196. *Let $G = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ and put $d = \mathrm{ord}_\ell(q)$. For cubic τ and every f, its G-pair has the semisimple sign $(a/r)^f$ and its field twists. Thus, the E_6T_1 involution gives $\succeq_c$ at $\mathrm{N}_G(P)$ in both forms at $d = 4, 8, 12$, with* Proposition 7.157 *at 4.*

Proof. At $d = 12$, we have $\ell \mid q^4 - q^2 + 1$ and $\ell \ge 13$. Moreover, $\ell \nmid q^2 - 1$, and P is cyclic because only degree 12 contributes. In ${}^2E_6(q)$, the unipotent block b_K has positive defect and contains $1, \mathrm{St}, \tau, \bar{\tau}$ by [48, Appendix H.6]. These have ℓ'-degree. Moreover, the central torus has ℓ'-order. Isogeny kernels have primes $2, 3$. All four lie in toral $\mathcal{E}_d$. Thus, the good-prime criterion gives b_K across the central changes.

In $G \le \widetilde{G}$, the dual centralizer of s is $\widetilde{K}^*$. Twist b_K by the central character κ_s. Jordan reduction gives one $\widetilde{G}$-block $\widetilde{b}$ by [18]. For the four labels ν, the parents satisfy $R_{\widetilde{K}}^{\widetilde{G}}(\kappa_s\nu) = \pm\widetilde{\chi}_\nu$. The covered blocks b form a $\widetilde{G}$-orbit. So do constituents of $\widetilde{\chi}_\nu|_G$. Thus, every parent meets every b. The block of cubic χ contains references ρ_1, ρ_2 over $1, \mathrm{St}$. Lemma 7.195 gives their permutation and twist μ_σ.

Set $t = f_2 f_3$, $m = f/t$, and $A = G \rtimes \langle \phi^m \rangle$, so $A/G = C_t$. The formula for $\mathrm{ord}_\ell(r^f) = 12$ gives 2-part $4f_2$ and 3-part $3f_3$ in $\mathrm{ord}_\ell(r)$. Hence, $t \mid \ell - 1$, $P \in \mathrm{Syl}_\ell(A)$, and $\mathcal{H}_p$ fixes $\mathrm{Irr}(C_t)$. Every base is ϕ-fixed. By [208, proofs of Lemma 11.14 and Proposition 11.18], we choose global extensions X with $\mathbb{Q}(X) \subseteq \mathbb{Q}(\zeta_{24rt})$. Thus, their restrictions to A are ℓ-rational. Other extensions differ by $\eta \in \mathrm{Irr}(C_t)$, also fixed by ℓ-inertia. This includes the references ρ_i.

Choose an extension X of χ to A, in block b'. Moreover, the block b of χ is A-stable. Over the splitting residue field k_0, the twists $\mathrm{Irr}(C_t)$ act on blocks above b. Given an orbit $\mathcal{O}$, the idempotent $e_\mathcal{O} = \sum_{c \in \mathcal{O}} e_c$ belongs to $k_0 G$, since every nontrivial C_t-degree has a nontrivial twist. Then $e_\mathcal{O}$ is central and lies under b, so $e_\mathcal{O} = b$. The action is transitive even at $t = 1$. Thus, reference extensions occur in b'. Given $\sigma \in \mathcal{H}_p$, we let δ^{j_σ} correct its base action, with $j_\sigma = 0$ if $2 \mid f$. If $2 \nmid f$, choose δ over $\mathbb{F}_r$. It normalizes A and fixes C_t. Next, we put $h(Y) = Y^{\sigma\delta^{j_\sigma}}\mu_\sigma^{-1}$, with μ_σ

restricted to C_t. Thus, h fixes $\widehat{\rho}_1, \widehat{\rho}_2$ even after quotient twists, since $\mathcal{H}_p$ fixes $\mathrm{Irr}(C_t)$. Thus, $h(b') = b'$. The reference degrees are prime to ℓ, so b' has nontrivial cyclic defect P.

Let $\mathcal{T}$ be the Brauer tree of b'. Moreover, on the residue field k_0, σ acts by a Frobenius power, and ℓ-inertia is trivial on k_0. The action is semilinear. At edge e, we have $h(S_e) \simeq S_{h(e)}$ and $h(\mathcal{P}_e) \simeq \mathcal{P}_{h(e)}$. Moreover, for projective $\mathcal{P}$, the layer $\mathrm{rad}^i(\mathcal{P})/\mathrm{rad}^{i+1}(\mathcal{P})$ maps to $\mathrm{rad}^i(h\mathcal{P})/\mathrm{rad}^{i+1}(h\mathcal{P})$. The layers give succ_v along uniserial branches at v, including the exceptional vertex [51]. Hence, $h(\mathrm{succ}_v(e)) = \mathrm{succ}_{h(v)}(h(e))$ on incident edges.

The references are ℓ-rational. At exceptional multiplicity 1, the vertices of $\widehat{\rho}_i$ are distinct. Otherwise, the I-orbits on $\mathrm{Irr}(P) \setminus \{1\}$ index exceptional characters, with inertial quotient I acting freely and $|I| = e \mid \ell - 1$. Unit powers from ℓ-inertia act transitively at order ℓ^j. A fixed orbit has $\ell^{j-1}(\ell-1) = e$, so $j = 1$ and $e = \ell - 1$. There is at most one ℓ-rational exceptional character [51]. Thus, $v_1 \neq v_2$ for the references.

Since $h(v_i) = v_i$, it fixes the path from v_1 to v_2. Choose an edge e_0 on this path. At either endpoint v, h fixes $\mathrm{succ}_v^j(e_0)$ for every j. By distance from e_0, all vertices are fixed. For inertia ι, $h(Y^\iota) = h(Y)^\iota$ since $\mu_\sigma^\iota = \mu_\sigma$. For nonexceptional X this gives $h(X) = X$. For exceptional X, uniqueness of the ℓ-rational character gives $h(X) = X$. Thus, $X^{\sigma\delta^{j\sigma}} = X\mu_\sigma$, and restriction to G gives $\chi^{\sigma\delta^{j\sigma}} = \chi$.

If $2 \mid f$, the extension X_m over C_m has $\mathbb{Q}(X_m) = \mathbb{Q}(\chi)$ by [208, proof of Proposition 11.19, Step 4]. Its twist is trivial since m is odd. If $2 \nmid f$, we choose σ_0 fixing $\mu_{2^\infty}, \mu_{3^\infty}$ and inverting μ_n for odd $(n, 3) = 1$. It fixes $\tau, \bar{s}$, so permutes the pair. Choose j_0 with $\chi^{\sigma_0\delta^{j_0}} = \chi$. Since $(m, 6) = 1$, $h_0 = \sigma_0\delta^{j_0}$ inverts $\mathrm{Irr}(C_m)$. On the extension torsor Y_m, $|Y_m^{h_0}| = 1$ gives X_m for mixed operations stabilizing χ. Moreover, in $\widetilde{G}$, $\delta^2 = xc_0$, with $x \in G$ and scalar c_0. The decomposition $C_f = C_t \times C_m$ gives X_f with $X_f|_A = X$ and $X_f|_{G\rtimes C_m} = X_m$. On the regular scalar subgroup, choose ν with $\nu(z) = k$ as locally. Thus, $\succeq_c$ holds at $d = 12$. At $d = 4, 8$ Lemmas 7.194 and 7.195 give $\succeq_c$. For the split form Proposition 7.193 gives $\succeq_c$. □

Fix split $\bar{s}$ of Lemma 7.192. Next, we put $W_6 = W(E_6)$, and let $\mathbf{K}$ be the split Levi of type E_6T_1. Next, we write $v = 2\omega_7^\vee$ and $j^2 = k = (-1)^{(q-1)/2}$, with $q = r^f$. Up to a power of 2, the Jordan index is $J = (q^5+1)(q^9+1)\frac{q^{14}-1}{q-1} = (\Phi_2^3\Phi_6\Phi_{10}\Phi_{18}\Phi_7\Phi_{14})(q)$. By Digne–Michel [58, Proposition 11.5.6], the unipotent label λ gives $\chi(1)_\ell = (J\lambda(1))_\ell$. The degrees of E_7 are $2, 6, 8, 10, 12, 14, 18$. For odd $\ell \neq r$, $\ell \mid \Phi_m(q)$ if and only if $m = d\ell^a$, with $d = \mathrm{ord}_\ell(q)$ and $a \geq 0$. The possible orders are $1, 3, 4, 5, 8, 9, 12$, excluding $(d, \ell) = (1, 7)$. In this domain $\ell \nmid J$, the index of Lemma 7.155.

Proposition 7.197. *Suppose that $\ell \mid q - 1$ and $\ell \neq 7$, and let $Q \in \mathrm{Syl}_\ell(W_6)$. The local labels are $\mathrm{Irr}_{\ell'}(N_{W_6}(Q))$ with the two component values $j, -j$. Specifically, they give $\succeq_c$ at $\mathrm{N}_G(P)$. For $\ell > 7$ there are 25 diagonal pairs.*

Proof. On the split torus T, we let θ be the quadratic character dual to $\bar{s}$. Moreover, its stabilizer is $W_6 \times \langle -I\rangle$. Indeed, W_6 fixes s and inversion takes s to zs. On the simple Tits lifts n_i for E_6, we extend θ by $\widehat{\theta}(n_i) = 1$. The coroot squares give $\theta(n_i^2) = 1$, so the braid relations suffice. Also $n_0n_i = n_in_0$ and $n_0^2 = z$, giving values $j, -j$ at n_0. By [166, Proposition 6.1 and Theorem 6.2], each eligible $\eta \in \mathrm{Irr}(W_6)$ gives its pair. Since $[W(E_7) : W_\theta] = 28$, the degree condition is $\ell \nmid \eta(1)$, including $\ell = 3$.

If $\ell > 7$, $Q = 1$, $P = T_\ell$ and $\mathrm{N}_G(P) = \mathrm{N}_G(T)$ by Proposition 7.121. All 25 labels of W_6 survive. If $\ell = 3, 5$, then $Q \in \mathrm{Syl}_\ell(W(E_7))$ since $[W(E_7) : W_6] = 56$. By [97, Theorem 3.8], choose a complement $\widehat{Q}$ over Q in the subsystem Tits group. F_r fixes $\widehat{Q}$. The split order formula gives $P = T_\ell\widehat{Q}$. By Proposition 7.121, $M = \mathrm{N}_G(P)$ lies in the torus normalizer. Put $T_* = T_\ell C_{T_{\ell'}}(Q)$. Given $t \in T$, we have $t \in M$ exactly when $t_{\ell'} \in C_{T_{\ell'}}(Q)$. Given $w \in N_W(Q)$, choose a Tits lift $\dot{w}$. Conjugacy of complements over Q gives t_0 in the sign torus with $\widehat{Q}^{t_0\dot{w}} = \widehat{Q}$. Thus, $M \cap T = T_*$ and $M/T_* = N_W(Q)$, with $W = W(E_7)$.

A Q-fixed $\xi \in \mathrm{Irr}(T)$ is determined by $\xi|_{T_*}$. Indeed, $[T_{\ell'}, Q] \leq \ker\xi$ and $T_{\ell'} = [T_{\ell'}, Q] \times C_{T_{\ell'}}(Q)$. Two Q-fixed weights in one W-orbit are $N_W(Q)$-conjugate by Sylow conjugacy in W_ξ. Thus, $I_M(\theta|_{T_*})/T_* = N_{W_6}(Q) \times \langle -I\rangle$. Since $\widehat{Q}^{n_0} = \widehat{Q}$, the restriction has $\widehat{\theta}_*(n_0) = j$. At 3, the three-plane calculation gives rational $\mathrm{Irr}_{3'}(N_{W_6}(Q))$. At 5 the Sylow Q is cyclic. By [142, §3], for $\sigma \in \mathcal{H}_5$ we have $|\mathrm{Irr}_{5'}(W_6)^\sigma| = |\mathrm{Irr}_{5'}(N_{W_6}(Q))^\sigma|$. Since $\mathrm{Irr}(W_6)$ is rational, $\mathcal{H}_5$ fixes every local label. At 3, by Lemma 7.155, $|\mathrm{Irr}_{3'}(W_6)| = |\mathrm{Irr}_{3'}(N_{W_6}(Q))|$. The Weyl labels satisfy $\eta^{F_r} = \eta^\delta = \eta$. Thus, any Weyl bijection b gives $(\eta, \pm j) \mapsto (b(\eta), \pm j)$.

On the inertia extension $\widehat{\rho}$, set $\widehat{\rho}(\phi) = I$. Its induction to $M\langle\phi\rangle$ has sign $\sigma(j)/j$. Since $q \equiv 1 \bmod \ell$, $\sqrt{q}$ is fixed by [156, Lemma 2.3(d)]. By Lemma 7.192, $\sigma(j\sqrt{q})/(j\sqrt{q}) = \sigma(j)/j$. If $2 \nmid f$, the diagonal δ_r of that lemma centralizes derived E_6 and so $\widehat{Q}$. Given $w \in N_{W_6}(Q)$, correct $\dot{w}$ in the subsystem Tits group. Then δ_r changes only the n_0-value, since $\theta([\delta_r, n_0]) = -1$. On both sides $\widehat{\chi}_\pm^{\delta_r} = \widehat{\chi}_\mp$. If $2 \mid f$, j, $\sqrt{q} \in \mathbb{Q}$. The diagonal swaps the $\mathcal{H}_p$-fixed bases. Thus, no element of $\delta\mathcal{H}_p$ stabilizes either base. Choose scalar ν with $\nu(z) = k$ on both sides by Proposition 7.193. □

Lemma 7.198. *At $d = 3, 9$, the split labels are $\eta \in \mathrm{Irr}(C_{W_6}(b))$ with component values $j, -j$. The local extensions match $\widehat{\chi}_\pm$ globally.*

Proof. At 3, put $b = b_1b_2b_3$ for the A_2-rotations $b_i \in W_6$. At 9, Proposition 7.166 gives $b = -c$ for the ambient Coxeter element c. The primitive space is the E_6 reflection space V_6. By the index test, $V_6^\perp$ is rootless. Choose Tits lift n in derived E_6 with $F = \mathrm{Int}(n)F_q$. Thus, the connected E_6 form stays split. The centralizer $C_{\mathbf{G}}(\mathbf{A}) = \mathbf{T}$ is a maximal torus. At $d = 3$, the contributing degrees $6, 12, 18$ have quotients by 3 prime to $\ell \geq 7$. If $d = 9$, the sole degree is 18 and $\ell \geq 19$. Hence, $P = (\mathbf{A}^F)_\ell$ is Sylow. Uniqueness of the Φ_d-torus in $\mathbf{T}$ and Sylow normalizer inclusion give $M = \mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{T})$.

The fixed line $V_6^\perp$ has pointwise stabilizer W_6. Thus, $M/T = B \times \langle -I \rangle$, with $B = C_{W_6}(b)$ and $\theta = \kappa|_T$. Then, $\theta^g = \theta$ for $g \in M$. The preimage of B lies in $K = \mathbf{K}^F$. By [114, corollary to Theorem 1] on the derived torus, each $a \in B$ lifts to $\dot{a}$ in derived E_6. The extension of θ on this preimage is κ from Lemma 7.192. Moreover, the operations n_0, F_r fix κ. Choose $\widehat{\theta}(n_0) = j$ and $\widehat{\theta}(U) = 1$, since $U^f = n^{-1}$ and $\kappa(n) = 1$ on the marked derived E_6.

The dual d-Levi $\mathbf{L}_d^*$ is a torus. Thus, $\eta \in \mathrm{Irr}(B)$ all have ℓ'-degree. The conductor bound in [156, Lemma 4.3] gives $\eta^\sigma = \eta$ for $\sigma \in \mathcal{H}_\ell$. If $r = 3$, choose b on the split form in characteristic 5. The quotient is still $C_{W_6}(b)$, so the conductor bound applies. Take $n_0 = I$ on the relative factor and choose $\alpha \in \mu_{df_3}$ with $\alpha^f = \eta(b^{-1})/\eta(1)$. Indeed, $v_3(\mathrm{ord}_\ell(r)) = v_3(d) + v_3(f)$, so $df_3 \mid \ell - 1$. Thus, $\alpha^\sigma = \alpha$. The only sign is $\sigma(j)/j$. By [156, Lemma 2.3(d)], odd d gives $\sqrt{q} \in \mathbb{Q}_\ell$, so both signs agree. On v, the diagonal centralizes the derived E_6-lifts. If $2 \nmid f$, its prime-field representative δ_r satisfies $\delta_r U = U\delta_r$ and exchanges $j, -j$. If $2 \mid f$, both labels are fixed by $\mathcal{H}_p$. The mixed comparison of Proposition 7.197 has $\nu(z) = k$. □

Proposition 7.199. *If $d = 5$, the split form has 10 diagonal pairs from 2 rank-one labels and $\mathrm{Irr}(C_5)$. They give $\succeq_c$ at $\mathrm{N}_G(P)$. Together with* Proposition 7.193*, this covers all d in the split domain.*

Proof. First, take $b = (1\,2\,3\,4\,5)$ in $A_4 \leq D_5 \leq E_6$. Its permutation lift $n \in \mathrm{SL}_5$ has order 5. The primitive space is $V = \{x \in \langle e_1, \ldots, e_5 \rangle : \sum_i x_i = 0\}$. By Proposition 7.191, the perpendicular roots give A_2 in $\mathbf{G}$. In E_6 they are just $\pm\beta$, since their first 5 coordinates must agree. This excludes the D_5-roots and leaves 2 opposite half-spin roots. Moreover, n centralizes the residual root groups. Put $L = C_{\mathbf{G}}(\mathbf{A})^F$, of derived type A_2. Then, $M = \mathrm{N}_G(P) = L\langle n, n_0 \rangle$ and $M/L = C_5 \times C_2$. We have $nn_0 = n_0n$ and $n_0^2 = z$.

A residual Weyl element fixing $\bar{s}$ stabilizes $\{\beta, -\beta\}$. Moreover, its stabilizer in $W(A_2)$ is $\{1, s_\beta\}$. Both elements lie in W_6 and fix s. If $w \in W_6$ acts on V as $-b^i$, $wb^{-i}(-I) \in W(A_2)$ is pointwise on V. It fixes $\bar{s}$ and takes s to zs, impossible in $\{1, s_\beta\}$. Thus, W_6 acts relatively by C_5. Also, the unipotent A_1-labels $1, \mathrm{St}$ each give 5 choices from $\mathrm{Irr}(C_5)$. The ambient component doubles these 10 labels.

Let Q_L be the A_2 root lattice of the Levi and Q_L^{sat} its saturation in $P(E_7)$. Since Q_L is primitive in $Q(E_7)$, the index $[Q_L^{\mathrm{sat}} : Q_L]$ divides 2. Coroot pairings give an inclusion $Q_L^{\mathrm{sat}}/Q_L \leq P(A_2)/Q(A_2) \simeq C_3$. Thus, the index is 1, so $\mathrm{Z}(\mathbf{L})$ is connected. Consequently, L maps onto its adjoint quotient $H = \mathrm{PGL}_3(q)$. In dual SL_3, put $s_0 = \beta^\vee(-1)$ and $t = \bar{s}s_0^{-1}$. Given any residual root α, $\alpha(\bar{s}) = \alpha(s_0)$. Thus, $t \in \mathrm{Z}(\mathbf{L}^*)$. Let κ_t be its linear character, with $\kappa_t^2 = 1$. For the quotient map $\pi : L \to H$, the sources are $\kappa_t(\chi_i \circ \pi)$ for the two χ_i over s_0. Proposition 7.191 gives the rational Borel extensions ρ_i over $\mathcal{A}_H = \langle \mathrm{Inn}(H), \gamma, F_r \rangle$.

The operations n, n_0, F_r fix t. Next, for $\widehat{\kappa}_t$ take values $1, j, 1$ on n, n_0, U, respectively. The relations are $n^5 = 1$, $n_0^2 = z$, $U^f = n^{-1}$, with $\langle n, n_0, U \rangle$ abelian. The H-model has z in its kernel. If $\eta \in \mathrm{Irr}(C_5)$, we choose $\alpha \in \mu_{5f_5}$ with $\alpha^f = \eta(b^{-1})$. Since $5f_5 \mid \ell - 1$, $\mathcal{H}_\ell$ fixes the relative extension. The diagonal δ on v is inner on H and $\delta n = n\delta$. Its commutator with n_0 is on the central line of $\mathbf{K}$. The weight of s_0 is trivial there since $\langle \beta, v \rangle = 0$. Thus, $\kappa_t([\delta, n_0]) = -1$ as for θ in Proposition 7.197. If $2 \nmid f$, the prime-field δ_r swaps the field extensions. If $2 \mid f$, the extensions are $\mathcal{H}_\ell$-fixed. Since q has odd order modulo ℓ, both signs are $\sigma(j)/j$. Thus, $\succeq_c$ holds at $d = 5$. If $d = 1$, apply Proposition 7.197, and at $d = 3, 9$ apply Lemma 7.198. Proposition 7.193 covers $d = 4, 8, 12$. □

Lemma 7.200. *Keep $q = r^f$ odd as in* Lemma 7.194*. If $e = \mathrm{ord}_\ell(-q)$, the split index at $x = -q$ gives the orders of* Section 7.6*. After $e = 1, 4, 8, 12$, we consider $e = 3, 5, 9$, with $d = \mathrm{ord}_\ell(q) = 2e$. At $e = 3, 9$, each $\eta \in \mathrm{Irr}(C_{W_6}(b))$ from* Lemma 7.198 *gives a pair. If $e = 5$, the rank-two Levi gives 10 pairs. If $\sigma \in \mathcal{H}_\ell$ has exponent a on μ_r, the base sign is $(a/r)^f$. If $2 \mid f$, the further quotient twist at U is (a/r). If $2 \nmid f$, the diagonal and coefficients permute $\widehat{\chi}_\pm$ as their bases $\chi_\pm$.*

Proof. The negative mark is $n_0 n$, where n_0 is the longest Tits lift. At $e = 3, 9$, $-b$ has primitive $2e$-eigenvalues on V_6, the E_6 reflection space. By Lemma 7.198, $M = \mathrm{N}_G(P)$ in this mark. Since $C_{W_6}(-b) = C_{W_6}(b)$, the inertia quotient of θ is $C_{W_6}(b) \times \langle -I \rangle$. The dual d-Levi is toral, with source 1. Moreover, the labels are $\eta \in \mathrm{Irr}(C_{W_6}(b))$ with the split conductor bound. In E_6T_1 extend θ by κ. At n_0 take $j, -j$ with $j^2 = k = (-1)^{(q+1)/2}$, with relative value 1 on $-I$. We have $F_r(n) = n$ and $n_0 n = n n_0$, so n is rational in twisted derived E_6 and $\kappa(n) = 1$.

If $e = 5$, by Proposition 7.199, we obtain n of order 5. The mark $n_0 n$ has polynomial Φ_{10} on V_4 and acts by opposition on residual A_2. Only degree 10 contributes to $|G|_\ell$, with $\ell \geq 11$. Thus, $M = L\langle n, n_0 \rangle$. The relative centralizer on V_4 and pointwise $W(A_2)$ are as for b, since n_0 gives $-I$ there. The residual subsystem is A_1, with C_5 and labels $(1, \eta), (\mathrm{St}, \eta)$ for $\eta \in \mathrm{Irr}(C_5)$. Moreover, this gives 10 pairs. The adjoint quotient is $H = \mathrm{PGU}_3(q)$. Next, we write $\bar{s} = ts_0$ as in Proposition 7.199. The Borel extensions ρ_i of Proposition 7.191 have $\mathbb{Q}(\rho_i) = \mathbb{Q}$ and $\rho_1(1) \neq \rho_2(1)$. Next, compose ρ_i with the conjugation action on H. The scalar model is κ_t times the rotation label ξ, with $\xi^5 = 1$. Moreover, it takes n to ξ and n_0 to $\pm j$.

The marked U acts by F_r, with $U^f = (n_0 n)^{-1}$. The rotation scalar is $u = \eta(b^{-1})/\eta(1)$ or ξ^{-1}, in μ_e. Next, we set $f_{[e]} = \prod_{p|e} f_p$, with f_p the p-part of f. The order formula gives $e f_{[e]} \mid \ell - 1$, so a root $\gamma^f = u$ exists in $\mu_{e f_{[e]}}$ and is $\mathcal{H}_p$-fixed. If $2 \nmid f$, put $V = n_0 U$, so $V^f = z^{(f-1)/2} n^{-1}$. On both labels take scalar $k^{(f-1)/2}\gamma$ at V. Its f-th power is $k^{(f-1)/2}u$, with sign part in $\mathbb{Q}$. Thus, the coefficient sign is $\sigma(j)/j$. The regular diagonal is $\delta = [v(t), c]$ with $t^{-r-1} = -1$, $c^{r-1} = -1$ and $|c| = 2(r-1)_2$. If f is odd, $\delta \in \widetilde{G}^F$ and $\delta V = V\delta$. It centralizes n and the derived E_6. The torus commutator with n_0 exchanges the values $j, -j$. If $e = 5$, its action on H is inner and s_0 is trivial on the cocharacter v. Thus, $\kappa_t([\delta, n_0]) = -1$. The extensions swap, with scalar at V fixed. If $2 \mid f$, $k = -1$. On the j-choice take $\alpha = \gamma\beta$ with $\beta^f = j^{-1}$ in μ_{4f_2}. Since $j = \pm i$, $|\beta| = 4f_2$. The condition $2f_2 \mid \ell - 1$ gives $\chi_\pm^\sigma = \chi_\pm$ and $\gamma^\sigma = \gamma$. By Section 2, $\sigma(\alpha)/\alpha = \sigma(\beta)/\beta = (a/r)$. If $2 \nmid f$, then $(a/r) = \sigma(j)/j$ by Section 2. □

Proposition 7.201. *At $d = 6, 10, 18$, the noncuspidal pairs match* Lemma 7.200*. At $d = 10, 18$ this covers all pairs. If $d = 6$,* Proposition 7.157 *covers the rational cusp τ_0, leaving $\tau, \bar{\tau}$.*

Proof. For principal $\chi_\pm$, Proposition 7.156 gives sources $\rho_\pm$ and Bruhat lines $\langle T_w \rangle$ independently of d. From $v_2(d) = 1$, $2f_2 \mid \ell - 1$. Thus, over μ_{2f_2} that proof gives sign $(a/r)^f$ and the even-field twist (a/r). The nonprincipal source 321 is on Spin_{12}. Since $d = 2e$ with e odd, Lemma 7.153 gives sign $(a/r)^f$ and twist (a/r) at even f. The restrictions of multiplicity 1 in Proposition 7.156 give these on $\chi_\pm$. If $2 \nmid f$, $\mathbb{Q}(\chi_+ + \chi_-) = \mathbb{Q}$ gives the normalization of $\widehat{\chi}_\pm$.

The rational cusp has degree D, with $D = q^7(q-1)^4\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{18}(q)/6$. At $d = 10, 18$, $\ell \mid D$. At $d = 6$, $\ell \geq 7$ and no index $6\ell^a$ occurs, so $\ell \nmid D$. On $A_2(q)A_1(q^3)\Phi_3$, take the matrices A_s, A_c of Proposition 7.157. Each column sum is ± 1, independently of ℓ. Next, for the relative mark c of order 3, we choose $c_0^{f_2} = c$ modulo the Levi. Since $f_2 \mid \ell - 1$, the cyclic action gives $\mu_\sigma(U) = (a/r)$ by that proposition. If $2 \nmid f$, the rational sum $\chi_+^{\mathrm{c}} + \chi_-^{\mathrm{c}}$ has the odd-field normalization.

Given $\tau, \bar{\tau}$, Lemma 7.155 gives degree D_{cub} at $x = -q$. The factor $\Phi_5(x)$ excludes $e = 5$. If $e \in \{3, 9\}$, no index $e\ell^a$ occurs and $\ell > 3$, so $\ell \nmid D_{\mathrm{cub}}$. If $d = 18$, [48, Appendix H.7] gives one positive-defect block containing $1, \mathrm{St}, \tau, \bar{\tau}$. Only degree 18 contributes to $|G|_\ell$, and P is cyclic. For Proposition 7.196, we have $\ell \geq 19$, $\ell \nmid q^2 - 1$, and $2f_2, 9f_3$ as the 2- and 3-parts of $\mathrm{ord}_\ell(r)$. Thus, $t = f_2 f_3 \mid \ell - 1$. The central kernels and tori have ℓ'-order. The principal references share twist μ_σ. By [208, proofs of Lemma 11.14 and Proposition 11.18], the extensions have values in $\mathbb{Q}(\zeta_{24rt})$, so they are ℓ-rational. Now Proposition 7.196 gives $X^{\sigma\delta^{j\sigma}} = X\mu_\sigma$ on $G \rtimes C_t$. Its $C_{f/t}$ and scalar arguments give X_f. Thus, $\succeq_c$ holds at $d = 18$. □

Lemma 7.202. *Let $G = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ odd and $d = \mathrm{ord}_\ell(q) = 6$. Next, we fix a cubic τ on ${}^2E_6(q)$, with values in $\mathbb{Q}(\mu_3)$, fixed by $\mathcal{H}_\ell$. The Sylow Φ_6-torus in E_6 has dimension 6. Its column determines the permutation over τ, with multiplicity divisible by 3. Thus, on C_{f_3} we choose the full E_6T_1 Levi. For either cubic τ, the sign at σ equals the semisimple sign when $\sigma|_{\mu_3} = 1$. If $\sigma \in \mathcal{H}_\ell$ has exponent a on μ_r, the sign is $(a/r)^f$.*

Proof. First, take the primitive D_4-space V and lines β_i of Proposition 7.193. Let $c \in W(F_4)$ be Coxeter and put $b = c^4$. The exponents $1, 5, 7, 11$ give $c^6 = -I$ on V, with primitive 3-eigenvalues for b. The short simple reflections s_1, s_2 give distinct transpositions in $W(F_4)/W(D_4)$, while the long reflections give 1. Thus, c induces a 3-cycle there. Hence, b has primitive 3-eigenvalues on $\langle \beta_1, \beta_2, \beta_3 \rangle \cap v^\perp$. The mark $-b$ gives order $\Phi_6(q)^3$ on twisted E_6, and a negative torus on v. Let $\mathbf{J}$ centralize the subtorus on V. Moreover, its root system is A_1^3. In the regular dual Levi $\widetilde{\mathbf{J}}^*$, the centralizer of s is toral. Opposition ω_0 descends by Lang on $\mathbf{T}$, giving the second component. Also, on $J = \mathbf{J}^F$, the

source has restriction $\psi_+ + \psi_-$, with $R^{\mathbf{J}}_{\mathbf{T}}(\theta) = \pm(\psi_+ + \psi_-)$. The sum has field $\mathbb{Q}$. The diagonal δ swaps $\psi_\pm$ and also $\chi_\pm$.

The torus column has $m_1 = 1$ and $m_\tau = m_{\bar\tau} = 9$ in absolute value. The cubic degree $D(q)$ is $D(X) = \frac{1}{3}X^7\Phi_1(X)^4\Phi_2(X)^6\Phi_4(X)^2\Phi_8(X)\Phi_{10}(X)$. Moreover, the full-defect torus $\mathbf{T}_6$ is self-centralizing with source 1. By [27, Theorems 4.8–4.9], m_τ is the relative Weyl degree. By [128, Theorem 7.3], $m_\tau = |D(x)|$ when $\mathrm{ord}(x) = d$. At $\mathrm{ord}(x) = 6$, $|x-1| = 1$ and $|x+1| = \sqrt{3}$, with $|\Phi_4(x)| = |\Phi_8(x)| = |\Phi_{10}(x)| = 1$. Thus, $|D(x)| = 9$.

By transitivity and regular restriction, $R^{\mathbf{G}}_{\mathbf{J}}$ has matrix $H_\chi = \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$ on source rows $\psi_\pm$ and target columns $\chi_\pm$. The diagonal swaps both pairs. Thus, $A + B = \pm 1$ for the semisimple pair and $A + B = \pm 9$ for the cubic pairs, with $A \neq B$. The source sum has field $\mathbb{Q}$. The cubic sums are fixed by $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}(\mu_3))$ by [208, Proposition 11.19, Step 3]. For ordinary Lusztig induction R, $R(\psi^\sigma) = R(\psi)^\sigma$. Let P_ψ, P_χ be the permutation matrices. Then $P_\psi H_\chi = H_\chi P_\chi$. Since $A \neq B$, we have $P_\psi = P_\chi$. By Proposition 7.201, we obtain the sign $(a/r)^f$. □

Lemma 7.203. *On C_n with $n = f_3$, the cubic extensions $\widehat{\rho}_\pm$ have twist* 1 *and the permutation of* Lemma 7.202. *If $2 \nmid f$, the same choices respect the diagonal action of δ.*

Proof. Let $n > 1$ and $q_0 = r^{f/n}$. On $\mathbf{K} = E_6T_1$, the mark is $F = \mathrm{Int}(n_0)F_q$. Next, we put $F_0 = \mathrm{Int}(n_0)F_{q_0}$. Since n is odd and $n_0^2 = z$, $F_0^n = F$. Lang in $\mathbf{G}$ gives the field subgroup of order n. The saturation quotient of the E_6 root lattice has order dividing 3 by its Cartan determinant and 2 by primitivity in $Q(E_7)$. Thus, $\mathrm{Z}(\mathbf{K})$ is connected, and $K = \mathbf{K}^F$ maps onto $H = {}^2E_{6,\mathrm{ad}}(q)$. For $\pi : K \to H$, set $\lambda = \kappa(\tau \circ \pi)$, where κ is dual to $\bar s \in \mathrm{Z}(\mathbf{K}^*)$. In $G \leq \widetilde{G}$, the lifted s has centralizer $\widetilde{\mathbf{K}}^*$. Moreover, the signed induction $R^{\widetilde{G}}_{\widetilde{K}}(\widetilde{\lambda})$ is the Jordan parent. On K, $\widetilde{\lambda}|_K = \lambda$ via the common quotient. Thus, $R^G_K(\lambda) = \pm(\rho_+ + \rho_-)$ with one sign.

Lang on H gives pinned graph–field F_0 and $F = F_0^n$. Thus, $B = F_0^2$ fixes the Weyl group and $B^n = F^2$. On H, $\langle B\rangle = \langle F_0\rangle$ since n is odd. By [61, proof of Prop. 2.7], we choose w with $\langle R_w, \tau\rangle = \pm 1$. Then, B fixes the variety $X(w)$. By [208, Lemma 4.11], the $(F^2)^*$-eigenvalues on the τ-part have form $\omega_\tau q^j$, $j \in \mathrm{Z}$. The root ω_τ lies in μ_3 by [61, §2.1 and Prop. 2.7]. Since $3n \mid \ell - 1$, take $\xi \in \mu_{3n}$ with $\xi^n = \omega_\tau$. Then, $(\xi q_0^j)^n = \omega_\tau q^j$ with $\xi q_0^j \in \mathbb{Q}_\ell$. By Corollary 2.18, we obtain $\widehat{\tau}$ fixed by $\mathcal{H}_\ell$. The extended quotient is $\widehat{\pi} : K\langle F_0\rangle \to H\langle F_0\rangle$. Take $\widehat{\tau} \circ \widehat{\pi}$ and $\widehat{\kappa}(F_0) = 1$, since $F_0^n = 1$ on K.

The C_n-action of Proposition 7.157 gives extended induction $\mathcal{V}$ with coefficient covariance. At fibre multiplicity ± 1, Lemma 2.17 gives the normalized extension $\widehat{\rho}_\pm$. Thus, $\widehat{\rho}^\sigma_\pm = \widehat{\rho^\sigma_\pm}$, with quotient twist 1. If $2 \nmid f$, Lemma 7.200 gives diagonal δ, with $F_0(\delta) = \delta$ since f/n is odd. Also, it centralizes K' and normalizes the parabolic $\mathbf{Q} \geq \mathbf{T}$. The rotation model gives $\widehat{\rho}^\delta_\pm = \widehat{\rho}_\mp$, with twist 1. Moreover, the regular scalar c is invariant under Lang transport. Lifts δ, δ' over c differ by $g \in G$, and $\delta' F_0 = F_0\delta'$. Thus, the local diagonal gives $\widehat{\rho}^{\delta'}_\pm = \widehat{\rho}_\mp$. □

Lemma 7.204. *Suppose that $2 \mid f$ and put $n = f_2$. On C_n, the twists satisfy $\mu^{\mathrm{cub}}_\sigma = \mu^{\mathrm{ss}}_\sigma$ at $d = 6$.*

Proof. With Δ from Lemma 7.194, Lang on $\mathbf{A}$ gives the lift t of c over $\mathbb{F}_r$, centralizing Δ. Next, we let m be its longest representative and set $a_0 = mt^4$. On V, t^4 has primitive 3-eigenvalues and $m|_V = 1$. Moreover, on the lines β_i, t^4 induces a 3-cycle and $m(\beta_i) = -\beta_i$. Thus, the six-space polynomial is $\Phi_3^2\Phi_6$, and $a_0(v) = -v$. This gives the twisted form with Φ_3-space V. The order polynomial gives $\mathbf{A}_3$ Sylow Φ_3 in ${}^2E_6$. Moreover, its centralizer in $\mathbf{K}'$ is toral. The ambient $\mathbf{J} = C_{\mathbf{G}}(\mathbf{A}_3)$ has roots A_1^3.

Given x of order 3, $|x-1| = \sqrt{3}$ and the other factors of $D(x)$ have absolute value 1. Thus, $|D(x)| = 3$. The cubic label has full Φ_3-defect, with toral source 1. By Lemma 7.202, the absolute sums are $m_1 = 1, m_\tau = 3$. On J, the pairings $\langle \beta_i, v\rangle = 2$ give $\beta_i(v(\iota)) = -1$ for $\iota^2 = -1$. The toral lifts are swapped by m. Moreover, its restriction gives ψ_+, ψ_-. The signed matrices H_1, H_τ have column sums $\pm 1, \pm 3$, respectively. On each row of H_1 and column of H_τ, an entry is odd. Thus, H_1 distinguishes both rows under σ, F_r.

With $q_0 = r^{f/n}$, choose k with $nk \equiv 1 \bmod 3$. Let $u = t^{4k}$. Then, $ua_0 = a_0u$, $u(v) = v$ and $u^n|_V = a_0|_V$. Thus, $a_0u^{-n} \in \mathbf{J}$. By [133, Corollary 21.8] in $\mathbf{J}$, the marking $\mathrm{Int}(a_0)F_q$ is conjugate to $(\mathrm{Int}(u)F_{q_0})^n$ as in Proposition 7.157. The transported multiplier $g \in \mathbf{J}u$ has norm a_0 over n factors. Thus, $a_0F_q(g)a_0^{-1} = g$. Since $ua_0 = a_0u$, $gu^{-1} \in J$. Both u and the q_0-power fix $\bar s$ of order 2. Thus, $U_0 = \mathrm{Int}(g)F_{q_0}$ fixes $\{\psi_+, \psi_-\}$. The references $\chi_\pm$ are U_0-fixed. Since H_1 separates the rows, $\psi_\pm^{U_0} = \psi_\pm$. Also $\psi^\sigma_\pm = \psi_\pm$ since $2 \mid f$.

For the rotation of Proposition 7.157, write $\mathcal{V}$ for the extended induction. Since $2n \mid \ell - 1$, $\mathcal{H}_p$ fixes $\mathrm{Irr}(C_n)$. For odd multiplicity m_0, choose a with $am_0 \equiv 1 \bmod n$. Proposition 7.157 then gives $\mu_{\mathrm{row}}(U_0) = \mu_{\mathrm{ss}}(U_0)$ on the odd

entry of H_1. The odd entry of H_τ gives $\mu_{\rm cub}(U_0) = \mu_{\rm row}(U_0)$. The root u exists for all $n = f_2$ since $a_0|_V$ has order 3. Moreover, the McKay order is $d = 6$. □

Proposition 7.205. *Let $\bar{s}$ be the involution of connected type E_6T_1 dual to $G = E_{7,\rm sc}(q)$, with q odd. At each odd $\ell \mid |G|$ with $\ell \neq r$, both forms give $\succeq_c$ at $\mathrm{N}_G(P)$ on their ℓ'-degree members.*

Proof. If $d = 6$, Lemma 7.203 gives the cubic extensions on C_{f_3}, and Lemma 7.204 on C_{f_2}. Put $m = f/(f_2f_3)$. If $2 \mid f$, [208, proof of Proposition 11.19, Step 4] gives $\mathbb{Q}(X_m) = \mathbb{Q}(\rho_\pm)$ for the extension over C_m. Hence, $\mu_\sigma|_{C_m} = 1$. If $2 \nmid f$, the coefficient involution σ_0 of Proposition 7.196 fixes τ. Correct it by δ when necessary to fix $\rho_\pm$. On the extension torsor Y_m, the corrected involution h_0 inverts the action of $\mathrm{Irr}(C_m)$. Since m is odd, $|Y_m^{h_0}| = 1$, giving X_m as in Proposition 7.196.

The decomposition $C_f = C_{f_2} \times C_{f_3} \times C_m$ gives the quotient character η with the given adjustments. On $\widehat{\rho}$, the adjustment $X_f = \widehat{\rho}\eta$ agrees on generators of C_f. Moreover, the regular scalar character ν has $\nu(z) = k$ on both sides. This gives $\succeq_c$ on the two cubic labels at 6. Proposition 7.199 covers split $\bar{s}$. For $-q$, the degree cut leaves $d \in \{2, 4, 6, 8, 10, 12, 18\}$ except $(d, \ell) = (2, 7)$. Propositions 7.156 and 7.157 give $d = 2$. Proposition 7.196 covers $d = 4, 8, 12$. By Proposition 7.201, $d = 10, 18$ and the remaining eligible labels at $d = 6$. □

Let $G = E_{7,\rm sc}(q)$, with $q = r^f$ odd. By Zhang [208, proof of Proposition 11.23, Step 1 and (11.19)], the lifts s, zs of $\bar{s}$ have $s^2 = z$ and $zs = s^{-1}$, and n_0 swaps them. The connected centralizers of $s, \bar{s}$ are SL_8/μ_2 and SL_8/μ_4, respectively. The signs are $\varepsilon = (-1)^e$, with $e = 0, 1$ for split and unitary. By Zhang [208, proof of Proposition 11.23, Step 2], each $\lambda \vdash 8$ gives a parent $\widetilde{\chi}_\lambda$ with restriction $\chi_{\lambda,+} + \chi_{\lambda,-}$ and z-scalar $k = (-1)^{(q-\varepsilon)/2}$. If $2 \nmid \ell$ and $\ell \mid |G|$ with $\ell \neq r$, we put $m = \mathrm{ord}_\ell(\varepsilon q)$.

Lemma 7.206. *The signed orders are $m = 1, 4, 5, 7, 8$, excluding $\ell = 3$ at $m = 1$. If $m \in \{4, 5, 7, 8\}$, we have that each form gives $28, 30, 14, 16$ characters, respectively. If $m = 1$, the counts are $30, 14, 44$ for $\ell = 5, 7$ and $\ell > 7$, respectively.*

Proof. Put $X = \varepsilon q$. The Jordan index away from $2, r$ has factor $J(X) = \frac{\prod_{j\in\{2,6,8,10,12,14,18\}}(X^j-1)}{\prod_{j=2}^{8}(X^j-1)}$. The numerator degrees are even, so J's numerator is independent of ε. Precisely at $\Phi_1, \Phi_4, \Phi_5, \Phi_7, \Phi_8$ all numerator factors cancel, with multiplicities $7, 2, 1, 1, 1$. At $m = 1$, the valuation formula gives index factor 72 by Lemma 7.152. If $m \in \{4, 5, 7, 8\}$, we have that $m\ell > 18$, so higher indices $m\ell^a$ are absent. The type A hook formula requires $\lfloor 8/m \rfloor$ hooks divisible by m, so the m-core has size 8 mod m. At $m = 5, 7$, Lemma 7.152 counts $15, 7$ pairs. At $m = 4, 8$, Proposition 7.188 gives the quotient counts $14, 8$. At $m = 1$, both forms have the hook criterion $\ell \nmid \dim S^\lambda$. Thus, at $\ell = 5, 7$, Lemma 7.152 gives $15, 7$ pairs, respectively. If $\ell > 7$, all 22 pairs in [208, proof of Proposition 11.23, Step 2] are eligible. □

Lemma 7.207. *Let $m = 5, 7$, with m-cycle $b \in W(A_7) = S_8$ and lift n of order m. Next, we set $F = \mathrm{Int}(n_0^e n)F_q$. The primitive torus $\mathbf{A}$ on the moving letters contains cyclic $P \in \mathrm{Syl}_\ell(G)$ in $\mathbf{A}^F$. If $L = C_{\mathbf{G}}(\mathbf{A})^F$ and $1 \neq u \in P$, we have $C_G(u) = L$ and $M = \mathrm{N}_G(P) = L\langle n, n_0\rangle$. Thus, we find $M/L = C_m \times C_2$. At 5, the adjoint factor is $\mathrm{PGL}_3(q)$ or $\mathrm{PGU}_3(q)$ for $\varepsilon = 1, -1$. At 7, we have that $L = T^F$ for a maximal torus T. Given $\mathbf{A}^*$, $C_{G^*}(u^*) = C_{\mathbf{G}^*}(\mathbf{A}^*)^F$ at every nonidentity ℓ-element $u^* \in (\mathbf{A}^*)^F$.*

Proof. Consider E_7 on the sum-zero hyperplane of 8 coordinates. Besides differences $e_i - e_j$, its half-roots have 4 positive and 4 negative coordinates. The primitive space V has coordinate sum zero on the m moving letters of b. Moreover, the mark εb has order $d = m$ or $2m$ according to ε. Only degree $2m$ contributes to $|G|_\ell$. The primitive eigenline modulo ℓ has multiplicity 1, so P is cyclic Sylow.

At $m = 7$, the perpendicular line has vector $\gamma = (1, 1, 1, 1, 1, 1, 1, -7)$ of norm squared 56. No rational multiple has norm squared 2. A root α perpendicular to one primitive eigenspace V_ζ is perpendicular to V, so $\alpha \in \mathbb{Q}\gamma$. Thus, b is regular, with $|C_W(b)| = 14$ by [178]. Moreover, it is $\langle b, -I\rangle$. Modulo ℓ, take ζ primitive of order 7. An eigenvector is $(1, \zeta, \ldots, \zeta^6, 0)$. Moreover, each difference root $e_i - e_j$ is nonzero there. A half-root α gives a sum S of $t = 3, 4$ distinct powers of ζ. Next, replace ζ by ζ_7. $S^\sigma \neq 0$ for $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ by Φ_7. Orthogonality gives $\sum_\sigma |S^\sigma|^2 = t(7-t) = 12$. Hence, $|\mathrm{Norm}(S)| \leq (12/6)^3 = 8 < \ell$, so none reduces to 0.

At $m = 5$, the perpendicular roots are $e_i - e_j$ for $i, j \in \{6, 7, 8\}$. No half-root α is constant on the first 5 coordinates. Also, its value on the ℓ-eigenline is proportional to $\sum_{j\in I} \zeta_5^j$ with $0 < |I| < 5$. Complementation leaves 1 or 2 powers. The sums $1 + \zeta_5^j$ are units for $5 \nmid j$, and the differences $1 - \zeta_5^j$ are units away from 5. Thus, $C_{\mathbf{G}}(P[\ell])$ also has root

system A_2. The roots in V form A_4, with isometry group $S_5 \times \langle -I \rangle$. Moreover, we have $C_{\mathrm{Isom}(A_4)}(b) = \langle b, -I \rangle$, and both b and $-I$ lift to $W(E_7)$, with kernel $W(A_2)$.

In $\mathbf{G}_{\mathrm{sc}}$, each $C_{\mathbf{G}_{\mathrm{sc}}}(u)$ is connected for semisimple u. Moreover, under $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{ad}}$, each u^* of odd order lifts uniquely to u of odd order. Thus, $C_{\mathbf{G}_{\mathrm{ad}}}(u^*)$ is the image of $C_{\mathbf{G}_{\mathrm{sc}}}(u)$. The roots above give $C_G(P[\ell]) = L$ and the dual formula. If $1 \neq u \in P$, $P[\ell] \leq \langle u \rangle$ and L centralizes P. Hence, $C_G(u) = L$. The Φ_d-torus of $\mathrm{Z}(\mathbf{L})^\circ$ is $\mathbf{A}$, so M normalizes it. Conversely, P is unique in $\mathrm{Syl}_\ell(\mathbf{A}^F)$, so $\mathrm{N}_G(\mathbf{A}) = M$. By [133, Proposition 23.2], $M/L = \langle b, -I \rangle$. The lift $n \in \mathrm{SL}_m$ satisfies $n^m = 1$, $nn_0 = n_0 n$, and $n_0^2 = z$. If $m = 5$, n centralizes the residual A_2, and n_0 acts by opposition on A_2. By Proposition 7.199, the saturation index of $Q(A_2)$ divides 2 and 3, so $\mathrm{Z}(\mathbf{L})$ is connected. Thus, $L/\mathrm{Z}(\mathbf{L})^F$ is $\mathrm{PGL}_3(q)$ for $e = 0$ and $\mathrm{PGU}_3(q)$ for $e = 1$. □

Proposition 7.208. *Let $m = 5, 7$, with $k = (-1)^{(q-\varepsilon)/2}$ as above. Each core gives $2m$ characters of M, with scalars (λ, j) satisfying $\lambda^m = 1$ and $j^2 = k$. Moreover, the field fixes (λ, j), and the diagonal gives $(\lambda, -j)$. If $\sigma \in \mathcal{H}_\ell$ of exponent $a \bmod r$ on μ_r, the base sign is $(a/r)^f$. At even f, the twist evaluated at F_r is 1 if $\varepsilon = 1$ and (a/r) if $\varepsilon = -1$.*

Proof. At $m = 7$, the quadratic weight θ on $L = T^F$ corresponds to $\bar{s}$ and is M-invariant. In Lemma 7.207, $n^m = 1$ and $n_0^2 = z$. Hence, the extensions have scalars λ on n and j on n_0. At $m = 5$, the perpendicular A_2 lies in the A_7 centralizer $C_{\mathbf{G}^*}(\bar{s})^\circ$, so $\bar{s} \in \mathrm{Z}(\mathbf{L}^*)$ yields a quadratic linear character κ on L. The Levi series consists of $\kappa\rho_\nu$ for $\nu \vdash 3$, where ρ_ν is inflated from $J = L/\mathrm{Z}(\mathbf{L})^F$.

If $J = \mathrm{PGL}_3(q)$, 1_J and St_J extend rationally, with St_J given by building homology H_1. The augmentation space V of the point permutation module affords $\rho_{(2,1)}$. A pinned polarity γ gives the incidence operator A with $A^2 = qI$ on V, since each point lies on $q+1$ lines and distinct points share exactly 1. Thus, $A/\sqrt{q}$ induces γ and has order 2. The involution is F_r-fixed, and $\mathcal{H}_\ell$ fixes $\sqrt{q}$ by [156, Lemma 2.3(d)], since $\mathrm{ord}_\ell(q)$ is odd. If $J = \mathrm{PGU}_3(q) = \mathbf{J}^F$, we write $F = \gamma F_r^f$ for the pinned involution γ. On J we have $F_r^{2f} = 1$ and $F_r^f = \gamma$. Since $\mathrm{ord}_\ell(-q)$ is odd, [156, proof of Proposition 8.5] gives an $\mathcal{H}_\ell$-fixed extension to $J \rtimes \langle F_r \rangle$, including $\gamma = F_r^f$. For $\langle F_r^a \rangle$, we apply restriction to the F_r-extension.

Compose the extension of ρ_ν with conjugation on J, including the inner components of n, n_0, U. The invariant κ has $\kappa(z) = k$, so the scalars λ, j on n, n_0 extend the linear factor. The quotient $C_m \times C_2$ yields precisely these $2m$ choices. With field lift U, we have $U^f = n^{-1}n_0^{-e}$, so its scalar α satisfies $\alpha^f = \lambda^{-1}j^{-e}$. Put $f_m = (f)_m$. An f-th root of λ^{-1} lies in μ_{mf_m}; the order of r mod ℓ has m-part mf_m. Hence, $\mathcal{H}_\ell$ fixes this root.

At even f with $e = 0$, this is the whole scalar α. At even f with $e = 1$, we have $k = -1$ and the 2-part of α has order $4f_2$. The 2-part of $\mathrm{ord}_\ell(r)$ is $2f_2$, so ℓ^h fixes j. By Lemma 7.200, for exponent ℓ^h off ℓ we get $\frac{\sigma(\alpha)}{\alpha} = (-1)^{h(\ell-1)/(2f_2)} = (a/r)$. This sign is independent of j. If $2 \nmid f$, we put $V = n_0^e U$, so $V^f = n^{-1}z^{e(f-1)/2}$. The sign $k^{e(f-1)/2}$ is its own f-th root. Hence, σ acts on j by the sign $\sigma(j)/j$. Since εq has odd order modulo ℓ, we get $(r/\ell) = (-1/\ell)^e$. Reciprocity yields $(\ell/r) = (-1/\ell)^{e+(r-1)/2}$. As $k = -1$ exactly at odd $e + (r-1)/2$, the exchange sign is (a/r).

For the diagonal at odd f, we put $y = \sum_{i=0}^{m-1} b^i(2\omega_7^\vee)$, so $y(-1) = z$. Choose t, c satisfying $t^{\varepsilon r - 1} = c^{r-1} = -1$. In the regular overgroup, $[y(t), c]$ is V-fixed and belongs to $\widetilde{G}^F$. Its scalar c^2 is nonsquare in $\mathbb{F}_q$ because f is odd. This representative centralizes n, and the commutator with n_0 has weight -1 by the nontrivial quadratic twist on n_0. On J, the extended ρ_ν accounts for the action on J. Thus, the extensions with labels (λ, j) and $(\lambda, -j)$ are exchanged for every λ. □

Lemma 7.209. *Take $m = 5, 7$ and G, L, M, P from* Lemma 7.207. *Fix an m-core ν, necessarily (1) at $m = 7$. Next, we write τ for the quadratic torus weight at $m = 7$ and for $\kappa\rho_\nu$ at $m = 5$. Then, we obtain that $P \leq \mathrm{Z}(L) \cap \ker\tau$, and the actions on τ are as in* Proposition 7.208. *The $2m$ characters over ν lie in a block B_ν with maximal Brauer pair $(P, \mathrm{bl}_L(\tau))$. Moreover, its Brauer correspondent b_ν contains the $2m$ extensions of τ to M. The complements in B_ν, b_ν correspond via the C_{2m}-orbits in $\mathrm{Irr}(P) \setminus \{1\}$, equivariantly for the field and $\mathcal{H}_\ell$.*

Proof. Since P is central Sylow in L, $L = P \times L_{\ell'}$ by [97, Theorem 3.8]. The block $\mathrm{bl}_L(\tau)$ has exactly one character with P in its kernel, and it is M-stable since τ extends to M. Put $d = \mathrm{ord}_\ell(q)$. The Levi $\mathbf{L}$ centralizes the Sylow d-torus, and its Φ_d-torus is central in $\mathbf{L}$. Moreover, the central $\bar{s}$ and the toral or A_2 source give d-cuspidality. The character τ belongs to an ℓ'-semisimple series of central ℓ-defect, with $\mathrm{Z}(L)_\ell = P$ and $C_{\mathbf{G}}(P) = \mathbf{L}$ by Lemma 7.207. Thus, [106, Theorem A and the discussion preceding Lemma 3.9] puts $R_L^G(\tau)$ in B_ν with pair $(P, \mathrm{bl}_L(\tau))$. Since P

is Sylow and $\mathrm{bl}_L(\tau)$ is M-stable, the pair $(P, \mathrm{bl}_L(\tau))$ is maximal with inertial index $|M:L| = 2m$. By Brauer's first main theorem, b_ν is the block of M over this pair. Every character above τ has full defect and lies in b_ν.

In $\widetilde{G}$, the connected centralizer has type A_7. If $m = 7$, its torus column gives $S^\lambda(7,1)$. At $m = 5$, transitivity and the torus expansion of ρ_ν give $[s_\lambda](p_5 s_\nu)$. By [182, Theorem 7.17.1], we obtain λ from ν by an m-hook H with sign $(-1)^{\mathrm{leg}(H)}$. The m-quotient has weight 1, so the hook is unique. If $\varepsilon = -1$, Ennola may multiply the coefficient by -1. The regular Levi character $\widetilde{\tau}$ restricts to τ, since it is linear at 7 and comes from the central $\bar{s}$ and the unipotent character of $J = L/\mathrm{Z}(\mathbf{L})^F$ at 5. On restriction to G we have $\big(R_{\widetilde{L}}^{\widetilde{G}}(\widetilde{\tau})\big)|_G = R_L^G(\tau)$. In [208, Proposition 11.23, Step 2], the parents $\widetilde{\chi}_\lambda$ restrict to disjoint pairs, so $2m$ constituents occur in B_ν.

If $1 \neq t \in \mathrm{Irr}(P)$, inflate t through $L \to P$. Its dual $t^* \in \mathrm{Z}(\mathbf{L}^*)$ has ℓ-power order and $C_{\mathbf{G}^*}(t^*) = \mathbf{L}^*$ by Lemma 7.207. The parts of $\bar{s}t^*$ have coprime orders, and $\bar{s}$ is central in $\mathbf{L}^*$. Hence, $C_{\mathbf{G}^*}(\bar{s}t^*) = \mathbf{L}^*$, and signed Jordan induction gives $\chi_t = \pm R_L^G(\tau t)$ irreducible. At ℓ-regular x, the semisimple arguments in $R_L^G(\tau t)(x)$ belong to $L_{\ell'}$, where $t = 1$; the unipotent parts also have $t = 1$. Thus, $R_L^G(\tau t)(x) = R_L^G(\tau)(x)$. This restriction is nonzero at 1_G, so disjoint Brauer supports give $\chi_t \in B_\nu$.

Conjugacy of $\bar{s}t^*$ and $\bar{s}u^*$ requires conjugacy of t^*, u^* through the normalizer of $\mathbf{L}^*$ and $\mathbf{A}^*$. On P this gives $M/L = C_{2m}$, which fixes ν. Moreover, its action on $P[\ell]$ is faithful. Since $\ell \nmid 2m$, the action on $\mathrm{Irr}(P) \setminus \{1\}$ is free. Locally, $\mathrm{Ind}_L^M(\tau t)$ is irreducible by trivial inertia in M/L. It has ℓ'-degree and lies above the M-stable block $\mathrm{bl}_L(\tau)$. Hence, it lies in b_ν. The cyclic-defect count is $k(B_\nu) = k(b_\nu) = 2m + \frac{|P|-1}{2m}$. Thus, the C_{2m}-orbits give the complements in B_ν, b_ν. The field fixes $\mathbf{L}$ and τ. By [53] and [16], R_L^G is covariant under $\mathcal{H}_\ell$ and under F_r with parabolic $F_r(\mathbf{Q})$, independent of $\mathbf{Q}$ at odd r. Both actions give $\chi_t(1) > 0$. Ordinary induction into M is covariant for $F_r, \mathcal{H}_\ell$. □

Proposition 7.210. *If $m \in \{5, 7\}$, each pair over ν has the actions on (λ, j) from* Proposition 7.208*, also on $\mathrm{Irr}(G\langle U\rangle)$ above the pair. At $\varepsilon = -1$ and even f, the quotient sign on U is (a/r).*

Proof. In $\mathcal{H}_\ell$, fix σ. Since $\tau^\sigma = \tau$, Brauer correspondence gives σ-stable B_ν, b_ν. Write k_σ for the fixed count. By [142, Theorem 3.4], $k_\sigma(B_\nu) = k_\sigma(b_\nu)$. Subtract the count on $t \neq 1$ from Lemma 7.209. The $2m$ selected characters have $2m$ or 0 fixed points according as $\sigma(j) = j$ or $-j$. Each global pair (χ_+, χ_-) is σ-stable by [208, Lemma 11.13 and Proposition 11.23], so each has sign $\sigma(j)/j$. Match m runners to $\lambda \in \mu_m$, with $j, -j$ related by the diagonal δ. The lemma gives $\chi_\pm^{F_r} = \chi_\pm$, and Proposition 7.208 fixes (λ, j) under U.

If $2 \nmid f$, the pair sum $\chi_+ + \chi_-$ is rational. Let ι fix μ_{2^∞} and invert odd-order roots. Replace ι by $\iota\delta$ if necessary to fix χ_+. It inverts $\mathrm{Irr}(C_f)$. By [208, proof of Lemma 3.1], the unique fixed extension is $\widehat{\chi}_+$. We have $\delta^2 \in G\mathrm{Z}(\widetilde{G})$, so the mixed transports of $\widehat{\chi}_+$ have the same normalization. On V, the chosen scalar gives $(\lambda, j) \leftrightarrow (\lambda, -j)$ on extensions by Proposition 7.208, with twist 1. Choose a common character of $\mathrm{Z}(\widetilde{G})$ with value k on z. At even f and $\varepsilon = 1$, Lemma 7.211 gives F_r-extensions fixed by $\mathcal{H}_\ell$ via principal induction with pinned F_r.

Suppose f is even and $\varepsilon = -1$. If $J = 2^j \mid f_2$, set $X_j = G \rtimes \langle \phi^{f/J} \rangle$, $\quad Y_j = \mathrm{N}_{X_j}(P) = M\langle U^{f/J}\rangle$. The lift U normalizes P and maps to ϕ. Moreover, both field quotients have order J prime to ℓ, and P is still Sylow. Each base $\chi_\pm$ has J extensions, as does (λ, j). All these bases are $\mathcal{H}_\ell$-fixed, as are $\mathrm{Irr}(C_J)$ since $J \mid \ell - 1$. Each orbit with $t \neq 1$ has field length J. Indeed, the 2-part of $\mathrm{ord}_\ell(r^{f/J})$ is $2J$, while the rotations and inversion in C_{2m} have 2-part 2. If $0 < a < J$, the power $r^{af/J}$ gives a distinct C_{2m}-orbit, already on the order-ℓ power of t. The inner component of $U^{f/J}$ acts within C_{2m}. By [43], each t-orbit induces a character in $\mathrm{Irr}(X_j)$ or $\mathrm{Irr}(Y_j)$. This orbit bijection is σ-equivariant by Lemma 7.209.

The field fixes B_ν, b_ν. By [88], Brauer correspondence at P matches the blocks of X_j, Y_j above B_ν, b_ν, since P is Sylow and $D(B_\nu) = D(b_\nu) = P$ for these pairs. Brauer correspondence is σ-equivariant. By [142, Theorem 3.4], $k_\sigma(B) = k_\sigma(b)$ for a σ-stable pair (B, b), and $k_\sigma(B) = k_\sigma(b) = 0$ otherwise. Subtract the counts at $t \neq 1$. Thus, the $2mJ$ extensions on both sides have equal σ-fixed counts.

Write $s_\sigma = (a/r)$. On Y_j the quotient twist has value $s_\sigma^{f/J}$ at $U^{f/J}$. Moreover, all extensions in Y_j are fixed at sign 1 and none at sign -1. If $J = f_2/2$, including $f_2 = 2$, the sign is 1, so each global restriction to X_j is fixed. Thus, at $J = f_2$ the quotient twist has order ≤ 2. Since f/f_2 is odd, equal counts give twist 1 for $s_\sigma = 1$ and quadratic twists for $s_\sigma = -1$. On $\langle \phi^{f_2} \rangle$, [208, Lemma 11.13 and Proposition 11.23] give an extension with field $\mathbb{Q}(\chi_\pm)$, fixed by $\mathcal{H}_\ell$. A unique character of C_f gives both restrictions to $C_{f_2}, C_{f/f_2}$. At even f, the bases are $\mathcal{H}_\ell$-fixed, so mixed stabilizer operations have δ-component 1. Next, choose a common character of $\mathrm{Z}(\widetilde{G})$. □

Lemma 7.211. *Keep $\bar{s}$ of type A_7, now with $\varepsilon = 1$ and $q = r^f$ odd. Moreover, both constituents over each $\lambda \vdash 8$ occur in the same rational $G\langle F_r\rangle$-module, independently of ℓ. Let $j^2 = (-1)^{(q-1)/2}$. Moreover, each pair over $\lambda \vdash 8$ has coefficient sign $\sigma(j\sqrt{q})/(j\sqrt{q})$, both on $\chi_\pm$ and on $\widehat{\chi}_\pm$ for the chosen extensions. If f is even, we have $\mathbb{Q}(\widehat{\chi}_\pm) = \mathbb{Q}$.*

Proof. Take the split torus T and its quadratic weight θ dual to $\bar{s}$, inflated to the split Borel B. In $\widetilde{G}$, torus induction contains all $\lambda \vdash 8$ of the connected A_7, with multiplicities $\dim S^\lambda > 0$. By [95, Problem 5.6], the restriction $I = \mathrm{Ind}_B^G \theta$ contains both children over each λ. The stabilizer is $W_\theta = S_8 \times \langle -I\rangle$. The positive-system complement of S_8 is generated by $y = w_{0,E_7} w_{0,A_7}$. Moreover, it acts on S_8 by inner opposition. In $W(E_7)$ its Coxeter length is odd since $\det(y) = (-1)^7(-1)^{28} = -1$. The torus extension takes value 1 on the subsystem representatives since the A_7 coroot tori lie in $\ker\theta$. The lift of y has value $\pm j$ by $n_0^2 = z$, and the lift of w_{0,A_7} centralizes n_0.

At y the extension factor is the sign $\sigma(j)/j$. Moreover, the length factor is $\sigma(\sqrt{q})/\sqrt{q}$ by the odd length of y. Every $\eta \in \mathrm{Irr}(S_8)$ has $\mathbb{Q}(\eta) = \mathbb{Q}$. By [166, Theorem 2.2, Lemma 3.1 and Proposition 5.4], Hecke and ordinary actions on $\mathrm{Irr}(W_\theta)$ agree. In type A_7 the extra hypothesis is vacuous. Since $W_\theta = S_8 \times C_2$, we have $\mathbb{Q}(\chi) = \mathbb{Q}$ for $\chi \in \mathrm{Irr}(W_\theta)$. Hence, the product sign is $\sigma(j\sqrt{q})/(j\sqrt{q})$ for each λ. The diagonal exchanges χ_+, χ_- by [208, Proposition 11.23]. At odd f, $\mathbb{Q}(j\sqrt{q}) = \mathbb{Q}\big(\sqrt{(-1)^{(r-1)/2}r}\big)$. If f is even, $j, \sqrt{q} \in \mathbb{Q}$.

The model of I consists of B-equivariant functions on G. The pinned F_r acts by pullback over $\mathbb{Q}$. Given a supporting coset BwB, a kernel in $\mathrm{End}_G(I)$ is determined by its value at $\dot{w}$, since θ is linear. The field fixes $\dot{w}$ and leaves both θ-covariance relations invariant, so F_r centralizes $\mathrm{End}_G(I)$ pointwise. Thus, on $I_\chi = V_\chi \otimes U_\chi$ its operator is $A_\chi \otimes I_{U_\chi}$ and $A_\chi^f = 1$. It gives one extension of V_χ repeated $\dim U_\chi$ times in I_χ. Rationality of the model gives $\widehat{\chi}^\sigma = \widehat{\chi^\sigma}$. □

Proposition 7.212. *If $\ell \mid q-1$ and $\ell \neq 2, 3$, the split A_7 series has the mixed comparison on $N_G(P)$, with 30 members at $\ell = 5$ and 14 at $\ell = 7$.*

Proof. Keep θ on T and the inertia extension $\widehat{\theta}$ from Lemma 7.211. If $\ell > 7$ we have $P = T_\ell$ and $N_G(P) = N_G(T)$, with inertia quotient $S_8 \times \langle -I\rangle$. Its factors give $\lambda \vdash 8$ and $j, -j$ on n_0, yielding 44 characters by Clifford induction. At $\ell = 5, 7$, we choose Q generated by an ℓ-cycle of S_8, Sylow in $W(E_7)$ since $[W(E_7) : S_8] = 72$. Take its prime-field odd complement $\widehat{Q}$ in the Tits group, and put $P = T_\ell \widehat{Q}$.

By Proposition 7.197, $M = N_G(P) \leq N_G(T)$ has torus kernel $T_* = T_\ell C_{T_{\ell'}}(Q)$ and quotient $N_W(Q)$. Restriction to T_* distinguishes the Q-fixed weights. Sylow conjugacy in W_θ gives one weight orbit of M, with inertia quotient $\mathrm{AGL}_1(\ell) \times S_{8-\ell} \times \langle -I\rangle$. Moreover, the $\ell - 1$ linear affine characters have values in $\mathbb{Q}(\mu_{\ell-1})$. The other character is induced from $1 \neq \xi \in \mathrm{Irr}(C_\ell)$, with rational values and degree $\ell - 1$. Hence, $\mathcal{H}_\ell$ fixes the ℓ affine characters. Match them to runners $i \in \mathbb{Z}/\ell\mathbb{Z}$ and the core ν to the $S_{8-\ell}$-character. The choices $j, -j$ remain on n_0, which centralizes $\widehat{Q}$. The inertia indices are prime to ℓ; Lemma 7.206 gives $30, 14$ at $\ell = 5, 7$.

Let the field lift act by 1 on $\widehat{\theta}$ and trivially on the quotient factors S_8 or $\mathrm{AGL}_1(\ell) \times S_{8-\ell}$, and C_2. Above $N_W(Q)$, the representatives in S_8 admit corrections by the sign torus $T[2]$ by Lemma 7.152. The diagonal commutator with $\widetilde{\delta}$ exchanges $j, -j$ on n_0. At odd f, we choose $[y(t), c]$ of Proposition 7.208 with $e = 0$, averaging the cocharacter over Q or over 1 when $\ell > 7$. The lift normalizes P and is F_r-fixed. The character $\widehat{\theta}$ is trivial on diagonal commutators from S_8 because S_8 fixes s. Thus, the exchange of $\widehat{\chi}_\pm$ has quotient twist 1. Since $\ell \mid q-1$, $\sqrt{q}$ is $\mathcal{H}_\ell$-fixed by [156, Lemma 2.3(d)]. Lemma 7.211 gives global sign $\sigma(j)/j$. The prime-field diagonal fixes B, θ and acts on I with the pinned F_r. Hence, it exchanges $\widehat{\chi}_+, \widehat{\chi}_-$ when $\sigma(j)/j = -1$. At even f, the bases and extensions are $\mathcal{H}_\ell$-fixed on G, M. Choose a common character of $\mathrm{Z}(\widetilde{G})$. Thus, the mixed comparison holds on $N_G(P)$. □

Proposition 7.213. *Both rational series of the projective A_7 involution in $E_{7,\mathrm{sc}}(q)$, for odd q, have the mixed comparison on $N_G(P)$ at every odd $\ell \mid |G|$ with $\ell \neq r$.*

Proof. Lemma 7.206 gives the eligible signed orders m. If $m = 1$, apply Proposition 7.212 for $\varepsilon = 1$ and Proposition 7.154 for $\varepsilon = -1$. Lemma 7.153 gives both extensions at 2-core $(3, 2, 1)$. At $m = 4, 8$, the ordinary order is m for $\varepsilon = \pm 1$, and the chosen λ have empty 2-core. Thus, Proposition 7.188 applies at $m = 4, 8$. At $m = 5, 7$, we apply Proposition 7.210 over each ν, with ordinary orders $5, 7$ in the split form and $10, 14$ for $\varepsilon = -1$. We have $B_\nu \neq B_{\nu'}$ for $\nu \neq \nu'$: conjugacy of $(P, \mathrm{bl}_L(\tau_\nu))$ and $(P, \mathrm{bl}_L(\tau_{\nu'}))$ would come from $N_G(P)$, which fixes each τ_ν. Hence, the comparisons combine on the ℓ'-degree characters. □

Corollary 7.214. *Let q be odd, $G = E_{7,\mathrm{sc}}(q)$, and $\ell \mid |G|$ with $\ell \neq 2, r$. Each rational dual involution $\bar{s}$ has the comparison on ℓ'-degree characters on $N_G(P)$ for $P \in \mathrm{Syl}_\ell(G)$.*

Proof. By [208, §11, Propositions 11.9, 11.19 and 11.23], the types A_1D_6, E_6T_1, A_7 have $1, 2, 2$ rational forms, respectively. In type A_1D_6, by Lemma 7.189, we obtain the eligible $d = \mathrm{ord}_\ell(q)$. By Proposition 7.136, $d = 1, 2$, and Propositions 7.190 and 7.191 give $d = 4, 8$ and $5, 10$, respectively. The E_6T_1 cases follow from Proposition 7.205 and the A_7 cases from Proposition 7.213. For characters of $G/\mathrm{Z}(G)$, these results determine the preimage in G. □

Lemma 7.215. *Next, we write $G = E_{7,\mathrm{sc}}(q)$, $D = E_{7,\mathrm{ad}}(q)$, $S = D'$, $q = r^f$, $r \geq 5$. For $\bar{s}$ of* Zhang [208, Proposition 11.21]*, the lifts s, zs have orders $3, 6$, and $C(s)$ has type A_5A_2. With $\varepsilon = (-1)^e \equiv q \bmod 3$, we put $X = \varepsilon q$. If $2 \nmid \ell$ and $\ell \neq r$ dividing $|G|$, we set $m = \mathrm{ord}_\ell(X)$. Both type A factors are split for $\varepsilon = 1$ and unitary for $\varepsilon = -1$. If $m = 1$, the eligible counts are $33, 15, 0$ for $\ell > 7$, $\ell = 5$ and $\ell = 3, 7$, respectively. At $m = 3, 5$, they are $27, 15$, respectively, and at other m they are 0. Each eligible character inflates $\chi|_S$, with $\chi \in \mathrm{Irr}(D)$ and $\chi|_S$ irreducible. Moreover, the parent χ extends rationally to $D\langle F_r\rangle$.*

Proof. For the isogeny π in [208, (11.11)], both $\ker(\pi^F)$ and $\mathrm{coker}(\pi^F)$ have order $|(\ker\pi)^F|$. Thus, the prime-to-r index is, up to sign, $\frac{\prod_{i\in\{2,6,8,10,12,14,18\}}(X^i-1)}{(X^2-1)^2(X^3-1)^2(X^4-1)(X^5-1)(X^6-1)}$. Precisely Φ_1, Φ_3, Φ_5 have equal positive multiplicities $7, 3, 1$ in numerator and denominator. At $m = 1$, the index factors Φ_9, Φ_7 exclude $\ell = 3, 7$. At $m = 3, 5$, $m\ell > 18$, so indices $m\ell^a$ for $a > 0$ are absent.

For the partitions $(\lambda, \nu) \vdash (6, 3)$, the hook formula at $m > 1$ requires both m-cores to have minimum size. At $m = 3$, the quotient weights are $2, 1$, giving $(2 \cdot 3 + \binom{3}{2}) \cdot 3 = 27$. If $m = 5$, the five runners on (1) pair with each $\nu \vdash 3$, giving 15. At $m = 1$, require $\ell \nmid \dim S^\lambda \dim S^\nu$. Thus, at $\ell > 7$, all 33 labels occur, and at $\ell = 5$ there are the same 15 five-hook pairs.

By [208, proof of Proposition 11.21], each parent χ is rational and F_r-fixed, and the $\chi|_S$ are irreducible and distinct. If $2 \nmid f$, [208, Lemma 3.2] gives an extension $\widehat{\chi}$ over C_f with values in $\mathbb{Q}$. If $2 \mid f$, q is square and $\varepsilon = 1$. Every (λ, ν) occurs in principal induction on D from the cubic weight θ, with neutral reflection type A_5A_2. Take the θ, θ^{-1} models over $\mathbb{Q}(\zeta_3)$. Next, write P_r for pinned field pullback and J for the longest intertwiner from $\sum_{u\in U_B} u$, where $U_B = R_u(B)^F$.

At $r \equiv 1 \bmod 3$, P_r acts on the θ-model by [208, Lemma 4.9(i)]. If $r \equiv -1 \bmod 3$, give the cubic weight value 1 on $n_0\phi$, with $(n_0\phi)^f = 1$ in $D\langle\phi\rangle$. On I_χ, divide JP_r by the scalar $(\sqrt{q})^h$ in [208, Proposition 4.15, Step 3], with $h \in \mathbb{Z}$ and $\sqrt{q} \in \mathbb{Q}$. In both cases I_χ is isotypic for one extension $\widehat{\chi}$. The projector onto I_χ is rational because χ is rational. Let C conjugate coefficients. The invertible antilinear map JC satisfies $(JC)A = A(JC)$ for the operators A of D and the field lift. Indeed, J is rational and P_r fixes $\dot{w}_0$ and permutes U_B. Thus, $\mathbb{Q}(\widehat{\chi}) \subseteq \mathbb{Q}(\zeta_3) \cap \mathbb{R} = \mathbb{Q}$. □

Lemma 7.216. *If $m = 3$, the irreducibles of $(C_3 \wr S_2) \times C_3$ label the 27 local characters, via the global pair $(\lambda^{(3)}, \nu^{(3)})$ of 3-quotients. They have $\mathcal{H}_\ell$-fixed extensions to $N_D(P)\langle U\rangle$.*

Proof. First, take $b = ((123)(456), (123)) \in S_6 \times S_3$. In the three A_2 subgroups, the Coxeter lifts n_i satisfy $n_i^3 = 1$ and $[n_i, n_j] = 1$. Thus, $n = n_1n_2n_3$ lifts b. Take the longest lift n_0 of [208, proof of Proposition 11.21], with $n_0^2 = 1$ in the adjoint group. Moreover, $n_0n = nn_0$, and we set $F = \mathrm{Int}(n_0^e n)F_q$. The fixed line of b is generated by $(1, 1, 1, -1, -1, -1)$ in the A_5-space, of squared norm 6, and $6a^2 \neq 2$ for $a \in \mathbb{Q}$. Thus, the primitive 6-space has centralizer $\mathbf{T}$. The torsion calculation at 3 is on the ℓ-localized lattice, identical under the 2-isogeny even at $\ell = 7$. It gives $M = N_D(P) = N_D(\mathbf{T})$.

Let $\mathbf{K}$ have roots A_5A_2 with the inherited torus lattices, so $\mathbf{K}^* = C(s)$. The central s defines $\kappa \in \mathrm{Irr}(K)$ of order 3, where $K = \mathbf{K}^F$. The inertia of $\kappa|_{\mathbf{T}^F}$ in M is $I = N_K(\mathbf{T})$, since its quotient by $\mathbf{T}^F$ is $B = C_{S_6\times S_3}(b) = (C_3 \wr S_2) \times C_3$ by Lang. By [99, Theorem 4.3.34], the 9 wreath-product labels and 3 cyclic labels give 27 characters with values in $\mathbb{Q}(\zeta_3)$. Inflate $\eta \in \mathrm{Irr}(B)$, tensor with $\kappa|_I$, and induce to M. Its degree is prime to ℓ since $|C_W(b)| = 6 \cdot 12 \cdot 18$ and $\ell \geq 7$.

