GENERALIZED CLUSTER STRUCTURES AND SIMPLE MODULES

Abstract. We prove Gleitz's conjecture, including simplicity of every mutable cluster monomial. We prove Fraser's geometric inclusions and identify the homogeneous coordinate rings of cyclic symmetry loci with their upper generalized cluster algebras. For every rank and period, we establish the independent-coefficient form of his categorical conjecture in type A. A transcendence-degree calculation gives the obstruction to the original palindromic formulation.

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1. Introduction

Fomin and Zelevinsky introduced cluster algebras in their study of dual canonical bases and total positivity [8]. Their generators occur in overlapping algebraically independent sets, related by binomial exchange relations. Chekhov and Shapiro replaced these binomials by higher-degree exchange polynomials in their work on decorated Teichmüller spaces of surfaces with orbifold points [4]. The resulting generalized cluster algebras retain the Laurent phenomenon. Their additional exchange coefficients occur naturally both in cyclic symmetry loci of Grassmannians and in representation theory at roots of unity. We study these two realizations in type A.

On the representation-theoretic side, Hernandez and Leclerc introduced monoidal subcategories of finite-dimensional quantum affine modules whose Grothendieck rings carry cluster structures [17]. This interprets products of compatible cluster variables as tensor products of simple modules. At a root of unity the restricted integral form retains divided powers and divided Cartan operators. Frenkel and Mukhin developed its character theory and quantum Frobenius decomposition [10]. The Frobenius factors account for coefficients in higher-degree exchanges, while the remaining spectral data provide the mutable variables. Thus the coefficient ring and the spectral orbit are part of the representation-theoretic statement.

Gleitz constructed this correspondence for $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_2)$ in her 2014 preprint, published in 2016 [16]. If $e = \mathrm{ord}(\varepsilon^2)$, the associated generalized cluster algebra has finite type C_{e-1}. She also proved the type A_2, $e = 2$ case, whose generalized cluster type is G_2. In these cases the full simple basis combines mutable cluster monomials with Chebyshev or Schur polynomials in the Frobenius coefficients. Her Conjecture 5.5 proposes the type A_2 correspondence for every e, including the assertion that every mutable cluster monomial represents a simple module. We use the theorem and conjecture numbers of the preprint throughout.

Subsequent character calculations give further evidence and partial results. An, Li, Luo and Zhang obtained path formulas for Kirillov–Reshetikhin characters and characters of highest ℓ-weight degree two, together with an irreducibility criterion for a product of two

fundamental modules [2]. An, Li and Luo then gave mutation sequences reaching the real Kirillov–Reshetikhin modules in type A_2 [1, Theorem 4.6]. The ring isomorphism in Gleitz's conjecture and equality of its lower and upper algebras already appear as Theorems 4.11 and 4.15 of their 2026 preprint. Section 5.3 also states the upper-algebra identification for $(k, e) = (4, 2)$. Theorem A includes these rank-three ring statements and proves simplicity of all mutable cluster monomials. Theorem D treats every $k, e \geq 2$ with integral coefficients. Its proof constructs the simple objects and their tensor products along every finite mutation sequence.

The geometric history begins with Scott's cluster structure on the homogeneous coordinate ring of the Grassmannian [22]. The signed cyclic shift preserves the Grassmannian and its totally nonnegative part. Karp determined the fixed points of this shift and identified the unique totally nonnegative fixed point [19]. Fraser studied the fixed loci of its arbitrary powers [9]. He described their components as products of smaller Grassmannians, constructed positivity tests by cyclic bridge decompositions, and singled out the component containing the totally nonnegative locus. In the range relevant here this component is a product of projective spaces, and its homogeneous coordinate ring is a Segre cone ring.

Fraser's cyclic seed has one exchange of degree k; its other exchanges are binomial. Its exceptional coefficients are symmetric q-binomial coefficients determined by the order of the cyclic symmetry. Theorem 7.5 of [9] proves regularity of the initial variables and their first replacements, and expresses every Plücker coordinate as a Laurent polynomial in the initial seed. Conjecture 7.6 asks for the inclusions between the full cluster algebra, the homogeneous coordinate ring, and the upper cluster algebra. The following paragraph further predicts equality of the latter two. Fraser proved the finite cluster type cases, several finite mutation type examples, and the geometric inclusions for $k = 3$ after inverting the ordinary frozen variables [9, Proposition 8.5 and the following discussion, Theorem 10.1, and Example 10.12]. Remark 7.7 identifies the remaining commutative-algebra issues: common divisor components of initial and adjacent variables, and upper-bound invariance after coefficient specialization.

Periodic band matrices provide another realization of the cyclic seed. Gekhtman, Shapiro and Vainshtein constructed generalized cluster structures on the Drinfeld double of GL_n and on periodic staircase matrices [13, 14]. Their subsequent completeness theorem identifies the upper algebra with the coordinate ring of the band-matrix space when the bandwidth parameter is at most the period and the two outer diagonals are nonzero [15, Theorem 5.1]. Fraser related this construction to his cyclic seeds by generalized cluster quasi-homomorphisms. The geometric theorems below prove his inclusions and upper-algebra equality throughout the prescribed range, retaining the frozen Plücker coordinates as polynomial functions. The proof computes the initial and adjacent charts after scalar spectral specialization and extends across their boundary divisors. An integral band-matrix argument then supplies the chart identities for every rank and period.

Fraser's Conjecture 8.8 connects this geometry to the root-of-unity Grothendieck ring in arbitrary type A. The categorical theorem below uses independent exchange coefficients, reversed by mutation. This convention already occurs in Gleitz's G_2 model and in [1], and is allowed by Fraser's Definition 2.7. Remark 1.1 explains the required correction to the palindromic convention in his Definition 2.6: its literal specialization has smaller transcendence degree than the proposed Grothendieck ring when $k \geq 3$.

