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ON THE LI–LIU SPECTRAL HYPOTHESES FOR THE BINARY GOLDBACH PROBLEM

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Abstract. Li and Liu (2026) proved that under the Generalized Riemann Hypothesis (GRH) for Dirichlet L-functions together with either of two spectral hypotheses, denoted S and R, every sufficiently large even integer is a sum of two primes. Both hypotheses bound the norm of a compact selfadjoint operator T_N constructed from the odd-character logarithmic-derivative correlation modulo N after subtracting its diagonal and a continuous Beta model. We establish four results concerning R and S: (1) Under GRH(N), rigorous ball-arithmetic evaluation of Rayleigh quotients in finite reproducing-kernel subspaces certifies $\|T_N\| > \delta_N = \sqrt{\mathfrak{S}(N)(\log N)^3/N}$ at the four dyadic moduli $N = 2^{15}, 2^{16}, 2^{17}, 2^{18}$, where the Goldbach singular series $\mathfrak{S}(N)$ attains its minimum $2C_2$. Thus the critical-line condition of Hypothesis R fails at these concrete moduli, complementing Yilin Liu's Maier-matrix proof that $\sqrt{N}\|T_N\| > (\log N)^D$ infinitely often under GRH. (2) Polarizing the Hardy quadratic-form identity at shifted smooth weights shows that Hypothesis S implies the Hardy–Littlewood asymptotic formula for prime pairs $(p, p+N)$ of even difference N (and more generally $m \mp n = jN$), a problem subject to the classical sieve parity barrier. (3) Along $N = 2^k$, Hypothesis S requires a specific two-term asymptotic expansion for the odd-character second moment of $(L'/L)(1/2 + c/\log N, \chi)$, which agrees with the unitary ratios recipe of Conrey, Farmer, and Zirnbauer; in numerical evaluations up to $N = 2^{28}$ the normalized residual $|E_N|/\log N$ falls from order 10^{-3} for $N \le 2^{18}$ to below 3×10^{-5} at the largest moduli. (4) Via the Fourier–Laplace representation of the strip Hardy space, T_N is unitarily equivalent to a weighted cumulative prime-pair discrepancy operator on $L^2((0,\infty), dX/X)$. Selberg's upper-bound sieve and Schur's test control the window $XY \le bN^2$, isolating the remaining difficulty of Hypothesis S in signed cancellations across $bN^2 < XY < N^{6\log\log N}$, where absolute-value kernel bounds provably diverge. Core finite identities and scalar inequalities (34 theorems) are formally verified in Lean 4.

1. Introduction and Statement of Results

1.1. **The Li–Liu operator and spectral hypotheses.** In a recent preprint, Li and Liu [5] introduced an operator-theoretic conditional approach to the binary Goldbach problem. Let $N > e^{10}$ be an even integer, and write GRH(N) for the assertion that every nontrivial zero of every primitive Dirichlet L-function whose conductor divides N has real part $1/2$. Denote by X_N^- the set of all $\varphi(N)/2$ odd Dirichlet characters modulo N (including imprimitive characters, extended by 0 on integers not coprime to N), and write

$$\mathfrak{S}(N) = 2C_2 \prod_{\substack{p>2 \\ p\mid N}} \frac{p-1}{p-2}, \qquad C_2 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right), \tag{1}$$

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for the Goldbach singular series. Setting $g_\chi(s) = (L'/L)(1/2+s,\chi)$, Li and Liu [5, (2.11)–(2.13)] consider the arithmetic residual kernel

$$E_N(s,z) = P_N(s,z) - D_N(s+z) - K_N(s,z), \tag{2}$$

where

$$P_N(s,z) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} g_\chi(s)\, g_{\bar\chi}(z), \tag{3}$$

$$D_N(w) = \sum_{\substack{n\geq 1 \\ (n,N)=1}} \frac{\Lambda(n)^2}{n^{1+w}}, \tag{4}$$

$$K_N(s,z) = \mathfrak{S}(N) \left(\frac{\pi}{N}\right)^{s+z} \mathcal{P}_N(s+z)\, \zeta(1-s-z)\, X_1(s)\, X_1(z), \tag{5}$$

with Euler product $\mathcal{P}_N(w) = \prod_{p>2,\, p\nmid N}\big(1+p^{-w}/(p-2)\big)$ and archimedean ratio $X_1(s) = \Gamma((3/2-s)/2)/\Gamma((3/2+s)/2)$. Under GRH$(N)$, $E_N(s,z)$ is holomorphic on $\{0 < \operatorname{Re} s < 3/2\} \times \{0 < \operatorname{Re} z < 3/2\}$.

Let $\alpha_N = 1/\log N$, $\beta = 1/8$, and $\Delta_N = \beta - \alpha_N > 0$. On the vertical strip $S_N = \{s \in \mathbb{C} : \alpha_N < \operatorname{Re} s < \beta\}$ with boundary $\Gamma_N = (\alpha_N + i\mathbb{R}) \sqcup (\beta + i\mathbb{R})$, let $H_N = H^2(S_N)$ be the Hardy space equipped with the unaveraged two-line boundary norm $\|h\|_{H_N}^2 = \int_{\mathbb{R}} |h(\alpha_N + it)|^2\, dt + \int_{\mathbb{R}} |h(\beta + it)|^2\, dt$. By [5, Proposition 5.3], the integral operator

$$(T_N h)(s) = \int_{\Gamma_N} \frac{E_N(s,\bar z)}{(1+s)(1+\bar z)}\, h(z)\, |dz| \tag{6}$$

is compact, selfadjoint, and Hilbert–Schmidt on H_N. The same kernel also defines an operator $\widetilde{T}_N$ on the full boundary space $L^2(\Gamma_N)$; it equals T_N on H_N and vanishes on the orthogonal complement, so $\|\widetilde{T}_N\| = \|T_N\|$. Writing $\{\mu_j(N)\}$ for its nonzero real eigenvalues (counted with multiplicity) and setting

$$\delta_N = \sqrt{\frac{\mathfrak{S}(N)(\log N)^3}{N}}, \tag{7}$$

Li and Liu package the spectrum into the slice-regular quaternionic product

$$X_N(q) = \prod_j \left(1 - \frac{4\mu_j(N)^2}{\delta_N^2}\, q(1-q)\right), \qquad q \in \mathbb{H}, \tag{8}$$

and formulate two hypotheses [5, Section 1.2]:

- **Hypothesis** S**.** Fix beforehand a positive function $h(N) \to \infty$ satisfying $\eta_N \leq 1/4$ eventually, where $\eta_N = h(N)^2(\log N)^3/(N\mathfrak{S}(N))$. For every sufficiently large even N satisfying GRH(N), $X_N(q)$ has no zeros in $\operatorname{Re} q > 1 - \eta_N$.
- **Hypothesis** R**.** For every sufficiently large even N satisfying GRH(N), every zero of $X_N(q)$ satisfies $\operatorname{Re} q = 1/2$.

By [5, Theorem 6.1] (verified at the scalar and quaternionic factor level in Section 2.1 below), the zero conditions at a fixed modulus N translate directly into operator-norm bounds: R(N) holds if and only if

$$\|T_N\| \leq \delta_N, \tag{9}$$

whereas S(N) with width $\eta_N \in (0, 1/4]$ holds if and only if

$$\|T_N\| \leq \frac{\mathfrak{S}(N)}{2h(N)\sqrt{1-\eta_N}}. \tag{10}$$

For comparison, GRH(N) alone gives $\|T_N\| \ll (\log N)^2$ [5, Theorem 6.2]. Along $N = 2^k$, where $\mathfrak{S}(N) = 2C_2$ is constant, Hypothesis S requires $\|T_{2^k}\| \to 0$ and Hypothesis R requires $\|T_{2^k}\| \le \sqrt{2C_2}(k \log 2)^{3/2} 2^{-k/2}$.

1.2. **Main results.** Our first result provides concrete, rigorously certified counterexamples to the norm inequality (9) at four powers of two.

Theorem 1.1 (Certified failure of R(N) at $N = 2^{15}, \dots, 2^{18}$)**.** *Assume* GRH($N$) *for* $N = 2^k$ *with* $k \in \{15, 16, 17, 18\}$. *At each of these four moduli, the Li–Liu operator* T_N *satisfies* $\|T_N\| > \delta_N$, *with certified enclosures recorded in Table 1. Consequently,* $X_N(q)$ *possesses real zeros in* $(0,1) \setminus \{1/2\}$. *Moreover, for* $k \in \{15, 16\}$ *one has* $\|T_N\| > \frac{2}{\sqrt{3}}\delta_N$, *so* $X_N(q)$ *also has zeros in* $\operatorname{Re} q > 3/4$ *(exceeding the threshold* (10) *at the largest admissible parameter* $\eta_N = 1/4$*).*

TABLE 1. Rigorous 192-bit ball-arithmetic enclosures for $N = 2^k$ ($k = 15, \dots, 18$) at accelerated prime cutoff 16,384. Here m is the number of interior reproducing-kernel nodes, $\|T_N\|_{\mathrm{lb}}$ is the certified lower bound for $\|T_N\|$, and $\delta_{N,\mathrm{ub}}$ is the certified upper bound for δ_N.

k	$N = 2^k$	m	$\|T_N\|_{\mathrm{lb}}$	$\delta_{N,\mathrm{ub}}$	$\|T_N\|_{\mathrm{lb}}/\delta_{N,\mathrm{ub}}$
15	32,768	706	0.253445290	0.21280977	1.19094
16	65,536	578	0.233644077	0.16577524	1.40940
17	131,072	514	0.133399661	0.12838021	1.03909
18	262,144	450	0.103145878	0.09890505	1.04287

Because Hypothesis R is stated for all sufficiently large even N, finitely many moduli do not by themselves refute R asymptotically. However, Theorem 1.1 puts into context the single numerical illustration in [5, Appendix B] at the primorial $N = 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, where a 2562-node boundary discretization gave $\|C\|/\delta_{30030} \approx 0.827 < 1$ (explicitly noted there as a finite-model illustration rather than a certification of R): primorials inflate the singular series to $\mathfrak{S}(30030) \approx 5.121 \approx 3.88 \times 2C_2$, nearly doubling δ_N relative to a dyadic modulus of comparable size ($2^{15} = 32{,}768$). Asymptotically, Yilin Liu [6] showed in a preprint posted on November 21, 2026, via Maier's matrix method, that under Dirichlet GRH, for every fixed $D > 0$ and $\varepsilon \in (0,1)$, every sufficiently large dyadic interval $[X, 2X]$ contains at least $X \exp(-(\log X)^{\varepsilon})$ even integers N satisfying $\sqrt{N}\|T_N\| > (\log N)^D$, refuting Hypothesis R; we review and verify the mechanism of [6] in Section 2.4. Neither Theorem 1.1 nor [6] conflicts with Hypothesis S for slowly growing $h(N)$ (for instance, at $h = 2$, the threshold (10) exceeds 0.333 at all four moduli in Table 1).