Put $j = 0, 1$ for $r \equiv 1, -1 \bmod 3$, respectively. Let U lift F_r in this mark and set $V = n_0^j U$. Since $jf \equiv e \bmod 2$, we have $V^f = n^{-1}$, and V acts trivially on B. On κ the exponent is $r(-1)^j \equiv 1 \bmod 3$, so $\kappa^V = \kappa$. On $\kappa|_I \otimes \eta$, n^{-1} has scalar in μ_3 since $b \in \mathrm{Z}(B)$ and $n^3 = 1$. Choose the V-scalar $\alpha \in \mu_{3f_3}$ with α^f equal to that scalar. We have $\mathrm{ord}_\ell(q) = 3, 6$, so $v_3(\mathrm{ord}_\ell(r)) = 1 + v_3(f)$. Hence, $3f_3 \mid \ell - 1$, and $\mathcal{H}_\ell$ fixes α and $\kappa|_I \otimes \eta$. Induction from $I\langle V\rangle$ to $M\langle U\rangle$ gives the extension of $\mathrm{Ind}_I^M(\kappa|_I \otimes \eta)$. □

Proposition 7.217. *For $\bar{s}$ in* Zhang [208, Proposition 11.21] *with $r \geq 5$, the mixed comparison holds on $N_G(P)$ at every odd $\ell \neq r$. Next, we restrict the characters from $D, N_D(P)$ to $S, N_S(P)$ and inflate to $G, N_G(\widehat{P})$. The adjoint field extensions are $\mathcal{H}_\ell$-fixed.*

Proof. At $m = 5$, we put a 5-cycle b on the first five A_5 letters and choose its order-5 lift n. Moreover, in the sum-zero 4-space V, the roots form A_4: coordinates of A_5-weights are congruent modulo $\mathbb{Z}$, and $x_6 = 0$. Norm squared 2 gives $e_i - e_j$. Moreover, a perpendicular root projects to $h\omega$ on A_5, with ω the end fundamental weight and $\omega^2 = 5/6$. Determinants give $[Q(E_7) : Q(A_5A_2)] = 3$. The class of ω mod $Q(A_5)$ has order 6, so $h \in 2\mathbb{Z}$. If $h \neq 0$, $(h\omega)^2 \geq 10/3 > 2$. Thus, the root lies in the A_2-space, where the nonzero classes of $P(A_2)/Q(A_2)$ have norm squared $2/3$ mod 2. The primitive torus $\mathbf{A}$ has centralizer $\mathbf{L}$ of residual type A_2. In the adjoint group, $\mathrm{Z}(\mathbf{L})$ is connected, so $L = \mathbf{L}^F$ surjects to its adjoint quotient H.

All ℓ-factors of $|D|$ occur in degree 10. The Sylow subgroup lies in $\mathbf{A}^F$, and $\mathbf{A}$ is the unique primitive torus of $\mathrm{Z}(\mathbf{L})^\circ$. Thus, $M = N_D(P) = N_D(\mathbf{A})$. The A_4 isometries on V centralizing b give $\langle b, -I\rangle$, with pointwise kernel $W(A_2)$. Thus, $M = L\langle n, n_0\rangle$ with lifts n, n_0 of orders $5, 2$ with $nn_0 = n_0 n$. Moreover, the mark is $n_0^e n$, and n centralizes H. We have $H = \mathrm{PGL}_3(q)$ for $e = 0$ and $\mathrm{PGU}_3(q)$ for $e = 1$.

The central dual s gives a cubic $\kappa \in \mathrm{Irr}(L)$ with $\kappa^n = \kappa$ and $\kappa^{n_0} = \kappa^{-1}$. Moreover, for the 3 unipotent characters ρ of H, $\kappa\rho$ has inertia $I = L\langle n\rangle$. Extend by value 1 on n and induce to M. This base is rational since ρ is rational and n_0 inverts κ and fixes ρ, n. It is U-fixed by the correction $V = n_0^j U$. If f is odd, M has odd index in $M\langle U\rangle$. By [143, Lemma 2.1 and Corollary 2.2], we choose the real extension $\widehat{\psi}$, unique and rational.

If $2 \mid f$, H is split. On $I\langle V\rangle$ give κ value 1 on n, V, with $V^f = n^{-1}$. Extend ρ along the conjugation action on H. If $H = \mathrm{PGL}_3(q)$, the models of Proposition 7.208 are over $\mathbb{Q}$ since $\sqrt{q} \in \mathbb{Q}$. The induced extension is rational since n_0 inverts κ, fixes n, V, and acts on ρ by its rational graph operator on H. At any f, tensor with the quotient character taking n to $\lambda \in \mu_5$, n_0 to 1, and U to an f-th root of λ^{-1} in μ_{5f_5}. We have $U^f = n^{-1}n_0^{-e}$. Since $\mathrm{ord}_\ell(q) = 5, 10$, we get $5f_5 \mid \ell - 1$. Thus, these 15 extensions are fixed by $\mathcal{H}_\ell$.

At $m = 1$, the mark n_0^e gives $V^f = 1$ for $\varepsilon = \pm 1$. The torus inertia quotient is $S_6 \times S_3$ at $\ell > 7$. If $\ell = 5$, by Proposition 7.138, we obtain $\mathrm{AGL}_1(5) \times S_3$ by restriction to T_* and the odd complement $\widehat{Q}$. Its proof depends on W_θ, the decomposition $T = T_\ell T_{\ell'}$, and the Tits 2-kernel, also at $e = 0$. Extend the weight by 1 on the subsystem Tits representatives, with V acting by identity. The characters of $S_6 \times S_3$ and $\mathrm{AGL}_1(5) \times S_3$ are $\mathcal{H}_\ell$-fixed. If σ inverts κ, n_0 corrects the inversion and fixes V, so the extensions are fixed at $m = 1$ for all f.

In every mark, $M = N_D(P)$ satisfies $M/(M \cap S) = D/S$ by Frattini, since $P \leq S \lhd D$ is Sylow in D. The nontrivial twist from D/S changes s of order 3 to zs of order 6. The torus orbits of s and zs are disjoint. At $(m, \ell) = (1, 5)$, restriction to T_* is injective on Q-fixed weights. If $m = 5$, the orders $3, 6$ separate the $\mathbf{L}^*$-classes. By Clifford theory [43] at index 2, the restrictions to $N_S(P) = M \cap S$ are irreducible and distinct. At $m = 3$ this includes $\mathrm{Ind}_I^M(\kappa|_I \otimes \eta)$ from Lemma 7.216. Let $\widehat{P}$ be the unique Sylow ℓ-subgroup of the preimage of P. Then, $N_G(\widehat{P})$ is the preimage of $N_S(P)$. Next, inflate the restrictions to $G, N_G(\widehat{P})$, with z in both kernels. The extensions over F_r are $\mathcal{H}_\ell$-fixed, have factor set 1, and give the same scalars on z. Now, match (λ, ν) via their quotients at $m = 3, 5$ and the scalar inertia characters at $m = 1$. □

Lemma 7.218. *Let $G = E_{7,\mathrm{sc}}(q)$, $q = r^f$, $r \geq 5$. By* Zhang [208, proof of Proposition 11.18, Step 1]*, the adjoint element $\bar{s}$ has order 6 and connected centralizer of type $A_2^3T_1$. We choose $\varepsilon = (-1)^e$ with $x = \varepsilon q \equiv 1$ mod 3. Next, we let $w = (1\,3)$ on the A_2-factors. Moreover, Frobenius acts by 1 or w, independent of ε. At an odd prime $\ell \mid |G|$ with $\ell \neq r$, set $m = \mathrm{ord}_\ell(x)$. The form with permutation w has eligible count 0. Moreover, the form with permutation 1 contributes at $m = 1$ with $\ell > 7$ and at $m = 3$, with 9 singletons and 9 diagonal pairs at either order. In $M = N_G(P)$, the torus inertia quotient is $J_0 \rtimes \langle w\rangle$, with $J_0 = S_3^3$ at $m = 1$ and $J_0 = C_3^3$ at $m = 3$. The involution w exchanges the outer factors.*

Proof. First, put $B_0 = \Phi_5\Phi_{10}\Phi_7\Phi_{14}\Phi_8\Phi_{12}\Phi_9\Phi_{18}$ at x. The ambient order divided by $|C_{\mathbf{G}^*}(\bar{s})^F|$ from Lemma 7.137 gives, up to sign, $\Phi_2^4\Phi_6^3\Phi_4^2B_0$, $\quad \Phi_1^3\Phi_2^3\Phi_3\Phi_6^2\Phi_4B_0$ for permutations $1, w$, respectively. The second contains Φ_m whenever $m \mid i$ for an ambient degree i. The first is an ℓ-unit only at $m = 1, 3$. At $m = 1$, the factors Φ_9, Φ_5, Φ_7 exclude $\ell = 3, 5, 7$. At $m = 3$, we have $3\ell > 18$. The A_2 degrees $1, q(q + \varepsilon), q^3$ are ℓ-units at $m = 1, 3$. By [208, proof of Proposition 11.18, Steps 1–2], the form with permutation 1 has 27 characters, with $k = 1$ at $q \equiv \pm 1$ mod 12 and $k = -1$ at $q \equiv \pm 5$ mod 12. Thus, all 27 are eligible at $m = 1, 3$.

Take the ambient longest lift n_0 and the lift y of w constructed in the proof of Lemma 7.139, with $y^2 = z$. The middle coroot correction fixes the A_2-pinning there. Take order-3 Coxeter lifts a_1, a_2 in the first and middle A_2 and set $a_3 = ya_1y^{-1}$. Then, $[a_i, a_j] = 1$, and conjugation by y exchanges a_1, a_3 and fixes a_2. Put $n_b = 1$ at $m = 1$ and $n_b = a_1a_2a_3$ at $m = 3$. The mark $n_0^e n_b$ satisfies $n_0^e n_b\, y = y\, n_0^e n_b$ and fixes $\bar{s}$. Moreover, it fixes each component of A_2^3.

The primitive torus is maximal at $m = 1$, and at $m = 3$ its space is the direct sum $V_1 \oplus V_2 \oplus V_3$ of the A_2-planes. Its perpendicular line at $m = 3$ has squared norm 6 for a primitive vector γ and $6a^2 \neq 2$ for $a \in \mathbb{Q}$. Thus, the centralizer is $\mathbf{T}$, the marked maximal torus. At $m = 3$, the degrees $6, 12, 18$ contribute the three Φ_3-directions, including $\ell = 7$. At either order the rational ℓ-part of $\mathbf{T}^F$ is Sylow in G by the degree list for E_7. By [63, Theorem 3.3], P normalizes a unique Sylow $\mathrm{ord}_\ell(q)$-torus $\mathbf{S}_d$, the one in $\mathbf{T}$. Since $\mathbf{T} = C_{\mathbf{G}}(\mathbf{S}_d)$, $M \leq N_G(\mathbf{T})$. Conversely, $P = (\mathbf{T}^F)_\ell$ gives $N_G(\mathbf{T}) \leq M$.

With $T = \mathbf{T}^F$, we let θ be the weight dual to $\bar{s}$. The geometric inertia quotient is $S_3^3 \rtimes \langle w \rangle$. By Lang on $\mathbf{T}$, we obtain the centralizer of $n_0^e n_b$ on the geometric quotient, namely $J_0 \rtimes \langle w \rangle$. At $m = 1$, extend θ by value 1 on subsystem representatives because $\theta = 1$ on neutral coroot tori. At $m = 3$, give value 1 on $\langle a_1, a_2, a_3 \rangle$, a complement because the a_i have order 3 and independent images in W. At either order give y the scalar j with $j^2 = k$.

For outer labels $\lambda_1 \neq \lambda_3$, induce from J_0. At $\lambda_1 = \lambda_3$, we extend with tensor flip $\pm$flip. Each factor has 3 labels, giving 9 singletons and 9 pairs. In the regular group, w changes $[s, t_0]$ by the quadratic quotient, while J_0 fixes $[s, t_0]$. Hence, the diagonal changes j to $-j$ on each pair $(\lambda_1, \lambda_2, \lambda_1)$. The torus weights over $\bar{s}$ form one M-orbit, since Sylow conjugacy in $C(\bar{s})$ lets the conjugator normalize the primitive ℓ-subgroup. All local degrees are ℓ'-numbers since $P \leq T$. □

Lemma 7.219. *At $m = 1, 3$, the local pairs have sign $\sigma(j)/j$ for $\sigma \in \mathcal{H}_\ell$, and each singleton is σ-fixed. The field extensions have sign $\sigma(j)/j$ on pairs and sign 1 on singletons. Both global forms $\varepsilon = \pm 1$ have this base action.*

Proof. First, choose $t = 0, 1$ with $(-1)^t r \equiv 1 \bmod 6$. Next, put $V = n_0^t U$, where U acts by F_r in the preceding mark. Then V fixes θ, y, a_i and $V^f = z^{(tf-e)/2} n_b^{-1}$. The sign $k^{(tf-e)/2}$ is 1 at even f since $q \equiv 1 \bmod 12$. At odd f this sign is its own f-th root. The relative scalar on n_b^{-1} is 1 at $m = 1$ and lies in μ_3 at $m = 3$. At $\lambda_1 \neq \lambda_3$, both blocks have one scalar on b^{-1} because b is central in J_0. If $m = 3$, take its f-th root in μ_{3f_3} as in Lemma 7.216. Choose one V-scalar for $\pm j$ at $\lambda_1 = \lambda_3$. Each σ gives $\theta^\sigma = \theta^{\pm 1}$. At $\theta^\sigma = \theta^{-1}$, conjugate by n_0, which centralizes V and the chosen lifts y, a_i. The relative values lie in $\mathbb{Q}$ at $m = 1$ and $\mathbb{Q}(\zeta_3)$ at $m = 3$, fixed since $\ell \equiv 1 \bmod 3$ at $m = 3$. The sign is then $\sigma(j)/j$. Given $\lambda_1 \neq \lambda_3$, the signs $(1, -1)$ implement it and fix the V-operator.

At $\varepsilon = 1$, principal induction from θ has all 27 labels of $S_3^3 \rtimes C_2$. By [166, Theorem 2.2, Lemma 4.3, Corollary 4.5 and Proposition 5.4], the Hecke action for A_2^3 is ordinary on $\mathrm{Irr}(S_3^3)$. The extension factors are signs $\eta_1 = \eta_2$, with ratio 1. The length factor is 1 since $\sqrt{q}$ is fixed by $\mathcal{H}_\ell$ at $\mathrm{ord}_\ell(q) = 1, 3$. Thus, each global pair has sign $\sigma(j)/j$. At $m = 3$, we have that the calculation on C_3^3 gives the same sign on M. Match the 3 labels of S_3 and C_3 by bijections f_1, f_2, f_3 with $f_1 = f_3$. Then, $(f_1 \times f_2 \times f_3) \circ w = w \circ (f_1 \times f_2 \times f_3)$.

At $\varepsilon = -1$, the detector at $-b$ in the proof of Proposition 7.140 is independent of ℓ, with rank-one group $\mathrm{SL}_2(q^3)$. Its two source rows transfer (a/r) to each target with $\lambda_1 = \lambda_3$. The A_2 Weyl values at a 3-cycle are $1, 1, -1$. Thus, the product for each label triple and its row sum are ± 1, hence odd in $\mathbb{Z}$, so the entries differ. For $a \bmod r$ on μ_r, the detected sign is (a/r). Since f is odd and $-q$ has odd order modulo ℓ, we get $(r/\ell) = (-1/\ell)$. Reciprocity and the value of k above give $(a/r) = \sigma(j)/j$. At $\lambda_1 \neq \lambda_3$, rationality follows from [208, Proposition 11.18, Steps 4–5]. □

Proposition 7.220. *For $\bar{s}$ in* Zhang [208, Proposition 11.18]*, the form with permutation 1 has the mixed comparison on $N_G(P)$ for all eligible odd $\ell \neq r$ and all f. The form with permutation w has zero eligible characters.*

Proof. Suppose $2 \nmid f$. By [208, proof of Proposition 11.18, Step 1], the central cocharacter v has $v(-1) = z$ and $w(v) = -v$. Moreover, $b(v) = v$. Take $h = [v(u), c]$ with $u^{\varepsilon r - 1} = c^{r-1} = -1$, adapting the proof of Lemma 7.139. Since v is perpendicular to all three A_2-factors, h lies in $\widetilde{G}^F$, induces the noninner diagonal, and centralizes V in both marks. Its commutator on y flips j by the quadratic quotient of $\widetilde{G}$. Given any $\chi_\pm$, compose complex conjugation with the diagonal h if necessary to fix $\chi_\pm$. It inverts differences in $\mathrm{Irr}(C_f)$ and fixes a unique extension $\widehat{\chi}_\pm$. The sign on both extensions is $\sigma(j)/j$ by Lemma 7.219. Choose one character of $\langle z, c^2 \rangle$ for both sides.

For the 9 singletons, order the parent labels as (λ_1, λ_3). Next, take the lift $[s, t_0]$ and the scalar b_A from Proposition 7.140. By [208, Proposition 11.18, Steps 4–5], the restrictions on G are irreducible and the twist on $\widetilde{G}$ is dual to b_A. The local calculation applies with J_0 because n_0 centralizes the rotation lifts a_i. At $A = r$, the factor is 1. Glue

the parent on $\widetilde{G}$ to the rational extension over C_f, and the local parent to its $M\langle V\rangle$-extension. On each restriction χ, the intertwiners differ by $\alpha \in \mathbb{C}^\times$, allowing the given power at V^f. The lift formula gives equal $\widetilde{G}/G$-ratios. The extensions give equal V-ratios.

Suppose f is even. Then $q \equiv 1 \bmod 12$ and $k = 1$. With $S = G/\mathrm{Z}(G)$, put $D = E_{7,\mathrm{ad}}(q)$. Next, we let $I_\theta, I_{\theta^{-1}}$ be split Borel inductions on S. Next, on $I_\theta \oplus I_{\theta^{-1}}$, take the longest Bruhat operator A before normalization, on U_B-coset functions with $U_B = R_u(B)^F$. It exchanges $e_\theta, e_{\theta^{-1}}$ and satisfies $A\,P_r = P_r\,A$. The simple sums A_i satisfy the positive braid relations by U_B-coordinates on length-additive cells BwB. In E_7 the longest braid is central, so $[A, A_i] = 0$. Also $A^* = A$ since the adjoint kernel has $n_0^{-1} = n_0$ in S.

Lift θ to the order-12 split weight dual to s on D. Since $D = B_D S$, restriction gives the principal modules on S, with the same longest sum on U_B. The reflection stabilizer is $W(A_2)^3$. The calculation in [208, proof of Proposition 4.15, Steps 1–2], independent of the field lift, gives the eigenvalues q^{63+c_ψ} of A^2, with $c_\psi \in \mathbb{Z}$ and $63 = |\Phi^+(E_7)|$. Their positive square roots lie in $\mathbb{Q}$ since q is square. Interpolate $p \in \mathbb{Q}[X]$ with $p(a) = a^{-1/2}$ on the distinct eigenvalues and put $J = Ap(A^2)$. The sum $I_\theta \oplus I_{\theta^{-1}}$ is rational, so J is rational and exchanges $I_\theta, I_{\theta^{-1}}$, with $J^2 = 1$.

Moreover, on I_θ, $J^t P_r$ implements the field and has f-th power 1. For a returning Bruhat corner $e_\theta A_w e_\theta$, P_r fixes A_w and transports e_θ, and J^t returns e_θ. Evaluation on the pinned representatives $\dot{w}$ shows these kernels span the commutant $\mathcal{A}$. Thus, $[J^t P_r, \mathcal{A}] = 0$, and the extension of χ occurs with its original multiplicity in I_θ. If σ inverts θ, correct by J, with $J(J^t P_r) = (J^t P_r)J$. By Lemma 7.219, $\chi^\sigma = \chi$, so the chosen extension is σ-fixed. Its local counterpart on $M\langle V\rangle$ is also σ-fixed. For the pair (χ_+, χ_-), the ordinary stabilizers over G and M are the field stabilizers, since δ exchanges χ_+, χ_-.

For an unequal singleton, take parents R on D and R' on $N_D(P)$ with the same outer ordering. By Lang and Clifford theory, $R'|_{N_S(P)}$ is irreducible, since y swaps the distinct outer labels. Let $\nu_A \in \mathrm{Irr}(D/S)$ be nontrivial exactly at $A \equiv \pm 5 \bmod 12$. With $t_0 = 1$, the formula at the lift t_0 gives twist ν_A under coefficients and ν_r under the field. Next, let $R|_S$ afford the base character χ and write T_0 for its operator on V. Under Lang transport $V \in S\langle\phi\rangle$. Thus, Schur's lemma on $R|_S$ gives $T_0 R(g) T_0^{-1} = \nu_r(g) R(VgV^{-1})$, $\quad T_0^f = R(V^f)$. Next, choose $R(g)T_0^i$ on gV^i for $0 \le i < f$. The multiplication factor on gV^i, hV^j is $\nu_r(h)^i$ also when $i + j \ge f$, by the power relation. For R' this gives $\nu_r(h)^i$ on its domain.

Given $\sigma \in \mathcal{H}_\ell$, the intertwiners for $\widehat{\chi}$ and the parent twist $\nu_A R$ are proportional on S since $R|_S$ is irreducible. Thus, ν_A is the coefficient ratio on D and 1 on V, with the same restrictions on $N_D(P)\langle V\rangle$. The factor set $\nu_r(h)^i$ also controls ordinary conjugation and simultaneous stabilizer actions. Inflate through $G \to S$ with z acting trivially; both models are trivial on the pullback centralizer. The eligible orders $m = 1, 3$ are from Lemma 7.218. □

Lemma 7.221. *Assume $G = E_{7,\mathrm{sc}}(q)$, with $q = r^f$ odd and $\mathrm{ord}_\ell(q) = 8$. Next, we choose $\bar{s}$ of* Zhang [208, Proposition 11.20] *with faithful central sign. Put $\mathrm{Z}(G) = \langle z\rangle$. Frobenius twists D_4 and exchanges the outer A_1's. Moreover, a global label has $x = 1, \mathrm{St}$ on $\mathrm{SL}_2(q^2)$, a toral D_4-label from a set of size 4, and a restriction sign. Locally, $\lambda^2 \in \mu_4$ records the toral D_4-label. There is a primitive torus $\mathbf{S}$ with $P = (\mathbf{S}^F)_\ell \in \mathrm{Syl}_\ell(G)$. Next, we write $M = N_G(P)$ and $\mathbf{J} = C_{\mathbf{G}}(\mathbf{S})$. A negative 4-cycle c gives $M/\mathbf{S}^F = (\mathbf{J}/\mathbf{S})^F \rtimes \langle c\rangle$, $\quad |c| = 8$. Moreover, the index-two subgroup $I = \mathrm{PSL}_2(q^2) \times \mathrm{SL}_2(q)$ of $(\mathbf{J}/\mathbf{S})^F$ is c-stable. Moreover, on I, we have that c gives F_q on $\mathrm{PSL}_2(q^2)$ and 1 on $\mathrm{SL}_2(q)$. The image of z is $(1, -I_2)$.*

Proof. Let V be the D_4-space on nodes $2, 3, 4, 5$, with perpendicular roots α_0, α_7, t from Proposition 7.142. Also, on V the component has sign parity -1. Thus, its $W(D_4)$-coset contains a negative 4-cycle. At $q \equiv -1 \bmod 4$ compose with $-I$, then correct the signs on α_0, α_7 by $s_{\alpha_0}, s_{\alpha_7}$. The mark on V is c or $-c$, both negative 4-cycles. Moreover, it exchanges α_0, α_7 positively and inverts t exactly at $q \equiv 1 \bmod 4$. Lang in $C_{\mathbf{G}^*}(\bar{s})^\circ$ fixes the rational class of $\bar{s}$.

Also, the lifts of Lemma 7.176 precede c. Transport the orientations on α_0, α_7, t by prime-field representatives of A_1^3. Next, choose a prime-field lift v of c exchanging the pinned groups for α_0, α_7 and centralizing $\mathrm{SL}_{2,t}$. By Lang in $\mathrm{SL}_{2,t}$, which centralizes v, absorb the longest representative on the t-line. In these coordinates $F = \mathrm{Int}(v)F_q$. Thus, the field lift U has $U^f = v^{-1}$ in the convention of Lemma 7.194.

On V, c has characteristic polynomial $X^4 + 1$. The torus accounts for $\Phi_8(q)$ with multiplicity 1 in $|G|$, and 8ℓ exceeds all ambient degrees. Thus, P is Sylow. By [63, Theorem 3.3], $\mathbf{S}$ is the unique Sylow 8-torus normalized by P, so $M \le N_G(\mathbf{S})$. Since P is characteristic in $\mathbf{S}^F$, $M = N_G(\mathbf{S})$. The relative Weyl group on V is $W(F_4)$ by Lemma 7.176. The image of $C_{W(F_4)}(c)$ centralizes a transposition in the diagram S_3. The preimage is $W(B_4)$, where a signed permutation v with $vc = cv$ is determined by the image of $+1$, since c is transitive on all 8. Thus,

$C_{W(F_4)}(c) = \langle c\rangle$. The pure lift has $v^8 \in \mathbf{S}^F$. By [133, Proposition 23.2], the quotient is $\mathbf{J}^F/\mathbf{S}^F \simeq (\mathbf{J}/\mathbf{S})^F$. Thus, $v\mathbf{S}^F$ gives the complement. The lattice on V is generated by the D_4-coroots, on which $\bar{s}$ is trivial. Hence, the weight on $\mathbf{S}^F$ is trivial.

Over $\overline{\mathbb{F}}_q$ we have $\mathbf{J}/\mathbf{S} = (\mathrm{SL}_2)^3/K$, where $K = \{(\epsilon_i) \in \mu_2^3 : \prod_i \epsilon_i = 1\}$. The half-sum $(\alpha_0 + \alpha_7 + t)/2$ is a weight trivial on $\mathbf{S}$ and takes value -1 on each single sign z_i. The residual Levi centre has order 2 by Lemma 7.176, so z_i represents z and each pair lies in K. On K, F exchanges the first two coordinates, so $K^F = \langle(-1,-1,1)\rangle$ and $|K/(F-1)K| = 2$. Thus, the fixed-point sequence gives $[(\mathbf{J}/\mathbf{S})^F : I] = 2$, with the stated z-image and c-action. □

Lemma 7.222. *The ℓ'-degree characters with faithful z-action have labels (x,λ), where $x = 1, \mathrm{St}$ and $\lambda \in \mu_8$. A diagonal pair is $(x,\lambda),(x,-\lambda)$, over the toral D_4-label $\lambda^2 \in \mu_4$. Moreover, the pinned field fixes (x,λ). If $\sigma \in \mathcal{H}_\ell$ of exponent $a \bmod r$ on μ_r, the permutation sign is $(a/r)^f$.*

Proof. On $\mathrm{SL}_{2,t}(q)$, we choose the quadratic constituents ξ_+, ξ_-. Moreover, we have that T_t is nonsplit at $q \equiv 1 \bmod 4$ and split at $q \equiv -1 \bmod 4$ in the original mark F. The weight on T_t takes value -1 on z, and a nonsquare diagonal exchanges ξ_+, ξ_-. The trace character η of U_t occurs with multiplicity 1 in ξ_+ and is F_r-fixed. Thus, each constituent is F_r-fixed. By [165, Theorem B], their field is $\mathbb{Q}(\sqrt{(-1)^{(r-1)/2}r})$ at odd f and $\mathbb{Q}$ at even f.

Extend x to $\mathrm{PGL}_2(q^2)\langle F_q\rangle$, with St on the augmentation space V_{St} of the projective line. Also, its F_q-operator Q satisfies $Q^2 = 1$. Extend $x \otimes \xi_+$ to $I\langle c\rangle$ by assigning λQ to c. Moreover, the other coset of $(\mathbf{J}/\mathbf{S})^F/I$ acts by a nonsquare diagonal on $\mathrm{SL}_{2,t}(q)$. Thus, the inertia is $I\langle c\rangle$, and induction gives 8 distinct irreducibles for each x. Their degrees $2x(1)\xi_+(1)$ are prime to ℓ since $\ell \nmid q(q-1)(q+1)$.

Represent the other coset by $j = (A,B,H)$ in the central quotient, where $B = F_q(A)$, $F_{q^2}(A) = -A$, and $F_q(H) = -H$. Moreover, these diagonal A, B, H exist in SL_2 by Lang on the tori T_i. The discrepancy under F is $(-1,1,-1) \in K$. Next, set $D_0 = AB^{-1}$. Then $F_{q^2}(D_0) = D_0$, and modulo K, $jc(j)^{-1} = (D_0, -F_q(D_0), 1) = (D_0, F_q(D_0), -1)$. The first pair conjugates Q by the projective class of A; the third sign acts by -1. Thus, changing ξ_+ to ξ_- changes λ to $-\lambda$. The even-parity subgroup is $\langle c^2\rangle$, with $(\lambda Q)^2 = \lambda^2$. The toral relative group in D_4 is $\langle c^2\rangle$. By [27, §§3–4] and [128, Proposition 7.3], $\lambda^2 \in \mu_4$ gives every eligible label.

Put $\mu = \alpha_0^\vee + \alpha_7^\vee + t^\vee$, with $\mu(-1) = z$. A regular-torus representative h has derived coordinate $\mu(\sqrt{b})$ with b a nonsquare and scalar coordinate c_0 with $F_q(c_0) = -c_0$. Moreover, it centralizes v, its action on $\mathrm{PSL}_2(q^2)$ centralizes Q, and its third factor exchanges ξ_+, ξ_-. Hence, it exchanges (x,λ) and $(x,-\lambda)$. Since $\ell \equiv 1 \bmod 8$, $\mathcal{H}_\ell$ fixes λ and has the rank-one sign $(a/r)^f$ on $\xi_\pm$. The parabolic on $\alpha_0 + \alpha_7$ in Proposition 7.181 has the full D_4-label λ^2. Restriction tests each pair by $\xi_\pm$ with multiplicity 1. Hence, the sign is $(a/r)^f$ for each λ^2. □

Proposition 7.223. *The* 16 *characters with labels (x,λ) in* Lemma 7.222 *have the comparison on $N_G(P)$. At even f, the twist at F_r is (a/r).*

Proof. Suppose f is even. Choose ξ_+ by the trace character η of $U_t \le \mathrm{SL}_2(q)$. Next, extend η with sign -1 on z and value 1 at F_r. Now, induce to $\mathrm{SL}_2(q)\langle F_r\rangle$. The multiplicity of ξ_+ at the base is 1, giving one extension $\widehat{\xi_+}$. Moreover, under σ, the additive character becomes $x \mapsto \eta(ax)$, corrected by the coroot at $\sqrt{a} \in \mathbb{F}_q^\times$. Its discrepancy under F_r is the central sign (a/r), and ξ_+ is faithful on z. Hence, the twist at F_r is (a/r).

On x, take the permutation F_r on $\mathbb{P}^1(\mathbb{F}_{q^2})$, with identity on $x = 1$. Moreover, its f-th power is Q. Multiply the tensor field operator by $\beta \in \mu_{8f_2}$ with $\beta^f = \lambda^{-1}$, where f_2 is the 2-part of f. The f-th power map on μ_{8f_2} has image μ_8. Since $\mathrm{ord}_\ell(q) = 8$, the order of $r \bmod \ell$ has 2-part $8f_2$, so $\mathcal{H}_\ell$ fixes β. The tensor operator A satisfies $A(\lambda Q) = (\lambda Q)A$ and $A^f = (\lambda Q)^{-1}$, since $U^f = c^{-1}$ modulo $\mathbf{S}^F$. Induction from $I\langle c, U\rangle$ gives the 16 extensions with twist (a/r).

Apply Proposition 7.181 with the parabolic on $\alpha_0 + \alpha_7$. Moreover, the corrections fix $\alpha_0 + \alpha_7$, and the outer factors give $1 + \mathrm{St}$ by q^2-Borel induction, with the twisted D_4-label still of multiplicity 1. In the split SO_{10} factor the sign spaces are $8^-, 2^-$. The hook argument requires row difference ± 2, so it covers all 4 toral symbols, from ordinary multiplicities at SO_{10} and lower ranks. At SO_4^+, each anisotropic plane contributes (a/r), so their tensor contributes 1. The t-root has nonsplit weight of $\xi_\pm$ with value -1 on z and contributes (a/r) with multiplicity 1. Moreover, both transfers have multiplicity 1 and give sign (a/r) at U_G. Both lifts project to F_r in the common quotient.

If $2 \nmid f$, choose $b \in \mathbb{F}_r^\times$ nonsquare and c_0 with $F_r(c_0) = -c_0$ of order $2(r-1)_2$. Moreover, the diagonal h centralizes the field, and h^2 is inner times c_0^2. Let ι act by 1 on μ_{2^∞} and $\zeta \mapsto \zeta^{-1}$ on odd-order roots. It fixes $\{\chi_+, \chi_-\}$ by the local calculation and [208, Proposition 11.20]. Replace ι by ιh if necessary to fix χ_+. On $\mathrm{Irr}(C_f)$ it inverts differences,

so one extension $\widehat{\chi}_+$ is fixed. The choice of $\widehat{\chi}_+$ is equivariant for coefficients and h, also with $U^f = c^{-1}$. At odd f, take a common character of $\langle z, c_0^2\rangle$ extending the value on z. At even f, choose $\widehat{\chi}_+$ and obtain $\widehat{\chi}_-$ by the diagonal δ. On $\mathrm{Z}(\widetilde{G})$, choose a common character ω with $\omega(z) = -1$ by Proposition 7.181. On $\langle z\rangle$, both operators have value $\omega(z)$ and field ratio ω^{F_r}/ω. Thus, the factor set α and functions μ_a coincide on the mixed stabilizer in $\succeq_c$. □

Lemma 7.224. *Suppose $q = r^f \equiv 1 \bmod 4$. With $G = E_{7,\mathrm{sc}}(q)$, we put $S = G/\mathrm{Z}(G)$ and $D = E_{7,\mathrm{ad}}(q)$. In both series, we have that $z \in \mathrm{Z}(G)$ acts trivially. Next, in D choose a split torus T and Borel $B = TU_0$, and put $M_D = N_D(T)$, $M_S = M_D \cap S$. Moreover, the reflection subsystem of s is $R = D_4A_1^2$ or $A_3^2A_1$. Let $\theta \in \mathrm{Irr}(T)$ correspond to either quartic lift s. Each constituent of $\mathrm{Ind}_B^D\,\theta$ has a rational extension over F_r. If $7 < \ell \mid q-1$, each local parent on M_D over θ has a rational extension over F_r. An extension restricts irreducibly to $S\langle F_r\rangle$ or $M_S\langle F_r\rangle$ if its parent restricts irreducibly to S or M_S.*

Proof. By [208, Proposition 11.20, Steps 3–4, and Proposition 11.22, Step 2], the principal constituents χ are rational and F_r-fixed. At odd f, we choose the real extension $\widehat{\chi}$, unique and rational by [143, Lemma 2.1 and Corollary 2.2]. Suppose f is even and put $\phi = F_r$. Next, on $V_\theta \oplus V_{\theta^{-1}}$, let $(Pv)(g) = v(\phi^{-1}(g))$, $\quad (Jv)(g) = \sum_{u\in U_0} v(n_0^{-1}ug)$. The invertible D-operator J exchanges $V_\theta, V_{\theta^{-1}}$. The coefficients of J are rational and F_r fixes n_0 and permutes U_0. Hence, J satisfies $JP = PJ$, $JC = CJ$ for coefficient conjugation C. The reflection algebra is $H_q(W(R))$ since $\theta(\alpha^\vee(t)) = 1$. Thus, apply [208, Proposition 4.15, Steps 1–3] for the reflection group $W(R)$. At $r \equiv 1 \bmod 4$, P acts on V_θ and gives one extension $\widehat{\chi}$ in each isotypic component V_χ, with its multiplicity by [208, Lemma 4.9(i)]. At $r \equiv -1 \bmod 4$, $n_0\phi$ fixes θ and $(n_0\phi)^f = 1$ in $D\langle\phi\rangle$. Give $n_0\phi$ value 1. On the χ-isotypic part of V_θ, the operator for ϕ is JP/q_χ by [208, Proposition 4.15, Step 3]. Specifically, q_χ is an integral power of $\sqrt{q} \in \mathbb{Q}$. The component is isotypic for $\widehat{\chi}$. The models and operators act over $\mathbb{Q}(i)$. Since χ is rational, the projector onto V_χ is rational. On V_χ, the invertible antilinear JC is D-linear and centralizes P or JP/q_χ. Thus, $\mathbb{Q}(\widehat{\chi}) \subseteq \mathbb{Q}(i) \cap \mathbb{R} = \mathbb{Q}$.

Suppose $7 < \ell \mid q-1$. The covering has kernel $\langle z\rangle$ of order 2, and $\ell \nmid |W(E_7)|$, so $P = T_\ell$ is Sylow in D and corresponds upstairs. A root α divisible by ℓ in the character lattice would give $\ell \mid \langle\alpha, \alpha^\vee\rangle = 2$. Thus, every root β has $\beta(T[\ell]) \neq 1$, and the connected centralizer of $T[\ell]$ is $\mathbf{T}$. Hence, $N_D(P) = M_D$. The same proof holds in S and G because the central 2-cover induces an isomorphism on ℓ-torsion.

The inertia quotient of θ in M_D is $W(R)$, by connectedness of $C(s)$. The coroot tori lie in $\ker\theta$. Thus, we extend θ by value 1 on subsystem Tits lifts. Tensor with $\eta \in \mathrm{Irr}(W(R))$ and induce to M_D. The labels η are rational, and n_0 corrects inversion of θ and fixes the Tits lifts n_α. Thus, the local parents χ are rational and F_r-fixed. At odd f, take its rational extension $\widehat{\chi}$ as above.

At even f, choose $Jv(g) = v(n_0^{-1}g)$ on the models induced from T, so $J^2 = 1$. Next, choose P or JP by the same congruences on r; its f-th power is 1. On an admissible $T\dot{w}T$, evaluation at $\dot{w}$ determines the kernel line. Both F_r and n_0 fix $\dot{w}$. Thus, P or JP centralizes the commutant $\mathcal{A}$. The component V_χ is isotypic for $\widehat{\chi}$, rational by the same JC argument. The nontrivial D/S-twist gives zs from s. Moreover, the conjugator between zs and s gives the based component action on $W(R)$ for both D, M_D. Each orbit of size 2 gives one singleton ρ. Its parent extension restricts to $\widehat{\rho}$. □

Proposition 7.225. *Suppose $q \equiv 1 \bmod 4$ and $7 < \ell \mid q-1$. The mixed comparison on $N_G(P)$ holds for 53 characters in the $D_4A_1^2$ series and 40 in the $A_3^2A_1$ series. The counts $(\#\text{singletons}, \#\text{pairs})$ are $(17, 18)$ and $(20, 10)$, respectively.*

Proof. For each singleton, choose $\widehat{\chi}$ by Lemma 7.224. Let ϑ be the pullback of θ to the torus in G. The bijection over ϑ in [166, Proposition 6.1] and Proposition 7.121 is equivariant for $\mathcal{H}_\ell, F_r, \delta$. Since $D = SB$ and the covering Borel maps onto $S \cap B$, $\mathrm{Ind}_B^D\,\theta$ restricts to $\mathrm{Ind}_{S\cap B}^S(\theta|_{S\cap B})$, with inflation to principal induction over ϑ. Singletons have δ-orbit length 1, and pair members have δ-orbit length 2. Thus, their images partition the eligible χ.

At even f, we have $8 \mid q-1$, so ϑ is trivial on $T[2]$ on the cover and so on K, the Tits kernel. Extend ϑ by value 1 on Tits lifts n_w. The relative quotient extends $W(R)$ by a based involution. If $R = D_4A_1^2$, this gives the graph transposition on D_4 and exchanges the two A_1-labels. A fixed D_4-label extends rationally in $W(B_4)$; the equal A_1-tensor extends by the flip. If $R = A_3^2A_1$, the two S_4-factors are exchanged with opposition w_0. After conjugation by (w_0, w_0), we obtain the flip τ. Moreover, a fixed tensor extends by $\pm\tau$, both rational. Each orbit of size 2 gives rational induction. The lift n_0 corrects inversion of ϑ, so each local χ' is rational. Equivariance gives $\chi^\sigma = \chi$, and [208, Lemma 11.14 and proofs of Propositions 11.20 and 11.22] gives field invariance of each χ. Restrict the parent extension $\widehat{\chi}$ from Lemma 7.224 to $S\langle F_r\rangle$ or $M_S\langle F_r\rangle$. The two base summands $\chi_\pm$ are distinct, invariant, and of

multiplicity 1. Thus, $\widehat{\chi}$ gives one extension of each $\chi_\pm$. Since $\widehat{\chi}$ is rational and $\chi_\pm$ are $\mathcal{H}_\ell$-fixed, so are their extensions $\widehat{\chi}_\pm$. Choose one orientation and transport by δ.

At odd f, we have $r \equiv 1 \bmod 4$, so F_r fixes ϑ. Take field pullback $\mathcal{F}$ on the modules for G and $N_G(P)$ over ϑ. Each kernel is determined at $\dot{w}$, fixed by F_r. Hence, pullback centralizes both commutants $\mathcal{A}_G, \mathcal{A}_M$, giving one extension per label. Thus, $\widehat{\chi}^\sigma = \widehat{\chi^\sigma}$ when $\vartheta^\sigma = \vartheta$. If $\vartheta^\sigma = \vartheta^{-1}$, correct by the longest intertwiner J after σ. On G, rank-one factorization gives J^{-1}, as does restriction from D. Also, locally J is left translation by n_0. Both models have $J\mathcal{F} = \mathcal{F}J$. Choose δ in the adjoint split torus over $\mathbb{F}_r$ with class $b \in \mathbb{F}_r^\times/\mathbb{F}_r^{\times 2}$ nontrivial. It stays nontrivial over $\mathbb{F}_q$ since f is odd. It also fixes B, ϑ, with $\delta\mathcal{F} = \mathcal{F}\delta$ in both modules V_ϑ. At each mixed stabilizer element a, the operators agree. The restrictions have z in their kernels, with the $(17, 18), (20, 10)$ orbit counts below. □

Proposition 7.226. *Let $G = E_{7,\mathrm{sc}}(q)$ with $q = r^f$ odd. Fix odd $\ell \mid |G|$ with $\ell \neq r$ and $P \in \mathrm{Syl}_\ell(G)$. Put $d = \mathrm{ord}_\ell(q)$ and $\varepsilon \in \{1, -1\}$ with $q \equiv \varepsilon \bmod 4$. In each geometric quartic class $\mathcal{C}$, the rational lift has z-value 1 and the other form has z-value -1. Specifically, for $\bar{s}$ of* Zhang [208, Proposition 11.20]*, the counts are for ℓ'-degree characters, with mixed comparison on $N_G(P)$ in the sense of $\succeq_c$.*

$D_4A_1^2$	$\ell > 7,\ \ell \mid q - \varepsilon$	$d = 4$	$d = 8$
rational lift	*53*	*38*	*0*
faithful form	*0*	*40*	*16*

At all other odd $\ell \neq r$, both counts are 0.

Proof. Write $P_j = q^j - 1$. The numerator away from r is $P_2P_6P_8P_{10}P_{12}P_{14}P_{18}$. By [208, proof of Proposition 11.20, Step 1], the denominators away from $2, r$ come from $(q - \varepsilon)P_2^3P_4^2P_6$ and $(q + \varepsilon)P_2P_4^2P_6(q^4 + 1)$, respectively. Full cancellation holds at $q - \varepsilon, \Phi_4$ in the first row and Φ_4, Φ_8 in the second. At $q = \varepsilon$ the cancelled index has absolute value 3780, excluding scalar $\ell = 3, 5, 7$. At $d = 4, 8$ we have $d\ell > 18$. Thus, the index lies in $\mathbb{Z}_\ell^\times$ exactly in the indicated cases.

In D_4, take the series of a Sylow Φ_d-torus $\mathbf{A}_d$. By [27, §§3–4] and [128, Proposition 7.3], the labels are $\mathrm{Irr}(W_d)$. We have $|W_d|_{\{2,3\}} = |W_d|$, and residual A_1-sources have degrees powers of q. Conversely, a remaining factor Φ_d makes the degree divisible by ℓ since the denominators are 2-powers. Both outer labels $1, \mathrm{St}$ survive. For the rational lift, let h be the number of eligible D_4-labels and j the number fixed by w, the graph involution. At $d = 1, 2$, the relative group is $W(D_4)$. By [208, proof of Proposition 11.20, Step 2], the principal series has 9 graph-fixed labels and 4 labels in exchanged pairs. Thus, $(h, j) = (13, 9)$. If $d = 4$, the two negative 2-cycles give $G(4, 2, 2)$ of even parity in $C_4 \wr S_2$. The latter has 8 extensions from equal colours and 6 inductions from colours $a \neq b$. Sign adds 2 to both colours, fixing $\{0, 2\}$ and $\{1, 3\}$. After restriction to $G(4, 2, 2)$, $(h, j) = (10, 6)$.

By [208, proof of Proposition 11.20, Step 1], the component acts by w on D_4 and exchanges the outer A_1's. Also, among the $4h$ triples, $2j$ split into $4j$ paired members. The orbits of size 2 give $2h - j$ singletons. The totals are $53, 38$ at scalar order and $d = 4$, respectively. At $\varepsilon = 1$ see Proposition 7.225, and at $\varepsilon = -1$ see Proposition 7.142. If $d = 4$, apply Proposition 7.178 to the lift with z-value 1.

In the faithful form at $d = 4$, a negative 2-cycle leaves $D_2 = A_1^2$. Next, correct the quarter-turn c by inversion on a perpendicular coordinate e_i. The lift in $W(D_4)$ exchanges the two residual A_1's. Equal labels have inertia C_4, while $(1, \mathrm{St})$ has inertia C_2, giving $4 + 4 + 2 = 10$ labels. At $d = 8$, Lemma 7.222 gives relative group C_4 in $W(D_4)$. Hence, there are 4 labels λ, each fixed by the global component w. Each outer label on $\mathrm{SL}_2(q^2)$ gives a restriction pair, yielding $40, 16$. Propositions 7.181 and 7.223 give the comparisons at $d = 4, 8$, respectively. □

Proposition 7.227. *Specifically, for $\bar{s}$ of* Zhang [208, Proposition 11.22]*, the mixed comparison on $N_G(P)$ holds for the ℓ'-degree characters counted below.*

$A_3^2A_1$	$\ell > 7,\ \ell \mid q - \varepsilon$	$d = 4$	$d = 8$
rational lift	*40*	*28*	*0*
faithful form	*0*	*20*	*16*

At all other odd $\ell \neq r$, both counts are 0.

Proof. Set $X = \varepsilon q$. Moreover, the rational-lift denominator is $(X^2 - 1)^3(X^3 - 1)^2(X^4 - 1)^2$. The numerator is invariant under $q \mapsto X$ since its degrees are even. Precisely $\Phi_1(X), \Phi_4(X)$ attain full multiplicity. At $X = 1 \bmod \ell$,

the cancelled index at $X = 1$ has absolute value $|W(E_7)|/(4!^2\,2) = 2520$, excluding $\ell = 3, 5, 7$. If $d = 4$ we have $4\ell > 18$.

By [208, Proposition 11.22(ii) and its proof], the rational lift has 40 characters on G. At scalar order, all 5 partitions occur on each A_3, so all 40 are eligible. At $d = 4$, the hook criterion gives 4 labels per A_3. Since the component acts by $(\lambda, \nu) \mapsto (\nu, \lambda)$, unequal pairs give $2\binom{4}{2} = 12$ characters and the equal pairs give 8 diagonal pairs. Thus, $12 + 16 = 28$. Next, apply Propositions 7.225 and 7.143 at $\varepsilon = 1, -1$, respectively, and Proposition 7.185 at $d = 4$.

For the faithful form, [208, proof of Proposition 11.22, Step 1] gives $\mathrm{SL}_4(q^2)$, $\mathrm{SL}_2(q)$ and denominator $P_2P_4P_6P_8$ away from $2, r$. Only Φ_4, Φ_8 attain full multiplicity, and $d\ell > 18$ at both orders. If $d = 4$, $\mathrm{ord}_\ell(q^2) = 2$. Every $\lambda \vdash 4$ has empty 2-core: the only even staircase size ≤ 4 is 0. If $d = 8$, $\mathrm{ord}_\ell(q^2) = 4$ leaves the four hook partitions. Each λ has two A_1-labels and each gives a pair $\{\chi_+, \chi_-\}$, with totals $20, 16$. Compare by Propositions 7.185 and 7.187. □

Corollary 7.228. *Fix q, ℓ, P. Moreover, the series for $D_4A_1^2, A_3^2A_1$ in* Propositions 7.226 *and* 7.227 *give disjoint $X_s \subseteq \mathrm{Irr}_{\ell'}(N_G(P))$. Moreover, their union has size* 93 *at the indicated scalar primes,* 126 *at $d = 4$, and* 32 *at $d = 8$.*

Proof. At $d = 1, 2$, choose a maximal torus of type ε. At $d = 4, 8$, we choose a Sylow Φ_d-torus $\mathbf{S}$. Such $\mathbf{S}$ are rationally conjugate with $(\mathbf{S}^F)_\ell \in \mathrm{Syl}_\ell(G)$. Conjugate that subgroup to P. Also, the choices differ by $M = N_G(P)$, which acts trivially on $\mathrm{Irr}(M)$. At $d = 1, 2$, both contributions have sign 1 on z. Each restriction to $T \lhd M$ lies over the orbit $W\theta$ of its covering weight θ. Conjugacy of weights implies conjugacy of s. Hence, the connected centralizers in $\mathbf{G}^*$ are conjugate. The types $D_4A_1^2, A_3^2A_1$ differ, so the two orbits on $\mathrm{Irr}(T)$ are disjoint. The value on z distinguishes the forms at each d. If $d \in \{4, 8\}$, we write $T = \mathbf{S}^F$ for the common primitive torus, $L = C_{\mathbf{G}}(\mathbf{S})^F$, and $M = N_G(P)$. At $d = 4$, the rational $D_4A_1^2$-characters are trivial on T by Lemma 7.177. The rational $A_3^2A_1$-sources have nontrivial quadratic weight on T by Lemma 7.183. Since $T \lhd M$, that weight has nontrivial M-conjugates. Thus, the images differ on T at central sign 1. On L, the constituents of an irreducible of M form a single M-orbit, with conjugate inertias. The faithful $D_4A_1^2$-inertia quotient over L is abelian by Lemma 7.179. For $A_3^2A_1$ it is $D_8 \circ C_4$ by Lemma 7.182, nonabelian since the embedded D_8 has commutator subgroup of order 2. Thus, the source orbits in $\mathrm{Irr}(L)$ are disjoint. At $d = 8$ only the faithful forms occur. The $D_4A_1^2$-characters factor through M/T by Lemmas 7.221 and 7.222. The $A_3^2A_1$-weight κ takes value -1 on the involution of T by Proposition 7.187. Hence, $\kappa \neq 1_T$ separates the two sets of 16. The column totals are $93, 126, 32$. □

Corollary 7.229. *Let $1 \neq s \in G^*$ be rational and quasi-isolated, with $r \nmid |s|$. Moreover, the ℓ'-degree characters of $\mathcal{E}(G, s)$ admit the series-wise comparison on $N_G(P)$ at every odd $\ell \neq r$ dividing $|G|$.*

Proof. The types for $|s| = 2$ in [208, §11.4, opening paragraph] are A_1D_6, E_6, A_7, with 5 forms covered by Corollary 7.214. At $|s| = 4$, the types $D_4A_1^2, A_3^2A_1$ each have 2 rational forms $\bar{s}$, covered by Propositions 7.226 and 7.227. Their images X_s are disjoint by Corollary 7.228. Given $r \geq 5$, A_5A_2 occurs at $|s| = 3$ and A_2^3 at $|s| = 6$. At $|s| = 3$, see Proposition 7.217. At $|s| = 6$, Proposition 7.220 gives count 0 at permutation w and the comparison for permutation 1. If $r = 3$, semisimplicity gives $3 \nmid |s|$. Each comparison takes the local image X_s. For $\bigcup_s X_s$, see Corollary 7.257 for disjointness of X_s. □

Proposition 7.230. *In* Proposition 7.172*, we suppose $y \neq 1$ and $\mathrm{C}(y) = \mathbf{K}^*$ has type D_4A_1. The cyclic comparison on the twisted D_4 and signed induction in G give the mixed comparison above y.*

Proof. Put $X = \mathrm{Spin}_8^-(q)$, $S = \mathbf{S}^F$ for its primitive torus, $P_X = S_\ell$, and $L_X = C_X(S)$. Next, we let F_X denote the bijection of Proposition 6.148 on X, constructed in Proposition 2.9. Fix $x \neq 1$ as the primitive π-part. Every aligned nonidentity π-part x in the dual lies in $\mathbf{S}^*$, since $\mathrm{SO}_4^+(q)$ has π'-order. Given any prime $v \mid |x|$, its order-v power x_v has eigenvalues a, a^q, a^{-1}, a^{-q} on the negative pair, distinct since $\mathrm{ord}_v(q) = 4$. It fixes the residual space V_0 of dimension 4. Isometries h with $hx_v = x_vh$ have reciprocal eigenvalues on the x_v-eigenspaces. Hence, h has determinant 1 off V_0 and on V_0. Under a central 2-isogeny π_0, x_v has a unique lift of order v, so its centralizer lifts. This gives $C_{\mathbf{X}^*}(x_v) = \mathbf{L}_X^*$. Since $x \in \mathrm{Z}(\mathbf{L}_X^*)$, also $C_{\mathbf{X}^*}(x) = \mathbf{L}_X^*$.

Let θ_x be the weight of S_π dual to x. Moreover, the isogeny $\mathbf{S} \times [\mathbf{L}_X, \mathbf{L}_X] \to \mathbf{L}_X$ has 2-primary kernel and rational cokernel. On S_π finite-torus duality gives θ_x. The primitive normalizer gives C_4, generated by a negative 2-cycle. Correct the quarter-turn c by a sign on V_0, then lift by Lang in $\mathbf{S}$. The C_4-action on S_π is by $1, q, -1, -q$ up to orientation, distinct modulo each $v \mid |x|$. Thus, θ_x has inertia L_X in $N_X(P_X)$. For each source τ of L_X above θ_x,

the full centralizer $C_{\mathbf{X}^*}(x) = \mathbf{L}_X^*$ gives $\pm R_{L_X}^X(\tau) \in \mathrm{Irr}(X)$. By Clifford induction, we obtain $\mathrm{Ind}_{L_X}^{N_X(P_X)}(\tau)$. Both are bijective modulo conjugacy of x.

Write $q = r^f$ and $F = \gamma\phi^f$, with diagram involution γ and pinned field map ϕ. Thus, $\phi^f = \gamma$ on rational points. In Lemma 7.4 take f positions with the wrap by γ. At a fixed position i, the ordinary stabilizer acts by $\tau\gamma^e$, with inner-diagonal τ and $e = 0, 1$. Moreover, it fixes $\mathbf{L}_X, \mathbf{S}, \theta_x$, and its action on $Y(\mathbf{S})$ is $\mathbb{Z}[i]$-linear for i acting by F/r^f. The lattice $Y(\mathbf{S})$ is free of rank 1 over the PID $\mathbb{Z}[i]$. Its $\mathbb{Z}[i]$-linear automorphisms are $\{\pm 1, \pm i\}$, acting by $1, q, -1, -q$ on S_π. Only 1 fixes θ_x, so the position stabilizer fixes $Y(\mathbf{S})$ pointwise. Choose $u \in Y(\mathbf{S})$ outside the complementary root hyperplanes H_α. Its parabolic $\mathbf{Q}_u$ has Levi $\mathbf{L}_X$ and is fixed by the i-stabilizer. By Lemma 7.4, even at exponent $f \bmod 2f$, this gives the factor set α, central scalars and the functions μ_a. On $N_X(P_X)$, Clifford induction gives α, μ_a and the same scalars. This defines f_X onto the characters of $N_X(P_X)$ nontrivial on S_π. The condition $x \neq 1$ is invariant under the transports of x. The condition $\theta_x \neq 1$ is also invariant on S_π. Thus, F_X, f_X meet the hypotheses of Lemma 2.12 for $X, N_X(P_X)$ with the marked automorphisms of X. This compares the complements, including $|P_X| = 5$. Moreover, the mixed stabilizer of χ is common even at $F_X(\chi) \in \mathrm{im}(f_X)$ for $x = 1$.

Tensor with the identity on A_1 and descend through Z_c with common central scalars. Next, in $K = \mathbf{K}^F$, we let H be the image of $X \times \mathrm{SL}_2(q)$ and $Z_0 = \mathrm{Z}(\mathbf{K})^{\circ F}$. Then, K/HZ_0 is a 2-group because the covering kernel Z_c is a 2-group. In particular, $P = (P \cap H)(Z_0)_\ell$. In the corrected transport group A of the source Levi, $C_A(H)$ centralizes $P \cap H$ and normalizes $(Z_0)_\ell$, so $C_A(H) \leq N_A(P)$. By Sylow conjugacy in H, $A = HN_A(P)$. Next, we restrict F_X to the induced action on X. Both overgroups have one image in $\mathrm{Aut}(X)$, as in [141, Theorem 2.9]. Apply Proposition 7.6 to $K, N_K(P)$ with the common quotient of $K, N_K(P)$.

Next, before the twist impose scalar 1 on $(Z_0)_\pi$. In $\mathbf{D}^F$ and $\mathbf{D}^{*F}$, the odd parts decompose by the 2-isogenies from Z° times the derived group. Thus, scalar 1 on $(Z_0)_\pi$ and the complement comparison select $x = 1$ globally and weight 1 on $\mathbf{D}_\pi^F$ locally. Restore κ_y by Lemma 7.7. Its inertia in M/L is the relative stabilizer of y by finite-torus duality, of order 4 by Proposition 7.172, with rational representatives in $N_K(P)$ from the negative 2-cycle of X. Given $C_{\mathbf{G}^*}(s_\pi)$, Corollary 7.3 gives a correction back to $\mathbf{L}$ under ordinary and mixed transports of s. Clifford induction to M and signed induction to G give the comparison above y. With Proposition 7.172 for regular y, this covers $y \neq 1$. □

Lemma 7.231. *Suppose $y = 1$ and exactly one u_i satisfies $u_i^2 \neq 1$. In the natural D_6-space, take sign multiplicities $1, 5$, or a sign of multiplicity 4 and the pair $i, -i$ of multiplicity 2. Then, we have that $C_{\mathbf{G}^*}(s)$ is a rational Levi $\mathbf{C}^*$ of type D_5 or D_4A_1, respectively. If $d = 4$, we have that $\mathcal{E}(G, s) \cap \mathrm{Irr}_{\ell'}(G)$ has the central mixed comparison on $N_G(P)$.*

Proof. By Proposition 7.172, $C_{\mathbf{G}^*}(s) \leq \mathbf{K}^*$, the Levi perpendicular to α, the moving root. In the E_8-model take E_7 perpendicular to $e_7 + e_8$, and moving root $e_7 - e_8$. The projection of $Q(E_7)$ onto the first six coordinates contains $Q(D_6)$. Its other projected roots are $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_6)$ with odd minus parity, all in $\omega_{\mathrm{spin}} + Q(D_6)$. Hence, $\mathrm{Spin}_{12} \to \mathbf{K}^*_{\mathrm{der}}$ has kernel of order 2, with generator mapping to $-I$ on the natural space since e_1 does not descend.

Lift s over $\overline{\mathbb{F}}_q$ to $\mathrm{Z}(\mathbf{K}^*)^\circ \times \mathrm{Spin}_{12}$. A commutator in $Z_c = \ker(\mathrm{Z}(\mathbf{K}^*)^\circ \times \mathrm{Spin}_{12} \to \mathbf{K}^*)$ has first coordinate 1. A nontrivial commutator gives $t_s \mapsto -t_s$, impossible by the unequal multiplicities of $+1, -1$. Thus, $C_{\mathbf{K}^*}(s)$ is the image of $C(\widetilde{s})^\circ$ by Steinberg, where $\widetilde{s}$ is the chosen lift. In row $(1, 5)$, the neutral roots are D_5 on the complement of $\mathbb{Q}e_j$ for the singleton coordinate j. In the quartic row orient coordinates $5, 6$ to have value i. The neutral roots are D_4 on the first four coordinates and $\pm(e_5 - e_6)$, precisely the roots perpendicular to $e_5 + e_6$ in D_6. A lattice multiple of e_j or $e_5 + e_6$ gives a Levi in $\mathbf{K}^*$ and so of $\mathbf{G}^*$. It is rational since it is $\mathbf{C}^* = C_{\mathbf{G}^*}(s)$.

Let $\mathbf{C}$ be the dual Levi containing $\mathbf{D}$, and put $\mathbf{L}_0 = \mathbf{C} \cap \mathbf{L}$, $J_0 = N_C(\mathbf{D})$. Both negative 2-cycles remain: in row $(1, 5)$, the singleton coordinate is 5 or 6. Thus, $\mathbf{D}$ carries $P \in \mathrm{Syl}_\ell(C)$ and $J_0 = N_C(P)$. Apply [156, Theorem D and its proof in §8] to the D_5 or D_4 unipotent series. On the additional factor $A_1(q)$, the comparison is the identity since $\ell \nmid |A_1(q)|$. Central isotypy gives irreducible pullbacks on $C, \mathbf{L}_0^F$ of the adjoint characters $\chi_{\mathrm{ad}}, \tau_{\mathrm{ad}}$. Moreover, it gives the same absolute W_{rel} and its F-action. As $C_{\mathbf{C}}(\mathbf{D})$ is connected, [133, Proposition 23.2] gives lifts of W_{rel}^F. Thus, the inertia quotients agree on W_{rel}^F, with the same Clifford labels η. Restrict the comparison on H_{ad} to the induced automorphisms.

Twist by κ_s on C dual to the central s, via Lemma 7.7. Signed induction from C to G and from $\mathbf{L}_0^F$ to L gives irreducibles since $C_{\mathbf{L}^*}(s) = \mathbf{C}^* \cap \mathbf{L}^*$ is the marked torus centralizer in $\mathbf{C}^*$. The L-series stabilizer in M/L is the image of $J_0/\mathbf{L}_0^F$. In fact a returning Weyl element w lifts into $\mathbf{C}^*$ after rational $\mathbf{L}^*$-conjugacy so it fixes s. Conversely, s is central in $\mathbf{C}^*$. The stabilizers for each χ coincide by injectivity of $R_{\mathbf{L}_0}^{\mathbf{L}}$. At a fixed position i, the inner-diagonal corrector has representative $g_i \in \mathbf{C}^*$. Hence, it fixes $Y(\mathrm{Z}(\mathbf{C}^*)^\circ)$ pointwise. A generic cocharacter u there gives

parabolics for the Levis $\mathbf{C}, \mathbf{L}_0$. Transport the chamber of u around the positions $0, \dots, f-1$. For a in the mixed stabilizer, the Galois power of s generates $\langle s \rangle$, so fixes the Levi $\mathbf{C}^*$. Apply Lemma 7.168 for $R_C^G, R_{\mathbf{L}_0^F}^L$. Clifford induction from the inertia in M gives the comparison of Proposition 7.172 above s. □

Lemma 7.232. *Next, for* $I(2,4), I(3,3), Q(2,2)$ *after* Lemma 7.231*, the factors of* $C_{\mathbf{G}^*}(s)^\circ$ *are as follows. Moreover, in* $n^\pm$*, the sign gives the quadratic type.*

row	$F(t)/t$	*rational sign spaces*
$I(2,4)$	$+I$	$4^+, 8^+$ *or* $4^-, 8^-$
$I(3,3)$	$+I$	$6^+, 6^+$ *or* $6^-, 6^-$
$I(3,3)$	$-I$	6^+ *over* $\mathbb{F}_{q^2}$
$Q(2,2)$	$+I$	$4^-, 4^-$
$Q(2,2)$	$-I$	4^+ *over* $\mathbb{F}_{q^2}$

t is the natural lift of s, with $F(t) = ct$, $c = \pm 1$. In $Q(2,2)$ there is an additional factor $\mathrm{GL}_2(q)$ *if* $q \equiv c \bmod 4$*, and* $\mathrm{GU}_2(q)$ *otherwise.*

Proof. The mark on D_6 has two negative 2-cycles and a residual space 4^+ because the ambient 12-space is split. At $c = 1$, each cycle has constant sign in t. In $I(2,4)$, the smaller space is either residual 4^+ or a negative cycle 4^-. The product of types is $+$, so 8 has the same sign as 4. In $I(3,3)$, each space is $4^- \perp V_i$ for a residual plane V_i. The planes V_1, V_2 have equal type. Moreover, in $Q(2,2)$, the $i, -i$ coordinates are residual, and $V_\pm$ both have type 4^- from one negative cycle.

If $c = -1$, F exchanges V_+, V_- of equal dimension. On V_+, F^2 acts by two negative singletons from c_1, c_2. In $I(3,3)$, the residual positive 2-cycle on 4^+ gives one positive singleton. Thus, F^2 has sign $+1$, giving split type over $\mathbb{F}_{q^2}$. On the natural eigenspaces F takes the i-space to the ci^q-space. This gives the type-A factor, of ℓ'-order by $\mathrm{ord}_\ell(q) = 4$.

Given $I(2,4)$, the test in Lemma 7.231 gives connected $C_{\mathbf{G}^*}(s)$ since the multiplicities $2, 4$ differ. Its neutral $D_2 D_4$ spans D_6, so its cocharacter annihilator is 0. On $4^+, 4^-$ there are $4, 2$ unipotent labels, respectively, all of ℓ'-degree. On 8^+ the quartic torus T_d is maximal. By [130, proof of Proposition 4.19], $W_d = G(4,2,2)$ in $C_4 \wr S_2$. Its rotation subgroup has order 8 and has 4 characters fixed by the flip, giving $2 \cdot 4 + (8-4)/2 = 10$ irreducibles. On 8^- the residual space 4^+ accompanies one negative cycle. Equal pairs $(1,1), (\mathrm{St}, \mathrm{St})$ have inertia C_4, and the orbit of $(1, \mathrm{St})$ has inertia C_2 since odd rotation flips $(1, \mathrm{St})$. This gives $2 \cdot 4 + 2 = 10$. By [156, §4 and Theorem D], these count the unipotent ℓ'-degree characters. The relative groups have ℓ'-order, with $P^* \le C_{\mathbf{G}^*}(s)^F$ for a dual Sylow subgroup P^*. By [58, Proposition 11.5.6], the Jordan degree multiplier is $[G^* : C_{G^*}(s)]_{r'}$, an ℓ-unit. Thus, the counts at $(4^+, 8^+), (4^-, 8^-)$ are $4 \cdot 10 = 40$ and $2 \cdot 10 = 20$.

Given $I(3,3)$ with $c = 1$, the hook formula gives 4 hooks of size 4 per factor, since $\mathrm{ord}_\ell(\pm q) = 4$. If $c = -1$, the split A_3 over $\mathbb{F}_{q^2}$ has cyclotomic order $2 = \mathrm{ord}_\ell(q^2)$. All 5 partitions $\lambda \vdash 4$ have empty 2-core and qualify. In $Q(2,2)$, each 4^- gives $1, \mathrm{St}$ of $A_1(q^2)$, while 4^+ over $\mathbb{F}_{q^2}$ gives two independent A_1-labels. Moreover, these counts concern $C_{\mathbf{G}^*}(s)^\circ$ and precede the outer actions on $\mathrm{Irr}(C_{\mathbf{G}^*}(s)^\circ)$. □

Lemma 7.233. *For* $I(2,4)$ *in* Lemma 7.232*, let* $\mathbf{K}$ *be the single-leg Levi and* $\mathbf{L}_K = C_{\mathbf{K}}(\mathbf{D})$*. In* $N_K(P)$*, the characters over the* $N_K(P)$*-orbit of* $\mathcal{E}(\mathbf{L}_K^F, s)$ *have* 40 *labels at* $(4^+, 8^+)$ *and* 20 *at* $(4^-, 8^-)$*.*

Proof. The two residual coordinates of t have equal signs since the 4-space is either residual 4^+ or an entire cycle 4^-. Thus, the residual A_1^2 lies in the neutral roots Ψ. Let $\mathbf{H} \le \mathbf{K}$ be generated by the marked torus $\mathbf{T}$ and root groups dual to $C_{\mathbf{K}^*}(s)$. The subsystem $\Psi \subset D_6$ is closed. Thus, $\mathbf{H}$ is rational with $\mathbf{H}^* = C_{\mathbf{K}^*}(s)$, the same marked torus $\mathbf{T}$, and $\mathbf{L}_K \le \mathbf{H}$. The central s gives a linear character κ_s on $H = \mathbf{H}^F$, and finite-torus duality gives s for its restriction to $\mathbf{L}_K^F$. Moreover, the four sources are $\kappa_s|_{\mathbf{L}_K^F}$ times the tensors of $1, \mathrm{St}$ on the residual A_1^2. The series stabilizer in $N_K(P)$ is $N_H(\mathbf{D})$. A returning dual coset has representative g fixing s after $\mathbf{L}_K^*$-conjugacy. Hence, $g \in \mathbf{H}^*$. Duality matches the relative cosets w, with rational lifts by Lang.

At $(4^+, 8^+)$ both active cycles lie in 8^+, with relative group $G(4,2,2)$ by [130, proof of Proposition 4.19]. Its image on 4^+ lies in $\mathrm{Inndiag}(A_1^2)$. Thus, every source has inertia quotient $G(4,2,2)$. At $(4^-, 8^-)$, the cycle in 4^- admits only even rotations by D_2-parity. The cycle in 8^- admits all powers of c. Moreover, at odd powers of c, the correction on

4^+ has determinant -1. Thus, the quotient $C_2 \times C_4$ acts by the C_4-generator exchanging the A_1's. Equal labels have inertia $C_2 \times C_4$, and $(1, \mathrm{St})$ represents the unequal orbit with inertia $C_2 \times C_2$.

Next, extend the unipotent factor ρ to I via its flag action, with constants V_1 and augmentation space V_{St}. Equal factors admit the flip τ. The model factors through $\mathrm{Aut}(A_1^2)$. Next, tensor with $\kappa_s|_I$ on the inertia I, giving the exact source extension $\widehat{\tau}$. By [95, Theorem 6.11 and Corollary 6.17], over the orbit of $\kappa_s|_{\mathbf{L}_K^F}\rho$ the characters are $\mathrm{Ind}_I^{N_K(P)}(\widehat{\tau}\eta)$ with $\eta \in \mathrm{Irr}(I/\mathbf{L}_K^F)$. Thus, there are $4 \cdot 10 = 40$ or $2 \cdot 8 + 4 = 20$ labels, respectively. The indices $[N_K(P) : I]$ and degrees $\eta(1)$ are prime to ℓ, giving the stated labels in $\mathrm{Irr}_{\ell'}(N_K(P))$. □

Proposition 7.234. *Take the single-leg Levi $\mathbf{K}^*$ of* Lemma 7.232. *Next, we let $F(t) = ct$ for the natural lift t, and $\mathbf{C}^* = C_{\mathbf{G}^*}(s)$. At $I(3,3), Q(2,2)$, the rational component group has order* 2. *Moreover, at $I(3,3)$, the ℓ'-degree count in $\mathcal{E}(G,s)$ is* 14 *for $c = 1$ and* 10 *for $c = -1$. At $Q(2,2)$, the count per type-A partition is* 8 *if $c = -1$ and $n_5 = n_6$, and* 5 *otherwise. Write the six torus entries i^{n_j} for i of order* 4. *At $c = -1$ orient the two cycles so that $(n_1, n_2, n_3, n_4) = (0, 2, 0, 2)$. In $Q(2,2)$, we compare the odd n_5, n_6 modulo* 4 *in these coordinates.*

Proof. First, let R_1, R_2 act by $(x, y) \mapsto (-y, x)$ on the active planes V_1, V_2. In the lattice of Lemma 7.231, t lifts to $n/4$ modulo $Q^\vee(D_6) = \{a \in \mathbb{Z}^6 : \sum_j a_j \in 2\mathbb{Z}\}$. Moreover, the derived dual kernel is $\langle v \rangle$ for $v = (1/2, \ldots, 1/2)$. Its pairings satisfy $\langle \omega_{\mathrm{spin}}, v \rangle \in \mathbb{Z}$ and $\langle e_1, v \rangle = 1/2$. Every $w \in W(D_6)$ fixes the central cocharacter. Thus, w fixes s if $w(n/4) - n/4$ is 0 or v mod $Q^\vee(D_6)$. Suppose $wn \equiv n + 2b\mathbf{1} \bmod 4$, $b = 0, 1$. The condition $wn - n - 2b\mathbf{1} \in 4Q^\vee(D_6)$ is $\sum_{j:\epsilon_j=-1} n_j \equiv 2b \bmod 4$, where $w(e_j) = \epsilon_j e_{\pi(j)}$. At $b = 0$, Steinberg gives connected centralizer in Spin_{12}. If $b = 1$, the component exchanges V_+, V_-, and $C_{\mathbf{G}^*}(s) \leq \mathbf{K}^*$ excludes further components.

At $c = 1$, orient the active exponents as $(0, 0, 2, 2)$. Let j exchange V_1, V_2 without rotation. Given $I(3,3)$, combine j with $e_5 \mapsto -e_6,\ e_6 \mapsto -e_5$, so $n_5 + n_6 = 2 \bmod 4$. Given $Q(2,2)$, combine j with inversion $-I_2$ on the residual axes. If $n_5 + n_6 \equiv 0 \bmod 4$, include R_1R_2, adding 2 mod 4 to the exponent sum A. Each operation v satisfies $vw_F = w_F v$, whose residual signs on e_5, e_6 coincide in $I(3,3)$. In $Q(2,2)$, $-I_2$ is scalar on the residual $\langle e_5, e_6 \rangle$. Consequently, $[-I_2, w_F|_{\langle e_5, e_6 \rangle}] = 1$. Thus, Lang gives rational lifts. Given $I(3,3)$, the component exchanges the hook lists (λ_+, λ_-). Graph automorphisms fix each unipotent A_3-partition. Thus, the 6 unequal orbits and 4 fixed labels give $6 + 2 \cdot 4 = 14$. Given $Q(2,2)$, the lists are $\{1, \mathrm{St}\}$, and duality fixes the partition on $\mathrm{GL}_2(q)$ or $\mathrm{GU}_2(q)$. The orbit $\{(1, \mathrm{St}), (\mathrm{St}, 1)\}$ and two fixed labels give $1 + 2 \cdot 2 = 5$.

If $c = -1$, reach $(0, 2, 0, 2)$ by powers of R_1, R_2 and a parity correction on e_5, and conjugate w_F with the coordinates. In $I(3,3)$, combine R_1R_2 with $e_5 \mapsto -e_6,\ e_6 \mapsto -e_5$. It centralizes $(5\,6)$ on $\langle e_5, e_6 \rangle$ and passes the test for $b = 1$. Moreover, on $A_3(\mathbb{F}_{q^2})$ it acts as F_q up to $\mathrm{Inndiag}(A_3)$ and the graph action. The partitions $\lambda \vdash 4$ are fixed, giving $2 \cdot 5 = 10$.

Given $Q(2,2)$, we choose the rational component representative $h = (R_1^{1+2e}R_2, -I_2)$, where $e = 0$ for $n_5 = n_6$ and $e = 1$ for $n_5 \neq n_6$. At $e = 0$ the active signs change at exponents $2, 2$, with residual contribution $n_5 + n_6$. The extra half-turn adds 2 mod 4, so h passes the test for both e. It satisfies $h\, w_F = w_F\, h$, with active square $R_1^2R_2^2$. Moreover, F exchanges V_+, V_-. Both h and F exchange the two axes of V_j. The discrepancy after F_q is R_1^{2e} on active coordinates. Moreover, on V_+ it gives $e \mapsto -e$ axes. At $e = 0$ the action on A_1^2 is inner-diagonal times F_q. At $e = 1$ the graph part exchanges the A_1-labels. Field fixes $1, \mathrm{St}$ on each factor. If $e = 0$, all 4 labels are fixed, giving 8. At $e = 1$, the two labels $(1,1), (\mathrm{St}, \mathrm{St})$ are fixed, so the total is $2 \cdot 2 + 1 = 5$.

The kernel of $\widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ is a torus. By [114, corollary to Theorem 1], s has a rational lift $\widetilde{s}$. The displayed representatives give the C_2-action by translates on $\mathcal{E}(\widetilde{G}, \widetilde{s})$. By [72, §2.6, especially 2.6.20–2.6.23], $\widetilde{\chi}|_G$ is multiplicity-free with squared norm $|(C_2)_{\widetilde{\chi}}| \in \{1, 2\}$. Thus, fixed labels give 2 constituents and size-2 orbits give 1, distinct on C_2-orbits. The indices and multipliers are ℓ'-numbers. Lemma 7.232 gives the labels on $C_{\mathbf{G}^*}(s)^\circ$. □

Proposition 7.235. *With $N = N_K(\mathbf{D})$, we put $L_K = C_K(\mathbf{D})$. For s in* Proposition 7.234, *we have that the L_K-series has one character at $I(3,3)$, and one per type-A partition at $Q(2,2)$. For such β, put $W_s = N_\beta/L_K$ and $e = [n_5 \neq n_6]$. At $I(3,3)$, $W_s = C_4 \wr S_2$ if $c = 1$ and $W_s = D_8 \circ C_4$ if $c = -1$. At $Q(2,2)$, $W_s = C_4 \times C_2$ if $(c, e) = (-1, 0)$ and $W_s = D_8$ otherwise. The group D_8 has order* 8, *and the central product $\circ$ equates the central involutions z. Induction to N gives the counts of* Proposition 7.234 *for β.*

Proof. The nonidentity component exchanges V_+, V_- and so acts nontrivially on $\mathbf{D}^*$. Therefore, $C_{\mathbf{L}_K^*}(s) = C_{(\mathbf{C}^*)^\circ}(\mathbf{D}^*)$. The derived type is trivial at $I(3,3)$ and A_1 on the quartic pair at $Q(2,2)$. Jordan decomposition gives the stated L_K-series. Each relative stabilizer fixes λ on the residual factor, since $1(1) \neq \mathrm{St}(1)$. Moreover, rational

correction in $\mathbf{L}_K^*$ gives the image of $N_{\mathbf{C}^*}(\mathbf{D}^*)^F$ on the active lattice. Before imposing s, torus duality matches the quotients W_{rel}^F, and Lang in $\mathbf{L}_K$ gives rational lifts. The coset in W_s comes from Proposition 7.234.

At $I(3,3)$, $c=1$, the connected quotient is $\langle R_1, R_2\rangle$. An odd power of R_i requires -1 on its same-sign singleton e_j. The extra coset acts by $R_1 \leftrightarrow R_2$, giving $C_4 \wr S_2$. If $c=-1$, F^2 on V_+ has signs $(-,-,+)$. On the two negative axes V_0, every signed permutation occurs in $W(B_2)$, with parity correction on e_j, of sign $+1$. Its image on the cycles is $\langle R_1^2, R_2^2, j\rangle \cong D_8$. The extra coset gives $h = R_1R_2$, central of order 4, with $h^2 = z = R_1^2R_2^2$. Moreover, each $\eta \in \mathrm{Irr}(D_8)$ extends by either scalar root of $\eta(z)/\eta(1)$ at h.

With $c=1$ at $Q(2,2)$, each 4^- requires even parity, giving quotient $\langle R_1^2, R_2^2\rangle$, and the extra coset j exchanges the generators. Its square is 1 or z. In the latter case multiply by R_1^2 to get square 1, giving D_8. If $c=-1$, the split 4-space over $\mathbb{F}_{q^2}$ has signs $(-,-)$ of even parity, giving connected quotient $\langle z, j\rangle$. The extra h from Proposition 7.234 has square z. Conjugation by j fixes h at $e=0$ and inverts h at $e=1$. Thus, the two groups are $C_4 \times C_2, D_8$.

With $\mathbf{X} = [\mathbf{K}, \mathbf{K}]$, we put $X = \mathbf{X}^F = \mathrm{Spin}_{12}^+(q)$ and $L_X = C_X(\mathbf{D})$ and $N_X = N_X(\mathbf{D})$. We have $N = L_K N_X$ since the derived subgroup has identical W_{rel} and F-action, and Lang on $C_{\mathbf{X}}(\mathbf{D})$ gives all rational lifts. The order $d=4$ has multiplicity 2 and is nonregular in X, with residual A_1^2. At this nonregular $d=4$, [35, Theorem 6.9(c)] gives the N_X-stable transversal $\mathcal{T}$ for the regular-diagonal orbits on $\mathrm{Irr}(L_X)$, with extensions to their N_X-inertias.