1.1. **Conventions.** We write $\mathbb{N} = \{0, 1, 2, \ldots\}$. We work over $\mathbb{C}$, except when an integral coordinate ring or Grothendieck ring is explicitly specified. A generalized seed consists of mutable variables $x_1, \ldots, x_N$, ordinary frozen variables, an extended integer matrix B, positive column degrees d_j, and coefficient strings

$$(z_{j;0}, \ldots, z_{j;d_j}), \qquad z_{j;0} = z_{j;d_j} = 1.$$

Write $\beta_{ij} = b_{ij}/d_j$; the columns of B are divisible by the corresponding d_j. Mutation in direction j replaces x_j by the rational function determined by

$$x_j x_j' = \sum_{s=0}^{d_j} z_{j;s} \prod_i x_i^{s[\beta_{ij}]_+ + (d_j - s)[-\beta_{ij}]_+}. \tag{1.1}$$

Here $[a]_+ = \max(a, 0)$, the product includes ordinary frozen variables, and B undergoes matrix mutation. The string in direction j is reversed; the other strings are unchanged. Equivalently the divided matrix mutates by

$$\beta'_{ab} = \begin{cases} -\beta_{ab}, & a = j \text{ or } b = j, \\ \beta_{ab} + d_j\big([\beta_{aj}]_+\beta_{jb} + \beta_{aj}[-\beta_{jb}]_+\big), & a, b \neq j. \end{cases}$$

In the scalar geometric specialization the strings are palindromic, so reversal has no effect. Independent strings require the reversal convention.

The *lower generalized cluster algebra* is generated by all mutable cluster variables and the specified polynomial coefficient ring. The *upper generalized cluster algebra* is the intersection, inside the common fraction field, of the Laurent rings of all seeds:

$$\mathcal{U} = \bigcap_t R_{\mathrm{coeff}}[x_{1;t}^{\pm 1}, \ldots, x_{N;t}^{\pm 1}]. \tag{1.2}$$

Ordinary frozen variables belong to R_{coeff} and are not inverted unless explicitly indicated. The Laurent phenomenon and upper-bound arguments used below are valid with this polynomial convention; the required coprimality and specialization arguments are proved in the text.

A *cluster monomial* means a product $\prod_{j=1}^{N} x_{j;t}^{a_j}$, $a_j \in \mathbb{N}$, of mutable variables from a single seed. It does not include arbitrary powers of the interior coefficients $z_{j;s}$.

Fix $k, e \geq 2$, and let ε satisfy $\mathrm{ord}(\varepsilon^2) = e$. Let $\mathcal{C}_{k,e}$ be the full tensor subcategory of finite-dimensional type-1 modules over $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_k)$ whose simple constituents have highest monomials in the free commutative monoid generated by the letters

$$\mathcal{Y}_{k,e} = \{Y_{i,\varepsilon^{\,i-1+2r}} : 1 \leq i < k,\ r \in \mathbb{Z}/e\mathbb{Z}\}. \tag{1.3}$$

We use the restricted quantum group and its full ε-character, including the divided Cartan data. For a dominant monomial m, $L(m)$ denotes the corresponding simple module. Our integral Grothendieck rings use addition from short exact sequences and multiplication from tensor product.

1.2. **Four main results.** The algebra $\mathcal{G}_e$ in the first theorem is Gleitz's polynomial-coefficient algebra over $\mathbb{Z}$. Its full initial matrix, coefficient polynomial, and labels are given in Section 13.2. The root chosen by Gleitz is $\varepsilon = \exp(\pi i/e)$ for even e, and $\varepsilon = \exp(2\pi i/e)$ for odd e.

Theorem A (Gleitz's conjecture)**.** *For every $e \geq 2$, Gleitz's assignment of the initial variables and the two coefficients to their simple-module classes extends to an isomorphism*

$$\mathcal{G}_e \xrightarrow{\ \sim\ } K_0(\mathcal{C}_{3,e}).$$

The lower and upper generalized cluster algebras coincide. Every mutable cluster monomial maps to the class of a simple module.

For the next two theorems let $n, \ell, k \geq 2$, and put

$$L = \gcd(n, \ell), \qquad p = n/L.$$

Assume $L \geq 2$ and $p \geq k$. Let $D_n(k, \ell)$ denote the distinguished component of the ℓ-fixed locus of the signed cyclic shift on $\mathrm{Gr}(k, n)$, as in [9, Definition 7.1]. Its projective variety is a product of k projective spaces of dimension $L - 1$. We write $R_n(k, \ell)$ for its homogeneous Plücker coordinate ring, or equivalently the coordinate ring of its affine cone. Thus $\mathbb{C}[D_n(k, \ell)]$ in the statements below always has this homogeneous meaning. The explicit eigenvector

model and the complete initial seed are given in Section 2.2. Denote the resulting lower and upper algebras by $\mathcal{A}_n(k,\ell)$ and $\mathcal{U}_n(k,\ell)$.

Theorem B (Fraser's geometric conjecture)**.** *For every $L \geq 2$ and $p \geq k \geq 2$ as above,*

$$\mathcal{A}_n(k,\ell) \;\subseteq\; R_n(k,\ell) \;\subseteq\; \mathcal{U}_n(k,\ell).$$

The inclusions hold without inverting the ordinary frozen Plücker coordinates.

Theorem C (The upper-algebra equality)**.** *In the same range,*

$$R_n(k,\ell) = \mathcal{U}_n(k,\ell).$$

Theorem C proves the expectation immediately following Fraser's Conjecture 7.6. The proof of Theorem B establishes regularity of every mutated variable.