We therefore turn to Hypothesis S: what arithmetic statements it entails, whether it is compatible with computation, and what analytic barrier obstructs its proof.

Theorem 1.2 (Shifted prime-pair asymptotic from Hypothesis S)**.** *Assume* GRH(N) *for all sufficiently large even* N, *together with Hypothesis* S *for an admissible function* $h(N) \to \infty$. *Let* $f \in C_c^\infty((1/4, 1/3))$ *be a nonzero real-valued smooth weight. Then as* $N \to \infty$ *through even integers,*

$$\frac{1}{N} \sum_{(n,N)=1} \Lambda(n)\Lambda(n+N)\, f\Big(\frac{n}{N}\Big)^2 = \mathfrak{S}(N) \int_0^\infty f(u)^2\, du + O_f\Big(\frac{\mathfrak{S}(N)}{h(N)}\Big). \tag{11}$$

In particular, Hypothesis S *under* GRH *implies the Hardy–Littlewood asymptotic formula for prime pairs* $(p, p+N)$ *with* $p \in (N/4, N/3)$.

Equation (11) is an open binary prime-pair problem obstructed by the classical sieve parity phenomenon in the same way as the Goldbach and twin-prime conjectures. Proving Hypothesis S is therefore at least as hard as establishing (11).

To test whether Hypothesis S is empirically plausible along powers of two, write $L = \log N$ for $N = 2^k$ and denote the normalized odd-character second moment on the real axis by

$$A_N(s) := P_N(s,s) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \left| \frac{L'}{L}\left(\frac{1}{2} + s, \chi\right)\right|^2. \tag{12}$$

Proposition 1.3 (Dyadic second-moment expansion and ratios comparison)**.** *Let $N = 2^k$, $L = \log N$, and fix $c > 1$ (so that $s = c/L \in S_N$ for all sufficiently large k). Under* GRH(N)*, Hypothesis* S *implies $E_N(c/L, c/L) = o(L)$, which is equivalent to the two-term expansion*

$$A_N(c/L) = \frac{1 - e^{-2c}}{4c^2}\, L^2 - \frac{e^{-2c}\big(\log(2\pi) - \psi_0(3/4)\big)}{2c}\, L + o(L) \qquad (k \to \infty), \tag{13}$$

where $\psi_0 = \Gamma'/\Gamma$ is the digamma function. Moreover, applying the Conrey–Farmer–Zirnbauer unitary ratios recipe [2] *across all primitive conductors $2^j \mid 2^k$ ($2 \le j \le k$) yields a prediction that differs from $D_N(2c/L) + K_N(c/L, c/L)$ by $O_c(L/N)$, matching both coefficients in* (13)*.*

Numerical evaluations of $A_{2^k}(c/L)$ up to $k = 28$ ($N = 268{,}435{,}456$, Section 4.2) show $|E_N(c/L, c/L)|/L < 3 \times 10^{-5}$ for $24 \le k \le 26$ at $c = 2$; at $c = 5/4$ the same ratio falls, though not monotonically, from 4.9×10^{-3} at $k = 16$ to 1.9×10^{-5} at $k = 28$. The continuous model K_N is thus highly accurate along the real diagonal; the failure of R(N) in Theorem 1.1 arises from off-diagonal correlations across heights in the strip rather than from real-axis diagonal error.

Our fourth result strips away the complex-analytic strip geometry and reduces Hypothesis S directly to an arithmetic integral operator on $(0, \infty)$. For $X, Y > 0$, define the signed cumulative prime-power correlation

$$C_N(X,Y) = \sum_{\substack{1 \le n \le X,\, 1 \le m \le Y \\ (nm,N)=1}} \Lambda(n)\Lambda(m)\sqrt{nm}\,\big(\mathbf{1}_{n \equiv m \bmod N,\, n \ne m} - \mathbf{1}_{n \equiv -m \bmod N}\big) \tag{14}$$

and its continuous counterpart

$$M_N(X,Y) = \sum_{j \ge 1} \mathfrak{S}(jN)\, J_{jN}(X,Y), \tag{15}$$

where $J_h(X,Y)$ integrates $\sqrt{uv}$ over $(0,X] \times (0,Y]$ along the two difference segments $u - v = h$ and $v - u = h$ minus the sum segment $u + v = h$:

$$\begin{aligned} J_h(X,Y) = {} & \int_0^{\min(Y,\, \max(0, X-h))} \sqrt{(v+h)v}\, dv + \int_0^{\min(X,\, \max(0, Y-h))} \sqrt{u(u+h)}\, du \\ & - \int_{\max(0,\, h-Y)}^{\min(X,h)} \sqrt{u(h-u)}\, du. \end{aligned} \tag{16}$$

Notice that $C_N(N,N) - M_N(N,N) = \frac{\pi}{8}\mathfrak{S}(N)N^2 - \sum_{n+m=N,\,(nm,N)=1} \Lambda(n)\Lambda(m)\sqrt{nm}$ is precisely a weighted Goldbach discrepancy. On $L^2((0,\infty), dX/X)$, define the symmetric kernel

$$\mathcal{B}_{\alpha_N}(X,Y) = 2\pi(XY)^{-1-\alpha_N}\big[C_N(X,Y) - M_N(X,Y)\big]. \tag{17}$$

Theorem 1.4 (Unitary cumulative-kernel reduction and window bounds)**.** *Assume* GRH(N) *for an even integer $N > e^{10}$, and put $L = \log N$, $\alpha_N = 1/L$, and $\Delta_N = 1/8 - \alpha_N$.*

(a) Exact unitary representation. *Under the Fourier–Laplace isomorphism of H_N, the operator T_N is unitarily equivalent to the integral operator $\mathcal{K}_N$ on $L^2((0,\infty), dX/X)$ with kernel*

$$\mathcal{K}_N(X,Y) = \sqrt{1+X^{-2\Delta_N}}\sqrt{1+Y^{-2\Delta_N}}\,\mathcal{B}_{\alpha_N}(X,Y). \tag{18}$$

Consequently,

$$\|\mathcal{B}_{\alpha_N}\| \le \|T_N\| \le 2\|\mathbf{1}_{X,Y\ge 1}\,\mathcal{B}_{\alpha_N}\| + O\big(\mathfrak{S}(N)N^{-1/2}\big). \tag{19}$$

(b) Outer-scale truncation. *Let $\mathcal{R}_N = \{X,Y \ge 1 : X+Y \ge N,\ N^2/L^7 \le XY \le N^{6\log L}\}$. Restricting $\mathcal{B}_{\alpha_N}$ to $\mathcal{R}_N$ changes the operator norm on $L^2((0,\infty),dX/X)$ by $O(L^{-1} + \mathfrak{S}(N)N^{-1/2})$.*

(c) Product-window upper bound. *Uniformly for $0 \le b_1 < b_2 < \infty$,*

$$\|\mathbf{1}_{b_1N^2<XY\le b_2N^2}\,\mathcal{B}_{\alpha_N}\| \ll \mathfrak{S}(N)\big(\sqrt{b_2}-\sqrt{b_1}\big) + L^3N^{-1/4}. \tag{20}$$

In particular, $\|\mathbf{1}_{XY\le bN^2}\mathcal{B}_{\alpha_N}\| \ll \mathfrak{S}(N)\sqrt{b} + L^3N^{-1/4}$, which is $o(\mathfrak{S}(N))$ as $b = b(N) \to 0$. Hence the existence of an admissible $h(N) \to \infty$ satisfying Hypothesis S *is equivalent to*

$$\big\|\mathbf{1}_{bN^2<XY<N^{6\log L}}\mathcal{B}_{\alpha_N}\big\| = o(\mathfrak{S}(N))$$

for some, equivalently every, choice of $b = b(N) \to 0$ with $b(N) \ge L^{-7}$. For fixed $b > 0$, part (c) bounds the omitted window $XY \le bN^2$ only by $O(\mathfrak{S}(N)\sqrt{b})$.

In Section 5.3 we also record the vertical height truncation $\|T_N - A_{N,L^5}\| = O(L^{-1/2})$ on Γ_N and prove three divergence facts showing that on the remaining intermediate scale $bN^2 < XY < N^{6\log L}$, any approach that replaces the kernel or its individual congruence families by absolute values (such as Schur's test) necessarily diverges. Proving Hypothesis S requires genuine sign cancellation in prime-pair correlations modulo N across intermediate scales, beyond single-modulus zero-location information supplied by GRH (including the formalized proof of GRH recently announced by Alpöge [1] and independently checked by Rubinstein [8]).

1.3. **Organization.** Section 2 proves the scalar/quaternionic norm equivalence, describes the reproducing-kernel interval certification, proves Theorem 1.1, and reviews the Maier-matrix argument of [6]. Section 3 proves Theorem 1.2. Section 4 proves Proposition 1.3 and presents the dyadic numerical experiments up to 2^{28}. Section 5 proves Theorem 1.4 and establishes the three absolute-value divergence obstructions. Section 6 summarizes the Lean 4 formalization, the replication package, and the provenance of the work.

2. Certified Counterexamples to Hypothesis R

2.1. **Scalar and quaternionic zero-free thresholds.** We first verify that quaternionic slice-regularity introduces no constraint beyond the real quadratic threshold.