Choose a regular overgroup $\widetilde{\mathbf{K}}$ of $\mathbf{K}$, which gives one of $\mathbf{X}$. In both overgroups the centralizer of $\mathbf{D}$ surjects onto L_{ad} by Lang on the central torus Z_c. Thus, their actions on L_X agree. The diagonal normalizes N, L_K. Choose $\xi \in \mathcal{T}$ below $\beta|_{L_X}$. The restriction from the regular Levi $\widetilde{L}_K$ to L_X is multiplicity-free since $\mathrm{Z}(\widetilde{\mathbf{L}}_K)$ is connected and $\widetilde{\mathbf{L}}_K, \mathbf{L}_X$ have common derived group. Hence, $\beta|_{L_X}$ is multiplicity-free.

Let α be the Clifford correspondent on $(L_K)_\xi$, with $\alpha|_{L_X} = \xi$. If $n \in (N_X)_\beta$, ξ^n lies in $\mathcal{T}$ and in the orbit of ξ. Thus, $\xi^n = \xi$, and n fixes α. Restrict the extension of ξ to $(N_X)_\beta$ on the space of α. Its matrix at n also intertwines α, since its restriction to L_X is proportional to any α-intertwiner at n. The product representation on $(L_K)_\xi(N_X)_\beta$ is well-defined by coincidence on L_X. Induce to $L_K(N_X)_\beta = N_\beta$. Mackey for the single coset gives β on L_K. Hence, we obtain an extension $\widehat{\beta}$. Induce $\widehat{\beta}\eta$, $\eta \in \mathrm{Irr}(W_s)$, to N. The four quotients give $14, 10, 5, 8$ characters, respectively, with $[N : N_\beta]\eta(1)$ prime to ℓ.

For (g,σ) in the mixed stabilizer, correct g in N so $\beta^{g,\sigma} = \beta$. Then $\widehat{\beta}^{g,\sigma} = \widehat{\beta}\lambda_{g,\sigma}$, $\quad \lambda_{g,\sigma} \in \mathrm{Irr}(W_s/[W_s, W_s])$. All $\eta \in \mathrm{Irr}(W_s)$ are valued in $\mathbb{Q}(i)$. Given $C_4 \wr S_2$, induce distinct pairs (λ, ν), and extend (λ, λ) by either sign. Next, for $D_8 \circ C_4$, we choose the rational D_8-representations and the scalar choices at h. Both D_8 and $C_4 \times C_2$ also have values in $\mathbb{Q}(i)$. Since $\ell \equiv 1 \bmod 4$, σ fixes $\mathbb{Q}(i)$, so the label becomes $\lambda_{g,\sigma}\eta^g$. The twist groups for $C_4 \wr S_2,\ D_8 \circ C_4,\ D_8,\ C_4 \times C_2$ are $C_4 \times C_2,\ C_2^3,\ C_2^2,\ C_4 \times C_2$, respectively. In $D_8 \circ C_4$, $h^2 = z$ generates $[W_s, W_s]$, so h has image of order 2 in the abelian quotient. With $\widehat{\beta}' = \widehat{\beta}\nu$ for a linear ν, we have $\lambda'_{g,\sigma} = \lambda_{g,\sigma}\nu^g/\nu$. In particular, $\lambda'_{1,\sigma} = \lambda_{1,\sigma}$. □

Proposition 7.236. *Given $I(2,4)$, we have that the* 40 *or* 20 *characters χ from* Lemma 7.232 *correspond to χ' from* Lemma 7.233 *under $\succeq_c$. This gives the ambient comparison on $N_G(P)$ for odd q with the given mark at $d=4$.*

Proof. Keep $\mathbf{K}, \mathbf{H}$ from Lemma 7.233, with connected $\mathbf{H}^* = C_{\mathbf{K}^*}(s)$. Next, match $\chi \in \mathcal{E}(K,s)$ to $\rho = \kappa_s(\delta \boxtimes \tau)$ on H, where (δ, τ) are its Jordan labels on the 4- and 8-spaces. Central isotypies match δ, τ on adjoint types D_2, D_4 with fixed bases. Rank distinguishes D_2 from D_4, and (δ, τ) are equivariant under γ, F_r. Jordan decomposition matches central isotypies, with $\mathcal{H}_\ell$-equivariance by [169, Theorems 1.1, 6.2 and Lemma 6.4]. Central multiplication by s corresponds to the κ_s-twist. Thus, $\chi \leftrightarrow \rho$ is equivariant under mixed operations returning s.

Choose rational opposite Borels $\mathbf{B}_+, \mathbf{B}_-$ in the 4-space factor of $\mathbf{H}$. Over $\overline{\mathbb{F}}_q$, the two A_1-flags determine the rulings through L_+ and its opposite L_-. In minus type, F exchanges the two rulings, so $L_\pm$ are rational. Take a cocharacter λ with equal positive pairings on α, β of the D_2 base and zero on the 8-space. A suitable lattice multiple gives λ rational. In orthogonal coordinates λ is the first hyperbolic direction e_1 of the 4-space. Its parabolics in $\mathbf{K}$ are $\mathbf{P}_1, \mathbf{P}_1^-$, and put $\mathbf{K}_5 = C_{\mathbf{K}}(\lambda)$, $\mathbf{K}_4 = C_{\mathbf{H}}(\lambda)$. Then, $\mathbf{K}_5$ has derived rank 5, and $\mathbf{K}_4$ contains the 8-space factor and the torus of D_2. Also, both contain $\mathbf{T}$. In the dual, we have that $C_{\mathbf{K}_5^*}(s) = \mathbf{K}_4^*$ by intersecting $\mathbf{H}^*$ with $C_{\mathbf{K}^*}(\lambda^*)$, for the torus dual to λ. Moreover, the remaining singleton e_j has opposite sign to the 8-space. Its axis exhibits $\mathbf{K}_4$ as a Levi of $\mathbf{K}_5$, even for an anisotropic plane V_0.

Put $\beta_4 = \kappa_s|_{K_4}\tau$ and $\beta_5 = \pm R_{K_4}^{K_5}(\beta_4)$, irreducible by $C_{\mathbf{K}_5^*}(s) = \mathbf{K}_4^*$. Moreover, the character χ occurs with multiplicity 1 in ordinary induction from P_1 with source β_5. The character ρ occurs with multiplicity 1 in induction from $P_1 \cap H$ with source β_4. Also, on D_2 the rational Borel induction gives $(1+\mathrm{St})^{\boxtimes 2}$ at 4^+ and $1+\mathrm{St}$ over $\mathbb{F}_{q^2}$ at 4^-. The large label τ is fixed. For the induction on K, we apply [72, Theorem 4.7.5] in the overgroup $\widetilde{K}$ with

connected centre. Dual centralizers in $\mathbf{K}^*, \mathbf{K}_5^*, \mathbf{K}_4^*$ are connected, so the lifts $\widetilde{\chi}$ within $\mathcal{E}(\widetilde{K}, \widetilde{s})$ give distinct irreducible restrictions to K. By Lang, $\widetilde{K} = K\widetilde{P}_1$ for the lifted parabolic $\widetilde{P}_1$. Thus, the multiplicities descend to K. On H, the induction has central twist $\kappa_s|_{P_1 \cap H}$.

Take the ambient transports on $\mathbf{K}$, with corrections in K. Next, for ordinary inertia, let B consist of the transports that normalize $\mathbf{H}$ and fix $\kappa_s, \delta \boxtimes \tau$. Then $B \cap K = H$, and the inertia on K is $A = KB$. These corrections cover the stabilizer of $\chi^{\mathcal{H}_\ell}$. For a mixed stabilizer element a, the transformed power of s is K^*-conjugate to s, so the image of $\mathbf{H}^*$ is a rational conjugate. Correct in $\mathbf{H}^*$ to align its maximally split torus $\mathbf{T}^*$. The dual operation on $\mathbf{T}^*$ gives a coset on $\mathbf{T}$ lifted by Lang. Thus, κ_s returns in that coset w. An inner operation returning the series fixes s after correction and lies in $\mathbf{H}^*$. Hence, it has primal representative in H. For $\widetilde{\mathbf{K}}$, Lang on $\mathbf{H}$ gives the extra diagonal cosets in $(\mathbf{H}\mathrm{Z}(\widetilde{\mathbf{K}}))^F$. These fix δ, τ, so equivariance gives the stabilizer of ρ.

Let B_0 stabilize $(\mathbf{B}_+, \mathbf{B}_-)$. By conjugacy of the flags $\mathbf{B}_\pm$, we get $B = HB_0$ and $H \cap B_0 = K_4$. With equal pairings, we have that B_0 fixes $\lambda, \mathbf{P}_1, \mathbf{P}_1^-$ even under the ruling exchange γ. At a fixed position j, the ordinary action on $\mathbf{K}_5$ has an invariant parabolic with Levi $\mathbf{K}_4$. Since E_7 is split, at field exponent 0 the inner part fixes $s, \mathbf{B}_+, \mathbf{B}_-$. By the normalizer argument, $g \in \mathbf{H}^*$, and g fixes the flags, so $g \in \mathbf{K}_4^*$. The external root inversion would invert u_3, and $u_3^2 \neq 1$. Moreover, the diagonal lies in $\widetilde{\mathbf{K}}_4$, the regular flag Levi. The cocharacters $Y(\mathrm{Z}(\mathbf{K}_4)^\circ)$ are fixed in $\mathbf{K}_5$. Next, choose a generic such cocharacter ν. Transport its parabolic to positions $0, \ldots, f-1$; the parabolic may be over $\overline{\mathbb{F}}_q$ for an anisotropic plane V_0.

By Lemma 7.168 on $\mathbf{K}_4 \subset \mathbf{K}_5$ in the split f-position product, $\beta_5 \boxtimes \beta_4^\vee$ extends to $(K_5 B_0) \times_{B/H} B_0$. The quotient kernels are K_5, K_4. A mixed stabilizer element a corrected on the flags returns β_4, since this choice of τ is invariant and $\kappa_s^a = \kappa_s$. It fixes the extension even when a changes the radical $\mathbf{U}_\nu$. Inflate through the flag radicals $U_1, U_1 \cap H$ and induce from $(P_1 B_0) \times_{B/H} ((P_1 \cap H) B_0)$ to $A \times_{B/H} B$. Moreover, the projections to $K_5 B_0, B_0$ are homomorphisms since B_0 stabilizes $\mathbf{B}_\pm$. The source subgroup surjects onto B/H. By [95, Problem 5.6] on $K \times H$, we obtain the tensor of $\mathrm{Ind}_{P_1}^K(\beta_5)$ and the dual of $\mathrm{Ind}_{P_1 \cap H}^H(\beta_4)$, with inflated sources. The component for $\chi \boxtimes \rho^\vee$ has multiplicity 1 and gives an extension $\widehat{\Psi}$. Given a mixed stabilizer element a, correct both sides by $h \in H$ to fix $\mathbf{B}_\pm$. Moreover, this is inner on $A \times_{B/H} B$. The corrected a fixes the induced model V and $V_{\chi \boxtimes \rho^\vee}$. Hence, it fixes $\widehat{\Psi}$. An element $c \in C_A(K)$ has field exponent 0 on the D_6 root groups. Thus, $c \in \mathrm{Z}(\widetilde{\mathbf{K}})^F \cap B_0$ in the corrected regular Levi $\widetilde{\mathbf{K}}$. It centralizes $U_1, U_1 \cap H$, so the scalar of c on V is 1. By the fibre-product criterion, $\chi \leftrightarrow \rho$ has common central scalars at $c \in C_A(K)$.

Apply [156, Theorem D] to the unipotent factors D_2, D_4 with their regular diagonal groups $\widetilde{D}_2, \widetilde{D}_4$. By Lemma 7.231, the comparison on H follows by central isotypy. On 4^+ take the identity. On 4^- apply the result to $A_1(q^2)$ at $\ell \mid q^2 + 1$. Since $P \in \mathrm{Syl}_\ell(H)$, we have $B = HN_B(P)$, and centralizers of H normalize P. Twist both models by κ_s on the common section of B/H by Lemma 7.7. Thus, the comparison on $H, N_H(P)$ has local sources $\kappa_s|_{\mathbf{L}_K^F} \rho_0$ for the unipotent sources ρ_0 of Lemma 7.233. Next, compose with $\chi \leftrightarrow \rho$ and Clifford induce from $N_H(P)$, the stabilizer of the exact L_K-series. This gives the comparison on $K, N_K(P)$. Next, for leg 3, apply Proposition 7.172 with signed induction to G. At fixed positions j, $u_3^2 \neq 1$ excludes inversion. For the central lifts s_1, s_2, we have $s_1^{-1} s_2 \in \mathrm{Z}(\mathbf{K}_{\mathrm{der}}^*)$ and $\alpha_{\mathrm{ext}}(s_1^{-1} s_2) = 1$ for the external root α_{ext}. Thus, the ratios u_i agree. The types and dimensions of $V_\pm$ also agree. Transfer the union over s_1, s_2 with their labels to $N_G(P)$. □

Proposition 7.237. *In* Proposition 7.235*, we put $L_0 = \mathbf{L}_K^F$, $N = N_K(P)$. Next, we let W_0 be the relative group of $(\mathbf{C}^*)^\circ$ and ϵ the nontrivial character of W_s/W_0. Next, we write $\widetilde{K}, \widetilde{N}$ for Levi and normalizer in $\widetilde{\mathbf{G}}$ with split central rank 1. A generator d of $\widetilde{K}/K$, represented in $\widetilde{N}$, acts on the Clifford labels by $\eta \mapsto \epsilon\eta$ after correction to β. Moreover, the fibres over β admit a comparison $\succeq_c$ for $\widetilde{K}, \widetilde{N}$ and for the subgroup of $\mathcal{H}_\ell$ fixing $\mu_{\ell'}$, the roots of order prime to ℓ.*

Proof. Take $\widetilde{\mathbf{G}} = (\mathbf{G} \times \mathbb{G}_m)/\langle (z, -1) \rangle$, with $\mathrm{Z}(\mathbf{G}) = \langle z \rangle$. By [114, corollary to Theorem 1] on $\mathbf{K}, \mathbf{L}_K$, the quotients are $\widetilde{K}/K = \widetilde{L}_0/L_0 = \mathbb{F}_q^\times$. The centre of $\mathbf{L}_K$ is connected. Its residual roots are $e_5 - e_6, e_5 + e_6$. Moreover, in their span, an E_7-weight λ has integral coordinates by pairing with $e_i \pm e_j$, $i \leq 4$, $j = 5, 6$. A half-root gives $\lambda_5 + \lambda_6 \in 2\mathbb{Z}$. Thus, $\lambda \in \langle e_5 - e_6, e_5 + e_6 \rangle_{\mathbb{Z}}$, so saturation holds. Since $\mathrm{Z}(\mathbf{L}_K)$ is connected, [114, corollary to Theorem 1] gives $\widetilde{L}_0 = L_0 \mathrm{Z}(\widetilde{\mathbf{L}}_0)^F$. Thus, $\widetilde{L}_0$ acts on L_0 by inner automorphisms. Choose $d \in \mathrm{Z}(\widetilde{\mathbf{L}}_0)^F$, where $\widetilde{\mathbf{L}}_0 = C_{\widetilde{\mathbf{K}}}(\mathbf{D})$.

By [95, Corollaries 6.17 and 11.22], the cyclic quotient $\widetilde{L}_0/L_0$ gives extensions of β differing by $\mathrm{Irr}(\widetilde{L}_0/L_0)$. If $n \in N_\beta$ lifts $w \in W_s$, conjugation by n twists that extension by $\mu_w \in \mathrm{Irr}(\widetilde{L}_0/L_0)$. For the dual quotient π with connected kernel $\mathbb{G}_m$, [133, Proposition 23.2] gives a rational lift $\widetilde{s}$ on the torus. A corrected w fixing s translates $\widetilde{s}$ by an element of $\ker \pi$. By [20, (2.2)], $\pi(C(\widetilde{s})^\circ) = C(s)^\circ$. By [40, §3.5, Steinberg's theorem], $C(\widetilde{s}) = C(\widetilde{s})^\circ$, since $[\widetilde{\mathbf{K}}^*, \widetilde{\mathbf{K}}^*]$

is simply connected. As $\ker\pi$ is central, $C(\widetilde{s}) = \pi^{-1}(C(s)^\circ)$. Correction by $\mathbf{L}_0^*$ leaves this test intact since $C_{\mathbf{L}_0^*}(s)$ is connected. The residual labels λ are fixed. Translation of $\widetilde{s}$ gives the twist μ_w. Thus, $\ker(w \mapsto \mu_w) = W_0$, the image has order 2, and $\mu_w(d) = \epsilon(w)$. Write ν for the central character of β. Since d centralizes L_0, the commutator $[n, d]$ lies in $\mathrm{Z}(L_0)$. Its value under ν gives $\mu_w(d)$ and the twist of $\widehat{\beta}$ on n, up to inversion. Since $\mu_w(d) \in \{\pm 1\}$, we get $\widehat{\beta}^d = \widehat{\beta}\epsilon$. Conjugation by d acts trivially on W_s, so induction gives $\eta \mapsto \epsilon\eta$ on N.

Globally a C_2-orbit of size 2 gives one d-fixed constituent. A fixed label yields χ_+, χ_- exchanged by d by Clifford restriction. We match the W_0-labels in each case. In $I(3,3)$ with $c = 1$, match the four hooks with $\mathrm{Irr}(C_4)$ with a common ordering on both C_4-factors. If $c = -1$, match the five partitions with $\mathrm{Irr}(D_8)$, taking $1, \mathrm{St}$ to 1 and the reflection sign $\varepsilon_{\mathrm{refl}}$. The extra coset centralizes D_8 and fixes every partition λ. At $Q(2,2)$, match $1, \mathrm{St}$ with the two characters of each C_2. If $c = -1$, the reflections are $j, z'j$, $z' = R_1^2 R_2^2$. The extra coset fixes both at $e = 0$ and exchanges them at $e = 1$. Thus, the match on W_0 is equivariant for W_s/W_0. Given an orbit of size 2, induction from W_0 gives η. A fixed label gives $\eta, \epsilon\eta$. Moreover, this gives the d-equivariant bijection per quartic λ.

Each global $\chi \in \mathcal{E}(K, s)$ has the corresponding scalar on $\mathrm{Z}(\mathbf{K})^F$. Locally ν has this restriction by duality on L_0. The inertia quotients on $\widetilde{K}, \widetilde{N}$ are cyclic and coincide. By [95, Corollary 11.22], χ, χ' extend to their inertias I, I'. Both have the same scalar on $\mathrm{Z}(\widetilde{\mathbf{K}})^F \cap K$. Thus, a character of I/K equalizes their scalars on $\mathrm{Z}(\widetilde{\mathbf{K}})^F$. By [66, Lemma 2.2], $C_{\widetilde{K}}(K) = \mathrm{Z}(\widetilde{\mathbf{K}})^F$, since K contains $[\widetilde{\mathbf{K}}, \widetilde{\mathbf{K}}]^F$.

The elements $s, \widetilde{s}$ have ℓ'-order since $\ell \nmid q - 1$. $\ell \geq 5$ is good for D_6 and prime to $|\mathrm{Z}(\mathbf{K})/\mathrm{Z}(\mathbf{K})^\circ|$. By [36, Theorem 14.4], restrictions to ℓ-regular elements are independent in each ℓ'-series on $K, \widetilde{K}$. If σ fixes $\mu_{\ell'}$, it fixes each series and the ℓ-regular restrictions. Hence, σ fixes each character χ. Take the Clifford correspondent $\widehat{\chi}$ of a character $\widetilde{\chi}$ above χ. Uniqueness gives $\widehat{\chi}^\sigma = \widehat{\chi}$. Locally $P \leq \mathrm{Z}(L_0) \cap \ker\beta$ since s has ℓ'-order. Thus, the models on I' factor through $I'/P \leq \widetilde{N}/P$ of ℓ'-order. The scalar twists have ℓ'-order as well, so σ fixes both extensions. □

Proposition 7.238. *Put $q = r^f$. Next, let $\sigma \in \mathcal{H}_\ell$ act by ℓ^k on the ℓ'-roots in question, and define $g_\sigma = (q/\ell)^k = (-1)^{k(\ell-1)/4}$. For σ stabilizing $\mathcal{E}(K, s)$, both pairs (χ_+, χ_-) and (χ_+^*, χ_-^*) have sign g_σ. For a mixed stabilizer element (g, σ), multiply by the sign of g on the two Whittaker orbits. Suppose $\beta^\sigma = \beta$. For an F-stable pinning of $\mathbf{L}_K$, we let $S \leq N_\beta$ be its stabilizer and $C = \mathrm{Z}(\mathbf{L}_K)^F$. Let $x \in S$ represent $w \in W_s$. Next, we let a be the order of w in $W_s/[W_s, W_s]$, and choose $y \in [S, S]$ above w^a. Then, we obtain that $c_x = x^a y^{-1} \in C$ and $\lambda_{1,\sigma}(w) = \epsilon(w)^{(1-g_\sigma)/2}\nu(c_x)^{(\ell^k-1)/a}$, where ν is the central character of β.*

Proof. In the split D_6 the positive coroot half-sum is $\rho_K^\vee = (5, 4, 3, 2, 1, 0)$. For the residual A_1^2 pinning, $\rho_L^\vee = e_5$. Moreover, changing positive systems changes $\rho^\vee$ by a vector in $Q(D_6)$. $\rho_K^\vee - \rho_L^\vee \in Q(D_6)$ since the coordinate sum is 14. Thus, $(2\rho_K^\vee)(-1) = (2\rho_L^\vee)(-1) = z_v$. The vector class z_v has nontrivial image in $\mathrm{Z}(\mathbf{K})/\mathrm{Z}(\mathbf{K})^\circ$. Indeed, the projection of $Q(E_7)$ onto the six coordinates is $Q(D_6)$ with coset $\omega_{\mathrm{spin}} + Q(D_6)$ by Lemma 7.231, excluding e_5. Modulo the line $Y(\mathrm{Z}(\mathbf{K})^\circ)_{\mathbb{Q}}$, the spin centre meets $\mathrm{Z}(\mathbf{K})^\circ$ in the half-spin subgroup. Hence, z_v generates the component group C_2, fixed by F.

On the pinned unipotent subgroup U, choose the character ψ from the absolute trace on U_α for each simple α. Next, put $b = \ell^k \bmod r$ and choose t over $\overline{\mathbb{F}}_q$ with $t^2 = b$. Moreover, the element $u_b = (2\rho_K^\vee)(t)$ multiplies each simple root coordinate by b, giving ψ^σ. We have $F(u_b)u_b^{-1} = 1$ or z_v according as b is square or nonsquare in $\mathbb{F}_q$. By Lang on $\mathrm{Z}(\widetilde{\mathbf{K}})$, we obtain a central correction c_b with $d_b^K = c_b u_b$ rational. Moreover, its boundary in $H^1(F, \mathrm{Z}(\mathbf{K})) = C_2$ is 1 for $\eta_q(b) = 1$ and z_v for $\eta_q(b) = -1$. Thus, d_b^K induces the diagonal exactly at $\eta_q(b) = -1$ by Proposition 7.237. Reciprocity gives $\eta_q(b) = (r/\ell)^{fk} = (q/\ell)^k$ since $\ell \equiv 1 \bmod 4$, with η_q quadratic. As $\mathrm{ord}_\ell(q) = 4$, Euler's criterion gives g_σ.

The rational component group of s has order 2. Moreover, each K-orbit of ψ gives a Gelfand–Graev representation Γ_ψ. By [31, Theorem 3.1], each Γ_ψ has one regular constituent with multiplicity 1 in $\mathcal{E}(K, s)$, distinct for the orbits of $\psi_\pm$. Thus, transport of s, ψ determines the sign at (g, σ). The pinned diagram and F_r fix ψ. The diagonal exchanges the orbits of $\psi_\pm$. The theorem gives $\chi_\pm^*$ from Alvis–Curtis duality D_K. Moreover, its formula in $R_L^K, {}^*R_L^K$ is equivariant, with $\chi_\pm^*(1) > 0$. Both pairs qualify because $1, \mathrm{St}$ have ℓ'-degrees on $C_{\mathbf{K}^*}(s)^\circ$.

For the regular source β of L_0, take the only label at $I(3,3)$ or the Steinberg partition at $Q(2,2)$. Its ψ-line has dimension 1. Since $\mathrm{Z}(\mathbf{L}_K)$ is connected, $N_\beta = L_0 S$ and $L_0 \cap S = C$. Indeed, correct $(\mathbf{B}, \mathbf{T})$ by L_0. The root scalings in T_{ad} lift by Lang on $\mathrm{Z}(\mathbf{L}_K)$. The pinning stabilizer S fixes ψ, including the root swap $\alpha \leftrightarrow \beta$. Thus, $S/C = W_s$, and S acts on the line of $\widehat{\beta}$ by a linear τ with $\tau|_C = \nu$.

Apply the same t and central correction to $(2\rho_L^\vee)(t)$, giving $d_b \in \widetilde{L}_0$. Its discrepancy equals that of u_b, so

$d_bK = d_b^K K$. The sum of positive coroots of $\mathbf{L}_K$ is pinning-invariant, so d_b centralizes S. Take $\widehat{\beta}^{d_b}(n) = \widehat{\beta}(d_b n d_b^{-1})$. Since the diagonal on L_0 is inner, both $\widehat{\beta}^{d_b}$ and $\widehat{\beta}^{\sigma}$ extend β. Their ψ^σ-lines afford τ, τ^σ on S, respectively. Hence, their ratio inflates τ^σ/τ. Proposition 7.237 gives the first factor in $\lambda_{1,\sigma}(w)$. Since $P \leq C$ and $\nu|_P = 1$, τ factors through the ℓ'-group S/P. Its coefficient ratio on x is $\tau(x)^{\ell^k-1}$. By Proposition 7.235, we obtain $\exp(W_s/[W_s, W_s]) \mid 4$, so $a \mid \ell^k - 1$. The map $S \to W_s$ is onto, so $[S, S]$ contains a lift y of w^a. Since $\tau(y) = 1$, we get $\tau(x)^a = \nu(c_x)$, independent of y. Replacing x by xc, $c \in C$, multiplies the ratio by $\nu^\sigma(c)/\nu(c) = 1$.

At $Q(2,2)$, we let R be the residual factor of rank 1 where s is central. On the other factor s is noncentral, so N_β fixes R. Denote the two sources by $\beta_1, \beta_{\mathrm{St}}$. Take the parabolic of L_0 with a Borel on R and full other factor R'. The Levi source is trivial on T_R since s is trivial on the coroot $\alpha_R^\vee$. By Lang on T_R, it descends to $(\mathbf{L}_K/\mathbf{R})^F$, then inflates to L_0. Its flag induction is $\beta_1(1 + \mathrm{St}_R)$. Both summands are irreducible because $\mathrm{St}_R|_R$ is irreducible and $R \leq \ker \beta_1$. The degrees then give $\beta_{\mathrm{St}} = \beta_1 \mathrm{St}_R$. The rational augmentation model on the R-flags extends St_R to the inertia. Thus, $\widehat{\beta}_1$ tensored with this extension gives $\widehat{\beta}_{\mathrm{St}}$, with common $\lambda_{1,\sigma}$ and ν. □

Corollary 7.239. *For a fixed source β in* Proposition 7.238*, we have that $\lambda_{1,\sigma} = \epsilon^{(1-g_\sigma)/2}$. The regular pair (χ_+, χ_-) and the semisimple pair (χ_+^*, χ_-^*) admit $\succeq_c$ for the diagonal overgroups and σ fixing β.*

Proof. Moreover, in $X = \mathrm{Spin}_{12}^+(q)$, the active natural space is $V_1 \perp V_2$ with two negative 4-spaces. The residual 4^+ is $W \otimes W$ for a symplectic plane W. Also, the tensor flip u is an improper rational involution with (-1)-space $\bigwedge^2 W$ of dimension 1. It exchanges identical pinnings of SL_2^2. Fix this pinning. Next, let S_0 consist of spin representatives acting by 1 or u on 4^+ and active Weyl image in W_s.

Next, for any active signed permutation $w \in W_s$, we choose an algebraic isometry $\dot{w}$ of image w, and pair it with $v_0 \in \{1, u\}$ to get determinant 1. Lift to the spin group. Since w centralizes the mark and $F(v_0) = v_0$, the discrepancy $F(\dot{w})\dot{w}^{-1}$ projects to the active torus T_{act} with residual matrix 1. The preimage of T_{act} is $\mathbf{D}$, which maps onto T_{act} and contains z_v. Lang in $\mathbf{D}$ gives rational $\dot{w}$. At $w = 1$, the representative has determinant 1 on $V_1 \perp V_2$. Hence, it acts trivially on 4^+ and belongs to $\mathbf{D}^F$. Thus, $S_0 \leq S$ stabilizes the pinning and torus, with $S_0/\mathbf{D}^F = W_s$. We may choose x, y in S_0 and get $c_x \in \mathbf{D}^F$ in the formula for $\lambda_{1,\sigma}$. The odd part of $\mathbf{D}^F$ lies in $\ker \nu$, since s has trivial π-part for the odd primes π of $q^2 + 1$. Restrict the torus weight to $\mathbf{D}^F$. The active mark J has $J^2 = -1$, so the exponent of $\mathbf{D}^F$ divides $q^2 + 1$. Since $v_2(q^2 + 1) = 1$, we get $\nu(\mathbf{D}^F) \subseteq \{\pm 1\}$. At $a \leq 2$, the second factor is 1 since $4 \mid \ell^k - 1$.

An element of order 4 in the abelian quotient requires $C_4 \wr S_2$ for $I(3,3), c = 1$, or $C_4 \times C_2$ for $Q(2,2), c = -1, e = 0$. Both have abelian quotient $C_4 \times C_2$. Next, we choose its order-4 generator $w = R_1$ or R_1R_2, respectively, and the order-2 generator from j. Since $w^4 = 1$, set $y = 1$. On each active space take $\mathbb{F}_{q^4}$ with form $\mathrm{Tr}_{q^2/q}(v^{q^2+1})$. Moreover, a coefficient $a_0 \neq 0$ of the norm is absorbed by $v \mapsto cv$ for a suitable c. Up to orientation, $v \mapsto v^q$ is the rotation, with spectrum $1, -1, i, -i$ by a normal basis. Inversion gives $1, -1, -i, i$. Next, choose R_1u or R_1R_2, with respective numbers $1, 2$ of reciprocal pairs $(i, -i)$.

A spin lift has fourth power z_v or 1, respectively. Indeed, with natural entries i^{m_j}, the lift is $m/4 \bmod Q^\vee(D_6)$. Also, its fourth power is 1 at even $\sum_j m_j$ and z_v at odd sum. Each pair $(i, -i)$ contributes 1 mod 2. Moreover, the isometry in $\mathrm{SO}_{12}(q)$ has lift discrepancy in $\{1, z_v\}$. Apply Lang on the lifted torus T_i of V_i, containing z_v. The four conjugates of the correction $a \in T_i$ have product 1 since $1 + R_i + R_i^2 + R_i^3 = 0$ on $Y(T_i)$. Thus, the correction leaves x^4 fixed. In the first case $F(t) = t$, so the dual class on adjoint D_6 has lift $t \in \mathrm{SO}_{12}(q)$. By duality, for the spin projection π and torus weight θ, the weight $\theta \circ \pi$ has central character ν, so $\nu(z_v) = 1$. In the second case $x^4 = 1$. Therefore, the second factor is 1 on abelian-quotient generators. Since $\lambda_{1,\sigma}$ and $\epsilon^{(1-g_\sigma)/2}$ are linear, they agree on W_s.

For each pair (χ_+, χ_-) or (χ_+^*, χ_-^*), the connected label is C_2-fixed. Its extensions over W_0 differ by ϵ. By Proposition 7.238, both sides have sign g_σ. Their inertia groups are $K\mathrm{Z}(\widetilde{\mathbf{K}})^F$ and $N\mathrm{Z}(\widetilde{\mathbf{K}})^F$, since the centre maps to $(\mathbb{F}_q^\times)^2$ and a nonsquare exchanges χ_+, χ_-. Next, extend the common character to $\omega \in \mathrm{Irr}(\mathrm{Z}(\widetilde{\mathbf{K}})^F)$ for both products $K\mathrm{Z}(\widetilde{\mathbf{K}})^F, N\mathrm{Z}(\widetilde{\mathbf{K}})^F$. For a mixed stabilizer element a, the ratio at $z_1 \in \mathrm{Z}(\widetilde{\mathbf{K}})^F$ is $\omega^\sigma(z_1)/\omega(z_1)$ on both sides. The restrictions of ω^σ, ω to $\mathrm{Z}(\mathbf{K})^F$ coincide at a. Thus, the factor set α, central scalars, and functions μ_a agree for both overgroups. □

Lemma 7.240. *At $I(3,3)$, every coefficient automorphism stabilizing $\mathcal{E}(K, s)$ fixes the local source β. At $Q(2,2)$, the only other case has $t_3^{\ell^k-1} = -1$. Then β moves and returns under R_1R_2 at $c = 1$ or R_1^2 at $c = -1$ before Clifford induction.*

Proof. Write $s = x\alpha_3^\vee(t_3)$ with $x \in \mathbf{K}^*_{\mathrm{der}}$. Its spin lift has order dividing 4, since $\sum_j n_j$ is even in both rows and $n \in Q(D_6)$. As $\ell^k \equiv 1 \bmod 4$, only t_3 varies. An action stabilizing the rational class fixes $s \bmod \mathbf{K}^*_{\mathrm{der}}$. Hence, $u_3 = t_3^2$. If $t_3^{\ell^k-1} = 1$, each local label is fixed since $1 \neq \mathrm{St}(1)$ on the residual A_1. In the other case $t_3^{\ell^k-1} = -1$. Its external central involution corresponds to z_v on the semispin torus: $(e_7 - e_8)/2 - e_1$ pairs integrally with every E_7-root, so belongs to $Y(\mathbf{T}^*)$ in adjoint type. The class z_v differs from the kernel class v.

At $I(3,3)$, take the residual $-I_2 \in W(D_2)$. It centralizes the residual mark $w_F|_{\langle e_5,e_6\rangle}$, so Lang gives a representative in L_0^*. The sign changes at $0, 2$ fix the natural t with sum $2 \bmod 4$, giving z_v on the spin lift. Thus, β returns by uniqueness. At $Q(2,2)$, the residual exponents n_5, n_6 satisfy $n_5 = \pm n_6 \bmod 4$. In $W(D_2)$ the stabilizer is generated by j_0, the swap at $n_5 = n_6$ and $-j_0$ otherwise. In each case, the chosen w gives $\sum_{j:w(e_j)\in-\{e_1,\dots,e_6\}} n_j \equiv 0 \bmod 4$, so the lift fixes x. The active entries are fixed by $W(D_2)$, excluding common negation. Thus, $x, z_v x$ are distinct L_0^*-classes over $\overline{\mathbb{F}}_q$. In Proposition 7.234's coordinates, R_1R_2 and R_1^2 in their respective marks fix t with the selected exponent sum $2 \bmod 4$. Both centralize the mark w_F and fix $\alpha_3^\vee$. Hence, the rational lifts fix λ on the residual factor. On W_s, R_1R_2 at $c = 1$ acts trivially. At $c = -1$, R_1^2 fixes h and acts by $j \mapsto z'j$. This is conjugation by h if $e = 1$ and an automorphism of $C_4 \times C_2$ if $e = 0$. Proposition 7.241 gives the scalar on $\widehat{\beta}$. □

Proposition 7.241. *For the extra stabilizing action in the $Q(2,2)$ case of* Lemma 7.240*, choose $\widehat{\beta}$ with local action $\eta \longmapsto \epsilon^{(1-g_\sigma)/2}\eta^{w_*}$. The action of w_* is trivial at $c = 1$ and at $(c,e) = (-1,1)$. At $(c,e) = (-1,0)$, it fixes h and gives $j \mapsto h^2 j$. Thus, the pairs in* Corollary 7.239 *have $\succeq_c$ for every σ returning the rational series, including when $\beta^\sigma \neq \beta$.*

Proof. Let $S_1 = S_0$ be as in Corollary 7.239. Then $S_1/\mathbf{D}^F = W_s$ and $S = CS_1$. The line character τ has $\tau(\mathbf{D}^F) \subseteq \mu_2$. If $w^2 \in [W_s, W_s]$, choose x above w and $y \in [S_1, S_1]$ above w^2. Then $x^2y^{-1} \in \mathbf{D}^F$ and $\tau(x)^4 = \tau(x^2y^{-1})^2 = 1$. For the generators R_1 in $C_4 \wr S_2$ and R_1R_2 in $C_4 \times C_2$, Corollary 7.239 also gives $\tau(x)^4 = 1$. Thus, $\tau(S_1) \subseteq \mu_4$ and $\tau|_{S_1}$ is $\mathcal{H}_\ell$-fixed. Next, take the correction n_* with Weyl image $w_* \notin W_s$ in the pinned spin normalizer N_{pin}. It normalizes S_1. Compare τ^{n_*} with τ on S_1. On C, the scalars agree because β returns.

Let A, B lift R_1, R_2 in the pin group over $\overline{\mathbb{F}}_q$. Both are odd and act on V_1, V_2, respectively. Hence, $AB = z_vBA$. The swap J_0 of the two 4-spaces has determinant 1. Choose lifts so that $J_1AJ_1^{-1} = B$, $J_1BJ_1^{-1} = A$, possible since J_1^2 is central. The Frobenius discrepancies of A, B are the same signs. Hence, AB, A^2, B^2 are rational. The negative eigenspace of J_0 is the anti-diagonal 4-space with quadratic form $2Q_-$ for the original form Q_-. Its determinant is nonsquare, so J_0 has nonsquare spinor norm. Given j, we choose TJ_1, where T belongs to the active torus and $F(T)T^{-1} = z_v$ by Lang. The correction also gives TJ_1AB rational.

Take AB as the correcting element at $c = 1$ and at $(c,e) = (-1,1)$. Specifically, in the latter case R_1R_2 differs from R_1^2 by an element of W_s, giving the same transport on $\mathrm{Irr}(N)$ after Clifford induction. Conjugation by AB centralizes $A^2, B^2, A^{1+2e}B$, and its commutator with J_1 is z_v. Moreover, for TJ_1, the extra commutator is $J(T)T^{-1}$, where $J = R_1R_2$ on the active torus.

If $c = 1$, $\nu(z_v) = 1$ since t has a rational lift. At $c = -1$, F exchanges $t, -t$, so $\{t, -t\}^F = \varnothing$. By duality, rational lifts give $\mathrm{Irr}(T_{\mathrm{spin}}/\langle z_v\rangle)$. Therefore, $\nu(z_v) = -1$. On the active torus take $Y = Q(D_4)$ and $F_0 = qJ$ on Y. The dual weight restricts to $\theta = (n_1, n_2, n_3, n_4)/4 \bmod Y^\vee$. Moreover, the rational point represented by v pairs with θ by $\langle\theta, (F_0 - 1)v\rangle \bmod \mathbb{Z}$, by pairing $Y/(F_0 - 1)Y$ with the fixed points for F_0^{t}. Since the entries are ±1, orientation changes fix the class θ. For T represented by t, we have that $(F_0 - 1)t \in e_1 + Y$. The commutator is represented by $(J - 1)t$, so the pairing is $\big\langle(J^{\mathrm{t}} - 1)\theta, (F_0 - 1)t\big\rangle$. The first vector is integral at $c = 1$ and lies in $(1/2 + \mathbb{Z})^4$ at $c = -1$. It lies in $Y^\vee$ in both cases, giving values $0, 1/2 \bmod \mathbb{Z}$ on $e_1 + Y$, respectively. Thus, their ν-values have product 1. At $c = 1$ the generators are A^2, B^2, TJ_1, or with TJ_1AB in place of TJ_1. At $(c,e) = (-1,1)$ they are A^3B, TJ_1. On $\mathbf{D}^F$, the correction acts by $\nu^\sigma \mapsto \nu$. Since $\nu^2 = 1$ there, it fixes ν. Thus, AB fixes $\tau|_{S_1}$. At $(c,e) = (-1,0)$, we choose A^2 instead. It centralizes AB, the rational lift of h. The ratio τ^{A^2}/τ on S_1 is trivial on $\mathbf{D}^F$ and so a character of $W_s = C_4 \times C_2$, trivial at h. Its value at j is ±1. Twist by $\xi \in \mathrm{Irr}(W_s)$ with $\xi(h) = i$ to change this sign, since $j \mapsto h^2 j$. Next, choose sign 1 for all these stabilizing actions.

Correct $\widehat{\beta}^\sigma$ by the chosen lift, returning β. Its ψ^σ-line has character $\tau|_{S_1}$ and the returned scalar ν on C. Comparison with $\widehat{\beta}^{d_b}$ in Proposition 7.238 gives the claimed $\lambda_{1,\sigma}$. The rational R-flag representation relates $\beta_1, \beta_{\mathrm{St}}$. At $(c,e) = (-1,0)$, the trivial and Steinberg labels η satisfy $\eta(h^2) = 1$ since they agree at $j, h^2 j$. Thus, w_* fixes both labels η. By Corollary 7.239, we obtain μ_a on $K\mathrm{Z}(\widetilde{\mathbf{K}})^F, N\mathrm{Z}(\widetilde{\mathbf{K}})^F$. The correction $n_* \in N$ induces $\mathrm{Int}(n_*)$ on both ordinary models. □

Lemma 7.242. *At $Q(2,2)$, take the residual $\mathbf{R}$ with projective dual class 1, and choose $(\mathbf{B}_+, \mathbf{B}_-)$ opposite and rational in $\mathbf{R}$. Moreover, the positive coroot λ defines a rational parabolic $\mathbf{P}_\lambda$ in $\mathbf{K}$, with Levi $\mathbf{K}' = C_{\mathbf{K}}(\lambda)$ of derived type D_4A_1. Each exact-series character on K' induces to one character of K per quartic partition γ, with multiplicity 1. If $N' = N_{K'}(P)$, induce the corresponding inflated characters from $\mathbf{P}_\lambda^F \cap N$ to N. Locally each γ occurs with multiplicity 1.*

Proof. Choose dual tori $\mathbf{T}, \mathbf{T}^*$ maximally split on $\mathbf{R}, \mathbf{R}^*$. Since s is central on $\mathbf{R}^*$, the construction holds for $c = \pm 1$. We have $\lambda = e_5 - e_6$ or $e_5 + e_6$. The perpendicular roots are D_4 and $\pm\alpha$ from the other residual pair. Thus, $\mathbf{D} \leq \mathbf{K}'$. In $C_{\mathbf{K}^*}(s)^\circ$, the cut replaces the quartic A_1 by its split torus $\mathbf{T}_R^*$ and includes the sign factors H_+, H_-. Each rational component normalizes $\mathbf{R}^*$. By flag transitivity in $\mathbf{R}^{*F}$, correct the representative to fix $\mathbf{B}_\pm^*$. Hence, it fixes λ^*. Connectedness of $C_{C_{\mathbf{K}^*}(s)^\circ}(\mathbf{T}_R^*)$ gives component kernel 1. Both component actions give the same $\rho \mapsto \rho^w$.

Moreover, in the regular overgroup, $C(\widetilde{s})$ is connected. By [72, Theorem 4.7.5], the Jordan label ρ induces to $\rho \boxtimes (1 + \mathrm{St})$, also at the unitary A_1. The parabolic $\widetilde{P}_\lambda$ surjects onto $\widetilde{K}/K$. With $P_\lambda = \widetilde{P}_\lambda \cap K$, [95, Problem 5.2] gives $(\widetilde{\rho}^{\widetilde{K}})|_K = (\widetilde{\rho}|_{P_\lambda})^K$ for $\widetilde{\rho}$ inflated to $\widetilde{P}_\lambda$. Parent restrictions to K', K are multiplicity-free. The label ρ splits at a fixed point of C_2. Both quartic partitions are fixed by C_2, so both lists have size $d_0 \in \{1, 2\}$. Induction of $\sum_{\xi \in X_0} \xi$ contains each upper character with multiplicity 1. The diagonal is transitive on X_0 and each upper list X_γ. Hence, each member of X_0 gives one in X_γ with multiplicity 1.

Locally put $N_1 = N_N(\mathbf{R})$. On the rational flag pair $\mathbf{B}_\pm$, the stabilizer in N_1 is N' since the pair determines λ, and R-transitivity gives $N_1 = RN'$, $R \cap N' = T_R$. By duality, the source on $\mathbf{L}_K^F \cap K'$ has T_R in its kernel because s has trivial projective root value on $\mathbf{R}$. The N'-characters also have kernel containing T_R, normal in N', and inflate from $N'/T_R \simeq N_1/R$ to N_1. The group $\mathbf{P}_\lambda^F \cap N$ normalizes $\mathbf{R}$, since $\mathbf{P}_\lambda$ has a Borel on $\mathbf{R}$ and the full other factor $\mathbf{R}'$. Its intersection with R is B_+, and $\mathbf{P}_\lambda^F \cap N = R_u(\mathbf{B}_+)^F N'$. Let η_1 denote the inflation. Induction to N_1 gives $\eta_1(1 + \mathrm{St}_R)$ by [95, Problem 5.3]. Since $1, \mathrm{St}_R$ restrict irreducibly to R, both products lie in $\mathrm{Irr}(N_1)$ by [95, Corollary 6.17]. Any $n \in N \setminus N_1$ exchanges the projective tags of $\mathbf{R}, \mathbf{R}'$. This gives disjoint series on R after exchange. By [95, Theorem 6.11], induction to N gives distinct irreducibles with multiplicity 1 for both γ. □

Proposition 7.243. *If $d = 4$ and odd q, we have that the $Q(2,2)$-union has $\succeq_c$ on $K, N_K(P)$. Transfer along the moving leg gives $\succeq_c$ on $G, N_G(P)$.*

Proof. First, we fix $\mathbf{K}, \mathbf{D}$ and a residual factor $\mathbf{R}$ of $\mathbf{L}_K$. Align the trivial projective tag on $\mathbf{R}$: choose an odd rotation on $\mathbf{D}$ with an improper residual isometry u, which exchanges the two A_1-factors. By Lang in $\mathbf{L}_K$, this coset lifts. Take the flags, $\lambda, \mathbf{K}', N'$ of Lemma 7.242. Let Γ_P consist of the chosen transports a on K with $P^a = P$, including N and the chosen outer operations on $\mathbf{K}$. Next, we put $\Gamma = K\Gamma_P$ and let $B \leq \Gamma_P$ fix $\mathbf{R}$ and its ordered flags. Then $\Gamma_P = NB, \quad B \cap K = N', \quad \Gamma = KB$. Indeed, Γ_P permutes the residual factors of $C_{\mathbf{K}}(\mathbf{D})$. The relative exchange and R-transitivity on flags allow correction by N. In N, the flags and λ have equal stabilizers. Moreover, the pinned F_r fixes λ and fixes $\mathbf{P}_\lambda, \mathbf{P}_{-\lambda}$.

With $Y = [\mathbf{K}', \mathbf{K}']^F$, put $Z_0 = \mathrm{Z}(\mathbf{K}')^{\circ F}$. The group $[\mathbf{K}', \mathbf{K}']$ is simply connected since $\mathbf{K}'$ is a Levi in simply connected E_7. Thus, $Y = X_4 \times X_1$, with $X_4 = \mathrm{Spin}_8^+(q)$ and $X_1 = \mathrm{SL}_2(q)$. The mark J is inner on D_4, and the remaining A_1 is split. The basis of $Y(\mathrm{Z}(\mathbf{K}')^\circ)$ is $(\lambda, \alpha_3^\vee)$. Their span V_Z has zero active coordinates, so $V_Z \cap Q(E_7)$ is $\mathbb{Z}\lambda \oplus \mathbb{Z}\alpha_3^\vee$. The torus factors have orders $q - 1, q \pm 1$, respectively. Moreover, the quotient $\mathbf{K}'/[\mathbf{K}', \mathbf{K}']$ is isogenous to $\mathrm{Z}(\mathbf{K}')^\circ$. By Lang on $[\mathbf{K}', \mathbf{K}']$, $\ell \nmid |K'/Y|$. Thus, $P \leq X_4$.

On X_4, take the three packets at $d = 4$ from Lemma 6.33 and Proposition 6.123 on X_4. The projected signs are $+, +, -, -$. If $c = 1$, the spaces $4^-, 4^-$ give the natural representative k. At $c = -1$, the other packets are triality translates k^γ. The mark on A_1^4 is nontrivial: a trivial mark gives ℓ'-order, contrary to $P \leq X_4$ of rank 2. Each mark has 10 branches by Lemma 6.33. On X_1 take the identity on $\mathcal{E}(X_1, s_1)$ with noncentral quadratic s_1, since $P \cap X_1 = 1$. The types D_4, A_1 differ, so the tensor comparison on Y is equivariant, including for simultaneous F_r. Moreover, the X_1-matrices coincide on both sides of Y.

Apply the comparison on $Y, N_Y(P)$ in $K'B, B$. By Sylow conjugacy, $K' = YN'$, and $C_{K'B}(Y)$ fixes P. In $K'B$, the normalizer of P is B, since $K' \cap N = N' \leq B$. Restrict to the images of $B, K'B$ in $\mathrm{Aut}(Y)$. By [141, Theorem 2.9] and Proposition 7.6, we obtain $\succeq_c$ on K', N' with the same multiplicity character μ over each pair of Y-orbits. If $Z_0 \leq C_{K'}(Y)$, both the lower scalars and μ-matrices coincide. The Clifford indices and $\mu(1)$ divide the order of K'/Y, an abelian ℓ'-group.

Impose on Z_0 the character ζ_0 dual to value 1 on λ and to the external ratio u_3 with $u_3^2 \neq 1$, allowing all transport values of u_3. Moreover, in torus duality, restriction to Z_0 corresponds to projection to Z_0^*. Restriction to Y gives dual adjoint torus labels on X_4, X_1. The projected series follow from $\mathrm{Res}^{\widetilde{K}'}_{K'}$ or central isotypy for $R_T^{K'}$. Furthermore, duality on the local torus T_0 gives the Clifford sources and their X_1-character. Moreover, the X_4-tag gives $+,+,-,-$ on the active coordinates, modulo $-I$. At x_5, x_6, the two residual root values are $x_5/x_6, x_5x_6$. One is 1 on λ, the other is -1 on the X_1-tag. Thus, x_5, x_6 have order 4 and the external ratio is u_3. Conversely, every aligned $Q(2,2)$ gives these Y-tags. Include all central translates with ζ_0.

Suppose $s_1^g = s_2$ for $g \in K^*$ and common central leg λ^*. The eigenspaces V_i, V_{-i} form a pair determined by s_2, also modulo $t \mapsto -t$ from the kernel $\langle v \rangle$. Thus, this pair determines the quartic A_1 in $C_{\mathbf{K}^*}(s_2)^\circ$. Its positive coroots $\lambda^*, (\lambda^*)^g$ lie on oriented split tori of A_1. The flag pairs in $\mathbf{R}^{*F}$ are conjugate, so correct g to fix λ^*. Then, $g \in K'^*$, with λ^* the simply laced dual, scaled into $Y(\mathbf{T}^*)$. Each upper class gives λ^* by choosing its quartic A_1-flags.

Moreover, on N the central tag selects the constituent ψ_1 on $N_1 = N_N(\mathbf{R})$. Any other coset exchanges ± 1. Cancel 1 or St_R to recover the N'-character ξ' of Lemma 7.242. Thus, each lower ξ' and $\gamma \in \{1, \mathrm{St}\}$ gives a unique upper ψ. Induction is equivariant for $B, \mathcal{H}_\ell$. The upper degree ratio is $1:q$ by Jordan degrees with equal splitting 1 or 2, and by the flag module of R. Thus, transport fixes γ.

Match lower characters ξ, ξ' at the same partition γ, giving upper characters χ, ψ. Next, we put $B_0 = B_{\xi'} = B_\xi$. Recovery and the factorizations of Γ and Γ_P give ordinary inertias KB_0, NB_0, with identical mixed stabilizers on B_0/N'. We have $N' \leq B_0$. Moreover, the lower comparison extends $\xi \boxtimes (\xi')^\vee$ to $(K'B_0) \times_{B_0/N'} B_0$ and the corrected mixed stabilizer elements a fix it. The tensor cancels the common factors α and functions μ_a.

Inflate along Levi projections from $\mathbf{P}_\lambda^F B_0$ and $(\mathbf{P}_\lambda^F \cap N)B_0$, which are homomorphisms since B_0 fixes both parabolics. Induce to $(KB_0) \times_{B_0/N'} (NB_0)$. The source subgroup surjects onto B_0/N', so Mackey on $K \times N$ gives $V_K \boxtimes V_N^\vee$ for the two flag inductions. The $\chi \boxtimes \psi^\vee$-component has multiplicity 1 by Lemma 7.242, giving the extension $\widehat{\Psi}$. For a mixed stabilizer element a, take the same $n \in N$ on both sides to correct into B. Conjugation by (n,n) is inner. The corrected a fixes the flags $\mathbf{B}_\pm$, the lower extension, and χ, ψ, so it fixes $\widehat{\Psi}$. Each $c \in C_{KB_0}(K)$ fixes the flags and belongs to B_0. Its diagonal action on $V_K \boxtimes V_N^\vee$ is trivial since it centralizes K, N and fixes the cosets $K/P_\lambda, N/(P_\lambda \cap N)$. Thus, the scalars at c agree and $\succeq_c$ holds on K, N. At $u_3^2 \neq 1$, we apply Proposition 7.172 on leg 3. Moreover, the union includes u_3 with all its central lifts t_3. Clifford induction to $N_G(P)$ and signed induction to G give $\succeq_c$ on the union. □

Proposition 7.244. *On each exact $I(3,3)$-series, by* Proposition 7.237*, we obtain $\succeq_c$ for $\widetilde{K}, \widetilde{N}$ and all σ returning the series. If $c = 1$, the six unequal-hook characters are σ-fixed. Each pair has sign g_σ from* Proposition 7.238*.*

Proof. Let V_+, V_- be the natural six-spaces over $\overline{\mathbb{F}}_q$. Next, we choose opposite isotropic lines $L_\pm, L_\pm^-$ in $V_\pm$, rational at $c = 1$. At $c = -1$, we choose L_+, L_+^- over $\mathbb{F}_{q^2}$ and their F-images in V_-. The space 6^- has Witt index 2 and also admits opposite lines $L_\pm, L_\pm^-$. Moreover, the planes $L_+ \oplus L_-$ and $L_+^- \oplus L_-^-$ are rational opposite planes of weights $1, -1$ and their common orthogonal complement weight 0. Choose a dual cocharacter λ^* on this ray. The parabolics for $\pm\lambda^*$ have common Levi of type D_4A_1. Next, write $\mathbf{J}$ for the primal Levi and $J = \mathbf{J}^F$.

On each $V_\pm$, $\mathrm{SO}(V_\pm)$ is transitive on opposite lines, with $L_\pm^0$ connected, the line Levi. Over $\overline{\mathbb{F}}_q$, by isotypy $C_{\mathbf{K}^*}(s)^\circ$ has the same flag action on $(L_\pm, L_\pm^-)$. Moreover, its stabilizer is $C_{C_{\mathbf{K}^*}(s)^\circ}(\lambda^*)$, connected. Thus, Lang gives a single $C_{\mathbf{K}^*}(s)^{\circ F}$-orbit even at $F(V_\pm) = V_\mp$. Correct the component in $C_{\mathbf{K}^*}(s)^\circ$ so it fixes λ^*. The map $C_{\mathbf{J}^*}(s) \to A_K(s)$ is onto with connected kernel $C_{C_{\mathbf{K}^*}(s)^\circ}(\lambda^*)$. Thus, the rational J^*-class is unique above the K^*-class for this distribution. If $s_1^g = s_2$ for $g \in K^*$, correct g by $C_{\mathbf{K}^*}(s_2)^{\circ F}$ to align the pairs. Then $g \in J^*$. Powers ℓ^k fix the unordered pair V_+, V_- and the line label λ^*. Thus, conjugacy in K^* gives conjugacy in J^*.

In the regular overgroup, $C(\widetilde{s})$ is connected. The Jordan map for $\widetilde{s}$ transfers Harish–Chandra induction to $C(\widetilde{s})$ by [72, Theorem 4.7.5]. If $c = 1$, each $V_\pm$ has line Levi $L_\pm^0$. In split $D_3 = A_3$, the middle parabolic has Weyl group $S_2 \times S_2$. Then, on the ordered partitions $(4), (3,1), (2,2), (2,1,1), (1^4)$, the column for $1 \boxtimes 1$ is $(1,1,1,0,0)$ and that for $\mathrm{sgn} \boxtimes \mathrm{sgn}$ is $(0,0,1,1,1)$. Column 1 is the permutation character on $\binom{\{1,2,3,4\}}{2}$ and the second twists by sgn. Moreover, at 6^-, the principal relative group is $W(B_2)$, with empty 2-core for each hook λ. The line Levi has subgroup $W(B_1)$ with labels $1, \mathrm{St}$. By Corollary 6.169, restriction to $W(B_1)$ gives one sign ± 1 for each η of degree 1 and both for η of degree 2, since a reflection has spectrum $1, -1$. Each hook has coefficient 1 in some column j, taken alike on V_+, V_- at equal hooks. If $c = -1$, both columns hold over $\mathbb{F}_{q^2}$, including $(2,2)$. The projective eigenvalues on the 2-plane are distinct. Its factor is T_2.

Moreover, an upper C_2-fixed label ρ has multiplicity 1 over an extremal lower label $\tau = 1$ or St on the connected factors. Next, take $\widetilde{s}$ on both levels J, K. By $\widetilde{K} = K\widetilde{J}$, Mackey gives Res Ind = Ind Res. The pairs (τ_+, τ_-), (ρ_+, ρ_-) are diagonal orbits. Translation on $\mathcal{E}(\widetilde{K}, \widetilde{s})$ acts through C_2, so the connected label ρ alone contributes to (ρ_+, ρ_-). Moreover, the sum of inductions of $\tau_\pm$ gives each $\rho_\pm$ with multiplicity 1. Diagonally equivariant incidence matches $\{\tau_\pm\}, \{\rho_\pm\}$. Induction is equivariant, and σ returns $\mathcal{E}(J, s)$ and the upper label ρ. Thus, the incidence is equivariant for σ. Next, for 6 unequal-hook labels, apply [169, Theorem 1.1]. Each λ is rational and fixed by γ, F_r. Moreover, swapping $V_\pm$ fixes $\{\lambda, \nu\}$, with one constituent on K.

For the extremal pair, take $\lambda^* = e_5 + e_6$ in split D_6 coordinates. Moreover, the planes span 4^+, with complement 8^+ in the split 12-space. Put $\delta = e_5 - e_6$. The positive coroot half-sum on D_4 is $(3, 2, 1, 0) \in Q(D_4)$. Thus, $(2\rho_J^\vee)(-1) = \delta^\vee(-1)$. It is nontrivial modulo $\mathrm{Z}(\mathbf{J})^\circ$: if $\delta/2$ belonged to $Q(E_7)$ modulo $\langle e_5 + e_6, e_7 - e_8\rangle_{\mathbb{Q}}$, its representative v would be integral since $v_1 = \cdots = v_4 = 0$. But $v_5 - v_6 = 1$ and $v_7 + v_8 = 0$ give odd coordinate sum, excluding $v \in Q(E_7)$. Since $Q(D_4A_1)$ is a direct summand of $Q(E_7)$, the torsion of $P(E_7)/Q(D_4A_1)$ injects into $P(E_7)/Q(E_7)$ of order 2. Thus, the diagonal action on $\psi_\pm$ factors through C_2.

By [31, Theorem 3.1], its nonidentity element exchanges $\tau_\pm$ since $|A_J(s)^F| = 2$. If $b = \ell^k \bmod r$, the trace scaling has discrepancy $(2\rho_J^\vee)(-1)$ at nonsquare $b \in \mathbb{F}_q^\times$, by Proposition 7.238. Alvis–Curtis duality gives g_σ on $\tau_\pm^*$, and incidence gives g_σ on $\rho_\pm$. Lemma 7.240 and Corollary 7.239 give sign g_σ locally. Each diagonal inertia has index 2. A common character ω of $\mathrm{Z}(\widetilde{\mathbf{K}})^F$ gives equal ratios ω^σ/ω on the central products $K\mathrm{Z}(\widetilde{\mathbf{K}})^F, N\mathrm{Z}(\widetilde{\mathbf{K}})^F$ by Corollary 7.239.

At $\lambda \neq \nu$, the singleton has inertia $\widetilde{K}$ globally and $\widetilde{N}$ locally. Lift s to $\widetilde{s}$ in the marked rational torus $\widetilde{T}_0^*$. Next, take the regular parent $\widetilde{\chi}$ with ordered hooks (λ, ν) and the source $\widetilde{\beta}$ at $\widetilde{s}$. Since $C_{\mathbf{L}_0^*}(s)$ is connected, $\widetilde{\beta}|_{L_0} = \beta$. Write $N_{\beta,0}$ for the preimage of W_0 in N_β. By Proposition 7.237, the inertia of $\widetilde{\beta}$ is $E = \widetilde{L}_0 N_{\beta,0}$. Extend it by $\widehat{\beta}|_{N_{\beta,0}}$. By Proposition 7.237 and its proof, $\widetilde{L}_0 = L_0\mathrm{Z}(\widetilde{\mathbf{L}}_0)^F$. Moreover, the scalar on $\mathrm{Z}(\widetilde{\mathbf{L}}_0)^F$ is $N_{\beta,0}$-fixed and has value $\beta(z)/\beta(1)$ at $z \in \mathrm{Z}(\widetilde{\mathbf{L}}_0)^F \cap L_0$. Thus, the models glue on E. Tensor with $(\lambda, \nu) \in \mathrm{Irr}(W_0)$ for $W_0 = C_4^2$ and induce to $\widetilde{N}$, giving $\widetilde{\psi}$. By [95, Problem 5.6], $\widetilde{\psi}|_N$ is induced from the unequal W_s-orbit. Also, $\widetilde{\chi}|_K$ is the singleton χ. The scalars agree by duality at $\widetilde{s}$.

For σ stabilizing the series, Lemma 7.240 corrects s^{ℓ^k} in L_0^* by 1 or $-I_2$ on $\langle e_5, e_6\rangle$. The central kernel is $\mathbb{G}_m$, so [114, corollary to Theorem 1] lifts the correction to $\widetilde{L}_0^*$. Moreover, the corrected power is $\widetilde{s}z_\sigma$ with quotient character ξ_σ dual to z_σ. Both corrections fix V_+, V_-. Hence, they fix the ordered hooks (λ, ν). The Jordan map gives $\widetilde{\chi}^\sigma = \xi_\sigma\widetilde{\chi}$ and $\widetilde{\beta}^\sigma = \xi_\sigma\widetilde{\beta}$. On $N_{\beta,0}$, the twist of $\widehat{\beta}$ is trivial since $\epsilon|_{W_0} = 1$, and the relative (λ, ν) has values in μ_4, fixed by σ. After cancelling ξ_σ, the two extensions on E agree on $\widetilde{L}_0, N_{\beta,0}$. Induction gives $\widetilde{\psi}^\sigma = \xi_\sigma\widetilde{\psi}$. Moreover, the mixed functions coincide on $\widetilde{K}/K = \widetilde{N}/N$, and inner conjugation fixes $\widetilde{\chi}, \widetilde{\psi}$. □

Corollary 7.245. *The $I(3,3)$-union has $\succeq_c$ for each normalizer action of exponent $i = 0$, including stabilizing pairs with $\mathcal{H}_\ell$. Thus,* Proposition 7.255 *applies over $\mathbb{F}_r$ at odd r.*

Proof. The central direction of $\mathbf{K}$ is $\alpha = e_7 - e_8$. In $W(E_7)$, $C_W(\alpha) = W(D_6)$ by the perpendicular roots, and s_α gives the other coset in $N_W(\mathbb{Q}\alpha)$. The rank-one group $\langle U_\alpha, U_{-\alpha}\rangle$ is rational even when $F(\alpha) = -\alpha$. Choose a by Lang on $\mathbf{T}_\alpha$, with a rational and inverting $\mathbf{T}_\alpha$. Root commutators give a centralizes the derived D_6. Hence, a normalizes P, L_0, J. It fixes the derived coordinates U_γ and $\mathrm{Z}(\widetilde{\mathbf{G}})$. Thus, $1, a$ represent the zero-field quotient modulo $\widetilde{K}$. The marked normalizer has kernel $\widetilde{N}$. The second coset inverts $u_3 = t_3^2$ with $u_3^2 \neq 1$. At exponent 0, by Proposition 7.244, we obtain I, I' and their extensions at $i = 0$.

Given (a, σ) in the joint stabilizer, a fixes the spin lift x with $x^4 = 1$, and $\ell^k \equiv 1 \bmod 4$. The external coordinate goes to $t_3^{-\ell^k} = \pm t_3$. At $-t_3$, the residual double inversion corrects z_v in L_0^* by Lemma 7.240. In both cases β returns and each rational $V_\pm$ is fixed by the correction in L_0^*. In Proposition 7.244, $t_3 \mapsto t_3^{-1}$ fixes flag type. Thus, the unique J-series returns. Since a centralizes $[\mathbf{J}, \mathbf{J}]$, it fixes the pinned ψ. Thus, (a, σ) has sign g_σ on $\tau_\pm$. Their incidences of multiplicity 1 give g_σ on $\rho_\pm$.

Locally take $S = CS_1$ and τ of Proposition 7.241. Since S_1 lies in the spin group, a centralizes S_1 and acts trivially on W_s. Both $\widehat{\beta}^{a\sigma}$ and $\widehat{\beta}^{d_b}$ extend β. On their scaled trace lines the S_1-characters are $\tau^{a\sigma}, \tau$, equal since $\tau(S_1) \subseteq \mu_4$. Their C-scalars also agree as ν returns. Thus, their quotient twist is 1 on $S/C = W_s$. By Proposition 7.238, the twist of $\widehat{\beta}$ is $\epsilon^{(1-g_\sigma)/2}$.

Given $\chi_\pm$, choose the common character ω on $\mathrm{Z}(\widetilde{\mathbf{K}})^F$. The nonsquare diagonal exchanges $\chi_\pm$ and fixes $Z = \mathrm{Z}(\widetilde{\mathbf{K}})^F$. Hence, both ratios are $\omega^{a\sigma}/\omega$. Given $\lambda \neq \nu$, lift the correction in L_0^*, giving $\widetilde{s}z_{a,\sigma}$. Proposition 7.244 gives the quotient twist. On $\widetilde{K}$, Jordan decomposition is a-equivariant, and a fixes (λ, ν). Moreover, the local twist on $N_{\beta,0}$ is 1. Thus,

both extensions acquire $\xi_{a,\sigma}$. The common centralizer fixes $\mathcal{E}(K,s)$ and lies in $\widetilde{K}$. Transport from $\mathcal{E}(K,s)$ across its orbit $\mathcal{O}$. Moreover, the stabilizers make transport on $\mathcal{O}$ independent of choice. Over $\mathbb{F}_r$, odd r, the outer action of $E_7(r)$ is diagonal. By Proposition 7.172, we obtain $\succeq_c$ on $G, N_G(P)$. □

Corollary 7.246. *Let $q = r^f$, and $m = |u_3| > 2$ on $I(3,3)$. Next, put $d_m = |\langle \overline{r} \rangle|$ in $(\mathbb{Z}/m\mathbb{Z})^\times / \langle -1, \ell \rangle$. Every joint stabilizer element has exponent $i \in d_m\mathbb{Z}$, and $d_m \mid f$. If $d_m = f$, the full orbit has $\succeq_c$ on $G, N_G(P)$. Ordinary stabilizer exponents are divisible by e_m, the order of r in $(\mathbb{Z}/m\mathbb{Z})^\times / \langle -1 \rangle$.*

Proof. The moving leg determines the unoriented $\mathbb{Q}\alpha$. Moreover, field exponent i acts on its ratio by r^i up to inversion, or r^{-i} under pullback. The coefficient exponent is ℓ^k since u_3 has ℓ'-order. The ratio u_3 is defined on the quotient by $\mathbf{K}^*_{\mathrm{der}}$, so its central translates fix u_3. Thus, joint stabilization requires $r^i \in \langle -1, \ell \rangle \bmod m$, and ordinary stabilization requires $r^i = \pm 1 \bmod m$. Rationality gives $u_3^q = u_3^{\pm 1}$. Hence, $d_m \mid f$. If $d_m = f$, each stabilizer has field exponent 0. By Corollary 7.245, we obtain $\succeq_c$ on $\mathcal{E}(K,s)$. Moreover, transport gives $\succeq_c$ on $\mathcal{O}$ independently of the stabilizers. The inertias have $i = 0$ on projective models. The transfer covers $\mathcal{O}$ under the induced action. At $(q, \ell, m) = (27, 5, 13)$, the subgroup $\langle -1, 5 \rangle$ has order 4, and 3 has order 3 modulo it. □

Lemma 7.247. *Given $I(3,3)$, we choose ϕ in the prime-field coset with $\phi(\mathbf{K}) = \mathbf{K}$ and $\phi(P) = P$. It acts as F_r on the rational pinning of residual A_1^2. Moreover, every $w \in W_s$ has a ϕ-fixed representative in the stabilizer S^F of the pinning. Moreover, on $X = \mathrm{Spin}^+_{12}(q)$, we have that ϕ is pinned field up to $\mathrm{Int}(x)$, $x \in X$.*

Proof. First, choose split hyperbolic coordinates over $\mathbb{F}_r$, with pairs p_i, q_i and quadratic form xy on $xp_i + yq_i$. On the first two pairs, let A_0 exchange the planes and act by $p_1 \leftrightarrow q_1$, in the order giving R_1. Take the identical B_0 on the next two pairs, giving R_2. Let J_0 act by $V_1 \leftrightarrow V_2$ via their bases p_i, q_i. Moreover, these are permutation matrices, and A_0B_0 satisfies $A_0B_0D = DA_0B_0$ for $D \in \{A_0, B_0, J_0\}$. On the last two planes let U_0 be reflection in $p_6 + q_6$, exchanging the residual A_1-factors. Next, we choose positive roots $e_5 - e_6, e_5 + e_6$ and generators exchanged by U_0. This pinning is defined over $\mathbb{F}_r$.

By [189, p. 165], the spinor norm is -1 for the plane swap with hyperbolic (-1)-eigenspace H_-. The swap $p_i \leftrightarrow q_i$ reflects in $p_i - q_i$ of norm -1. Thus, A_0, B_0 have square spinor norm. The element J_0 has two plane swaps, while U_0 reflects in $p_6 + q_6$ of norm 1. Hence, $A_0U_0, B_0U_0, J_0 \in \mathrm{SO}_{12}(\mathbb{F}_r)$ have spinor norm 1 modulo squares. By [87, Theorem 7], choose lifts A, B, J_1 in $\mathrm{Spin}_{12}(\mathbb{F}_r)$. Put $T = AB$, whose image is A_0B_0.

Twist split F_q by T, with an external A_1-lift over $\mathbb{F}_r$ if its torus T_{ext} is nonsplit. Also, this lift centralizes X. The active torus has cycles R_1, R_2, both negative. Moreover, the residual A_1^2-pinning is split. Lang on $\mathbf{T}$ conjugates the rational representatives of $w = R_1R_2$. Next, let F denote the twist. The split r-power map ϕ satisfies $\phi F = F \phi$. If $g^{-1}F_q(g)$ is the twist element over $\mathbb{F}_r$, $g\phi(g)^{-1}$ is F_q-fixed, so ϕ represents the ambient field coset. On X, Lang chooses $g \in \mathrm{Spin}_{12}$ for T. Thus, $g\phi(g)^{-1} \in X$.

Given $w \in C_4 \wr S_2$, choose a word n_w in A, B, J_1 above w on the active blocks. On V_0, its image is 1 or U_0 according to rotation parity. Its matrix N_w on the natural module has $N_wA_0B_0 = A_0B_0N_w$, so $F(n_w)n_w^{-1} \in \{1, z_v\}$. In the active spin torus put $c = (e_1 + e_2)^\vee(-1)$. Specifically, its additive coordinate is $(e_1 + e_2)/2$ modulo $Q^\vee(D_6)$. The quarter-turn changes it by $-e_1$, representing z_v. Since $c^2 = 1$, we have $F(c)c^{-1} = z_v$ for odd q, and $\phi(c) = c$. Thus, n_w or cn_w is fixed by both F, ϕ. The lift permutes the A_1^2-pinning and normalizes $\mathbf{D}$. If $w \in W_s$, it lies in N_β. □

Proposition 7.248. *The matching on $I(3,3)$ is equivariant under ordinary transports and $\mathcal{H}_\ell$. The tensor extension U_0 on H_0 extends to H, the fibre product of ordinary inertias I, I', and is fixed by every (g, σ) in the joint stabilizer. Moreover, the extensions form an $\mathrm{Irr}(H/H_0)$-torsor, where H/H_0 is cyclic of order $t \mid f/d_m$.*

Proof. First, choose the field lift ϕ of Lemma 7.247 and external inversion a of Corollary 7.245. Modulo $\mathrm{Inndiag}(K)$, each transport is $\phi^i a^e$. Subtract its field exponent i, and apply Corollary 7.245 at exponent 0. Suppose it returns $\mathcal{E}(K,s)$ jointly with $\sigma \in \mathcal{H}_\ell$. The source returns as an L_0-character. Choose a rational dual torus $\mathbf{T}^*$ containing s, with residual type T^*_{res}. Next, in Lemma 7.247, we choose n to change the split residual torus $\mathbf{T}_{\mathrm{res}}$. Next, choose n as a D_2 Weyl lift over $\mathbb{F}_r$. Thus, $\phi(n) = n$. By [114, corollary to Theorem 1], we choose h in the connected residual group with $h^{-1}F(h) = n$. Then, $h\phi(h)^{-1}$ is a rational residual correction. Modulo L_0^*-conjugacy, the torus action is $r^{\pm i}$. Duality matches marks and roots on $\mathbf{T}, \mathbf{T}^*$. Moreover, the joint odd power fixes $0, 2$, the binary entries of x and changes its lift by 1 or z_v. The external ratio may contribute another z_v in $\mathbf{K}^*_{\mathrm{der}}$. Residual double inversion corrects z_v by Lemma 7.240, and fixes $V_\pm$ separately at $c = 1$. The exact L_0-series contains only β.

By Proposition 7.244, each two-plane flag $\mathcal{F}$ gives a unique lower series. Take $\mathbf{P}$ as the standard parabolic for $\mathcal{F}$ in split coordinates of $[\mathbf{K},\mathbf{K}]$. An inner spin correction gives pinned F_r by Lemma 7.247. Also, it fixes $\mathbf{P}$, the Levi pinning, and ψ. The split model conjugates F_r on $[\mathbf{K},\mathbf{K}]$. Its actions on $[\mathbf{K},\mathbf{K}]$ and $\mathrm{Z}(\mathbf{K})^\circ$ determine the action on $\mathbf{K}$. Moreover, the external inversion centralizes $[\mathbf{K},\mathbf{K}]$. Thus, each global pair has sign g_σ, and the six singletons return under $\phi^i a^e$. A nonsquare diagonal exchanges χ_+,χ_-.

Locally, $\phi^i a^e$ fixes the pinned trace character ψ. Next, take the pinning stabilizer $S = CS_1$ of Proposition 7.238, with residual reflection U_0 from Lemma 7.247. The group S_1 is generated by $\mathbf{D}^F$ and the fixed lifts of W_s: its active Weyl kernel is $\mathbf{D}^F$ by Corollary 7.239. The line character τ takes signs on $\mathbf{D}^F$. Given $w^2 \in [W_s,W_s]$, choose $x,y \in S_1$ above w, w^2, with $y \in [S_1,S_1]$. Then $x^2y^{-1} \in \mathbf{D}^F$, so $\tau(x)^4 = \nu(x^2y^{-1})^2 = 1$.

If $W_s = C_4 \wr S_2$, the remaining generator is R_1. Its natural lift has quartic pair $(i,-i)$ and two eigenvalues -1, from R_1, U_0. Its spin fourth power is z_v, with $\nu(z_v) = 1$ by Corollary 7.239. Correction by c fixes this power since $1 + R_1 + R_1^2 + R_1^3 = 0$ on V_1. In the exchanged case, $W_s = D_8 \circ C_4$ and $W_s/[W_s,W_s] \simeq C_2^3$. Thus, $\tau(S_1) \subseteq \mu_4$ at $c = \pm 1$.

On the chosen lifts n_w, field acts trivially, a centralizes the spin group, and σ fixes μ_4. On $\mathbf{D}^F$, odd powers fix the signs of τ. Moreover, the joint action fixes ν on C. Proposition 7.238 compares the scaled ψ-lines and gives twist $\epsilon^{(1-g_\sigma)/2}$. The quotient action on W_s is 1. Moreover, a nonsquare diagonal exchanges χ_+,χ_-. On a pair, U_0 has its base tensor and 1 on $Z = \mathrm{Z}(\widetilde{\mathbf{K}})^F$. The image of Z is the diagonal inertia quotient D_0, so the joint stabilizer fixes U_0. If $\lambda \neq \nu$, Proposition 7.244 gives ordered lifts. The kernel Z_c is connected. Lift the correction through Z_c, giving quotient twist ξ. Moreover, both global hooks λ,ν return. Locally there is no γ-label. Thus, ξ twists the source and $\widetilde{\chi}$. The additional local twist is 1 on $N_{\beta,0}$, and σ fixes the ordered relative character (λ,ν). Glue on $\widetilde{L}_0 N_{\beta,0}$. Proposition 7.244 gives the local twist ξ. On $\widetilde{K},\widetilde{N}$, the characters $\widetilde{\chi},\widetilde{\psi}$ are invariant under conjugation. Hence, ξ cancels on H_0 for every i.

The stabilizers define transport on $\mathcal{O}$ and H-invariance of U_0. By Corollary 7.246, H/H_0 is cyclic of order $t \mid f/d_m$. By [95, Corollaries 6.17 and 11.22], we obtain an $\mathrm{Irr}(H/H_0)$-torsor of extensions. If $c \in C_H(K)$, its field exponent is 0, since a nonidentity field operation on $X/\mathrm{Z}(X)$ is outer modulo $\mathrm{Inndiag}(X/\mathrm{Z}(X))$. Hence, the scalars on $C_H(K)$ are those of U_0. □

Corollary 7.249. *An $I(3,3)$ pair with ordinary field inertia order t has $\succeq_c$ if $2 \nmid t$. Thus, $2 \nmid (f/e_m)$ gives $\succeq_c$ throughout the series. In particular, $2 \nmid f$ gives the fourth-order case for $E_7(r^f)$.*

Proof. Write $s = uz$, with $u = s_2$ and $z = s_{2'}$. In the adjoint derived group, s has two-power image, with lift x on the six coordinates satisfies $x^4 = 1$ by Lemma 7.240. In $s = x\alpha_3^\vee(t_3)$ only the external factor contributes to z. The cocharacter $\alpha_3^\vee$ is orthogonal to derived roots of $\mathbf{K}^*$. Thus, $z \in (Z(\mathbf{K}^*)^\circ)^F$, also rational as an element of $\langle s\rangle$. Under duality for the quotient torus $\mathbf{K}/[\mathbf{K},\mathbf{K}]$, z gives a linear character κ of K, trivial on $X = [\mathbf{K},\mathbf{K}]^F$. Moreover, $\mathbf{K}/[\mathbf{K},\mathbf{K}]$ is dual to $Z(\mathbf{K}^*)^\circ$. By [133, Proposition 23.2], $K/X = (\mathbf{K}/[\mathbf{K},\mathbf{K}])^F$. The κ^{-1}-twist takes the s-series to the u-series. The corresponding torus translation for L_0 twists the local source by $\kappa^{-1}|_{L_0}$, and its induced partner by $\kappa^{-1}|_N$. Moreover, these κ^{-1}-twists follow from a lift of z to $\widetilde{\mathbf{K}}^*$. At a lift $\widetilde{z}$, each root has value 1, so $\widetilde{z}$ is central. Lang gives a rational $\widetilde{z}$. Write $\widetilde{\kappa}$ for the character from $\widetilde{z}$. By [169, Theorem 6.2 and Lemma 6.4], the translated lift for $\widetilde{\xi} \in \mathrm{Irr}(\widetilde{K})$ has restriction $(\widetilde{\xi}\widetilde{\kappa}^{-1})|_K = (\widetilde{\xi}|_K)\kappa^{-1}$. Moreover, both $\mathbf{K}^*$ and $\mathbf{L}_0^*$ have the same centralizers at s and at u.