For arbitrary $k, e \geq 2$, let $\mathcal{U}^{\mathrm{ind}}_{k,e}$ be the upper generalized cluster algebra over $\mathbb{Z}[F_1, \ldots, F_{k-1}]$ of the determinant-one seed in Section 4.4. This sets every $D_h = 1$ and the auxiliary normalization variable $T = 1$. Its mutable rank is $N = (k-1)(e-1)$, only the first direction has degree k, and its interior exchange coefficients $F_1, \ldots, F_{k-1}$ are algebraically independent polynomial variables. In the initial seed the exceptional exchange is

$$Q\phi = \sum_{s=0}^{k} F_s U^s V^{k-s}, \qquad F_0 = F_k = 1, \tag{1.4}$$

with $Q = x_1$, $U = x_2$, $V = x_{e+1}$, and $x_j = 1$ for $j > N$. Subsequent exceptional mutations reverse the coefficient string as in (1.1).

Theorem D (The categorical structure with independent coefficients)**.** *For all $k, e \geq 2$, there is an integral ring isomorphism*

$$\mathcal{U}^{\mathrm{ind}}_{k,e} \;\xrightarrow{\sim}\; K_0(\mathcal{C}_{k,e}).$$

Every mutable cluster monomial maps to a simple-module class. The coefficient F_s maps to a Frobenius fundamental simple class. Under the band-coordinate convention of the proof,

$$a_{j,h} \longmapsto [L(Y_{j,\mathcal{E}^{\,2h+j-1}})] \qquad (1 \leq j < k,\ h \in \mathbb{Z}/e\mathbb{Z}).$$

In particular the assertion also holds after complexification, as in Fraser's categorical formulation.

Remark 1.1 (The coefficient correction in Fraser's Conjecture 8.8)**.** Theorem D uses independent coefficients. Fraser's Definition 2.6 imposes $z_{j;s} = z_{j;d_j-s}$; Definition 7.2 uses that convention, and Conjecture 8.8 sets all ordinary frozen variables equal to 1. Consequently its literal coefficient ring in the exceptional direction has only $\lfloor k/2 \rfloor$ independent parameters. Every element of that upper algebra lies in a field of transcendence degree at most

$$(k-1)(e-1) + \lfloor k/2 \rfloor.$$

By the fundamental-generator theorem for the restricted Grothendieck ring [10, Section 3.1], or by the explicit based comparison in Section 13.1, the latter is a polynomial ring in $(k-1)e$ algebraically independent fundamental classes. For $k \geq 3$ the difference is

$$(k-1)e - \big((k-1)(e-1) + \lfloor k/2 \rfloor\big) = k - 1 - \lfloor k/2 \rfloor > 0.$$

Thus the literal palindromic version cannot be isomorphic to the stated Grothendieck ring. Already $k = 3, e = 2$ gives three independent seed parameters versus four fundamental classes. Removing the palindromic relations and reversing the independent string on mutation changes the coefficient ring. Fraser's reversal convention in Definition 2.7 allows the corrected formulation used here. The scalar geometric coefficients are palindromic, so this issue does not alter Theorems B and C.

Remark 1.2 (The meaning of cluster monomial)**.** The coefficient simples need not be real. For example in type A_2, if F_1 is the pullback of the defining representation under quantum Frobenius, write $V(\lambda)$ for the classical simple module of highest weight λ, and ω_i for the fundamental weights. Then

$$F_1^2 = [\mathrm{Fr}^* V(2\omega_1)] + F_2.$$

Thus adding arbitrary coefficient powers to the cluster-monomial assertion would make it false. Ordinary determinants in the retained GL_k model are one-dimensional and may be multiplied or inverted without this problem.

1.3. **Structure of the proof.** The geometric argument constructs the initial and all adjacent charts with their polynomial coefficient rings. A canonical-form calculation controls the divisors omitted by these charts, and upper-bound invariance gives the full upper algebra. An integral band-matrix version provides the regular functions used in the categorical argument.

We then use all orbit intersection-cohomology complexes of the cyclic quiver, including periodic labels. Strict partial flags construct the bounded basis and its Miura character map. Right-adjoint restriction, completed affine Schur algebras, and the integral double-Hall Hopf algebra give an actual exact tensor category. An equivariant polynomial deformation supplies normalized intertwining maps. Their head and socle properties yield a signed-support lemma: a regular virtual class which is a logarithmic Poisson eigenvector against a real simple factor has support consisting of strongly commuting simples.

Ordinary determinantial modules give the initial real family. The first exceptional mutation is proved separately, using its regularity, a joint Poisson and degree calculation, and the complete highest Miura word. The argument proves simplicity of every positive power and hence reality. A filtration argument then propagates the construction through every mutation. Finally the based GL_k-to-SL_k quotient identifies the target category; in rank three an explicit mutation word recovers Gleitz's precise seed, and two further mutation blocks reach all fundamental generators.

2. Cyclic symmetry loci

We retain the ordinary frozen variables as polynomial functions on the homogeneous coordinate cone, including its frozen boundary.

2.1. **Two local principles.** We use the following form of a standard consequence of Zariski's Main Theorem.