Lemma 2.1. *Let $\eta < 1/2$ and $c \in \mathbb{R}$. The following are equivalent:*

(i) $q(1-q) \ne c$ for all $q \in \mathbb{H}$ with $\operatorname{Re} q > 1-\eta$;
(ii) $x(1-x) \ne c$ for all $x \in \mathbb{R}$ with $x > 1-\eta$;
(iii) $\eta(1-\eta) \le c$.

In particular, if $\delta, \sigma, h > 0$ and $\eta \in (0,1/2)$ satisfy $\delta^2h^2 = \eta\sigma^2$, then for $\mu \in \mathbb{R}\setminus\{0\}$, the factor $1-(4\mu^2/\delta^2)q(1-q)$ has no zeros in $\operatorname{Re} q > 1-\eta$ if and only if $4\mu^2h^2(1-\eta) \le \sigma^2$.

Proof. Write $q = x + y_1i + y_2j + y_3k \in \mathbb{H}$ with $x, y_r \in \mathbb{R}$. Since $q^2 = x^2 - (y_1^2+y_2^2+y_3^2) + 2x(y_1i+y_2j+y_3k)$, the three imaginary components of $q(1-q)$ are $(1-2x)y_r$ $(r=1,2,3)$. On $\operatorname{Re} q = x > 1-\eta > 1/2$ we have $1-2x < 0$, so $q(1-q) = c \in \mathbb{R}$ forces $y_1 = y_2 = y_3 = 0$, giving (i) $\Leftrightarrow$ (ii). On $(1/2,\infty)$, $x \mapsto x(1-x)$ is strictly decreasing with image $(-\infty, 1/4)$, so

$x(1-x) \neq c$ for all $x > 1-\eta$ if and only if $c \geq \eta(1-\eta)$, giving (ii) $\Leftrightarrow$ (iii). Setting $c = \delta^2/(4\mu^2)$ and multiplying $\eta(1-\eta) \leq \delta^2/(4\mu^2)$ by $4\mu^2h^2/\eta > 0$ using $\delta^2h^2 = \eta\sigma^2$ yields $4\mu^2h^2(1-\eta) \leq \sigma^2$. Taking $\eta \uparrow 1/2$ similarly recovers $|\mu| \leq \delta$ for $\operatorname{Re} q = 1/2$. □

2.2. **Reproducing-kernel Rayleigh quotients.** Truncating the boundary integral (6) to a finite quadrature mesh (as in [5, Appendix B.4]) does not compress T_N to a subspace of H_N, so quadrature eigenvalues do not automatically bound $\|T_N\|$ from below without an analytic truncation bound of size $< \delta_N$. To obtain rigorous lower bounds without boundary integration, we evaluate T_N on finite linear combinations of interior reproducing kernels.

By [5, Proposition 5.4], point evaluation at any $z \in S_N$ is represented in H_N by

$$k_z(s) = \frac{1}{4\Delta_N} \csc\left(\frac{\pi(s + \bar{z} - 2\alpha_N)}{2\Delta_N}\right), \qquad h(z) = \langle h, k_z\rangle_{H_N}, \tag{21}$$

and the action of T_N satisfies $\langle T_N k_z, k_s\rangle_{H_N} = E_N(s,\bar{z})/((1+s)(1+\bar{z}))$. Given nodes $z_1, \dots, z_m \in S_N$, define the $m \times m$ Hermitian matrices

$$G_{ij} = \langle k_{z_j}, k_{z_i}\rangle_{H_N} = k_{z_j}(z_i), \qquad A_{ij} = \langle T_N k_{z_j}, k_{z_i}\rangle_{H_N} = \frac{E_N(z_i,\bar{z}_j)}{(1+z_i)(1+\bar{z}_j)}. \tag{22}$$

For any coefficient vector $c = (c_1, \dots, c_m)^\top \in \mathbb{C}^m$ with $u = \sum_{j=1}^m c_j k_{z_j} \neq 0$, selfadjointness of T_N gives

$$c^*Gc = \|u\|_{H_N}^2 > 0, \qquad \frac{|c^*Ac|}{c^*Gc} = \frac{|\langle T_N u, u\rangle_{H_N}|}{\|u\|_{H_N}^2} \leq \|T_N\|, \tag{23}$$

while $c^*Ac/(c^*Gc) \leq \lambda_{\max}(T_N)$ and $c^*Ac/(c^*Gc) \geq \lambda_{\min}(T_N)$. Certifying a lower bound on $\|T_N\|$ therefore requires no eigenvalue perturbation theory: one uses floating-point generalized eigenvectors of (A, G) only to select Gaussian-integer vectors $c \in \mathbb{Z}[i]^m$, and then encloses the scalar quotient $(c^*Ac)/(c^*Gc)$ directly in interval arithmetic.

2.3. **Ball-arithmetic enclosures and proof of Theorem 1.1.**

Proof of Theorem 1.1. Fix $k \in \{15, 16, 17, 18\}$ and $N = 2^k$; note that $N \geq 32{,}768 > e^{10}$, so S_N is nonempty (Δ_N increases from 0.02882 at $k = 15$ to 0.04485 at $k = 18$). Set $J = 4\lceil 5/(4\Delta_N)\rceil$ and $t_j = 5j/J$ for $-J \leq j \leq J$, so $\Delta_N/2 < 5/J < \Delta_N$. Our node set consists of the two vertical layers

$$z_{a,j} = \alpha_N + \frac{a}{4}\Delta_N + it_j, \qquad a \in \{1, 3\}, \quad -J \leq j \leq J,$$

giving $m = 2(2J+1) = 706{,}578{,}514{,}450$ interior nodes for $k = 15, 16, 17, 18$, respectively.

Step 1: Certified logarithmic derivatives for all $N/4$ odd characters. For $N = 2^k$ ($k \geq 3$), the unit group $(\mathbb{Z}/N\mathbb{Z})^\times \cong \{\pm 1\} \times \langle 5\rangle$ has cyclic factor of order $M = N/4$. Writing $a_r \equiv 5^r \pmod N$ ($0 \leq r < M$), every odd character satisfies $\chi(-1) = -1$ and $\chi(5) = e^{-2\pi i f/M}$ for a unique frequency $0 \leq f < M$. Expressing $L(u,\chi)$ in terms of the Hurwitz zeta function $\zeta(u,x) = \sum_{n=0}^\infty (n+x)^{-u}$ gives, whenever $L(u,\chi) \neq 0$,

$$\frac{L'}{L}(u,\chi_f) = -\log N + \frac{\sum_{r=0}^{M-1} e^{-2\pi i f r/M}\, \partial_u\big[\zeta(u, a_r/N) - \zeta(u, 1 - a_r/N)\big]}{\sum_{r=0}^{M-1} e^{-2\pi i f r/M}\, \big[\zeta(u, a_r/N) - \zeta(u, 1 - a_r/N)\big]}, \tag{24}$$

which is formally verified in Lean (`odd_log_deriv_eq_transform`). Since $\overline{g_\chi(\sigma - it)}$ equals $g_{\bar\chi}(\sigma + it)$, values at negative heights $t_{-j} = -t_j$ are obtained from t_j by conjugation and frequency reversal $f \mapsto (-f) \bmod M$.

For each nonnegative node $s = z_{a,j}$ ($j \geq 0$), we evaluate $u = 1/2 + s$ using $\zeta(u,x) = x^{-u} + \zeta(u, 1+x)$ for $x \in (0,1)$, partitioning $(0,1)$ into 16 equal bins of width $1/16$ with

midpoints $x_\nu = (2\nu+1)/32$ ($0 \le \nu < 16$). On each bin we expand $\zeta(u, 1+x)$ in a degree-20 Taylor polynomial in $\delta = x - x_\nu$ ($|\delta| \le 1/32$). Because $u \in (1/2 + \alpha_N, 5/8) + i[0,5]$, every point v in the closed disk $|v-u| \le 1/8$ satisfies $\operatorname{Re} v > 1/8 > 0$ and $|v| < 6$. For $n \ge 21$, the n-th Taylor coefficient satisfies

$$\left| \frac{(v)_n}{n!}\, \zeta(v+n, 1+x_\nu) \right| \le 2\binom{n+5}{5},$$

since $|(v)_n|/n! \le \binom{n+5}{5}$ for $|v| < 6$ and $|\zeta(v+n, 1+x_\nu)| \le 1 + \sum_{m\ge 2} m^{-21} < 2$. With $|\delta| \le 1/32$, consecutive majorants have ratio $< \frac{27}{22} \cdot \frac{1}{32} < \frac{1}{2}$, so the omitted tail for $\zeta(v, 1+x)$ is bounded in modulus by $4\binom{26}{5}/32^{21} < 6.5 \times 10^{-27}$; Cauchy's integral formula on $|v-u| = 1/8$ multiplies this bound by 8 for the u-derivative. Evaluating the 16 polynomials and the discrete Fourier transforms (24) in 112-bit FLINT complex ball arithmetic (`python-flint` 0.9.0 [4]) certifies that none of the M denominator balls contains 0, and produces outward-rounded error radii $< 1.18 \times 10^{-15}$ for every weighted row $g_\chi(z_p)/(1+z_p)$.

Step 2: Integer-quantized correlation quadratic form. Given $c \in \mathbb{Z}[i]^m$, the correlation term in $c^* A c$ equals the nonnegative character average

$$P(c) := \frac{1}{M} \sum_{\chi \in X_N^-} \left| \sum_{p=1}^{m} \bar{c}_p \frac{g_\chi(z_p)}{1+z_p} \right|^2 = \frac{1}{M} \sum_{\chi \in X_N^-} \left| \sum_{p=1}^{m} c_p \overline{\left(\frac{g_\chi(z_p)}{1+z_p} \right)} \right|^2 .$$

Quantizing the floating-point row midpoints by 2^P ($P \le 40$ chosen so all intermediate products fit strictly below 2^{62}) allows exact accumulation of the quantized square sum MP_0 in arbitrary-precision Python integers. If each row has certified ball radius e_p, then each character sum has total error bounded by $e = \sum_{p=1}^m (|\operatorname{Re} c_p| + |\operatorname{Im} c_p|)(e_p + 2^{-P})$, and Cauchy–Schwarz gives $|P(c) - P_0| \le 2\sqrt{P_0}\, e + e^2$. Across all witnesses in Table 1, this quantization and row radius contributes less than 9.6×10^{-9} to $|c^* A c|/(c^* G c)$.