Define $\chi_2 = \chi\kappa^{-1}$ and $\psi_2 = \psi(\kappa|_N)^{-1}$. Take the same inertias and $H_0 \leq H$ from Proposition 7.248. Each ordinary stabilizer of χ fixes the rational class of s, so it also fixes z and κ. The same reasoning fixes κ under the joint stabilizer. If $(x,y) \in H$, we put $D_\kappa(x,y) = \kappa(xy^{-1})$. $xy^{-1} \in K$, and invariance of κ under the ordinary inertia makes D_κ a linear character of H. Indeed, the second difference $x_2y_2^{-1}$ is conjugated by x_1 before multiplication by $x_1y_1^{-1}$. For (a,σ) in the joint stabilizer, $\kappa^{(a,\sigma)} = \kappa$, so $D_\kappa^{(a,\sigma)} = D_\kappa$. Moreover, $D_\kappa(x,x) = 1$. Next, put $V_0 = U_0(D_\kappa|_{H_0})^{-1}$. It extends $\chi_2 \boxtimes \psi_2^\vee$, is fixed by H and by the joint stabilizer. Its scalars match U_0.

Next, choose θ trivial on μ_{2^∞} and inverting odd-order roots. Thus, θ fixes u, the relative labels and the μ_4-values at pinned spin lifts. Let $d \in \widetilde{N}$ represent the nonsquare diagonal. Put $j = 0$ if -1 is square in $\mathbb{F}_q$, and $j = 1$ otherwise. Then, $v = (d^j,\theta)$ fixes the reduced pair. Indeed, s and u give the same flags on $V_\pm$. By Proposition 7.244, trace separates the regular u-pair. Inversion on μ_r scales trace by -1. Thus, θ has sign $(-1)^j$ on the lower pairs $\tau_\pm,\tau_\pm^*$. Hence, the sign on $\rho_\pm$ is $(-1)^j$. Rational hooks $\{\lambda,\nu\}$ label unequal fibres. Choose flags in K, even if the ambient centralizer of u exceeds K^*. The reduced L_0-series has one source τ_0. With original source τ, we have $\tau_0|_{S_1} = \tau|_{S_1}$

since $S_1 \leq X$. By Proposition 7.238, comparison with the (-1)-scaled line gives twist ϵ^j. On C, the reduced central character returns. On S_1, θ fixes the fourth roots. Thus, d^j cancels both permutations.

The operation v normalizes H, H_0. Indeed, d fixes κ on the quotient torus, and $\kappa^\theta = \kappa^{-1}$. Within the stabilizer of κ, fixing χ, ψ is equivalent to fixing χ_2, ψ_2. Moreover, on this reduced pair, d^j acts as θ, and normalizes both stabilizers. Inertia fixes κ on both sides and fixes i. Also v fixes V_0. For a pair, its base tensor $\chi_2 \boxtimes \psi_2^\vee$ and trivial diagonal-centre action determine V_0, since the centre surjects onto D_0. If $\lambda \neq \nu$, Proposition 7.244 gives ordered extensions. Next, let $\widetilde{\kappa}$ be their common lift of κ. The lift and its dual give D_κ on the fibre product. Division by D_κ gives V_0. Next, take the two-part $\widetilde{u}$ of a rational torus lift of u. Other lifts give equal quotient twists. θ fixes $\widetilde{u}$, the hooks (λ, ν) and reduced source. Moreover, the twist ϵ^j is trivial on $N_{\beta,0}$. Thus, the local extension in $\widetilde{N}$ is fixed after induction from $\widetilde{L}_0 N_{\beta,0}$. On $\widetilde{K}, \widetilde{N}$, regular diagonal conjugation fixes V_0.

By [95, Corollary 11.22], V_0 extends to H. Its extensions form a torsor for $\mathrm{Irr}(H/H_0) \cong C_t$ by [95, Corollary 6.17]. The order t is odd. On H/H_0, the ordinary part of v acts trivially and θ inverts characters. Thus, the v-fixed extension V is unique. Let (a, σ) lie in the joint stabilizer, with a marked modulo inner corrections. Then $c = [(a, \sigma), v]$ has coefficient part 1 and fixes the reduced pair and κ, since d fixes the a-stable quotient torus. Thus, c fixes (χ, ψ). Its diagonal image lies in H_0, since its field exponent is 0. It and base corrections act on H by inner conjugation. Thus, uniqueness fixes V. Thus, VD_κ extends with the given scalars on H_0. By Corollary 7.246, $t \mid f/e_m$. □

Proposition 7.250. *On $I(3,3)$ write $t = t_2 t_o$, with t_2 a power of two and t_o odd. In* Proposition 7.248*, we let H_2/H_0 be the subgroup of order t_2 in H/H_0. Next, we fix a generator yH_0 and choose any extension U_2 of U_0 to H_2. For each h stabilizing the pair jointly, define $\delta(h) \in \mu_{t_2}$ by ${}^h U_2 = U_2 \lambda_h$, $\delta(h) = \lambda_h(yH_0)$, $\lambda_h \in \mathrm{Irr}(H_2/H_0)$. δ is a homomorphism independent of U_2. The extension U_0 has a joint-invariant extension to H exactly when $\delta(h) = 1$ for each h in that stabilizer whose field component has 2-power order. Test h modulo ordinary inertia.*

Proof. The group H_2/H_0 is characteristic in the cyclic quotient H/H_0. The joint stabilizer fixes U_0, so the twist λ_h is unique. Ordinary transport fixes H/H_0. If $n = \mathrm{ord}_\ell(r)$, we have $n/\gcd(n, f) = 4$. Hence, $v_2(\ell - 1) \geq v_2(n) = v_2(f) + 2$. Thus, the coefficient operations fix μ_{t_2}. The translation by h is independent of the choice of U_2. Moreover, composition multiplies the values $\delta(h)$. The condition $\delta(h) = 1$ is independent of yH_0. By [95, Corollary 11.22], U_0 extends to H. Moreover, its restriction to H_2 is H-invariant. Thus, ordinary inertia fixes U_2. Given h in the joint stabilizer, the power h^{f/f_2} has two-primary field component. Since f/f_2 is odd, $\delta(h^{f/f_2}) = 1$ detects $\delta(h) = 1$.

Necessity follows by restriction to H_2. Suppose $\delta = 1$ and choose D_κ, V_0, v of Corollary 7.249. The construction up to V_0 holds at every order t. The element v fixes V_0 and acts identically on H/H_0. Moreover, the actions of v and an element a of the joint stabilizer on H give $[v][a] = [a][v]$ in $\mathrm{Out}(H)$. Next, we put $V_2 = U_2(D_\kappa|_{H_2})^{-1}$. It is fixed by H and all relevant stabilizer elements. Next, write ${}^v V_2 = V_2 \lambda$, with $\lambda \in \mathrm{Irr}(H_2/H_0)$. Restriction is an isomorphism. Thus, λ has a unique two-primary extension $\bar{\lambda}$ to H/H_0. Its values in μ_{t_2} are fixed by the joint stabilizer. By [95, Corollary 11.22], V_2 extends to H. The map $V \mapsto {}^v V \bar{\lambda}^{-1}$ fixes $V|_{H_2} = V_2$ and inverts twists in $\mathrm{Irr}(H/H_2)$. Since $|H/H_2| = t_o$ is odd, it fixes a unique V. Given a in the joint stabilizer, we have $V_2^a = V_2$. Thus, $V^a = {}^v(V^a)\bar{\lambda}^{-1}$, so uniqueness gives $V^a = V$. Then, VD_κ extends U_2 to H. □

Lemma 7.251. *In the flag Levi $\mathbf{J}$ of* Proposition 7.244*, we let $\mathsf{V}_+, \mathsf{V}_-$ be the sign spaces, with $\dim \mathsf{V}_\pm = 4$. Next, we choose opposed isotropic lines $\mathcal{L}_\pm, \mathcal{L}'_\pm \leq \mathsf{V}_\pm$. If $\mathsf{V}_+, \mathsf{V}_-$ are rational, we choose $\mathcal{L}_\pm, \mathcal{L}'_\pm$ rational. If $F(\mathsf{V}_+) = \mathsf{V}_-$, we choose $\mathcal{L}_+, \mathcal{L}'_+$ over $\mathbb{F}_{q^2}$ and set $(\mathcal{L}_-, \mathcal{L}'_-) = (F(\mathcal{L}_+), F(\mathcal{L}'_+))$. Then, $V_2 = \mathcal{L}_+ \oplus \mathcal{L}_-$ and $V'_2 = \mathcal{L}'_+ \oplus \mathcal{L}'_-$ are rational opposed isotropic planes. Let $\nu_2 \in X_*(\mathbf{J}) \otimes_{\mathbb{Z}} \mathbb{Q}$ have weights $1, -1, 0$ on $V_2, V'_2, (V_2 \oplus V'_2)^\perp$. Next, we set $\mathbf{J}_2 = \mathrm{C}_{\mathbf{J}}(\nu_2)$ and $J_2 = \mathbf{J}_2^F$. Given any upper component-fixed label χ, there exist a regular J_2-character ρ_2 and a constituent $\rho_1 \in \mathrm{Irr}(J)$. The flag multiplicities are one, from ρ_2 to ρ_1 and from ρ_1 to χ. Now, in a regular overgroup $\widetilde{\mathbf{K}}$, put $\widetilde{K} = \widetilde{\mathbf{K}}^F$. Next, we write $\widetilde{J}, \widetilde{J}_2$ for the rational Levi groups. Choose $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{K})$ with unequal hooks (λ, μ). Moreover, there exist regular $\widetilde{\rho}_2 \in \mathrm{Irr}(\widetilde{J}_2)$ and a constituent $\widetilde{\rho}_1 \in \mathrm{Irr}(\widetilde{J})$. The flag multiplicities are one, from $\widetilde{\rho}_2$ to $\widetilde{\rho}_1$ and from $\widetilde{\rho}_1$ to $\widetilde{\chi}$.*

Proof. The minus 4-spaces have Witt index 1. Let ν_1 be the first flag direction. Over $\overline{\mathbb{F}}_q$, the connected centralizer of ν_1, ν_2 on $\mathsf{V}_\pm$ is a line-flag Levi. The (ν_1, ν_2)-flags form one orbit. Match the ν_1-lines, then take their Levi on $\mathsf{V}_\pm$. By [133, Theorem 21.11(b)], we obtain one F-rational orbit, also if $F(\mathsf{V}_+) = \mathsf{V}_-$. The exchange component has an F-fixed lift by Proposition 7.244, fixing ν_1, ν_2. The rational components agree at $\mathbf{J}_2, \mathbf{J}, \mathbf{K}$. In $\mathbf{K}$, the directions split off two hyperbolic 4-spaces. The derived type of $\mathbf{J}_2$ is split $D_2 A_1 A_1$. Moreover, the A_1 factors act on the ν_1-plane

and on V_2. The projective eigenvalues differ on each 2-plane. The remaining 4-space has sign spaces of dimension 2. Thus, the connected centralizer in $\mathbf{J}_2^*$ is a torus. Next, for (λ, μ), we apply Proposition 7.244 on its 4-space. Its labels are $1, \mathrm{St}$. Opposite lines in a split 4-space give Borels in both A_1 factors, with flag character $1 + \mathrm{St}$. Thus, the torus induction contains $1 \boxtimes 1$ and $\mathrm{St} \boxtimes \mathrm{St}$, each with multiplicity 1. In a minus 4-space, F exchanges the factors. The flag gives $1 + \mathrm{St}$ over $\mathbb{F}_{q^2}$. Moreover, for exchanged 6-spaces, the split calculation is over $\mathbb{F}_{q^2}$. Moreover, on $\widetilde{K}$, the two Jordan steps have multiplicity 1 along $\widetilde{\rho}_2 \to \widetilde{\rho}_1 \to \widetilde{\chi}$.

On $\mathsf{V}_\pm$, match the component-fixed columns. At J_2, J, K, both diagonal conjugates occur, and $\widetilde{J}/J, \widetilde{K}/K$ are parabolic images. Thus, $\mathrm{Res}\,\mathrm{Ind} = \mathrm{Ind}\,\mathrm{Res}$ by [95, Problem 5.2]. Moreover, at each of $J_2 \leq J$ and $J \leq K$, only the chosen label contributes. The 2×2 matrix follows by diagonal transitivity. On J_2, the torus centralizer gives regularity. At $M = J_2, L_0$, put $z_M = (2\rho_M^\vee)(-1)$ for the split derived pinning. At z_{J_2}, z_{L_0}, both scalars agree. Indeed, the D_2 half-sum is a vector coordinate e_i, giving z_v in the spin group, as for L_0. The other coroots contribute at -1. Moreover, in each simply connected A_1, the dual image has order 2. The eigenlines on the ν_1-plane and V_2 have matching F-actions. Also, both tori have order $q-1$ or both $q+1$. By duality for SL_2, both characters are quadratic, agreeing at -1 coincide, so their product is 1. The scalar on z_v agrees by restriction of s, also with a regular lift $\widetilde{s}$. □

Proposition 7.252. *Component-fixed $I(3,3)$-labels give $\succeq_c$ at all degrees, including exchanged $I(3,3)$.*

Proof. Assume $2 \mid t$, with $n = t_2 > 1$ and $j = f/n$. The external torus splits because r^j returns u_3 up to inversion. Since n is even, this gives $u_3^q = u_3$. As $|u_3| > 2$, the mark splits. Lemma 7.247 gives $T = AB$. Next, in the split external A_1, we choose a root representative a over $\mathbb{F}_r$, with $a^2 = \alpha^\vee(-1)$. We have $F_r(a) = a$ and $ax = xa$ for $x \in X$. The normalized local representative ϕ and a global representative Φ differ by an element of X. Moreover, in the ambient overgroup, $\phi^f = T^{-1}$ and $\Phi^f = 1$. Indeed, untwist by $g^{-1}F_q(g) = T$. The image of ϕ is $g\phi(g)^{-1}F_r$, whose f-th power is $gT^{-1}g^{-1}$. Moreover, in untwisted coordinates, $\Phi = F_r$ fixes pinning and flag parabolics. Both ϕ, Φ satisfy $\phi a = a\phi,\ \Phi a = a\Phi$.

Next, choose $e \in \{0, 1\}$ so that $\Phi^j a^e$ fixes u_3. Representatives are $y_G = \Phi^j a^e$ and $y_N = \phi^j a^e$. Proposition 7.248 gives these ordinary stabilizer elements. Put $z = a^{en} \in Z(\mathbf{K})^F$, with $z^2 = 1$. Then, $y_G^n = z$ and $y_N^n = T^{-1}z$. The image R_1R_2 of $T \in S_1$ is central on both sides. Let c_η multiply the Whittaker scalar of $\widehat{\beta}$ and the scalar on η. Moreover, both scalars lie in μ_4.

Choose the pinned-trace label ρ from Lemma 7.251. Moreover, its mate is ρ's regular diagonal conjugate. Write ψ for the match on N and ω for the common scalar character on $Z(\mathbf{K})^F$. The lower series returns under y_G by flag transitivity, and y_G fixes the pinned trace. Hence, y_G fixes ρ. Choose an intertwiner with scalar d_G on the trace line, where $d_G^n = \omega(z)$. By Schur's lemma, we obtain the n-th power as the action of z. The local trace gives $\widehat{\beta}^{\,y_N} = \widehat{\beta}$. Tensor its intertwiner with 1_η, since y_N acts trivially on W_s. Next, we choose its trace scalar d_N with $d_N^n = \omega(z)c_\eta^{-1}$. By comparison on the trace, its n-th power is the operator of $T^{-1}z$. There T acts by c_η. Moreover, on the Clifford source, $T^{-1}z$ and the power implement y_N^n and agree on the trace. From $N_\beta\langle y_N\rangle$, induce to $N\langle y_N\rangle$ by field-quotient surjectivity. Both d_G, d_N lie in μ_{4n}, fixed by $\mathcal{H}_\ell$ by Proposition 7.250.

With base $J_2 \times N$, tensor over C_n with the local dual. Adjoin $\Delta Z(\mathbf{K})^F$ with action 1. Moreover, it centralizes $J_2 \times N$. Scalars agree on their intersection. On $\Delta Z(\mathbf{K})^F$, the representatives y_G, y_N act identically. Moreover, the generators are $J_2 \times N$, $\Delta Z(\mathbf{K})^F$ and (y_G, y_N). At zero field, paired inertia on J_2 has square diagonals only, giving full 2-primary inertia on J_2, with identical lifts on J, K.

Ordinary parts of joint stabilizer elements have representatives $\Phi^k a^h, \phi^k a^h$. Moreover, both have h modulo inner and diagonal corrections. Since q is square, $g_\sigma = 1$. Paired inertia excludes nonsquare diagonals. Thus, $\Phi^k a^h, \phi^k a^h$ satisfy $\Phi^k a^h y_G = y_G \Phi^k a^h,\ \phi^k a^h y_N = y_N \phi^k a^h$ and fix their traces. J_2 acts transitively on flags. Next, on N_β, choose trace comparison. Let $b_0 \in \mathbb{F}_r^\times$ be the power of σ on μ_r. Next, we choose $v_0 \in \mathbb{F}_q$ with $v_0^2 = b_0$. The elements $(2\rho_{J_2}^\vee)(v_0)$ and $(2\rho_{L_0}^\vee)(v_0)$ scale both traces identically. With y_G, y_N, their commutators in split pinning are $1, 1$ if $v_0^{x^j} = v_0$, and z_{J_2}, z_{L_0} otherwise. Lemma 7.251 equates their scalars. Thus, (y_G, y_N) has scalar d_G/d_N on both traces. Transport gives $(d_G/d_N)^\sigma = d_G/d_N$. Moreover, locally the stabilizing action fixes β and its label η. Read scalars before induction from N_β. The base intertwiner matches G, N-traces. Moreover, on (y_G, y_N), the quotient twist is trivial. On $\Delta Z(\mathbf{K})^F$, conjugation is inner.

Next, along $J_2 \leq J \leq K$, inflate and induce. Both flags admit y_G and the joint stabilizer, giving surjectivity onto H_2/H_0. The restriction gives flag induction along $J_2 \leq J \leq K$, tensored with $\psi^\vee$. Lemma 7.251 gives multiplicity 1 for the chosen J- and then K-constituent. Incidence fixes their spaces, giving U_2 on H_2. Its restriction to H_0 has the

original tensor and trivial $\Delta Z(\widetilde{\mathbf{K}})^F$-action. Thus, $U_2|_{H_0} = U_0$. By Proposition 7.250, we obtain $\succeq_c$. □

Proposition 7.253. *On separately rational $I(3,3)$, the six singleton fibres give $\succeq_c$ at every field degree.*

Proof. Assume $2 \mid t$ and choose $n, j, z, y_G, y_N, J_2, J$ from Proposition 7.252. Moreover, at zero field, H_0 contains the regular diagonal. Order $\mathsf{V}_+, \mathsf{V}_-$ and lift rationally to $\widetilde{s}$. Moreover, the corresponding regular extensions are $\widetilde{\chi}, \widetilde{\psi}$ from Proposition 7.244, with $\widetilde{\psi}|_N = \psi$ irreducible. On both flags take $\widetilde{s}$ up to conjugacy over $\mathbb{F}_q$. Lemma 7.251 gives the bottom label $\widetilde{\rho}$.

After inner and diagonal corrections for h, we set $g_G = \Phi^k a^l$, $g_N = \phi^k a^l$. Moreover, the diagonal lies in H_0 and acts by conjugation. In L_0^*, the correction of Proposition 7.248 stabilizes $\mathsf{V}_+, \mathsf{V}_-$ separately. The central translation of $\widetilde{s}$ gives quotient character ξ_h. By Lemma 7.251, $C_{\mathbf{K}^*}(s)^\circ$ corrects both flags. By [20, (2.2)], $C_{\widetilde{\mathbf{K}}^*}(\widetilde{s})^\circ$ maps onto $C_{\mathbf{K}^*}(s)^\circ$. Since Z_c is connected, [133, Proposition 23.2] gives rational lifts in $C_{\widetilde{\mathbf{K}}^*}(\widetilde{s})^\circ$. Thus, the corrections fix $\widetilde{s}$ on both flag Levis $\mathbf{J}_2, \mathbf{J}$. Ordered partitions return on J_2, J, K with quotient twist ξ_h. Moreover, locally ξ_h twists $\widetilde{\beta}$ and $\widetilde{\psi}$. The relative label on $W_0 = C_4^2$ returns, with source twist 1 on W_0. Thus, ${}^{g_G,\sigma}\widetilde{\rho} = \xi_h \widetilde{\rho}$, ${}^{g_N,\sigma}\widetilde{\psi} = \xi_h \widetilde{\psi}$, with ξ_h read on the common quotient $\widetilde{K}/K$.

For $M = J_2, J, K$, we put $D_M = \widetilde{M} \times_{\widetilde{K}/K} \widetilde{N}$. Specifically, for fixed $\widetilde{\psi}$ and $\lambda \in \mathrm{Irr}(\widetilde{M})$, the characters $(\lambda \boxtimes \widetilde{\psi}^\vee)|_{D_M}$ are orthonormal. Put $\alpha_\lambda = \lambda \boxtimes \widetilde{\psi}^\vee$. By [95, Lemma 5.2], $\langle \alpha_\lambda|_{D_M}, \alpha_{\lambda'}|_{D_M} \rangle = \sum_{\xi \in \mathrm{Irr}((\widetilde{M}\times\widetilde{N})/D_M)} \xi(1) \langle \alpha_\lambda, \xi \alpha_{\lambda'} \rangle$. Only the trivial twist fixes $\widetilde{\psi}$, since its restriction to N is irreducible. This holds even when $\lambda|_M$ splits. If $M = K$, we have $D_K = H_0$. Given $y = (y_G, y_N)$, let ξ_y be the twist. Next, choose operators A_G, A_N for y_G, y_N with correction ξ_y^{-1} and arbitrary scalar factors. Norm ξ_y over n powers is 1. Indeed, it stabilizes $\widetilde{\psi}$, since $y_N^n = T^{-1}z$ is inner, and $\widetilde{\psi}$ has no self-twist. Thus, the n-th powers act by conjugation by $z, T^{-1}z$, respectively. The unipotent radical lies in $\ker \xi_y$, so the global trace line of $\widetilde{\rho}$ is fixed. The $\widetilde{\beta}$-block in $\widetilde{\psi}$ has multiplicity 1, since $\eta_0 \in \mathrm{Irr}(W_0)$ is linear. Moreover, the torus identity ${}^{y_N}\widetilde{\beta} = \xi_y \widetilde{\beta}$ fixes the $\widetilde{\beta}$-block and its trace. There T acts by $\tau(T)\eta_0(R_1R_2) \in \mu_4$. Next, take the scalar choices d_G, d_N from Proposition 7.252, with this value for c_η. On simple modules, traces give the n-th powers. Moreover, on D_{J_2}, their tensor cancels these twists. Its n-th power is the operator for y^n on the base. Next, extend to $D_{J_2}\langle y \rangle$.

For a joint stabilizer element h, the intertwiners with twist ξ_h carry the tensor on D_{J_2} back. The identity for $\widetilde{\beta}$ matches the ordered source, scaling by b_0. Moreover, the representatives h_G, h_N satisfy $h_G y_G = y_G h_G$ and $h_N y_N = y_N h_N$. The transported scalar is $(d_G/d_N)^\sigma$, and the scaled original has d_G/d_N. The scaling tori are derived, so ξ_h takes 1, and Lemma 7.251 matches commutator scalars. Since σ fixes μ_{4n}, the scalars agree. Thus, $D_{J_2}\langle y \rangle$ carries a joint-invariant extension.

Along $J_2 \le J \le K$, choose parabolics with full diagonal quotient $\widetilde{K}/K$. Next, inflate and induce with y. Coset representatives are in M. Thus, on D_M Harish–Chandra induction is tensored with $\widetilde{\psi}^\vee$. Orthogonality and ordered columns give multiplicity 1 for the selected constituent on D_J and then on D_K. Both return by cancellation on D_M. Their isotypic spaces extend U_0 to $H_2 = D_K\langle y \rangle$ with joint invariance. Proposition 7.250 gives $\succeq_c$. □

Corollary 7.254. *For odd q and $\mathrm{ord}_\ell(q) = 4$, the balanced single-leg $I(3,3)$ series gives $\succeq_c$ on the Sylow normalizer.*

Proof. If $2 \nmid t$, we apply Corollary 7.249. If $2 \mid t$, Propositions 7.252 and 7.253 cover both forms at u_3 and finite central lift $\widetilde{s}$. Proposition 7.172 transfers the pair on K, N by signed induction to G and Clifford induction to $\mathrm{N}_G(P)$, by Proposition 7.243. Thus, the pairs on K, N yield $\succeq_c$ for the induced characters on $G, \mathrm{N}_G(P)$ in both parities of t. Moreover, the pair (ρ, η) on K, N has matching functions $\mu_{a,\sigma}$ before induction. □

Proposition 7.255. *Let $d = 4$, $y = 1$, and take the primitive sign a, leg coordinates t_i and projective ratios u_i of* Proposition 7.172. *Moreover, other series give $\succeq_c$ on $\mathrm{N}_G(P)$. Each union includes rational classes of s and local series before induction.*

Proof. Orient $u_1 = u_2 = u$ with $u^2 \neq 1$. If $[u_3] \neq [u]$, the added glues oppose legs $1, 2$. The cocharacter centralizer has type D_5 if u_3 moves, otherwise D_5A_1. By triality, natural coordinate 5 is $t_1/t_2 = \pm 1$. The first four come from a, either $(+,+,+,+)$ or $(+,+,-,-)$ up to permutation. Each nonzero a bisects the 8 minuscule weights. Moreover, modulo negation, s is central or has row $I(1,4), I(2,3)$. Involution tags give primitive pairs (quadratic, quadratic) and (quadratic, 1), respectively. Next, at central s, apply [156, Theorem D]. For involutions in either rational D_5, apply Corollary 6.169. Next, on the remaining A_1, take the identity. Row $Q(2,1)$ is absent.

By Sylow conjugacy in D_5 align the row. With fixed scalar κ_s, the discrepancy is central in derived D_5. The remaining root span opposes t_1, t_2, giving $(b, b^{-1}, 1)$. Coordinate 5 gives the central sign b^2, so $b^4 = 1$. The centre of the remaining A_1 fixes its projective ratio. Thus, u_1, u_2 change by a common sign, with u_3 and the D_4-root values of a fixed. The π-part returns. The union returns when $u_3^2 = 1$ or $[u_3] \notin \{[u], [-u]\}$.

At $[u_3] = [-u] \neq [u]$, we put $\mathbf{H}^* = (\mathrm{SL}_2)^3/\operatorname{diag}\mu_2$ and $B = \mathbf{D}^* \cap \mathbf{H}^* = \mu_2^3/\operatorname{diag}\mu_2$. The mark J is inner on D_4 and fixes the axes, so fixes B pointwise. Since $((q^2+1)^2)_2 = 4$, the image of $\mathbf{D}^{*F}$ in $(\mathbf{D}^*/B)^F$ has odd order. Moreover, in $(\mathbf{D}^* \times \mathbf{H}^*)/B$, the class in $H^1(F, B)$ records the primitive class c. Moreover, up to convention, c^{-1} records projective determinants in B. Equal nonquadratic ratios are conjugate in $\mathrm{PGL}_2(q)$, with torus centralizers. Determinants agree in B. Moreover, the primitive lift gives $a = 1$. If $a \neq 1$, the union includes the sign-linked $([u_1], [u_2], [u_3])$. The condition $[u_1] = [u_2] = [u_3]$ would give $a = 1$. At each excluded position i, transport Borels equivariantly along loops u_i and exchanges with inversion. Induce to L with sign, then by Clifford to M by Proposition 7.172.

At $a = 1$ with equal ratios $[u]$, centralize the oriented line $u_1 = u_2 = u_3$. At the strict boundary $[u], [-u]$, we consider s^2, with equal squared classes $[u^2]$. These are non-self-inverse since $u^2 = -1$ gives $[u] = [-u]$. Consistent orientation gives type E_6 from D_4 and opposite-leg glues. Since $|Z(E_{6,\mathrm{sc}})| = 3$, it misses the kernel of size 2. Thus, the derived E_6 is simply connected. By Steinberg, $C(s)$ is connected in E_6. Neutral roots are D_4 and opposite-leg glues with $t_i = t_j$. For each distinct value of t_i, choose one coordinate of S. Thus, $C(S)$ gives the Levi. Twist by κ_s. Transfer the local sources to obtain $\succeq_c$ on the union.

Suppose u_3 alone moves. Its line centralizer is D_6. The unoriented stabilizer is $W(D_6)$ times its line reflection s_{α_3}, trivial perpendicularly. Thus, D_6 splits. The natural orthogonal entries are $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, t_1t_2, t_2/t_1)$ modulo Weyl action and negation. Signs ϵ_i come from a. If $u_1 = u_2$, the last entries lie in $\{\pm 1\}$. In the central row, let $\mathbf{L}_\perp^*$ be the perpendicular Levi. Roots of $\mathbf{L}_\perp^*$ lie in $\mathrm{C}(s)$, and $\mathrm{C}(s) \leq \mathbf{L}_\perp^*$ by Proposition 7.172. Thus, $\mathrm{C}(s) = \mathbf{L}_\perp^*$. Twist by κ_s and transfer as in Lemma 7.231. At $u_1 = -u_2$ both are quartic. For $a \neq 1$, row $Q(2,2)$ has equal sign spaces. If $a = 1$, its quartic pair and $I(1,5)$ apply Lemma 7.231. For either form of $I(2,4)$, we apply Proposition 7.236. On $I(3,3)$ and $Q(2,2)$, Proposition 7.234 gives component action. Proposition 7.235 treats W_s. Diagonal comparisons on $\widetilde{K}, \widetilde{N}$ follow from Proposition 7.237. Proposition 7.238 gives g_σ for each pair. On the source $\widehat{\beta}$, take the central character. Corollary 7.239 gives g_σ on both series. For σ apply Lemma 7.240 and Proposition 7.241, including when β moves. Thus, both pairs give $\succeq_c$ on $\widetilde{K}, \widetilde{N}$. For $Q(2,2)$, Lemma 7.242 gives multiplicity 1 for both partitions on D_4A_1-flags. Proposition 7.243 transfers them to $G, \mathrm{N}_G(P)$ with full ordinary inertia. On $I(3,3)$, Proposition 7.244 handles σ on $\widetilde{K}, \widetilde{N}$. At field exponent 0, by Corollary 7.245, we obtain the external u_3-inversion. Joint exponents lie in $d_m\mathbb{Z}/f\mathbb{Z}$, and ordinary exponents in $e_m\mathbb{Z}/f\mathbb{Z}$ by Corollary 7.246. Lemma 7.247 gives F_r-fixed spin lifts, with equivariance on $I(3,3)$ from Proposition 7.248. At odd t, by Corollary 7.249, we obtain $\succeq_c$ via the central twist for $s_{2'}$. If t is even, the flag $J_2 \leq J \leq K$ gives a regular label with multiplicity 1. At t_2-inertia, Proposition 7.252 handles pairs. Singletons $\lambda \neq \nu$ apply Proposition 7.253. Corollary 7.254 gives $\succeq_c$ on $\mathrm{N}_G(P)$. The first two blocks carry P, the other rank-two factor having ℓ'-order.

The primitive α_3 generates the quotient character lattice. For κ_s, the invariant is $u_3 = \alpha_3(s)$. Next, align the marked D_6-class over $\mathbb{F}_q$. A central discrepancy lies on $\alpha_3^\vee$, with α_3-value 1. The sole nonidentity element $\alpha_3^\vee(-1)$ fixes a, u_1, u_2, u_3, so both lifts return. In $K \cap L$, keep the first two A_1-factors. The complementary torus at u_3 is nonquadratic. Position loops fix u_3, and its Borels. Transfer from $K \cap L$ by Lemma 7.168.

If $u_1 = u_2 = u_3 \in \{1, -1\}$, $s^2 = 1$ since central negation is trivial. At $s = 1$, we apply [156, Theorem D]. At $|s| = 2$, we apply Corollary 7.229. Otherwise, $n = 1, 2$ counts negative ratios. Unlike legs give -1, so majority glues alone survive. Its D_4 module is vector by triality. Leg i contributes A_1 when $u_i = 1$. At $a \neq 1$, sign planes give $D_2 + D_2$, with four weights per sign. If $n = 2$, orient as i, i. The glues with signs $+, +$ and $+, -$ have distinct planes, extending D_2 to D_3. The third leg gives A_1. Thus, $R_s = A_3^2A_1$ is isolated quartic. If $n = 1$, neutral glues have one plane. Its D_2 and majority-leg roots extend to D_4. The other D_2 remains. Moreover, the full centralizer C has image $\{\pm 1\}$ on the minority line. If $a = (e_1 + e_2)(-1)$, flip coordinates $1, 3$ for the vector central class. Exchanging $\{1, 2\}$ and $\{3, 4\}$ gives the spinor class. These generate $\mathbf{D}^* \cap \mathbf{H}^*$. Next, on minority leg i, combine negation and inversion of t_i. This fixes s and inverts the line. Thus, $R_s = D_4A_1^2$ is quasi-isolated. At central scalar 1, Proposition 7.178 compares all 38 eligible labels. For central scalar -1, Proposition 7.181 takes the primitive quadratic weight θ and both rotation stabilizers. Moreover, at $A_3^2A_1$, Proposition 7.185 treats both diagonal inertias.

At $a = 1, n = 1$ with unequal majority coordinates, $R_s = D_4A_1^2$ has no glues. Exchange majority legs $1, 2$ by triality and invert leg 3, fixing s modulo simultaneous negation. Next, apply $D_4A_1^2$-comparison. Equal signs give $\alpha^\vee(\pm i)$ on

the minority root, perpendicular to D_6. Moreover, inversion exchanges $i, -i$. The centralizer is the D_6-Levi. At $a = 1, n = 2$, orient as (i, i, b), $b = \pm 1$. Moreover, opposite glues extend D_4 to D_5, with third leg A_1. This Levi centralizes the majority diagonal $t_1 = t_2$. Each component representative fixes sign leg 3 and fixes b. Thus, it fixes the lifted triple (i, i, b) in the simply connected cover, where $C(s)$ is connected. The same test takes (b, b, i) for D_6. Moreover, the adjoint torus gives $W(E_7)$-orbits, their component orders $|C/C^\circ|$, and tuple counts with repetitions.

$\|s\|$	R_s	$\|W.s\|$	$\|C/C^\circ\|$	tuples
1	E_7	1	1	2
2	D_6A_1	63	1	54
2	A_7	36	2	48
2	E_6	28	2	8
4	$D_4A_1^2$	1890	2	156
4	$A_3^2A_1$	1260	2	144
4	D_5A_1	756	1	24
4	D_6	126	1	12

$C = \mathrm{C}_{\mathbf{G}^*}(s)$. Take roots perpendicular to $e_7 + e_8$ in the usual E_8 coordinates, with ordered simple roots $(e_1 + e_8 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7)/2$, $e_1 + e_2$, followed by $e_{j+1} - e_j$ for $1 \le j \le 5$. Next, with Cartan matrix C_{ij}, we put $n = (a_1, a_2, a_3, a_4, k+l, l-k, h, -h)$ and $b_j = (n, \alpha_j) \pmod 4$. Independently $k, l, h \in \{0, 1, 2, 3\}$. The primitive tuple (a_1, a_2, a_3, a_4) is zero or permutes $(2, 2, 0, 0)$. Moreover, the three exponents lie on $e_5 - e_6, e_5 + e_6, e_7 - e_8$. The 448 tuples give 128 distinct b. Next, for each uncounted b, close under $b_j \mapsto b_j - b_i C_{ij} \pmod 4$ for $1 \le i \le 7$. Count tuples in each orbit. Roots $\sum m_j\alpha_j$ are those with $\sum m_j b_j = 0 \bmod 4$. Thus, the component order is $2^{10} \cdot 3^4 \cdot 5 \cdot 7/(|W.s|\,|W(R_s)|)$. Next, at neutral rank 6, choose $0 \ne v \in R_s^\perp$ over $\mathbb{Q}$. Apply these reflections to (b, v). For $E_6, D_4A_1^2$, this doubles the b-orbit size. Thus, C inverts the line. For D_5A_1, D_6, orbit sizes coincide. Specifically, C is connected, with perpendicular roots R_s. Corollary 7.229 separates rational forms. On D_5A_1, D_6, twist centrally on $K \cap L$. Next, keep two A_1-factors for D_6, one for D_5A_1. Keep only nonquadratic ratios. Lifts fix i at ordinary position loops. Moreover, both i-legs keep oriented Borels. Next, on $K \cap L$, apply Lemma 7.168. The full centralizer in $K \cap L$ gives signed irreducible induction on L. Then Clifford-induce to M. □

Corollary 7.256. *Let $S = E_7(q)$ be simple and $\ell \mid |S|$ odd nondefining. If $\mathrm{ord}_\ell(q) \in \{1, 5, 7, 8, 9, 10, 12, 14, 18\}$, then $\mathrm{iMN}(S, \ell)$ holds.*

Proof. If $d = 1$, Theorem 7.123 gives the comparison through $\mathrm{N}_G(T)$ for the split torus T. Otherwise, $d \in \{5, 7, 8, 9, 10, 12, 14, 18\}$. By Corollary 7.169, the comparison at $\mathrm{N}_G(P)$ gives $\mathrm{iMN}(S, \ell)$ for these orders. Thus, set $M = \mathrm{N}_G(T)$ at $d = 1$ and $M = \mathrm{N}_G(P)$ at $d \in \{5, 7, 8, 9, 10, 12, 14, 18\}$. Moreover, ℓ'-characters correspond under an $\mathcal{H}_\ell$-equivariant bijection giving $\succeq_c$. □

Corollary 7.257. *Let $S = E_7(q)$ be simple. For its universal cover, $\mathrm{iMN}(S, \ell)$ holds for each prime $\ell \mid |S|$.*

Proof. At $\ell = r$, apply [162, Theorem 1.3]. If $\ell = 2$ and q is odd, apply [158, Theorem A]. Otherwise, we put $d = \mathrm{ord}_\ell(q)$. Fundamental degrees $2, 6, 8, 10, 12, 14, 18$ give $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18$. Corollary 7.169 gives the full comparison on $\mathrm{N}_G(P)$ at $d = 5, 7, 8, 9, 10, 12, 14, 18$. At $d = 4$, we fix the π-part y. If $y \ne 1$, the regular and singular cases give $\succeq_c$ by Propositions 7.172 and 7.230, respectively. At $y = 1$, fusion acts on $([u_1], [u_2], [u_3])$ by signed permutations. Correct within $C_{\mathbf{G}^*}(s)$ at P. Distinct moving classes give $\succeq_c$ by Proposition 7.172. Other $([u_1], [u_2], [u_3])$ apply Proposition 7.255. If $a = 1$, combine equal and strictly sign-linked $([u_1], [u_2], [u_3])$. Boundary tests give invariance. When $L^* = C_{\mathbf{G}^*}(s)$, we choose $\mathcal{E}(L, s)$. Otherwise, take its $Z(L^*)$-translates. Coprime powers give the same $C_{\mathbf{G}^*}(s)$ and the marked spans up to M-conjugacy. Moreover, on M, the Clifford source determines the L-series. This exhausts the labels of $\mathrm{N}_G(P)$. The D_6A_1-involutions give $\succeq_c$ at nonscalar d by Propositions 7.190 and 7.191. In type E_6T_1, Proposition 7.205 includes field extensions of the ${}^2E_6$ cubic labels. Corollary 7.214 covers all $|s| = 2$. For cubic A_5A_2 at $r \ge 5$, we apply Proposition 7.217 at the signed d. At $|s| = 6$ and $R_s = A_2^3$, Proposition 7.220 holds for each factor. If $2 \mid f$, include projective factors on diagonal-fixed χ. At odd q, nonidentity quasi-isolated series give $\succeq_c$ by Corollary 7.229. If $2 \mid q$, apply Corollary 7.118. At odd q, Theorem 7.175 covers $d = 3, 6$. If $2 \nmid q$, Proposition 7.121 covers $d = 1$. If $d = 2$, we apply Corollary 7.160 for $\ell = 3, 5, 7$. Otherwise, Theorem 7.165 is on $\mathrm{N}_G(T)$. If $d = 4$, Corollary 7.254 treats balanced involutions at every f. □

7.7 Series in E_8

Let $G = E_8(q)$ with $q = r^f$ and $\ell \neq r$ odd. Next, we put $d = \mathrm{ord}_\ell(q)$. $\mathrm{Z}(G) = 1$ and $\mathrm{Out}(G) = \langle F_r \rangle$. On $\mathbf{G}^{F'}$, where $F' = \mathrm{Int}(n)F_q$ and $F_r(n) = n$, write $\Phi = F_r|_{\mathbf{G}^{F'}}$. Then $\Phi^f = \mathrm{Int}(n^{-1})$. Given an integer m, we write m_b for its b-part. Next, for genuine extensions choose μ_h of Corollary 2.5. Both multiplication maps take $(u, \mathrm{Int}(u^{-1}))$ to 1 in the common conjugation kernel.

Proposition 7.258. *Let $(d, \delta) = (5, 1), (10, -1)$. On $\mathbf{T}$ of* Section 7.1*, isolated s exists exactly when $\delta q \equiv 1 \bmod 5$. Its connected roots are A_4^2, with 25 eligible hook pairs. Specifically, they give $\succeq_c$ at $\mathrm{N}_G(P)$, with simultaneous field swaps.*

Proof. $\exp(T) \mid \Phi_5(\delta q)$, coprime to 6. By [208, Table 12.2], the isolated order is $|s| = 5$, with roots A_4^2. This order occurs when $\delta q \equiv 1 \bmod 5$. $\Phi_5(\delta q)_5 = 5$, with 25 points of order dividing 5. Next, we write $a = (c, c)$, with $c \in W(A_4)$ a 5-cycle. A common central generator s is rational. Connectedness gives stabilizer $\mathrm{C}_{W(A_4)^2}(a) = C_5^2$. The relative group has order 600, so the orbit has size 24 and exhausts the nonidentity 5-torsion.

Both rational factors have sign δ. By [57, Proposition 13.20], the two A_4 factors give the unipotent degrees. Since $\mathrm{ord}_\ell(\delta q) = 5$, Lemma 3.2 gives one 5-hook in each partition of 5. Hence, there are 25 ordered pairs. By [208, Lemma 12.7], $b(x, y) = (y, -x)$ is a Weyl element with $b^2 = -1$, acting faithfully on $\langle s \rangle$. Moreover, $ba = ab$. A field action corrected by b^k to fix s exchanges the hooks exactly when k is odd. Thus, χ is real and $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\zeta_5)$ by [208, proof of Proposition 12.8, Steps 2–3]. Given a hook pair, put $E_\chi = \mathrm{Stab}_E(\chi)$ with $E = \langle F_r \rangle$ of order f. Write $n = |E_\chi|_2$. If $n > 1$, put $q_0 = r^{f/n}$. On the dual lattice choose j, k with $jn \equiv 1 \bmod 5$ and $b^k(s^{q_0}) = s$, since $\langle b \rangle|_{\langle s \rangle} = \mathrm{Aut}(\langle s \rangle)$. If $x = a^j b^k$, the power b^{kn} acts on $\langle s \rangle$ by $q^{-1} \equiv \delta \bmod 5$. Hence, $b^{kn} = \delta$ and $x^n = \delta a$.

On primal cocharacters take x^{-1}. Its Tits lift $\dot{x}$ satisfies $F_r(\dot{x}) = \dot{x}$. Next, put $F_0 = \mathrm{Int}(\dot{x})F_{q_0}$, $F' = F_0^n$. Thus, $F' = \mathrm{Int}(\dot{x}^{\,n})F_q$, with primal mark $(\delta a)^{-1}$. Two lifts differ by $t \in \mathbf{T}$. By [114, corollary to Theorem 1], choose $z \in \mathbf{T}$ with $t = z^{-1}F'(z)$. Conjugation by z fixes $\mathbf{T}$ pointwise. Hence, we choose $\mathbf{G}^{F'}$ for the marked pair. On $\mathbf{G}^{F'}$ we have $F_0^n = 1$. Its split outer image is F_{q_0}, of order n. Thus, $\langle G, F_0 \rangle$ is the preimage of $E_{\chi,2}$. Since $x(s^{q_0}) = s$, $\theta_s^{F_0} = \theta_s$ and $\chi^{F_0} = \chi$.

The Jordan torus formula expresses $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta_s), \chi \rangle$ up to sign as $u_1 u_2$, where u_i is the unipotent coefficient for the i-th A_4. The split mark is c. Moreover, the unitary mark is $-c$; opposition leaves the class $c \in \mathfrak{S}_5$ up to sign. At leg j_0, its trace at c is $(-1)^{j_0}$ by [182, Theorem 7.17.1 and Corollary 7.17.4]. Hence, $|u_i| = 1$. Lemma 2.16 with $F = F'$ and $\gamma = A = 1$ gives an extension with values in $\mathbb{Q}(\zeta_5)$ on $\langle G, F_0 \rangle$. If $n = 1$, we choose χ on G. On $O = O_{2'}(E_\chi)$, [208, Proposition 12.8] gives an extension ψ_{odd} over $\mathbb{Q}(\chi)$. Thus, $\mathcal{H}_\ell$ fixes ψ_{odd}. Since E_χ is cyclic, χ extends to $G \rtimes E_\chi$ by [95, Corollary 11.22]. By [156, Lemma 7.1], Ψ is the unique common extension across $E_{\chi,2} \times O$. Since $5 \mid \ell - 1$, both restrictions are $\mathcal{H}_\ell$-fixed. Thus, $\Psi^\sigma = \Psi$ for $\sigma \in \mathcal{H}_\ell$.

In both factors take $n_i \in \mathrm{SL}_5$, permutation matrices for c^{-1} on the respective factors of a. A prime-field lift h of b acts by $(g_1, g_2) \mapsto (g_2, g_1^{-\mathsf{T}})$. Indeed, the Tits lift signs on $Q(A_4^2)$ extend to $X(\mathbf{T})$, because $[X(\mathbf{T}) : Q(A_4^2)] = 5$. Thus, $\mathbf{T}[2]$ gives the correction to h. This action on SL_5^2 stabilizes the central gluing of [208, Lemma 12.7]. Since h^4 centralizes the subsystem, $h^4 \in C_5$, its centre. Since h acts faithfully on C_5, $hh^4 = h^4 h$ gives $h^4 = 1$. Since $n_i^{-\mathsf{T}} = n_i$, we have $hn_1h^{-1} = n_2$ and $hn_2h^{-1} = n_1$.

Write $\delta = (-1)^t$ and $\dot{w} = h^{2t} n_1 n_2$. Then, n_1, n_2, h satisfy $v\dot{w} = \dot{w}v$ for $v \in \{n_1, n_2, h\}$ and are rational for $\mathrm{Int}(\dot{w})F_q$. Set $M = \mathrm{N}_G(\mathbf{T}) = \mathrm{N}_G(P)$. The inertia of θ_s in M is $I = T \rtimes \langle n_1, n_2 \rangle$; these lifts give $I/T = C_5^2$. If $\lambda_i \in \mu_5$, write $\tau_{\lambda_1,\lambda_2}$ for the linear extension with $\tau_{\lambda_1,\lambda_2}(n_i) = \lambda_i$. The characters $\mathrm{Ind}_I^M \tau_{\lambda_1,\lambda_2}$ exhaust 25 labels over $M \cdot \theta_s$. $\Phi^f = \mathrm{Int}(\dot{w}^{-1})$. Next, choose $v = h^{k'}\Phi$ fixing θ_s. Its image generates the field quotient $\langle F_r \rangle$ on weight inertia. It acts on (λ_1, λ_2) by the swap to power k'. Also, the dual correction $b^{-k'}$ gives the same swap parity $k' \bmod 2$. Since Φ fixes h, n_1, n_2, $v^f = h^{k'f-2t}(n_1 n_2)^{-1} = (n_1 n_2)^{-1}$. Indeed, v^f and $n_1 n_2$ fix θ_s, whereas h^j fixes s only for $4 \mid j$.

Write $m \in \{1, 2\}$ for the orbit length of $\tau = \tau_{\lambda_1,\lambda_2}$. Moreover, its inertia over θ_s is $I\langle v^m \rangle$, with quotient of order f/m. The extension scalar α at v^m has the sole relation $\alpha^{f/m} = (\lambda_1\lambda_2)^{-1}$. The power f/m maps μ_{5f_5} onto μ_5, since $5 \nmid m$. Put $e_r = \mathrm{ord}_\ell(r)$. Then $e_r/(e_r, f) = d \in \{5, 10\}$ gives $v_5(e_r) = v_5(f) + 1$. Thus, $5f_5 \mid \ell - 1$ and $\mathcal{H}_\ell$ fixes $\theta_s, \lambda_1, \lambda_2, \alpha$. Write $\widehat{\tau}$ for this extension. Since $M\langle v^m \rangle = M(I\langle v^m \rangle)$, the induced character restricts to $\mathrm{Ind}_I^M \tau$ by [95, Problem 5.6]. By Clifford theory, $\xi = \mathrm{Ind}_I^M \tau$ is irreducible. Its stabilizer fixes $M \cdot \theta_s$ setwise, so M corrects its action to fix θ_s. The corrected operation must fix (λ_1, λ_2). Hence, it belongs to $I\langle v^m \rangle$. Thus, its full inertia is $M\langle v^m \rangle$.

Match hook leg j_0 with $\zeta_5^{j_0}$ with the same ζ_5 twice. Both base labels are $\mathcal{H}_\ell$-fixed. Their E-actions agree, so $(E \times \mathcal{H}_\ell)_\chi = (E \times \mathcal{H}_\ell)_\xi$. Both extensions are $\mathcal{H}_\ell$-fixed there. Next, at χ^γ, take the paired conjugates by γ. Compose both models with $(u, \alpha) \mapsto u\alpha$. On the common centralizer pair $(u, \mathrm{Int}(u^{-1}))$ with $u \in M$, both maps give 1. Thus,

the central scalars equal 1, as do factors c and twists μ_h. Section 7.1 and Corollary 7.278 treat other s. □

Corollary 7.259. *Let* $(d,\delta) = (5,1),(10,-1)$. *Series with proper Levi* $\mathbf{K}$ *in* Section 7.1 *give* $\succeq_c$ *at* $\mathrm{N}_G(P)$*, including when* $5 \mid q-\delta$.

Proof. Only rank 4 needs another source. Section 7.1 gives roots A_4. The diagram action is trivial for $d=5$. For $d=10$ it is nontrivial, since $\mathfrak{S}_5$ has no eigenvalue of order 10. Thus, for the adjoint factor J of K we have $J = \mathrm{PGL}_5(q)$ or $\mathrm{PGU}_5(q)$, respectively. Its mark on the rank-4 lattice has characteristic polynomial Φ_d, so the torus is Sylow d. The derived centre can have order 5. Let $\pi : \mathbf{K} \to \mathbf{J} = \mathbf{K}/Z(\mathbf{K})$. Since $X(\mathbf{T}) = Q(E_8)$, the simple roots of $\mathbf{K}$ lie in a basis of $X(\mathbf{T})$. Thus, their span is saturated and $Z(\mathbf{K})$ is connected. If $\overline{\mathbf{T}} = \pi(\mathbf{T})$, we have $\pi^{-1}(\overline{\mathbf{T}}) = \mathbf{T}$. By [133, Proposition 23.2], we obtain the surjections $K \longrightarrow J$, $\quad \mathrm{N}_K(\mathbf{T}) \longrightarrow \mathrm{N}_J(\overline{\mathbf{T}})$, $\quad T \longrightarrow \overline{\mathbf{T}}^F$. Indeed, a lift in K of $j \in \mathrm{N}_J(\overline{\mathbf{T}})$ normalizes $\mathbf{T} = \pi^{-1}(\overline{\mathbf{T}})$. By [57, Proposition 13.20], the unipotents of K are $\chi \circ \pi$ for unipotent χ of J. Locally, a character over 1_T has kernel containing $Z(\mathbf{K})^F$. Moreover, the compositions with $\pi|_{\mathrm{N}_K(\mathbf{T})}$ give all labels over 1_T from labels over $1_{\overline{\mathbf{T}}^F}$, with the same degrees.

Apply [156, §8 and proof of Theorem D] on $\overline{\mathbf{T}}$ with $1_{\overline{\mathbf{T}}^F}$. The extensions are on overgroups of J and $\mathrm{N}_J(\overline{\mathbf{T}})$. Also, a geometric transport with $\mathbf{K}$ invariant gives $\alpha \in \mathrm{Aut}(J)$ of inner, field or diagram type. $k \in K$ acts by $\mathrm{Int}(\pi(k))$. Next, choose the section on J from Proposition 7.11 and restrict to B_P. On base characters the maps are $\chi \mapsto \chi \circ \pi$ above. Moreover, both section twists μ_h agree. The conjugation kernel gives 1. With κ_s, Corollary 7.278 gives $\succeq_c$. For regular semisimple s, we set $K = T$. □

Proposition 7.260. *If* $d=8$*, write* $s = ty$ *with* y *of odd order and* t *of two-power order. Given* $y \neq 1$*, take a torus* T*, the rank-one comparison over* $\mathbb{F}_{q^4}$*, or the cyclic Sylow comparison for* $D_4^-(q)$*. The twist gives odd part* 1*. If* $y = 1$ *and odd* q*, the nonunipotent series is the split* D_8 *involution series* $\mathcal{E}$*. Both* F_r*-extensions are joint-invariant.*

Proof. Odd primes $v \mid q^4+1$ satisfy $\mathrm{ord}_v(q) = 8$ and $v \geq 17$. Thus, $K^* = \mathrm{C}(y)$ is a Levi by Section 7.1. Given $K^* \neq T^*$, root rank is 4. Indeed, the primitive 4-space is irreducible. Moreover, a rank-8 isolated vertex has no such divisor v. The mark permutes components transitively. Type A_4 has no root automorphism of order 8 and two exchanged A_2 factors cannot give an action of order 4 on one factor after a full cycle. Thus, the roots are D_4 or four cyclically permuted A_1's. Moreover, a primitive negative 4-cycle lies outside $W(D_4)$, giving minus type. The A_1 orbit gives $\mathrm{SL}_2(q^4)$. Next, put $H = (\mathbf{K}_{\mathrm{der}})^F$, $P_H = P \cap H$, $Z = Z^\circ(\mathbf{K})^F$ and $P_Z = Z_\ell$. The central isogeny $Z^\circ(\mathbf{K}) \times \mathbf{K}_{\mathrm{der}} \to \mathbf{K}$ has kernel A_0. This kernel has 2-power order. By [133, Theorem 21.7 and Proposition 23.2], $K/(ZH) \cong A_0/(F-1)A_0$. The isogeny on $Z^\circ(\mathbf{K}) \times \mathbf{T}_H$ has the same kernel for $\mathbf{T}_H = \mathbf{T} \cap \mathbf{K}_{\mathrm{der}}$. Thus, the inclusion gives $T/(ZT_H) \cong K/(ZH)$, with $T_H = \mathbf{T}_H^F$. Since ℓ is odd, $P = P_H P_Z$. Twist away the central linear character κ_y dual to y. Thus, the lower odd part is 1. For $H = \mathrm{SL}_2(q^4)$, $\ell \mid q^4+1$. Thus, on $H, \mathrm{N}_H(P_H)$, [186, Theorems 4.2 and 4.3] gives the nonsplit comparison. Next, consider $D_4^-(q)$. Since the mark is a negative 4-cycle, $\det(qw-1) = q^4+1$. Thus, $\mathbf{T}_H$ is Sylow Φ_8 for $H = \mathrm{Spin}_8^-(q)$ by the order formula. By [133, Theorem 25.14], $P_H = (T_H)_\ell$ is cyclic. By [130, Proposition 2.2], $M_H := \mathrm{N}_H(P_H) = \mathrm{N}_H(\mathbf{T}_H)$ and $\mathrm{N}_H(P_H[\ell]) = M_H$. The centralizer of w in signed permutations is $\langle w \rangle$, so $M_H/T_H = \langle w^2 \rangle$ has order 4.

Let $1 \neq z \in T_H^*$ have odd order and lift z to a torus coordinate a in SO_8. The isogenies have 2-primary kernels and cokernels. Moreover, on SO_8, the eigenvalues are $a^{\pm q^i}$ for $0 \leq i \leq 3$, with $a^{q^4+1} = 1$. Since each prime $v \mid |a|$ satisfies $\mathrm{ord}_v(q) = 8$, the root values $a^{\pm q^i \pm q^j}$ are nontrivial for $i < j$. By [20, Corollary 2.9], the component group of $C(z)$ has 2-power order and exponent dividing $|z|$. Thus, $C(z) = \mathbf{T}_H^*$. For a 2-element $t \in T_H^*$, the centralizer of tz is also $\mathbf{T}_H^*$ since $z \in \langle tz \rangle$. Hence, the weight θ dual to tz has trivial Weyl stabilizer. Orthogonality gives $\chi_\theta = \epsilon_H \epsilon_{T_H} R_{T_H}^H(\theta) \in \mathrm{Irr}(H)$. By Clifford theory, $\psi_\theta = \mathrm{Ind}_{T_H}^{\mathrm{N}_H(P_H)} \theta$ belongs to $\mathrm{Irr}(\mathrm{N}_H(P_H))$. If r is the field prime, then $\chi_\theta(1) = |H|_{r'}/|T_H|$ is prime to ℓ, since T_H contains a Sylow ℓ-subgroup. Also, $\psi_\theta(1) = |M_H : T_H| = 4$. Hence, both degrees are ℓ'. Orthogonality and Clifford theory [43] give M_H/T_H-orbits on both sides. Let a satisfy $a\,F = F\,a$ and give $a(\mathbf{T}_H) = \mathbf{T}_H$. For coefficient σ, functoriality of $R_{T_H}^H$ and Ind gives $\chi_\theta^{a\sigma} = \chi_{\theta^{a\sigma}}$, $\psi_\theta^{a\sigma} = \psi_{\theta^{a\sigma}}$. Thus, both joint stabilizers fix the θ-orbit. Below Ω gives factors and correcting functions.

Since H is perfect, Proposition 2.9 gives Ω with Euler signs $\delta_\chi \in \{1,-1\}$. Moreover, the singular-value comparison gives $\chi(x) = \delta_\chi\, \Omega(\chi)(x)$ $(x \in T_H,\ x_\ell \neq 1,\ \chi \in \mathrm{Irr}_{\ell'}(H))$. x_ℓ is the ℓ-part of x. Since $1 \neq x_\ell \in P_H$ has centralizer $\mathbf{T}_H$ by the same root values, x is regular. By [40, 7.5.4], $\chi_\theta(x) = \epsilon_H \epsilon_{T_H} \sum_{\eta \in \mathcal{O}_\theta} \eta(x)$ for $\mathcal{O}_\theta = (M_H/T_H) \cdot \theta$. Given any $\psi \in \mathrm{Irr}(M_H)$, Clifford theory [43] gives $\psi|_{T_H} = \sum_{\eta \in \mathcal{O}_\lambda} \eta$ for a weight λ, with $|\mathcal{O}_\lambda| \leq 4$: the cyclic inertia quotient allows λ to extend.

Write $T_H = P_H \times C_H$, with C_H the Hall ℓ'-subgroup. Put $B = \epsilon_H \epsilon_{T_H} \sum_{\eta \in \mathcal{O}_\theta} \eta - \delta_{\chi_\theta} \Omega(\chi_\theta)|_{T_H}$. By the singular-value comparison, B is supported on $\{1\} \times C_H$ and has at most 8 Fourier characters. For each $\nu \in \mathrm{Irr}(C_H)$, the ν-coefficient is $a_\nu \mathbf{1}_{\{1\}}$ on P_H. Its Fourier coefficients on P_H are constant. Since $|P_H| \geq \ell \geq 17 > 8$, some Fourier coefficient is 0. Hence, $a_\nu = 0$, and $B = 0$. At 1, both orbit sums are positive. Therefore, $\delta_{\chi_\theta} = \epsilon_H \epsilon_{T_H}$. Since θ has trivial inertia in M_H/T_H, $\Omega(\chi_\theta) = \psi_\theta$. Thus, Ω gives $\succeq_c$ on full joint inertia. The union $z \neq 1$ is stable under both actions. Thus, Ω restricts to weights on T_H of 2-power order, including order 2. The isogeny gives $[T : T_H Z]$ a power of 2. Thus, odd weights on T are determined by restrictions to T_H and $Z \cap T$. Also, their triviality gives odd part 1 on K. At rank 1, $H = \mathrm{SL}_2(q^4)$ has nonsplit $|T_H| = q^4 + 1$ and Weyl quotient C_2. By [130, Proposition 2.2], $N_H(P_H[\ell]) = N_H(P_H) = N_H(\mathbf{T}_H)$. The root value a^2 gives regularity for nontrivial a of odd order. Thus, on T_H, the signed difference has at most 4 Fourier characters. Since $|P_H| \geq 17 > 4$, it gives $\Omega(\chi_\theta) = \psi_\theta$ for each regular θ. Thus, Ω restricts to odd part 1. In both cases keep K with its central-isogeny cokernel. Put $D = N_{G \rtimes E}(\mathbf{K})$, $C = N_D(P_H)$, $L = K \cap C$, for pinned field group E. Since $H, K \lhd D$, Frattini gives $D = HC$ and $K = HL$. The group P_Z is characteristic in Z, which is D-invariant. Thus, C normalizes $P = P_H P_Z$. The converse follows from $P_H = P \cap H$. Hence, $C = N_D(P)$, $L = N_K(P)$, $H \cap C = M_H$, $C_D(H) \leq C$. This includes diagonals from K and field operations from D.

Take the virtual permutation character $\widehat{\Theta}$ for H from Proposition 2.9. If $\theta \in \mathrm{Irr}_{\ell'}(H)$ and $\eta = \Omega(\theta)$, equivariance gives $C_\theta = C_\eta$ and $D_\theta = HC_\eta$. The fibre product $\mathcal{D}_\theta = D_\theta \times_{D_\theta/H} C_\eta$ acts by $\rho(d, c)(x) = dxc^{-1}$ on H. Indeed, $dc^{-1} \in H$ and c acts in $\mathrm{Aut}(H)_{P_H}$. Next, on $H \times M_H$, take left and inverse-right translations. $\rho^* \widehat{\Theta}$ has coefficient ± 1 at $\theta \boxtimes \eta^\vee$. Explicitly, the kernel is $\{(z, z) : z \in C_D(H)\}$. If $(a, \sigma) \in C \times \mathcal{H}_\ell$ fixes θ, η, its action fixes $\rho^* \widehat{\Theta}$ by conjugation on permutations and $\widehat{\Theta}^\sigma = \widehat{\Theta}$. Thus, Proposition 2.4 with $\pm \rho^* \widehat{\Theta}$ gives V. By Corollary 2.5, the common factor is c and the corrections agree on D_θ, C_η. Since $\Delta C_D(H) = \ker \rho$, scalars agree on $C_D(H)$. Next, apply Proposition 7.6 with $H \leq K \lhd D$, $H' = M_H$ and $L = N_K(P)$. The inverse factors on $K_\theta/H = L_\eta/M_H$ give $\beta \in \mathrm{Irr}(K), \nu \in \mathrm{Irr}(L)$, with matching ℓ'-degree cuts. Since $Z(K) \leq L \cap C_D(K)$, the scalars of β, ν agree on $Z(K)$, including odd parts on $Z = Z^\circ(\mathbf{K})^F$.

The simple roots of $\mathbf{K}$ extend to an E_8-basis. Thus, $Z(\mathbf{K})$ is connected and $\mathbf{K}_{\mathrm{der}} \leq \mathbf{K}$ is regular. By [72, §2.6], restriction to H projects the dual element. Given $\lambda \in \mathrm{Irr}(T)$, the projected weight is $\lambda|_{T_H}$. The central character is $\lambda|_Z$. Thus, test $\lambda_{\mathrm{odd}}|_{T_H}$ and $\lambda_{\mathrm{odd}}|_Z$. Since $[T : T_H Z]$ is a power of 2, both are trivial precisely at $\lambda_{\mathrm{odd}} = 1$. The map on K keeps the 2-primary cokernel and multiplicity space.

Restore κ_y on the matched sources. Next, put $B = D_{\kappa_y}$ and $B_P = B \cap C$. Then, $B = KB_P$, $K \cap B_P = L$, with $J = B/K = B_P/L$. By Lemma 7.7, both factors on J gain c_{κ_y} and the correcting functions agree. If $b_j \in B_P$ is the section lift of j, the power factor is $\kappa_y(b_j^{|j|})^{-1}$. Since $y \in \langle s \rangle$, we have $C_{\mathbf{G}^*}(s) \leq K^* = C(y)$. Rational and joint transport fix K^*. Corollary 7.3 induces on ambient inertia. Over y, inertia is $L = \mathrm{N}_K(P)$, so Clifford induction gives $\succeq_c$ on the proper Levi packet.

At $y = 1$ with even q, only $s = 1$ remains. At odd q, the exponent bound gives $(q^4 + 1)_2 = 2$. By [120, $E_8(q)$], only split D_8 has two factors $\Phi_8(q)$ at an involution. Since $\ell \geq 17$, each other factor is ℓ'. Thus, one involution class has ℓ'-index, with one $C_W(w)$-orbit by [132, Proposition 5.11]. In $W(D_8)$, the polynomial Φ_8^2 gives two negative 4-cycles. Moreover, the stabilizer B_t is the even-sign subgroup of $C_8 \wr \mathfrak{S}_2$, of order 64 and exponent dividing 8. For rotation exponents a, b, a swap squares to $a + b$ on both blocks, with $2 \mid a + b$. Write $q = r^f$, $f_2 = 2^{v_2(f)}$, and $k = \mathrm{ord}_\ell(r)$. Since $k/(k, f) = 8$, one has $v_2(k) = v_2(f) + 3$. In particular, $8 f_2 \mid \ell - 1$. Since $C_{G^*}(t)$ has split type D_8 over $\mathbb{F}_r$, choose a representative t in the split 2-torsion. By [208, Proposition 12.4, Steps 1–2], the standard source (L, λ) of $\chi \in \mathcal{E}(G, t)$ has $\lambda = \epsilon_t \rho$ with quadratic ϵ_t. The derived roots of L lie in D_8. Thus, $\rho = 1$ on a torus or is split D_4-cuspidal. The same proposition gives $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\lambda) = \mathbb{Q}$.

Next, we put $\phi = F_r|_G$, $E = \langle \phi \rangle$ of order f, and $\widehat{G} = G \rtimes E$. Next, extend ϵ_t by 1 on E. For adjoint D_4, apply [156, Proposition 7.3] with eigenvalue ω_ρ equal to 1 or -1. Choose an f-th root $\zeta_\rho \in \mu_{2f_2}$, taking 1 at sign 1 and a primitive $2f_2$-th root at sign -1, since f/f_2 is odd. Root changes twist by $\mathrm{Irr}(E)$. Since $2f_2 \mid \ell - 1$, each $\sigma \in \mathcal{H}_\ell$ has correcting value $\sigma(\zeta_\rho)/\zeta_\rho = 1$ at ϕ. Compose with the quotient to $(\mathbf{L}/Z(\mathbf{L}))^F$ by [208, proof of Corollary 4.13], giving an extension of ρ. Hence, $\widehat{\lambda}$ is $\mathcal{H}_\ell$-fixed. For a torus take ϵ_t alone. Hecke operators are fixed by ϕ. By [208, Lemma 4.9(i),(ii)], each χ extends to $\widehat{\chi} \in \mathrm{Irr}(\widehat{G})$ with the correcting character of $\widehat{\lambda}$. Hence, $\widehat{\chi}^\sigma = \widehat{\chi}$ for $\sigma \in \mathcal{H}_\ell$, and χ is E-fixed.

Locally take split D_8 subgroup K dual to $\mathrm{C}(t)$, with torus T and weight θ_t. Let $M = N_G(P)$, $I = M_{\theta_t} = N_K(T)$, $\widehat{M} = N_{\widehat{G}}(P)$ and $\widehat{I} = (\widehat{M})_{\theta_t}$. Next, we choose marked coordinates $F' = \mathrm{Int}(n) F_q$, with Chevalley lift n of w over $\mathbb{F}_r$.

The lift Φ normalizes T, P and fixes θ_t. Indeed, $\theta_t^r = \theta_t$, since θ_t is quadratic and r is odd. The automorphism Φ represents the generator of $\widehat{G}/G$, and so $\widehat{M} = M\langle\Phi\rangle$, $\widehat{I} = I\langle\Phi\rangle$, $\Phi^f = n^{-1}$, $\Phi b T \Phi^{-1} = bT$ ($b \in I$). Write $G = \mathrm{Inn}(G)$. Specifically, $n^{-1}T = w^{-1}$ in $I/T = B_t$. On K, the quadratic character κ_t is Φ-fixed and restricts to θ_t. Its value at n is 1, since the lift in simply connected D_8 is fixed by F'. Thus, n lies in that image and in $\ker(\kappa_t)$. Invariance and $\kappa_t(n) = 1$ give the rational extension $\widehat{\kappa}_t(i\Phi^a) = \kappa_t(i)$ on $\widehat{I}$.

Given $\eta \in \mathrm{Irr}(B_t)$, we choose a model R_η. Since $w^{-1} \in Z(B_t)$, it acts by $c_\eta \in \mu_8$. Choose $\beta_\eta \in \mu_{8f_2}$ with $\beta_\eta^f = c_\eta$, possible since f/f_2 is odd. On $\widehat{I}/T$, choose R_η on I/T and give ΦT scalar β_η. The relations of $\widehat{I}/T$ give a representation, with character $\widehat{\eta}$. Moreover, its values are powers of β_η times values of η, in $\mathbb{Q}(\zeta_{8f_2})$. Thus, $\mathcal{H}_\ell$ fixes $\widehat{\eta}$. Inflate to $\widehat{I}$ and induce $\widehat{\psi}_\eta = \mathrm{Ind}_{\widehat{I}}^{\widehat{M}}(\widehat{\kappa}_t\,\widehat{\eta})$. Since $\widehat{M} = M\widehat{I}$ and $M \cap \widehat{I} = I$, restriction to M gives $\psi_\eta = \mathrm{Ind}_I^M(\kappa_t\eta)$, irreducible by Clifford theory. Thus, $\widehat{\psi}_\eta$ extends ψ_η and is fixed by $\mathcal{H}_\ell$.

In split simply connected D_8, the marked torus centralizes a Sylow Φ_8-torus and has relative group B_t. The unipotent correspondence of [156, Theorem D and §4.4] is above 1_T. Its relative labels have ℓ'-degree since B_t is a 2-group. Central isotypy matches unipotents on $C(t)$. Since $P \le K$, the Jordan multiplier is prime to ℓ, giving all eligible labels.

In $\widehat{G}$, we choose the model $\widehat{M}$ of $\Gamma = \mathrm{Aut}(G)_P$. Both base labels are fixed by $\Gamma \times \mathcal{H}_\ell$. Next, compose $\widehat{\chi}, \widehat{\psi}_\eta$ with multiplication $G \rtimes \Gamma \to \widehat{G}$ and $M \rtimes \Gamma \to \widehat{M}$, respectively. The factors are 1. The joint stabilizer fixes both extensions, giving 1 by [141, Lemma 1.4]. The common centralizer consists of $(m, \mathrm{Int}(m^{-1}))$, $m \in M$. Both maps take them to 1. Hence, $\succeq_c$ holds on full joint inertia. □

Corollary 7.261. *If $d = 8$, proper centralizer Levis give $\succeq_c$ on $\mathrm{N}_G(P)$. If $2 \mid q$, these and unipotents exhaust $\mathrm{Irr}_{\ell'}(G)$.*

Proof. A proper nontoral Levi has rank 4. By Proposition 7.260, the irreducible polynomial Φ_8 gives twisted D_4 or four cyclic A_1's. On D_4, Weyl order 8 requires a negative 4-cycle outside $W(D_4)$. On A_1-orbits, the action on a root line after a full orbit is -1. Thus, the rational adjoint factor is ${}^2D_4(q)$ or $\mathrm{PGL}_2(q^4)$. The first torus has type Φ_8. However, the second is nonsplit, with $\mathrm{ord}_\ell(q^4) = 2$. Thus, $\mathbf{K}$ is a minimal cyclotomic Levi. Put $\mathbf{C} = Z(\mathbf{K})$ and $C = \mathbf{C}^F$. Next, write $\pi : \mathbf{K} \to \mathbf{J} = \mathbf{K}/\mathbf{C}$ for the quotient map, and $J = \mathbf{J}^F$. Let $\overline{\mathbf{T}} = \pi(\mathbf{T})$ and $S = \overline{\mathbf{T}}^F$. Moreover, the simple roots of $\mathbf{K}$ span a saturated E_8-sublattice after standard conjugation. Thus, $\mathbf{C}$ is connected, and [133, Proposition 23.2] gives $K/C \simeq J$ and $T/C \simeq S$. Put $L = N_K(\mathbf{T})$ and $R = N_J(\overline{\mathbf{T}})$. Since $\mathbf{T} = \pi^{-1}(\overline{\mathbf{T}})$, the preimage of R in K is L, giving $L/C \simeq R$ and $L/T \simeq R/S$. Next, we choose all of K, even at derived index 2 in J, including the full J. By [57, Proposition 13.20], the unipotents of K inflate from J. Local labels over 1_T have C in their kernels, so inflate from 1_S-labels on R, with the same degrees for any $|C|$.

Apply [156, §8 and proof of Theorem D] on S, matching τ_0, η_0 on J, R at ℓ'-degree, with η_0 over 1_S. Let $\mathcal{U}_J$ be the field-diagram overgroup of $J = \mathrm{Inn}(J)$. Next, we put $A_{\tau_0} = (\mathcal{U}_J)_{\tau_0}$, $A_{\eta_0} = N_{\mathcal{U}_J}(\overline{\mathbf{T}})_{\eta_0}$. The image $\pi(P)$ is Sylow in J. Its algebraic centralizer is $\overline{\mathbf{T}}$ by Proposition 7.260 for minus D_4, or by the root value a^2 on the nonsplit torus for A_1 since $\ell \ge 17$. Hence, $N_{\mathcal{U}_J}(\overline{\mathbf{T}}) = N_{\mathcal{U}_J}(\pi(P))$, giving $N_K(P) \le L$. Since P is characteristic in T, $L = N_K(P)$. By Sylow conjugacy, $A_{\tau_0} = JA_{\eta_0}$, $J \cap A_{\eta_0} = R$, and $A_{\tau_0}/J \simeq A_{\eta_0}/R$. The cited construction gives ordinary extensions U, V with matching correcting functions. Inflate τ_0, η_0 to τ, η on K, L. Next, we let κ_s be dual to s. Next, in the split field overgroup Γ, put $D_K = N_\Gamma(\mathbf{K})$, $B = (D_K)_{\kappa_s\tau}$, $B_P = N_B(P)$. Moreover, the adjoint action gives $c : D_K \to \mathcal{U}_J$, with $c(k) = \mathrm{Int}(\pi(k))$ for $k \in K$. For four components over q^4, projection records both operations in $c(D_K)$.

An ordinary stabilizer fixes κ_s and τ separately by central isotypy. Thus, $c(B) \le A_{\tau_0}$ and $c(B_P) \le A_{\eta_0}$. By Sylow conjugacy, $B = KB_P$ and $K \cap B_P = L$. Hence, $U \circ c, V \circ c$ extend τ, η on B, B_P. Their quotient map is $B/K \simeq B_P/L \to A_{\tau_0}/J \simeq A_{\eta_0}/R$. Compose both functions with $B_P/L \to A_{\tau_0}/J$. Also $c(C_B(K)) = 1$, with $C_B(K) \le B_P$, so the untwisted centralizer scalars are 1.

For a joint stabilizer operation, correct by K to a representative $a \in D_K$ with $P^a = P$. By $c(aba^{-1}) = c(a)c(b)c(a)^{-1}$, pullback is equivariant. Choose a section $b_j \in B_P$ for $B/K = B_P/L$. By Lemma 7.7, both twists have factor c_{κ_s} and equal corrections. For $x \in C_B(K)$, their scalars agree. If $b = b_j$ with $|j| = m$ and $b^m = u \in L$, then $c(b)^m = c(u)$, and the twist contributes $\kappa_s(u)^{-1}$. Corollary 7.278 transfers the series to $N_G(P)$. If $\mathbf{K} = \mathbf{T}$, the common source is θ_s. If $2 \mid q$, the torus exponent divides odd $q^4 + 1$. Its prime divisors satisfy $\mathrm{ord}_a(q) = 8$. Hence, $a \ge 17$. Thus, we have $\gcd(|s|, 30) = 1$ and Section 7.1 gives a Levi. At $s = 1$, we apply [156, Theorem D], with fusion on T and Clifford theory [43] by Proposition 7.11. □

Proposition 7.262. *Let $q = r^f$ be odd and $\mathrm{ord}_\ell(q) = 8$. On the marked torus let $s = ty$ have nontrivial odd part y. If $C(s)$ is non-Levi, $\mathcal{E}(G, s)$ has regular and semisimple labels with $\succeq_c$ over θ_s on $M = \mathrm{N}_G(P)$, on full inertia.*

Proof. Put $\mathbf{K}^* = C_{\mathbf{G}^*}(y)$, a Levi by Proposition 7.260, with dual $\mathbf{K}$. The toral exponent has 2-part at most 2. Coprimality gives $R_s = \{\alpha \in R_y : \alpha(t) = 1\}$. If $R_y = \varnothing$, $C(s)$ is toral. For four cyclic A_1's, the mark permutes R_s, with 0 or 4 lines, giving Levis. By Steinberg, $\mathbf{G}^*$ has connected centralizers. In D_4, we consider a negative 4-cycle. Up to diagram conjugacy, w is signed with polynomial Φ_8 and so a negative 4-cycle. If t is adjoint-trivial, its centralizer in $\mathbf{K}^*$ is $\mathbf{K}^*$. Otherwise, its adjoint image generates $T^*_{\mathrm{ad}}[2]$, since $(q^4 + 1)_2 = 2$. Moreover, on the adjoint lattice $D_4 = \{x \in \mathbb{Z}^4 : \sum_i x_i \equiv 0 \bmod 2\}$, this involution has value $(-1)^{x_1+x_3}$ at x. The functional is nonconstant. Under w, $x_1 + x_3$ becomes $x_2 + x_4$ modulo 2, equal by the lattice condition. Since q is odd, the remaining roots are $\pm e_1 \pm e_3$ and $\pm e_2 \pm e_4$. The four orthogonal lines cycle under w, since w^2 swaps each parity with sign. Thus, R is A_1^4 and their parabolic closure is D_4. Also, the non-Levi stabilizer of s in $C_W(w)$ is $W(R)^w$, by connectedness. Their product v generates this order-2 stabilizer. It is -1 on their span and $+1$ perpendicularly. By duality, the inertia is $I = M_{\theta_s}$ with $I/T = \langle v \rangle$. The rational factor is $A_1(q^4)$. By [57, Proposition 13.20], the degrees at $1, \mathrm{St}$ are $1, q^4$. Since $P^* \leq T^*$, the Jordan multiplier is prime to ℓ. Degrees distinguish both labels. On U_B, we choose the Whittaker character $\lambda(u) = \psi_0(\mathrm{Tr} \sum_i x_i(u))$, with ψ_0 nontrivial on $\mathbb{F}_r$. Next, we extend λ by 1 on E, and induce to $G \rtimes E$. Let W afford this character. For power a on μ_r, choose split $t_a \in \mathbf{T}^{F_r}_{\mathrm{spl}}$ with every simple-root value a, in the adjoint basis. Then t_a gives coefficient transport of λ up to inversion. It centralizes E. Thus, W has rational character.