Lemma 2.1. *Let $Y \to X$ be a separated, quasi-finite, birational morphism of integral Noetherian schemes, with X normal. It is an open immersion. Suppose in addition that X and Y are affine and X is factorial. The open immersion is an isomorphism if their coordinate rings have the same units. It is also an isomorphism if, for some nonzero $P \in \Gamma(X, \mathcal{O}_X)$, its image contains $D(P)$ and meets every height-one component of $V(P)$.*

Proof. Zariski's Main Theorem factors the morphism as an open immersion into a finite scheme over X. Replace that finite scheme by the reduced closure of the integral scheme Y. The resulting finite scheme is integral and birational over X. On an affine open of X, its ring is an integral extension of $\mathcal{O}(X)$ contained in the common function field, so normality forces equality. This proves the first assertion. In the first additional situation, a missing prime divisor has an irreducible equation g in the factorial coordinate ring of X. Its restriction would be a unit on Y, although g is not a unit on X. Thus no prime divisor is missing. In the second situation a missing prime divisor must divide P, and meeting that divisor contradicts its absence: a nonempty open subset of an irreducible divisor contains its generic point. In either case the complement has codimension at least two. A normal affine domain is the intersection of its local rings at height-one primes inside its fraction field. Hence restriction identifies the two coordinate rings. Affineness now proves that the open immersion is an isomorphism. This argument applies over $\mathbb{Z}$ as well; in particular, it detects a missing vertical prime divisor. □

Here and below a divisor is reduced when each of its height-one multiplicities is one.

Lemma 2.2. *Let R be a connected graded normal $\mathbb{C}$-domain of dimension $d \geq 2$ whose punctured cone is smooth. Suppose that it has a nowhere-zero top form ω of scaling degree κ. Let $x_1, \dots, x_d$ be independent degree-one functions, and put $N = d - \kappa$. Assume that their divisors are reduced and have no pairwise common component. For $1 \leq i \leq N$, suppose that*

$$x_i y_i = P_i(x_j : j \neq i), \qquad y_i \in R,$$

and that at every generic component of $x_i = 0$, one of the partial derivatives of P_i is a unit. Then

$$dx_1 \wedge \cdots \wedge dx_d = c\Big(\prod_{i=1}^{N} x_i\Big)\omega, \qquad c \in \mathbb{C}^\times. \tag{2.1}$$

Moreover x_i and y_i have no common divisor component.

Proof. Write the left side as $J\omega$. The regular function J on the punctured cone extends to R by normality and has degree $d - \kappa = N$. At a generic component of $x_i = 0$, differentiating the exchange relation gives a nontrivial relation among the initial differentials. Consequently J vanishes there. Reducedness and the absence of shared components show that $J/(x_1 \cdots x_N)$ has no height-one pole. It belongs to R and has degree zero. Algebraic independence in characteristic zero gives $J \neq 0$, proving (2.1).

Let v be the valuation at a component of $x_i = 0$. Suppose $v(y_i) = r > 0$. Since x_i is a local parameter, the exchange relation has the form $P_i = x_i^{r+1} u$ with u a unit. Replacing a differential dx_j by dP_i, where the corresponding partial derivative is a unit, changes the full wedge by a unit. But

$$dx_i \wedge dP_i = x_i^{r+1} dx_i \wedge du.$$

Thus $v(J) \geq r + 1 \geq 2$, contrary to (2.1). Therefore $v(y_i) = 0$. □

2.2. **The cone and its initial seed.** Let $L = \gcd(n, \ell) \geq 2$ and $p = n/L \geq k \geq 2$. Set

$$\begin{gathered} N = (k-1)(L-1), \qquad d = N + L = k(L-1) + 1, \\ \lambda_a = \exp\Big(\frac{2\pi i}{p}\Big(a - \frac{k-1}{2}\Big)\Big) \quad (0 \leq a < k). \end{gathered} \tag{2.2}$$

The λ_a are distinct, their product is one, and their multiset is invariant under inversion. The distinguished cyclic symmetry component selects a line in each of these k eigenspaces, all of dimension L. Its homogeneous coordinate ring is therefore the Segre cone ring

$$\begin{gathered} R = S^H = \bigoplus_{r \geq 0} S_{(r,\dots,r)}, \qquad S = \mathbb{C}[z_{a,s} : 0 \leq a < k, 1 \leq s \leq L], \\ H = \{(t_a) \in (\mathbb{C}^\times)^k : \prod_a t_a = 1\}. \end{gathered} \tag{2.3}$$

Here H rescales rows. To see the asserted embedding, the exterior product of the chosen eigenlines identifies their tensor product with a linear subspace of the Plücker space. Thus the restricted Plücker coordinates span the full degree-one part of (2.3). In particular R is the homogeneous coordinate ring denoted $\mathbb{C}[D_n(k, \ell)]$ in the geometric statement.

Define column vectors by

$$(v_{s+tL})_a = \lambda_a^t z_{a,s} \qquad (1 \leq s \leq L, t \in \mathbb{Z}).$$

For a finite ordered index set I of size k, write $\Delta_I = \det(v_i : i \in I)$, with columns in increasing order. An integer interval $[a, b]$ denotes $\{a, a+1, \dots, b\}$, and is empty if $a > b$. For $1 \leq m \leq d$, put

$$\begin{gathered} h_m = \left\lceil \frac{m-1}{L-1} \right\rceil, \qquad w_t = 1 + tL, \\ I_m = [m, m + h_m - 1] \cup \{w_{h_m}, \dots, w_{k-1}\}, \\ x_m = \det(v_i : i \in I_m). \end{gathered} \tag{2.4}$$

Empty intervals are omitted and determinants use increasing column order. The last L sets are consecutive k-subsets; these give the ordinary frozen variables. We keep $x_{N+1}, \dots, x_d$ polynomial.