Step 3: Accelerated Euler tails for D_N and K_N. For $w = z_p + \bar{z}_q$ ($\operatorname{Re} w \ge 2\alpha_N + \Delta_N/2 > 0$), write $x_p = p^{-w}$ and $a_p = p^{-1-w}$. In $D_N(w)$, subtracting $(\log\zeta)''(2+2w) + 2(\log\zeta)''(3+3w)$ from $(\log\zeta)''(1+w)$ cancels the a_p^2 and a_p^3 terms at odd primes:

$$D_N(w) = (\log\zeta)''(1+w) - (\log\zeta)''(2+2w) - 2(\log\zeta)''(3+3w) - \frac{(\log 2)^2 a_2}{1-a_2} - \sum_{p\ge 2} (\log p)^2 r(a_p),$$

where $r(a) = (2a^3 + a^4)/(1-a^2)^2 - 2a^3/(1-a^3)^2$ satisfies $|r(a)| \le 8|a|^4$ for $|a| < 1/4$. Summing explicitly over $p \le P_{\mathrm{cut}}$ leaves a tail bounded by $8\sum_{n > P_{\mathrm{cut}}} (\log n)^2 n^{-4} \le 8\big((\log P_{\mathrm{cut}})^2 + 2\log P_{\mathrm{cut}} + 2\big)/P_{\mathrm{cut}}^3$. Similarly, for $N = 2^k$, the identity (verified in Section 4.1)

$$\mathfrak{S}(N)\mathcal{P}_N(w) = \zeta(1+w)(2-2^{-w}) \prod_{p>2} F_p(w), \qquad F_p(w) = 1 - \frac{(1-x_p)^2}{(p-1)^2},$$

is accelerated by extracting $\zeta(2+w)^2/(\zeta(2)\zeta(2+2w)) = \prod_p B_p(w)$ with $B_p(w) = 1 - p^2(1-x_p)^2/(p^2-x_p)^2$. For $p > P_{\mathrm{cut}} \ge 5$ and $|x_p| \le 1$, $|B_p(w)| \ge 1/2$ and $|F_p(w)/B_p(w) - 1| \le 16/(p-1)^3$, giving a relative product tail bounded by $\exp(8/(P_{\mathrm{cut}}-1)^2) - 1$. The constant C_2 in δ_N is enclosed on $[1 - 1/(P_{\mathrm{cut}}-1),\, 1]$ times the finite product up to P_{cut}.

Step 4: Witness evaluation and consistency checks. On nested subgrids with height strides $4, 2, 1$, the compressed extreme eigenvalues expand monotonically, with finest-grid Gram condition numbers below 28. Rounding the extreme floating-point generalized eigenvectors at stride 1 to Gaussian integers and replaying the enclosures at 192-bit precision with $P_{\mathrm{cut}} = 16{,}384$ certifies $c^* G c > 0$ and produces the strict inequalities $\|T_N\|_{\mathrm{lb}} > \delta_{N,\mathrm{ub}}$ in Table 1 (achieved by $\lambda_{\max}(T_N)$

for $k = 15, 16$ and by $-\lambda_{\min}(T_N)$ for $k = 17, 18$). By (23) and Lemma 2.1, R(2^k) fails for $k = 15, 16, 17, 18$, and at $k = 15, 16$ the ratios 1.19094 and 1.40940 also exceed $2/\sqrt{3} \approx 1.15470$, the threshold (10) at $\eta_N = 1/4$. □

Remark 2.2 (Independent numerical cross-checks). To guard against implementation or indexing errors in the certification scripts, four independent checks were executed across $k = 15, \dots, 18$: (i) direct numerical quadrature of the boundary integral $\int_{\Gamma_N} k_z(s)\overline{k_w(s)}\,|ds|$ agreed with $k_z(w)$ to $< 5.4 \times 10^{-15}$; (ii) accelerated Euler enclosures overlapped direct 2×10^6-prime enclosures; (iii) 4,496 evaluations of induced characters via `mpmath` and 16 evaluations of primitive characters via FLINT's native `dirichlet_l` fell strictly inside the cached balls (max midpoint discrepancy $< 4.0 \times 10^{-16}$); and (iv) real-point evaluations $A_{ii}(1+s)^2$ at $s = 1.25/\log N$ agreed with the independent Euler–Maclaurin code of Section 4. Refining the floating-point grid and adding interior layers at $k = 15$ gave $\|T_N\|/\delta_N \in [1.21, 1.25]$, indicating that the certified lower bound 1.191 is close to the true norm ratio.

2.4. **Asymptotic failure of Hypothesis R via Maier's matrix method.** We briefly record how Theorem 1.1 relates to Yilin Liu's asymptotic disproof [6] of Hypothesis R under GRH, whose proof we checked step by step against [5]. Polarizing [5, Theorem 7.1] at a weight b supported on small integers m and a second weight constantly equal to 1 near $u = 1$ (so that the continuous model cancels identically, by the same mechanism as Section 3 below) bounds the short-shift prime-pair difference

$$B_N(b) = \sum_{(m,N)=1} \Lambda(m)\, b(m) \big[\Lambda(N-m) - \Lambda(N+m)\big]$$

in terms of $\sqrt{N}\|T_N\|$ times the L^2-scale norm of b. Averaging N over an arithmetic progression $N \equiv a \pmod{Q}$ with $Q = \prod_{p \le z} p$ as in Maier's matrix method [7], the sifted densities of $N-m$ and $N+m$ are governed by Buchstab's function $\omega(u)$ and $\omega(u+1)$ at suitable logarithmic ratios u, and choosing u with $\omega(u) > \omega(u+1)$ produces $\ge X \exp(-(\log X)^{\varepsilon})$ even moduli $N \in [X, 2X]$ with $\sqrt{N}\|T_N\| > (\log N)^D$ for any fixed $D > 0$. Because u, a, and N are selected nonconstructively by averaging, that argument yields no explicit N, whereas Theorem 1.1 exhibits concrete failures at $N = 2^{15}, \dots, 2^{18}$.

3. Shifted Prime Pairs from Hypothesis S

We now prove Theorem 1.2 by polarizing the quadratic-form identity of [5, Section 7].

Proof of Theorem 1.2. Fix a nonzero real function $f \in C_c^\infty((1/4, 1/3))$ and set $g(u) = f(u-1) \in C_c^\infty((5/4, 4/3))$. Define $\Phi_1(u) = u^{1/2} f(u)$ and $\Phi_2(u) = u^{1/2} g(u)$, with Mellin transforms $\widetilde{\Phi}_r(s) = \int_0^\infty \Phi_r(u) u^{s-1}\, du$ $(r = 1, 2)$. For even $N > e^{10}$ satisfying GRH(N) and $\sigma = 1/9 \in S_N$, consider the bounded linear functionals on H_N defined as in [5, Section 7.2] by

$$\ell_{N,r}(h) = \frac{1}{2\pi i} \int_{(\sigma)} N^s \widetilde{\Phi}_r(s)(1+s)\, h(s)\, ds, \qquad r \in \{1, 2\},$$

and let $v_{N,r} \in H_N$ be their Riesz vectors, $\ell_{N,r}(h) = \langle h, v_{N,r} \rangle_{H_N}$. Setting $V_r(u) = \Phi_r(u) - u\Phi_r'(u)$, the exact norm formula [5, Proposition 7.4, (7.15)] gives

$$\|v_{N,r}\|_{H_N}^2 = \frac{1}{2\pi} \int_0^\infty \frac{(Nu)^{2\alpha_N} |V_r(u)|^2}{1 + (Nu)^{-2\Delta_N}} \frac{du}{u}.$$

On $\operatorname{supp} V_r \subset (0, 4/3]$, since $\alpha_N = 1/\log N$ we have $(Nu)^{2\alpha_N} = e^2 u^{2/\log N} \le e^2 (4/3)^{2/\log N} \ll 1$, while $1 + (Nu)^{-2\Delta_N} \ge 1$. Hence $\|v_{N,r}\|_{H_N} \ll_f 1$ for $r = 1, 2$.

Repeating the Bochner-integral calculation of [5, Theorem 7.1] for $\langle T_N v_{N,1}, v_{N,2}\rangle_{H_N}$ (using real-valuedness of Φ_1, Φ_2, so that $\widetilde{\Phi}_2(\sigma - iy)$ is the complex conjugate of $\widetilde{\Phi}_2(\sigma + iy)$) yields the exact bilinear identity

$$\langle T_N v_{N,1}, v_{N,2}\rangle_{H_N} = \frac{1}{(2\pi i)^2} \int_{(\sigma)} \int_{(\sigma)} N^{s+z} \widetilde{\Phi}_1(s) \widetilde{\Phi}_2(z)\, E_N(s,z)\, ds\, dz. \tag{25}$$

We evaluate the three components P_N, D_N, and K_N of (25) in turn.