On G, this is Gelfand–Graev. The constituent χ_{reg} has multiplicity 1 by [58, §§12.3–12.4]. Thus, $W_{\chi_{\mathrm{reg}}}$ affords χ_{reg} on G and an extension to $G \rtimes E_{\chi_{\mathrm{reg}}}$. Transport by (a, σ) acts on the projector by $e_{\chi_{\mathrm{reg}}} \mapsto e_{\chi^{a\sigma}_{\mathrm{reg}}}$. On the joint stabilizer, the projectors agree. Since W is rational, a coefficient intertwiner on $G \rtimes E$ exists. Next, compose with a and restrict to $W_{\chi_{\mathrm{reg}}}$. The extension is invariant even when a, σ separately move χ_{reg}.

Take the signed dual of χ_{reg}. Next, apply alternating parabolic induction on $G \rtimes E_{\chi_{\mathrm{reg}}}$. Given any Q, we choose U_Q-invariants, inflate, and induce from $Q \rtimes E_{\chi_{\mathrm{reg}}}$. Moreover, the pinned field group normalizes Q, U_Q. The U_Q-average is Galois equivariant. Thus, Ξ is joint-invariant and restricts to $\pm\chi_{\mathrm{ss}}$. Proposition 2.4 at second base 1 gives an invariant extension. Both stabilize $\mathcal{E}(G, s)$ by degree separation.

Next, put $\mathcal{A} = \mathrm{N}_{G \rtimes E}(P)$, $I^+ = \mathcal{A}_{\theta_s}$, and write $\theta_s = \theta_y \theta_t$ on the two primary parts. In $F' = \mathrm{Int}(n)F_q$, choose a prime-field Tits lift n of w. Next, let V be generated by Tits lifts and $\mathbf{T}[2^\infty]$. Specifically, it is F'-stable with 2-primary torus kernel. Every element of $C_W(w)$ lifts to $V^{F'}$. A Tits lift has discrepancy in $\mathbf{T}[2^\infty]$. Moreover, the isogeny $F' - 1$ surjects there. By [133, Theorem 21.7], choose g with $F_q(g) = gn$. Conjugation $\iota = \mathrm{Int}(g)$ gives $\mathbf{G}^{F'} \cong G = \mathbf{G}^{F_q}$. Moreover, it carries $\mathbf{T}, P$ to position up to G-conjugacy. The marked F_r becomes aF_r, with $a = gF_r(g)^{-1}$. As $F_r(n) = n$, we have $F_q(a) = a$. Moreover, in $G \rtimes E$, telescoping gives $(aF_r)^f = gF_q(g)^{-1} = \iota(n^{-1})$, including torus component. Also aF_r normalizes $\iota(V^{F'})$, which contains $\iota(n^{-1})$. Put $D = \langle \iota(V^{F'}), aF_r \rangle$. Then, $TD = \mathcal{A}$, since these images give $C_W(w)$ and generate the field quotient. On $T \cap D$, field exponents are multiples of f. The relation puts them in $T \cap \iota(V^{F'})$, a 2-group.

Since $\theta_y|_{T \cap D} = 1$, the formula $\lambda_y(ud) = \theta_y(u)$ for $u \in T, d \in D_{\theta_y}$ defines an extension on $\mathcal{A}_{\theta_y} = TD_{\theta_y}$. It is multiplicative because D_{θ_y} fixes θ_y. Transport by D and coefficients fixes the formula for λ_y. Conjugation by T is inner. Thus, the joint stabilizer fixes λ_y, even when both parts move θ_y. Given θ_t, we choose the subsystem $\mathbf{J}_t$ on $\mathbf{T}$ dual to $C_{\mathbf{G}^*}(t)$. It has type D_8 by Proposition 7.260. The D_8 root lattice has index 2 in the E_8 lattice. Thus, a D_8-normalizer fixes t, the nontrivial quotient character. Since $C(t)$ is connected, its stabilizer is $W(D_8)$ on both sides. In $\mathbf{J}_t$, torus conjugacy gives $N_{\mathbf{G}}(\mathbf{J}_t) = \mathbf{J}_t$. A transporter g to the split subsystem satisfies $g^{-1}F(g) \in \mathbf{J}_t$. Lang gives a rational model with $J_t \rtimes E$, with $J_t = \mathbf{J}_t^F$. The nontrivial central element gives κ_t on J_t with $\kappa_t|_T = \theta_t$, by Proposition 7.260. Uniqueness makes κ_t fixed by E. Extend by 1 on E and transport back to λ_t, rational on its normalizer. The joint stabilizer of θ_s fixes θ_t, so they normalize $\mathbf{J}_t$ and fix λ_t by conjugation.

The product $\lambda_y \lambda_t$ on I^+ extends θ_s. Put $U = (P \cap \mathbf{K}^F_{\mathrm{der}})[\ell]$. Moreover, this has order ℓ: the derived D_4 torus has polynomial Φ_8 with simple selected eigenvalue modulo ℓ, and the central isogeny has 2-primary kernel. Put $S_y = \{a \in \mathcal{A} : \langle \theta_y^a \rangle = \langle \theta_y \rangle\}$. The group S_y normalizes $\mathbf{K}$, since unit powers give $R_{y^i} = R_y$ for $\gcd(i, |y|) = 1$. They normalize P and so U. Compose conjugation on U with a faithful character $\xi : \mathrm{Aut}(U) = \mathbb{F}_\ell^\times \to \mu_{\ell-1}$ to obtain $\nu : S_y \to \mu_{\ell-1}$. Then, $\nu|_T = 1$ and $\nu(v) = -1$, since v inverts U. For elements jointly stabilizing θ_s, coefficient powers of θ_y put the geometric part in S_y. Conjugation fixes ν and coefficients fix $\mu_{\ell-1}$. Thus, $(\lambda_y \lambda_t)|_{I^+}$ and $(\lambda_y \lambda_t \nu)|_{I^+}$ are joint-invariant. Their restrictions to I give both labels over θ_s.

Next, induce to MI^+, with irreducible restriction to M by [95, Problem 5.2]. Each local operation stabilizing a

label induced from these sources fixes $(\theta_s)^M$, and a correction in M makes the operation fix θ_s and both extensions. Proposition 7.11 matches $\mathcal{E}(G,s)$. Next, pair the semisimple label with $\lambda_y\lambda_t$, and the regular label with $\lambda_y\lambda_t\nu$, via induction. Transport along orbits. Given χ,ψ, we put $J=\mathcal{A}_{(\theta_s)^M}$. Moreover, every element of J is corrected by M to an element of I^+. Thus, $J=MI^+$ and $\mathcal{A}_\chi=\mathcal{A}_\psi=J$. The extensions are on GJ and J, respectively, with $GJ=G\rtimes E_\chi$ since inner automorphisms fix χ. Since $\mathrm{Z}(G)=1$ and $\mathrm{Out}(G)=E$, $\Gamma=\mathrm{Aut}(G)_P=\mathcal{A}$. Compose with multiplication $G\rtimes J\to GJ$ and $M\rtimes J\to J$. If $D_0=\Gamma_{\chi^{\mathcal{H}_\ell}}$, the centralizer $C_{G\rtimes D_0}(G)$ consists of $(u,\mathrm{Int}(u^{-1}))$, with $u\in M$. The pullbacks are trivial on $C_{G\rtimes D_0}(G)$. Factors and corrections are 1. Thus, $\succeq_c$ holds on D_0. □

Corollary 7.263. *Given any q with $\mathrm{ord}_\ell(q)=8$, the inductive comparison for $E_8(q)$ holds on $\mathrm{N}_G(P)$.*

Proof. First, fix the marked torus T, its Sylow subgroup P, and $M=N_G(P)=N_G(T)$. By [58, Proposition 11.5.6], an ℓ'-label in $\mathcal{E}(G,s)$ gives $\ell\nmid[G^*:C_{G^*}(s)]$. Thus, by [132, Theorem 5.9], the class of s meets T^*, since $\mathbf{T}^*$ is a maximal Sylow d-torus. By [132, Proposition 5.11], the intersection is one $N_{G^*}(\mathbf{T}^*)$-orbit, dual to an M-orbit on $\mathrm{Irr}(T)$. Write $\bar{s}$ for a common index, and put $\mathcal{X}_{\bar{s}}=\mathcal{E}(G,s)\cap\mathrm{Irr}_{\ell'}(G)$, $\mathcal{Y}_{\bar{s}}=\mathrm{Irr}_{\ell'}(M\mid(\theta_s)^M)$. The $\mathcal{X}_{\bar{s}}$ partition by rational series. The $\mathcal{Y}_{\bar{s}}$ partition by Clifford theory over T, with weights in $(\theta_s)^M$. The group $\Gamma=\mathrm{Aut}(G)_P$ normalizes $T=C_G(P)$ and M. Torus duality matches its action, and $\mathcal{H}_\ell$ powers s,θ_s by the same unit. Consequently, $\bar{s}$ may move under individual operations.

If $s=1$, apply [156, Theorem D] on the minimal cyclotomic torus T. The source 1_T gives $\mathcal{Y}_{\bar{1}}=\mathrm{Irr}_{\ell'}(M\mid 1_T)$. Given a proper Levi $C_{\mathbf{G}^*}(s)$, we apply Corollary 7.261 on ℓ'-labels. If q is even, $\exp(T^*)\mid q^4+1$ is odd. This exhausts all $s\neq 1$. If q is odd, $(q^4+1)_2=2$. Write $s=ty$ with $t^2=1$ and y of odd order. If $y=1\neq t$, Proposition 7.260 gives split D_8 and quadratic weights over θ_t. Then, $I=M_{\theta_t}$ and $I/T=B_t$ has order 64. Induction of $\kappa_t|_I$ times $\eta\in\mathrm{Irr}(B_t)$ gives all labels on M over θ_t, matched with global relative labels. If $y\neq 1$ with non-Levi $C(s)$, Proposition 7.262 gives both labels. Quotient C_2 yields both weight extensions in $\mathcal{Y}_{\bar{s}}$. Both actions on $\bar{s}$ keep its type and carry t,y separately. Thus, each case gives $\succeq_c$ on joint stabilizers. Lemma 9.4 combines the comparisons on M. Finally, $M<G$ since $P\neq 1$ and G is simple. □

Proposition 7.264. *Suppose $d=4$. The torus centralizing its ℓ-part has mark $J\in W(D_8)$ of four negative 2-cycles and relative group $B=\mathrm{C}_W(J)$ of order $2^{10}\cdot 3^2\cdot 5$. For $\ell>5$ $\mathrm{N}_G(T)=\mathrm{N}_G(P)$. If $\ell=5$, we let $R\in\mathrm{Syl}_5(B)$ and take P over R. Then, with $M=\mathrm{N}_G(P)$ and $T_5=(\mathbf{T}^F)_5$ the Sylow 5-subgroup of T, $M/T_5\cong\mathrm{N}_B(R)=\langle J\rangle\times(C_5\rtimes C_4)$ with faithful C_4-action. The unipotent and isolated $|s|=5$ series exhaust $\mathrm{Irr}_{5'}(G)$, with $20,5$ labels respectively and $\succeq_c$ on M.*

Proof. The mark satisfies $J^2=-1$, so $\mathbf{T}$ is Sylow Φ_4 and $|T|=(q^2+1)^4$. By [178, Theorem 4.2], $|B|=8\cdot 12\cdot 20\cdot 24$. By [63, Proposition 2.19 and Lemma 2.20], $T\simeq(C_{q^2+1})^4$ and B acts faithfully on T_ℓ. The $J=q^{-1}$ eigenspace modulo ℓ is its own orthogonal complement. Since $(\alpha,\alpha)=2$, every root restricts nontrivially to T_ℓ. Thus, $C_{\mathbf{G}}(T_\ell)^\circ=\mathbf{T}$, and $C_G(T_\ell)=T$ by the faithful action of B. At $\ell>5$, the order formula gives $P=T_\ell$. Hence, $\mathrm{N}_G(P)=\mathrm{N}_G(\mathbf{T})$.

At $\ell=5$, put $a=v_5(q^2+1)$ and let $\rho:\mathrm{N}_G(\mathbf{T})\to B$ be Weyl projection. The degrees $8,12,20,24$ are divisible by 4. By [63, Lemma 4.1], $v_5(q^m-1)=a+v_5(m/4)$ for these degrees m, so $|G|_5=5^{4a+1}$. Thus, P is Sylow in $\rho^{-1}(R)$ with $P\cap T=T_5\cong(C_{5^a})^4$. By [63, Theorem 3.3, inclusion (3.4)], $M\leq\mathrm{N}_G(\mathbf{T})$. By [127, Table 1], $|M|=80|T_5|$. On the 4-space over $\mathbb{Q}(i)$, a generator u of R has irreducible polynomial Φ_5, since $\mathbb{Q}(i)\cap\mathbb{Q}(\zeta_5)=\mathbb{Q}$. On the chosen 5-adic eigenspace, its lattice has rank 1 over $\mathcal{O}=\mathbb{Z}_5[\zeta_5]$. The lattice is an $\mathcal{O}$-ideal, with quotient T_5 modulo 5^a. Moreover, the action $u-1$ is multiplication by the uniformizer ζ_5-1, with kernel of order 5 on T_5. The dual calculation gives $|T_5^R|=|\mathrm{Irr}(T_5)^R|=5$. On the full lattice, u has polynomial Φ_5^2, so $\det(u-1)=25$. Thus, $u-1$ has kernel of order 25 on $\mathbf{T}$, and $T_{5'}^R=1$. Given $t\in M\cap T_{5'}$, the commutators with P lie in $T_{5'}\cap T_5=1$. Thus, $t\in T_{5'}^R=1$ and $\ker(\rho|_M)=T_5$. Hence, $|\rho(M)|=80$. By the Frattini argument [97, Theorem 1.12] in $\rho^{-1}(\mathrm{N}_B(R))$ with normal subgroup $\rho^{-1}(R)$, $\rho(M)=\mathrm{N}_B(R)$.

On the irreducible $\mathbb{Q}(i)R$-module, $C_B(R)$ embeds in $\mathbb{Q}(i,\zeta_5)^\times=\mathbb{Q}(\zeta_{20})^\times$. Its finite image lies in μ_{20}. The subgroup $R\times\langle J\rangle$ already has order 20, so it equals $C_B(R)$. Thus, $[N_B(R):C_B(R)]=80/20=4$ since $|N_B(R)|=80$. On the four nonzero R-fixed dual weights, J acts by a primitive 4-th root modulo 5. Moreover, the point stabilizer H has order 20 and meets $C_B(R)$ in R. The quotient $H/R=C_4$ acts faithfully on R, and the coprime extension splits. Thus, $H=C_5\rtimes C_4$ and $N_B(R)=\langle J\rangle\times H$. Choose eligible s centralizing P^*. Thus, $s\in C(T_5^*)=T^*$ is fixed by R. If $s\neq 1$, the elliptic $R\leq W_s$ gives the full-rank A_4^2 vertex at order 5. Conversely, it centralizes P^*. By Clifford theory [43], a $5'$-degree character of M lies over a T_5-weight orbit of size prime to 5. Since $R\lhd M/T_5$, each such weight is R-fixed.

At $s=1$, [156, Theorem D and §4.4] pairs unipotents with $\mathrm{Irr}_{5'}(B)$ on the torus normalizer. On M the labels over 1_{T_5} inflate from $\mathrm{Irr}_{5'}(N_B(R))$. By [35, Theorem B], $|\mathrm{Irr}_{5'}(B)| = |\mathrm{Irr}_{5'}(N_B(R))| = 20$. Indeed, $C_5 \rtimes C_4$ has degrees $1,1,1,1,4$, with rational nonlinear label, and $|\mathrm{Irr}(\langle J\rangle)| = 4$. Choose a prime-field Tits lift $\dot{J}$, with marked field Φ satisfying $\Phi^f = \dot{J}^{-1}$. Its Weyl action is trivial. Thus, $\Phi(P)$ and P are Sylow in $\rho^{-1}(R)$. A Sylow conjugator has a P-translate in T. Choose $t_0 \in T$ and put $v = t_0\Phi$ fixing P. Then, $N_{G\rtimes E}(P) = \langle M, v\rangle$ and $v^f = t_0\Phi(t_0)\cdots\Phi^{f-1}(t_0)\dot{J}^{-1} \in M$. Its Weyl image is J^{-1}, and v acts trivially on M/T_5. As $\mathrm{ord}_5(q) = 4$, the degree f is odd. If $\eta \in \mathrm{Irr}(M/T_5)$, give v the unique scalar $\beta_\eta \in \mu_4$ satisfying $\beta_\eta^f = \eta(J^{-1})/\eta(1)$. The values of η lie in $\mathbb{Q}(i)$, so this extension is $\mathcal{H}_5$-fixed. Globally F_r fixes the 20 unipotents, with $\mathcal{H}_5$-fixed extensions by [156, §8.2, proof of Theorem D].

Given $s \neq 1$, put $C = \mathrm{C}_{\mathbf{G}^*}(s)$. It has type A_4^2 and $\mathrm{Z}(C) = \langle s\rangle$ of order 5. By [208, Lemma 12.7], the quotient $N_{\mathbf{G}^*}(C)/C$ acts on $\langle s\rangle$ as $C_4 = \langle w\rangle$. Since $\mathrm{ord}_5(q) = 4$, the mark represents $w^{\pm 1}$. Apply [208, proof of Proposition 12.8, Step 2] to the exchanging mark. Thus, C^{F^*} has one unitary A_4 factor over $\mathbb{F}_{q^2}$ up to isogeny, since w^2 induces opposition. The Jordan index is $5'$. By [57, Proposition 13.20], C^{F^*} and $\mathrm{GU}_5(q^2)$ have the same unipotent degrees. Since $-q^2 \equiv 1 \bmod 5$, Lemma 3.2 gives $v_5(5!) = \sum_{h\in \mathrm{Hook}(\lambda)} v_5(h)$. Thus, the 5 eligible partitions are the hooks of 5. By [208, proof of Proposition 12.8, Steps 2–3], the partition λ is fixed by $\langle w\rangle$. On $\langle s\rangle$, the group acts by $s \mapsto s^i$, $1 \le i \le 4$. Thus, χ is rational and fixed by F_r. Since $4 = \mathrm{ord}_5(q) \mid \mathrm{ord}_5(r) \mid 4$, we have $2 \nmid f$. By [143, Lemma 2.1 and Corollary 2.2], the real extension $\widehat{\chi}$ over $\langle F_r\rangle$ is unique and rational.

Put $I = M_{\theta_s}$ for the weight on T_5, extended trivially over $T_{5'}$ in $N_G(T)$. The image H of I fixes s, so I lies in the primal subsystem of C. The central character κ_s restricts to θ_s on T_5. Put $\psi_0 = \mathrm{Ind}_I^M(\kappa_s|_I)$. Either operation replaces s by a nonzero power. Take J^j in M to restore the pair. Thus, ψ_0 is $\mathbb{Q}$-valued and F_r-fixed. Its real extension X_0 on $\langle M, v\rangle$ is rational by [143, Lemma 2.1 and Corollary 2.2]. Extend $M \to N_B(R) \to H$ to $\pi : \langle M, v\rangle \to H$ by $\pi(v) = 1$, since the image of v^f is J^{-1}. If $\eta \in \mathrm{Irr}(H)$, the tensor identity gives $(X_0(\eta\circ\pi))|_M = \mathrm{Ind}_I^M(\kappa_s|_I\ \mathrm{Inf}_H^I\,\eta)$. These exhaust the 5 local labels, with extensions over $\mathbb{Q}(i)$. Both sides have twists 1, as for the 20 unipotents. Next, put $\mathcal{A} = N_{G\rtimes E}(P) = \langle M, v\rangle$. Since $G = \mathrm{Inn}(G)$ and $\mathrm{Out}(G) = E$, $\mathcal{A} = \Gamma = \mathrm{Aut}(G)_P$. Base labels are fixed by $\Gamma \times \mathcal{H}_5$. Its two ordinary extensions are on $G\rtimes E$ and $\mathcal{A}$, and are fixed by $\mathcal{H}_5$. Next, compose on $G\rtimes\Gamma$ with $(g,b) \mapsto gb \in G\rtimes E$ and on $M\rtimes\Gamma$ with $(m,b) \mapsto mb \in \mathcal{A}$. Invariant extensions give factors and corrections 1. Also, both maps are trivial on $C_{G\rtimes\Gamma}(G) = \{(m, \mathrm{Int}(m^{-1})) : m \in M\}$. Thus, $\succeq_c$ holds on Γ. Weights 1_{T_5} and $\theta_s \neq 1_{T_5}$ separate the local sectors. Globally $s = 1$ and $|s| = 5$ exhaust the eligible series. □

Proposition 7.265. *Assume $d = 4$ and $\ell > 5$. For a proper Levi closure K^*, either $C(s) = K^*$ or K^* has type $D_4, {}^2D_6$ with two signs. Full rank gives D_8 and, at $\mathrm{ord}_5(q) = 4$, A_4^2 of order 5. Also, both give $\succeq_c$ on $\mathrm{N}_G(T)$.*

Proof. Since $J^2 = -1$, neutral rank is even. Under J, components are fixed or exchanged. In A, full rank gives one eigenvalue. If J fixes A_n for $n \ge 2$, $J^2|_{A_n}$ is inner, but $-1 \notin W(A_n)$. Root lines cannot support $J^2 = -1$. Outer opposition excludes E_6. Thus, non-Levi factors lie in D_4, D_6. Moreover, the D_4 node and neighbours exclude two disjoint A_1's in a Levi. Fixed A_2 is excluded. At even rank ≤ 6, the sign ranks are $(2,2),(2,4),(3,3)$. Moreover, in D_4, triality conjugates involutions. Moreover, the negative 2-cycles have even sign, giving split D_4. Three yield minus D_6. The respective neutral systems are $A_1^4,\ D_2D_4,\ D_3^2$, with $D_2 = A_1^2$ and $D_3 = A_3$. By Proposition 7.266, we obtain $\succeq_c$ on $\mathrm{N}_G(T)$. For Levi centralizers apply Proposition 7.279.

Since $3 \nmid q^2+1$ and $(q^2+1)_2 \le 2$, isolated orders are $2, 5$. At odd q, the orders in [120, $E_8(q)$] give only split D_8 with four factors $\Phi_4(q)$. Since $\ell > 5$, each other factor is ℓ'. The mark J in $W(D_8)$ has four negative 2-cycles. Thus, B_t is the even-sign subgroup of $C_4 \wr \mathfrak{S}_4$, of order $3072 = 2^{10}\cdot 3$. Next, put $q = r^f$, $f_2 = 2^{v_2(f)}$, and $e = \mathrm{ord}_\ell(r)$. Since $e/(e,f) = 4$, we have $v_2(e) = v_2(f)+2$, so $4f_2 \mid \ell - 1$. At t, Proposition 7.260 has source $\epsilon_t\rho$. Take roots in $\mu_{2f_2} \subset \mathbb{Q}_\ell$ as there. Pullback and [208, Lemma 4.9(i),(ii)] give $\mathcal{H}_\ell$-fixed extensions of eligible χ on $\widehat{G} = G \rtimes E$.

Locally, Proposition 7.260 sets $n = \dot{J}$ in split D_8 subgroup K. Put $I = M_{\theta_t} = N_K(\mathbf{T})$, $\widehat{M} = M\langle\Phi\rangle$, $\widehat{I} = I\langle\Phi\rangle$. The lift is rational in $D_{8,\mathrm{sc}}$, so $\kappa_t(n) = 1$. As before, $\widehat{\kappa}_t(i\Phi^a) = \kappa_t(i)$ on $\widehat{I}$. On $B_t = I/T$, Φ acts trivially and $(\Phi T)^f = J^{-1}$. Moreover, the central element J^{-1} has order 4, with scalar $c_\eta \in \mu_4$ on the model of $\eta \in \mathrm{Irr}(B_t)$. Choose scalar $\beta_\eta \in \mu_{4f_2}$ on ΦT, where $\beta_\eta^f = c_\eta$, possible since f/f_2 is odd. By [156, Corollary 4.4], η has values in $\mathbb{Q}(i)$. Thus, the extended $\widehat{\eta}$ has values in $\mathbb{Q}(\zeta_{4f_2})$, fixed by $\mathcal{H}_\ell$. Clifford induction as in Proposition 7.260 gives $\widehat{\psi}_\eta = \mathrm{Ind}_{\widehat{I}}^{\widehat{M}}(\widehat{\kappa}_t\,\widehat{\eta})$, $\widehat{\psi}_\eta|_M = \mathrm{Ind}_I^M(\kappa_t\eta) \in \mathrm{Irr}(M)$. Indeed, $\widehat{M} = M\widehat{I}$ and $M \cap \widehat{I} = I$. The extension is fixed by $\mathcal{H}_\ell$.

The minimal cyclotomic Levi in D_8 has source 1_T. By [156, Theorem D and Corollary 4.7], take $\mathrm{Irr}(B_t)$. All relative degrees are ℓ-units since $|B_t| = 3072$, and $[M:I] = 2^{10}\cdot 3^2\cdot 5/3072 = 15$ is also an ℓ-unit. By [57, Proposition 13.20] and the ℓ'-Jordan index, these exhaust the eligible $\mathcal{E}(G,t)$-labels. Take these paired extensions. The 5-point has

$\mathrm{ord}_5(q) = 4$. Moreover, the exchange-opposition operator b corrects the q-power on s, has $b^2 = -1$, and has polynomial Φ_4^4. Its primitive eigenspace is regular, so b is conjugate to J by [178]. The index is prime to ℓ. Fusion gives one orbit, so choose $A_4(q^2)$ of unitary type. Since $-q^2 \equiv 1 \bmod \ell$ and $\ell > 5$, all partitions of 5 are eligible. Under J, the fixed subgroup is $H = \mathrm{C}_{W(A_4)^2}(J) \cong \mathfrak{S}_5$. The line stabilizer is $H \times \langle J \rangle$, since J induces all four powers of s prime to 5.

Let D, I lift line and point stabilizers in M. Then, $I/T = H$ and $D/I \simeq \langle J \rangle$ has order 4. The marked Φ fixes the line setwise and has $\Phi(d)T = dT$ for $d \in D$. The primal subsystem character κ_s extends θ_s to I. Next, put $\psi_D = \mathrm{Ind}_I^D(\kappa_s|_I)$. After transport, J^j restores the source up to D-conjugacy. Thus, ψ_D is rational and field-fixed. As $\mathrm{ord}_5(q) = 4$, the quotient of $D^+ = \langle D, \Phi \rangle$ by D has odd order f. Its unique real extension X_D of ψ_D is rational by [143, Lemma 2.1 and Corollary 2.2]. The map $D \to D/T \to H$ extends to $\pi : D^+ \to H$ by $\pi(\Phi) = 1$, since $\dot{J}^{-1}$ projects trivially to H. Given $\eta \in \mathrm{Irr}(H)$, induce $X_D(\eta \circ \pi)$ from D^+ to $\langle M, \Phi \rangle$. The full field quotient of D^+ gives one coset. Restriction to M is $\mathrm{Ind}_I^M(\kappa_s|_I \, \mathrm{Inf}_H^I \eta)$, irreducible by Clifford theory. Also, these 7 extensions are rational. Match partitions of 5 by Proposition 7.264 and [143, Corollary 2.2]. Moreover, extensions on $\widehat{G}, \widehat{M}$ are fixed by $\mathcal{H}_\ell$. The base labels are fixed by $\Gamma \times \mathcal{H}_\ell$, with $\Gamma = \mathrm{Aut}(G)_P = \widehat{M}$. Next, apply Proposition 7.264 with $\mathcal{H}_\ell$ in place of $\mathcal{H}_5$. Moreover, both factors and corrections are 1. The common centralizer has elements $(m, \mathrm{Int}(m^{-1}))$, $m \in M$. Both maps take them to 1, giving $\succeq_c$ on Γ. □

Proposition 7.266. *Next, we let $q = r^f$ be odd, let $\mathrm{ord}_\ell(q) = 4$ with $\ell > 5$, and set $M = \mathrm{N}_G(P)$ in $G = E_8(q)$. Moreover, in* Proposition 7.265*, the three R-types A_1^4, D_2D_4, D_3^2 admit extensions of χ, ψ to inertia with joint invariance.*

Proof. Write $F' = \mathrm{Int}(\dot{J})F_q$, with Tits lift $\dot{J}$ of J over $\mathbb{F}_r$. Next, we let $T = \mathbf{T}^{F'}$, $B = C_{W(E_8)}(J)$, and let $\mathcal{A} = N_{G \rtimes E}(P)$ for pinned $E = \langle F_r \rangle$. The marked Φ normalizes T, P. By Proposition 7.262, $\Phi^f = \dot{J}^{-1}$ in $\mathcal{A}$. In particular $\mathcal{A}/M = E$, $M/T = B$, and Φ acts trivially on B. In each series at s, let R be its neutral roots, θ_s its weight, and $I = M_{\theta_s}$. By connectedness and torus duality, $I/T = B_s = W(R)^J$. Corollary 7.263 gives fusion in T^* by its rational normalizer. The Sylow d-torus centralizer in $C(s)$ is $\mathbf{T}^*$. Thus, [156, Cor. 4.7 and §4.4] pairs unipotents with $\mathrm{Irr}(B_s)$. Their degrees are prime to ℓ since $\ell \nmid |B|$.

Write $s = ty$, with $|t| = 2$ and y of odd order. This case requires $t \neq 1$, and $(q^2 + 1)_2 = 2$. Next, we put $I_s^+ = \mathcal{A}_{\theta_s}$. Next, choose V generated by Tits lifts and $\mathbf{T}[2^\infty]$ as in Proposition 7.262. Surjectivity of $F' - 1$ gives B-lifts in $V^{F'}$. Put $D = \langle V^{F'}, \Phi \rangle$, giving $\mathcal{A} = TD$. Since Φ normalizes $V^{F'}$ and $\Phi^f \in V^{F'}$, $D \cap T$ is a 2-group. Thus, $\lambda_y(xd) = \theta_y(x)$ $(x \in T,\ d \in D_{\theta_y})$ is a character on $\mathcal{A}_{\theta_y} = TD_{\theta_y}$, covariant under D and coefficients. Conjugation by T is inner. Involution type is D_8, since $J^2 = -1$ precludes the E_7A_1 root line. Let $\mathbf{K}_t$ be the primal subsystem on $\mathbf{T}$. By Proposition 7.262, $N_{\mathbf{G}}(\mathbf{K}_t) = \mathbf{K}_t$. By Lang, take $\mathbf{K}_t$ split over $\mathbb{F}_r$, with normalizer $K_t \rtimes E$. The central quadratic character restricts to θ_t and extends by 1 on E. Conjugate back to λ_t. Since transporters differ by K_t, choices agree. For elements jointly stabilizing θ_s, the geometric part fixes θ_t, since coefficients fix signs. It normalizes $\mathbf{K}_t$ and fixes λ_t by conjugation. Coefficients fix the rational λ_t. Moreover, covariance of λ_y gives invariant $\lambda_s = (\lambda_y\lambda_t)|_{I_s^+}$. If $(j, |\theta_s|) = 1$, $I_{\theta_s^j} = I_{\theta_s}$.

If $\eta \in \mathrm{Irr}(B_s)$, pair its Jordan label χ with $\psi = \mathrm{Ind}_I^M(\lambda_s|_I \, \mathrm{Inf}_{B_s}^I \eta)$. This is the map of [156, §4.4] with λ_s. By Clifford theory, recover each η. When one of these labels returns, correct by M, uniquely modulo I acting on $\mathrm{Irr}(B_s)$. Covariance of λ_s gives equal transported labels. Thus,

$$\mathcal{J} = \{(a, \sigma) : \chi^{a\sigma} = \chi\} = \{(a, \sigma) : \psi^{a\sigma} = \psi\},\ D_0 = \mathrm{pr}_{\mathcal{A}} \mathcal{J},\ D_1 = \mathcal{A}_\chi = \mathcal{A}_\psi.$$

We have $D_0 = \mathcal{A}_{\chi^{\mathcal{H}_\ell}} = \mathcal{A}_{\psi^{\mathcal{H}_\ell}}$. By commutation of actions, $\chi^a = \chi^\tau$ for $a \in D_0$ has inertia D_1. Thus, $D_1 \lhd D_0$. Let E_χ be the image of D_1 in E, of order h. Next, we put $\mathcal{B}_G = GD_1 = G \rtimes E_\chi$ and $\mathcal{B}_M = D_1$, with $\mathcal{B}_G/G \simeq \mathcal{B}_M/M \simeq E_\chi$. Both are normalized by D_0, as base labels move. The element $-1 = J^2$ takes s^{-1} to s. Also, it acts by conjugation on D_2, D_4 and by opposition on $D_3 = A_3$. Moreover, both fix partitions, including exchanged A_3's. It fixes the lines of A_1^4. Classical unipotents are real [72, Cor. 4.5.6]. Thus, [180] gives $\bar{\chi} = \chi$.

Locally let σ_0 fix μ_{2^∞} and invert odd roots. Thus, $\theta_s^{\sigma_0} = \theta_s^{-1}$. A lift of J^2 restores the weight and $\lambda_s|_I$, centralizing B_s. The relative character η is fixed by σ_0. Indeed, A_1^4 gives C_2^2. In type D_2D_4, the D_2-lines give C_2 and D_4 gives the even-sign subgroup of $C_4 \wr \mathfrak{S}_2$, a 2-group. Moreover, on D_3^2, a J-centralizer is determined on one factor. $B_s \simeq W(D_3) \simeq \mathfrak{S}_4$ has rational characters. Consequently, $\psi^{\sigma_0} = \psi$.

Suppose h is odd. By [143, Lemma 2.1], χ has the unique real extension $\widehat{\chi}$ on $\mathcal{B}_G$. Since σ_0 fixes ψ and inverts μ_h, $\mu_a \notin \mathbb{Q}(\psi)$ for each prime $a \mid h$. By [208, Lemma 3.2], ψ has a unique extension $\widehat{\psi}$ over $\mathbb{Q}(\psi)$, which is fixed by

σ_0. Let ι be complex conjugation. Each $j = (a, \sigma) \in \mathcal{J}$ satisfies $j\iota = \iota j$ and $j\sigma_0 = \sigma_0 j$. Hence, uniqueness gives $j(\widehat{\chi}, \widehat{\psi}) = (\widehat{\chi}, \widehat{\psi})$ throughout D_0.

Suppose h is even. A torus corrector $u \in B$ acts by $s \mapsto s^{r^{f/h}}$. Consequently, $u^h(s) = s^q$. Rationality gives the same power by $J^{\pm 1}$, up to convention. Thus, $u^h, J^{\pm 1}$ differ by $W(R)$ and have equal component permutations. On $D_2 D_4$ this transposes the D_2-lines, on D_3^2 the factors. Neither is even in $\mathfrak{S}_2$. Moreover, on A_1^4, J gives two transpositions. Its root in $\mathfrak{S}_4$ is a 4-cycle exchanging J-orbits. If inertia is even, only $(1, 1)$ or $(\mathrm{St}, \mathrm{St})$ occurs. Generator signs coincide on C_2^2.

Moreover, the constant labels are χ_{reg} and its signed dual χ_{ss}. The module of Proposition 7.262 contains χ_{reg} with multiplicity 1 on G. Its isotypic space gives joint-invariant $\widehat{\chi}_{\mathrm{reg}}$. Parabolic duality and Proposition 2.4 give $\widehat{\chi}_{\mathrm{ss}}$ on the same inertia. The joint stabilizer of $\mathcal{E}(G, s)$ fixes both. Let $\mathbf{K}_R$ be the primal subsystem on $\mathbf{T}$ with roots R. Next, put $U_R = P[\ell] \cap [\mathbf{K}_R, \mathbf{K}_R]^{F'}$. Each root-line pair under J contributes one dimension to U_R. Moreover, its coroot polynomial is $X^2 + 1$ and F' acts by qJ on the ℓ-torsion. Thus, only the $J = q^{-1}$ eigenline returns. By coroot Gram determinant 2^4, the saturation index is a 2-power. Both planes exhaust U_R. Unit-power transport of θ_s stabilizes R and U_R.

Next, we choose a faithful $\iota : \mathbb{F}_\ell^\times \to \mu_{\ell-1}$. Next, on $S_R = N_{\mathcal{A}}(U_R)$, put $\nu_R(b) = \iota(\det_{\mathbb{F}_\ell}(b|U_R))$. Then $\nu_R|_T = 1$. Each generator of $B_s = C_2^2$ acts by -1 on its eigenline alone. Thus, $\nu_R|_I$ has both signs negative. On U_R, Φ acts by r and $\dot{J}^{-1}$ by q. Hence, $\nu_R(\Phi)^f = \iota(q^2) = \nu_R(\dot{J}^{-1})$. The group $\mu_{\ell-1}$ is fixed by $\mathcal{H}_\ell$. For a corrected (a, σ) in the joint stabilizer, the weight θ_s^a is a coprime power of θ_s. Then, a gives $a(R) = R,\ a(\mathbf{K}_R) = \mathbf{K}_R,\ a(U_R) = U_R$, so $a \in S_R$. It normalizes I_s^+. Given $b \in I_s^+$, determinant invariance gives $\nu_R(a^{-1}ba) = \nu_R(b)$, and $\nu_R(b)^\sigma = \nu_R(b)$ since values lie in $\mu_{\ell-1}$. Thus, λ_s and $\lambda_s \nu_R$ are joint-invariant on I_s^+.

Constant labels have inertia MI_s^+, as M corrects their stabilizing operations to fix θ_s. Induction to MI_s^+ restricts to ψ on M, by [95, Problem 5.2] since $M \cap I_s^+ = I$. For $(a, \sigma) \in \mathcal{J}$, we choose $m \in M$ so that (ma, σ) returns θ_s. It fixes the source extension on I_s^+. Hence, induction is fixed on $MI_s^+ = D_1 = \mathcal{B}_M$. The correction m acts by inner conjugation. Thus, $\widehat{\psi}^{a\sigma} = \widehat{\psi}$. The global construction gives $\widehat{\chi}^{a\sigma} = \widehat{\chi}$ on $\mathcal{B}_G$.

Next, we choose the model $\mathcal{A}$ of $\Gamma = \mathrm{Aut}(G)_P$. Each parity has projected stabilizer D_0, and ordinary subgroups have automorphism part D_1. Next, compose $\widehat{\chi}, \widehat{\psi}$ with multiplication $G \rtimes D_1 \to \mathcal{B}_G$ and $M \rtimes D_1 \to \mathcal{B}_M$. The factors are 1, and $\mathcal{J}$-invariance gives correcting functions 1 by [141, Lemma 1.4]. The common centralizer is $C_{G \rtimes D_0}(G) = \{(m, \mathrm{Int}(m^{-1})) : m \in M\} \leq M \rtimes D_1$. Both maps are trivial there. Thus, $\succeq_c$ holds on D_0. □

Proposition 7.267. *Let $(d, \delta) = (3, 1), (6, -1)$. The dual elements of order 3 form one marked E_6A_2-orbit when $\delta q \equiv 1 \bmod 3$. With nontrivial $3'$-part, take the centralizer Levi or A_2^3 in an E_6-Levi.*

Proof. With Coxeter element c, the mark is δa for $a = c^{10}$. Thus, the characteristic polynomial of δa is Φ_d^4. By [178, Theorem 4.2], δa is regular. By [27, Table 3], T has order $\Phi_3(\delta q)^4$ and contains $P \in \mathrm{Syl}_\ell(G)$ by the order formula. Moreover, $\exp(T) \mid \Phi_3(\delta q)$ by [63, proof of Lemma 2.20]. By [132, Proposition 5.13], $P[\ell]$ determines $\mathbf{T}$. Thus, $M = \mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{T})$. For each root α, the pair $\alpha, a\alpha$ spans an A_2 lattice of determinant 3, since $\alpha + a\alpha + a^2\alpha = 0$. Specifically, among $2, 3, 5$, the only possible divisor is 3, with valuation 1 at $\delta q \equiv 1 \bmod 3$. In E_6A_2, put $a' = (c_6^4, c_2)$, with the two Coxeter elements c_6, c_2. Moreover, its polynomial is Φ_3^4. The A_2-test gives regularity. Thus, a', a are W-conjugate. By [120, $E_8(q)$], one class of 3-elements has ℓ'-index. Indeed, only $E_6^\delta A_2^\delta$ has four factors $\Phi_d(q)$, and no listed factor has index $d\ell^j$ with $j \geq 1$. By Sylow conjugacy, choose z centralizing P^*. Hence, $z \in \mathbf{T}^{*F^*}$ by regularity. By [132, Proposition 5.11], the intersection with $\mathbf{T}^{*F^*}$ is one $N_{G^*}(\mathbf{T}^*)$-orbit. With mark a', the cubic element z of [208, Table 12.2] is fixed by a'. Hence, it is rational when $\delta q \equiv 1 \bmod 3$.

Write $s = yz_0$, with $3 \nmid |y|$ and $z_0^3 = 1$. At $y \neq 1$, Section 7.1 gives Levi $\mathbf{H}^* = C_{\mathbf{G}^*}(y)$. If $C_{\mathbf{G}^*}(s)$ is a Levi, twist by κ_s. Otherwise, one factor has vertex z_0. Highest-root marks avoid primes dividing $|y|$. The span has even rank ≤ 6, and A, D marks exclude 3. Thus, the closure has type E_6 with vertex A_2^3, and equals $\mathbf{H}^*$ by rank. The action on E_6 is inner with rational sign δ. Put $\mathbf{D}^* = [\mathbf{H}^*, \mathbf{H}^*]$. By Lemma 7.268, $z_0 \in \mathbf{D}^{*F^*}$. By Proposition 7.269, we obtain $\succeq_c$ for the A_2^3-series. □

Lemma 7.268. *In* Proposition 7.267*, put $\mathbf{H}^* = C_{\mathbf{G}^*}(y)$ and $\mathbf{D}^* = [\mathbf{H}^*, \mathbf{H}^*]$. Moreover, the marked cubic part z_0 belongs to $\mathbf{D}^{*F^*}$. Divide the primal weight by $\kappa_y|_T$. It then inflates from the adjoint E_6-quotient.*

Proof. Put $\mathbf{T}_D^* = \mathbf{T}^* \cap \mathbf{D}^*$ and let $\bar{z}_0$ be the image of z_0 in the adjoint quotient of $\mathbf{H}^*$. By [20, Theorem 4.5 and Table III], the isolated class of type A_2^3 has representative $\omega_4^\vee(\zeta_3)$ in the adjoint torus, where ζ_3 has order 3.

By [208, (10.23)], the Cartan matrix takes $(2,3,4,6,4,2)$ to e_4. Hence, $\omega_4^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 4\alpha_5^\vee + 2\alpha_6^\vee$ belongs to the coroot lattice. Since $\mathbf{D}^*$ is simply connected, Weyl transport on $\mathbf{T}_D^*$ gives a lift t of order 3 of $\bar{z}_0$.

The marked torus has exponent dividing $q^2+\delta q+1$. Since z_0 has order three, $\delta q \equiv 1 \pmod 3$. Write δa for its normalized mark. Rationality of z_0 then gives $a(z_0) = z_0$, so a gives $a(R) = R$. On the derived 6-space, a has no eigenvalue 1. A cycle of arms would give $0 \neq v$ in one arm and fixed vector $v + av + a^2v$. Thus, a fixes each arm setwise. On A_2, its diagram image has order dividing 2 and 3. Hence, it is trivial. Since R spans, $a \in W(R)$ there. Every reflection in $W(R)$ fixes t, since the roots of R take value 1 at t. Thus, $F^*(t) = t^{\delta q} = t$.

Next, we put $c = z_0 t^{-1} \in Z(\mathbf{H}^*)^{F^*}[3]$. The Levi roots span a saturated E_8-sublattice. Its connected centre has polynomial Φ_d, giving $|Z(\mathbf{H}^*)^{F^*}| = q^2+\delta q+1$. Thus, writing $\delta q = 1+3b$ gives $3(1+3b+3b^2)$, with 3-part 3. Moreover, on $Z(\mathbf{D}^*)$, the Weyl part is trivial, giving δq. Thus, every element of $Z(\mathbf{D}^*)$ is rational. Since $Z(\mathbf{D}^*) \leq Z(\mathbf{H}^*)$, $Z(\mathbf{H}^*)^{F^*}[3] = Z(\mathbf{D}^*)^{F^*}$. In particular, $z_0 = tc \in \mathbf{D}^{*F^*}$. On the primal side let $\pi : \mathbf{H} \to \mathbf{D} = \mathbf{H}/Z(\mathbf{H})$ and $\overline{\mathbf{T}} = \pi(\mathbf{T})$. The same lattice argument gives connected $Z(\mathbf{H})$. By [133, Proposition 23.2], $T/Z(\mathbf{H})^F \simeq \overline{\mathbf{T}}^F$. The dual torus inclusion has image $\mathbf{T}_D^*$. As $z_0 \in \mathbf{T}_D^{*F^*}$, torus duality gives $\theta_{z_0} = \bar{\theta}_{z_0} \circ \pi$ on T. Also, the central element y gives κ_y on $\mathbf{H}^F$, with $\theta_s = (\kappa_y|_T)\theta_{z_0}$. Thus, $\theta_s(\kappa_y|_T)^{-1}$ inflates from $\overline{\mathbf{T}}^F$. □

Proposition 7.269. *For non-Levi $C(s)$ in* Proposition 7.267*, we write $s = yz_0$ with $1 \neq y$ of $3'$-order. The* 27 *ordered partition triples give $\succeq_c$ on $\mathcal{E}(G,s)$ and the θ_s-orbit in $M = \mathrm{N}_G(P)$.*

Proof. Write $\mathbf{H}^* = C_{\mathbf{G}^*}(y)$, with dual $\mathbf{H}$ and quotient $\pi : \mathbf{H} \to \mathbf{D} = \mathbf{H}/Z(\mathbf{H})$. Next, we put $H = \mathbf{H}^F$, $D = \mathbf{D}^F$, $\overline{\mathbf{T}} = \pi(\mathbf{T})$, and $L = \mathrm{N}_H(\mathbf{T})$. By [133, Proposition 23.2], the connected kernel gives $H \twoheadrightarrow D$, also on T and L. Lemma 7.268 puts z_0 in the dual image. By [180, Theorem 4.1(6)], inflation $\mathcal{E}(D, z_0) \to \mathcal{E}(H, z_0)$ fixes Jordan labels. Dually, $\mathbf{H}^* = \mathbf{D}^* Z(\mathbf{H}^*)$, so $C_{\mathbf{H}^*}(z_0) = C_{\mathbf{D}^*}(z_0) Z(\mathbf{H}^*)$. Both have the adjoint quotient of $C_{\mathbf{D}^*}(z_0)$. Central isotypy matches unipotents. Thus, π gives bijective inflation with $(\xi \circ \pi)(1) = \xi(1)$. Locally θ_{z_0} inflates from $\overline{\mathbf{T}}^F$. Each weight is trivial on $Z(\mathbf{H})^F$. Thus, inflation is bijective. Lemma 7.268 shows a acts on each arm by a 3-cycle by Proposition 7.104. The sign is δ. Moreover, the affine rotation normalizes $\mathbf{T}$, cycles arms and centrally translates z_0. The longest element restores z_0^{-1}. Lang gives rational corrections on $\mathbf{T}$. The lifts fix each $\lambda \vdash 3$. Each arm partitions 3 of degrees $1, q(q+\delta), q^3$, prime to ℓ. Since $P^* \leq T^*$, the index is prime to ℓ. In D, relative inertia C_3^3 gives 27 linear labels. Match by Proposition 7.104.

Let D_0 be the connected cover image. Take ν of $\mathrm{Irr}(D/D_0) \cong C_3$. Moreover, the ν-twist rotates the labels in C_3^3 for both sets. By Proposition 7.104 and [208, proof of Proposition 10.9, Steps 4 and 7], γ reflects the labels. On the labels, ϕ acts as γ when $r \equiv -1 \bmod 3$ and fixes them when $r \equiv 1 \bmod 3$. On ν, ϕ acts by exponent $r \bmod 3$. Take pinned E from Proposition 7.104. Moreover, its generators satisfy $\phi^f = \gamma^t$ and $\gamma^2 = [\gamma, \phi] = 1$, with $t = 0$ in split type and $t = 1$ in twisted type. Since $\delta q \equiv 1 \pmod 3$, the twisted case has $r \equiv -1 \pmod 3$ and odd f. If $q = 2$, the exponent divides $\Phi_6(2) = 3$, contradicting $1 \neq y$ of $3'$-order. Let E_0 be the sign kernel. At v^{-1}, the extension is from Proposition 7.104. All 27 base labels have values in $\mathbb{Q}(\zeta_3)$, fixed by $\mathcal{H}_\ell$ since $\ell \equiv 1 \bmod 3$.

On E-fixed words apply Propositions 7.104 and 7.105 above D_0. Distinct-letter orbit representatives extend over E_0 by those proofs. Moreover, the map $de \mapsto \nu(d)$ extends ν on $D \rtimes E_0$ and is fixed by $\mathcal{H}_\ell$. Twist both sides by ν^j, on both sides. Moved words have E_0. Specifically, there are 3 constant, 18 repeated-letter and 6 distinct-letter words. Transport pairs with $\mathcal{H}_\ell$-invariance.

Put $R_D = N_D(\overline{\mathbf{T}})$, $\mathcal{U}_D = D \rtimes E$, $\mathcal{U}_D' = N_{\mathcal{U}_D}(\overline{\mathbf{T}})$. The adjoint group D is centreless, so $\mathcal{U}_D$ acts faithfully by automorphisms on D. Given a corresponding adjoint pair $\rho \in \mathcal{E}(D, z_0)$, $\eta \in \mathrm{Irr}(R_D)$, take models U, V on $A_\rho = (\mathcal{U}_D)_\rho$, $A_\eta = (\mathcal{U}_D')_\eta$. Moreover, these models are $\mathcal{H}_\ell$-fixed before restriction to D_0. The image $\bar{P} = \pi(P)$ is Sylow in D. By Lemma 7.98, the torus calculation survives 3-primary isogenies since $\ell \geq 7$. Thus, $N_D(\bar{P}) = R_D$. Equivariance and Sylow conjugacy give $A_\rho = DA_\eta$, $D \cap A_\eta = R_D$, and $A_\rho/D \simeq A_\eta/R_D$.

Let E_G be the ambient pinned field group. $\mathcal{N}_H = N_{G \rtimes E_G}(\mathbf{H})$. Next, inflate to $\rho_H = \rho \circ \pi$, $\eta_L = \eta \circ (\pi|_L)$. Set $\beta = \kappa_y \rho_H$, $\beta_L = (\kappa_y|_L)\eta_L$. The adjoint action gives $\alpha : \mathcal{N}_H \to \mathcal{U}_D$, including inner, graph and field actions. $h \in H$, $\alpha(h) = \mathrm{Int}(\pi(h))$. Moreover, the rational surjections give $\alpha(H) = D$ and $\alpha(L) = R_D$, including rational diagonal cosets. Put $B = (\mathcal{N}_H)_\beta$, $B_P = N_B(P)$. Next, let A_0 be the projected joint stabilizer of β and $C_0 = N_{A_0}(P)$. Since $(\chi^a)^\sigma = (\chi^\sigma)^a$, $B \lhd A_0$. By Sylow conjugacy in H, $A_0 = HC_0$, $H \cap C_0 = L$, $B \cap C_0 = B_P$. An ordinary stabilizer element of β fixes the central $3'$-element y. Hence, it fixes κ_y and ρ_H separately. Thus, $\alpha(B) \leq A_\rho$ and $\alpha(B_P) \leq A_\eta$. Again $B = HB_P$ and $H \cap B_P = L = N_H(P)$. The pullbacks $U \circ \alpha$ and $V \circ \alpha$ extend ρ_H, η_L on B, B_P. Moreover, both have $B/H \simeq B_P/L \to A_\rho/D \simeq A_\eta/R_D$. Also $C_B(H) \leq B_P$ and $\alpha(C_B(H)) = 1$, giving untwisted centralizer scalars 1.

For (a,σ) in the joint stabilizer of β, the class of y returns, so $\kappa_y^{a\sigma}=\kappa_y$. Because the untwisted ρ_H is $\mathcal{H}_\ell$-fixed, $\alpha(a)$ fixes ρ ordinarily. Correct a by H to normalize P. Now a normalizes B_P and $\alpha(a)\in A_\eta$. Conjugation and powers pass through α under pullback, including inner powers. Thus, $\alpha(a)$ and σ fix the untwisted source, giving corrections 1. Both factor sets are 1.

Conversely, a local marked operation stabilizing β_L fixes θ_y jointly, since it is the $3'$-part of every weight of $\beta_L|_T$. Thus, κ_y returns. The same operation fixes η_L by division by κ_y. Its coefficient part fixes η_L, so the induced automorphism fixes η and so ρ by equivariance and injectivity. Hence, β returns. Thus, C_0 is the local projected joint stabilizer. B_P is ordinary inertia. Also, $C_{A_0}(H)=C_B(H)$ lies in $\ker\alpha$.

If $B/H=B_P/L$, choose common lifts $b_j\in B_P$, with $b_1=1$. Restore $\kappa_y,\kappa_y|_L$ by Lemma 7.7. Thus, the twist gives c_{κ_y} and equal corrections and scalars. If $b=b_j$ has $|j|=m$ and $b^m=h\in L$, then $\alpha(b)^m=\alpha(h)$, with twist factor $\kappa_y(h)^{-1}$. Thus, the two source triples over A_0,C_0 satisfy $\succeq_c$. Since $y\in\langle s\rangle$ is characteristic in s, $C_{\mathbf{G}^*}(s)\le\mathbf{H}^*$. At fixed y, ambient rational conjugacy is conjugacy in H^*. The Weyl stabilizer of s lies in that of y, the image of L. Thus, Clifford induction from L is irreducible and injective at fixed y. Both indices are prime to ℓ. Correct a joint stabilizer element first at y. Sylow conjugacy in $C(s)$ aligns corrections modulo W_y. Their ambiguity is inner by H and L, respectively. Moreover, a source stabilizer admits H^*-correction to centralize s, giving trivial coset. Proposition 7.277 takes y the $3'$-part. By Proposition 7.2 on G and Clifford induction to M, the twists coincide. □

Examples occur at $q=4,d=3$ and at $q=5,d=6$, with $\ell=7$. Take arm rotations on E_6 and a Coxeter rotation on perpendicular A_2. The product w has polynomial Φ_3^4 and is regular by Proposition 7.267. At $d=6$, take $-w$. Moreover, the Levi centre has order $q^2+\delta q+1=21$ on finite points. Take central $|y|=7$. The highest-root test makes $C(y)$ a proper Levi containing E_6. By even rank, $C(y)=\mathbf{H}^*$. Lemma 7.268 gives rational z_0, with $R_{yz_0}=A_2^3$.

Proposition 7.270. *At isolated $|s|=3$ in* Proposition 7.267*, the rational factors give $\succeq_c$ as in* Definition 2.1*, with* 3 *conjugate pairs from E_6 cuspidals.*

Proof. First, take K dual to $\mathrm{C}(z)$ and put $H=\mathrm{C}_{W(E_6)}(c_6^4)\times C_3$. Both factors have sign δ. The element -1 restores z^{-1}, and opposition fixes labels by [208, proof of Proposition 12.8, Steps 2–3]. Thus, F_r fixes labels with values in $\mathbb{Q}(\zeta_3)$. Only the E_6 cuspidal pair gives nonreal labels. Put $M=N_G(T)$ and $I=M_{\theta_z}$. The preimage of H in M lies in K, since the Weyl lifts and kernel T lie there. Then, $I/T=H$ and $\kappa_z|_I$ extends θ_z. Fix the primal mark $w=\delta a$ from Proposition 7.267. The component c_6^4 is regular of order 3, with a regular eigenvector of c_6. Moreover, its centralizer is the relative 3-torus group in E_6. By [156, Lemma 4.3], the conductors divide 6. Changing signs stabilizes this group. Include C_3, giving $\eta\in\mathrm{Irr}(H)$ valued in $\mathbb{Q}(\zeta_3)$. The d-torus correspondence the result of [156, Cor. 4.7, §4.4] gives these labels since $|H|=6\cdot9\cdot12\cdot3$ is prime to ℓ and T contains the Sylow subgroup. Take $\dot w$ over $\mathbb{F}_r$, and set $F=\mathrm{Int}(\dot w)F_q$. On $G=\mathbf{G}^F$, the marked field Φ satisfies $\Phi^f=\dot w^{-1}$ in the automorphism group. Put $e=0$ if $r\equiv1 \bmod 3$, and $e=1$ otherwise. By Lang choose $h\in M$ over $(-1)^e$. Then, $v=h\Phi$ fixes θ_z and κ_z by duality. Since $(-1)^e$ is Weyl-central, v acts trivially on I/T. The element $v^f\in I$ has Weyl image $(-1)^{ef}w^{-1}=a^{-1}$, because $(-1)^{ef}\equiv r^f\equiv\delta \bmod 3$. Take only the Weyl image of $h\Phi(h)\cdots\Phi^{f-1}(h)$. On a model of η, a^{-1} acts by $\beta_\eta\in\mu_3$. Also $\kappa_z(v^f)\in\mu_3$ since $|z|=3$. Give v an f-th root of each scalar in μ_{3f_3}. This is possible since the f-th powers give μ_3. If $d\in\{3,6\}$, the relation $\mathrm{ord}_\ell(r)/\gcd(\mathrm{ord}_\ell(r),f)=d$ gives $3f_3\mid\ell-1$. Thus, both scalars are fixed by $\mathcal{H}_\ell$ with the given powers. Tensor extends $\kappa_z|_I\ \mathrm{Inf}_H^I\,\eta$ to $\langle I,v\rangle$. Induction to $\langle M,v\rangle$ gives an invariant extension irreducible on M by Clifford theory. Real labels are $\mathbb{Q}$-valued in $\mathbb{Q}(\zeta_3)$. Apply [143, Lemma 2.1] on $E_{2'}$, covering odd f and twisted type. Now let $n=f_2>1$ and $q_0=r^{f/n}$. The form is split and q is square. Its real source $\kappa_z\rho$ is on a torus or Levi L with split D_4, by [208, Proposition 12.6, Case 2, and Table 12.3]. $\rho=1$ or inflated cuspidal. Its eigenvalue matches E_6. Its order divides $d=3$ by [156, proof of Theorem 8.5]. Since ρ is real, its eigenvalue lies in $\{1,-1\}$. Hence, it is 1. By [156, Proposition 7.3], the adjoint D_4-extension with f-th root 1 for F_r is fixed by $\mathcal{H}_\ell$. Thus, its restriction remains fixed for $F_{q_0}=F_r^{f/n}$. Corrections fix powers in $L\rtimes\langle F_{q_0}\rangle$.

If $q_0\equiv1 \bmod 3$, extend κ_z trivially on F_{q_0} and apply [208, Lemma 4.9(i),(ii)]. If $q_0\equiv-1 \bmod 3$, take the prime-field Tits representative h_0 of $w_0w_{0,L}$ and $v_0=h_0F_{q_0}$ from [208, proof of Proposition 12.6]. It fixes κ_z, since both operations invert z. On the D_4 quotient h_0 is inner as both elements give -1. Thus, v_0 maps to the field overgroup of D_4. Compose along this map. The power $v_0^n=h_0^n$ is toral 2-torsion, $\kappa_z(v_0^n)=1$. Next, extend by $\kappa_z(v_0)=1$. Take reflection generators of [208, proof of Proposition 12.6, Case 2]. At square q, [208, Proposition 4.16(ii)] gives rationality. Each constituent inherits twist 1. By Hall restriction, twist on E. For nonreal labels, take an E_6-Levi K_0. Next, take the source $\lambda=\kappa_z(\tau-\bar\tau)$, with τ a nonreal cuspidal. Proper Levis of K_0 have real unipotents [40, §13.9]. Restrictions

of $\tau, \bar{\tau}$ agree, so λ restricts to zero. On the perpendicular A_2, the root automorphism group is $\langle W(A_2), -1\rangle$. The pointwise stabilizer is $W(E_6)$. Thus, the normalizer quotient is $\mathrm{Aut}(A_2)$. The stabilizer of z is $W(A_2)$, fixing unipotent labels.

On a nontrivial 3-part take $n = f_3$, $q_0 = r^{f/n}$ and sign δ on both spaces. Moreover, the signed q_0-map fixes z, with the chosen n-th power. At 2-part take $n = f_2$ in split type and $q_0 = r^{f/n}$. Choose a Coxeter rotation c on the plane. Choose $jn \equiv 1 \bmod 3$ and sign δ_0 with $\delta_0 q_0 \equiv 1 \bmod 3$. Next, take marks $\delta_0 c^j$ on the plane and δ_0 on E_6. Their n-th powers give c and split E_6.

Let F_0 lift the primal q_0-mark with its dual. Next, put $F = F_0^n$. The ambient lift differs by $\mathbf{T}$. By Lang, correct in $\mathbf{T}$ while stabilizing $\mathbf{K}_0, \mathbf{T}^*$. On $G = \mathbf{G}^F$, the image u of F_0 has $u^n = 1$ and outer order n. Since $F_0^*(z) = z$, we extend κ_z to $\widehat{\kappa}_z$ by 1 on $\langle u\rangle$. Connectedness gives rational conjugacy of z in both models. The normalizer modulo K_0 is $W(A_2)$ for the first model and $\langle c\rangle$ for the second.

Next, put $D = R_{K_0}^G(\lambda)$. Proper intersections contribute 0. Normalizer cosets contribute $\|\lambda\|^2 = 2$, giving squared norms $12, 6$. Moreover, a rational lift of -1 corrects inversion of z and fixes cuspidals by opposition. Hence, $\bar{D} = -D$. The three nonreal pairs contribute coefficients $m_i, -m_i$. Thus, $\sum_{i=1}^3 m_i^2 = 6$ or 3. Their absolute values are $2, 1, 1$ on the 3-part and $1, 1, 1$ on the 2-part, all prime to n.

Next, we put $\mathbf{J} = \mathbf{K}_0/Z(\mathbf{K}_0)$ and $J = \mathbf{J}^F$. The Levi roots extend to an E_8-lattice basis. Thus, the centre of $\mathbf{K}_0$ is connected. By [133, Proposition 23.2], $K_0 \to J$ is onto. Cuspidals inflate from J. On $\mathbf{J}$, Lang conjugates F_0 to $\gamma_0 F_{q_0}$ for a pinned diagram operation γ_0, and also conjugates $F = F_0^n$. Put $b = 1$ in split type and $b = 2$ in twisted type. In twisted type n is odd. Thus, $B = F_0^b$ generates the same group as u on J, with $B^n = F^b$.

For $\rho = \tau, \bar{\tau}$, choose a torus character R_{w_1} with coefficient ± 1 at ρ, by [61, proofs of Props. 2.4, 2.7]. In split type take w_1 Coxeter and diagram-fixed. Otherwise, take $B = F_0^2$, trivial on the Weyl group. Thus, B acts on $X(w_1)$ in both cases. By [208, Lemma 4.11 and the paragraph following its proof], the F^b-eigenvalues at ρ have form $\mu = \omega_\rho q^j$, $j \in \mathbb{Z}$. Split E_6 has integral exponents. In twisted type $b = 2$. By [61, §2.1 and proofs of Props. 2.4, 2.7], $\omega_\rho \in \mu_3 \setminus \{1\}$. Choose $\xi^n = \omega_\rho$ in μ_3 if $n = f_2$, or in μ_{3f_3} if $n = f_3$. Since $3f_3 \mid \ell - 1$, each μ has the root $\xi q_0^j \in \mathbb{Q}_\ell$. Corollary 2.18 gives the invariant extension of ρ over $\langle u^b\rangle$. Inflate to K_0, since $\langle u^2\rangle = \langle u\rangle$ when twisted. Multiply the difference by $\widehat{\kappa}_z$. Moreover, this gives the $\mathcal{H}_\ell$-fixed virtual extension $\widehat{\lambda}$ on $K_0 \rtimes \langle u\rangle$.

Choose a parabolic of $\mathbf{G}$ with Levi $\mathbf{K}_0$ and radical $\mathbf{U}$. Take $\mathbf{G}^n$ with rotation τ_0 and $\mathcal{F} = \tau_0^{-1}\,\mathrm{diag}(F_0)$, by [208, proofs of Proposition 4.16, Step 4, and Lemma 4.19], with fixed points G, K_0 under projection. Let Θ be the alternating character on $(G \times K_0) \rtimes \langle u\rangle$ of the variety for $\mathbf{U}^n$, with inverse right translation and u-rotation. By [53, Prop. 3.3], Θ is integer-valued. Tensor it with $\widehat{\lambda}$ and take diagonal K_0-invariants. The averaging projector gives $\Xi(gu^i) = \frac{1}{|K_0|}\sum_{k\in K_0}\Theta((g,k)u^i)\widehat{\lambda}(ku^i)$. Thus, $\mathcal{H}_\ell$ fixes Ξ. By [72, Theorem 3.3.8], we may take radical $\prod_{j=0}^{n-1} F_0^j(\mathbf{U})$ for $\Xi|_G$. Indeed, $\mathcal{F}$ cycles $n > 1$ factors of $\mathbf{G}^n$, so [16, Theorem 3.8(4)] covers even $q_0 = 2$. By [208, proof of Lemma 4.19(b)], projection gives induction from K_0. Hence, $\Xi|_G = D$. The Mackey norm on G is covered by [16, Thm. 3.8(3)], since $q = q_0^n > 2$. Lemma 2.17 with coefficients m_i gives the extensions on the 2- and 3-parts. On $E_{6'}$, [208, Lemma 3.2] gives unique extensions over $\mathbb{Q}(\zeta_3)$, since $|E_{6'}|$ is prime to 6. By [95, Corollary 6.17], these determine the extensions over E. Thus, $\mathcal{H}_\ell$ fixes them. Pullbacks give $\succeq_c$ by Corollary 2.5, with common centralizer scalars 1. □

For either scalar mark $\delta = \pm 1$ and $\ell > 7$ dividing $q - \delta$, the torus $\mathbf{T}$ of order polynomial $(q-\delta)^8$ contains $P \in \mathrm{Syl}_\ell(G)$ in T. By Malle [132, Proposition 5.13], $P[\ell]$ determines $\mathbf{T}$, so $\mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{T})$. By Enguehard–Michel [63, Lemma 2.20(1)], W acts faithfully on $P[\ell]$. Thus, $\mathrm{C}_G(P) = T$ and $\mathrm{N}_G(P) = \mathrm{N}_G(T)$. For Levi centralizers choose twist κ_s as in Section 7.1. Moreover, this includes $\gcd(|s|, 30) = 1$. Choose common Weyl generators for $\delta = \pm 1$. Next, let T_δ be the torus marked by 1 or $w_0 = -1$, with rational algebraic normalizer N_δ. The positive simple lifts n_i and the points $h_i = \alpha_i^\vee(-1)$ are rational for both δ. Also, the longest braid gives $n_0 n_i = n_i n_0$ for the positive lift n_0 of w_0. Together with T_δ the n_i generate N_δ. Adjointness makes the coroot half-sum $\rho^\vee$ a cocharacter. If $\theta \in \mathrm{Irr}(T_\delta)$ let e_θ be its weight idempotent in $K[T_\delta]$, where $K = \mathbb{Q}^{\mathrm{ab}}$, and put

$$c_\theta = \sqrt{\theta(\rho^\vee(-1))},\ d_i(\theta) = \sqrt{\theta(h_i)},\ b_i(\theta) = d_i(\theta)c_\theta/c_{s_i\theta},\ \sqrt{1} = 1,\ \sqrt{-1} = i.$$

The identity $s_i\rho^\vee = \rho^\vee - \alpha_i^\vee$ gives $b_i(\theta) \in \{1, -1\}$ and $b_i(s_i\theta)b_i(\theta) = \theta(h_i)$. Define $B_i e_\theta = b_i(\theta) n_i e_\theta$. Then $B_i^2 = 1$. In a reduced expression for w the c-factors telescope, while the d_i-product runs over coroots inverted by w. Thus, the B_i satisfy braid and act on T_δ by s_i, giving the isomorphism $K[T_\delta \rtimes W] \xrightarrow{\sim} K[N_\delta]$, $t \longmapsto t$, $s_i \longmapsto B_i$. Indeed, the images of we_θ are nonzero multiples of $n_w e_\theta$, a basis. In particular, if m_θ counts the positive coroots $\beta^\vee$ with

$\theta(\beta^\vee(-1)) = -1$, then $B_{w_0} e_\theta = \epsilon_\theta n_0 e_\theta$, $\epsilon_\theta = i^{m_\theta} \in \{1, -1\}$. $c_{\theta^{-1}} = c_\theta$, and m_θ is even by the coroot sum $2\rho^\vee$. Since opposite coroots agree at -1, m_θ is invariant on W-orbits of $\theta|_{T_\delta[2]}$.

Proposition 7.271. *For every f and every odd $\ell \mid q-1$, the monodromic Hecke algebra on G gives a principal-series correspondence to $N = \mathrm{N}_G(T)$, with mixed twist equivariance. Moreover, on ℓ'-characters it is bijective, and N is admissible.*

Proof. Take pinned $B = TU$ and put $e = |U|^{-1} \sum_{u \in U} u$ and $v = \sqrt{q} > 0$. The map $x \mapsto xe$ gives $K[T] \cong K[T]e$ in $eK[G]e$. Given s_i, the operator $a_i = q e n_i e$ satisfies $a_i^2 = q h_i e + a_i \sum_{x \in \mathbb{F}_q^\times} \alpha_i^\vee(x) e$ by [123, §35.3]. Indeed, $[U : U \cap {}^{n_i}U] = q$ gives $a_i = \mathbf{1}_{U n_i U}$. Set $A_i e_\theta = b_i(\theta) v^{-1} a_i e_\theta$. If $\theta \circ \alpha_i^\vee = 1$, $s_i\theta = \theta$ and $b_i(\theta) = 1$. Otherwise, the coroot sum acts by 0. Thus, $A_i^2 e_\theta = e_\theta + \mathbf{1}_{\theta \circ \alpha_i^\vee = 1}(v - v^{-1}) A_i e_\theta$. Both coroot orders give braid for A_i by length additivity. Moreover, specializations are $eK[G]e$ at $v = \sqrt{q}$ and $K[N]$ at $v = 1$. The description of $eK[G]e$ is by U-double cosets [123, §35.4]. At $v = 1$ the corresponding elements are $B_w e_\theta$.

Given θ, put $R_\theta = \{\beta : \theta \circ \beta^\vee = 1\}$ and $W_\theta = W(R_\theta)$. This is the full stabilizer in W by dual connectedness. Next, choose positives R_θ^+. The reflection in a simple root β of R_θ inverts one positive coroot in $R_\theta^\vee$. That step gives label v by [124, §1.4(f)–(h)]. The corner at e_θ is $H_u(W_\theta)$ for $u = 1, v$. A based transporter $z : \theta \to \theta'$ maps R_θ^+ to $R_{\theta'}^+$. Also, its normalized operator acts by $w \mapsto zwz^{-1}$ on neutral generators. Transporters multiply as $z'z$ by [124, §§1.5–1.6].

Let J_θ be the asymptotic algebra of W_θ. If $u \in \{1, v\}$, the canonical map $\phi_{\theta,u} : H_u(W_\theta) \to K \otimes_{\mathbb{Z}} J_\theta$ has nilpotent kernel I_u by [125, Thm. 18.9, Prop. 18.12(a) and §§15.1–15.7]. Semisimplicity of $H_u(W_\theta)$ gives kernel 0. Both dimensions are $|W_\theta|$. Moreover, $\phi_{\theta,u}$ is equivariant under based isomorphisms. On $H_v(W_\theta)$, take $\phi_{\theta,1}^{-1} \phi_{\theta,v}$. Match each transporter by its index z. Moreover, the composition law gives $\Psi : eK[G]e \to K[N]$ fixing every e_θ.

Pinned fields permute θ and fix n_i. Since $q \equiv 1 \bmod \ell$ with ℓ odd, the two roots of $X^2 - q$ lie in $\mathbb{Q}_\ell$. Hence, $\mathcal{H}_\ell$ fixes v. Both actions transport the signs $\theta(h_i)$ and $\theta(\rho^\vee(-1))$. Hence, both actions transport $b_i(\theta), R_\theta^+$. The specialization coefficients lie in $\mathbb{Q}(u)$. Thus, Ψ is equivariant on transporters. Next, for D in the pinned field group, extend Ψ by $x[d] \mapsto \Psi(x)[d]$ on each d-component. This gives $eK[G \rtimes D]e \cong K[N \rtimes D]$. Now, for a local label ψ, let D be field inertia. Extend cyclically to $N \rtimes D$. Transfer to the corner and apply $K[G \rtimes D]e \otimes_{eK[G \rtimes D]e} -$. Because e is fixed by D, move field factors right. On G, take $K[G]e \otimes_{eK[G]e} -$, affording the principal-series partner χ. A twist $\lambda \in \mathrm{Irr}(D)$ multiplies the d-component by $\lambda(d)$ on both sides. Hence, on the joint stabilizer of (χ, ψ), the twists agree.

Let V_χ afford χ, take a weight θ in V_χ^U, and put $m = \dim e_\theta V_\chi^U$. The weights form one W-orbit with isomorphic spaces. Since Ψ fixes e_θ, the local degree is $\psi(1) = [W : W_\theta] m$. By reciprocity, m is the multiplicity in $\mathrm{Ind}_{TU}^G \theta$. Tits deformation gives the relative degree m by [166, §2.3]. Thus, $\psi(1)$ is given by [166, Formula (2)]. By [166, Theorem 6.2], the ℓ'-bijection is on admissible N, including Weyl primes. If $P \in \mathrm{Syl}_\ell(N)$, its field images are N-conjugate. The Sylow corrections act by N on both extensions. Pullback gives $\succeq_c$. □

Proposition 7.272. *Assume $\ell \mid q+1$ is odd and $M = \mathrm{N}_G(\mathbf{T})$ for the torus marked by $w_0 = -1$. The equivariant ℓ'-bijection is indexed by W-orbits of (s, η), $s \in \mathbf{T}^{*F^*}$, $\ell \nmid [W : W_s]$, $\eta \in \mathrm{Irr}_{\ell'}(W_s)$. Moreover, this includes Weyl primes and $\ell \mid |s|$.*

Proof. The scalar mark centralizes W and each W_s. The minimal Levi is $\mathbf{T}$ with source 1_T. The Sylow Φ_2-torus order formula gives Syl_ℓ-subgroups in both normalizers. The factor $(q+1)^8$ reduces the index condition to $\ell \nmid [W : W_s]$. Apply [156, §4.4, proof of Theorem B] for relative degrees. By [132, Thm. 5.14], M is admissible and contains $\mathrm{N}_G(P)$. Take K from the primal subsystem dual to $C_{\mathbf{G}^*}(s)$. Connectedness gives $\mathrm{N}_K(T) = M_{\theta_s}$. Moreover, the central character κ_s restricts to θ_s, and the local label is $\mathrm{Ind}_{\mathrm{N}_K(T)}^M(\kappa_s \eta)$. Central duality makes κ_s joint-equivariant. Globally, take unipotent d-series of $C_{\mathbf{G}^*}(s)$ by [156, Prop. 4.6 and Cor. 4.7]. □

Proposition 7.273. *Each local ψ in* Proposition 7.272 *extends to marked inertia $\mathcal{A}_\psi$ with joint invariance.*

Proof. First, write $q = r^f$, $T = \mathbf{T}^{F'}$ and $\mathcal{A} = \mathrm{N}_{G \rtimes \langle F_r \rangle}(\mathbf{T})$. In $F' = \mathrm{Int}(n_0) F_q$, the positive lift n_0 of $w_0 = -1$ satisfies $n_0\, n_i = n_i\, n_0$ by the longest braid. Its square $(2\rho^\vee)(-1)$ is central and so 1 in E_8. Thus, the Tits group V is F'-fixed and $M = TV$ maps onto W with kernel T. The marked Φ fixes V, acts by the r-power on T, and has $\Phi^f = n_0^{-1}$ in $\mathcal{A}$ by Proposition 7.266. Next, put $D = \langle V, \Phi \rangle$, we get $\mathcal{A} = TD$ and $D \cap T = V \cap T$, a 2-group. Given a local label ψ over s, we put $I = M_{\theta_s}$, $Y = \mathcal{A}_\psi$ and $h = |Y/M|$. Moreover, the group $W_s = I/T$ has rational characters. Complex conjugation gives $\kappa_s \mapsto \kappa_{s^{-1}}$. The lift n_0 returns the latter and centralizes W_s. By Proposition 7.272, $\bar{\psi} = \psi$. At odd h, apply [143, Lemma 2.1] on Y for the invariant real extension.

If $2 \mid h$, $2 \mid f$. The torus exponent is $q+1$, odd for $r=2$. If $2 \nmid r$, $q \equiv 1 \bmod 8$, so $(q+1)_2 = 2$. Thus, $s = ty$ with $t^2 = 1$ and $|y|$ odd. Because $\mathrm{ord}_\ell(r^f) = 2$, the order formula gives $2f_2 \mid \ell - 1$. Thus, $\mathcal{H}_\ell$ fixes μ_{2f_2}. On $\mathcal{A}_{\theta_y} = TD_{\theta_y}$, put $\lambda_y(xd) = \theta_y(x)$, with $x \in T$, $d \in D_{\theta_y}$. This is well defined since $\theta_y|_{D \cap T} = 1$, and multiplicative by D_{θ_y}-invariance. The formula is equivariant for σ and D-transport, while T acts by inner conjugation. Put $I_t = M_{\theta_t}$. Both Φ, n_0 fix θ_t, so $\mathcal{A}_{\theta_t} = I_t\langle\Phi\rangle$. By Proposition 7.272, we obtain the Φ-fixed character $\kappa_t|_{I_t}$, of order at most 2. Choose $b \in \mu_{2f_2}$ with $b^f = \kappa_t(n_0^{-1})$. Then $\lambda_t(m\Phi^j) = \kappa_t(m)b^j$ extends θ_t by the field relation. At θ_s, the geometric part fixes θ_t, since coefficients fix signs. It fixes λ_t by conjugation in $\mathcal{A}_{\theta_t}$, and coefficients fix its values. Thus, $\lambda_s = (\lambda_t\lambda_y)|_{\mathcal{A}_{\theta_s}}$ is joint-invariant. This gives unique $\eta \in \mathrm{Irr}(W_s)$ with $\psi = \mathrm{Ind}_I^M(\lambda_s|_I \ \mathrm{Inf}_{W_s}^I \eta)$. Next, put $C = \mathrm{N}_W(\langle\theta_s\rangle)$. Chamber correction gives $C = W_s \rtimes C^+$, with C^+ permuting the simple roots of W_s. On $\langle\theta_s\rangle$, the kernel is W_s, so C^+ is abelian. Moreover, the representation of W_s with character η has a rational model extending to $C_\eta = W_s \rtimes (C^+)_\eta$. On each factor R_i, diagram action is abelian. Given A, E_6, take the longest element. Also, in D_n, the extension group is $W(B_n)$, split over $\mathbb{Q}$. Diagram transpositions in D_4 are conjugate. For C_3 on D_4, [143, Corollary 2.2] gives a rational extension. Its multiplicity 1 in induction gives a model of $W(D_4) \rtimes C_3$. Moreover, a trivial diagram acts by 1. Transport each model along its $(C^+)_\eta$-orbit by tensor permutation. Choose model $\widetilde{\rho}$ of C_η.