The initial divided exchange columns, in the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_d$, are

$$\begin{aligned} b_1 &= \mathbf{e}_2 - \mathbf{e}_{L+1}, && (2.5)\\ b_m &= \mathbf{e}_{m+1} + \mathbf{e}_{m+L-1} - \mathbf{e}_{m-1} - \mathbf{e}_{m+L} && (2 \le m \le \min(L, N)),\\ b_m &= \mathbf{e}_{m-L} + \mathbf{e}_{m+1} + \mathbf{e}_{m+L-1} - \mathbf{e}_{m-L+1} - \mathbf{e}_{m-1} - \mathbf{e}_{m+L} && (L < m \le N). \end{aligned}$$

The actual exchange matrix has column kb_1 at vertex 1 and column b_m elsewhere. The degree at vertex 1 is k and all other degrees are one. Its exceptional exchange polynomial is

$$P_1(U, V) = \prod_{a=0}^{k-1} (U + \lambda_a^{-1} V), \qquad U = x_2, \quad V = x_{L+1}. \tag{2.6}$$

It is the symmetric q-binomial specialization in Fraser's seed. All other exchanges are the binomials specified by (2.5).

The bridge completion in [9, Theorem 7.5 and Lemma 6.10] gives

$$\mathbb{C}[x_1, \dots, x_d] \subseteq R \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]. \tag{2.7}$$

We recall precisely which assertion is used. Completing the bridge collection to a Grassmannian cluster gives a Laurent expression for every Plücker coordinate. On this component each added member becomes a nonzero scalar multiple of an initial member. When $p = k$ there are no added members. Restricting these Laurent expressions proves the second inclusion in (2.7). Consequently the x_m generate $\operatorname{Frac} R$. Their number is $\dim R = d$, so they are algebraically independent.

Remark 2.3. Formula (2.4) records the global indices given by the bridge construction. It also fixes the consecutive frozen intervals, whose lower endpoints must depend on the index. The sign in (2.6) is a plus sign: for example, when $k = 2, p = 3$, the polynomial is $U^2 + UV + V^2$. These conventions correct the corresponding indexing and factorization displays in [9, Definition 7.1 and Remark 7.4]; they do not alter the geometric seed defined by its bridge collection.

2.3. **Initial divisors and their neighbors.**

Lemma 2.4. *The ring R is normal and its punctured cone is smooth. The divisors of the x_m are reduced and have no pairwise common component. For $m \ge 2$, x_m is prime in R. The divisor of x_1 has k distinct components.*

Proof. The quotient of $\prod_a (\mathbb{A}^L \setminus \{0\})$ by H is the punctured cone and is a principal torus bundle. This proves smoothness. An element of $\operatorname{Frac} R$ integral over R is integral over S, belongs to the integrally closed ring S, and is H-invariant. It therefore belongs to R, proving normality.

Group the columns of x_m by their residues modulo L. In each group the exponents of the λ_a are consecutive. To check the only possible join, let $h = h_m$. The interval part ends before $1 + hL$, and its residue-1 powers end at $h - 1$ whenever present; the tail starts with power h. Expansion by residue groups therefore has every row assignment with the prescribed residue counts. Each coefficient is a product of nonzero Vandermonde determinants in subsets of the distinct λ_a, multiplied by nonzero powers of them.

For $m \ge 2$, at least two residues occur. Suppose $x_m = fg$ is a nontrivial factorization in S. The row and residue tori preserve the polynomial up to character; their connectedness implies that each irreducible factor is homogeneous for both tori. Since every row has degree one, the factors use disjoint nonempty subsets of rows. Choose one row in each subset and two active residues. Full assignment support contains two monomials obtained by swapping these residues on the chosen rows. They give different residue degrees in the first factor, a contradiction. Disjoint row-variable sets make cancellation between the two factors impossible.

Thus x_m is irreducible in S. Since it is invariant, $(x_m)S \cap R = (x_m)R$: if $x_m g$ is invariant, so is g. Hence it is prime in R.

The remaining determinant is

$$x_1 = c \prod_{a=0}^{k-1} z_{a,1}, \qquad c \in \mathbb{C}^\times. \tag{2.8}$$

No other x_m has any of these row factors. Independence excludes association of two different initial variables. The omitted row-zero sets have codimension $L \geq 2$, and the torus bundle detects both divisor multiplicities and common components. Thus the row factors in (2.8) descend to k distinct reduced components, and all the asserted divisor properties follow. □

Lemma 2.5. *Every initial exchange quotient $y_m = P_m/x_m$ belongs to R.*

Proof. Put $r(t) = t + \lceil t/(L-1) \rceil$. Substitution in (2.4) gives

$$I_{m+1} = (I_m \setminus \{m\}) \cup \{r(m)\}, \qquad I_{m-L} + L = (I_m \setminus \{r(m-1)\}) \cup \{w_k\}; \tag{2.9}$$

the second identity is used only for $m > L$. At a ceiling jump the first tail member is exactly the interval member required to preserve the equality. In particular $r(L) = L + 2$.

In the first ordinary strip,

$$P_m = x_{m+1}x_{m+L-1} + x_{m-1}x_{m+L} \qquad (2 \leq m \leq \min(L, N)).$$

Let $K = \{w_1, \ldots, w_{k-2}\}$, $d_0 = m-1$, $a = m$, $e_0 = r(m)$ and $c = w_{k-1}$. After shifting the appropriate entire column sets by L, which does not change their determinants, the two products become

$$\Delta_{Ke_0c}\Delta_{Kd_0a} + \Delta_{Kd_0c}\Delta_{Kae_0} = \Delta_{Kac}\Delta_{Kd_0e_0}.$$

This is the rank-two Plücker relation in the quotient by K. After returning all column sets to increasing order, the first factor on the right is x_m and the common ordering sign is absorbed in the other factor.