First, rapid vertical decay of $\widetilde{\Phi}_1, \widetilde{\Phi}_2$ and polynomial vertical growth of g_χ under GRH(N) [5, Lemma 3.4] allow shifting both contours in the P_N and D_N integrals to $\operatorname{Re} s = \operatorname{Re} z = 3/4 > 1/2$, where the Dirichlet series converge absolutely. Termwise Mellin inversion and odd-character orthogonality [5, (2.1)] give

$$\begin{aligned} I_P - I_D = \frac{1}{N} \sum_{\substack{n,m \geq 1 \\ (nm,N)=1}} \Lambda(n)\Lambda(m)\, f\Big(\frac{n}{N}\Big)\, g\Big(\frac{m}{N}\Big) \left(\mathbf{1}_{n \equiv m \bmod N} - \mathbf{1}_{n \equiv -m \bmod N}\right) \\ - \frac{1}{N} \sum_{(n,N)=1} \Lambda(n)^2\, f\Big(\frac{n}{N}\Big)\, g\Big(\frac{n}{N}\Big). \end{aligned}$$

Since $\operatorname{supp} f \subset (1/4, 1/3)$ and $\operatorname{supp} g \subset (5/4, 4/3)$, the diagonal product $f(n/N)g(n/N)$ vanishes identically. Moreover, whenever $f(n/N)g(m/N) \neq 0$, we have

$$\frac{m-n}{N} \in \left(\frac{11}{12}, \frac{13}{12}\right), \qquad \frac{m+n}{N} \in \left(\frac{3}{2}, \frac{5}{3}\right).$$

Thus $N \nmid (m+n)$, while $m \equiv n \pmod N$ holds if and only if $m = n + N$, in which case $g(m/N) = f(n/N)$ and $(m, N) = (n, N)$. Consequently,

$$I_P - I_D = \frac{1}{N} \sum_{(n,N)=1} \Lambda(n)\Lambda(n+N)\, f\Big(\frac{n}{N}\Big)^2.$$

Second, for the model integral I_K, using $K_N(s,z) = N^{-s-z} S_N(s+z) K_-(s,z)$ on $\operatorname{Re} s = \operatorname{Re} z = \sigma = 1/9$ as in [5, Lemma 7.2], Euler's Beta integrals transform the (s,z) integral at fixed $w = s + z$ into $\frac{1}{2\pi i} \int_{(2\sigma)} S_N(w) \widetilde{J}_{f,g}(w)\, dw$, where for $y > 0$,

$$J_{f,g}(y) = \int_0^\infty \big[f(u)g(u+y) + g(u)f(u+y)\big]\, du - \int_0^y f(u)g(y-u)\, du.$$

Integrating $J_{f,g}(y)$ over $y \in (0, \infty)$ by Tonelli's theorem, the first two terms partition $\{u > 0, v > 0, u \neq v\}$ into $v > u$ and $u > v$, contributing $\big(\int_0^\infty f\big)\big(\int_0^\infty g\big)$, which cancels the convolution integral $\int_0^\infty \int_0^y f(u)g(y-u)\, du\, dy$. Thus $\widetilde{J}_{f,g}(1) = \int_0^\infty J_{f,g}(y)\, dy = 0$, removing the simple pole of $S_N(w) = \mathfrak{S}(N)\zeta(w)\mathcal{P}_N(w)$ at $w = 1$. Shifting the w-contour to $\operatorname{Re} w = 3/2$ and inverting termwise yields $I_K = \sum_{k=1}^\infty \mathfrak{S}(kN) J_{f,g}(k)$. For integer $k \geq 1$: $f(u+k) = 0$ for $u > 0$ since $u + k > 1/3$; $f(u)g(k-u) = 0$ since $u + (k-u) = k \notin (3/2, 5/3)$; and $f(u)g(u+k) = f(u)f(u+k-1)$ vanishes for $k \geq 2$ and equals $f(u)^2$ for $k = 1$. Hence $I_K = \mathfrak{S}(N) \int_0^\infty f(u)^2\, du$.

Combining $E_N = P_N - D_N - K_N$ with Cauchy–Schwarz and (10) under Hypothesis S gives

$$\begin{aligned} \left| \frac{1}{N} \sum_{(n,N)=1} \Lambda(n)\Lambda(n+N)\, f\Big(\frac{n}{N}\Big)^2 - \mathfrak{S}(N) \int_0^\infty f(u)^2\, du \right| &\leq \|T_N\|\, \|v_{N,1}\|_{H_N} \|v_{N,2}\|_{H_N} \\ &\ll_f \frac{\mathfrak{S}(N)}{h(N)}, \end{aligned}$$

proving (11). Proper prime powers p^r $(r \geq 2)$ in either n or $n+N$ up to $2N$ contribute $O((\log N)^2/\sqrt{N})$ after dividing by N, leaving $\gg_f \mathfrak{S}(N)N/(\log N)^2$ prime pairs. □

Remark 3.1 (Multiples jN and lack of residue-class averaging). For any fixed integer $j \geq 1$, taking $g(u) = f(u-j)$ in the same argument isolates the $k = j$ term of $\sum_{k\geq 1} \mathfrak{S}(kN)J_{f,g}(k)$ and yields

$$\frac{1}{N}\sum_{(n,N)=1} \Lambda(n)\Lambda(n+jN)\, f\Big(\frac{n}{N}\Big)^2 = \mathfrak{S}(jN)\int_0^\infty f(u)^2\, du + O_{f,j}\Big(\frac{\mathfrak{S}(N)}{h(N)}\Big),$$

while for $j \geq 2$ taking $g(u) = f(j-u)$ similarly yields the Goldbach-type asymptotic for $n+m = jN$ with $(nm, N) = 1$ and $n \in (N/4, N/3)$, again with main term $\mathfrak{S}(jN)\int_0^\infty f(u)^2\, du$ ($j = 1$ being the Goldbach weight itself). Conceptually, odd-character orthogonality modulo N only ties the two variables through $n \equiv \pm m \pmod N$; once smooth weights fix both variables at scale $O(N)$, each congruence class contains at most a single line $m - n = jN$ or $n + m = jN$ in the support, so no averaging across distinct shifts or moduli remains available.

4. Dyadic Second Moment and Numerical Tests of Hypothesis S

4.1. Proof of Proposition 1.3.

Proof of Proposition 1.3. Let $N = 2^k$, $L = \log N$, and fix $c > 1$. For sufficiently large k, $s = c/L$ lies in $S_N = (1/L, 1/8)$ at distance $d_N(s) = \min(c-1, L/8 - c)/L = (c-1)/L$ from Γ_N. By [5, (5.8)],

$$|E_N(s,s)| \leq \frac{(1+s)^2}{8d_N(s)}\,\|T_N\| = \frac{(1+c/L)^2}{8(c-1)}\,L\,\|T_N\|.$$

Along $N = 2^k$, $\mathfrak{S}(N) = 2C_2$ is constant, so Hypothesis S implies $\|T_N\| = o(1)$ and hence $E_N(s,s) = o(L)$.

Write $w = 2s = 2c/L \to 0^+$. First, $D_N(w) = \sum_{n \text{ odd}} \Lambda(n)^2 n^{-1-w}$ is independent of k and differs from $(\log\zeta)''(1+w) = w^{-2} + O(1)$ by an analytic Dirichlet series near $w = 0$, so $D_N(w) = w^{-2} + O(1) = L^2/(4c^2) + O(1)$. Second, factoring $\mathcal{P}_N(w) = \zeta(1+w)R_N(w)$ for $N = 2^k$ gives

$$\mathfrak{S}(2^k)R_{2^k}(w) = (2 - 2^{-w})\prod_{p>2}\left(1 - \left(\frac{1-p^{-w}}{p-1}\right)^2\right),$$

using $(p(p-2) + 2p^{-w} - p^{-2w})/(p-1)^2 = 1 - ((1-p^{-w})/(p-1))^2$ (`dyadic_local_factor`). For each $p \geq 3$ and $w \geq 0$, $0 \leq (1-p^{-w})/(p-1) \leq w\log p/(p-1)$, and induction over finite sets combined with passage to the limit (`dyadic_infinite_product_bound`) gives

$$0 \leq 1 - \prod_{p>2}\left(1 - \left(\frac{1-p^{-w}}{p-1}\right)^2\right) \leq w^2 \sum_{n=2}^{\infty}\left(\frac{\log n}{n-1}\right)^2 = O(w^2).$$

Since $2 - 2^{-w} = 1 + w\log 2 + O(w^2)$, we obtain $\mathfrak{S}(2^k)R_{2^k}(w) = 1 + w\log 2 + O(w^2)$. Third, in (5), Euler's constant cancels in $\zeta(1+w)\zeta(1-w) = (w^{-1} + \gamma + O(w))(-w^{-1} + \gamma + O(w)) = -w^{-2} + O(1)$, while differentiating $\pi^w X_1(w/2)^2 = \pi^w[\Gamma(3/4 - w/4)/\Gamma(3/4 + w/4)]^2$ at $w = 0$ gives $1 + w[\log\pi - \psi_0(3/4)] + O(w^2)$ (`odd_gamma_factor_deriv`). Since $N^{-w} = e^{-2c}$ is exact, multiplying these expansions yields

$$K_N(s,s) = -\frac{e^{-2c}}{4c^2}\,L^2 - \frac{e^{-2c}\big(\log(2\pi) - \psi_0(3/4)\big)}{2c}\,L + O(1).$$

Adding $D_N(w)$ and $E_N(s,s) = o(L)$ gives (13). At $c = 2$, the leading coefficient $(1 - e^{-4})/16$ agrees with the Goldston–Gonek–Montgomery coefficient [3] at $a = 2$.

For the Conrey–Farmer–Zirnbauer comparison [2, §5.1], raising the modulus from 2^j to 2^k ($2 \le j \le k$) adds no prime divisors and leaves every nonprincipal Dirichlet L-function unchanged (`same_prime_divisors_LFunction`). Applying $\partial_\alpha\partial_\beta|_{\alpha=\beta=\gamma=\delta=s}$ to the unitary ratios recipe for $L(1/2+\alpha,\chi)L(1/2+\beta,\bar\chi)/[L(1/2+\gamma,\chi)L(1/2+\delta,\bar\chi)]$ over primitive odd characters of conductor $q=2^j$, the zero-swap term gives $D_N(w)$, while in the double-swap term (root-number product 1) extracting the 2-adic zeta ratio $Z(1-\alpha-\beta)Z(1+\gamma+\delta)/[Z(1-\beta+\delta)Z(1-\alpha+\gamma)]$ ($Z(t)=(1-2^{-t})\zeta(t)$, residue $1/2$ at $t=1$) leaves at each odd prime the local factor $((1-p^{-1-w})(1-2/p+p^{-1-w}))/(1-1/p)^2 = 1-((1-p^{-w})/(p-1))^2$ (`ratios_odd_prime_factor`), identical to $R_{2^k}(w)$, together with $(N/q)^w(2-2^w)K_N(s,s)$. Modulo 2^k ($k\ge 3$), there are 2^{j-3} primitive odd characters of conductor 2^j for $3\le j\le k$ and 1 of conductor 4, out of 2^{k-2} total. Averaging $(N/q)^w$ with $m=k-2$ and $v=2^w$ gives (`finite_conductor_average`)

$$(2-v)\left(\frac{1}{2}\sum_{d=0}^{m-1}\left(\frac{v}{2}\right)^d+\left(\frac{v}{2}\right)^m\right)=1+(1-v)\left(\frac{v}{2}\right)^m.$$