Next, put $L = (\mathcal{A}_{\theta_s})_\eta$. The field Φ acts trivially on W and powers the weights, so $m\Phi^j \in L$ implies $mT \in C_\eta$. Moreover, the element $w_0^{-1} \in Z(C_\eta)$ acts in $\widetilde{\rho}$ by $c \in \{1, -1\}$. Next, we choose $a \in \mu_{2f_2}$ with $a^f = c$. The map $m\Phi^j \mapsto a^j\widetilde{\rho}(mT)$ is a representation of L: the Weyl coordinates multiply, and $\Phi^{j+f} = n_0^{-1}\Phi^j$ gives the scalar c. Now, tensor with $\lambda_s|_L$ and induce to Y. Correct ordinary operations stabilizing ψ by M to fix θ_s, after which they fix η. Thus, $Y = ML$. Since $L \cap M = I$, the restriction to M is ψ. For a joint operation stabilizing ψ, the same M-correction returns θ_s. The corrected Weyl coordinate lies in C and fixes η, since λ_s is invariant and η rational. Thus, conjugation is inner on C_η. Field exponents stay fixed before $\Phi^f = n_0^{-1}$, while coefficients fix a and η. Thus, induction on Y is invariant. □

Proposition 7.274. *On the negative mark, χ extends for type A, isolated involutions and real χ at odd inertia D. Proper Levi series give $\succeq_c$. Moreover, this covers odd f.*

Proof. Write $q = r^f$, $E = \langle F_r\rangle$, $C = C_{\mathbf{G}^*}(s)$, and let J_s be Jordan decomposition on $\mathcal{E}(G, s)$. Next, we fix an eligible character χ and write $D = E_\chi$. The local extension at s follows from Proposition 7.273. Suppose C_{der} has type A. The mark -1 acts by opposition on each C_i. By the proof of [208, Proposition 12.19], type A unipotents ρ_i are rational and uniform. Thus, ρ is rational and uniform on C^{F^*} by central isotypy. Conjugation inverts s. Next, a rational -1-lift returns it. Opposition on $\mathfrak{S}_m$ is inner. Thus, J_s fixes partitions and gives $\bar{\chi} = \chi$ [180, Thm. 5.1]. The linear J_s matches $R_T^G(\theta_s)$ with $R_{T^*}^{C^{F^*}}(1)$ up to sign [180, Thm. 4.1(1)]. Uniformity of $J_s(\chi)$ gives uniformity of χ, also for toral C.

If $2 \nmid |D|$, then [143, Lemma 2.1] gives an extension of χ to $G \rtimes D$. Uniqueness gives joint invariance. At even $|D|$ with real uniform χ, set $j = F_r^{f/2} \in D$. Then, $\chi^j = \bar{\chi}$, and [105, Thm. 3.1.3(ii)] gives $\langle\chi, \mathrm{Ind}_{G^j}^G 1\rangle = 1$. Since D centralizes j, apply [208, proof of Lemma 3.13] to the trivial character on $G^j \rtimes D$. The unique constituent above χ is fixed by the joint stabilizer on $G \rtimes D$. At involution $s = t$, [120, $E_8(q)$] gives the two split types D_8, E_7A_1. Since both centralizers are split over $\mathbb{F}_r$, take t in split 2-torsion. Locally put $w_0 = -1$ on $\mathbf{T}$. The Weyl indices are 135 and 120, respectively. Thus, Proposition 7.272 excludes $\ell = 3, 5$. The Jordan multiplier is $[G^* : C^{F^*}]_{r'}$, and $2f_2 \mid \ell - 1$ since $\mathrm{ord}_\ell(r^f) = 2$. In type D_8, we apply Proposition 7.260 on $\lambda = \varepsilon_t\rho$. The global root lies in $\mu_{2f_2} \subset \mathbb{Q}_\ell$, giving twists 1 by [156, Proposition 7.3] and [208, Lemma 4.9(i),(ii)], including multiplicities > 1.

In type E_7A_1, [208, proof of Lemma 12.5] gives a parabolic $P = LU$ and a quadratic character ε_t of L. For the unipotent source ρ_L, that proof gives $\mathrm{Ind}_P^G(\varepsilon_t\rho_L) = \chi_{\rho,0} + \chi_{\rho,1}$ with $\chi_{\rho,1}(1) = q\chi_{\rho,0}(1)$. The Levi L maps onto the split adjoint group $J = (\mathbf{L}/\mathrm{Z}(\mathbf{L}))^F$ of type E_7. The Levi roots extend to an E_8-basis, so $\mathrm{Z}(\mathbf{L})$ is connected. By [133, Proposition 23.2], ρ_L is inflated from $\rho \in \mathrm{Irr}(J)$. The common degree factor is $[G^* : C^{F^*}]_{r'}\rho(1)$. Thus, $\ell \nmid \rho(1)$.

On J, F_r acts by Frobenius, and $\mathcal{H}_\ell$ fixes ρ by [156, Corollary 4.7]. At $d = 2$, the eigenvalues lie in $\{\pm 1, \pm\sqrt{-q}\}$. By [156, Lemma 8.1(a)–(b)], they have f-th roots in $\mathbb{Q}_\ell$. Then, [156, Theorem 7.5] gives an $\mathcal{H}_\ell$-fixed extension to $J \rtimes E$. Inflate to $L \rtimes E$, tensor with the extension of ε_t trivial on E, and inflate to $P \rtimes E$. By [208, proof of Lemma 12.5], this parabolic extension gives $\widehat{\chi}_{\rho,i}$ on $G \rtimes E$ fixed by $\mathcal{H}_\ell$. On M, apply Proposition 7.273 at $|t| = 2$. Next, at other proper centralizers for $d = 2$, only E_7 may have nonreal unipotents [156, proof of Prop. 4.6 and Table 4.1]. On W, opposition fixes labels by [156, Prop. 4.6 and Cor. 4.7]. A rational -1 returns s^{-1} and fixes each factor. Thus, χ is

real. On odd D, apply [143, Lemma 2.1]. A remaining E_7 fills rank 7. By the simply laced list, E_7 is a Levi with twist κ_s at weight θ_s with inertia $\mathrm{N}_K(T)$. If $\ell > 7$ apply Proposition 7.279. At $\ell = 3, 5, 7$, we apply Proposition 7.280 by [156, §4.4]. If $s = 1$, apply [156, Thm. D]. □

Proposition 7.275. *Outside* Proposition 7.274*, r is odd, field inertia order h is even, and $\ell > 7$. Next, write $s = ty$, with $|t| = 2$ and $1 \neq y$ of $6'$-order. Projected sign ranks are $2, 4$ in a D_6-Levi with centre dimension 2. If $k = f/h$, $h \equiv 2 \bmod 4$ and $|y| \mid r^{2k} + 1$. The generator correction acts with order 4 centrally, with diagram involution on D_6 modulo $W(D_6)$.*

Proof. At $\ell = 5$, the product of Coxeter rotations in A_4^2 is elliptic of order 5. Since $5 \nmid [W : W_s]$, some conjugate w lies in W_s. Hence, the rank is full, and $1 + w + \cdots + w^4 = 0$ gives $s^5 = 1$. Thus, $s = 1$ or the isolated A_4^2, by Proposition 7.274. At even inertia, f is even. Thus, $3 \nmid q + 1$ and $(q + 1)_2 \leq 2$. Also $7 \nmid q + 1$, since -1 is a nonsquare modulo 7. Among odd Weyl primes only 5 can occur, already treated. Write $s = ty$, with $t^2 = 1$ and y of odd order. If $y \neq 1$ has full rank, its type is A_4^2 by [208, Table 12.2]. Otherwise, take its closure $\mathbf{K}^*$. Each adjoint projection of y has full rank and order dividing $2, 3$ or 4. Since $\gcd(|y|, 6) = 1$, these projections are trivial. Thus, $K^* = \mathrm{C}(y)$. At isolated A_4^2, the centralizer of t has type A factors by coordinate differences. The cases $s = 1$, pure involutions, Levis and type A products are covered. Also $t = 1$ in characteristic 2. The remaining case has odd characteristic, $t, y \neq 1$ and K^* proper Levi, with $C(s)$ non-Levi and a non-A component.

Let h be the even inertia order and $k = f/h$. Moreover, a Weyl correction of y^{r^k} normalizes K^*. Its simple roots extend to an E_8-basis, giving connected centre. In dimension 1, $y^{r^k} = y^{\pm 1}$. Even h would imply $y^q = y$, whereas the mark gives $y^q = y^{-1}$. Since $|y| > 1$ is odd, rank is ≤ 6. In type A, both eigenspaces give a Levi. In D_n, its special orthogonal lift squares to $-I$ or I. Moreover, at $-I$, the centralizer is general linear. At I, it has type $D_a D_b$ with $a + b = n$. A non-Levi requires $a, b \geq 2$, and a non-A block requires $\max(a, b) \geq 4$. Since $n \leq 6$, this gives $n = 6$ and $\{a, b\} = \{2, 4\}$. Two non-A ambient factors require rank at least 8. Moreover, in type E_6, a proper closure has classical rank ≤ 5. At rank 6, an involution t has a unique lift of order 2 through C_3. By [208, proof of Proposition 10.7, Step 1], its isolated centralizer has type $A_5 A_1$. Nested Levi cocharacters combine. Only D_6 remains.

Next, take $D_6 = \{\pm e_i \pm e_j : 1 \leq i < j \leq 6\}$ in E_8. For generic $v \in \langle e_7, e_8 \rangle$, the roots perpendicular to v form D_6. The arm lengths $1, 2, 4$ give one standard D_6, of arms $1, 1, 3$. Moreover, the plane has roots $\pm e_7 \pm e_8$, and its pointwise stabilizer is $W(D_6)$. The normalizer acts on $\langle e_7, e_8 \rangle$ by signed permutations. Each extends to $W(D_8)$ by even sign parity. Next, choose w with $w(y) = y^{r^k}$. Its action on y factors through the plane group, since $y \in Z(\mathbf{K}^*)^\circ$. If the image of w had order at most 2, then $y^{r^{2k}} = y$ and $y^q = y$, a contradiction. The order is 4 and square $-I$, so $y^{r^{2k}} = y^{-1}$. With $w^h(y) = y^q = y^{-1}$, this gives $h \equiv 2 \bmod 4$. An order-4 plane permutation changes one sign. In $W(D_8)$, its lift changes odd signs among $1, \ldots, 6$, and induces the diagram involution of D_6. Moreover, lifts differ by $W(D_6)$. □

Let $\mathbf{T}$ have the negative scalar mark, let $\ell > 7$ divide $q + 1$, and let P be its rational ℓ-part. Next, we write $G = E_8(q)$, $M = \mathrm{N}_G(P)$. In Proposition 7.275, fix the odd part $y \neq 1$ and take $\mathbf{K}$ dual to $\mathbf{K}^* = \mathrm{C}_{\mathbf{G}^*}(y)$. Next, we write $K = \mathbf{K}^F$, $L = \mathrm{N}_K(P)$, and $T = \mathbf{T}^F$. On the dual torus let $\mathcal{U}_y$ be the union of the rational K^*-classes of the lifts $t'y$ of the projected involution. The lifts t' include the allowed central translates of 2-power order. Moreover, elements stabilizing $\mathcal{U}_y$ jointly may move t'.

Lemma 7.276. *Each joint stabilizer element of $\mathcal{U}_y$ admits correction to y. Moreover, the correcting Weyl coset modulo $W_y = W(\mathbf{K}^*, \mathbf{T}^*)$ is unique. With ordinary part fixing P, corrections differ by K globally and L locally. At fixed y, signed induction from K is injective. Moreover, induction from L is irreducible and injective at fixed y on weight orbits.*

Proof. With mark -1, [133, Proposition 25.3(a)] gives $\mathrm{N}_G(\mathbf{T})/T = W$. The mark gives $P[\ell] = T[\ell]$. In unimodular E_8, root primitivity gives nontrivial restrictions. Thus, $C_G(P) = T$ and $M = \mathrm{N}_G(\mathbf{T})$. The same calculation on $\mathbf{K}$ gives $L = \mathrm{N}_K(\mathbf{T})$. Both local groups have kernel T and Weyl images W, W_y. By connectedness of $C(y)$, $W_y = \mathrm{Stab}_W(y)$. A coefficient operation powers T^* by a unit. Field transport moves T^* within its rational class. Thus, $s \mapsto y$ is equivariant. For (a, σ) in the joint stabilizer, align in T^* and restore P in its centralizer via W. Moreover, corrections of y differ by W_y on the target. Lifts differ by L. In G^*, stabilizers of y lie in K^*. Moreover, a series stabilizer admits K^*-correction to fix $t'y$. The corrected element lies in $C_{\mathbf{G}^*}(t'y) \leq \mathbf{K}^*$, since y is a power of $t'y$. The coset is trivial.

At fixed y, conjugacy in G^* equals conjugacy in K^*. Moreover, the inclusion $C_{\mathbf{G}^*}(t'y) \leq \mathbf{K}^*$ gives signed irreducibles [72, Theorem 3.3.22]. Distinct K-series have disjoint images. Given $\theta_{t'y}$, inertia in M has Weyl image $W_{t'y} \leq W_y$, so lies in L. Clifford transitivity [43] gives irreducible induction from L to M. Equal inductions have M-conjugate weights. At fixed y, their weight conjugator lies in L. The Clifford correspondent then recovers the L-label. The local index $[M : L] = [W : W_y]$ is prime to ℓ, and $[G : K]$ is prime to ℓ since $P \leq K$. □

Proposition 7.277. *Suppose the bijection on K, L matches the indicated unions over $\mathcal{U}_y$ and gives $\succeq_c$ on the induced inertias. Then K and ordinary induction from L give $\succeq_c$ over the transported union, at $M = \mathrm{N}_G(P)$.*

Proof. Take the twisted source pair β, ν on K, L. Next, put $\chi = \epsilon_{\mathbf{G}}\epsilon_{\mathbf{K}} R_{\mathbf{K}}^{\mathbf{G}}(\beta)$ and $\psi = \mathrm{Ind}_L^M \nu$. In the inner-field group, let A be the inertia of χ and $B \leq A$ the Levi normalizer fixing β. By Lemma 7.276, $A = GB$: correct the action at y, then apply injectivity at fixed y. The relative-coset calculation gives $G \cap B = K$, since K acts by inner conjugation on β. Put $B_P = \mathrm{N}_B(P)$. By Sylow conjugacy in K, $B = KB_P$. Hence, $\mathrm{N}_A(P) = MB_P$, since gb with $b \in B_P$ normalizes P exactly when $g \in M$. Thus, $B_P/L = B/K = A/G$. Include the coefficient stabilizer. Moreover, at fixed y, Lemma 7.276 recovers ν from ψ. Equivariance matches inertias. In Proposition 7.2 put $\widetilde{\mathbf{G}} = \mathbf{G}$. By Corollary 7.3, y determines the Levi canonically, with $C_{\mathbf{G}^*}(t'y) \leq \mathbf{K}^*$ throughout. Lang conjugates $\mathbf{T}, \mathbf{K}$ to the split model. In $F' = \mathrm{Int}(n_0)F_q$ the marked field power is $\mathrm{Int}(n_0^{-1})$ at exponent f. Moreover, its conjugate is $x\phi$. In the field product, $(\mathrm{Int}(\widehat{x})\tau)^f$ conjugates by the fixed tuple for $x\phi(x)\cdots\phi^{f-1}(x)$. Also, the lift at y is the same.

Put $J = B/K$. The proposition gives invariant X on $A \times_J B$. If the lower model P_β on V_β has factor c on J, $V_\chi = (X \otimes V_\beta)^K$ has factor c by Corollary 2.5. Its corrections match P_β. Moreover, the geometric part may move y, restored by coefficients. Induce P_ν from B_P to $MB_P = \mathrm{N}_A(P)$ in the c-twisted algebra. Lemma 7.276 gives irreducible ψ on M. Conjugation and coefficient transport act on P_ν, c. For a scalar function c_1 on J, we have $\mathrm{Ind}(c_1 P_\nu) = c_1 \mathrm{Ind}(P_\nu)$ because M contains coset representatives. Thus, P_ν, P_β have the same functions on J. Moreover, the diagonal $C_B(K)$ acts trivially on the variety. In P_ν, its scalar is constant on induced M-blocks. Thus, pullback matches scalars. □

Corollary 7.278. *Let $\mathbf{T}$ be an E_8-mark with $C_G(P) = T$ and $M = \mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{T})$, and $P \leq T$. Assume the dual identities. Let $s \in \mathbf{T}^{*F^*}$ have full centralizer a proper Levi $\mathbf{K}^*$. Assume unipotent $\succeq_c$ on K for induced inertias, with local ℓ'-labels over 1_T on $\mathrm{N}_K(\mathbf{T})$. Then the central twist gives $\succeq_c$ on $\mathcal{E}(G, s)$ and its weight orbit in M. If $\mathbf{K} = \mathbf{T}$, both sources are θ_s.*

Proof. Put w for the torus mark and $W_{\mathrm{rat}} = C_W(w)$. The rational group on $\mathbf{T}$ in K is $W_{\mathrm{rat}} \cap W(\mathbf{K}^*, \mathbf{T}^*)$ by [133, Proposition 25.3(a)] and rational duality. Since $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$, this is $(W_{\mathrm{rat}})_s$. Thus, $N_K(\mathbf{T})$ is the inertia of θ_s. It equals $\mathrm{N}_K(P)$ by the hypotheses on G. The central linear character κ_s of K restricts to θ_s. Lemma 7.7 gives the twisted comparison over θ_s, before induction to M. The inverse twist on inertia is $\kappa_s^{-1}|_{\mathrm{N}_K(\mathbf{T})}$.

Next, apply Lemma 7.276 and Proposition 7.277 with s. Sylow correction gives one $(W_{\mathrm{rat}})_s$-coset in W_{rat}. Series stabilizers have trivial coset since $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$. These steps at s are those at y. Form the field product in pinned $\mathbf{G}$. Also, its position stabilizer acts in K, and marked transport fixes its f-th power. For joint operations corrected at s, both parabolics are available. Apply Lemma 7.168 to s. Contraction and twisted induction give $\succeq_c$. At $K = T$, the unipotent source is 1_T. Local inertia is exactly T. Moreover, this gives $\epsilon_{\mathbf{G}}\epsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta_s)$ on G and $\mathrm{Ind}_T^M \theta_s$ on M. □

Proposition 7.279. *In* Corollary 7.278*, assume $\ell > 5$, $\ell \nmid |W_{\mathrm{rat}}|$, and mark eigenvalues of order $d = \mathrm{ord}_\ell(q)$. For a proper Levi $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$, the whole eligible series gives $\succeq_c$, including permutations of rational K-factors.*

Proof. First, we set $\pi : \mathbf{K} \to \mathbf{J} = \mathbf{K}/Z(\mathbf{K})$ and $\overline{\mathbf{T}} = \pi(\mathbf{T})$. The simple roots of $\mathbf{K}$ extend to an E_8-basis. Thus, $Z(\mathbf{K})$ is connected. By [133, Proposition 23.2], π gives surjections onto $J = \mathbf{J}^F$ and $\overline{\mathbf{T}}^F$. Since $\pi^{-1}(\overline{\mathbf{T}}) = \mathbf{T}$, it also gives $L = \mathrm{N}_K(\mathbf{T}) \to \mathrm{N}_J(\overline{\mathbf{T}})$ onto. Central isotypy gives inflation from J. Labels over 1_T factor through L/T. Thus, we obtain the quotient $L/T \simeq \mathrm{N}_J(\overline{\mathbf{T}})/\overline{\mathbf{T}}^F$. Write $J = \prod_i J_i$ over simple-factor F-orbits. Moreover, at length m_i, projection onto $\mathbf{J}_0$ gives $J_i \simeq \mathbf{J}_0^{F^{m_i}}$, over q^{m_i}. On the projected torus $\overline{\mathbf{T}}_i$, the mark is w^{m_i}. Its eigenvalue order is $d_i = d/\gcd(d, m_i) = \mathrm{ord}_\ell(q^{m_i})$. Thus, $\overline{\mathbf{T}}_i$ is a self-centralizing Sylow d_i-torus, with minimal Levi source 1. Next, we apply [156, §8, proof of Theorem D] on simply laced J_i with field and diagram operations. Write ξ_i, η_i for paired eligible labels on $J_i, L_i = \mathrm{N}_{J_i}(\overline{\mathbf{T}}_i)$.

Identical models follow from Sylow d_i-torus conjugacy. Now, put $A_i = J_i E_i$, where E_i acts by field and diagram operations. $A_i' = \mathrm{N}_{A_i}(\overline{\mathbf{T}}_i)$. The extensions from [156, proof of Theorem D] give tensor U_{ξ_i} on $(A_i)_{\xi_i} \times_{A_i/J_i} (A_i')_{\eta_i}$,

extending $\xi_i \boxtimes \eta_i^\vee$. On the stabilizer the twists cancel. Choose U_{ξ_i} on one label per orbit under $A_i' \times \mathcal{H}_\ell$, then transport equivariantly under $A_i' \times \mathcal{H}_\ell$. Inner changes conjugate. Inflate products to τ, η. Next, for the ordinary inertia B of $\kappa_s \tau$ in Corollary 7.278, set $B_P = \mathrm{N}_B(P)$ and $\Delta = B \times_{B/K} B_P$. Each element of B fixes κ_s by duality and the unipotent label τ. Thus, $B = KB_P$ and $K \cap B_P = L$. Both entries of Δ permute factors alike. Moreover, on Δ_i, the actions on J_i differ by conjugation from K. The second normalizes $\overline{\mathbf{T}}_i$. Next, compose U_{ξ_i} with the projection from Δ_i.

Tensor-induce each orbit to Δ. Next, we choose h_j carrying position i to j. Moreover, an element $b \in \Delta$ acts on factors by $j \mapsto bj$ with $h_{bj}^{-1} b h_j$. Now, tensor over all orbits. On $K \times L$ all positions are fixed, giving $\tau \boxtimes \eta^\vee$. Moreover, changing h_j changes bases. Elements corrected to stabilize P permute positions. Also, the chosen U_{ξ_i} transports to U_{ξ_j} at the image position j. Rational permutations stabilize the tensor. Diagonal $C_B(K)$ fixes positions and centralizes each J_i. Thus, the kernel contains $C_B(K)$. The quotient B/K is cyclic. Next, we extend η to B_P and contract over L to obtain an extension of τ to B. Tensor invariance gives equal twists on B/K, with equal diagonal scalars. Twist by κ_s by Lemma 7.7 and Corollary 7.278. At $K = T$, both sources are θ_s. □

Proposition 7.280. *Let $\ell \in \{3, 5, 7\}$ divide $q + 1$. Next, take the negative torus in $G = E_8(q)$. If $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$ is proper Levi, the ℓ'-characters in $\mathcal{E}(G, s)$ give $\succeq_c$ on $M = N_G(\mathbf{T})$.*

Proof. Take $K = \mathbf{K}^F$ and $L = N_K(\mathbf{T})$. By Proposition 7.272, $L = M_{\theta_s}$ and the index condition is $\ell \nmid [W(E_8) : W(\mathbf{K})]$. By [156, §4.4], the remaining ℓ'-cuts are on unipotents of K and characters of L/T. Next, choose $\mathbf{K}/Z(\mathbf{K})$ from Proposition 7.279. The mark acts by -1 on each factor's torus lattice. The 2-split torus has source 1. By [156, §8, proof of Theorem D], factor comparisons include ℓ dividing the relative Weyl order.

Let A, B be ambient and source inertias. Since s determines $\mathbf{K}^* = C(s)$, Corollary 7.3 gives $A = GB$ and $G \cap B = K$, also for joint corrections. Put $B_T = N_B(\mathbf{T})$. All tori of this mark in $\mathbf{K}$ form one K-class, invariant under induced operations. Hence, $B = KB_T$, $K \cap B_T = L$, $N_A(\mathbf{T}) = MB_T$. To obtain $N_A(\mathbf{T}) = MB_T$, write $a = gb$ with $b \in B_T$. Joint corrections can lie in B_T with ambiguity L.

Next, substitute B_T for B_P in Proposition 7.279. The entries over B/K share outer operations on J_i and include the inner parts. From Δ_i, tensor-induce pullbacks. Now, twist by κ_s on the section B/K via Lemma 7.7. Induce with signs to G by Corollary 7.3. Clifford induction from L gives partners on M. Both extensions have quotient $B_T/L = B/K$. Induction is equivariant by Proposition 7.277. Choose $P \in \mathrm{Syl}_\ell(G)$ in M with $N_G(P) \leq M$, as in Proposition 7.272. Let a stabilize P. By rational conjugacy, choose $g \in G$ with $(ga)(\mathbf{T}) = \mathbf{T}$. Since ${}^{ga}P, P \in \mathrm{Syl}_\ell(M)$, we choose $m \in M$ such that mga normalizes P. Then, $mg \in N_G(P) \leq M$. Both mg, mga normalize $\mathbf{T}$. Hence, $a(\mathbf{T}) = \mathbf{T}$. Restrict to the Sylow stabilizer $\mathrm{Aut}(G)_P$ in the inner-field group. □

Lemma 7.281. *Let $q = r^f$ be odd with $2 \mid f$ and odd $\ell \mid q + 1$. Next, we let J be split adjoint $D_6(q)$, with rational involution t in its dual J^*, of sign ranks $2, 4$. Each $\chi \in \mathcal{E}(J, t)$ has an $\mathcal{H}_\ell$-fixed inertia extension under J and outer $\langle F_r, \gamma \rangle$, with equivariant choices.*

Proof. Fix an $\mathbb{F}_r$-torus in Spin_{12}, with cocharacter lattice D_6. Next, write $t = b(-1)$; odd coordinates of b give negative signs. The shift $2e_i$ yields the other rational lift. Since $C(t)$ is connected, classes descend to $\mathbb{F}_q$ per geometric class. Place ranks $2, 4$ in order. Let J_5, J_4 be Levis on the final $5, 4$ coordinates and full torus. First parabolic P_5 has Levi J_5. In J_5, $P_4 = J_4 U$ fixes the second isotropic line and $\overline{P}_4 = J_4 \overline{U}$ its opposite. Dually, t is central in $\mathbf{J}_4^*$, and the roots joining positions 2 and $3, \ldots, 6$ have value -1. By simple connectedness, $C_{\mathbf{J}_5^*}(t) = \mathbf{J}_4^*$. On J_4, take $\beta_4 = \kappa_t \tau$, with τ inflated from the adjoint D_4 and κ_t dual to t. By [72, Thm. 3.3.22], induction from P_4 gives irreducible β_5, since the dual centralizer lies in $\mathbf{J}_4^*$. Next, induce to J. In the dual centralizer $D_2 D_4$, the step fixes D_4 with a split torus in $D_2 = A_1^2$. Each rank-one flag character is $1 + \mathrm{St}$. Thus, under Jordan decomposition each A_1^2-label has multiplicity 1. Vary τ over unipotents to exhaust $\mathcal{E}(J, t)$ by central isotypy.

Represent γ by the swap in hyperbolic plane 6 for a positive large block, otherwise 2. The isometry over $\mathbb{F}_r$ is fixed by pinned F_r. Together with J, it generates the graph overgroup. As b_i is even, sign change lies in $2D_6$, giving $t^\gamma = t$. The involution normalizes J_5, J_4 and fixes the first parabolic. On D_4, plane 6 gives graph action, plane 2 trivial action. The character τ has an $\mathcal{H}_\ell$-fixed inertia extension. If γ fixes τ, apply Lemma 6.216. Otherwise, apply [156, Theorem 7.5 and proof of Proposition 8.5] on adjoint D_4. The Frobenius sign has f-th root in μ_{2f_2}, fixed since $2f_2 \mid \ell - 1$. Compose with the adjoint D_4 quotient and extend κ_t by 1 on split operations. Moreover, this extends β_4 invariantly. For a positive block, P_4 is stable. Induction extends β_5 on that inertia. Otherwise, γ exchanges $P_4, \overline{P}_4$. The opposite radicals have order q^8, from the root counts of D_5, D_4. Let $\Theta_+ = |U| e_U$ with left average e_U. By

Lemma 7.92, γ acts by $q^{-4}\Theta_+ D$, where $Dv(x) = \rho(\gamma)v(\gamma^{-1}(x))$. Thus, $\widehat{\beta}_5^{\sigma} = \widehat{\beta}_5$ for $\sigma \in \mathcal{H}_\ell$. Each χ determines its D_4-label. Its inertia fixes β_5 and stabilizes P_5. Since $|t| = 2$ with rational unipotents, Jordan decomposition gives $\chi^{\sigma} = \chi$ for $\sigma \in \mathcal{H}_\ell$. By [208, Lemma 3.13], the multiplicity-1 induction from P_5 gives a fixed extension of χ. Hence, conjugate extensions are $\mathcal{H}_\ell$-fixed. □

Proposition 7.282. *At ranks* $2, 4$*, the permitted lifts give* $\succeq_c$ *on* $\mathrm{N}_G(P)$ *of* Proposition 7.275.

Proof. First, take $\mathbf{K}$ dual to $\mathbf{K}^* = C_{\mathbf{G}^*}(y)$. The mark is -1 on both spaces. Since $-1 \in W(D_6)$, the derived dual group is split Spin_{12}, with rational centre of order 4. The connected centre contains it, with $(q+1)^2$ points. Since q is an odd square, $(q+1)_2 = 2$. The derived centre exhausts $(Z(\mathbf{K}^*)^{F^*})_2$. Next, evaluate its sign vector at -1. With 2 or 4 ones, it belongs to the D_6 lattice. The scalar mark gives rational t_0. Moreover, its ratio t/t_0 is derived-central with order dividing 2. Thus, all allowed translates lift. Put $\mathbf{J} = \mathbf{K}/Z(\mathbf{K})$. All t lie in $\mathbf{J}^*$. By [180, Thm. 4.1(6)], inflation gives a bijection $\mathcal{E}(J,t) \to \mathcal{E}(K,t)$. The dual map from $\mathbf{J}^*$ matches labels by Proposition 7.269. Saturation makes $Z(\mathbf{K})$ connected.

Take the source inertia B of Proposition 7.277 and D of Proposition 7.273. Its intersection $D \cap T$ is two-primary, and $D \cap K$ maps onto $W(D_6)$ with kernel $D \cap T$, since the preimage of $W(D_6)$ lies in K. The character κ_y is trivial on the 2-kernel and simple lifts of $W(D_6)$ by odd order. Thus, $\kappa_y|_{D\cap K} = 1$. The stabilizer D_{θ_y} normalizes $\mathbf{K}$ and fixes κ_y by duality. Consequently, $kd \mapsto \kappa_y(k)$ is an extension on KD_{θ_y}. We have $B \leq KD_{\theta_y}$ by Sylow correction in K and $T \leq K$. Moreover, a stabilizer operation corrected at T has ordinary part in TD. Its D-part transports powers, with corrections in K that fix the extension.

Choose h, k from Proposition 7.275 and marked field Φ. In B, choose a lift of exponent k fixing P and so $T = C_G(P)$. Choose $v = n\Phi^k$ modulo the torus, where n is a rational Tits lift and $w = nT$. On the central plane w has order 4. Since $h/2$ is odd, $v^{h/2}$ induces the graph involution on D_6 and represents $F_r^{f/2}$ modulo G. Its square is $v^h = n^h n_0^{-1}$, since Φ fixes the Tits lifts and $\Phi^f = n_0^{-1}$. The image $w^h w_0^{-1}$ fixes the plane. Hence, $w^h w_0^{-1} \in W(D_6)$. Thus, $v^h \in D \cap K$, with κ_y-value 1, acting on J by conjugation by its image.

Write $\beta = \kappa_y \tau_t$. By Lemma 7.281, compose the τ_t-extension with $B \to \mathrm{Aut}(J)$, including inner corrections. Correct stabilizer elements at y, t. Since $\mathcal{H}_\ell$ fixes τ_t, its adjoint label returns. Twist by κ_y gives $\widehat{\beta}$ with twists 1. Contract via Proposition 7.277 with $\widehat{\beta}$. Thus, $\widehat{\chi}$ has trivial twists on G-inertia. Proposition 7.273 cancels twists on M for Proposition 7.272. Inertias coincide. Since $C_{\mathrm{Aut}(G)}(G) = 1$, pullbacks agree on the kernel, giving $\succeq_c$. □

Corollary 7.283. *Let* $G = E_8(q)$*, let* ℓ *be an odd nondefining prime of* G*, with* $\mathrm{ord}_\ell(q) \in \{7, 9, 14, 15, 18, 20, 24, 30\}$*. Then* $\mathrm{iMN}(E_8(q), \ell)$ *holds on* $\mathrm{N}_G(P)$ *with* $P \in \mathrm{Syl}_\ell(G)$*.*

Proof. Each indicated $d = \mathrm{ord}_\ell(q)$ divides one degree among $2, 8, 12, 14, 18, 20, 24, 30$. Since $d \mid \ell - 1$ and $d \geq 7$, $\ell \nmid q|W(E_8)|$. By Lemma 2.11, $P \in \mathrm{Syl}_\ell(G)$ is cyclic and $\mathrm{N}_G(P[\ell]) = \mathrm{N}_G(P)$. Corollary 2.10 gives $\mathrm{iMN}(G, \ell)$ on $\mathrm{N}_G(P)$. □

Theorem 7.284. *For each prime power* q *and prime* $\ell \mid |E_8(q)|$*,* $\mathrm{iMN}(E_8(q), \ell)$ *holds.*

Proof. First, we put $q = r^f$. The universal cover is $G = E_8(q)$. At $\ell = r$, [162, Theorem 1.3] gives $\mathrm{iMN}(G, \ell)$. If $\ell = 2$ and q is odd, apply [158, Theorem A]. If $q = r$ and $\ell \mid q - 1$ is odd, then $\mathrm{iMN}(G, \ell)$ follows from [166, Theorem 6.2 and the paragraph following Theorem A]. Otherwise, put $d = \mathrm{ord}_\ell(q)$. The fundamental degrees $2, 8, 12, 14, 18, 20, 24, 30$ give $d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30\}$. At $d = 7, 9, 14, 15, 18, 20, 24, 30$, Corollary 7.283 compares all ℓ'-labels on $\mathrm{N}_G(P)$.

If $d \in \{5, 8, 10, 12\}$, choose $N_G(T)$ of Section 7.1. If $d = 12$, Proposition 7.11 gives $\succeq_c$ on the rank-2 Sylow normalizer. At $d = 5, 10$, apply [156, Theorem D] for 1_T and Corollary 7.259 for Levis. At $|s| = 5$, apply Proposition 7.258. If $d = 8$, apply Corollary 7.263. At odd q, Proposition 7.262 distinguishes both labels on $P_H[\ell]$. If $d = 4$, apply Proposition 7.264 at $\ell = 5$, with $M/T_5 = N_B(R)$. If $\ell > 5$, we apply Proposition 7.265 on $N_G(T)$ for Levis and isolated s. Apply Proposition 7.266 for A_1^4, D_2D_4, D_3^2. At $d = 3, 6$, the $3'$-part defines Levi by Proposition 7.267. Given A_2^3, Proposition 7.269 transfers 27 ordered pairs from adjoint E_6. Next, for isolated E_6A_2, apply Proposition 7.270. Unipotents on the marked T apply [156, Theorem D].

If $d = 1$, take $N_G(T)$ in Proposition 7.271. At $d = 2$, Proposition 7.272 gives the choices of M even at $\ell = 3, 5, 7$. Apply Propositions 7.273 and 7.274 for $\succeq_c$. On Levi K, we apply Proposition 7.280 at $\ell \leq 7$ and Proposition 7.279

when $\ell > 7$. Proposition 7.282 handles Proposition 7.275 at $\ell > 7$ and $M = \mathrm{N}_G(P)$. Take joint orbits of these θ_s. Next, choose the complement on M. □

7.8 Tori in F_4 and G_2

Proposition 7.285. *Let $G = F_4(q)$ with $q = p^f > 2$, and let $\ell \neq p$ be odd. If $d = \mathrm{ord}_\ell(q) \in \{1, 8, 12\}$, then* iMN$(G, \ell)$ *holds.*

Proof. At $d = 1$ and $q = p$, [166, Theorem 6.2 and the paragraph following Theorem A] gives iMN(G, ℓ). At $d = 1$ and $f > 1$, put $N = \mathrm{N}_G(T)$ for split T. By [166, Theorem 6.2], the equivariant ℓ'-bijection to N includes $\ell = 3$. Take outer inertia D. The pinned F_p fixes n_i and permutes the Hecke generators qen_ie. If $p = 2$, the isogeny γ, with $\gamma^2 = F_2$, permutes the n_i and the subgroups $\alpha_i^\vee(\mathbb{F}_q^\times)$. Thus, the condition $\theta|_{\alpha_i^\vee(\mathbb{F}_q^\times)} = 1$ is transported with i, and the Tits signs are 1. By [166, Corollary 3.3], $\mathcal{H}_\ell$ fixes $\sqrt{q}$. Thus, the specializations at $\sqrt{q}$ and 1 in Proposition 7.271 extend over D, with matching twists by $\mathrm{Irr}(D)$. Since $Z(G) = 1$, the central characters are trivial. At $d \in \{8, 12\}$, the degrees $2, 6, 8, 12$ give multiplicity 1 of Φ_d. Since $d \mid \ell - 1$, $\ell \nmid |W(F_4)| = 2^7 3^2$. By Lemma 2.11, $P \in \mathrm{Syl}_\ell(G)$ is cyclic and $\mathrm{N}_G(P[\ell]) = \mathrm{N}_G(P)$. By Corollary 2.10, iMN$(G, \ell)$ holds with $M = N_G(P)$. □

Proposition 7.286. *At odd q and $\ell \mid q + 1$, the negative torus normalizer gives* iMN$(F_4(q), \ell)$.

Proof. Take $M = \mathrm{N}_G(T)$ at the mark $w_0 = -1$, so $\mathbf{T}$ is a Sylow 2-torus. By [132, Theorem 5.14], $\mathrm{N}_G(P) \leq M$ for every odd $\ell \mid q + 1$ when P is chosen in M. By [132, Theorems 7.5 and 7.8], the common labels are W-orbits of (s, η), with $s \in T^*$, $\ell \nmid [W : W_s]$ and $\eta \in \mathrm{Irr}_{\ell'}(W_s)$. The scalar mark gives $W_s = W(R_s)$. Next, apply the isomorphism for $K[N_\delta]$ for $\delta = -1$, with both root lengths. Adjointness makes $\rho^\vee$ integral. The positive lift n_0 centralizes the simple representatives and has $n_0^2 = 1$. Thus, over the orbit of $\theta = \theta_s$, the marked prime-field lift has f-th power $\epsilon_\theta w_0^{-1}$ in the algebra model.

If $\eta \in \mathrm{Irr}(W_\theta)$, we choose the rational extension $\widetilde{\rho}$ to C_η, with $C = N_W(\langle\theta\rangle)$, as in Proposition 7.273. Diagram stabilizers on B, C fix lengths and are trivial, including B_2. Next, permute equal tensor factors. Let $c \in \{1, -1\}$ be the scalar of w_0^{-1} on $\widetilde{\rho}$. Next, we choose $a \in \mu_{2f_2}$ with $a^f = \epsilon_\theta c$. $2f_2 \mid \ell - 1$, since $\mathrm{ord}_\ell(q) = 2$. Extend θ by 1 on stabilizing Weyl–field pairs. On (w, Φ^j) take $a^j\widetilde{\rho}(w)$ for the field relation. Moreover, at θ, corrected stabilizer operations conjugate in C_η and fix a. Thus, local extensions are invariant. In $\mathbb{Q}^{\mathrm{ab}}[T \rtimes W]$ the extension by 1 on W_θ gives the map in [156, Theorem B] at $Z(G) = 1$. On the θ-corner, scalars cancel by Proposition 7.300. If $s = 1$, local labels lie over 1_T. Apply [156, Theorem D] on $\mathcal{E}(G, 1)$.

Given $s \neq 1$, eligible global labels are real. Indeed, -1 corrects inversion of s, and opposition fixes unipotents in A, B, C. At isolated s, rationality follows from [208, proof of Proposition 9.13(i)]. At odd inertia D, [143, Lemma 2.1] gives invariant real extension on $G \rtimes D$. At even $|D|$, type A gives subfield multiplicity 1 by Section 7.1. At even inertia outside$A$ components, q is an odd square, $(q + 1)_2 = 2$, and $3 \nmid q + 1$. Next, write $s = ty$, with $t^2 = 1$ and y of odd order. If $y \neq 1$, primes dividing $|y|$ exceed 3. Highest-root coefficients in F_4 and its Levis have primes ≤ 3. Moreover, the vertex test makes $\mathbf{K}^* = C_{\mathbf{G}^*}(y)$ a Levi. The roots in $\Delta(\mathbf{K}^*)$ extend to a basis of $X(\mathbf{T}^*)$. Thus, $Z(\mathbf{K}^*)$ is connected. In central rank 1, we let r be the even inertia order and $k = f/r$. The Weyl correction of F_p^k acts on the central line by ± 1. Thus, $y^{p^k} = y^{\pm 1}$ would give $y^q = y$, whereas $y^q = y^{-1}$. Hence, the semisimple rank is ≤ 2. In the non-A case, R_s is the whole B_2 of K^*, so $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$.

The simple roots of $\mathbf{K}$ extend to a lattice basis. Thus, $Z(\mathbf{K})$ is connected. Put $\mathbf{J} = \mathbf{K}/Z(\mathbf{K})$. By [133, Proposition 23.2], $K/Z(\mathbf{K})^F \simeq J = \mathbf{J}^F$. The preimage of $\bar{\mathbf{T}}$ is $\mathbf{T}$, so $\mathrm{N}_K(\mathbf{T})$ maps onto $\mathrm{N}_J(\bar{\mathbf{T}})$. The adjoint B_2 splits by inner marking. Moreover, on J, corrections are inner times field on B_2, whose length-stable diagram group is 1. Take $\bar{T}$ at the negative mark of J. By [156, §4.4], the series at $s = 1$ matches eligible unipotents with $\mathrm{Irr}(W(B_2))$ over $1_{\bar{T}}$. $\ell \geq 5$, so $\ell \nmid 8$. All 5 labels are rational and F_p-fixed. At ± 1, we choose f-th roots in μ_{2f_2}. By [156, Theorem 7.5], global extensions are $\mathcal{H}_\ell$-fixed. Locally inflate from $W(B_2)$. The marked field centralizes $W(B_2)$, with f-th power the longest element w_0. Next, take its scalar in μ_{2f_2} with the given f-th power. Inflate the extensions from J. Then P contains $T[\ell]$ and $\ell \nmid |W(F_4)|$. Thus, $\mathbf{T} = C_{\mathbf{G}}(P)^\circ$ and $M = \mathrm{N}_G(P)$. The Weyl stabilizer of s is $W(\mathbf{K})$, giving $L = \mathrm{N}_K(P) = M_{\theta_s}$. Now, on K, take κ_s dual to s. Operations corrected to return the source fix κ_s. Since $K \lhd B$ and $P \in \mathrm{Syl}_\ell(K)$, the Frattini argument gives $B = KB_P$, with $B_P = \mathrm{N}_B(P)$. Moreover, $K \cap B_P = L$. Thus, Lemma 7.7 gives c_{κ_s} on the common quotient B/K. Transfer by Corollary 7.3. Induce from L and recover its source on the θ_s-space. This exhausts the packet over θ_s. The B_2-labels are fixed by $\mathrm{Inn}(J), F_p$ and $\mathcal{H}_\ell$, so choices transport. If $y = 1$

and $s \neq 1$, [208, proof of Proposition 9.13, Case 1] gives split torus or B_2 sources over $\mathbb{F}_p$. Extend the quadratic twist trivially on F_p. On adjoint B_2, the cuspidal eigenvalue -1 has an f-th root in $\mu_{2f_2} \subseteq \mathbb{Q}_\ell$. Thus, [156, Proposition 7.3] gives an extension over $\mathbb{Q}_\ell$. Inflation and the quadratic twist give $\widehat{\lambda}$. By [208, Lemma 4.9(i),(ii)], $\widehat{\chi}$ is fixed by $\mathcal{H}_\ell$. □

Lemma 7.287. *Let $q = 2^f > 2$, $d = 2, 4$ and $\ell > 3$. Suppose $s \in T^*$ has full centralizer $\mathbf{T}^*$ or a B_2-Levi. For the Levi assume $\succeq_c$ on marked $B_2(q)$ for induced actions. Signed induction gives $\succeq_c$ on $\mathcal{E}(G, s)$ and the θ_s-orbit in $M = \mathrm{N}_G(\mathbf{T})$. In particular, both sources are θ_s when $\mathbf{K}^* = \mathbf{T}^*$.*

Proof. Put $\mathbf{K}^* = C_{\mathbf{G}^*}(s)$, with dual Levi $\mathbf{K}$ containing $\mathbf{T}$. Since $\ell \nmid |W(F_4)|$, the ℓ-part P of T is Sylow in G. The marked $\mathbf{T}$ is a Sylow d-torus. By [130, proof of Proposition 2.1], $M = \mathrm{N}_G(P)$. Put $L = \mathrm{N}_K(\mathbf{T}) = \mathrm{N}_K(P)$. By [133, Proposition 25.3(a)], L/T has dual $C_W(w) \cap W(\mathbf{K}^*) = C_W(w)_s$. By [18, Lemma 11.6], the stabilizer of s corresponds to the stabilizer of θ_s. Thus, $L = M_{\theta_s}$.

For a B_2-Levi, we put $H = [\mathbf{K}, \mathbf{K}]^F$ and $Z = Z^\circ(\mathbf{K})^F$. The isogeny's kernel has scheme order ≤ 2. If $p = 2$, it is geometrically bijective and F-equivariant. Thus, $K = H \times Z$, $H \simeq B_2(q)$ and $L = \mathrm{N}_H(T_H) \times Z$, where $T_H = H \cap T$. The derived mark is -1 at $d = 2$ and a quarter rotation at $d = 4$. Hence, T_H is as in [100, Table 5.1], with relative group D_8 or C_4, respectively.

Let $\beta = \kappa_s \nu$, with unipotent ν inflated from H. Then, $\kappa_s|_H = 1$ and $\kappa_s|_T = \theta_s$. Put $B = N_\Gamma(\mathbf{K})_\beta$ in Proposition 7.5, and $B_P = \mathrm{N}_B(P)$. Since $P \in \mathrm{Syl}_\ell(K)$, the Frattini argument [97, Theorem 1.12] gives $B = KB_P$, with $B_P \cap K = L$. Hence, the corrected stabilizer operation in B_P normalizes H, Z, T_H, including root-length exchanges. On Z, coefficients return the weight. Take the $H, \mathrm{N}_H(T_H)$ preimages under conjugation on H. Since $Z(H) = 1$, $hz \in HZ$ acts by h. Quotient actions agree modulo $\mathrm{Inn}(H)$.

If $j \in B/K = B_P/L$, take the same $b_j \in B_P$, with factor multiplier $\kappa_s(b_i b_j b_{ij}^{-1})^{-1}$. Moreover, at b_j, Lemma 7.7 matches twists and action on Z. Proposition 7.5 induces with sign to G. Locally induce from $L = M_{\theta_s}$ to M. Moreover, weight corrections differ by L. Coset representatives permute weights. On the θ_s-space, source intertwiners give twists on L. By [18, Remark 10.12], the global degree multiplier is $[G : K]_{2'}$, prime to ℓ since $P \leq K$. The local multiplier $[M : L]$ divides $|W|$. The packet is exhausted. When $\mathbf{K} = \mathbf{T}$, we have $L = T$, and take θ_s on both sides before induction. □

Lemma 7.288. *Let $q = p^f$ be odd and $\mathrm{ord}_\ell(q) = 4$. Take the mark with two negative 2-cycles. Moreover, each torus weight θ extends to marked inertia, equivariantly under $\mathcal{H}_\ell$. Quadratic weights have extensions valued in μ_4.*

Proof. Take Torus 22 of [69, Table 1 and the calculation for Torus 22]. In $M = TA$, the supplement $A = \langle a, b, c\rangle$ has prime-field Tits generators, and $V = A \cap T = T[2] = \langle u, v\rangle$, $u = h_3$, $v = h_4$. Indeed, $T \simeq C^2_{q^2+1}$ and $(q^2+1)_2 = 2$. The presentation of $A/V = C_W(w)$ has relators a^4, b^4, c^3, $[a, b]$, $a^3b^2cb^{-1}c^{-2}$. Moreover, in A their values are $u, u, 1, u, u$. Also, both a, b conjugate (u, v) to (u, uv), and $(u^c, v^c) = (uv, u)$, with $x^c = c^{-1}xc$. By [69, Section 4, Torus 22], these relations present A.

Next, write $\mathbf{e} = (e_1, e_2, e_3)$ and let a, b, c act componentwise. Then, $a\mathbf{e} = (ie_1, ie_3, e_2)$, $b\mathbf{e} = (ie_1, e_3, ie_2)$, and $c\mathbf{e} = (e_2, e_3, e_1)$. Take $u = \mathrm{diag}(1, -1, -1)$ and $v = \mathrm{diag}(-1, 1, -1)$. They satisfy the relators on generators a, b, c and conjugations on u, v. Thus, they represent A. The three lines afford nontrivial V-characters. Moreover, each stabilizer acts by μ_4, extending its V-character. Monomiality gives equivariance under A.

Keep $n = h_2n_8n_{16}n_3n_2$ over w. By [69, calculation for Torus 22], we have $ng = gn$ for $g \in \{a, b, c\}$. A word in a, b, c over w differs from n by a Tits-kernel element in $T[2] = V$. Thus, $n \in A$ fixes all three lines setwise. Since c permutes them transitively, n acts by one $\xi \in \mu_4$. The determinants are 1. Hence, $\xi^3 = 1$ and $\xi = 1$. The marked field U fixes A, with $U^f = n^{-1}$. Extend to $A_0 = \langle A, U\rangle$ by $U \mapsto 1$, since the outer order is f. In particular, $A_0 \cap T = V$. On $(A_0)_\theta$, choose the line character λ over $\theta|_V$, or the trivial character if $\theta|_V = 1$. Moreover, the formula $\widehat{\theta}(tk) = \theta(t)\lambda(k)$ on $T(A_0)_\theta$ agrees on V and is multiplicative by θ-invariance. Transport permutes the lines. Their μ_4-values are fixed by $\mathcal{H}_\ell$. Given $t_0 \in T$, the extra factor is $\theta([k, t_0]) = 1$ because k fixes θ. Thus, transport is equivariant. □

Proposition 7.289. *Suppose $d = 4$. Given $s \neq 1$, $\mathcal{E}(F_4(q), s)$ has $\succeq_c$ on $\mathrm{N}_G(T)$, where T has two negative 2-cycles.*

Proof. The mark $w \in W(B_4)$ satisfies $w^2 = -1$. Thus, $\mathbf{T}$ is a Sylow 4-torus of order polynomial Φ_4^2 and T has exponent dividing $q^2 + 1$. Since $4 \mid \ell - 1$, $\ell \nmid |W|$ and $P = T_\ell \in \mathrm{Syl}_\ell(G)$ by the order formula. By [130, proof of Proposition 2.1], $\mathrm{N}_G(P) = \mathrm{N}_G(T) = \mathrm{N}_G(\mathbf{T})$. By [132, Proposition 5.11], $\mathrm{N}_{G^*}(T^*)$ controls G^*-fusion in T^*. The w-stable span of R_s has rank $0, 2$ or 4. Since $3 \nmid q^2 + 1$ and $(q^2 + 1)_2 \leq 2$, if $s \neq 1$ has full-rank root subsystem,

then $|s| = 2$ by [208, proof of Proposition 9.13]. The involution centralizers have types B_4 and C_3A_1 by the same classification. At C_3A_1, w cannot stabilize the unique A_1-line. Thus, the vertex is B_4 and q is odd.

At rank 2, the odd part y is nontrivial. By Proposition 7.286, $C(y)$ is a proper Levi. Its w-stable system R_y has rank 2. A quarter-turn excludes the A_2 hexagon and an unequal-length A_1^2. Thus, $R_y = B_2$. Complementary coordinates give another B_2 perpendicularly. After alignment, the F_p^k-corrector centralizes w and fixes lengths, with angle in $(\pi/2)\mathbb{Z}$. Thus, $p^{4k} \equiv 1 \bmod |y|$. But $q^2 \equiv -1 \bmod |y|$ and $|y| > 1$ is odd. Since $k \mid f$, the quotient f/k is odd. At $p = 2$ and even f, even inertia contains γ^f, also excluded.

We have $R_s = B_2$ or a same-length A_1^2. Moreover, these give rotations $B_s = C_4$ and exchanges $B_s = C_2$, respectively. On the genuine B_2-Levi, κ_s extends θ_s. If $2 \nmid p$, we apply Lemma 7.288 and tensor by $\eta \in \mathrm{Irr}(B_s)$. Field correctors centralize B_s by the rotation description. If $p = 2$, $s = y$ and $R_s = B_2$. Its corrected exceptional similarity centralizes w. Next, on B_2, apply [156] at odd p and [100, Lemma 5.1 and Table 5.1] for $p = 2$. Exchanged A_1's give $1, \mathrm{St}$ over $\mathbb{F}_{q^2}$, matched with $\mathrm{Irr}(C_2)$. Thus, degree cuts match. For odd $|D|$, apply [143, Lemma 2.1]. Since -1 corrects inversion of s by inner opposition on B_2, unipotents are real there, and $1, \mathrm{St}$ are real on exchanged A_1^2. Locally take τ inverting odd roots and fixing μ_{2^∞}. Then τ inverts the odd part of θ_s and fixes the quadratic part, μ_4-scalars and labels of B_s. The element -1 restores the weight and centralizes B_s. Thus, the induced label is τ-fixed. On the odd quotient D, τ inverts twists on $\mathrm{Irr}(D)$. Bijective squaring gives the unique extension $\widehat{\psi}^\tau = \widehat{\psi}$. Thus, stabilizer operations fix it. At rank 2, even inertia requires $p = 2$ and odd f. Specifically, $s = y$, and the Levi is $B_2(q)$ times its centre. By [100, Lemma 5.1, Table 5.1 and Theorem 5.4], the marked $B_2(q)$ comparison includes all automorphisms. It pairs 1_{T_H} with unipotents on H. Lemma 7.287 transfers the twisted pairs to G, M. Moreover, at rank 0, both inductions have the source θ_s. The unique label has inertia T. Next, apply Lemma 7.287 for $p = 2$, and Corollary 7.3 with $\mathbf{K} = \mathbf{T}$ for odd p.

At the B_4 vertex, we have $4f_2 \mid \ell - 1$. Its cycle polynomials are powers of $X^2 + 1$. Thus, we obtain two negative 2-cycles. Thus, the inertia quotient is $C_4 \wr \mathfrak{S}_2$. Also, it has ℓ'-order, central w and values in $\mathbb{Q}(i)$. Lemma 7.288 extends the weight over U. Moreover, modulo T, U fixes the quotient and $U^f = w^{-1}$. At each quotient label take the f-th root of the w^{-1}-scalar in μ_{4f_2}. Next, tensor and induce to obtain $\mathcal{H}_\ell$-fixed extensions. By [156, §4.4], global labels are $d = 4$ unipotents of B_4. Take quadratic torus and split B_2-cuspidal twists over $\mathbb{F}_p$ by Proposition 7.286 and [208, proof of Proposition 9.13(i), Case 1]. Extend twists by 1 on F_p. On adjoint B_2 the Frobenius eigenvalue is -1, with an f-th root in $\mu_{2f_2} \subset \mathbb{Q}_\ell$. Thus, the result of [156, Proposition 7.3] gives fixed pullbacks. Case 1 gives rational constituents fixed by F_p. By [208, Lemma 4.9(i),(ii)], quotient twists are 1. The vertex gives one W-orbit. □

For $\ell \neq p$ dividing $|G|$, the degree factors $q^m - 1$ with $m \in \{2, 6, 8, 12\}$ give $d \in \{1, 2, 3, 4, 6, 8, 12\}$. If q is odd, apply Ruhstorfer et al. [156, Theorem D] on $\mathcal{E}(G, 1)$. At $\ell = p$, apply Ruhstorfer [162, Theorem 1.3] and Johansson [101, Theorem 1.1], with the latter covering $F_4(2^f)$. If $q = 2$, apply Ruhstorfer–Schaeffer Fry [157, Lemma 4.1(2)]. If $\ell = 2$ and q is odd, apply Ruhstorfer–Schaeffer Fry [158, Theorem A]. Next, assume $q = 2^f > 2$.

Lemma 7.290. *At $p = 2, d = 2, s \neq 1$ with R_s of type A componentwise, the series gives $\succeq_c$ on $\mathrm{N}_G(T)$.*

Proof. Put $F' = \mathrm{Int}(n_0)F_q$. The Tits group in characteristic 2 is W, with central $n_0 = w_0 = -1$. Moreover, its representatives are F'-fixed, with the diagram involution γ. In the mark $\gamma^{2f} = n_0$. Next, take a formal g of order $4f$ that induces this action on $M = T \rtimes W$. The marked quotient has kernel $\langle z \rangle$, with $z = g^{2f} w_0$. On W_s, take rational models for stable η. Moreover, on A_n the longest element gives opposition if $n \geq 2$. Permute equal factors. At (w, g^j) in the stabilizer of (θ_s, η), take its operator. Next, extend θ_s by 1 on the stabilizing (w, g^j). Induce to ordinary inertia. Restriction to M recovers its base label. Next, when the label returns, restore θ_s in W. The symmetry fixes η and the extension by conjugacy.

On A_n with $n \geq 2$, inversion acts trivially as w_{0,A_n} times opposition. Each A_1 contributes its reflection sign to z, since the weight is trivial on W_θ. Odd g-powers exchange lengths. Their A_1-cycles are even with constant signs ± 1, with product 1. Otherwise, inertia has order $r \mid f$. Since $q = (2^{f/r})^r$ and $d = 2$, the proof of [156, Lemma 8.1(a)] gives $2r_2 \mid \ell - 1$. At z, correct -1 by $\zeta \in \mu_{2r_2}$ with $\zeta^r = -1$. By [156, Lemma 2.3(a)], $\mathcal{H}_\ell$ fixes ζ, so the model descends. On A-factors take partitions with opposition from W and factor permutations by [208, Proposition 9.13]. At $-q \equiv 1 \bmod \ell$, hooks give the symmetric-group ℓ-part. The index cut is $\ell \nmid [W : W_s]$, including $\ell = 3$. Moreover, in isolated A_2A_2, take unequal partitions with $q \equiv -1 \pmod 3$. The six characters χ have inertia group $G \rtimes \langle F_2 \rangle$ in $\mathrm{Aut}(G)$ and rational extensions $\widehat{\chi}$ by [208, Proposition 9.13(ii)–(iii) and its proof]. At equal partitions (λ, λ), [208, Proposition 9.14(ii)] gives extensions over $\mathbb{Q}(\zeta_3)$, which are $\mathcal{H}_\ell$-fixed if $3 \mid \ell - 1$. In the other cases -1 restores s^{-1}, so $\overline{\chi} = \chi$. If $|D|$ is odd,

apply [143, Lemma 2.1]. At even $|D|$, take the subfield module for γ^f, with $(\gamma^f)^2 = F_q$, by [105, Theorem 3.1.3(ii)]. □

Lemma 7.291. *Let $q = p^f$, with odd prime $\ell \neq p$ and $\mathrm{ord}_\ell(q)_2 = 2$. Next, we put $h = f_2$ and take primitive $\alpha \in \mu_{4h}$. Then, we obtain that $\alpha^{2f} = -1$ and, under the chosen map, $\sqrt{p}\,\alpha \in \mathbb{Q}_\ell$. Moreover, both square roots of $-q$ have f-th roots in $\mathbb{Q}_\ell$.*

Proof. The order of $-q$ modulo ℓ is odd, so [156, Lemma 8.1(b)] gives $u_0 \in \mathbb{Q}_\ell$ with $u_0^{2f} = -q$. Put $\beta = u_0/\sqrt{p}$. From $\beta^{2f} = -1$, its 2-part β_2 has order $4h$. The odd part $\beta_{2'}$ belongs to $\mathbb{Q}_\ell$, since it is a power of $\beta^2 = u_0^2/p$. Now, $\sqrt{p}\,\beta_2 = u_0/\beta_{2'} \in \mathbb{Q}_\ell$. Both α/β_2 and α^2 lie in $\mu_{2h} \subset \mathbb{Q}_\ell$, as $\mathrm{ord}_\ell(p)_2 = 2h$ divides $\ell - 1$. Hence, $u = \sqrt{p}\,\alpha \in \mathbb{Q}_\ell$ and $\alpha^{2f} = -1$. The elements u and $\alpha^2 u$ have opposite f-th powers, both squaring to $-q$. □

For $B_2(2^f)$ at $d = 2$, Zhang [208, Proposition 6.25, Step 3] gives the cuspidal cohomology operator A satisfying $A^{2f} = (-q)^{-(f+1)}$. Next, we multiply by $(\sqrt{2}\,\alpha)^{f+1}$ to give a genuine extension over $\mathbb{Q}_\ell$. Steps 1, 2 give rational extensions for $1_G, \mathrm{St}$ and the exchanged pair over $\langle F_2 \rangle$. Moreover, the remaining degree $q(q+1)^2/2$ is ineligible. The local quotient is D_8. Moreover, the marked u exchanges the simple reflections and has $u^{2f} = w_0$. Linear labels take w_0 to 1. Extend by 1 on stabilizing generators. Next, we set $S = \mathrm{diag}(1, -1)$, with T swapping coordinates. The involution $J = (S + T)/\sqrt{2}$ exchanges S, T. Give u the matrix αJ. Its $2f$-th power is $-I = w_0$, with entries in $\mathbb{Q}_\ell$ by Lemma 7.291. Given any f, by Johansson [100, Lemma 5.1 and Table 5.1], we obtain the map over 1_T.

Proposition 7.292. *For $q = 2^f > 2$ at $d = 2$ the remaining series at $s \neq 1$ give $\succeq_c$ on M.*

Proof. By [208, Proposition 9.13], isolated odd $s \neq 1$ gives A_2A_2. Apply Lemma 7.290. Otherwise, $C_{\mathbf{G}^*}(s)$ has a proper Levi closure $\mathbf{K}^*$. Highest-root coefficients are powers of 2. Thus, the vertex test gives $C_{\mathbf{G}^*}(s) = \mathbf{K}^*$. Also, its non-$A$ types are B_2, B_3, C_3. At rank 3, unequal length counts give even γ-exponents. Moreover, the centre has rank 1, and even field inertia would map s to $s^{\pm 1}$, implying $s^q = s$. But $s^q = s^{-1}$ and $1 < |s|$ is odd. Moreover, opposition fixes B_3, C_3 unipotents, and w_0 corrects inversion of θ_s. Both labels are real. For labels χ, ψ, apply [143, Lemma 2.1] at odd D. Also, the result of [156, §4] gives field equivariance. The isogeny γ exchanges lengths, with $\gamma^2 = F_2$. The Weyl index excludes $\ell = 3$. At rank 2 the Levi is $B_2(q)$ times Z by Proposition 7.289. Twist over 1_T by κ_s. The index $[W(F_4) : W(B_2)] = 144$ excludes $\ell = 3$. Lemma 7.287 gives signed induction to G and Clifford induction to M, on the same central section. □

Lemma 7.293. *Let $q = 2^f > 2$, let $\gamma^2 = F_2$, and take the torus normalizer M from above. Next, we write $B = C_W(w)$ for the relative Weyl group at $d = 2, 3, 4, 6$. The ℓ'-characters in $\mathcal{E}(G, 1)$ and $\mathrm{Irr}_{\ell'}(B)$ are outer-equivariantly bijective, with $\mathrm{Irr}_{\ell'}(B)$ inflated to M. Both sets are pointwise $\mathcal{H}_\ell$-fixed.*

Proof. By [157, proof of Proposition 6.12], regularity of d gives condition (1) of [157, Theorem 5.1]. At $s = 1$, [156, §4.4 and proof of Theorem B] gives the tuples $(1, 1_T, \eta)$ with $\eta \in \mathrm{Irr}_{\ell'}(B)$. Choose 1_M to extend 1_T. Then, the local image is $\mathrm{Inf}_B^M\,\eta$. By [156, Corollary 4.7 and Lemma 4.3], $\mathcal{H}_\ell$ fixes both labels. At $d = 2, \ell = 3$, $\mathrm{Irr}_{\ell'}(B)$ excludes degrees $6, 9, 12$ by [73, Table 1]. □

Proposition 7.294. *At $d = 4$ pairs in* Lemma 7.293 *admit genuine $\mathcal{H}_\ell$-fixed extensions.*

Proof. Choose positive w-quarter rotations from Proposition 7.289. Then, $D = (1 + w)/\sqrt{2}$ is an isometry, with $D^2 = w$ and $[D, B] = 1$. Each $D(e_i)$ is a normalized long root. For $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$, it has one entry $\pm 1/\sqrt{2}$ per plane. Thus, D exchanges normalized lengths and $D = v\iota$, with $v \in W$ and the diagram involution ι. Next, we choose the Tits lift $h = v\gamma$ in characteristic 2. It is rational on the w-marked group and centralizes B modulo T. There $h^{2f} = D^{2f}w^{-1} = w^{f-1}$, with w^{-1} from the marked Frobenius wF_q. Moreover, the choice of D^{-1} gives w^{-f-1}. All B-labels have full inertia $\langle \gamma \rangle$. By Lemma 7.293, $\langle \gamma \rangle$ also fixes each global label. If $2 \nmid f$, one exponent is in $4\mathbb{Z}$, so h can act by 1. If $2 \mid f$, $\mathrm{ord}_\ell(2)_2 = 4f_2$. Hence, $\ell \equiv 1 \bmod 8$ and 2 is a quadratic residue, giving $8f_2 \mid \ell - 1$. Choose an h-scalar in μ_{8f_2} whose $2f$-th power is the scalar of w^{f-1}. It exists by $w^4 = 1$ and is fixed by $\mathcal{H}_\ell$. The Weyl values lie in $\mathbb{Q}(i)$, so extensions are fixed.

Since $\ell \equiv 1 \pmod 8$ at even f, [208, proof of Proposition 9.8] gives principal series extensions over $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}_\ell$. By [40, p. 479], the cubic cuspidal degrees contain $\Phi_4(q)^2/3$. Thus, the eligible nonreal cuspidals are $F_4[\pm i]$. In [208, proof of Proposition 9.9], take $\xi = \pm i$ and $\alpha^{2f} = \xi^{f+1}$. If f is even, choose $\alpha \in \mu_{8f_2} \subset \mathbb{Q}_\ell$. If f is odd, choose $\alpha \in \mu_4 \subset \mathbb{Q}_\ell$, since $\xi^{f+1} \in \mu_2$. Thus, the scalar $2^{f+1}\alpha$ and the eigenvalue ξq^2 lie in $\mathbb{Q}_\ell$. Moreover, the same proof

gives $\xi = -1$ and $j = 1, 2, 3$ for the three stable B_2-labels. At even f, choose $\alpha \in \mu_{4f_2}$ with $\alpha^{2f} = -1$. At odd f, choose $\alpha = 1$. Thus, $2^{j(f+1)/2}\alpha \in \mathbb{Q}_\ell$. By [40, p. 479], both exchanged B_2-degrees contain $\Phi_4(q)/4$. Consequently, ℓ divides both. By [40, p. 479], the degrees of $\phi_{16,5}, F_4^{II}[1], F_4[-1]$ contain $\Phi_4(q)$ in the numerator, with denominators $4, 24, 4$. Since $\ell \equiv 1 \pmod 4$, ℓ divides all three degrees. At even f, we have $\ell \equiv 1 \pmod 8$. Thus, [208, proof of Proposition 9.11, Step 2] gives a $\mathbb{Q}_\ell$-valued extension of $F_4^I[1]$. If f is odd, [61, Proposition 4.3] gives the extensions on $\langle \gamma^f \rangle$. Glue over $\langle \gamma^f \rangle$ and $\langle F_2 \rangle$ for rational labels. □

Lemma 7.295. *Let $\Gamma = W(F_4) \rtimes \langle \iota \rangle$, where ι is the based diagram involution, $z = -I \in W(F_4)$, and $e : \Gamma \to \mathbb{Z}/2$ the coset parity. Next, for k prime to $|\Gamma|$, we write $\epsilon = 0$ or 1 according as the k-th cyclotomic power acts by $\sqrt{2} \mapsto \pm\sqrt{2}$. Then, we have that x^k is conjugate in Γ to $z^{\epsilon e(x)}x$ for all $x \in \Gamma$.*

Proof. By [108, §III, (12)–(15)], Γ acts on $\mathbb{H}$ by pairs (a, b) from the binary octahedral group O and by quaternion conjugation t. The pair acts by $x \mapsto axb^{-1}$, and t swaps a, b. Moreover, the kernel on pairs is $\langle(-1, -1)\rangle$, with z represented by $(-1, 1)$. Take the binary tetrahedral subgroup H and parity $e_0 : O \to O/H$. Then, W is given by $e_0(a) = e_0(b)$ in either t-coset. By [184, Tables A.7 and A.9], irrational values of O occur at traces $\pm\sqrt{2}$. These two odd classes differ by -1. Consequently, a^k is conjugate in O to $(-1)^{\epsilon e_0(a)}a$. For $x = (a, b)$, the signs give $z^{\epsilon e(x)}$ in the quotient. At $x = (a, b)t$, conjugate x to $(ab, 1)t$. Its odd k-th power is conjugate to $((ab)^k, 1)t$. Also, zx is conjugate to $(-ab, 1)t$. Thus, x^k and $z^{\epsilon e(x)}x$ are conjugate. □

Proposition 7.296. *An extension from W to Γ changes outside W by the z-scalar when $\sqrt{2}$ changes sign. A z-positive extension is rational, with multiplicity 1 in induction from its rational W-model. The rational idempotent gives a $\mathbb{Q}$-model on Γ. At $d = 2, 3, 6$ each eligible trivial-weight character of M has an $\mathcal{H}_\ell$-fixed ordinary inertia extension.*

Proof. If $d = 2$, the marked exceptional lift g induces the diagram involution on W, with $g^{2f} = z \bmod T$. For stable labels take Γ, with involution J. If z acts by 1, we choose J for g. If z acts by -1, take αJ with α from Lemma 7.291. By Lemma 7.295, on odd g-powers the sign cancels with α^σ/α. Even powers have scalar α^2, which is fixed. On moved labels, g^2 acts trivially on W. Next, choose its scalar in μ_{2f_2} with the given f-th power. Inflate to inertia, including $\ell = 3$.

At $d = 3, 6$, take $w = z^u L_a$, with $u = 0, 1$ respectively, with L_a left multiplication by $a \in H$ of order 3. Moreover, the polynomial Φ_d^2 gives a rational conjugate. The O-class of a has size 8 by [184, Table A.7], so $C_O(a) = \langle a, -1\rangle$. The swap $L_a \mapsto R_b$ has $b \notin \mathrm{Z}(O)$. Thus, $C_\Gamma(w)$ has no swap component, and

$$B = (C_6 \times H)/\langle(-1, -1)\rangle \simeq C_3 \times H,\ B^+ = (C_6 \times O)/\langle(-1, -1)\rangle \simeq C_3 \times O.$$

$B^+ = C_\Gamma(w)$. By [156, Lemma 4.3], values on B lie in $\mathbb{Q}(\zeta_3)$. For $\sigma \in \mathcal{H}_\ell$, the left C_3-label is fixed. If σ moves $\sqrt{2}$, the right O-class of $b \in O \setminus H$ goes to that of $-b$ by [184, Tables A.7 and A.9]. This gives the z-scalar on $B^+ \setminus B$.

Put $D = (1, (i + j)/\sqrt{2}) = v\iota$, with $D^2 = z$. Moreover, the marked lift $h = v\gamma$ acts on B by D, with $h^{2f} = z^f w^{-1} \bmod T$. Given an extension $\widehat{\eta}$ to B^+, let h act by $\beta\widehat{\eta}(D)$. Next, we choose β^{2f} as the scalar of w^{-1}. Choose the cubic part in μ_{3f_3}, since $\mathrm{ord}_\ell(2^f) = 3$ or 6 gives $3f_3 \mid \ell - 1$. This suffices when z acts by 1. At sign -1 and $d = 6$, we choose $\alpha^{2f} = -1$ by Lemma 7.291 to cancel odd h-powers. At $d = 3$, we choose a $2f$-th root of 1. If $\sqrt{2}$ is fixed take 1. Otherwise, 2 is nonsquare modulo ℓ, so $\mathrm{ord}_\ell(2)_2 = (\ell - 1)_2$. Since 2^f has odd order, $(\ell - 1)_2 \mid f_2$. Moreover, a primitive $2(\ell - 1)_2$-root ζ has order dividing $2f$. Frobenius acts by $\zeta \mapsto -\zeta$ and $\sqrt{2} \mapsto -\sqrt{2}$. Thus, $\beta\zeta$ cancels the sign. For moved labels, U centralizes B and satisfies $U^f = w^{-1}$. Next, we choose its scalar in μ_{3f_3}, with an additional μ_{2f_2}-factor at negative sign for $d = 6$. □

Proposition 7.297. *At $d = 2, 3, 6$ the eligible unipotents of $F_4(2^f)$ have ordinary extensions fixed by $\mathcal{H}_\ell$.*

Proof. The moved principal labels extend over $\mathbb{Q}$ to $G \rtimes \langle F_2 \rangle$ by [208, proof of Proposition 9.8]. The stable labels extend over $\mathbb{Q}(\sqrt{2})$, which suffices if $\sqrt{2} \in \mathbb{Q}_\ell$. Otherwise, on the multiplicity space take J_0 over $\mathbb{Q}$ with $(cJ_0)^2 = 1$ and $c \in \mathbb{Q}(\sqrt{2})$ by the same proof. Since J_0^2 is rational scalar, $c \in \mathbb{Q} \cup \sqrt{2}\,\mathbb{Q}$. Thus, on odd powers of γ the coefficient twist is $\sigma(c)/c \in \{1, -1\}$. If $d = 3$ and $c \in \sqrt{2}\,\mathbb{Q}$, choose $\zeta \in \mu_{2f}$ as in Proposition 7.296. Thus, $\sigma(\zeta)/\zeta = \sigma(c)/c$ for $\sigma \in \mathcal{H}_\ell$. The quotient character with value ζ on γ cancels the twist.

If $d = 2, 6$ and $\sqrt{2}$ moves, [61, Proposition 4.3] gives rational extensions at odd f on $\langle \gamma^f \rangle$ except at $\phi_{16,5}$. The latter's degree contains $\Phi_2\Phi_6^2/4$, so its degree is divisible by ℓ. By [208, Lemma 3.1], choose the rational extension $\widehat{\chi}_{\mathrm{odd}}$

over $\langle F_2\rangle$. By [95, Corollary 6.17], the restrictions determine $\widehat{\chi}$ over $\langle\gamma\rangle$. If f is even, $\sqrt{q}\in\mathbb{Q}$. Apply [74, Corollary 2.5] at $v=\sqrt{q}$ and $v=1$. Proposition 7.296 gives the rational involution for z-scalar 1. At $\phi_{n,b}$, occurrence in S^b gives the scalar $(-1)^b$. In the stable list of [61, Proposition 4.3], odd b gives $\phi_{4,1},\phi_{4,13},\phi_{16,5}$. By [40, §13.9], the degrees of $\phi_{4,1},\phi_{4,13}$ contain $\Phi_2^2\Phi_6^2\Phi_8/2$. Thus, at $d=2,6$, these degrees are divisible by ℓ.

By [40, p. 479], all five B_2-degrees contain $\Phi_3(q)$, with denominators 2 or 4. Thus, ℓ divides those degrees at $d=3$. Moreover, $B_2:r$ has degree $q^4\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6\Phi_8/4$, divisible by ℓ at $d=2,6$. For other stable B_2-labels, [208, proof of Proposition 9.9] gives the scalar $\beta=2^{j(f+1)/2}\alpha$, with j odd and $\alpha^{2f}=(-1)^{f+1}$. Choose $\alpha=\alpha_0^{f+1}$ with $\alpha_0^{2f}=-1$. By Lemma 7.291, take $\sqrt{2}\,\alpha_0\in\mathbb{Q}_\ell$. Thus, $\beta=2^{(j-1)(f+1)/2}(\sqrt{2}\,\alpha_0)^{f+1}\in\mathbb{Q}_\ell$. For swapped labels, we apply [208, Propositions 4.8 and 9.9] on B_2 with root in μ_{2f_2}.

The quartic cuspidal degrees exclude $d=2,3,6$, while the cubic cuspidal degrees exclude $d=2$ by [40, §13.9]. Moreover, $3f_3\mid\ell-1$. Thus, in [208, proof of Proposition 9.9], take $\alpha\in\mu_{3f_3}$ with $\alpha^{2f}=\xi^{f+1}$. Since $j=2$, the scalar is $2^{f+1}\alpha\in\mathbb{Q}_\ell$. If f is odd, [61, Proposition 4.3] gives extensions to $G\rtimes\langle\gamma^f\rangle$ with rational values for $F_4^I[1],F_4^{II}[1]$ and with values in $\mathbb{Q}(\sqrt{-2})$ for $F_4[-1]$. At $d=2,6$, Lemma 7.291 gives $\sqrt{-q}\in\mathbb{Q}_\ell$. Hence, $\sqrt{-2}\in\mathbb{Q}_\ell$. At $d=3$, the factor $\Phi_3/4$ excludes $F_4[-1]$ by [40, §13.9]. Glue over $\langle\gamma^f\rangle$ and $\langle F_2\rangle$. At even f, [208, proof of Proposition 9.11, Step 2] gives $\mathbb{Q}_\ell$-valued extensions of $F_4^I[1],F_4^{II}[1]$. By [40, p. 479], $\ell\mid F_4[-1](1)$ at $d=3$. At $d=2,6$, put $b=f/f_2$ and $m=2f_2$. By Lemma 7.291, $\beta=2^b(\sqrt{2}\,\alpha_0)^b\in\mathbb{Q}_\ell$ and $\beta^m=-q^3$. Thus, [208, proof of Proposition 9.11, Step 2] with this scalar gives the $\mathbb{Q}_\ell$-valued extension of $F_4[-1]$ on $G\rtimes\langle\gamma^b\rangle$. Moreover, [208, proof of Proposition 9.11, Step 2] gives the rational extension on $G\rtimes\langle\gamma^m\rangle$ before the twist by $-\alpha^m$. Therefore, the two restrictions give the $\mathbb{Q}_\ell$-valued extension on $G\rtimes\langle\gamma\rangle$. □

Corollary 7.298. *Let $S=F_4(q)$ for a prime power q. For its universal cover,* $\mathrm{iMN}(S,\ell)$ *holds for each $\ell\mid|S|$.*

Proof. If $q=2$, [157, Lemma 4.1(2)] gives $\mathrm{iMN}(S,\ell)$ for every ℓ. Suppose $q=p^f>2$. Then, S is its own cover. At $\ell=p$, [162, Theorem 1.3] and [101, Theorem 1.1] give $\mathrm{iMN}(S,\ell)$. If $\ell=2$ and q is odd, apply [158, Theorem A]. If $q=p$ and $\ell\mid q-1$ is odd, apply [166, Theorem 6.2 and the paragraph following Theorem A]. Otherwise, put $d=\mathrm{ord}_\ell(q)$. The degrees $2,6,8,12$ give $d\in\{1,2,3,4,6,8,12\}$. By Proposition 7.285, we obtain $\mathrm{iMN}(S,\ell)$ at $d=1,8,12$. At $d=3,6$, take signed A_2^2 from Section 7.1. Given $s\neq1$, centralizers are tori, type A_2, or cubic. In the first two, χ is uniform and real. Let E_χ be its outer inertia group. If $|E_\chi|$ is even, take its involution j with $j^2=F_q$ on $\mathbf{G}$. By [105, §2.2 and Theorem 3.1.3(ii)], χ occurs with multiplicity 1 in $\mathrm{Ind}_{S^j}^{S}1$, including the exceptional isogeny at $p=2$. Since E_χ centralizes j, take $H=S^j\rtimes E_\chi$. By [208, Lemma 3.13], the trivial character of H gives an extension of χ with values in $\mathbb{Q}(\chi)$. The A_2-label has the indicated $3f_3$-th root. A cubic weight on $T_{2'}$ extends trivially across its complement. If $p=2$, γ transports cyclic labels and partitions.