For $m > L$, the numerator is

$$P_m = x_{m-L}x_{m+1}x_{m+L-1} + x_{m-L+1}x_{m-1}x_{m+L}.$$

Set

$$d_0 = m-1, \quad a = m, \quad b = r(m-1), \quad e_0 = r(m),$$
$$c = w_{k-1}, \quad f = w_k, \quad K = I_m \setminus \{a, b, c\}.$$

Their order is $d_0 < a < b < e_0 < c < f$. In the quotient by K, use

$$\begin{aligned}\Delta_{acf}\Delta_{be_0c}\Delta_{d_0ab} + \Delta_{bcf}\Delta_{d_0ac}\Delta_{abe_0} \\ = \Delta_{abc}(\Delta_{d_0ac}\Delta_{be_0f} - \Delta_{d_0af}\Delta_{be_0c}).\end{aligned} \tag{2.10}$$

Here is an elementary verification. On $\Delta_{abc} \neq 0$, choose a, b, c as the coordinate basis. After division by Δ_{abc}^3, both sides equal $f_2(e_0)_1(d_0)_3 - f_1(d_0)_2(e_0)_3$. Polynomiality extends the identity to all six vectors. Restoring the common columns K gives the same total sign in every term, since their complete column multiplicities agree. Equations (2.9) now identify the left side with P_m and the first right factor with x_m. This proves regularity of all ordinary quotients. In the later strips the quotient is generally quadratic in Plücker coordinates.

For the special direction write $a_r = z_{r,1}$, $b_r = z_{r,2}$, and $T = \mathrm{diag}(\lambda_r)$. Then

$$U = \det(b, Ta, \ldots, T^{k-1}a), \qquad V = \det(a, b, Ta, \ldots, T^{k-2}a).$$

The coefficients of $b_r \prod_{s \neq r} a_s$ in these two determinants have ratio $-\prod_{s \neq r} \lambda_s = -\lambda_r^{-1}$. Indeed, deleting row r shifts all Vandermonde powers by one and changes the cofactor sign once. Hence

$$U + \lambda_r^{-1} V = a_r G_r, \tag{2.11}$$

where G_r is independent of the whole rth row. If $s \neq r$, the polynomial G_s has a term containing b_r but no a_r, with nonzero coefficient proportional to $\lambda_s^{-1} - \lambda_r^{-1}$. Therefore

$$y_1 = c^{-1} \prod_r G_r \in R$$

and none of the components $a_r = 0$ divides y_1. This also covers $p = k$, when the exceptional polynomial is $U^k + V^k$. □

2.4. **Complete charts and the upper ring.**

Proposition 2.6. *The initial chart and every initial adjacent chart are complete:*

$$R[(x_1 \cdots x_N)^{-1}] = \mathbb{C}[x_{N+1}, \ldots, x_d][x_1^{\pm 1}, \ldots, x_N^{\pm 1}], \tag{2.12}$$

$$R[(y_i \prod_{j \neq i, j \leq N} x_j)^{-1}] = \mathbb{C}[x_{N+1}, \ldots, x_d][y_i^{\pm 1}, x_j^{\pm 1} : j \neq i, j \leq N]. \tag{2.13}$$

Proof. Contract $\bigwedge_{a,s} dz_{a,s}$ by the $k-1$ vertical vector fields $E_a - E_{k-1}$, where E_a is row scaling. The resulting horizontal form is H-invariant, because its character is $(\prod t_a)^L = 1$. It descends to a nowhere-zero top form ω on the punctured cone. Scaling one row effects cone scaling and gives ω degree L. Thus $d - L = N$.

At a generic component of an ordinary $x_m = 0$, the derivative with respect to x_{m+L} is respectively x_{m-1} or $x_{m-L+1}x_{m-1}$, a unit. At a special component, exactly one factor in (2.6) vanishes, by (2.11); its derivative with respect to U is a unit. Lemma 2.2 applies, and gives both (2.1) and own-neighbor coprimality.

The morphism from the left-hand affine scheme in (2.12) to the abstract polynomial/Laurent chart on the right is birational by (2.7) and étale by the Jacobian identity. It is therefore an open immersion. Compute its units in the Cox localization: they are products of $x_2, \ldots, x_N$ and the factors a_r of x_1, times nonzero constants. Invariance under H forces all a_r exponents to agree. The source unit group is consequently $\mathbb{C}^\times x_1^{\mathbb{Z}} \cdots x_N^{\mathbb{Z}}$, exactly the target unit group. Lemma 2.1 proves (2.12).

Replacing x_i by y_i changes the top differential to

$$\pm c\, y_i \prod_{j \neq i, j \leq N} x_j\, \omega.$$

The neighboring map is again a birational étale open immersion. Its image contains $D(P_i)$, where the relation recovers $x_i = P_i/y_i$ and the initial chart applies. For an ordinary exchange, the exponent difference is primitive: its x_{i+L} coordinate has difference one. After inverting all coordinates, a unimodular monomial change makes P_i a unit times $1 + z$. Its endpoint monomials have disjoint support, so no coordinate factor was removed; P_i is prime in the actual polynomial-frozen target. The generic divisor $x_i = 0$ meets this prime, since all other mutables and y_i are units there. For the special exchange the k prime factors are $U + \lambda_r^{-1} V$, and the component $a_r = 0$ meets precisely its corresponding factor. Every possible missing divisor is thus met. Lemma 2.1 proves (2.13). □

Theorem 2.7. *For the full parameter range $L = \gcd(n, \ell) \geq 2$ and $p = n/L \geq k \geq 2$, with polynomial ordinary frozen variables, one has*

$$\mathcal{A}_n(k, \ell) \subseteq \mathbb{C}[D_n(k, \ell)] = \mathcal{U}_n(k, \ell).$$

Proof. First check the hypotheses for invariance of the finite upper bound. In the divided extended matrix (2.5), rows $L+1, \ldots, L+N$ give a triangular minor with diagonal -1: column m has lowest possible selected row $m + L$, with coefficient -1. Thus the actual extended matrix has full column rank. Matrix mutation preserves this rank, and divisibility of the special column by k is preserved as well.