Hence the full-family ratios expression equals $D_N(w)+K_N(s,s)+K_N(s,s)(1-2^w)2^{(w-1)(k-2)}$. For $w=2c/L$, $K_N(s,s)=O_c(L^2)$, $1-2^w=O_c(L^{-1})$, and $2^{(w-1)(k-2)}=O_c(N^{-1})$, so the discrepancy between the two models is $O_c(L/N)$. □

4.2. **Numerical evaluation up to $N=2^{28}$.** Using sixth-order Euler–Maclaurin summation for $\zeta(1/2+s,a)$ and $\partial_s\zeta(1/2+s,a)$ coupled with real-input FFTs over conjugate character pairs (validated to 3.6×10^{-15} against 45-digit `mpmath` at $N=32$, and cross-checked between adjacent levels $k-1$ and k via $g_{k,2r}=g_{k-1,r}$ to $<5.4\times10^{-12}$), we evaluated $A_{2^k}(c/L)$ up to $k=28$ ($N=268{,}435{,}456$). Table 2 lists every computed level at which c/L lies in S_N: for $c=1.25$ this holds from $k=15$, and for $c=2$ from $k=24$. In all cases $|E_N|/L$ is small and decreases overall, with primitive and induced subfamilies individually matching their conductor-weighted models $D_N+(2-2^w)K_N$ and $D_N+2^wK_N$. These computations reveal no conflict with Hypothesis S.

TABLE 2. Odd-character second moment $A_N(c/\log N)$ versus D_N+K_N for $N=2^k$ ($L=k\log 2$), decomposed into primitive ($q=2^k$) and inherited ($q\le 2^{k-1}$) normalized residuals.

c	k	$A_N(c/L)$	D_N+K_N	E_N/L	Primitive / Inherited res/L
1.25	15	13.520189826	13.528404577	-7.90×10^{-4}	-1.06×10^{-2} / $+9.00\times10^{-3}$
1.25	16	15.561466904	15.615675817	-4.89×10^{-3}	-8.64×10^{-3} / -1.13×10^{-3}
1.25	17	17.831161501	17.846284971	-1.28×10^{-3}	$+2.40\times10^{-3}$ / -4.97×10^{-3}
1.25	18	20.251146658	20.220043869	$+2.49\times10^{-3}$	$+6.05\times10^{-3}$ / -1.07×10^{-3}
1.25	21	28.208757584	28.198570117	$+7.00\times10^{-4}$	$+3.75\times10^{-3}$ / -2.35×10^{-3}
1.25	24	37.458509192	37.460115323	-9.66×10^{-5}	-9.81×10^{-4} / $+7.88\times10^{-4}$
1.25	25	40.830368353	40.831894608	-8.81×10^{-5}	-7.35×10^{-5} / -1.03×10^{-4}
1.25	26	44.341510784	44.345822518	-2.39×10^{-4}	-3.71×10^{-4} / -1.07×10^{-4}
1.25	27	48.001311793	48.001821760	-2.73×10^{-5}	$+2.20\times10^{-4}$ / -2.75×10^{-4}
1.25	28	51.799450637	51.799822310	-1.92×10^{-5}	$+3.65\times10^{-6}$ / -4.20×10^{-5}
2.00	24	15.611425598	15.611843830	-2.51×10^{-5}	-1.40×10^{-4} / $+9.01\times10^{-5}$
2.00	25	17.024431323	17.024269649	$+9.33\times10^{-6}$	$+4.63\times10^{-5}$ / -2.77×10^{-5}
2.00	26	18.496733956	18.496668023	$+3.66\times10^{-6}$	-1.70×10^{-7} / $+7.48\times10^{-6}$

5. Unitary Cumulative-Kernel Reduction and Sieve Bounds

5.1. Proof of Theorem 1.4(a)–(b): Fourier–Laplace unitary representation.

Proof of Theorem 1.4(a)–(b). Write $\alpha = \alpha_N = 1/L$ and $\beta = 1/8$. By [5, Lemma 5.2], the Fourier–Laplace map $(U_H\psi)(s) = (2\pi)^{-1/2}\int_{\mathbb{R}} \psi(\tau)e^{s\tau}\,d\tau$ is unitary from $L^2(\mathbb{R}, w_{H,N}(\tau)\,d\tau)$ onto H_N, where $w_{H,N}(\tau) = e^{2\alpha\tau} + e^{2\beta\tau}$. Under the change of variables $X = e^{-\tau}$ ($d\tau = -dX/X$) and $\xi(X) = \sqrt{X^{-2\alpha} + X^{-2\beta}}\,\psi(-\log X)$, H_N is unitarily isomorphic to $L^2((0,\infty), dX/X)$.

We identify the Fourier–Laplace density of $(s,z) \mapsto E_N(s,z)/((1+s)(1+z))$ on $S_N \times S_N$. First, for $\operatorname{Re} s > 1/2$, writing $S_\chi(X) = \sum_{n\le X}\Lambda(n)\chi(n)\sqrt{n}$ and integrating by parts gives

$$\frac{g_\chi(s)}{1+s} = -\int_1^\infty X^{-1-s}S_\chi(X)\,\frac{dX}{X}.$$

Under GRH(N), $g_\chi(s)/(1+s)$ is holomorphic on $\operatorname{Re} s > 0$ and uniformly square-integrable on vertical lines in $[\alpha, 3/4]$ [5, Lemma 3.4], so [5, Lemma 5.2] preserves this Fourier–Laplace density down to S_N. Averaging over $\chi \in X_N^-$ and subtracting the rank-one diagonal vectors $n^{-s}/(1+s) = \int_n^\infty X^{-1-s}n\,dX/X$ removes the $n = m$ terms (`odd_rectangle_projection_sub_diagonal`), yielding on $S_N \times S_N$ (in the L^2 sense on vertical lines)

$$\frac{P_N(s,z) - D_N(s+z)}{(1+s)(1+z)} = \int_0^\infty\int_0^\infty X^{-1-s}Y^{-1-z}C_N(X,Y)\,\frac{dX}{X}\frac{dY}{Y}. \tag{26}$$

Second, for $M_N(X,Y) = \sum_{j\ge1}\mathfrak{S}(jN)J_{jN}(X,Y)$ in (15)–(16), note that $J_h(X,Y) = 0$ for $h > X+Y$ and $\int_0^\infty J_h(X,Y)\,dh = 0$ (the two difference regions $u > v$ and $v > u$ together cover $(0,X]\times(0,Y]$ once, matching the sum region). Set $c_N = N/\varphi(N)$ and $H_N(h) = \sum_{jN\le h}\mathfrak{S}(jN) - c_N h/N$. Writing $\mathfrak{S}(jN)/\mathfrak{S}(N) = \sum_{d\mid j} b_N(d)$ with b_N supported on odd squarefree integers coprime to N and $b_N(p) = 1/(p-2)$, Mertens' theorem gives $\sum_{d\le J} b_N(d) \ll \log(2+J)$, whence

$$|H_N(h)| \ll \mathfrak{S}(N)\log(2+h/N) \quad (h\ge0), \qquad H_N(h) = -c_N\frac{h}{N} \quad (0\le h < N). \tag{27}$$

For $X \ge Y$, each of the three integrands in $J_h(X,Y)$ at fixed $v \in [0,Y]$ is monotone on its support in h of height $\le \sqrt{Xv}$, so $\operatorname{Var}_h J_h(X,Y) \ll Y\sqrt{XY}$ and $h \mapsto J_h(X,Y)$ is absolutely continuous. Integrating by parts against $\int_0^\infty J_h\,dh = 0$ gives

$$\begin{aligned} M_N(X,Y) &= -\int_0^\infty H_N(h)\,\partial_h J_h(X,Y)\,dh, \\ \frac{|M_N(X,Y)|}{XY} &\ll \mathfrak{S}(N)\sqrt{\frac{\min(X,Y)}{\max(X,Y)}}\log\left(2+\frac{X+Y}{N}\right). \end{aligned} \tag{28}$$

For $0 < \operatorname{Re} s, \operatorname{Re} z < 1/4$ (which contains $S_N \times S_N$, where $2\alpha < \operatorname{Re}(s+z) < 1/4 < 1$), (27) and (28) justify Fubini's theorem. Since $\iint X^{-1-s}Y^{-1-z}J_h(X,Y)\,\frac{dX}{X}\frac{dY}{Y} = h^{-s-z}K_-(s,z)/((1+s)(1+z))$ by [5, Lemma 4.1], integrating by parts in h identifies the double Mellin transform of $M_N(X,Y)$ with

$$\left[(s+z)\int_0^\infty H_N(h)h^{-s-z-1}\,dh\right]\frac{K_-(s,z)}{(1+s)(1+z)} = \frac{N^{-s-z}S_N(s+z)K_-(s,z)}{(1+s)(1+z)} = \frac{K_N(s,z)}{(1+s)(1+z)},$$

because on $\operatorname{Re} w > 1$ one has

$$N^{-w}S_N(w) = w\int_N^\infty \big(H_N(h) + c_N h/N\big)h^{-w-1}\,dh = w\int_N^\infty H_N(h)h^{-w-1}\,dh + \frac{wc_N}{w-1}N^{-w},$$

and on $0 < \operatorname{Re} w < 1$ the last term equals $w\int_0^N(-c_N h/N)h^{-w-1}\,dh = w\int_0^N H_N(h)h^{-w-1}\,dh$.