At $d=4$ the two negative 2-cycles give the mark. Proposition 7.289 treats $s\neq1$. At $p=2$ with even inertia, transfer B_2 by Lemma 7.287. At odd p, Lemma 7.288 extends the B_4-weight to marked inertia. If $d=2$ and odd q, apply Proposition 7.286. At $d=2$ and $p=2$, apply Lemma 7.290 for type A and Proposition 7.292 for other labels. At B_2, transfer by Lemma 7.287.

If $d\in\{2,3,4,6\}$, the remaining labels lie over 1_T on M. At odd q and $d=3,4,6$, apply [156, Theorem D]. If $p=2$, Lemma 7.293 gives the bijection with $\mathcal{H}_\ell$-fixed characters. At $d=2,3,6$, Propositions 7.296 and 7.297 give fixed extensions. If $d=4$, we apply Proposition 7.294. Each coefficient operation σ fixes base characters. Thus, an element (a,σ) of the joint stabilizer has a in ordinary inertia, with correcting functions 1. Combine the weight orbits on M. □

Lemma 7.299. *Let $G=G_2(q)$, $q=p^f$, and let $d=\mathrm{ord}_\ell(q)$ for an odd nondefining prime ℓ. Moreover, the fundamental degrees $2,6$ give $d\in\{1,2,3,6\}$. Suppose $d=1$, the Schur multiplier is generic, and $q\equiv1$ mod 9 if $\ell=3$. Then the split torus normalizer gives* $\mathrm{iMN}(G_2(q),\ell)$.

Proof. Put $N=\mathrm{N}_G(T)$ for split T. At $q=p>3$, [166, Theorem 6.2 and the paragraph following Theorem A] gives $\mathrm{iMN}(G,\ell)$ since $\mathrm{Out}(G)=1$. If $p=3$ and f is odd, [100, §3.1 and Theorem 4.5] gives $\mathrm{iMN}(G,\ell)$ on $N=\mathrm{N}_G(\mathbf{S}_1)$. For the remaining cases with $f>1$, [166, Theorem 6.2] gives the admissible N and the equivariant bijection $\Omega_\ell\colon\mathrm{Irr}_{\ell'}(G)\to\mathrm{Irr}_{\ell'}(N)$. Next, put $B=TU$. Then the pinned outer group E is generated by F_p at $p\neq3$ and by γ at $p=3$. The chosen n_i lie over $\mathbb{F}_p$, so F_p fixes them. Connectedness gives $W_\theta=W(R_s)$. Thus, at θ, apply the specializations at $\sqrt{q},1$ from Proposition 7.271. If $p=3$, γ exchanges the simple roots and satisfies $\gamma^2=F_3$ by [100, §3.1]. Both maps fix $1,-1$ and permute $\mathbb{F}_q^\times$. Thus, $\gamma(n_i)=n_{\iota(i)}$, and both $\theta|_{\alpha_i^\vee(\mathbb{F}_q^\times)}=1$ and $\theta(\alpha_i^\vee(-1))$ are transported with i. On $T[2]$,

cubing is trivial, so γ exchanges the simple roots there. At $\rho^\vee(-1)$, both simple-root values equal -1, so γ fixes it. Therefore, the sign in $K[N_\delta]$ satisfies $b_i(\theta) = b_{\iota(i)}(\theta^\gamma)$. The neutral maps are ι-equivariant. By [166, Corollary 3.3], $\mathcal{H}_\ell$ fixes $\sqrt{q}$ at $d = 1$. It also fixes the signs ± 1. Thus, ϕ is equivariant for both actions.

Put $K = \mathbb{Q}^{\mathrm{ab}}$, $e = |U|^{-1} \sum_{x \in U} x$. Given $D \leq E$, we choose $a[d] \mapsto \phi(a)[d]$ from $K[N \rtimes D]$ to $eK[G \rtimes D]e$. If D is the outer inertia of ψ, extend ψ to $N \rtimes D$ by [95, Corollary 11.22]. Transport it via ϕ and the idempotent. On G, this gives its principal-series partner χ. Moreover, the twist $\lambda \in \mathrm{Irr}(D)$ on each component $d' \in D$ is $\lambda(d')$ in both models. Since $Z(G) = 1$, the diagonal action is trivial on D. A Sylow correction by $n \in N$ acts on both extensions by n-conjugation. This gives the comparison on $\mathrm{Aut}(G)_P$ for a Sylow subgroup $P \leq N$. □

Proposition 7.300. *Suppose $d = 2$, the Schur multiplier is generic, and $q \equiv -1 \pmod 9$ if $\ell = 3$. Then $s \neq 1$ gives $\succeq_c$ on the negative scalar torus normalizer M.*

Proof. For $w_0 = -I$, let T be the corresponding torus and put $M = \mathrm{N}_G(T)$. By [70, Theorem 1.1 and Table 1, torus 4], $T \cong C_{q+1}^2$ and M has a complement $W \cong W(G_2)$. At $p = 3$ and odd f, the comparison on M follows from [100, Theorem 4.5]. Otherwise, put $R_s = \{\alpha : \alpha(s) = 1\}$ for $s \neq 1$. By [20, Theorem 5.1 and Table I], full rank gives A_1A_1 at $|s| = 2$ or A_2 at $|s| = 3$. In lower rank, R_s is A_1 or $\varnothing$. At $p = 3$, [100, Table 3.1] gives the rational types for $d = 2$. Thus, each unipotent Jordan label λ is uniform on its type-A centralizer, so χ is uniform. By [208, proof of Proposition 9.6], $\widehat{\chi}$ is rational on its inertia at $|s| = 2$ and at $|s| = 3$ with $q \equiv -1 \pmod 3$. For the remaining labels, w_0 inverts s, and opposition fixes the partitions. Thus, $\bar{\chi} = \chi$. If $|E_\chi|$ is odd, apply [143, Lemma 2.1]. At even inertia, $j = F_p^{f/2}$ if $p \neq 3$, and $j = \gamma^f$ if $p = 3$. Thus, $j^2 = F_q$ on $\mathbf{G}$, and $j|_G$ is the involution of E_χ. Since $\chi^j = \chi = \bar{\chi}$, the multiplicity in $\mathrm{Ind}_{G^j}^G 1$ is 1 by [105, Theorem 3.1.3(ii)]. Since E_χ centralizes j, take $H = G^j \rtimes E_\chi$. By [208, Lemma 3.13], the trivial character of H gives an extension of χ with values in $\mathbb{Q}(\chi)$. Pairs (θ_s, η) have $\eta \in \mathrm{Irr}(W(R_s))$, with the cuts in [156, Section 4.4]. At $-q \equiv 1 \bmod \ell$, the hook formula for A gives the degree cut, including $\ell = 3$ for generic M.

At $p = 3$, take Ω from [100, Proposition 3.2]. By [100, Lemma 3.1], its extension map is Λ. By [100, proof of Proposition 3.2], at $d = 2$, the relative Weyl characters are $\mathbb{Q}$-valued. For the lift on the θ-space, take $F' = \mathrm{Int}(n_0)F_q$ and ϕ from (7.7). The cocharacter $\rho^\vee$ is integral. At $p = 3$, the isogeny fixes $\rho^\vee(-1)$ and fixes the braid lift n_0. Thus, the model $K[T \rtimes W] \simeq K[M]$ from ϕ is equivariant for both actions. At $p \neq 3$, $(t, w) \mapsto \theta(t)$ defines the equivariant map Λ via ϕ. Since $Z(G) = 1$, [156, Theorem B] with this Λ gives Ω at $p \neq 3$. In both cases, $\mathcal{E}(G, 1)_{\ell'}$ corresponds to $\mathrm{Irr}_{\ell'}(M \mid 1_T)$.

The sign at n_0 is 1 on every weight. If $2 \nmid p$, reduce the coroot lattice modulo 2. The Coxeter square has polynomial $X^2 + X + 1$, cycling the 3 nonzero vectors. Moreover, both lengths cover $Q^\vee/2Q^\vee \setminus \{0\}$. A nonzero parity functional takes 1 twice per length, giving $i^4 = 1$. The zero functional and characteristic 2 give phase 1. Thus, the field lift U has $U^f = w_0$, with $g^{2f} = w_0$ for the exceptional lift when $p = 3$.

Choose rational factor models for W_s. Permute equal tensor factors. Next, for F_p, apply Proposition 7.273. If $p = 3$, we apply Lemma 7.290 with $g^{2f} = w_0$. On A_2, inversion is opposition times w_{0,A_2}, acting by 1 in the extended model. Each A_1 contributes its reflection sign. If g^j returns with $2 \nmid j$, its corrector a satisfies $\|a\alpha\| = \|\alpha\|$ for each root α. As g exchanges lengths, the A_1-cycles are even with constant signs ± 1. Their product is 1. Otherwise, ordinary inertia has order $r \mid f$. Since $q = (p^{f/r})^r$ and $d = 2$, the proof of [156, Lemma 8.1(a)] gives $2r_2 \mid \ell - 1$. If the sign is -1, take $\zeta \in \mu_{2r_2}$ with $\zeta^r = -1$. By [156, Lemma 2.3(a)], $\mathcal{H}_\ell$ fixes ζ. Thus, the mixed stabilizer in $E \times \mathcal{H}_\ell$ fixes the descended model. □

Proposition 7.301. *With $q = 3^f$, we let ℓ be an odd prime dividing $q + 1$. Moreover, on the generic cover G, the unipotent ℓ'-characters pair with those over 1_T on the negative torus normalizer M. The comparison includes full $\langle G, \gamma \rangle$-inertia.*

Proof. Since $\ell \mid 3^f + 1$ is odd, $\ell \geq 5$. At odd f, [100, Theorem 4.5] gives the full comparison. At even f, take the bijection of [100, Proposition 3.2] on $\mathcal{E}(G, 1)_{\ell'}$ and $\mathrm{Irr}_{\ell'}(M \mid 1_T)$. Let S, T_1 be the simple reflections of $I_2(6)$. Moreover, the lift g exchanges them and satisfies $g^{2f} = w_0 = (ST_1)^3 \bmod T$. A stable linear label agrees on S, T_1 and extends by $g \mapsto 1$. A moved label has field generator g^2. By [156, proof of Lemma 8.1(a)], $2f_2 \mid \ell - 1$. Give g^2 an f-th root of its w_0-sign in μ_{2f_2}, fixed by $\mathcal{H}_\ell$ by [156, Lemma 2.3(a)]. On each rational 2-dimensional reflection module, $J = S + T_1$ satisfies $JS = T_1J$, $JT_1 = SJ$. If the matrix ST_1 has order 3, $J^2 = 2 + ST_1 + T_1S = I$ and $w_0 = I$. Take J for g. If

ST_1 has order 6, $J^2 = 3I$ and $w_0 = -I$. Take $\alpha J/\sqrt{3}$ for g, with α from Lemma 7.291. Its $2f$-th power is $-I$ and its entries lie in $\mathbb{Q}_\ell$, since $\alpha/\sqrt{3} = (\sqrt{3}\,\alpha)/3$.

For split Hecke labels $\eta(1) = 2$, [72, Table 4.10] gives $\ell \mid \chi_\eta(1)$. For linear η, [208, proof of Proposition 9.3] gives rational extensions. If $\eta(S) = \eta(T_1)$, their inertia is $G \rtimes \langle\gamma\rangle$. Otherwise, it is $G \rtimes \langle F_3\rangle$. By [100, proof of Lemma 4.4], at $d = 2$, the cubic cuspidals are ineligible. On $G_2[-1]$, [208, proof of Proposition 9.4, (9.2)] gives the $\mathbb{Q}_\ell$-operator A for γ on the $(-q)$-eigenspace, with $A^{2f} = (-q)^{-(f+1)}I$. By Lemma 7.291, $(\sqrt{3}\,\alpha)^{2f} = -q$. Thus, $(\sqrt{3}\,\alpha)^{f+1}A$ gives an invariant extension. Given $G_2[1]$, [208, proof of Proposition 9.5] gives rational full-inertia extensions when $2 \mid f$. Also, both correcting functions are 1 on the joint stabilizer. □

Corollary 7.302. *Let q be a prime power, and let $S = G_2(q)$ be simple. For its universal cover,* $\mathrm{iMN}(S, \ell)$ *holds for each prime $\ell \mid |S|$.*

Proof. At $\ell = p$, [162, Theorem 1.3] and [101, Theorem 1.1] give $\mathrm{iMN}(S, \ell)$. At $q = 4$, [103, Theorem A(iii)] gives $\mathrm{iMN}(G_2(4), 3)$. By [157, Lemma 4.1(2)], $\mathrm{iMN}(G_2(4), \ell)$ holds at the other primes. If $\ell = 2$ and q is odd, apply [102, Theorem 8.3] at $q = 3^f$ and [158, Theorem A] at $3 \nmid q$. If $\ell = 3$ and $q \equiv 2, 4, 5, 7 \bmod 9$, apply [103, Theorem A(iii)]. At $q = p > 3$ and the remaining odd $\ell \mid q - 1$, apply [166, Theorem 6.2 and the paragraph following Theorem A]. If $q = 3^f$ with odd f, apply [100, Theorem 4.5]. At even f, [100, Proposition 3.2] pairs $\mathrm{Irr}_{\ell'}(S)$ equivariantly with $\mathrm{Irr}_{\ell'}(M)$ for the Sylow d-torus normalizer M. At $d = 1, 2$, the comparisons below give $\succeq_c$. In the remaining cases S is its own universal cover, and $d \in \{1, 2, 3, 6\}$ by the degrees $2, 6$. Lemma 7.299 treats $d = 1$, including $\ell = 3$ when $q \equiv 1 \bmod 9$. At $d = 3, 6$, $d \mid \ell - 1$ gives $\ell \nmid |W(G_2)| = 12$. Only degree 6 is divisible by d. By Lemma 2.11, $P \in \mathrm{Syl}_\ell(S)$ is cyclic and $\mathrm{N}_S(P[\ell]) = \mathrm{N}_S(P)$. By Corollary 2.10, $\mathrm{iMN}(S, \ell)$ holds.

If $d = 2$, let T be the negative torus and put $M = \mathrm{N}_S(T)$. By [132, Theorem 5.14], M contains $\mathrm{N}_S(P)$ for some $P \in \mathrm{Syl}_\ell(S)$. By Clifford theory [43], M-orbits on $\mathrm{Irr}(T)$ index the local sectors. If $p \neq 3$, [156, Theorem D] pairs the unipotent ℓ'-characters with $\mathrm{Irr}_{\ell'}(M \mid 1_T)$. If $p = 3$, apply Proposition 7.301 to 1_T. At $\theta \neq 1_T$, Proposition 7.300 gives the ℓ'-degree cuts and $\succeq_c$. The two sectors exhaust $\mathrm{Irr}_{\ell'}(M)$. By [45], $G_2(2)' \cong \mathrm{PSU}_3(3)$, treated in Section 3. □

7.9 Proof of Theorem 7.1

Proof. Let X be the universal cover of S. By [103, Theorem A(i)], Suzuki and Ree groups satisfy $\mathrm{iMN}(S, p)$. For $S = {}^2F_4(2)'$, [103, Proposition 5.11 of the arXiv version] gives $\mathrm{iMN}(S, p)$. At $p = \mathrm{char}\,\mathbb{F}_q$, apply [162, Theorem 1.3] and [101, Theorem 1.1]. At $p = 2$, [159, Theorem A(3)] treats $E_6(q), {}^2E_6(q)$. By [158, Theorem A], the condition holds for $F_4(q), E_7(q), E_8(q)$. The same theorem covers $G_2(q)$ at $3 \nmid q$, and [102, Theorem 8.3] covers $G_2(3^f)$. By [157, Lemma 4.1(2)], $\mathrm{iMN}(S, p)$ holds with exceptional $\mathrm{Mult}(S)$. At the remaining p, put $d = \mathrm{ord}_p(q)$. Let $\mathbf{S}_d$ be a Sylow d-torus. Except for $F_4(2^f)$ and $G_2(3^f)$, [156, Theorem D] gives $\succeq_c$ for the pairs on $\mathcal{E}(X, 1)_{p'}$ and its image in $\mathrm{Irr}(\mathrm{N}_X(\mathbf{S}_d))$ under the bijection of [132]. If $S = E_6(q)$ or ${}^2E_6(q)$, Corollary 7.114 gives $\mathrm{iMN}(S, p)$. Moreover, Corollary 7.257 and Theorem 7.284 cover $E_7(q)$ and $E_8(q)$, respectively. Finally, Corollaries 7.298 and 7.302 cover $F_4(q)$ and $G_2(q)$, including the exceptional isogenies in characteristic 2 or 3. □

8 Sporadic simple groups

Let S be a sporadic simple group and $p \mid |S|$. Write X for the universal cover of S and choose $R \in \mathrm{Syl}_p(X)$. Moreover, write $Z = \mathrm{Z}(X)$, $N = \mathrm{N}_X(R)$ and $\Gamma = \mathrm{Aut}(X)_R$. Schaeffer Fry [168, slide 22] attributes the 2018 verification of $\mathrm{iMN}(S, p)$ to C. Vallejo. At $p = 2$, Ruhstorfer–Schaeffer Fry [159, Proposition 2.1] gives $\mathrm{iMN}(S, 2)$. The inductive Isaacs–Navarro condition at $\mathcal{H}_0$ follows from Ruhstorfer–Schaeffer Fry [157, Lemma 4.2]. At odd p, the comparisons for $\mathcal{H}_p$ are on central-character fibres. Put $\bar{R} = RZ/Z$. Since R is characteristic in RZ, $N/Z = \mathrm{N}_S(\bar{R})$. By Turull [192, Theorem 3.1], pair $S, N/Z$. Inflate to Ω_{1_Z} on X, N. At $\lambda \in \mathrm{Irr}(Z)$, write $\Omega_\lambda : \mathrm{Irr}_{p'}(X \mid \lambda) \to \mathrm{Irr}_{p'}(N \mid \lambda)$. By Section 8.5, on $\Gamma \times \mathcal{H}_p$-orbits of λ, these maps give Ω.

For table calculations at odd p, write the group exponent as $m = ab$ with $a = m_p$. By Navarro [142, Theorem 2.1], the local action on μ_m is $(\mathrm{Z}/a\mathrm{Z})^\times \times \langle p \bmod b\rangle$. Choose u generating $(\mathrm{Z}/a\mathrm{Z})^\times$ and take $e \equiv (u, 1)$, $f \equiv (1, p)$ modulo (a, b). In each table, the powers e, f act on a row by $\chi(g) \mapsto \chi(g^e), \chi(g^f)$, respectively. Thus, $\chi(g) \in \mathbb{Q}_p$ if both powers fix χ.

Theorem 8.1. *Let S be a sporadic simple group. Then* $\mathrm{iMN}(S, p)$ *holds for every prime $p \mid |S|$.*

8.1 The Baby Monster cover

Let $S = B$ be the Baby Monster, $X = 2.B$, and z the central involution of X. By Conway et al. [45], $\mathrm{Out}(B) = 1$. A faithful $\chi \in \mathrm{Irr}(X)$ has $\chi(z) = -\chi(1)$. At $p = 7$, the Sylow normalizer in X has table `2.BN7` (Breuer [23]).

Proposition 8.2. *Let $V \in \mathrm{Syl}_7(X)$ and $U = \mathrm{N}_X(V)$. If $\chi \in \mathrm{Irr}(U)$ satisfies $\chi(z) = -\chi(1)$, then $\chi(g) \in \mathbb{Q}_7$ for every $g \in U$.*

Proof. By [23, `2.BN7`], $\exp(U) = 168$ and the 27 faithful rows satisfy $\chi(g^{73}) = \chi(g^{127}) = \chi(g)$. The powers $73, 127$ give $(3,1),(1,7)$ modulo $(7,24)$. The image of $\mathcal{H}_p$ on μ_{168} is generated by $73, 127$ by Section 8. Thus, $\chi(g) \in \mathbb{Q}_7$. □

Lemma 8.3. *Let S be a sporadic simple group with $\mathrm{Out}(S) = 1$, and let X be its universal cover. Suppose that $Z = \mathrm{Z}(X) = \langle z \rangle$ has order 2, and let p be an odd prime dividing $|S|$. Choose $R \in \mathrm{Syl}_p(X)$, and put $N = \mathrm{N}_X(R)$. Suppose that $\chi(g) \in \mathbb{Q}_p$ for every $\chi \in \mathrm{Irr}_{p'}(X) \cup \mathrm{Irr}_{p'}(N)$ with $\chi(z) = -\chi(1)$, and every g in the domain of χ. Then, $\mathrm{iMN}(S,p)$ holds.*

Proof. By Section 8, $N/Z = \mathrm{N}_S(RZ/Z)$ and inflation gives an $\mathcal{H}_p$-equivariant bijection on the 1_Z-fibres. Write λ for the character of Z with $\lambda(z) = -1$. The central-character part of [35, Theorem B] gives $|\mathrm{Irr}_{p'}(X \mid \lambda)| = |\mathrm{Irr}_{p'}(N \mid \lambda)|$. Since $\chi(g) \in \mathbb{Q}_p$ by hypothesis, each χ in either λ-fibre is $\mathcal{H}_p$-fixed. Consequently, any bijection $F_\lambda : \mathrm{Irr}_{p'}(X \mid \lambda) \to \mathrm{Irr}_{p'}(N \mid \lambda)$ is $\mathcal{H}_p$-equivariant. Together, the two maps give $\Omega : \mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(N)$ over each $\nu \in \mathrm{Irr}(Z)$. Since R is characteristic in N, $\mathrm{N}_X(N) = N$ and $\mathrm{Aut}(X)_N = \Gamma$. Sylow conjugacy gives intravariance of N. Since $\mathrm{Aut}(X) = \mathrm{Inn}(X)$, Γ acts through N on X, N. Thus, for $\psi = \Omega(\chi)$, $(\Gamma \times \mathcal{H}_p)_\chi = (\Gamma \times \mathcal{H}_p)_\psi$ by equivariance. Since χ, ψ lie over ν, [186, Theorem 2.6(b)] gives $\succeq_c$ and hence $\mathrm{iMN}(S,p)$. □

Lemma 8.4. *Let $X = 2.B$, $Z(X) = \langle z \rangle$ and $p \in \{3,5,7\}$. If $\chi \in \mathrm{Irr}_{p'}(X)$ is faithful and $g \in X$, we find $\chi(g) \in \mathbb{Q}_p$.*

Proof. From $\mathrm{CT}(X)$ we compute [23] $\exp(X) = m = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$. Then, Section 8 gives the powers e_p, f_p, with $(a_3, a_5, a_7) = (27, 25, 7)$ and $(u_3, u_5, u_7) = (2, 2, 3)$. Write $n_p = |\{\chi \in \mathrm{Irr}_{p'}(X) : \chi(z) = -\chi(1)\}|$. By [23, `2.B`], $(n_3, n_5, n_7) = (27, 25, 27)$. On these rows the table gives $\chi(g^{e_p}) = \chi(g^{f_p}) = \chi(g)$. Thus, $\chi(g) \in \mathbb{Q}_p$. □

Lemma 8.5. *Let T be a finite group. Suppose that $\mathrm{CT}(T) = \mathrm{CT}(\mathrm{N}_{\mathrm{Ly}}(Q))$ for $Q \in \mathrm{Syl}_5(\mathrm{Ly})$ (Breuer [23]). Then, we find $\chi(g) \in \mathbb{Q}_5$ for every $\chi \in \mathrm{Irr}_{5'}(T)$ and $g \in T$.*

Proof. By [23, `LyN5`], $N_{\mathrm{Ly}}(Q)$ has exponent 100. Its $5'$-degree rows have degrees $1, 4, 16$ with multiplicities $16, 8, 1$, respectively. The table gives $\psi(h^{77}) = \psi(h)$ on each of these rows. Since $77 \equiv (2,1) \pmod{(25,4)}$, the power 77 generates the local action on μ_{100} by Section 8. Thus, in $\mathrm{CT}(T)$, the corresponding entries lie in $\mathbb{Q}_5$. □

Proposition 8.6. *The condition $\mathrm{iMN}(B,p)$ holds at each $p \mid |B|$.*

Proof. Given $P \in \mathrm{Syl}_p(X)$, put $U = \mathrm{N}_X(P)$ and $N = \mathrm{N}_B(P\langle z \rangle / \langle z \rangle)$. At $p = 3$, choose P with $\bar{P} = P\langle z \rangle / \langle z \rangle$ in Fi_{23}. By [23, `Fi23N3`], $\mathrm{N}_{\mathrm{Fi}_{23}}(\bar{P})$ and N have the same table. Thus, $N = \mathrm{N}_{\mathrm{Fi}_{23}}(\bar{P})$. Since Fi_{23} has split preimage by [3, §3], $U = \langle z \rangle \times N$. The $3'$-degree rows of $\mathrm{CT}(N)$ are integral [23]. Thus, on U, the faithful $3'$-degree rows are integral. At $p = 5$, [23, `2xBN5`] gives $\mathrm{CT}(U) = \mathrm{CT}(C_2) \otimes \mathrm{CT}(N)$. The alias `BN5` denotes `LyN5`. The 25 faithful rows of $5'$-degree have the form $\varepsilon \otimes \psi$, where ε is the nontrivial character of C_2. By Lemma 8.5, $\psi(n) \in \mathbb{Q}_5$ for $n \in N$. Thus, $(\varepsilon \otimes \psi)(u) \in \mathbb{Q}_5$ for $u \in U$. If $p = 7$, Proposition 8.2 gives local values in $\mathbb{Q}_7$. By Lemma 8.4, $\mathbb{Q}_p(\chi) = \mathbb{Q}_p$ for faithful $\chi \in \mathrm{Irr}_{p'}(X)$ and $p \in \{3,5,7\}$. By Lemma 8.3, we obtain $\mathrm{iMN}(B,p)$ for $p \in \{3,5,7\}$. At every other odd $p \mid |B|$, a Sylow p-subgroup of B has order p [45]. By Corollary 2.10, we obtain $\mathrm{iMN}(B,p)$. At $p = 2$, [159, Proposition 2.1] gives $\mathrm{iMN}(B,2)$. □

8.2 Higman–Sims, Mathieu and Janko

Let X cover HS or M_{22}. Suppose that $X \lhd Y$ and $[Y : X] = 2$. Choose $P \in \mathrm{Syl}_p(X)$ and put $N = \mathrm{N}_X(P)$, $U = \mathrm{N}_Y(P)$. By Isaacs [97, Lemma 1.13], we obtain that $Y = XU$, so $U/N \cong Y/X$. Write c for the action on $\mathrm{Irr}(X)$ and $\mathrm{Irr}(N)$ of a representative $u \in U \setminus N$. Let $\varepsilon \neq 1$ be the character of Y/X and its restriction to U. By Proposition 2.13, we compare each ν-fibre. Given $Z \leq \mathrm{Z}(X)$ with $Z \lhd Y$, the preimage of PZ/Z is $PZ = P \times Z_{p'}$. Since P is the unique Sylow p-subgroup of PZ, we have $\mathrm{N}_{X/Z}(PZ/Z) = N/Z$ and $\mathrm{N}_{Y/Z}(PZ/Z) = U/Z$. If $Z \leq \ker\chi \cap \ker\psi$, the models come from $Y/Z, U/Z$ or $X/Z, N/Z$. Thus, Proposition 2.13 gives $\succeq_c$ over 1_Z.

Proposition 8.7. *Let $X = M_{22}$ or* HS*, $Y = \mathrm{Aut}(X)$, and $p = 3$. The principal blocks $B_0(X), B_0(N)$ admit both bijections of* Proposition 2.13*: $\chi \leftrightarrow \psi$ and $\widehat{\chi} \leftrightarrow \widehat{\psi}$. If $X = M_{22}$, then these exhaust the 1_Z-sector.*

Proof. In both cases $P \cong C_3^2$. By [49, §5.1.2 and Proposition 5.3], for $X = M_{22}$, the automizers in X, Y are Q_8, SD_{16}, respectively. For $1 \neq u \in P$, the centralizers in X, Y are $C_3 \times A_4, C_3 \times S_4$. Since $C_Y(P) = P$, Schur–Zassenhaus gives $N = P \rtimes Q_8$ and $U = P \rtimes \mathrm{SD}_{16}$. If $X =$ HS, [49, §5.6.4.3] gives $N = C_2 \times (P \rtimes \mathrm{SD}_{16})$. Its principal block lies over 1_{C_2}.

Now, let $E \in \{Q_8, \mathrm{SD}_{16}\}$. The characters of $P \rtimes E$ are $\mathcal{H}_3$-fixed. Indeed, $\mathrm{Irr}(Q_8)$ is rational-valued. In SD_{16}, conjugation on C_8 acts by $x \mapsto x^3$, the Frobenius power on μ_8. Thus, the irreducibles $\mathrm{Ind}_{C_8}^{E}(\theta)$ are fixed, and each invariant $\theta \in \mathrm{Irr}(C_8)$ has rational extensions. Moreover, Q_8 acts regularly on $\mathrm{Irr}(P) \setminus \{1_P\}$, since its unique involution acts by $-I$. Hence, each nonzero weight λ satisfies $|E_\lambda| \leq 2$. Extend by $\widehat{\lambda}(xe) = \lambda(x)$ on $P \rtimes E_\lambda$. By Clifford theory, over λ^E these are $\mathrm{Ind}_{P \rtimes E_\lambda}^{P \rtimes E}(\widehat{\lambda}\,\eta)$, with $\eta \in \mathrm{Irr}(E_\lambda)$. Each $\sigma \in \mathcal{H}_3$ gives $\lambda^\sigma \in \{\lambda, \lambda^{-1}\}$. Conjugation by $-I$ gives inversion and fixes η.

For $M_{22}, M_{22}.2,$ HS, [49, §§5.2, 5.4.3, 5.6.3–5.6.4] give splendid equivalences over $\mathbb{F}_3$. By [49, Theorem 3.5], these lift over $\mathbf{Z}_3$. If $L/\mathbb{Q}_3$ is cyclotomic and splits X, Y, N, U, the bijections are $\mathrm{Gal}(L/\mathbb{Q}_3)$-equivariant by [107, Theorem 1.3]. Thus, $\mathcal{H}_3$ fixes the corresponding $\chi \in \mathrm{Irr}(B_0(X)) \cup \mathrm{Irr}(B_0(Y))$. Since the defect group P is abelian, height zero gives $3 \nmid \chi(1)$ for those χ.

For the Mathieu normalizers N, U, we have $\mathbb{F}_3 N = B_0(N)$ and $\mathbb{F}_3 U = B_0(U)$ by [49, §5.2]. Thus, in X, Y, there is one block of defect P by Brauer correspondence. Write a, b for the lower and upper $3'$-degree counts and f for the number of c-fixed lower characters. By Clifford theory, $b = 2f + (a - f)/2$. The two equivalences give equal a, b for (X, Y) and (N, U), so $f = (2b - a)/3$ coincides. Match $\{\widehat{\chi}, \widehat{\chi}\varepsilon\}$ over each fixed parent and $\{\chi, \chi^c\}$ over each moving orbit. The group $\mathcal{H}_3$ fixes both sets pointwise.

If $X =$ HS, the upper automizer A_Y has $3 \nmid |A_Y|$ and contains $\mathrm{SD}_{16} \in \mathrm{Syl}_2(\mathrm{GL}_2(3))$. Therefore, it equals the lower automizer SD_{16}. Thus, $U = N\mathrm{C}_Y(P)$ and $Y = X\mathrm{C}_Y(P)$. We have restriction bijections $\mathrm{Irr}(B_0(Y)) \to \mathrm{Irr}(B_0(X))$ and $\mathrm{Irr}(B_0(U)) \to \mathrm{Irr}(B_0(N))$ by [1]. Thus, χ has a unique principal extension $\widehat{\chi}$. By uniqueness, $\widehat{\chi}$ and $\widehat{\chi}\,\varepsilon$ are $\mathcal{H}_3$-fixed. Match $\widehat{\chi} \leftrightarrow \widehat{\psi}$ and $\widehat{\chi}\varepsilon \leftrightarrow \widehat{\psi}\varepsilon$, then inflate over 1_Z. □

Proposition 8.8. *With $X =$ HS, we put $Y = \mathrm{Aut}(X)$ and $p = 3$. Let A, B be the nonprincipal blocks of X, N with defect P. Next, we put $K = N \times X$ and $D = U \times_{Y/X} Y$. There is a virtual character Θ of D over $\mathbb{Q}_3$ with $\Theta|_K = \sum_{\chi \in \mathrm{Irr}(A)} e_\chi\, \psi_\chi \boxtimes \chi^\vee$. Moreover, we have that $e_\chi = \pm 1$ and $\chi \mapsto \psi_\chi$ gives a bijection $\mathrm{Irr}(A) \to \mathrm{Irr}(B)$. On extensions it gives $\widehat{\chi} \leftrightarrow \widehat{\psi}$ as in* Proposition 2.13*.*

Proof. Over $k = \mathbb{F}_3$, we write A, B for the unique blocks of defect P other than $B_0(X), B_0(N)$. By [110, §§5.4, 5.7], ${}_BV_A$ gives a stable Morita equivalence. Since $\mathrm{N}_{X \times X}(\Delta P) \leq K$, the restriction of the block summand A to K contains V with multiplicity 1. The group D acts on kX by $(u, y) \cdot x = uxy^{-1}$. Moreover, the summand A is D-stable, and V is fixed up to isomorphism by D and residue Frobenius. In $E/\operatorname{rad} E$ for $E = \mathrm{End}_{kK}(A)$, the central idempotent $\bar{e}$ for V descends to k by finite-field descent. Since $|D/K| = 2$, averaging gives $E^{D/K} \twoheadrightarrow (E/\operatorname{rad} E)^{D/K}$ with nilpotent kernel. Thus, $\bar{e}$ lifts to an idempotent e_D, and $V_D = e_D A$ restricts to V. On the permutation lattice $\mathbf{Z}_3 X$, the orbit matrices on $X \times X$ lift the D-endomorphisms of kX. Hence, over $\mathbf{Z}_3$ the idempotent lifts with image $\widetilde{V}_D$.

Let I be the two dual simple A-modules of dimension 49, exchanged by c. Next, for a simple S, we write $C_S = V \otimes_A S$. The perverse equivalence F above the Green equivalence gives $F(S) = C_S[0]$ for $S \notin I$, with C_S simple. If $S \in I$, it gives $[P_B(T_S) \to C_S]$, where T_S is the simple head of C_S and $P_B(T_S)$ its projective cover. The heads T_S are distinct and split over k, since SD_{16} is split over k. Green correspondence is Frobenius-equivariant on $S \mapsto T_S$, so S splits over k as well. See [49, §§3.2.1, 5.6.4.3].

Put $Q = \bigoplus_{S \in I} P_B(T_S) \otimes_k P_A(S)^*$, with right projective $P_A(S)^*$. Next, we choose $d : Q \to V$ such that $d \otimes_A 1_S$ gives the cover map on C_S for $S \in I$. Indeed, for $P_A(S) = Ae_S$, symmetry gives $P_A(S)^* = e_S A$. Thus, on this summand choose $P_B(T_S) \to Ve_S$. Lift the projective-cover map through $Ve_S \twoheadrightarrow V \otimes_A S = C_S$. Given a different simple S', we have $e_S A \otimes_A S' = 0$.

Next, we put $C = [Q \xrightarrow{d} V]$. Next, let $L = F^{-1} \circ (C \otimes_A^{\mathbf{L}} -)$. By the calculation on simple modules, $L(S) \cong S[0]$ for each S. Moreover, L induces a stable equivalence since Q is bimodule-projective and V induces one. The isomorphisms $L(S) \cong S[0]$ give t-exactness by induction on length. Since Q, V are projective on both sides, C^* gives both adjoints of $C \otimes_A^{\mathbf{L}} -$ by symmetry. Consequently, R is left and right adjoint to L. Hence, R is t-exact. On projectives J, both

$L(J), R(J)$ are projective, so adjunction descends to $\underline{A}$-mod. For each simple S, the unit $S \to RL(S)$ is a stable isomorphism. Since S is nonprojective, Krull–Schmidt gives $RL(S) \cong S \oplus J$ for projective J. Moreover, for each simple T, adjunction gives $\mathrm{Hom}(T, RL(S)) \cong \mathrm{Hom}(L(T), L(S)) \cong \mathrm{Hom}(T, S)$. Thus, the socle is S, so $J = 0$. Thus, the unit is invertible on S. Hence, it is invertible on $D^b(A)$. The images of the simples generate $D^b(A)$, so L is essentially surjective. Consequently, C induces a derived equivalence.

Lift Q to a projective $\mathbf{Z}_3K$-lattice $\widetilde{Q}$. Projectivity gives a lift $\widetilde{d} : \widetilde{Q} \to \widetilde{V}_D|_K$. Moreover, the lifts $\widetilde{Q}, \widetilde{V}_D|_K$ are projective on both sides. Write C_u, C_v for the adjunction cones on $\widetilde{A}, \widetilde{B}$. Both are bounded and $\mathbf{Z}_3$-free, acyclic modulo 3. Thus, Nakayama's lemma gives acyclicity over $\mathbf{Z}_3$ and the splitting field L. Over L, semisimplicity gives the Euler character $\sum_\chi e_\chi \, \psi_\chi \boxtimes \chi^\vee$ on K. The outer coset exchanges $\widetilde{Q}_1, \widetilde{Q}_2$ in $\widetilde{Q}$. If $\widetilde{Q}_1$ is one summand, $\mathrm{Ind}_K^D \widetilde{Q}_1$ restricts to $\widetilde{Q}$. Hence, set $\Theta = \mathrm{ch}(\widetilde{V}_D) - \mathrm{ch}(\mathrm{Ind}_K^D \widetilde{Q}_1)$.

The matching from $\Theta|_K$ is equivariant for $c, \mathcal{H}_3$. Next, for a c-fixed pair, we choose extensions η' to U and η to Y. On D the two extensions of $\psi_\chi \boxtimes \chi^\vee$ are $(\eta' \boxtimes \eta^\vee)|_D$ and its twist by ε_D. Their coefficients m_+, m_- in Θ satisfy $m_+ + m_- = e_\chi = \pm 1$, so $m_+ \neq m_-$. If $\sigma \in \mathcal{H}_3$ fixes the base pair, then σ fixes both extensions on D. Thus, the twists on η', η have equal signs. Extend along $\mathcal{H}_3$-orbits of the pairs (η', η) by Lemma 2.14. The moving pairs (χ, ψ_χ) come from $\Theta|_K$. Since P is abelian, height zero gives $3 \nmid \chi(1)$. With Proposition 8.7, this gives $\succeq_c$ over 1_Z at 3. □

Proposition 8.9. *With $X = 2.\mathrm{HS}$, we put $p = 3$. Next, we set $Y = X \rtimes \langle c \rangle$ for the lifted outer involution c of* HS*. Let A, B be the faithful blocks of defect P in X, N, respectively. With $D = U \times_{Y/X} Y$, we have that the virtual character Θ gives both bijections of* Proposition 8.8 *for A, B. Thus, both central characters have $\succeq_c$ on N.*

Proof. Conjugation gives $Y/\mathrm{Z}(X) \cong \mathrm{Aut}(X)$. By [111, §4], the faithful blocks A, B of defect P are unique. Thus, conjugation fixes A, B, as does residue Frobenius. Over $\overline{\mathbb{F}}_3$, the Green summand V of vertex ΔP has multiplicity 1. By [111, Lemma 5.7(i)], V gives a stable Morita equivalence. On $\mathbb{F}_3 X$, the idempotent gives V_D and the lift $\widetilde{V}_D$ over $\mathbf{Z}_3 D$.

By [111, Lemmas 6.5–6.7], exactly one simple S has nonsimple $L = V \otimes_A S$. Its Loewy series is T, U_1, T with $U_1 \neq T$. Moreover, all other images $T' \neq T$ are distinct simples. By [111, Lemma 4.6(v)], $T = 2\gamma$ has three entries 1 in its decomposition column. Hence, $c_{TT} = 3$. Thus, residue Frobenius fixes S, T. Simultaneous conjugation also fixes S, T, which descend absolutely irreducibly to $\mathbb{F}_3$.

Over $k = \mathbb{F}_3$, choose a projective cover P_D of V_D and restrict it to $K = N \times X$. It remains a cover since $\mathrm{rad}(kD) = \mathrm{rad}(kK)kD$. Indeed, $I_D = \mathrm{rad}(kK)kD$ is nilpotent. The quotient kD/I_D is a crossed product of $kK/\mathrm{rad}(kK)$ by D/K of order 2, invertible in k. Thus, kD/I_D is semisimple over k and $\overline{k}$. Tensor–Hom adjunction gives multiplicity $\dim \mathrm{Hom}_B(L, T) = 1$ for $T \boxtimes S^*$ in the head of V. The cover of $T \boxtimes S^*$ is defined over k and fixed by D/K. By Proposition 8.8, lift the averaged idempotent $\bar{e}$ in $\mathrm{End}_{kK}(P_D)$. The image Q_D is a D-summand and $Q = Q_D|_K$ has type $P_B(T) \otimes P_A(S)^*$ with multiplicity 1 over $\overline{k}$.

The cover map gives $d : Q \to V$. Thus, $d \otimes_A 1_S$ still gives a surjection onto L. Only the summand over T survives, since L has head T. Thus, $d \otimes_A S$ is the cover $P_B(T) \twoheadrightarrow L$. Given any other simple S', we have $Q \otimes_A S' = 0$. Next, let $J = \ker(P_B(T) \to L)$. The kernel contains the essential socle T, and $[J : T] = c_{TT} - [L : T] = 1$. Thus, J is the largest submodule containing T with multiplicity 1. Indeed, every nonzero image in L contains T, while the socle of $P_B(T)$ maps to zero. For perversity 1 at T and 0 elsewhere, the canonical complex at T is $[P_B(T) \to L]$, and the other complexes are $T'[0]$. In the stable category these give $V \otimes_A S'$. The perverse lift F follows from [49, §§3.1.2, 3.2.1]. Put $L' = F^{-1} \circ ([Q \to V] \otimes^{\mathbf{L}}_A -)$. By the biadjoint argument of Proposition 8.8, L' is an equivalence. Thus, $[Q \to V]$ gives a derived equivalence.

Lift Q_D and its map over $\mathbf{Z}_3 D$ by projectivity. The modular cones $\overline{C}_u, \overline{C}_v$ are acyclic over k by faithful scalar extension to $\overline{k}$. Hence, the regular-block cones C_u, C_v lift acyclically. Thus, $\Theta = \mathrm{ch}(\widetilde{V}_D) - \mathrm{ch}(\widetilde{Q}_D)$ restricts to a signed permutation on K. The diagonal central involution acts by $(-1)(-1) = 1$. On fixed pairs, we have $m_+ + m_- = \pm 1$. Moreover, on moving pairs take $\Theta|_K$. By Proposition 8.8, we obtain both bijections $\Omega, \widehat{\Omega}$. If $p = 3$, Propositions 8.7–8.9 treat every block of defect P. Thus, height zero gives all of $\mathrm{Irr}_{3'}(X)$ and $\mathrm{Irr}_{3'}(N)$. □

Lemma 8.10. *Let $X = 2.\mathrm{HS}$, $p = 5$, and $\mathrm{Z}(X) = \langle z \rangle$. Then, we have that $\mathcal{H}_5$ fixes every $\psi \in \mathrm{Irr}_{5'}(N)$ for both central characters of $\langle z \rangle$.*

Proof. Put $\bar{P} = P\langle z \rangle / \langle z \rangle$. The Sylow normalizer $\mathrm{N}_{\mathrm{HS}}(\bar{P})$ lies in $\mathrm{PSU}_3(5).2$ and has order $5^3 \cdot 16$ [45, pp. 34, 80] [23]. Since $\mathrm{Syl}_5(P\langle z \rangle) = \{P\}$, we have $N/\langle z \rangle = \mathrm{N}_{\mathrm{HS}}(\bar{P})$. The perfect subgroup $J = \mathrm{PSU}_3(5)$ has multiplier of order 3, so

its preimage splits uniquely. In $\mathrm{Out}(J) \cong S_3$, the involutions are conjugate. Thus, choose $x \mapsto x^5$ on entries over $\mathbb{F}_{25}$ for the outer involution. Its lift s has $s^2 \in \langle z \rangle$.

On upper unitriangular P, the quotient $W = P/P'$ is $(\mathbb{F}_{25}, +)$. Moreover, the element $\mathrm{diag}(t, t^4, t^{-5})$ acts by t^{-3} on the $(1,2)$-coordinate. Thus, the projective torus $\langle h \rangle$ acts by scalars of order 8. We have $N = P \rtimes E$ with $|E| = 32$ and $E = \langle h, z, s \rangle$. Moreover, $|h| = 8$, $|z| = 2$, $shs^{-1} = h^5$ and $s^2 \in \langle z \rangle$. Hence, $\langle h^2 \rangle \leq \mathrm{Z}(E)$ acts on W by $\mathbb{F}_5^\times$. By Clifford theory, each $\psi \in \mathrm{Irr}_{5'}(N)$ comes from $\lambda \in \mathrm{Irr}(W)$. Extend it by $\widehat{\lambda}(xe) = \lambda(x)$ on $W \rtimes E_\lambda$. The characters are the inflations of $\mathrm{Ind}_{W \rtimes E_\lambda}^{W \rtimes E}(\widehat{\lambda}\,\tau)$, with $\tau \in \mathrm{Irr}(E_\lambda)$, up to E-conjugacy. If $\lambda \neq 1_W$, we have that $|\langle h \rangle \cdot \lambda| = 8$, so $|E_\lambda| \leq 4$. Thus, τ has values in $\mathbb{Q}_5$. Every $\sigma \in \mathcal{H}_5$ changes λ by a scalar in $\mathbb{F}_5^\times$, from the action of $\langle h^2 \rangle \leq \mathrm{Z}(E)$. Hence, σ fixes the induced character ψ. If $\lambda = 1_W$, set $E_0 = \langle h, z \rangle$ of index 2 in E. On $\mathrm{Irr}(E_0)$, the action of $\mathcal{H}_5$ is generated by $\theta \mapsto \theta^5 = \theta^s$. Thus, for moving $\theta \in \mathrm{Irr}(E_0)$, the character $\mathrm{Ind}_{E_0}^E \theta$ is fixed. For invariant θ, we have $\theta(h)^4 = 1$ and $\theta(z)^2 = 1$. The extensions satisfy $\widehat{\theta}(s)^2 = \theta(s^2) \in \{1, -1\}$. Thus, $\widehat{\theta}(h), \widehat{\theta}(z), \widehat{\theta}(s) \in \mu_4 \subseteq \mathbb{Q}_5$. □

Proposition 8.11. *Let $X =$ 2.HS, $Y =$ 2.HS.2 and $p = 5$, with Y as in Suleiman–Wilson. The group $\mathcal{H}_5$ fixes $\mathrm{Irr}_{5'}(X)$ pointwise. Moreover, the moving parents χ, ψ of X, N admit a matching equivariant for $c, \mathcal{H}_5$ over each $\nu \in \mathrm{Irr}(\langle z \rangle)$. If a faithful $\chi \in \mathrm{Irr}_{5'}(X)$ extends to $\eta \in \mathrm{Irr}(Y)$, then $\eta(y) = 0$ for 5-regular $y \in Y \setminus X$.*

Proof. If $z \in \ker \chi$, the bijection Ω_{1_z} of Section 8 and Lemma 8.10 give $\chi^\sigma = \chi$ for $\sigma \in \mathcal{H}_5$. For faithful χ, write $d(\chi)$ for restriction to the 5-regular elements. The faithful Brauer characters φ have values in $\mathbb{Q}(i, \sqrt{-11})$. By Suleiman–Wilson, Theorem 3 and Lemmas 5–10, $d(\chi)$ is distinct except at degree 2520 (Tables X–XI). The row χ_0 of defect zero has degree 1000. Thus, faithful $\chi \in \mathrm{Irr}_{5'}(X)$ have singleton fibres under d.

Since $-1 \equiv -11 \equiv 4 \pmod 5$, [156, Lemma 2.2] gives $i, \sqrt{-11} \in \mathbb{Q}_5$. Given $\sigma \in \mathcal{H}_5$, we have $d(\chi^\sigma) = d(\chi)^\sigma = d(\chi)$. Thus, $\chi^\sigma = \chi$, and the ordinary McKay correspondence is $\mathcal{H}_5$-equivariant on the moving c-orbits over each ν. Equivalently, count on X, Y and their quotients by $\langle z \rangle$ by ordinary McKay [35]. Thus, $a_- = a_X - a_{X/\langle z \rangle}$ and $b_- = b_Y - b_{Y/\langle z \rangle}$. The index-two count gives $f_\nu = (2b_\nu - a_\nu)/3$ on each fibre ν. Thus, both sides have $(a_\nu - f_\nu)/2$ free orbits. Since $\mathcal{H}_5$ fixes every parent, the c-orbit pairing is equivariant on each ν-fibre.

The only c-invariant faithful φ has degree 1000 and defect 0 by Suleiman–Wilson, p. 241. Hence, φ is absent from $d(\chi)$, in the full-defect block of χ. Since $d(\eta)|_X = d(\chi)$, each constituent γ of $d(\eta)$ lies over a moving constituent of $d(\chi)$. By modular Clifford theory, $\gamma = \mathrm{Ind}_X^Y(\gamma_0)$ for some Brauer constituent γ_0. Consequently, $d(\eta)(y) = 0$ for 5-regular $y \in Y \setminus X$. □

Theorem 8.12. *Keep X, Y, N, U from* Proposition 8.11. *Next, we write α for the action of $\mathcal{H}_5$ on $\sqrt{5}$ and β for the unramified quadratic character. On Y, U, all extension pairs over $1_{\langle z \rangle}$ have sign* 1. *Moreover, over the faithful character of $\langle z \rangle$, the signs are $\alpha, \alpha\beta, \alpha\beta$. Thus, the matchings on X, N satisfy* the $\succeq_c$ relation *on both central fibres.*

Proof. Write I for coefficient inertia. Next, we let F fix the 5-power roots and act by $\zeta \mapsto \zeta^5$ on roots of order prime to 5. The images of I, F generate the finite decomposition group. Thus, $\alpha(F) = 1$, $\beta(F) = -1$, and $\beta|_I = 1$. The group P is extraspecial of exponent 5, with $W = P/P' = P/\Phi(P)$. By Lemma 8.10, $N = P \rtimes E$ with $|E| = 32$ and $z \in E$. In $\mathrm{Syl}_2(U)$, we choose L with $E \leq L$. Since $[U : N] = 2$, we have $|L| = 64$, $L \cap N = E$, and $U = P \rtimes L$. Put $K = L/\langle z \rangle$. Moreover, its subgroup $E/\langle z \rangle$ acts faithfully on W. A nontrivial kernel of $K \to \mathrm{GL}(W)$ would have order 2 and contain an exterior involution. Since $\ker(\mathrm{Aut}(P) \to \mathrm{Aut}(W))$ is a 5-group, this involution would centralize P in HS.2. Every involution t there has $|\mathrm{C}_{\mathrm{HS.2}}(t)|_5 \leq 5$ [45, pp. 79–81] [23]. Hence, $K \in \mathrm{Syl}_2(\mathrm{GL}_2(5))$.

Take $K = (\mathbb{F}_5^\times \times \mathbb{F}_5^\times) \rtimes \langle w \rangle$, with coordinate swap w. The element h of order 8 is off diagonal, and $h^2 = c_0 I_2$ for a generator c_0 of $\mathbb{F}_5^\times$. If s were off diagonal, hs would be diagonal, but $(hs)^2 = h^6$ has order 4. Thus, s is diagonal. We choose coordinates with $s = \mathrm{diag}(1, -1)$, $h = \left(\begin{smallmatrix} 0 & c_0 \\ 1 & 0 \end{smallmatrix}\right)$, $d = \mathrm{diag}(1, c_0)$. Next, we write h, s, d for lifts to L, with h, s as in Lemma 8.10. The preimage of $\mathrm{PSU}_3(5).2$ in X is the fibre product $(\mathrm{PSU}_3(5).2) \times_{C_2} C_4$ over the outer quotient. Thus, $s^2 = z$. Since $d^2 = s \bmod \langle z \rangle$, we have $d^4 = z$. Moreover, $dhd^{-1} = h^3 s \bmod \langle z \rangle$, so $dh^2 d^{-1} = (h^3 s)^2 = h^2 z$. Hence, h^2 centralizes E and conjugates $x \in L \setminus E$ to xz. See the Character Table Library, 2.HS, subgroups of index 176. At zero weight, the characters over $1_{\langle z \rangle}$ are the characters of K. Moreover, the diagonal subgroup K_0 has exponent 4, and each w-fixed $\theta \in \mathrm{Irr}(K_0)$ extends with $\widehat{\theta}(w) = \pm 1$. Thus, both $\widehat{\theta}$ and $\mathrm{Ind}_{K_0}^K \theta$ are $\mathbb{Q}_5$-valued. At zero weight and faithful $\chi \in \mathrm{Irr}(E)$, the scalar at h^2 changes by -1 under d. Thus, no such parent extends to L.

The nonzero weights have two K-orbits. Specifically, the axis representative $(1, 0)$ has stabilizer $\langle d \rangle$, and $(1, 1)$ has stabilizer $\langle w \rangle$. The subgroup $E/\langle z \rangle$ is transitive on both. Indeed, $\langle h \rangle$ has one orbit on axis weights λ. For ratios

$u \in \mathbb{F}_5^\times$, s gives $u \mapsto -u$ and h gives $u \mapsto c_0/u$ after choice of ratio. Thus, $\langle h, s\rangle$ acts transitively on $\mathbb{F}_5^\times$. Hence, L_λ surjects onto L/E and is abelian since $L_\lambda/\langle z\rangle$ is cyclic. Next, on $P \rtimes L_\lambda$, extend by $\widehat{\lambda}(xe) = \lambda(x)$.

Over $1_{\langle z\rangle}$, the stabilizer characters have exponent dividing 4 and scalar transport in K fixes them. Next, for faithful θ, we write $\sigma(\zeta_5) = \zeta_5^a$. A power $(h^2)^j$ returns λ^σ to λ, with $(-1)^j = (a/5) = \alpha(\sigma)$. If $g \in L_\lambda \setminus E$, the commutator is $[(h^2)^j, g] = z^j$, giving $\alpha(\sigma)$. At full support, $|L_\lambda| = 4$ and both faithful labels θ have values in μ_4. Thus, one pair has sign α. On an axis, we have that $L_\lambda = \langle d\rangle$ with $d^4 = z$, so faithful θ has $\theta(d) \in \mu_8 \setminus \mu_4$. Hence, $\theta^F(d) = -\theta(d)$ and $\theta^F(d^2) = \theta(d^2)$. The four faithful θ give two pairs of sign $\alpha\beta$. The local multiplicities are

faithful weights	number of extension pairs	coefficient sign
full support	1	α
one coordinate	2	$\alpha\beta$

By Proposition 8.11, the counts a_-, b_- match by ordinary McKay on $X, Y, X/\langle z\rangle, Y/\langle z\rangle$. Thus, by the index-two count, $f_- = 3$ in X. For $1_{\langle z\rangle}$, take Frame's table of HS.2. Its nonsingleton rational power orbit is $\{20\mathrm{C}, 20\mathrm{D}\}$. Moreover, the extension η_{2520} takes $\sqrt{5}, -\sqrt{5}$ there [45, pp. 79–81] [23]. Thus, a swap by σ exchanges two rows of degree 2520. By Brauer's permutation lemma, σ moves only two rows. Thus, $\mathcal{H}_5$ fixes all upper $5'$-degree characters over $1_{\langle z\rangle}$.

Suppose that a faithful extension η is I-fixed. Then, its values lie in $\mathcal{O}_E$, with $E/\mathbb{Q}_5$ unramified. If $g \in Y \setminus X$, the 5-regular part t also lies outside X. By Proposition 8.11, $\eta(t) = 0$. Reduction of eigenvalues gives $\eta(g) \equiv \eta(t)$ modulo the maximal ideal. Thus, $v_5(\eta(g)), v_5(\eta(g^{-1})) \geq 1$. Some power of g maps to an exterior involution in HS.2, so $|\mathrm{C}_Y(g)|_5 \leq 5$. Hence, each exterior summand lies in $\mathfrak{m}_E$, whereas $\sum_{[g]\subseteq Y\setminus X} \frac{\eta(g)\eta(g^{-1})}{|\mathrm{C}_Y(g)|} = \langle \eta, \eta\rangle_Y - \frac{1}{2}\langle \eta|_X, \eta|_X\rangle_X = \frac{1}{2}$. Hence, each faithful sign restricts to $\alpha|_I$.

Let V span the three differences $\eta - \varepsilon\eta$ over faithful fixed parents of $5'$-degree. Next, we choose x of order 40 over 20B, with fifth power over 4F, and y of order 30 over 30A. These can be classes $50, 56$ of 2.HS.2. By the power maps [45, pp. 79–81], $x^{20} = z$, $y^{11} \sim zy$, and $y \nsim zy$ (classes 56, 57). See the Character Table Library, 2.HS.2, projectives and power maps. On x, y, the exponents for F are $21, 11$, respectively. Thus, the evaluations at x, y on V^* have F-eigenvalue -1.

For the degree-1792 extension, $\eta(x) \neq 0$. Moreover, $y \nsim zy$ gives positive faithful column norm at y. Each faithful base degree a with $5 \mid a$ has defect zero at 3 except $a = 1000$, which has defect zero at 5. These defects persist at index 2, and y is both 3- and 5-singular. Thus, faithful rows with $5 \mid \eta(1)$ satisfy $\eta(y) = 0$. For induced η, we also have $\eta(y) = 0$ since $y \notin X$. Since $x \nsim y, zy$, the faithful columns C_x, C_y are orthogonal. Thus, on extensions of $5'$-degree we have $\langle C_x, C_y\rangle = 0$ and $C_x, C_y \neq 0$. Passing to $\eta - \varepsilon\eta$ rescales C_x, C_y uniformly. Hence, the evaluations $\mathrm{ev}_x, \mathrm{ev}_y$ are independent in $(V^*)_{-1}$ [45, pp. 79–81]. See Suleiman–Wilson, pp. 226–227.

A degree-56 extension has a nonzero exterior value $\sqrt{5}$ fixed by F. Thus, its difference has eigenvalue $+1$, and the multiplicities of $-1, +1$ are $2, 1$. Together with inertia, the signs are $\alpha, \alpha\beta, \alpha\beta$, with fields $\mathbb{Q}_5(\sqrt{5})$ for α and $\mathbb{Q}_5(\sqrt{10})$ for $\alpha\beta$. Match the pairs $\{\eta, \eta\varepsilon\}$ over each ν by sign. Next, we choose $\widehat{\Omega}$ with $\widehat{\Omega}(\eta\varepsilon) = \widehat{\Omega}(\eta)\varepsilon$. By Proposition 8.11, the parents with $\chi^c \neq \chi$ also match. By Proposition 2.13, we obtain $\succeq_c$. For the values on Y, see Suleiman–Wilson, Theorem 2, Lemmas 2, 4. If $p = 3$, apply Propositions 8.7–8.9. At other odd p, we have $|P| = p$ [45, pp. 79–81], so apply Corollary 2.10. At $p = 2$, apply [159, Proposition 2.1]. □

Proposition 8.13. *With $X = 4.M_{22}$, we put $Z = \langle z\rangle = \mathrm{Z}(X) \cong C_4$ and $p = 3$. Next, we put $\Lambda = \{\lambda \in \mathrm{Irr}(Z) : \lambda(z) = \pm i\}$. There is an $\mathrm{Aut}(X)_P \times \mathcal{H}_3$-equivariant bijection $\bigsqcup_{\lambda\in\Lambda} \mathrm{Irr}_{3'}(X \mid \lambda) \longrightarrow \bigsqcup_{\lambda\in\Lambda} \mathrm{Irr}_{3'}(N \mid \lambda)$ over Z. Its pairs satisfy $\succeq_c$, including (γ, σ) in the mixed stabilizer with $\lambda^\gamma = \lambda^\sigma = \lambda^{-1}$.*

Proof. By [139, §§2.1–2.3], the normalizer is $N = P \rtimes S$ with $S = \langle a, b\rangle$, $|a| = 8$, $|b| = 4$ and $aba^{-1} = b^{-1}$. Write $z = b^2a^2$ and $h = b^2$. Moreover, $S/Z \cong Q_8$ acts regularly on $P \setminus \{1\}$, so it acts transitively on lines $R \leq P$. Thus, the central involution hZ inverts P. The preimage of $\mathrm{Z}(N/PZ)$ is $T = Z \times (P \rtimes \langle h\rangle)$. Over $k = \overline{\mathbb{F}}_3$, take V from [139, §§2.2–2.3], with P acting through P/R. Its generator has matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, while h acts by $\pm\,\mathrm{diag}(1, -1)$ and z by $\zeta_4 I_2$. Take the unsigned tensor induction $W = \mathrm{Ten}_T^N(V)$.

For group transport γ, conjugate in N so that $R^\gamma = R$. A field transport also fixes R. There is one indecomposable length-two $k[P/R]$-module. In an h-eigenbasis, inversion gives eigenvalues $1, -1$, and rescaling normalizes the Jordan entry 1. Thus, the signs on h may differ, also for another lift h' of inversion. At z, the scalar choices are $\zeta_4, -\zeta_4$. Hence, the transport gives $V \otimes \delta$ up to conjugacy with $\delta^2 = 1_T$.

We have $\mathrm{Ten}_T^N(\delta) = 1_N$. Indeed, it is trivial on the 3-group P. On S, the tensor-induction character is δ evaluated on the transfer to $T \cap S = \langle z, h\rangle$. The subgroup $\langle z, h\rangle$ is central with quotient C_2^2. If $g \in S$, the coset cycles of lengths $1, 2$ contribute g, g^2, respectively, with product g^4. Since $g^4 \in \langle a^4\rangle = \langle z^2\rangle$, we have $\delta(g^4) = 1$. Thus, $W^{(\gamma,\sigma)} \cong W \otimes \mathrm{Ten}_T^N(\delta) \cong W$ by multiplicativity and conjugation invariance. On W, the central scalar is $\zeta_4^4 = 1$.

Given a faithful simple M, let $G(M)$ be its Green correspondent. By [139, §§2.5, 3.1], $W \otimes_k G(M)$ has a unique indecomposable summand $L(M)$ of $3'$-dimension, with multiplicity 1. Since $W^{(\gamma,\sigma)} \cong W$, uniqueness gives $L(M)^{(\gamma,\sigma)} \cong L(M^{(\gamma,\sigma)})$. By [139, Tables 1 and 3 and §3.1], degrees $56, 56, 160, 160$ have distinct simple reductions $56_a, 56_b, 160_a, 160_b$. For $M = 56_a, 56_b$, the head or socle of $L(M)$ is linear, giving $1_a, 1_b$, respectively. Moreover, $L(160_a) = 1_c$ and $L(160_b) = 1_d$. Thus, the assignment $M \mapsto 1_j$ is equivariant for both transports. Each 1_j lifts uniquely to $\psi_j \in \mathrm{Irr}(N)$ of $3'$-order. By [139, Table 1], pair $176 \leftrightarrow 2$ and $560 \leftrightarrow 8$ in each λ-fibre. Thus, degree gives equivariance. Since U/N inverts Z, no faithful parent is c-fixed. Thus, f_D matches these parents over λ. Thus, Proposition 2.13 gives $\succeq_c$. □

Proposition 8.14. *Let $X = 2.M_{22}$, $Y = 2.M_{22}.2$ as in* Danz–Külshammer [52, §3]*, $Z = \mathrm{Z}(X) \cong C_2$, and $p = 3$. By* Proposition 2.13*, the faithful $3'$-sets $\mathcal{E}_Y$ and $\mathcal{D}_X$ have both matchings. Inflation to $4.M_{22}$ gives $\succeq_c$ over its order-two central character ν_2.*

Proof. First, we write $(J, H) = (X, N)$ or (Y, U), with faithful full-defect blocks B, b. On J/Z the centralizer of PZ/Z is PZ/Z. Since the odd-order lifts in PZ are unique, $\mathrm{C}_J(P) = P \times Z$. Thus, the maximal Brauer pair (P, e_P) over $\nu \neq 1_Z$ is unique up to conjugacy, so B, b are unique. Given $T \in \mathrm{Syl}_2(H)$, we write $H = P \rtimes T$. Each $\lambda \in \mathrm{Irr}(P)$ extends by $\widehat{\lambda}(xe) = \lambda(x)$ on $P \rtimes T_\lambda$. Clifford theory gives degrees $[T : T_\lambda]\tau(1)$ for $\tau \in \mathrm{Irr}(T_\lambda)$, which are powers of 2. Thus, every block of H has full defect. Since P is abelian, height zero gives $\mathrm{Irr}(B) = \mathrm{Irr}_{3'}(J \mid \nu)$.

By [52, Theorems 4.3–4.4], every simple S of B has trivial source and vertex P. Its Green correspondent $G(S)$ is a simple of b, since it is indecomposable in semisimple $k[H/P]$, with $k = \overline{\mathbb{F}}_3$. The $G(S)$ give distinct characters exhausting $\mathrm{IBr}(b)$ since the source equivalences give $l(B) = l(b)$. We have $S^{\gamma\sigma}|_H = (S|_H)^{\gamma\sigma}$ and $(\varepsilon S)|_H = (\varepsilon|_H)(S|_H)$. Each operation carries the vertex-P summand to its counterpart. Thus, $G(S)$ is equivariant. The trivial-source lift $\widetilde{S}$ has irreducible character since S is simple. Since $\varepsilon|_P = 1$ and σ induces residue Frobenius, $S \mapsto \widetilde{S}$ is equivariant. On H, they inflate from H/P over ν. Thus, the lifts $\widetilde{S}$ give the matching on J, H.

The remaining $\psi \in \mathrm{Irr}(H \mid \nu)$ lie over nonzero weights λ, since Q_8 is regular on $\mathrm{Irr}(P) \setminus \{1_P\}$. Moreover, for such λ, the stabilizer T_λ is Z downstairs and abelian of order 4 above, containing Z. Induction of $\widehat{\lambda}\tau$ from $P \rtimes T_\lambda$ for faithful linear τ gives one character on N and an ε-exchanged pair on U. Ordinary McKay on $J, J/Z$ gives $|\mathrm{Irr}(B)| = |\mathrm{Irr}(b)|$ by subtraction [35]. Thus, the remainder is $\{\chi_0\}$ downstairs.

On the upper sets $\mathrm{Irr}(B), \mathrm{Irr}(b)$, compare permutation characters π_B, π_b of the finite image of $\mathcal{H}_3 \times \langle\varepsilon\rangle$. For (u, b_u), Brauer's second main theorem gives d_u. This map evaluates f on us for 3-regular $s \in \mathrm{C}_J(u)$ and projects to b_u. Also, the combined map $\bigoplus_u d_u$ is injective by 3-sections, and surjective by $k(B) = \sum_{(u,b_u)} l(b_u)$. The same holds for b. If $u = 1$, Green correspondence matches $\mathrm{IBr}(B), \mathrm{IBr}(b)$. If $1 \neq u \in P$, the subpair on $\langle u\rangle$ below (P, e_P) is unique, and the automizer contains Q_8 transitive on $P \setminus \{1\}$. Thus, the sole nonidentity subsection (u, b_u) has defect P since P is abelian.

In X, the centralizer of u is $C_3 \times C_2 \times A_4$. Indeed, on $X_4 = 4.M_{22}$, a centralizer lift fixes $\widehat{u}$, the unique order-three lift of u. Thus, $V = [\mathrm{C}_X(u), \mathrm{C}_X(u)] \cong C_2^2$ is normal in $\mathrm{C}_Y(u)$ and normalized by $\mathrm{N}_Y(\langle u\rangle)$. At either level, $[V \cap \mathrm{C}_H(u), P] \leq V \cap P = 1$, since P normalizes V and $\mathrm{C}_H(u)$ normalizes P. Since $\mathrm{C}_X(P) = P \times Z$ and $V \cap Z = 1$, the intersection $V \cap \mathrm{C}_H(u)$ is trivial. Modulo Z, the ambient centralizers are $C_3 \times A_4$ and $C_3 \times S_4$. The H-orbit of u has size 8, so $[\mathrm{C}_J(u) : \mathrm{C}_H(u)] = 4$. Consequently, $\mathrm{C}_J(u) = V \rtimes \mathrm{C}_H(u)$. See [139, §§2.1–2.2] [45, M_{22}].

Inflate b_u from $\mathrm{C}_H(u)$. Indeed, the average $e_V = |V|^{-1} \sum_{v \in V} v$ has Brauer image 1 at P, since $\mathrm{C}_V(P) = 1$ and $|V| \equiv 1 \pmod 3$. Thus, the inflated block contains (P, e_P) and is unique over ν. If $n \in H$ gives $\langle u^n\rangle = \langle u\rangle$, it normalizes $V, \mathrm{C}_H(u)$. Thus, the projection $\mathrm{C}_J(u) \to \mathrm{C}_H(u)$ is equivariant for n. The coefficient exponent j acts linearly by $f(g) \mapsto f(g^j)$. On the subsection it gives $u \mapsto u^j$, corrected by n on both sides. The sign contributes $\varepsilon(s)$, which descends since $\varepsilon|_V = 1$. Hence, both actions on b_u agree. The subsection dimensions agree, so $k(B) = k(b)$ also gives $l(B) = l(b)$.

Subtract the lift permutation characters from $\pi_B = \pi_b$. Given a two-point set A, trace 0 gives a swap and trace 2 gives the identity. Thus, both 2-point remainders are isomorphic for $\mathcal{H}_3 \times \langle\varepsilon\rangle$, with ε free. Restrict the upper bijection to $\mathcal{E}_Y$ and the lower one to $\mathcal{D}_X$. By Proposition 2.13, the pairs satisfy $\succeq_c$. The projective evaluations inflate to $4.M_{22}$,

since the central involution lies in $\ker\chi \cap \ker\psi$. □

Propositions 8.7, 8.13 and 8.14 give the comparison at 3 over every central character of $4.M_{22}$. Let $\widehat{X} = 12.M_{22}$, choose $R \in \mathrm{Syl}_3(\widehat{X})$, and put $N = \mathrm{N}_{\widehat{X}}(R)$. By Section 2, $\mathrm{Z}(\widehat{X})_3 \leq \ker\chi$ for $\chi \in \mathrm{Irr}_{3'}(\widehat{X}) \cup \mathrm{Irr}_{3'}(N)$. Thus, the comparison inflates from $4.M_{22}$. At each other odd prime p, we have that the prime-to-p covers have Sylow subgroup of order p. Thus, by Corollary 2.10, we obtain $\mathrm{iMN}(M_{22}, p)$ there. Together with the case $p = 2$ (Ruhstorfer–Schaeffer Fry [159, Proposition 2.1]), this gives $\mathrm{iMN}(M_{22}, p)$ for all $p \mid |M_{22}|$ (Conway et al. [45, M_{22}]).

Proposition 8.15. *Write $X \lhd Y$ for the cover and extension. Given $P \in \mathrm{Syl}_p(X)$, we set $N = \mathrm{N}_X(P)$, $U = \mathrm{N}_Y(P)$. In* Proposition 2.13*, we have that c acts on the lower characters and $\varepsilon \neq 1$ is the character of Y/X and its restriction to U/N. If $X = 2.M_{12}$ and $p = 3$, both central fibres have comparisons on $\mathrm{N}_X(P)$ satisfying $\succeq_c$.*

Proof. First, take the standard $Y = 2.M_{12}.2$ with $Z = \langle z \rangle$ of order 2. We have $P = 3_+^{1+2}$ and $N/Z = P : T$, with $T = C_2^2$ in $3^2 : \mathrm{GL}_2(3)$ [116, §6.2]. On P/P', the two coordinates have weights $t, s/t$ under $\mathrm{diag}(s,t)$. Thus, we have that T acts by $\mathrm{diag}(\pm 1, \pm 1)$. Also $U/Z = P : K$ with $K = D_8$ [45]. The kernel on P/P' intersects T trivially and has order at most 2. A normal subgroup of order 2 in K is central and lies in T. Thus, K acts by all signed permutations on P/P'.

Choose a Sylow 2-complement in U, which contains Z. Next, lift the coordinate swap to w over class 2C, with $w^2 = 1$ for both lifts [153, §2.6]. The preimage $\widetilde{T}$ is elementary abelian by splitting over $3^2 : \mathrm{GL}_2(3)$ [22, §3.1]. Take a over the first sign and set $b = waw^{-1}$. We have $ab = ba$ and independent images of a, b in T, and $w^2 = 1$. Thus, $\langle a, b, w \rangle \cong K$, giving $U = Z \times (P : K)$ and $N = Z \times (P : T)$.

If $L \in \{K, T\}$, Clifford theory puts each $\psi \in \mathrm{Irr}_{3'}(P : L)$ over a weight $\lambda \in \mathrm{Irr}(P/P')$. Its constituents arise from $\widehat{\lambda}(xe) = \lambda(x)$ on $P : L_\lambda$ and a character $\tau \in \mathrm{Irr}(L_\lambda)$. If $\lambda = 1$, the characters of L are rational. Otherwise, the stabilizer in $\mathrm{GL}_2(3)$ has order 6, so $|L_\lambda| \leq 2$. Thus, we have that τ is rational and $\lambda^\sigma \in \{\lambda, \lambda^{-1}\}$. Conjugation by $-I$ inverts λ and fixes τ. Thus, the induced ψ are rational for both $\nu \in \mathrm{Irr}(Z)$.

By Section 8.3, take the two powers $881, 883$ on X, Y. In [23, `2.M12` and `2.M12.2`], each cut over ν has size 9. By `GaloisCyc`, both powers fix $\mathrm{Irr}_{3'}(X \mid \nu), \mathrm{Irr}_{3'}(Y \mid \nu)$. Ordinary McKay for $X, Y, X/Z, Y/Z$ gives equal counts a, b on each fibre ν [35], subtracting quotient counts for $\nu \neq 1_Z$. By the index-two count, $f = (2b - a)/3$ parents are fixed and $(a - f)/2$ orbits are free. Match free c-orbits equivariantly. With fixed χ, pair η, η' and $\eta\varepsilon, \eta'\varepsilon$. The extensions are $\mathcal{H}_3$-fixed, so Proposition 2.13 gives $\succeq_c$. □

Proposition 8.16. *For $2.J_2 \lhd 2.J_2.2$ and $p = 3, 5$, we have that both fibres have $\succeq_c$.*

Proof. If $\sigma \in \mathcal{H}_p$, let $u_p(\sigma)$ be its Frobenius exponent modulo 2 on the p'-roots in question. Next, let $\sigma(\sqrt{5}) = (-1)^{d_5(\sigma)}\sqrt{5}$. On both sides, σ acts on the base rows by $c^{u_3(\sigma)}$ for $p = 3$ and $c^{d_5(\sigma)}$ for $p = 5$. Moreover, the extension twist is 1 at $z = +1$ and is $\varepsilon^{u_p(\sigma)}$ at $z = -1$. The index-two count gives equal n_c, n_ε. Next, match the free c-orbits and extension pairs over each ν. Thus, the match is $\mathcal{H}_p$-equivariant.

By Section 8.3, the powers at 3 are $281, 283$, and at 5 they are $337, 341$. By [23, `2.J2` and `2.J2.2`], compute $\eta|_X$ by fusion and the twist $\varepsilon\eta$. Each irreducible restriction gives a c-fixed parent. Otherwise, the restriction is $\chi + \chi^c$. Below, the actions are $1, c$ at 3 and $c, 1$ at 5. On extensions the first gives 1. The second gives $1, \varepsilon$ over $1_Z, \nu \neq 1_Z$, respectively.

If $p = 3$, the point stabilizer $\mathrm{U}_3(3)$ contains $P : C_8$, and $U/Z = P : \mathrm{SD}_{16}$ [116, §6.4]. On $W = P/P' \cong \mathbb{F}_9^+$, the torus acts by $\mathbb{F}_9^\times$ and c cubes C_8. Thus, at zero weight downstairs σ acts by $c^{u_3(\sigma)}$. The nonzero weights λ form one orbit with lower stabilizer 1. If $\lambda = 1$, then upper linears are rational, and $\mathrm{Ind}_{C_8}^{\mathrm{SD}_{16}}\theta$ is fixed by cubing. For nonzero λ, the upper stabilizer has order 2. Moreover, the scalar $-I$ inverts λ and fixes its stabilizer label τ_λ. Thus, $\mathrm{Irr}(U/Z)$ is $\mathcal{H}_3$-fixed.

Downstairs the preimage of $\mathrm{U}_3(3)$ is $Z \times \mathrm{U}_3(3)$. By contrast, upstairs the preimage of $\mathrm{U}_3(3) : 2$ is the isoclinic twist of $Z \times (\mathrm{U}_3(3) : 2)$ [22, §3.12]. Modulo the embedded $\mathrm{U}_3(3)$, the quotient is C_4 with squares Z. Also, a faithful $\tau \in \mathrm{Irr}(C_4)$ is nontrivial on Z, with exterior values $\pm i$. The products with $\tau|_U, \tau|_N$ give the faithful fibres. Thus, on N the action is $c^{u_3(\sigma)}$ while the upper sign is $\varepsilon^{u_3(\sigma)}$.

If $p = 5$, we have $P = \mathbb{F}_5^2$, $U/Z = P : (C_4 \times \mathfrak{S}_3)$ and $N/Z = P : (C_2 \times \mathfrak{S}_3)$. The upper complement acts faithfully on P. Indeed, an element of 5A has centralizer $C_5 \times A_5$ of the same order in $\mathrm{Aut}(J_2)$. Centralizing a second independent element leaves C_5 in the A_5 factor [173, §2] [45, pp. 42–43]. The subgroup C_3 acts irreducibly, with endomorphism field

$\mathbb{F}_{25}$; the inverting reflection acts by $x \mapsto x^5$. Thus, $\mathrm{C}_{\mathrm{GL}(P)}(\mathfrak{S}_3) = \mathbb{F}_5^\times$, from the central C_4. The dihedral subgroup is $C_2 \times \mathfrak{S}_3$. Indeed, the sign kernel is $C_4 \times A_3$, and the product-sign kernel K_1 has involutions in C_4 only.

Choose a complement K containing Z, put $K_0 = K \cap N$, and lift a scalar generator to s. Then, $s^4 = z$. In $2.A_5 \times D_{10}$, the scalar $-I$ is the image of uv, where u, v invert independent Sylow lines, with orders $4, 2$ respectively. All lifts of $-I$ in its P-coset are P-conjugate since $1 - (-I)$ is invertible. Thus, s^2 has square z. The image of s is in 4C, whose square is in 2B. The faithful degree-14 extension has trace $\pm\sqrt{2}$ there [45, p. 43, row 24]. Thus, s and zs are not conjugate. Since sZ is central in K/Z, this gives $s \in \mathrm{Z}(K)$.

Let L be the preimage of $\mathfrak{S}_3$. Then, $K = \langle s \rangle L$, $K_0 = \langle s^2 \rangle L$ and $\langle s \rangle \cap L = Z$. The preimage of A_3 is C_6, with a normal C_3 in L and quotient of order 4. Thus, each central character of Z extends linearly to L over $\mathbb{Q}(i)$. Tensor with $\mathrm{Irr}(L/Z)$ to obtain the fibre $\mathrm{Irr}(L \mid \nu)$. At zero weight, $\mathrm{Irr}(K_0)$ are $\mathcal{H}_5$-fixed since $i \in \mathbb{Q}_5$. An extension to K has scalar on s in μ_4 over 1_Z and $\mu_8 \setminus \mu_4$ over faithful ν. Moreover, the power map $g \mapsto g^5$ fixes μ_4 and gives $\zeta_8^5 = -\zeta_8$ for primitive ζ_8.