In any seed, temporarily invert all extended coordinates. Every exchange polynomial has a nondegenerate Newton segment in its extended column direction. The Newton polytope of a product is the Minkowski sum of those of its factors. A common nonunit factor of two

exchange polynomials would therefore have a positive-dimensional Newton polytope parallel to both column directions, impossible since the columns are independent. A factor with point polytope is a Laurent unit. The endpoint coefficients are one and the endpoint monomials have disjoint support, so passing to the Laurent ring did not remove a common coordinate factor. Hence the exchange polynomials are pairwise coprime in the polynomial-frozen ring at every seed.

For $p > k$, the q-binomial expansion of (2.6) gives its intermediate coefficients as

$$\prod_{j=1}^{s} \frac{\sin((k+1-j)\pi/p)}{\sin(j\pi/p)} > 0 \qquad (1 \le s < k).$$

Apply the rank-two proof of [13, Theorem 4.1 and Lemma 4.2] directly to these specialized exchange polynomials. The endpoint monomials remain nonzero, and the two-step mutation identities, polynomial in the coefficient entries, survive their evaluation. Pairwise coprimality holds at every specialized seed by the preceding paragraph. The local proof temporarily introduces monomial denominators in frozen and unchanged mutable coordinates. Each such monomial is coprime to the new exchange polynomial, so the divisibility obtained in the localized ring descends to the polynomial-frozen ring by unique factorization. This proves upper-bound invariance in the specified scalar ring. If $p = k$, the exceptional exchange has only its two endpoint terms and the pattern is its ordinary right companion. The ordinary upper-bound argument gives the same conclusion. Thus the upper algebra is the intersection of the initial chart and its N adjacent chart rings.

Proposition 2.6 puts R in that intersection. Conversely, let f belong to the intersection and let $\mathfrak{p}$ be a height-one prime of R. If no initial mutable lies in $\mathfrak{p}$, the initial chart makes f regular there. Otherwise exactly one does, by Lemma 2.4, and its own neighbor is a unit by Lemma 2.2. The corresponding adjacent chart again makes f regular. Normality gives $f \in R$. Finally the generalized Laurent phenomenon [4, 13] gives $\mathcal{A}_n(k,\ell) \subseteq \mathcal{U}_n(k,\ell) = R$. □

3. Periodic bands and integral coordinates

We now replace the fixed characteristic coefficients by independent variables. The resulting integral model will also supply the exact charts needed for the representation-theoretic argument.

3.1. **The band, its core, and vector monodromy.** Fix $k, e \ge 2$ and put $N = (k-1)(e-1)$. All second indices below are periodic modulo e. Work in

$$R_{\mathbb{Z}} = \mathbb{Z}[D_h^{\pm 1}, a_{j,h} : 1 \le j < k, h \in \mathbb{Z}/e\mathbb{Z}]. \tag{3.1}$$

Set $a_{0,h} = 1$, $a_{k,h} = D_h$, and $a_{j,h} = 0$ outside $0 \le j \le k$. The infinite band has entry $a_{c-r,r}$ in row r, column c. Its finite core uses rows and columns

$$R_j = j + \left\lceil \frac{j}{e-1} \right\rceil, \qquad C_j = k + j \qquad (1 \le j \le N). \tag{3.2}$$

Let x_m be the trailing principal minor on positions $m, \dots, N$, and put $x_m = 1$ for $m > N$. The rows in (3.2) avoid residue 1. Hence every x_m is independent of all phase-1 entries, including D_1.

Take $w_1, \dots, w_k$ to be the standard frame and extend it by

$$w_c = \sum_{i=1}^{k} (-1)^{i-1} a_{i,c-i} w_{c-i}. \tag{3.3}$$

Because the D_h are units this defines vectors in both directions. For $f_r = \det(w_r, \dots, w_{r+k-1})$, expansion using the last recurrence coefficient gives

$$f_1 = 1, \qquad f_{r+1} = D_r f_r. \tag{3.4}$$

Thus every consecutive frame is invertible. Periodicity defines a unique invertible monodromy M by $w_{r+e} = Mw_r$. Define its characteristic coefficients by $\det(tI - M) = \sum_{s=0}^{k}(-1)^s e_s(M)t^{k-s}$. Write

$$\Delta = \prod_h D_h = \det M, \qquad F_s = e_s(M) \quad (1 \le s < k), \qquad F_0 = 1, \quad F_k = \Delta. \tag{3.5}$$

Lemma 3.1. *Using (2.4) with $L = e$, put $\Pi_m = \det(w_i : i \in I_m)$. Then*

$$x_m = \Pi_m / f_m, \qquad a_{i,r} = \frac{\det(w_j : j \in [r+i-k, r+i] \setminus \{r\})}{f_{r+i-k}}. \tag{3.6}$$

Proof. On the finite row interval $m, \dots, k+N$, form the relation matrix with columns $c = k+m, \dots, k+N$ and entries $(-1)^{c-r}a_{c-r,r}$. These columns are relations among the w_r by (3.3). The square block in rows equal to the column indices is unit upper triangular; the complementary k rows are the frame $m, \dots, m+k-1$. Complementary minors of the relation matrix and vector matrix therefore differ by the factor f_m. The complement of rows $R_m, \dots, R_N$ is exactly I_m. The sign in the band minor is $(-1)^{\sum_{j=m}^{N} C_j - \sum_{j=m}^{N} R_j}$, and the complementary-minor sign is the same, so they cancel. This proves the first identity. For the second identity, solve for the omitted coefficient of w_r in its recurrence frame. The recurrence sign $(-1)^{i-1}$ is canceled by the $i-1$ swaps moving that vector to the omitted position. □