Subtracting M_N from (26) and setting $X = e^{-\tau}$, $Y = e^{-\upsilon}$ gives on $S_N \times S_N$ the real symmetric Fourier–Laplace density $\Theta(\tau,\upsilon) = 2\pi e^{\tau+\upsilon}[C_N(e^{-\tau},e^{-\upsilon}) - M_N(e^{-\tau},e^{-\upsilon})]$. For $h = U_H\psi \in H_N$, Plancherel along the two boundary lines $z = x + it$ ($x \in \{\alpha,\beta\}$) yields

$$\sum_{x\in\{\alpha,\beta\}} \int_{\mathbb{R}} e^{(x-it)\upsilon} h(x+it)\,dt = \sqrt{2\pi}\,(e^{2\alpha\upsilon} + e^{2\beta\upsilon})\psi(\upsilon) = \sqrt{2\pi}\,w_{H,N}(\upsilon)\psi(\upsilon).$$

Substituting this into (6) gives $(T_N(U_H\psi))(s) = (U_H\tilde{\psi})(s)$, where

$$\tilde{\psi}(\tau) = \int_{\mathbb{R}} \Theta(\tau,\upsilon)\psi(\upsilon)w_{H,N}(\upsilon)\,d\upsilon.$$

Passing to the orthonormalized density $\xi(X) = \sqrt{X^{-2\alpha}+X^{-2\beta}}\,\psi(-\log X)$ in $L^2((0,\infty), dX/X)$ multiplies the kernel $\Theta(-\log X, -\log Y)$ by

$$\sqrt{X^{-2\alpha}+X^{-2\beta}}\sqrt{Y^{-2\alpha}+Y^{-2\beta}} = (XY)^{-\alpha}\sqrt{1+X^{-2\Delta_N}}\sqrt{1+Y^{-2\Delta_N}},$$

proving (18).

Since $\mu_N(X) := \sqrt{1+X^{-2\Delta_N}} \geq 1$ for every $X > 0$, multiplication by $\mu_N(X)^{-1}$ is a contraction on $L^2((0,\infty), dX/X)$, which gives the exact lower bound $\|\mathcal{B}_\alpha\| \leq \|\mathcal{K}_N\| = \|T_N\|$ with no error term (note that by Plancherel along $\operatorname{Re} s = \alpha$, $\mathcal{B}_\alpha$ is unitarily equivalent to the left-boundary compression $T_{\alpha\alpha}$ of $\widetilde{T}_N$). On $\{X \geq 1, Y \geq 1\}$ we have $\mu_N(X) \leq \sqrt{2}$, so $\|\mathbf{1}_{X,Y\geq 1}\mathcal{K}_N\| \leq 2\|\mathbf{1}_{X,Y\geq 1}\mathcal{B}_\alpha\|$. On $\{\min(X,Y) < 1\}$, $C_N(X,Y) = 0$ and $M_N(X,Y) = 0$ unless $\max(X,Y) > N-1$; for $0 < X < 1$ and $Y > N-1$, we have $\mu_N(X)X^{-\alpha} \leq \sqrt{2}\,X^{-\beta}$ and $\mu_N(Y)Y^{-\alpha} \leq \sqrt{2}\,Y^{-\alpha}$, so (28) bounds

$$|\mathcal{K}_N(X,Y)| \ll \mathfrak{S}(N)X^{1/2-\beta}Y^{-1/2-\alpha}\log(2+Y/N) = \mathfrak{S}(N)X^{3/8}Y^{-1/2-\alpha}\log(2+Y/N),$$

whose $L^2(dX/X\,dY/Y)$ norm on $\{0 < X < 1, Y > N-1\}$ (and symmetrically on $\{0 < Y < 1, X > N-1\}$) is $O(\mathfrak{S}(N)N^{-1/2})$. The same bound holds without μ_N for $\mathcal{B}_\alpha$. This proves (19) and completes (a).

For (b), both C_N and M_N vanish when $X + Y < N$. On $\{X, Y \geq 1,\ X+Y \geq N,\ XY < N^2/L^7\}$ with $X \geq Y$ (so $X \geq N/2$), for each $m \leq Y$ the congruence $n \equiv \pm m \pmod N$ has $O(X/N)$ solutions $n \leq X$, giving $|C_N(X,Y)|/(XY) \ll \sqrt{XY}L^2/N$, while $|M_N(X,Y)|/(XY) \ll \mathfrak{S}(N)\sqrt{XY}/N$. Integrating $(XY)^{1-2\alpha}(L^4/N^2)$ over $\log X + \log Y \leq 2L - 7\log L$ yields a Hilbert–Schmidt norm $O(L^{-1})$. Finally, at $\gamma = \alpha/2$, [5, Proposition 6.5] gives $\|\mathcal{B}_\gamma\|_{\mathrm{HS}} \ll L^2 + \gamma^{-2} \ll L^2$. On $\{X, Y \geq 1,\ XY > N^{6\log L}\}$ we have $\mathcal{B}_\alpha(X,Y) = (XY)^{-\alpha/2}\mathcal{B}_\gamma(X,Y)$ with $(XY)^{-\alpha/2} < e^{-3\log L} = L^{-3}$, contributing $O(L^{-1})$ in Hilbert–Schmidt norm. Finally, if $\|T_N\|/\mathfrak{S}(N) \leq u_N \to 0$, choosing $h(N) = \min\bigl(1/(2u_N), \sqrt{N\mathfrak{S}(N)}/(2L^{3/2})\bigr) \to \infty$ gives $\eta_N \leq 1/4$ and satisfies (10); conversely, (10) with $\eta_N \leq 1/4$ implies $\|T_N\|/\mathfrak{S}(N) \leq (\sqrt{3}\,h(N))^{-1} \to 0$. □

5.2. **Proof of Theorem 1.4(c): Selberg upper bounds and Schur's test.**

Proof of Theorem 1.4(c). We establish the pointwise majorant

$$\frac{|C_N(X,Y) - M_N(X,Y)|}{XY} \ll \mathfrak{S}(N)\frac{\sqrt{XY}}{N}$$

outside the small corner where $\min(X,Y) < N^{1/4}$ and $\max(X,Y) < N^{5/4}$. For M_N, summing $|J_{jN}(X,Y)| \ll \min(X,Y)\sqrt{XY}$ over $j \leq (X+Y)/N$ using $\sum_{j\leq J}\mathfrak{S}(jN) \ll \mathfrak{S}(N)J$ already gives $|M_N(X,Y)|/(XY) \ll \mathfrak{S}(N)\sqrt{XY}/N$ whenever $X + Y \geq N$.

Case 1: $N^{1/4} \leq X, Y \leq N^{5/4}$ ($X + Y \geq N$). Set $W = \min(X,Y) \geq N^{1/4}$ and $z = W^{1/8}$. Each line $n \pm m = h$ ($h = jN$ even) meets $(0,X] \times (0,Y]$ in an interval of length $\ell \leq W$. Sieving $n(n \pm h)$ by squarefree $d \leq z$ with local exclusion count $\rho_h(p) = 1$ ($p \mid h$)

and $\rho_h(p) = 2$ ($p \nmid h$), Selberg's quadratic weights λ_d ($|\lambda_d| \le 1$, $\lambda_1 = 1$) bound pairs of primes $> z$ by $\ell/G(z) + O(z^4)$, where $G(z) = \sum_{d\le z} \mu(d)^2 \prod_{p\mid d} \rho_h(p)/(p-\rho_h(p))$. Normalizing $g(d) = \prod_{p\mid d} \rho_h(p)/(p-\rho_h(p))$ on $\{p \mid d \implies p \le z^\theta\}$ into a probability measure, the expected value of $\log d$ is $\le 2\sum_{p\le z^\theta}(\log p)/p \le \frac{1}{2}\log z$ for a small absolute $\theta > 0$, so Markov's inequality and Mertens' product give $G(z) \gg (\log z)^2/\mathfrak{S}(h)$. Primes $\le z$ add $O(z)$ pairs, and higher prime powers $p^r \le 2N^{5/4}$ ($r \ge 2$) add $O(\sqrt{W}\log N)$ pairs. Multiplying by $\Lambda(n)\Lambda(m) \ll L^2$ and $\sqrt{nm} \le \sqrt{XY}$, each family contributes $\ll \mathfrak{S}(h) W\sqrt{XY}$, and summing over $j \le (X+Y)/N$ gives $|C_N(X,Y)|/(XY) \ll \mathfrak{S}(N)\sqrt{XY}/N$.

Case 2: $\max(X,Y) \ge N^{5/4}$. By symmetry assume $X \ge Y$ and $X \ge N^{5/4}$. We first show uniformly for $(a,N) = 1$ that

$$\sum_{\substack{n\le X\\ n\equiv a \bmod N}} \Lambda(n) \ll \frac{X}{\varphi(N)}. \tag{29}$$

Writing $n = a + kN$ ($0 \le k \le T := X/N$, $T \ge N^{1/4}$), Selberg's upper-bound sieve on k modulo primes $p \nmid N$ up to $z = T^{1/8}$ gives $\ll \frac{TN}{\varphi(N)\log z} + z^4 + z \ll \frac{X}{\varphi(N)\log X}$ primes, contributing $\ll X/\varphi(N)$ after weighting by $\log X$. For prime powers $p^r \equiv a \pmod N$ with $2 \le r \le 7$, the r-th power map on $(\mathbb{Z}/N\mathbb{Z})^\times$ has kernel size $\le r^{\omega(N)+1} \ll_r N^{1/16}$, giving $\ll_r N^{1/16}(X^{1/r}/N + 1)\log X = o(X/N)$ since $X \ge N^{5/4}$, while all $r \ge 8$ contribute $O(X^{1/8}(\log X)^2) = o(X/N)$, proving (29).

Fixing $m \le Y$ with $(m,N) = 1$, both nonnegative sums of $\Lambda(n)\sqrt{n}$ over $n \le X$ with $n \equiv \pm m \pmod N$ are bounded by $M \ll X^{3/2}/\varphi(N)$, so after deleting $n = m$ their signed difference still has modulus $\le M$ (`abs_signed_rectangle_le_of_progression_bounds`). Summing against $\sum_{m\le Y}\Lambda(m)\sqrt{m} \ll Y^{3/2}$ and using $N/\varphi(N) \ll \mathfrak{S}(N)$ again yields $|C_N(X,Y)|/(XY) \ll \mathfrak{S}(N)\sqrt{XY}/N$.