For nonzero λ, a nonsquare scalar requires an eigenvalue ζ_4 in $\mathfrak{S}_3$. Its stabilizer in $K_0/Z \cong D_{12}$ has order dividing $\gcd(12, 20) = 4$. Its action on $P/\ker\lambda$ gives an injection into $\mathbb{F}_5^\times$, since the kernel is a 5-group. Also, D_{12} has no element of order 4. Thus, its order is ≤ 2, and its preimage K_λ is abelian of exponent ≤ 4. Next, we extend by $\widehat{\lambda}(xe) = \lambda(x)$ on $P : K_\lambda$. The stabilizer labels τ are fixed, and λ^σ is conjugate to λ by a power of s. Squares act from N, and a nonsquare moves $N \cdot \lambda$. Thus, the lower action is $c^{d_5(\sigma)}$, with extension signs from $\lambda = 1$. By Proposition 2.13, the matchings give $\succeq_c$. □

Proposition 8.17. *With $S = J_3$, we put $p = 3$. Next, the permutation character on $\mathrm{Syl}_3(S)$ selects an $\mathcal{H}_3$-fixed extension for each c-fixed $\chi \in \mathrm{Irr}_{3'}(S)$. These extensions give the comparison on $3.S$.*

Proof. Put $A = S : 2$, $\widetilde{S} = 3.S$, $Z = \mathrm{Z}(\widetilde{S})$, and $R \in \mathrm{Syl}_3(\widetilde{S})$. By Section 2, the $3'$-degree sets on $\widetilde{S}$ and $\mathrm{N}_{\widetilde{S}}(R)$ inflate modulo Z. Put $P = R/Z$. Then, $P/P' = \mathbb{F}_3^2$, $N = P : C_8$, and $U = P : \mathrm{SD}_{16}$. The group C_8 acts regularly on nonzero weights λ by [134, §2] and [113, Lemmas 3.1.3 and 3.3.2–3.3.4]. On $\mathrm{Irr}(C_8)$, cubing fixes only $0, 4$ modulo 8. The nonzero λ-orbit adds one fixed character ψ_λ. Thus, u_3 determines the action on N with 3 fixed rows, and $\mathrm{Irr}_{3'}(U)$ is fixed pointwise.

By Ω_{1_Z} of Section 8, $\mathrm{Irr}_{3'}(S)$ also has 3 fixed points and action kernel $\ker u_3$. By [113, Lemma 3.2.1], the c-fixed degrees are $1, 1615, 2432$, each with one χ. Each remaining degree gives $\{\chi, \chi^c\}$. Degree invariance fixes the three singleton χ's under $\mathcal{H}_3$, exhausting its fixed points. Hence, $u_3(\sigma) = 1$ swaps every $\{\chi, \chi^c\}$, giving $c^{u_3(\sigma)}$. Put $\pi = \mathrm{Ind}_U^A 1$. Since $A = SU$, we have $\pi|_S = \mathrm{Ind}_N^S 1$. Its degree is $2^4 \cdot 5 \cdot 17 \cdot 19$. By [24, J_3.2, action 2], each parent $1a, 1615a, 2432a$ has one extension η in π, with multiplicity 1. The two extensions are $\eta, \eta\varepsilon$. By [23, `J3.2`], the powers $25841, 64603$ fix all 9 upper rows of $3'$-degree. Also, both base actions are c^{u_3}, with $\eta^\sigma = \eta$. Match $\mathrm{Irr}_{3'}(S)$ and $\mathrm{Irr}_{3'}(N)$ as in Proposition 8.15. The automorphisms of $\widetilde{S}$ induce those of S. Since $Z \leq \ker\chi \cap \ker\psi$ on the cover, the factors c and mixed functions μ inflate. □

8.3 Held, McLaughlin, Rudvalis and Conway

Let $X \lhd A$ have index 2 and let p be odd. Given $P \in \mathrm{Syl}_p(X)$, we put $N = \mathrm{N}_X(P)$ and $U = \mathrm{N}_A(P)$. By Isaacs [97, Lemma 1.13], we obtain $A = XU$. Write c for the action of $u \in U \setminus N$ on $\mathrm{Irr}(X)$ and $\mathrm{Irr}(N)$. Next, we let ε denote the nontrivial character of A/X and its restriction to U. An irreducible character of X or N is called a parent. Thus, each c-fixed parent has extensions $\eta, \varepsilon\eta$ on A or U, as in Proposition 2.13. Each c-orbit of size 2 induces one irreducible character.

Lemma 8.18. *Suppose $U = N \times \langle t \rangle$ with $t^2 = 1$ and $t \notin N$. Given an $\mathcal{H}_p$-equivariant bijection $b : \mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(N)$, all these parents are fixed by c. Their extensions admit a matching above b equivariant under $\mathcal{H}_p$ and $\eta \mapsto \varepsilon\eta$. If b matches the given central characters on X, N, the extensions have matching characters on $\mathrm{Z}(X), \mathrm{Z}(N)$.*

Proof. Given $\eta \in \mathrm{Irr}_{p'}(A)$, the height formula gives defect P for its block B. Brauer correspondence acts by $B \mapsto b_B$ in U. In characteristic p, the idempotents $e_\pm = (1 \pm t)/2$ split kU into two summands. Moreover, each b_B has sign $s = \pm 1$ from e_s, the value at t on its characters. The sign is $\mathcal{H}_p$-fixed and flips under ε. Give B the sign of b_B. Next, in a cyclotomic splitting field K, let $\mathfrak{p} \mid p$. By [142, Theorem 2.1], $\mathcal{H}_p$ acts through the decomposition group at $\mathfrak{p}$. Thus, Brauer correspondence is $\mathcal{H}_p$-equivariant [142, p. 1134]. For the sign twist, put $\epsilon(g) = \varepsilon(g)g$ on kA. The Brauer

map deletes coefficients outside $C_A(P)$, so $\mathrm{Br}_P(\epsilon u) = \epsilon\,\mathrm{Br}_P(u)$. Hence, the sign of b_B changes under ε, and the blocks of η and $\varepsilon\eta$ have opposite signs.

If $\chi \in \mathrm{Irr}_{p'}(X)$ moves under c, $\mathrm{Ind}_X^A \chi$ is irreducible of degree $2\chi(1)$. It is fixed by ε, contradicting the opposite block signs. On N, every parent is fixed by c since $U = N \times \langle t\rangle$. Moreover, for a c-fixed χ, exactly one extension η_+ has block sign $+1$. If $\sigma \in \mathcal{H}_p$, the choice for χ^σ is η_+^σ. For $b(\chi)$, choose the extension ψ_+ trivial on t. Next, pair η_+ with ψ_+ and pair $\varepsilon\eta_+$ with $\varepsilon\psi_+$. Restriction to X, N recovers b and its central labels. □

Lemma 8.19. *Suppose* $\mathrm{Irr}_{p'}(X)$ *and* $\mathrm{Irr}_{p'}(N)$ *are isomorphic as* $\mathcal{H}_p$*-sets. Suppose* $\mathcal{H}_p$ *fixes* $\mathrm{Irr}_{p'}(A), \mathrm{Irr}_{p'}(U)$ *pointwise and* $|\mathrm{Irr}_{p'}(A)| = |\mathrm{Irr}_{p'}(U)|$*. Then there is a parent matching* c*- and* $\mathcal{H}_p$*-equivariant. If* $\chi^c = \chi$*, the extensions match equivariantly for* ε *and* $\mathcal{H}_p$*.*

Proof. If $\chi^c = \chi$, the fixed extension η gives $\chi^\sigma = \eta^\sigma|_X = \chi$ for $\sigma \in \mathcal{H}_p$. If $\chi^c \neq \chi$, its irreducible induction is fixed, so $\mathcal{H}_p \cdot \chi \subseteq \{\chi, \chi^c\}$. The same holds on N. Thus, c is the unique transposition on each $\mathcal{H}_p$-orbit of length 2. Set $a = |\mathrm{Irr}_{p'}(X)|$, $b = |\mathrm{Irr}_{p'}(A)|$, and let f count the c-fixed parents. Thus, $b = 2f + (a-f)/2$. Since a, b match locally, so does f. Keep the bijection on two-point $\mathcal{H}_p$-orbits. Next, match the c-fixed singletons and then the free c-pairs of the singletons left. Since $\mathcal{H}_p$ fixes extensions, choose any ε-equivariant matching. □

Take $X = \mathrm{He}$, $A = \mathrm{He}{:}2$, and $p = 3$. Let $Q = \langle x\rangle$ with x in class 3A. By Conway et al. [45, pp. 104–105], $H = \mathrm{N}_X(Q)$ has shape $3.S_7$. Moreover, $\mathrm{N}_A(Q) = H \times \langle t\rangle$ for an exterior involution t (Breuer [23, `He.2`, maximal subgroup 6]). A Sylow 3-subgroup P of H is Sylow in X and extraspecial of order 27. Since $Q \leq P$ centralizes P, we have $Q = \mathrm{Z}(P)$. Thus, $U = \mathrm{N}_{\mathrm{N}_A(Q)}(P) = N \times \langle t\rangle$. The multiplier of X is trivial by Conway et al. [45]. By Section 8, Ω_{1_Z} pairs $\mathrm{Irr}_{3'}(X)$ with $\mathrm{Irr}_{3'}(N)$. By Lemma 8.18, both matchings exist. Thus, $\mathrm{iMN}(X, 3)$ follows from Proposition 2.13.

By Section 8, choose e, f for A, U in Breuer [23]. With `GaloisCyc`, both powers fix $\mathrm{Irr}_{p'}(A)$ and $\mathrm{Irr}_{p'}(U)$ pointwise. The rightmost column gives $|\mathrm{Irr}_{p'}(A)| = |\mathrm{Irr}_{p'}(U)|$.

p	Table of A	Table of U	(e, f) on A	(e, f) on U	Size
5	`He.2`	`He.2N5`	$(17137, 17141)$	$(97, 101)$	20
7	`He.2`	`He.2N7`	$(20401, 24487)$	$(31, 1)$	25
3	`McL.2`	`McL.2N3`	$(15401, 24643)$	$(29, 19)$	18
5	`McL.2`	`McL.2N5`	$(22177, 22181)$	$(97, 101)$	20
5	`3.McL.2`	`3.McL.2N5`	$(22177, 22181)$	$(97, 101)$	33

Thus, the map Ω_{1_Z} pairs the lower sets at $p = 5, 7$ for He and $p = 3, 5$ for McL. By Lemma 8.19, both matchings give $\succeq_c$ via Proposition 2.13. For He, these primes and $p = 3$ exhaust the odd primes with noncyclic P (Conway et al. [45, pp. 104–105]). If $X = \mathrm{McL}$ and $p = 3$, the comparison inflates to 3.McL by Johansson [101, Lemma 5.1].

At $p = 5$, take $X = 3.\mathrm{McL}$ and $A = 3.\mathrm{McL}.2$. Put $Z = \mathrm{Z}(X) \cong C_3$. By An–Eaton [6, Proposition 7.4 and Tables I–II], $N = Z \times \mathrm{N}_{\mathrm{McL}}(\bar{P})$ for $\bar{P} = PZ/Z$. By Breuer [23], both classes in $Z \setminus \{1\}$ fuse under $X \hookrightarrow A$ and $N \hookrightarrow U$. Thus, c inverts Z at each level. In Breuer [23, `3.McL` and `3.McLN5`], each faithful $\alpha \in \mathrm{Irr}(Z)$ gives 13 characters of $5'$-degree in each lower table. For $(J, L) = (X, A), (N, U)$, let $\chi \in \mathrm{Irr}_{5'}(J \mid \alpha)$. Since $c\alpha = \alpha^{-1} \neq \alpha$, we have $\mathrm{Ind}_J^L \chi \in \mathrm{Irr}_{5'}(L)$. By the table, $\mathcal{H}_p$ fixes its restriction $\chi + \chi^c$. Thus, α^σ determines the image of χ for all $\sigma \in \mathcal{H}_p$. A bijection over α extends by c over α^{-1}. Thus, the bijection is equivariant for $c, \mathcal{H}_p$. Together with the inflated comparison over 1_Z, Proposition 2.13 gives $\succeq_c$ at N.

Proposition 8.20. *Let* $S = \mathrm{Ru}$ *with* $p \in \{3, 5\}$*, or* $S = \mathrm{Co}_1$ *with* $p \in \{3, 5, 7\}$*. Put* $X = 2.S$*,* $Z = \mathrm{Z}(X)$ *and* $M = \mathrm{N}_X(P)$ *for* $P \in \mathrm{Syl}_p(X)$*. Then for each* $\lambda \in \mathrm{Irr}(Z)$*, the* $\mathcal{H}_p$*-sets* $\mathrm{Irr}_{p'}(X \mid \lambda)$ *and* $\mathrm{Irr}_{p'}(M \mid \lambda)$ *are isomorphic.*

Proof. At $\lambda = 1_Z$, take Ω_{1_Z} from Section 8, with $M/Z = \mathrm{N}_S(PZ/Z)$. For $\lambda \neq 1_Z$, we put $\mathcal{X} = \mathrm{Irr}_{p'}(X \mid \lambda)$ and $\mathcal{Y} = \mathrm{Irr}_{p'}(M \mid \lambda)$. By [23], at $p = 3, 5$ the Ru tables for M are `(2x3^(1+2)_+:8):2` and `5^(1+2):(4x4):4`, respectively. At $p = 3, 5, 7$ the Co_1 tables are `2xCo1N3`, `2xCo1N5`, `2x7^2:(3x2A4)`, respectively. The class orders give $m = \exp(X) = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 29$ for Ru and $m = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23$ for Co_1. In all cases $\exp(M) \mid m$. By Section 8, take $u = 2$ at $3, 5$ and $u = 3$ at 7 for the actions σ_e, σ_f. In the global tables of [23], z has class number 2. Under the stored fusions, 2 maps to 2. Thus, the rows r satisfy $r_2 = -r_1$ and $p \nmid r_1$. By [23], σ_e fixes every selected row. If $(S, p) = (\mathrm{Ru}, 3)$, σ_f fixes 3 characters and has 3 orbits of size 2 on each set. If $(S, p) = (\mathrm{Ru}, 5)$, it fixes all 20 characters on either side. Moreover, for $S = \mathrm{Co}_1$, σ_f fixes all $27, 25, 27$ characters for $p = 3, 5, 7$, respectively. Since σ_e

acts trivially, the image of $\mathcal{H}_p$ on $\mathcal{X} \sqcup \mathcal{Y}$ is cyclic. At orbit length d, the stabilizer is the preimage in $\mathcal{H}_p$ of the unique subgroup of index d. Thus, the matching counts give an $\mathcal{H}_p$-equivariant bijection $\mathcal{X} \to \mathcal{Y}$. □

8.4 The Suzuki cover at $3, 5$

Proposition 8.21. *Let $X = 6.\mathrm{Suz}$ and $Y = 6.\mathrm{Suz}.2 \leq \mathrm{Co}_0$. Then at $p = 5$, every $\zeta \in \mathrm{Irr}(\mathrm{Z}(X))$ admits the comparison of* Proposition 2.13*, with $\zeta^c = \zeta^{-1}$ for the outer action c.*

Proof. With $S = \mathrm{Suz}$, we put $A = S : 2$ and $P = C_5^2$. Given $\sigma \in \mathcal{H}_5$, we let m be its unramified exponent and set $\sigma(\sqrt{5}) = (-1)^{h(\sigma)}\sqrt{5}$, with $h(\sigma) \in \{0, 1\}$. If t is exterior of class 2C, $\mathrm{C}_A(t) = (J_2.2) \times \langle t \rangle$, with $J_2.2 \leq S$. By [45] and [23], $\mathrm{N}_S(P) = \mathrm{N}_{J_2.2}(P) = P : (C_4 \times \mathfrak{S}_3)$ and $\mathrm{N}_A(P) = \mathrm{N}_S(P) \times \langle t \rangle$. Moreover, the preimage of $J_2.2$ in X is $C_3 \times (2.J_2.2)$, locally $C_3 \times Q$ with $Q = \mathrm{N}_{2.J_2.2}(P)$. The outer action inverts C_3 and normalizes $2.J_2.2$. Indeed, $2.J_2$ is perfect with index 2, so $\mathrm{Hom}(2.J_2.2, C_3) = 1$.

Write z for the central involution of $2.J_2.2$ and δ for its quotient sign. In 2.Suz.2, a lift $\widetilde{t}$ has centralizer order $2^9 \cdot 3^3 \cdot 5^2 \cdot 7 = |\mathrm{C}_A(t)|$. Conjugation by $\widetilde{t}$ is trivial modulo z, so $g \mapsto [\widetilde{t}, g]$ defines a sign on $2.J_2.2$. If the sign were trivial on $2.J_2.2$, we have that the preimage of $\mathrm{C}_A(t)$ would centralize $\widetilde{t}$, giving order $2|\mathrm{C}_A(t)|$. Thus, the sign is δ. By [97, Lemma 1.13], $2.J_2.2 = (2.J_2)Q$, since $P \in \mathrm{Syl}_5(2.J_2)$. Hence, $\delta|_Q \neq 1$. Given $\psi \in \mathrm{Irr}_{5'}(Q)$ with central parity $a \in \{0, 1\}$, this gives $\psi^c = \delta^a\psi$. By Proposition 8.16, $\psi^\sigma = \delta^{am}\psi$. If the J_2-normalizer parent is invariant, δ^{am} is the extension sign on ψ. Otherwise, its induction is δ-fixed, and the pair is σ-stable. Since $5 \equiv -1 \pmod 3$ on C_3, the full action on $C_3 \times Q$ is c^m.

Let $|\zeta| \in \{3, 6\}$. Since c inverts ζ, each induction to Y is irreducible. The table exponent is 720720 [23, `6.Suz.2`]. By Section 8, the Galois exponents at 5 are $576577, 576581$. By the stored fusion $X \to Y$, the 64 parents in these fibres give 32 inductions. Both powers fix these 32 rows by `GaloisCyc`. Thus, $\chi^\sigma \in \{\chi, \chi^c\}$, and the central label gives $\chi^\sigma = c^m\chi$.

For $d \mid 6$, we let $Z_{6/d} \leq \mathrm{Z}(X)$ have order $6/d$. The McKay count N_d on $X/Z_{6/d}$ includes all labels of order dividing d. Thus, $n_d = N_d - \sum_{e|d,\ e<d} n_e$ counts labels of exact order d by their characters. The same N_d locally gives the same n_d. Since c exchanges ζ, ζ^{-1}, the counts coincide over each ζ. Match over ζ and transport by c to ζ^{-1}. If $\zeta = 1$, $\mathrm{N}_A(P) = \mathrm{N}_S(P) \times \langle t \rangle$. Apply Lemma 8.18 with Turull's bijection on χ, ψ. If $|\zeta| = 2$, put $X_2 = 2.\mathrm{Suz}$ and $Y_2 = 2.\mathrm{Suz}.2$. By the fusion in [23], restrict the $5'$-degree rows of Y_2 to X_2. By scalar products, c moves 12 spin parents and fixes 4. For moving χ, the first power fixes χ, and the second gives χ^c. Moreover, both powers fix the other 4. On their extensions, 576577 swaps the rows and 576581 fixes both. Since $576577 \equiv 2 \pmod 5$ and $576581 \equiv 1 \pmod 5$, the extension sign is $\varepsilon^{h(\sigma)}$. Thus, the spin parents satisfy $\chi^\sigma = \chi^{c^m}$.

Locally, take the central scalar lift s in the complement of P in Q from Proposition 8.16. Then, $s^4 = z$, and s acts by a nonsquare on P. If $\lambda = 1$, the spin value v at s satisfies $|v| = 8$ in μ_8 and $c(v) = -v$. Thus, these parents move under c. For nonzero weight λ, the stabilizer K_λ is abelian of exponent dividing 4 and is contained in $\ker\delta$. Since $\widetilde{t}$ centralizes P, K_λ and $\widetilde{t}^2 \in \langle z \rangle$, the enlarged complement $\langle K_\lambda, \widetilde{t} \rangle$ is abelian of exponent dividing 4. Extend λ trivially across $\langle K_\lambda, \widetilde{t} \rangle$, then induce with a complement label ν. Since ν is $\mathbb{Q}_5$-valued, only λ changes initially. A power s^j restores λ^σ with $j \equiv h(\sigma) \pmod 2$. Moreover, s^j centralizes K_λ and acts by $\widetilde{t} \mapsto z^j\widetilde{t}$. Thus, the local extension sign is $\varepsilon^{h(\sigma)}$. The spin counts a, b follow from McKay for X_2, Y_2 by subtracting counts for S, A. Thus, the index-two count gives f, the number of c-fixed parents. Match $\{\chi, \chi^c\}$ with c^m and $\{\eta, \varepsilon\eta\}$ with $\varepsilon^{h(\sigma)}$. Thus, Proposition 2.13 gives the comparison at $p = 5$ for every ζ. □

Proposition 8.22. *If $p = 3$, the comparison on $X = 2.\mathrm{Suz}$ holds over both central characters λ and inflates to* $6.\mathrm{Suz}$.

Proof. On 6.Suz and its normalizer, Section 2 gives $C_3 \leq \ker\chi$ for each $3'$-degree character χ. Hence, we set $X = 2.\mathrm{Suz}$ and $Y = 2.\mathrm{Suz}.2$. Put $V = 3^5$ and $D \in \mathrm{Syl}_3(M_{11})$. In $H = V : (M_{11} \times \langle t \rangle) \leq \mathrm{Suz}.2$, we set $P = V : D$. Then, $\mathrm{N}_H(P) = V : (\mathrm{N}_{M_{11}}(D) \times \langle t \rangle)$, with $\mathrm{N}_{M_{11}}(D) = 3^2 : \mathrm{SD}_{16}$. Since $|\mathrm{N}_H(P)| = 2^5 \cdot 3^7 = 2|\mathrm{N}_{\mathrm{Suz}}(P)|$, it equals $\mathrm{N}_{\mathrm{Suz}.2}(P)$. By [23], Ly has a maximal subgroup of shape H with the same action on V. Its Sylow normalizer has order $2^5 \cdot 3^7$, so Ostermann gives the Sylow 3-normalizer table for Ly.

The lift of $V : M_{11}$ in X is $\langle z \rangle \times (V : M_{11})$. Since V is nontrivial irreducible under perfect M_{11}, $V : M_{11}$ is perfect. Thus, its complement to $\langle z \rangle$ is unique. The exterior involution $t \in 2\mathrm{D}$ centralizes M_{11}, with $11 \nmid |\mathrm{C}(2\mathrm{C})|$ excluding 2C. In Y, the square map acts by $63 \mapsto 1$, so $\widetilde{t}^2 = 1$. It normalizes the unique complement $V : M_{11}$, and adjoining $\widetilde{t}$ extends it over H. Thus, both local splittings over $\langle z \rangle$ are compatible.

By [23, LyN3], its exponent is 72. By GaloisCyc, powers 65, 19 fix all 27 rows of 3′-degree. Thus, for either local sign, $\mathcal{H}_p$ fixes the 3′-degree rows. By [23, 2.Suz.2], the exponent of Y is 720720. The exponents 320321, 80083 fix all 54 rows of 3′-degree. Each sign on X has 18 such rows, with 12 fixed by 320321 and all fixed by 80083 [23, 2.Suz]. Since the inductions to Y are fixed, each $\mathcal{H}_p$-orbit on X has size at most 2. Thus, the two sectors are $\mathcal{H}_p$-isomorphic.

Transport Turull's map for $S = \mathrm{Suz}$ with the local sign $\lambda(z)$. In each sector, both sides give $(a, b) = (18, 27)$ [89, §6.6 for the global counts]. Thus, Lemma 8.19 and Proposition 2.13 give $\succeq_c$ on X over both signs. After inflation across C_3, the scalar at a lift of x is the scalar at x. If $p \in \{7, 11, 13\}$, the Sylow subgroups of Suz have order p [45], so Corollary 2.10 applies. At $p = 2$, apply [159, Proposition 2.1]. □

8.5 Covers of sporadic simple groups

Suppose $S \lhd A$ is sporadic simple with $[A : S] = 2$ as in Proposition 2.13. If $P \in \mathrm{Syl}_p(S)$, we set $N = \mathrm{N}_S(P)$ and $U = \mathrm{N}_A(P)$. Write c for the action of $u_0 \in U \setminus N$ on $\mathrm{Irr}(N)$, and ε for the sign of U/N. Let $Y \to A$ have kernel $Z = \mathrm{Z}(X)$ and preimage X of S. If $R \in \mathrm{Syl}_p(X)$ maps to P, then R is characteristic in RZ. Thus, the preimages of N, U are $\mathrm{N}_X(R), \mathrm{N}_Y(R)$. For CT, see Breuer [23, version 1.3.11] and Conway et al. [45]. The counts over $\lambda \in \mathrm{Irr}(Z)$ agree by Cabanes–Späth [35, Theorem B].

Take $S = \mathrm{O}'N$, $A = S.2$, $X = 3.S$ and $Y = 3.S.2$. At $p = 7$, choose $R \in \mathrm{Syl}_7(X)$ and put $M = \mathrm{N}_X(R)$. Section 8 gives powers $(3109921, 1554967)$ on A, X, $(73, 127)$ on U, and $(73, 43)$ on M. By Breuer [23, ON.2, ON.2N7, 3.ON, 3.ONN7], each pair fixes all 7′-degree rows of its table. The 25 rows on each of A, U give Lemma 8.19 over $1_{\mathrm{Z}(X)}$. By the fusions, c inverts $\mathrm{Z}(X)$, and each faithful fibre has 20 rows on X, M. Thus, pair the ν-fibres and extend by c over ν^{-1}.

At $p = 3$, Section 2 gives $\mathrm{Z}(X) \le \ker \chi$ for 3′-degree rows on $X, \mathrm{N}_X(R)$, with $R \in \mathrm{Syl}_3(X)$. By Clifford theory, this holds on $Y, \mathrm{N}_Y(R)$. Thus, take the quotient tables of S, A, N, U in Breuer [23, ON, ON.2, ONN3, ON.2N3]. By Section 8, the exponents on A, U are $3628241, 3628243$ and $41, 43$, respectively. Since e fixes the rows, take Frobenius F. By Gómez-Taberner [78, Theorem 4.3], restriction from A, U gives both types on S, N. By Breuer [23], F acts as follows. On S, N, take two c-pairs with $F = 1$ and two with $F = c$. The extension rows give four ε-pairs with $F = 1$, four with $F = \varepsilon$, and one regular orbit under $\langle F, \varepsilon \rangle \cong C_2^2$. Thus, Proposition 2.13 gives $\succeq_c$, which inflates to X.

For $S = \mathrm{HN}$, take $X = S$ and $A = S.2$ by Conway et al. [45]. By Breuer [23, HN.2], $292601, 2048203$ fix the 27 rows of 3′-degree, and $1685377, 1158701$ fix the 25 of 5′-degree. Locally the 27 rows of 3′-degree are integral by Breuer [23, HN.2N3]. By Breuer [23, alias HN.2N5], U has table LyN5 at 5. Thus, Lemma 8.5 gives 25 fixed rows. On X, N, take Ω_{1_Z} from Section 8. Thus, Lemma 8.19 applies at $3, 5$.

Next, we put $S = \mathrm{Fi}_{22}$, $A = S.2$, $X = 2.S$ and $Y = 2.S.2$. By Johansson [101, Lemma 5.1], at $p = 3$ the p'-cover $X = 2.S$ suffices. By Breuer [23, 2xFi22N3 and 2xFi22N5], the preimages split at $3, 5$. By fusions in $6.S.2$ (Breuer [23]), the quotients by subgroups of $\mathrm{Z}(6.S)$ of orders $3, 2$ give $2.S.2, 3.S.2$. At $p = 3$, An–O'Brien [4, Table 11, row $k(1, d, u)$] gives 18 faithful global parents at defect 9, with 4 at $u = 2$. At $p = 5$, An–O'Brien [4, proof of Theorem 7.1, case (1)] gives 20 faithful parents of defect 2 on each side. Of these, 2 per residue have $u = 2$. In these counts, $u = 2$ means that c fixes the parent.

By Section 8, take e, f on X, Y in Breuer [23]. At 3, the powers are $320321, 80083$. At 5, they are $576577, 576581$. By GaloisCyc, both fix $\mathrm{Irr}_{p'}(X \mid \lambda)$. By the fusion $X \to Y$, compute the twist $\varepsilon\eta$ of each upper η. On faithful rows of Y, each listed power k acts by $\eta \mapsto \eta$ if $k \equiv 1 \pmod 3$ and by $\eta \mapsto \varepsilon\eta$ if $k \equiv -1 \pmod 3$. Thus, $\sigma(\zeta_3)$ determines the extension sign.

At 3 the Borel $P \rtimes H$ of $\Omega_7(3)$ has order $4 \cdot 3^9$. The good-prime commutator formula gives $W = P/P' = \bigoplus_{i=1}^{3} W_{\alpha_i}$ for the simple roots of B_3. The spinor norm equation $\alpha_1\alpha_2^2\alpha_3^3 = 1$ gives torus signs (u, v, u) on W. Next, let $a, b \in H$ act by $-I$ on the u-lines and v-line, respectively. The parabolic for α_1 has radical V of order 3^5. By An–O'Brien [4, Table 1, row 3^5], the lower group is $V : \mathrm{U}_4(2).2$. By Breuer [23, Fi22.2M10], the upper group is $V : (2 \times \mathrm{U}_4(2).2)$. Moreover, its complementary central involution t acts scalarly on irreducible orthogonal V. Exterior centralizers exclude $P \le \mathrm{C}_S(t)$, so $t|_V = -I$. t centralizes the lower Levi L_0, with L_0 as a graph subgroup. Hence, t gives $W_{\alpha_1} \mapsto -W_{\alpha_1}$ alone. The complement $H \times \langle t \rangle$ gives all signs on W. All stabilizer labels have values in $\{1, -1\}$. Moreover, $ab = -I$ returns $-\beta$ to β and fixes its label. Thus, the 3′-degree rows of U are rational. By Breuer [23, Fi22.2], $320321, 80083$ fix the 27 rows of $\mathrm{Irr}_{3'}(A)$. By Lemma 8.19, the quotient satisfies $\succeq_c$.

At faithful λ, lift t to Y and put $\mathrm{Z}(X) = \langle z \rangle$. In the quotient parabolic $V : (C_2 \times L_0)$, with $L_0 \simeq \mathrm{U}_4(2).2$, let ξ be the sign on $L_0/\mathrm{U}_4(2)$, trivial on t, V. On the B_2 Levi the spinor norm is $\alpha_2\alpha_3^2 = \alpha_2$, so $\xi(a) = 1$ and $\xi(b) = -1$.

For exterior involutions with $\xi = 1$, columns $5, 8, 16$ of `Fi22.2M10` fuse to $61, 60, 61$ in A. Those with $\xi = -1$ have columns $71, 77$, both fusing to 62 by Breuer [23]. The lifts in Y have orders $2, 2, 4$ at $60, 61, 62$, respectively. Thus, $t^2 = (at)^2 = 1$ and $(bt)^2 = z$. Since $a^2 = b^2 = 1$ in the split preimage, $[t, h] = z^{d(h)}$ has $d(a) = 0, d(b) = 1$. If a weight orbit returns under t, at most one u-coordinate is nonzero. Then, extension requires $H_\beta \leq \ker d$. Each u-axis with $v \neq 0$ has $H_\beta = 1$. A u-axis alone has stabilizer $\langle b \rangle$, the v-axis has $\langle a \rangle$, and weight 0 has H. Thus, there are 4 local fixed parents. Thus, $v \neq 0$ for extending weights. The lift is t or at as the u-axis varies. Moreover, both square to 1 and centralize the stabilizer in $\langle a \rangle$. All stabilizer signs lie in $\{\pm 1\}$. The inversion transporter ab has commutator z with either lift. Thus, $\sigma \in \mathcal{H}_p$ swaps the two extensions exactly when $\sigma(\zeta_3) = \zeta_3^{-1}$. By Breuer [23, `2xFi22N3`], the 18 faithful parents are $\mathcal{H}_p$-fixed. Both sides have 4 fixed parents and 7 free c-pairs. Match the c-orbits and then the ε-pairs by their shared $\mathcal{H}_p$-action. Thus, Proposition 2.13 applies.

At 5, An–O'Brien [4, Table 1, row 5^2] gives $N = V : K$, $V = C_5^2 = \mathrm{C}_S(V)$, and $U = N \times C_2$. Thus, $K = 4.\mathfrak{S}_4$ acts faithfully on V. Since $5 \nmid |K|$, a reducible V splits as $V_1 \oplus V_2$ into lines. An endomorphism field $\mathbb{F}_{25}$ puts K in $\mathbb{F}_{25}^\times$. Both cases contradict $K' \neq 1$. Thus, V is absolutely irreducible. The normal C_4 acts by scalars, since otherwise K permutes its two eigenlines and $|K| \leq 32$. Hence, $\mathrm{Z}(K) = \mathbb{F}_5^\times I$. The 20 rows per fibre are $\mathcal{H}_p$-fixed by Breuer [23, `2xFi22N5`]. By Lemma 8.18, $U = N \times C_2$ gives $\succeq_c$ over $1_{\mathrm{Z}(X)}$.

Lift the exterior involution to t in Y. On the split N, its action is multiplication by $\delta : N \to \langle z \rangle$, with $\delta \neq 1$ by the parent count. Since $K' \leq \mathrm{SL}_2(5)$ maps onto $\mathfrak{A}_4$, the quaternion preimage of V_4 gives $-I$ as a commutator of lifts. Hence, $K' = \mathrm{SL}_2(3)$ as the binary tetrahedral subgroup. Since $\mathfrak{S}_4$ lies outside $\mathrm{PSL}_2(5)$, the determinant meets a nonsquare in $\mathbb{F}_5^\times$, so $\det(K) = \mathbb{F}_5^\times$. It follows that $K/K' \simeq C_4$. Scalar action gives $V \leq N'$. Thus, $\delta = \det^2$ in sign notation and $\ker(\delta|_K) = C_4 \circ \mathrm{SL}_2(3)$.

Put $L = \ker \delta$ and $R = \langle z, t \rangle$. Given an extension ξ of λ to R, the values are in $\mu_4 \subset \mathbb{Q}_5$. Its inertia is $R \times L$, so $\mathrm{Irr}_{5'}(L) \longrightarrow \mathrm{Irr}_{5'}(\mathrm{N}_Y(V) \mid \lambda)$, $\quad \rho \longmapsto \mathrm{Ind}_{R \times L}^{\mathrm{N}_Y(V)}(\xi \otimes \rho)$ is the Clifford bijection. The twist ε acts there by conjugation by N/L, since it exchanges the extensions on R. A nontrivial cubic ω on $\mathrm{SL}_2(3)/Q_8$ gives a character of L via the quotient by $\langle V, C_4 \rangle$. For each scalar label choose an N-invariant, $\mathcal{H}_p$-fixed ρ on $C_4 \circ \mathrm{SL}_2(3)$, viewed on L. By contrast, even labels factor through $L/\mathrm{SL}_2(3)$. Next, take Brauer lifts of V, V^* for odd labels as K-modules. Their restrictions to $\mathrm{SL}_2(3)$ are irreducible, with trace -1 at order 3. Hence, we find $\rho\omega \neq \rho\omega^{-1}$. Frobenius at 5 swaps $\rho\omega, \rho\omega^{-1}$, as does N/L by inversion on $\mathfrak{A}_4/V_4$. Moreover, the 4 scalar labels account for all 4 extension fibres. Their sign agrees on ζ_3 with the global sign. Proposition 2.13 gives $\succeq_c$.

Now let $\lambda \in \mathrm{Irr}(\mathrm{Z}(6.S))$ have order 3 or 6. Both c and the Frobenius F at 5 invert λ. Moreover, the projective fields involve μ_6 and quadratic periods of conductors $8, 11, 13$. Over $\mathbb{Q}_5$ they are unramified of degrees ≤ 2. Thus, we have that $T = cF$ is an involution on the λ-fibre. A T-equivariant bijection extends by c to the inverse fibre equivariantly for $F = cT$. Both λ-fibres have size 20 by An–O'Brien [4, proofs of Theorems 8.1 and 9.1].

The lower cover is $\mathrm{Z}(6.S) \times N$, with $N = C_5^2 : 4.\mathfrak{S}_4$. An outer lift centralizing N modulo $\mathrm{Z}(6.S)$ twists the splitting by $N_{\mathrm{ab}} \to \mathrm{Z}(6.S)$. Since $N_{\mathrm{ab}} = C_4$, its 3-part is trivial. Its 2-part is δ above. Also $\mathrm{Irr}(N)$ is $\mathcal{H}_p$-fixed. Thus, we find $T = 1$ for $|\lambda| = 3$, and $T : \eta \mapsto \delta\eta$ for $|\lambda| = 6$. The double-cover count gives 4 fixed by δ. Thus, locally the counts are $20, 4$, respectively.

Take the fusions into $3.S.2, 6.S.2$ from Breuer [23]. For $J = 3.S, 6.S$ and $\chi \in \mathrm{Irr}_{5'}(J \mid \lambda)$, put $\eta = \mathrm{Ind}_J^{J.2}\chi$. By the fusion, $c\chi = \eta|_J - \chi$. By Section 8, the powers $576577, 576581$ give inertia and F, respectively. Inertia is trivial and $F^2 = 1$ on each cut. Moreover, $F\chi = c\chi$ holds for all 20 rows over the order-3 label and for 4 over the order-6 label. Thus, match the T-orbits over λ. Extend by c and apply Proposition 2.13.

With $S = \mathrm{Fi}_{24}'$, we write $A = \mathrm{Fi}_{24}$ and $X = 3.S$ and $Y = 3.S.2$. By Conway et al. [45, Monster, class 3A], the normalizer of $\mathrm{Z}(X)$ in the Monster is Y, and its centralizer is X. Thus, c inverts $\mathrm{Z}(X)$. By Breuer [23, `F3+.2`], take $\mathrm{Irr}_{p'}(A)$. At $p = 3$, $12712379681, 23608705123$ fix all 81 rows. At $p = 5$, $9806692897, 9806692901$ fix all 100. At $p = 7$, the table has 35 rows of $7'$-degree. For each χ in these rows and every $g \in A$, the power maps give $\chi(g^k) = \chi(g)$ for $k \in \{3502390321, 14009561287\}$. Thus, $\mathcal{H}_p$ fixes the upper cut.

At 3, we have that the automizers of P in S, A are C_2^3, C_2^4 by Gautam [71, Table 2]. By An et al. [5, Proposition 4.1 and Table 10, chain 36], $\mathrm{C}_S(P) = \mathrm{Z}(P)$ and $|N : P| = 8$. Since $[U : N] = 2$, the automizer has order $|U : P| = 16$. Thus, $\mathrm{C}_A(P) = \mathrm{Z}(P)$ and a complement C_2^4 acts faithfully on P. By Gautam [71, §5, subgroup M_4], choose P in $3^7.\mathrm{SO}_7(3)$. Write V for the natural 7-space and $\bar{P} = P/V$. Since $(V^*)^P$ is the highest-weight line, $\dim_{\mathbb{F}_3} V/[V, P] = 1$. For the simple roots $\alpha_1, \alpha_2, \alpha_3$ of B_3 we have $\bar{P}/\bar{P}' \simeq \prod_{i=1}^3 X_{\alpha_i} \simeq C_3^3$. Hence, $V/[V, P] \longrightarrow P/P' \longrightarrow \bar{P}/\bar{P}' \longrightarrow 1$ is exact, and $|P/P'| \leq 3^4$. By Gautam [71, Lemma 2.5], C_2^4 acts faithfully on $P/\Phi(P)$, requiring 4 independent signs.

Thus, $P/P' = P/\Phi(P) = \mathbb{F}_3^4$, with all coordinate signs in U/P. An upper $3'$-row lies over a linear $\beta \in \mathrm{Irr}(P)$, with a stabilizer label in $\{\pm 1\}$. The scalar $-I$ returns $-\beta$ to β, and fixes the sign ± 1. Thus, these values lie in $\mathbb{Q}$. By Breuer [23, `F3+.2N5`], $97, 101$ fix all 100 rows of $5'$-degree on U. At 7, the powers $73, 43$ fix the 35 rows of $7'$-degree in Breuer [23, `F3+.2N7`]. The exponents are 120 at 5 and 84 at 7. Thus, apply Lemma 8.19 on S, N at $3, 5, 7$.

On the cover, write N_X, U_Y for the preimages of N, U. Given a faithful $\nu \in \mathrm{Irr}(\mathrm{Z}(X))$, the action of c exchanges ν, ν^{-1}. By Gómez-Taberner [78, Appendix A], both have 40 rows at 5 on both X, N_X. By An et al. [5, proofs of Theorems 6.1 and 7.1, case (1)], each fibre has 25 rows at 7. At 7, the powers $3502390321, 14009561287$ fix the 75 rows of $7'$-degree in Breuer [23, `3.F3+`]. Locally, the power 31 fixes the 75 rows in Breuer [23, `3xF3+N7`] of exponent 42. Thus, $\mathcal{H}_p$ fixes both fibres at 7.

At 5, the powers $9806692897, 9806692901$ fix the 140 rows of $5'$-degree on Y by Breuer [23, `3.F3+.2`]. The powers $97, 101$ fix the 140 rows of $5'$-degree on U_Y by Breuer [23, `3.F3+.2N5`]. Hence, $\mathcal{H}_p$ fixes $\mathrm{Ind}\,\chi$ for each faithful parent χ. By Gómez-Taberner [78, Theorem 4.3], restriction gives $\chi + \chi^c$ above ν, ν^{-1}. Thus, on both sides, $\chi^\sigma = \chi$ if $\nu^\sigma = \nu$, and $\chi^\sigma = \chi^c$ otherwise. By Gómez-Taberner [78, Appendix A], the bijection is c-equivariant with the same central character ν on each pair. Hence, it is $\mathcal{H}_p$-equivariant on the faithful fibres. At 7, pair the ν-fibres and extend by c. By Proposition 2.13, these maps give $\succeq_c$ on both faithful fibres. At $p = 3$, Section 2 gives $\mathrm{Z}(X)_p \leq \ker\chi$ for each $\chi \in \mathrm{Irr}_{p'}(X) \cup \mathrm{Irr}_{p'}(\mathrm{N}_X(R))$, where $R \in \mathrm{Syl}_p(X)$. Thus, all central fibres of $3.\mathrm{Fi}'_{24}$ are covered at $p = 3, 5, 7$.

Corollary 8.23. *Let S be a sporadic simple group, $p \mid |S|$ and $Q \in \mathrm{Syl}_p(S)$. Then* $\mathrm{iMN}(S, p)$ *holds if $p = 2$, Q is cyclic, or* $\mathrm{Out}(S) = 1$. *It also holds for all $p \mid |S|$ when $S \in \{M_{12}, M_{22}, J_2, J_3, \mathrm{HS}, \mathrm{Suz}\}$.*

Proof. At $p = 2$, apply [159, Proposition 2.1]. At odd p, let G be the universal p'-cover of S. A Sylow p-subgroup of G maps isomorphically to Q. By [157, proof of Lemma 4.2], $Q/[Q, Q]$ is elementary abelian. Thus, cyclic Q has order p. Corollary 2.10 gives $\mathrm{iMN}(S, p)$ in this case. Suppose now Q is noncyclic. Write X for the universal cover, $R \in \mathrm{Syl}_p(X)$ and $N = \mathrm{N}_X(R)$. Suppose $\mathrm{Out}(S) = 1$, and put $Z = \mathrm{Z}(X)$. By [45], $Z = 1$ except for $S \in \{\mathrm{Ru}, \mathrm{Co}_1, B\}$, where $|Z| = 2$. If $S = B$, Proposition 8.6 gives $\mathrm{iMN}(B, p)$. Suppose $S \neq B$. If $Z = 1$, take $\Omega = \Omega_{1_Z}$ from Section 8. If $S = \mathrm{Ru}$, then $p = 3, 5$, and if $S = \mathrm{Co}_1$, then $p = 3, 5, 7$ [45]. On these double covers, Proposition 8.20 gives Ω on both Z-fibres. Thus, with Ω, apply [186, Theorem 2.6(b)] as in Lemma 8.3 to obtain $\mathrm{iMN}(S, p)$.

Given M_{12}, J_2, J_3, Propositions 8.15, 8.16 and 8.17 cover p with noncyclic odd Sylow subgroup. For Suz, Propositions 8.22 and 8.21 cover both signs modulo $\mathrm{Z}(X)_3$ at $p = 3$ and all six labels at $p = 5$. Moreover, the local subgroup is N in each case. If $S = M_{22}$, $p = 3$, every $\chi \in \mathrm{Irr}_{p'}(X) \cup \mathrm{Irr}_{p'}(N)$ has $\mathrm{Z}(X)_p \leq \ker\chi$ by Section 2. Thus, $X/\mathrm{Z}(X)_p = 4.M_{22}$. Propositions 8.7, 8.14 and 8.13 cover $|\lambda| = 1, 2, 4$, respectively. By [101, Lemma 5.1], inflation gives maps on X, N. If $S = \mathrm{HS}$, $p = 3$, Propositions 8.7–8.9 exhaust blocks of defect R on $2.\mathrm{HS}, N$. If $p = 5$, Proposition 8.11 matches the moving pairs $\{\chi, \chi^c\}$. Theorem 8.12 matches the extensions $\{\eta, \varepsilon\eta\}$ over both signs of $\mathrm{Z}(X)$. The maps intertwine c, ε in Proposition 2.13, giving the $\succeq_c$ relation. At all remaining odd p in these six groups, $|Q| = p$ by [45]. □

8.6 Proof of Theorem 8.1

Proof. By Corollary 8.23, only odd p and noncyclic $Q \in \mathrm{Syl}_p(S)$ remain, with $\mathrm{Out}(S) \neq 1$ and $S \notin \{M_{12}, M_{22}, J_2, J_3, \mathrm{HS}, \mathrm{Suz}\}$. Thus, the list in [45] leaves $S \in \{\mathrm{He}, \mathrm{McL}, \mathrm{O'N}, \mathrm{HN}, \mathrm{Fi}_{22}, \mathrm{Fi}'_{24}\}$. Let X be the universal cover, Y its overgroup of index 2, and $R \in \mathrm{Syl}_p(X)$. Put $N = \mathrm{N}_X(R)$, $U = \mathrm{N}_Y(R)$. By [97, Lemma 1.13], $U/N \simeq Y/X$. As in Proposition 2.13, write c for conjugation by $u \in U \setminus N$ and ε for the sign of U/N.

Put $C = \mathrm{Z}(X)_p$. Since R is the unique Sylow p-subgroup of RC, its normalizer is $U = \mathrm{N}_Y(RC)$. Thus, $U/C = \mathrm{N}_{Y/C}(RC/C)$ and $N/C = \mathrm{N}_{X/C}(RC/C)$. Inflation from $Y/C, U/C$ is equivariant for $\mathcal{H}_p, c, \varepsilon$. By Section 2, $C \leq \ker\chi$ for all $\chi \in \mathrm{Irr}_{p'}(X) \cup \mathrm{Irr}_{p'}(N)$. Thus, by Clifford theory, $C \leq \ker\chi$ for $\chi \in \mathrm{Irr}_{p'}(Y) \cup \mathrm{Irr}_{p'}(U)$. At $(S, p) = (\mathrm{He}, 3), (\mathrm{He}, 5), (\mathrm{He}, 7), (\mathrm{McL}, 3), (\mathrm{McL}, 5)$, apply Section 8.3. Section 8.5 gives them for $\mathrm{O'N}$ at $3, 7$, HN and Fi_{22} at $3, 5$, and Fi'_{24} at $3, 5, 7$. The comparisons over $\langle c, \mathcal{H}_p\rangle$-orbits in $\mathrm{Irr}(\mathrm{Z}(X))$ cover all p'-degree characters. After inflation to X, N, Y, U where needed, their union gives $\mathrm{iMN}(S, p)$ by Proposition 2.13. □

9 Proof of the main theorem

Theorem 1.2 follows from Navarro et al. [141, Theorem A] if $\mathrm{iMN}(S, p)$ holds for every nonabelian simple S with $p \mid |S|$. Fix such an S and write X for the universal cover of S, choose $R \in \mathrm{Syl}_p(X)$, and put $\Gamma = \mathrm{Aut}(X)_R$. If

$S = A_n$ and p is odd, Brunat–Nath [30, Theorem 1.1] gives the $\mathcal{H}_p$-equivariant correspondence on A_n. At odd p, Brunat–Nath [29, Corollary 5.3] gives $G(p)$-equivariance on the spin characters of $2.A_n$, where $G(p) \leq \mathcal{H}_p$ fixes all roots of p'-order. The comparisons below give $\succeq_c$ on the universal cover X and its local subgroup.

Lemma 9.1. *Let $n \geq 7$ and let $p \leq n$ be an odd prime with $p \nmid n$. If $n = 7$, suppose that $p = 3$. Then* iMN(A_n, p) *holds.*

Proof. We put $r \equiv n \pmod p$, with $1 \leq r < p$, and $m = n - r$. Moreover, let $Y = 2.\mathfrak{S}_n$ and $X = 2.A_n$, with $Z = \langle z \rangle = \mathrm{Z}(X)$. Let $\mathfrak{S}_r$ act on $\{1, \dots, r\}$, and let $\mathfrak{S}_m$ act on $\{r+1, \dots, n\}$. By [76, §2.2], choose $P \in \mathrm{Syl}_p(\mathfrak{S}_m)$ as a Sylow subgroup of both $\mathfrak{S}_n$ and A_n. Its fixed letters are precisely $\{1, \dots, r\}$, from the digit r of n. Next, choose $R \in \mathrm{Syl}_p(X)$ with $RZ/Z = P$. Let V be the preimage of $\mathfrak{S}_r \times \mathfrak{S}_m$ in Y and put $M = V \cap X$. The same normalizer formula gives $\mathrm{N}_Y(R) \leq V$. Since X is perfect, conjugation gives $\mathrm{Aut}(X) \cong Y/Z$ by $\mathrm{Aut}(A_n) \cong \mathfrak{S}_n$. Thus, $M < X$ satisfies Proposition 2.13.

Put $K = \mathfrak{S}_r \times \mathrm{N}_{\mathfrak{S}_m}(P)$, $K^+ = K \cap A_n$ and $J = (V/Z) \cap A_n$. By [30, Proposition 2.3 and Section 5], take the maps ω_n, ω_n^+ from $\mathfrak{S}_n, A_n$ to K, K^+, compatible with restriction. Moreover, on V/Z, put $\omega_V = \mathrm{id} \times \omega_m$ with ω_m the same parent map on $\mathfrak{S}_m$. Thus, ω_n, ω_V are $\langle \varepsilon \rangle$-equivariant. If $r = 1$, take ω_m^+ on $J = A_m$. For $r \geq 2$, let $\eta = \eta_r \otimes \eta_m$ have split restriction to J. By [30, Proposition 3.2], $e_\sigma(\eta) = e_\sigma(\eta_r) e_\sigma(\eta_m)$ for $\sigma \in \mathcal{H}_{p_\eta}$, where e_σ is the permutation sign. By $\mathcal{H}_p$-equivariance of ω_V, both pairs have stabilizer $\mathcal{H}_{p_\eta}$. By [30, Lemma 5.1], their signs agree on $\mathcal{H}_{p_\eta}$. Therefore, Lemma 2.14 matches the pairs for $\mathcal{H}_p$ and odd conjugation. Given $\eta|_J$ irreducible, [43] gives its image $\omega_V(\eta)|_{K^+}$. Write ω_V^+ for the map to K^+. Then, the compositions $\omega_V^{-1}\omega_n$ and $(\omega_V^+)^{-1}\omega_n^+$ give the comparisons over 1_Z.

Suppose λ is a spin label and put $\nu = \lambda^{[p]}$. By [29, Section 5, (36)], full defect in Y requires p-bar cores κ of size r. At full defect in V, the second factor has weight m/p and core $\varnothing$. Since $[Y : X] = [V : M] = 2$, the full-defect spin blocks of X, M lie under these blocks of Y, V. By [28, Theorems 6.3–6.4 and 7.4], $\lambda \mapsto (\kappa, \nu)$ gives $\mathcal{H}_p$-equivariant bijections Ω_Y, Ω_X on the spin parts of $\mathrm{Irr}_{p'}(Y)$ and $\mathrm{Irr}_{p'}(X)$. By [28, (41)], $\Omega_X(\eta|_X) = \Omega_Y(\eta)|_M$ by linear extension. It also gives $\Omega_X(\chi^v) = \Omega_X(\chi)^v$ for $v \in V \setminus M$ and $\Omega_Y(\varepsilon\eta) = \varepsilon\Omega_Y(\eta)$. Moreover, z acts as 1 or -1 on the representations of Y, V, X, M. Thus, Proposition 2.13 gives Ω and the triple comparison on (X, M). Given $n \geq 8$, the group X is the universal cover of A_n. If $(n, p) = (7, 3)$, the universal cover is $6.A_7$, and $X = 2.A_7$ is its $3'$-part. Thus, by [102, Lemma 4.15], both cases give iMN(A_n, p). □

Lemma 9.2. *Let $n \geq 8$, and let $p \leq n$ be an odd prime. Suppose that $p \mid n$ and that n has at least two nonzero base-p digits. Then* iMN(A_n, p) *holds.*

Proof. We put $t = p^a$, where $a \geq 1$ and $t \leq n < pt$. Write $n = m + r$, with $m = t\lfloor n/t \rfloor$ and $0 < r < t$. Moreover, let $Y = 2.\mathfrak{S}_n$ and $X = 2.A_n$, with $Z = \mathrm{Z}(X) = \langle z \rangle$. Let $\mathfrak{S}_r$ act on $\{1, \dots, r\}$, and let $\mathfrak{S}_m$ act on $\{r+1, \dots, n\}$. The sum $m + r$ has no base-p carries. Thus, Kummer's theorem in [81, Section 1] gives $p \nmid \binom{n}{r}$. By [76, §2.2], take $P = P_r \times P_m$ for $P_i \in \mathrm{Syl}_p(\mathfrak{S}_i)$, $i = r, m$. Thus, P is Sylow in $\mathfrak{S}_n$ and A_n. The normalizer is $K = \mathrm{N}_{\mathfrak{S}_r}(P_r) \times \mathrm{N}_{\mathfrak{S}_m}(P_m)$ by the same formula. Next, choose $R \in \mathrm{Syl}_p(X)$ with $RZ/Z = P$. For the preimage V of $\mathfrak{S}_r \times \mathfrak{S}_m$ in Y, put $M = V \cap X$. Thus, $\mathrm{N}_Y(R) \leq V$ by $K \leq V/Z$. As in Lemma 9.1, $M < X$ satisfies Proposition 2.13.

Put $J = M/Z$. By [30, Proposition 2.3 and Section 5], take the maps ω_n, ω_n^+ to $K, K \cap A_n$, respectively. The same construction on $\mathfrak{S}_r, \mathfrak{S}_m$ gives the product map ω_V on V/Z to K, equivariant for $\mathcal{H}_p$ and the sign ε. With $p \mid r, m$, the factor signs are nontrivial. For ω_V, combine [30, Lemma 5.1] on $\mathfrak{S}_r, \mathfrak{S}_m$ as in Lemma 9.1. Thus, ω_V^+ on J is equivariant for $\mathcal{H}_p$ and odd conjugation. Moreover, $\omega_V^+(\mathrm{Res}_J\, \eta) = \mathrm{Res}_{K \cap A_n}\, \omega_V(\eta)$ by linear extension. The compositions $\omega_V^{-1}\omega_n, (\omega_V^+)^{-1}\omega_n^+$ give the comparisons over 1_Z.

Given a bar partition τ, let $w_s(\tau)$ be its s-bar weight for $s = p^j$. By [80, Sections 1–2], there are $w_s(\tau)$ bars of length divisible by s. Thus, the bar formula gives $v_p(\xi_\tau(1)) = \sum_{j \geq 1}(\lfloor |\tau|/p^j \rfloor - w_{p^j}(\tau))$, with nonnegative summands. In particular, at p'-degree, $w_t(\tau) = \lfloor |\tau|/t \rfloor$. For $|\tau| = n$, the t-bar core has size r. For $|\tau| = m$, that core is empty.

By [28, Section 3], write (κ, ν) for the t-bar core and cocore of λ, respectively. Since $r < t$, every κ of size r is a t-bar core. Thus, (κ, ν) of sizes r, m with $\mathrm{core}_t(\nu) = \varnothing$ determines a unique λ. By [80, Corollary 4.3], $\xi_\lambda(1)$ equals $\binom{n}{r}\, \xi_\kappa(1)\xi_\nu(1)$ up to a power of 2. The degree on V with label (κ, ν) is also a power of 2 times $\xi_\kappa(1)\xi_\nu(1)$ by [28, Section 5]. Since $p \nmid \binom{n}{r}$, these degrees have the same p-valuation. Thus, these labels match the p'-degree spin classes of Y, V and of X, M.

For τ a bar partition, put $d_\tau = |\tau| - \ell(\tau)$ and $b_\tau \in \{0, 1\}$ with $b_\tau \equiv d_\tau \pmod 2$. Moreover, set $D(\tau) = 2^{b_\tau}(-1)^{\lceil d_\tau/2 \rceil} \prod_{h \in \tau} h$. The pair (ξ_τ^+, ξ_τ^-) lies on $2.\mathfrak{S}_{|\tau|}$ or $2.A_{|\tau|}$, according to b_τ. For $\sigma \in \mathcal{H}_p$, Schur's formulas give on

(ξ_τ^+, ξ_τ^-) the sign $\sigma(\sqrt{D(\tau)})/\sqrt{D(\tau)}$ [29, Section 2.2]. Moreover, put $L = \prod_{h\in\lambda} h/(\prod_{h\in\kappa} h \prod_{h\in\nu} h)$. By [28, proof of Lemma 3.11, (10)–(11)], the zero runner cancels from L. The same bead count gives $\ell(\lambda) = \ell(\kappa) + \ell(\nu) - 2d_0$, where $d_0 = \sum_{1\le i<t/2} |D_i|$ for the bead sets D_i there. Thus, matching modulo t leaves $|D_i|$ denominator pairs at residues $i, -i$. Since $t = p^a$ and $v_p(i) < a$, matched parts have a ratio in $1 + p\mathbb{Z}_p \subseteq (\mathbb{Q}_p^\times)^2$. A denominator pair has product $p^{2v_p(i)}uv$ with $u \equiv -v \pmod p$ and $p \nmid uv$. Thus, its square class is -1. Consequently, $(-1)^{d_0}L \in (\mathbb{Q}_p^\times)^2$.

By [28, Section 5 and (46)], the Humphreys product has discriminant $D_{\mathrm{loc}} = D(\kappa)D(\nu)(-1)^{b_\kappa b_\nu}$ modulo $(\mathbb{Q}^\times)^2$. The length formula gives $D(\lambda)/D_{\mathrm{loc}} \equiv (-1)^{d_0}L \pmod{(\mathbb{Q}^\times)^2}$. By [141, Section 1], $\mathcal{H}_p$ fixes $\sqrt{D(\lambda)/D_{\mathrm{loc}}}$. Thus, both signs agree on $\mathcal{H}_p$. By [28, Corollary 3.12 and Section 5], the split types of λ and (κ, ν) agree. Thus, Lemma 2.14 gives Ω_Y, Ω_X equivariant for $\mathcal{H}_p$ on each pair. Moreover, $\Omega_X(\eta|_X) = \Omega_Y(\eta)|_M$ by linear extension. The map Ω_Y is equivariant for sign twist and Ω_X for odd conjugation. In each pair, z acts by -1. Consequently, for M, Proposition 2.13 gives $\mathrm{iMN}(A_n, p)$. □

Theorem 9.3. *For every finite nonabelian simple group S and every $p \mid |S|$, let X be the universal cover, $R \in \mathrm{Syl}_p(X)$, and $\Gamma = \mathrm{Aut}(X)_R$. There is one Γ-stable subgroup $\mathrm{N}_X(R) \le M < X$ and a $\Gamma \times \mathcal{H}_p$-equivariant bijection $\Omega : \mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(M)$. Thus, $\Omega(\chi^{a\sigma}) = \Omega(\chi)^{a\sigma}$ for $(a, \sigma) \in \Gamma \times \mathcal{H}_p$. Moreover, each pair $(\chi, \Omega(\chi))$ satisfies* the $\succeq_c$ relation *on $\Gamma_{\chi^{\mathcal{H}_p}}$.*

Lemma 9.4. *Next, fix X, R, Γ as above. Moreover, choose one Γ-stable subgroup $\mathrm{N}_X(R) \le M < X$. Put $K = \Gamma \times \mathcal{H}_p$. Suppose the p'-degree characters of X, M have disjoint partitions indexed by I, $\mathrm{Irr}_{p'}(X) = \coprod_{i\in I} \mathcal{X}_i$, $\mathrm{Irr}_{p'}(M) = \coprod_{i\in I} \mathcal{Y}_i$. Assume I is a right K-set. Moreover, both partitions are K-equivariant. For one i in each orbit, take K_i as its stabilizer. Suppose that $b_i : \mathcal{X}_i \longrightarrow \mathcal{Y}_i$ is a K_i-equivariant bijection. Choose one χ per K_i-orbit in $\mathcal{X}_i$, and put $\psi = b_i(\chi)$ and $D = \mathrm{pr}_\Gamma(K_\chi)$, with K_χ the stabilizer in K. Assume that (χ, ψ) satisfies* the $\succeq_c$ relation *for this D. Then the maps b_i extend uniquely to a K-equivariant bijection Ω. Moreover, Ω satisfies* the $\succeq_c$ relation *at each character χ. In particular,* $\mathrm{iMN}(S, p)$ *holds for $S = X/\mathrm{Z}(X)$.*

Proof. For each representative $i \in I$, the union of the fibres over i^K is the induced K-set $\mathcal{X}_i \times_{K_i} K$ via $[x, k] \mapsto x^k$. The other union is $\mathcal{Y}_i \times_{K_i} K$ by the same rule. Thus, b_i extends uniquely by $[x, k] \mapsto [b_i(x), k]$, with inverse induced by b_i^{-1}. Thus, their union is Ω. Since the partition $(\mathcal{X}_i)_{i\in I}$ is K-equivariant, $\chi \in \mathcal{X}_i$ has $K_\chi \le K_i$. Equivariance of Ω gives $K_\chi = K_{\Omega(\chi)}$. Moreover, $\gamma \in \Gamma_{\chi^{\mathcal{H}_p}}$ exactly when $\chi^\gamma = \chi^\tau$ for some $\tau \in \mathcal{H}_p$. Thus, $D = \mathrm{pr}_\Gamma(K_\chi) = \Gamma_{\chi^{\mathcal{H}_p}}$. By [141, Lemmas 2.1 and 2.3], the comparison for one pair $(\chi, \Omega(\chi))$ gives it on the whole K-orbit. Consequently, M and Ω give $\mathrm{iMN}(S, p)$. □

Lemma 9.5. *Assume p is odd and $t = p^a$, with $a \ge 1$. Let $Y = 2.\mathfrak{S}_t$, $Z = \langle z \rangle$, $R \in \mathrm{Syl}_p(Y)$, and $V = \mathrm{N}_Y(R)$. Moreover, let Λ be a linear spin character of V/R', and let ε be the sign character of V. Put $h = (p-1)/2$, $H = hp^{a-1}$, and $C = (-1)^{H(H+1)/2}$. Let $\sigma_p \in \mathcal{H}_p$ fix roots of p-power order and act by the p-th power on roots of p'-order. Then $\Lambda^{\sigma_p} = \Lambda$ if $C = 1$, and $\Lambda^{\sigma_p} = \varepsilon\Lambda$ if $C = -1$. Moreover, $C = (\frac{2}{p})(\frac{-1}{p})^{(t+1)/2}$.*

Proof. By [29, Sections 3.1–3.2], the quotient $(V/R')/Z$ is L_p^a for $L_p = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$. Since $[L_p, L_p] = \mathbb{F}_p$, we have $R \le [V, V]Z$. As $|R|$ is odd, $\Lambda|_R = 1_R$. Thus, $\Lambda(V)$ has exponent dividing $2(p-1)$. For $1 \le i \le a$, let $x_i \in V$ lift a primitive multiplier on digit i of the standard action of R. The other digits are fixed. Since $h = (p-1)/2$, the permutation of x_i^h has H disjoint transpositions. By the relations in [29, Section 2], $x_i^{p-1} = z^{H(H+1)/2}$. Also, $\varepsilon(x_i) = -1$. Therefore, $\Lambda^{\sigma_p}(x_i)/\Lambda(x_i) = \Lambda(x_i^{p-1}) = C$. Since $R, z, x_1, \ldots, x_a$ generate V/R' by their images, $\Lambda^{\sigma_p}/\Lambda$ is 1 or ε as stated. Put $q = p^{a-1}$. Since $H = hq$ and $t = pq$, $(H(H+1) - h(h+1))/2 \equiv h(q-1)/2 \equiv h(t+1)/2 \pmod 2$. Thus, the supplementary laws $(\frac{2}{p}) = (-1)^{h(h+1)/2}$ and $(\frac{-1}{p}) = (-1)^h$ give the formula for C. □

Lemma 9.6. *Let p be an odd prime, let $t = p^a$ with $a \ge 1$, and let $1 \le r < p$. Let ρ be an elementary spin representation of $N_{\mathfrak{S}_t}(P)^+$, with $P \in \mathrm{Syl}_p(\mathfrak{S}_t)$, and suppose that ρ is not self-associate. Let $\mu \vdash r$, and let $\Theta_{\rho,\mu}$ be the representation of $(N_{\mathfrak{S}_t}(P) \wr \mathfrak{S}_r)^+$* Brunat–Nath [29, Section 3.3.1]. *Put $d = r \bmod 2$ and $D_\circ = 2^d(-1)^{\lceil rt/2 \rceil}$. Moreover, let $\sigma_p \in \mathcal{H}_p$ fix p-power roots and act on p'-roots by the p-th power. Then σ_p fixes the association class of $\Theta_{\rho,\mu}$. Moreover, σ_p acts on the pair attached to $\Theta_{\rho,\mu}$ with sign $\sigma_p(\sqrt{D_\circ})/\sqrt{D_\circ}$.*

Proof. Put $h = (p-1)/2$, $H = hp^{a-1}$, and $C = (-1)^{H(H+1)/2}$. Moreover, put $L_p = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$. By [29, Section 3.2], we have $\rho = \Lambda \otimes \mathrm{Inf}(\phi)$, with linear spin Λ and $\phi \in \mathrm{Irr}(L_p^a)$. The L_p^a-representation has matrices over $\mathbb{Q}(\zeta_p, \zeta_{p-1})$. Moreover, Lemma 9.5 gives $\Lambda^{\sigma_p}/\Lambda = \varepsilon^e$, with $C = (-1)^e$. Thus, we choose matrices $\rho(x)$ with $\rho(x)^{\sigma_p} = C^{\partial(x)}\rho(x)$, where $\varepsilon(x) = (-1)^{\partial(x)}$. Put $u = (\frac{2}{p})$, $v = (\frac{-1}{p})$, and $g = v^{(t-1)/2}$. Moreover, Lemma 9.5 gives $C = uvg$.

By [29, Propositions 3.3–3.4], the j-th base factor acts by $\rho(x)^{(j)} \otimes F_j^{\partial(x)}$ after reindexing. The lift of s_j acts by $P_j \otimes \gamma_t D_j \otimes S^\mu(s_j)$, where $\gamma_t = 1$ for $t \equiv 1 \pmod 4$ and $\gamma_t = i$ otherwise. The Specht matrices $S^\mu(s_j)$ are rational. Since D_j and $i(F_j + F_{j+1})/\sqrt{2}$ have square -1 and give the same conjugations on F_i, $D_j = \pm i(F_j + F_{j+1})/\sqrt{2}$ by Schur's lemma. Thus, σ_p acts on $\gamma_t i/\sqrt{2}$ with sign $gvu = C$.

For $w \in \{1, -1\}$, seek B with $BF_j^{\sigma_p}B^{-1} = CwF_j$ for all j. By [117, Chapter I, Theorem 4.3], either w gives B for even r. For $r = 2k+1$, the scalar of $\Omega = F_r \cdots F_1$ distinguishes the two simple modules. Since $\Omega^2 = (-1)^k$, its σ_p-sign is v^k. Thus, B exists precisely for $Cw = v^k$. The elementary matrices contribute $C^{\partial(x)}$, and the F_j-comparison contributes $(Cw)^{\partial(x)}$. Moreover, each $\gamma_t D_j$ contributes $C^2 w = w$. Thus, the comparison acts by $\Theta_{\rho,\mu}^{\sigma_p} \mapsto \Theta_{\rho,\mu}$ if $w = 1$, and to its ε-associate if $w = -1$. If $r = 2k+1$, the σ_p-sign on the associated pair is Cv^k.

If $r = 2k$, choose $w = 1$. Moreover, an associator for $\Theta_{\rho,\mu}$ is $J = i^k\Omega$ on the Clifford factor W_r [29, Proposition 3.4]. Moreover, under the comparison, J^{σ_p} has scalar $v^k C^r = v^k$ relative to J. Thus, the σ_p-sign on the even constituents is v^k. If $r = 2k+1$, the character attached to D_{o} has value $uv^{k+(t+1)/2} = Cv^k$ at σ_p. If $r = 2k$, the corresponding value is v^k. Consequently, D_{o} gives the sign for $\Theta_{\rho,\mu}$. □

Lemma 9.7. *Let p, t, r, σ_p be as in* Lemma 9.6. *Let R_t be the elementary spin representation of $N_{\mathfrak{S}_t}(P)^+$ with $\varepsilon R_t = R_t$. Moreover, let α be a strict partition of r, and write $\ell = \ell(\alpha)$ and $d \equiv r - \ell \pmod 2$, with $d \in \{0, 1\}$. Moreover, let Θ_α be the homogeneous representation attached to R_t, α in* Brunat–Nath [29, Section 3.3.2]. *Put $D_{\mathrm{s}} = 2^d(-1)^{\lceil (rt-\ell)/2 \rceil} t^\ell \prod_{j=1}^{\ell} \alpha_j$. Then σ_p fixes the association class of Θ_α. Moreover, σ_p acts on the pair attached to Θ_α with sign $\sigma_p(\sqrt{D_{\mathrm{s}}})/\sqrt{D_{\mathrm{s}}}$.*

Proof. Let C, g be as in Lemma 9.6, and write $C = (-1)^e$. By [29, Lemma 3.5], the elementary associator J is rational, with $J^2 = 1$. As above, $R_t(x)^{\sigma_p} = C^{\partial(x)} R_t(x)$, so J^e gives the elementary intertwiner. Put $J_E = J^{\otimes r}$. By [29, Corollaries 3.11–3.13], a block generator has matrix $\mathcal{A}_j \otimes \gamma_t \rho_\alpha(\widetilde{s}_j)$. Moreover, $\mathcal{A}_j$ is rational and $J_E \mathcal{A}_j J_E^{-1} = \mathcal{A}_j$. Thus, J_E^e handles the elementary action, and $\gamma_t^{\sigma_p} = g\gamma_t$ gives sign g at each s_j.

Put $D_\alpha = 2^d(-1)^{\lceil (r-\ell)/2 \rceil} \prod_j \alpha_j$. By [29, Section 2.2], the outer sign is $\sigma_p(\sqrt{D_\alpha})/\sqrt{D_\alpha}$. If $d = 1$, the extra sign contributes g on the pair of associates. If $d = 0$, [29, Corollary 3.13] gives the full associator $J_E \otimes J_\alpha$. The extra twist on ρ_α is afforded by J_α, whose conjugation fixes $J_E \otimes J_\alpha$. Hence, the sign on Θ_α is $g^d \sigma_p(\sqrt{D_\alpha})/\sqrt{D_\alpha}$. Finally, $D_{\mathrm{s}}/D_\alpha = (-1)^{r(t-1)/2} t^\ell$. With $p^* = (-1)^{(p-1)/2} p$, the quadratic Gauss sum gives $\sqrt{p^*} \in \mathbb{Q}(\zeta_p)$. Thus, σ_p fixes $\sqrt{p^*}$, so its sign on $\sqrt{t}$ is g. Therefore, the sign on $\sqrt{D_{\mathrm{s}}/D_\alpha}$ is $g^{r+\ell} = g^d$. □

Lemma 9.8. *Let p be an odd prime, let $a \geq 1$, and let $1 \leq b < p$. Put $n = bp^a$, and suppose that $n \geq 8$. Then* $\mathrm{iMN}(A_n, p)$ *holds.*

Proof. Put $t = p^a$, $X = 2.A_n$, and $Y = 2.\mathfrak{S}_n$. Let $Z = \langle z \rangle = \mathrm{Z}(X)$, and choose $R \in \mathrm{Syl}_p(X)$. Moreover, put $V = \mathrm{N}_Y(R)$, $M = V \cap X$, and $P = RZ/Z$. Since $1 < P < A_n$ and A_n is simple, the normalizer M is proper in X. Moreover, X is perfect and $\mathrm{Aut}(A_n) \cong \mathfrak{S}_n$, so $\mathrm{Aut}(X) \cong Y/Z$. Given $z \in \ker \chi$, the maps on Y, V and X, M come from [30, Proposition 2.3 and Section 5]. Thus, the maps ω_Y, ω_X are $\mathcal{H}_p$-equivariant. Moreover, $\omega_Y(\varepsilon\eta) = \varepsilon\omega_Y(\eta)$ and $\omega_X(\eta|_X) = \omega_Y(\eta)|_M$ by linear extension.

Let $G(p) \leq \mathcal{H}_p$ fix all roots of p'-order. By [29, Theorem 1.1 and Corollary 5.3], take the $G(p)$-equivariant spin bijections for Y, V and X, M. By [29, Section 5], the labels λ have empty t-bar core, and their t-bar quotients $(\alpha, \mu_1, \ldots, \mu_{(t-1)/2})$ label the local classes. Write $\ell = \ell(\alpha)$. For $P_t \in \mathrm{Syl}_p(\mathfrak{S}_t)$, put $K = \mathrm{N}_{\mathfrak{S}_t}(P_t)^+$. By [29, Theorem 3.20], α labels the self-associate factor from K, and each μ_i labels a distinct ε-orbit of size 2. By [29, Lemma 5.1], both pairs lie on Y, V if $b - \ell$ is odd and on X, M otherwise.

Put $d \equiv b - \ell \pmod 2$, with $d \in \{0, 1\}$, and let $D = 2^d(-1)^{\lceil (bt-\ell)/2 \rceil} t^\ell \prod_j \alpha_j$. By [80, proof of Theorem 4.1], the parts of λ divisible by t are $t\alpha_1, \ldots, t\alpha_\ell$. By [28, proof of Lemma 3.11], the empty t-bar core gives equal counts on $(i, -i)$. If the nonzero runners contain $2u$ parts, each $(i, -i)$ pair has product of square class -1 in $\mathbb{Q}_p^\times$. Thus, their product has square class $(-1)^u$. Since $\ell(\lambda) = \ell + 2u$, Schur's discriminant for λ has the square class of D in $\mathbb{Q}_p^\times$ [29, Section 2.2]. Thus, the $\mathcal{H}_p$-sign on the global pair is $\sigma(\sqrt{D})/\sqrt{D}$ for $\sigma \in \mathcal{H}_p$.

Let $\sigma_p \in \mathcal{H}_p$ fix the p-power roots and act by the p-th power on the p'-roots. For a nonempty component μ_i of size r_i, Lemma 9.6 gives the σ_p-sign attached to $D_i = 2^{r_i \bmod 2}(-1)^{\lceil r_i t/2 \rceil}$. Moreover, for $r_0 = |\alpha| > 0$, Lemma 9.7 gives the σ_p-sign attached to $D_0 = 2^{d_0}(-1)^{\lceil (r_0 t - \ell)/2 \rceil} t^\ell \prod_j \alpha_j$, with $d_0 \in \{0, 1\}$ and $d_0 \equiv r_0 - \ell \pmod 2$. If $r_0 = 0$, put $D_0 = 1$ and $d_0 = 0$. Put $d_i = r_i \bmod 2$ for $i \geq 1$, with $D_i = 1$ if $r_i = 0$. By [28, Section 5 and (46)], the Humphreys product has σ_p-sign given by the square class $D_{\mathrm{loc}} = (-1)^{\sum_{i<j} d_i d_j} \prod_i D_i$. Let $e_0 = r_0 t - \ell$ and $e_i = r_i t$ for $i \geq 1$.

Then, $e_i \equiv d_i \pmod 2$, and $\sum_i e_i = bt - \ell$. Since $\sum_i \lceil e_i/2 \rceil - \lceil \sum_i e_i/2 \rceil = \lfloor \sum_i d_i/2 \rfloor \equiv \sum_{i<j} d_i d_j \pmod 2$, we have $D_{\mathrm{loc}}/D \in (\mathbb{Q}^\times)^2$. By [29, proof of Theorem 4.8], induction gives the same σ_p-sign.

By definition of $\mathcal{H}_p$, each $\sigma \in \mathcal{H}_p$ satisfies $\sigma\sigma_p^{-k} \in G(p)$ for some $k \in \mathbb{Z}$. Thus, the signs agree on $\mathcal{H}_p$. By Lemma 2.14, the maps Ω_Y, Ω_X are $\mathcal{H}_p$-equivariant. Moreover, Ω_Y is equivariant for sign twist and Ω_X for odd conjugation. Since λ and (α, μ_i) have the same split type, $\Omega_X(\eta|_X) = \Omega_Y(\eta)|_M$ by linear extension. Each pair has $z \mapsto -1$. Together with the maps over 1_Z, Proposition 2.13 gives $\mathrm{iMN}(A_n, p)$. □

Proof of Theorem 9.3. By [157, Lemma 4.1(5)], the condition $\mathrm{iMN}(S, p)$ holds for $p = 2$. Moreover, [157, Lemma 4.1(1)–(3)] covers $p \mid q$, exceptional Schur multipliers of S in Lie type, and S of Suzuki or Ree type, including ${}^2F_4(2)'$. At odd p with $p \nmid q$, [157, Lemma 4.1(4)] covers the exceptions X in [132, Theorem 5.14] with non-generic Sylow normalizer $\mathrm{N}_X(R)$. Thus, the remaining Lie type pairs (S, p) have odd $p \nmid q$, generic multiplier and generic $\mathrm{N}_X(R)$.

Next, for $S = \mathrm{PSL}_n(q)$ or $S = \mathrm{PSU}_n(q)$, Theorem 3.1 gives $\mathrm{iMN}(S, p)$. Moreover, for a simple group S of type B_n with $n \geq 3$ or type C_n with $n \geq 2$, Theorems 4.1 and 5.1 give $\mathrm{iMN}(S, p)$. For $S = {}^3D_4(q)$ or $S = \mathrm{P}\Omega^\varepsilon_{2n}(q)$ with $n \geq 4$, Theorem 6.1 gives $\mathrm{iMN}(S, p)$. Finally, in Section 7, a simple group S of exceptional Lie type satisfies $\mathrm{iMN}(S, p)$ by Theorem 7.1.

Suppose $S = A_n$ and p is odd. By [186, Theorem C], $\mathrm{iMN}(S, p)$ holds for $A_5 \cong \mathrm{PSL}_2(5)$. Moreover, [186, Theorem 4.4] covers $A_6 \cong \mathrm{PSL}_2(9)$. When $p \leq n < 2p$, a Sylow p-subgroup of A_n has order p, so Corollary 2.10 covers this range. At $n = 7$, Lemma 9.1 covers $p = 3$. At $n \geq 8$, Lemma 9.1 covers $p \nmid n$. When $p \mid n$, Lemma 9.2 covers n with at least two nonzero base-p digits, and Lemma 9.8 covers $n = bp^a$ with $a \geq 1$ and $1 \leq b < p$. By Theorem 8.1, $\mathrm{iMN}(S, p)$ holds for sporadic S. Thus, all pairs satisfy $\mathrm{iMN}(S, p)$. □

Proof of Theorem 1.2. If $p \nmid |G|$, then $P = 1$, and the identity on $\mathrm{Irr}(G)$ suffices. Suppose $p \mid |G|$. By Theorem 9.3, each nonabelian simple section S of G with $p \mid |S|$ satisfies $\mathrm{iMN}(S, p)$. Therefore, [141, Theorems A and 4.4], with $Z = 1$, gives an $\mathcal{H}_p$-equivariant bijection $\mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(\mathrm{N}_G(P))$. □

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