Completion via Schur's test. On the remaining corner $\{1 \le \min(X,Y) < N^{1/4},\ N/2 \le \max(X,Y) < N^{5/4}\}$, we have $XY < N^{3/2}$; the trivial progression bound $O(X/N+1)$ gives $|\mathcal{B}_\alpha(X,Y)| \ll L^2 N^{-1/4}$ over logarithmic area $O(L^2)$, contributing $O(L^3N^{-1/4})$ in Hilbert–Schmidt norm, while $\min(X,Y) < 1$ contributes $O(\mathfrak{S}(N)N^{-1/2})$. Outside these negligible pieces, $(XY)^{-\alpha} \le 1$ gives $|\mathcal{B}_\alpha(X,Y)| \le C\mathfrak{S}(N)\sqrt{XY}/N$. For fixed X,

$$\int_{b_1N^2/X}^{b_2N^2/X} \frac{\sqrt{XY}}{N}\,\frac{dY}{Y} = 2\big(\sqrt{b_2} - \sqrt{b_1}\big),$$

and identically in X for fixed Y. Applying Schur's test (`schur_bound_unit_pair`) gives (20). □

5.3. **Height truncation and three obstructions to absolute-value bounds.** In vertical boundary coordinates on Γ_N, restricting both variables of $\widetilde{T}_N$ to $|\operatorname{Im} s| \le H$ ($H \ge 1$) defines a finite-height operator $A_{N,H}$ satisfying

$$0 \le \|T_N\| - \|A_{N,H}\| \le C\left(\frac{L\big(L + \log(2+H)\big)}{\sqrt{H}} + \frac{L\log(2+H)}{H^{3/8}}\right), \tag{30}$$

which is $O(L^{-1/2})$ at $H = L^5$. Indeed, using the explicit formula on $\operatorname{Re} s = 2/L$ and Poisson/Cayley transfer to α_N and β as in [5, (6.10)–(6.15)], the Cauchy-weighted character mean square $\int_{\mathbb{R}} \frac{H}{\pi(H^2+t^2)} a_x(t)\,dt$ ($x \in \{\alpha_N, \beta\}$) is $O((L+\log(2+H))^2)$, giving Hilbert–Schmidt tail $O(L(L+\log(2+H))/\sqrt{H})$ for P_N (`kernel_cutoff_mass_le_of_cauchy_moments`), $O(L^2/\sqrt{H})$ for D_N, and $O(L^2/\sqrt{H} + L\log(2+H)H^{-3/8})$ for K_N after separating the polar part $-1/w$ of $\zeta(1-w)$ from the Euler-summation remainder (`model_height_factor_le`).

On the retained intermediate scale $bN^2 < XY < N^{6\log L}$ (or bounded height windows on $\operatorname{Re} s = \alpha_N$), three facts show why triangle-inequality or Schur-type bounds cannot prove Hypothesis S:

(1) *Row divergence of the difference families taken separately.* Fix N and let $X \geq 8N$. For $Y \in [X, 2X]$ and each $h = jN \in [N, X/4]$, integrating $\sqrt{uv}$ along $v = u + h$ over $u \in [X/4, X/2]$ gives $\gg X^2$, so the difference part of $M_N(X, Y)$ alone exceeds $c\,\mathfrak{S}(N)X^3/N$, and its dY/Y-integral over $[X, 2X]$ grows like $\mathfrak{S}(N)X^{1-2\alpha_N}/N \to \infty$ as $X \to \infty$.

(2) *Logarithmic growth of the left-boundary model integral.* Along $N = 2^k$ at $s = \alpha_N$, set $K_N^{\#}(s, z) = K_N(s, z)/((1 + s)(1 + z))$. Putting $z = \alpha_N - iu/L$ gives $w = (2 - iu)/L$, $|N^{-w}| = e^{-2}$, and $L^{-2}|K_N^{\#}(\alpha_N, \alpha_N - iu/L)| \to e^{-2}/(4 + u^2)$. Dominated convergence on $|u| \leq HL$ yields, for every fixed $H > 0$,

$$\int_{-H}^{H} |K_N^{\#}(\alpha_N, \alpha_N - iv)|\, dv = \left(\frac{\pi e^{-2}}{2} + o(1)\right) \log N,$$

ruling out bounding K_N and $P_N - D_N$ separately in Schur's test.

(3) *Full-line absolute divergence of the residual.* For fixed $s = \alpha_N + it$, the single-variable Fourier density of $v \mapsto E_N^{\#}(s, \alpha_N - iv)$ at $Y = e^{-v}$ has arithmetic jumps $ma_m(s)/(1+s)$ at integers m coprime to N, whereas the model density is continuous in Y. For an odd prime $m \nmid N$, expanding $a_m(s) = -\Lambda(m)m^{-1/2}\langle g_\chi(s), \chi(m)\rangle - \Lambda(m)^2 m^{-1-s}$ on $\operatorname{Re} s > 1/2$ removes $n = m$ and retains the nonzero Dirichlet term $n = m^{1+\varphi(N)}$, so $a_m(s) \not\equiv 0$ and vanishes only on a null set of $t \in \mathbb{R}$. By the Riemann–Lebesgue lemma, $\int_{\mathbb{R}} |E_N^{\#}(\alpha_N + it, \alpha_N - iv)|\, dv = \infty$ for almost every $t \in \mathbb{R}$.

Numerically, trapezoidal quadrature over $v \in [-5, 5]$ at $t = 0$ for $N = 2^k$ ($k = 15, 16, 17, 18$) confirms that $\int_{-5}^{5} |K_N^{\#}(\alpha_N, \alpha_N - iv)|\, dv$ increases steadily $(3.123, 3.275, 3.426, 3.578)$, whereas on the interior line $x = \alpha_N + \Delta_N/4$ the model integral stays near 2.5 $(2.552, 2.523, 2.489, 2.451)$ while the subtracted residual $\int_{-5}^{5} |E_N^{\#}(x, x - iv)|\, dv$ drops sharply $(0.470, 0.374, 0.263, 0.169)$. Substantial cancellation between $P_N - D_N$ and K_N is thus visible numerically, but establishing it analytically across $bN^2 < XY < N^{6\log L}$ remains open.

6. Formal Verification, Replication Package, and Provenance

6.1. **Lean 4 formalization and interval-certificate archive.** The companion public repository

https://github.com/whenpoem/li-liu-spectral-hypotheses

contains:

- *Lean 4 library (`Spectral/`, `Spectral.lean`, `Audit.lean`).* Built under Lean `v4.37.0-rc1` against Mathlib commit `81cb93dd9d`, verifying 34 theorems with zero warnings and depending solely on the three standard axioms `propext`, `Classical.choice`, and `Quot.sound` (checked via `Verify.ps1`). Formalized statements include Lemma 2.1, the finite Rayleigh-quotient bound (23), odd-character rectangle projections and progression bounds, explicit-formula cutoff inequalities, Chebyshev higher-prime-power bounds and divisor-cutoff splits, the dyadic infinite-product remainder bound, the Hurwitz logarithmic-derivative transform (24), the CFZ local factor and conductor-average identities, and the Schur and Cauchy-moment integral lemmas.
- *Interval certificates and reproduction scripts.* The JSON certificates for $k = 15, \ldots, 18$, independent check logs, and real-diagonal moment outputs are generated and replayed sequentially via `kernel_rows.py`, `kernel_grid_certificate.py`,

`kernel_grid_checks.py`, and `experiment_dyadic.py` using `python-flint` 0.9.0, `NumPy`, `SciPy`, `mpmath`, and `threadpoolctl`.

6.2. **Provenance and human comprehension.** This work was produced through human-directed sessions with frontier language models:

- **OpenAI GPT-6.2 Astra and GPT-6.2 Sol (via Codex):** carried out analytical explorations and derivations (the polarized prime-pair argument, dyadic ratios comparison, cumulative-kernel reduction, and sieve/Schur bounds), implemented and executed the Python/FLINT ball-arithmetic certification pipeline and numerical experiments, and authored and checked the 34 Lean 4 theorems.
- **Anthropic Claude Opus 5.5:** cross-checked analytical derivations and error budgets, audited external preprints (including step-by-step verification of [6] against [5]), rebuilt and audited the Lean library, helped consolidate and prune the research notes, and proofread the final manuscript against the certificates and numerical outputs.
- **Google Gemini 4 (Hydrogen, Neon, and Carbon):** reviewed the analytical arguments, simplified the cumulative-kernel reduction in Theorem 1.4(a) via the exact Fourier–Laplace unitary map, and drafted and typeset the present LaTeX manuscript and repository documentation.
- **Human contributor and comprehension level:** defined the research scope, directed model iterations, ran local Lean 4 kernel and interval-arithmetic replays, and checked and understood *some, but not all, parts* of the proofs and code (Hexagon comprehension level: *some parts*).

References

[1] L. Alpöge, remark announcing a proof of the Riemann Hypothesis and the Generalized Riemann Hypothesis together with a Lean formalization, November 20, 2026, https://alpo.ge/remark.pdf.

[2] J. B. Conrey, D. W. Farmer, and M. R. Zirnbauer, *Autocorrelation of ratios of L-functions*, Commun. Number Theory Phys. **2** (2008), no. 3, 593–636, arXiv:0711.0718.

[3] D. A. Goldston, S. M. Gonek, and H. L. Montgomery, *Mean values of the logarithmic derivative of the Riemann zeta-function with applications to primes in short intervals*, J. Reine Angew. Math. **537** (2001), 105–126.

[4] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), no. 8, 1281–1292; used via `python-flint` 0.9.0.

[5] J. Li and J. Liu, *A conditional proof of the Goldbach conjecture*, preprint, Mathematical Research Center, Shandong University, 2026.

[6] Y. Liu, *Failure of the Li–Liu spectral hypothesis R under Dirichlet GRH*, Hexagon preprint `2611.00788v1`, November 21, 2026, https://hexagonmath.org/2611.00788v1.

[7] H. Maier, *Primes in short intervals*, Michigan Math. J. **32** (1985), no. 2, 221–225.

[8] M. Rubinstein, *An independent check of the Lean proof of RH and GRH*, November 21, 2026, https://mrubinst.github.io/grh-lean-verification/.