3.2. **Universal initial exchanges.** The rank-two and rank-three identities used in Lemma 2.5, applied to (3.6), give

$$x_m y_m = D_m x_{m+1} x_{m+e-1} + x_{m-1} x_{m+e} \qquad (2 \le m \le \min(e, N)), \tag{3.7}$$

$$x_m y_m = x_{m-e} x_{m+1} x_{m+e-1} + x_{m-e+1} x_{m-1} x_{m+e} \qquad (e < m \le N). \tag{3.8}$$

They are identities in $R_{\mathbb{Z}}$, with $y_m \in R_{\mathbb{Z}}$. To verify the coefficient in the first strip, the frame ratio is

$$\frac{f_{m+1} f_{m+e-1}}{f_{m-1} f_{m+e}} = \frac{D_{m-1} D_m}{D_{m+e-1}} = D_m;$$

the second-strip ratio is one. A shift of an entire column set by e contributes Δ, with the same total shift in both full terms. Thus the determinant identities introduce only the units already present in (3.1).

For the exceptional exchange put

$$Q = x_1, \quad U = x_2, \quad B = x_{e+1}, \quad D_* = \Delta / D_1, \quad V = BD_*. \tag{3.9}$$

The following explicit identity constructs its quotient before any cluster-algebra argument is applied.

Lemma 3.2. *There is an element $\phi \in R_{\mathbb{Z}}$ such that*

$$Q\phi = \sum_{s=0}^{k} F_s U^s V^{k-s}. \tag{3.10}$$

More precisely, set $u = w_1$, $v = M^{-1}w_2$ and

$$K = [u, Mu, \dots, M^{k-1}u], \quad K_1 = [v, u, Mu, \dots, M^{k-2}u], \quad K_2 = [Mv, u, Mu, \dots, M^{k-2}u].$$

If w is the last row of $\operatorname{adj} K_1$ *and K^* has rows $w, wM, \dots, wM^{k-1}$, then*

$$\phi = (-1)^{k(k-1)/2} D_*^k \det K^*.$$

Proof. The complementary-minor identities give

$$\det K = Q, \qquad \det K_1 = D_*^{-1} U, \qquad \det K_2 = -D_*^{-1} V.$$

We spell out the polynomial identity behind the calculation [14, Proposition 3.6]. For a diagonal matrix with distinct eigenvalues λ_j, write $d_1 = \det K_1$, $d_2 = \det K_2$. Cofactor expansion gives

$$\lambda_j d_1 - d_2 = u_j w_j \prod_{i \neq j} (\lambda_j - \lambda_i).$$

Multiplying over j, the product of the difference factors is $(-1)^{k(k-1)/2}$ times the squared Vandermonde. The other factors are the row and column factors of $\det K \det K^*$, giving

$$\prod_j (\lambda_j d_1 - d_2) = (-1)^{k(k-1)/2} \det K \det K^*.$$

Under $(M, u, v) \mapsto (gMg^{-1}, gu, gv)$, both d_i and $\det K$ acquire the factor $\det g$, while $w \mapsto (\det g) w g^{-1}$ and $\det K^* \mapsto (\det g)^{k-1} \det K^*$. Both sides of the identity therefore acquire the same factor $(\det g)^k$. Diagonalization proves it for every simple-spectrum matrix. This locus is dense, and both sides are polynomials in the entries of M, u, v, so the identity holds for every matrix, including repeated spectra. Substituting the displayed d_1, d_2 gives (3.10). The adjugate formula expresses ϕ as a polynomial in the band entries and inverse determinants. Since the identity has integer coefficients after these units are cleared, it holds over $\mathbb{Z}$. □

3.3. **The missing phase.** Let $R_{0,\mathbb{Z}}$ be the ring obtained from $R_{\mathbb{Z}}$ by omitting $a_{1,1}, \dots, a_{k-1,1}$, but retaining every D_h.

Lemma 3.3. *The F_s are affine-linear simultaneously in the omitted variables and*

$$\det \frac{\partial(F_1, \dots, F_{k-1})}{\partial(a_{1,1}, \dots, a_{k-1,1})} = \pm Q. \tag{3.11}$$

In particular

$$R_{\mathbb{Z}}[Q^{-1}] = R_{0,\mathbb{Z}}[Q^{-1}][F_1, \dots, F_{k-1}]. \tag{3.12}$$

Proof. Let C_r be the companion matrix with subdiagonal entries one and top row

$$D_r^{-1}(a_{k-1,r}, -a_{k-2,r}, \dots, (-1)^{k-2} a_{1,r}, (-1)^{k-1}).$$

Its determinant is D_r^{-1}. Put $T_0 = C_e \cdots C_1$, $H_r = C_{r-1} \cdots C_1$, $b_{r,j} = H_r^{-1} \mathbf{e}_j$, and $u_r = b_{r,k}$. The columns of the companion inverse give

$$\begin{aligned} b_{r+1,j} &= b_{r,j-1} + (-1)^{j-k} a_{k-j+1,r} u_r \qquad && (2 \le j \le k), \\ b_{r+1,1} &= (-1)^{k-1} D_r u_r. \end{aligned}$$

Eliminating these columns gives exactly (3.3) for u_r. Their first frame is anti-triangular, with anti-diagonal entries one, and determinant $\epsilon_k = (-1)^{k(k-1)/2}$. Periodicity gives $H_{r+e} = H_r T_0$ and $u_{r+e} = T_0^{-1} u_r$. Thus M is conjugate to T_0^{-1} and

$$M \sim T_0^{-1}, \qquad F_s = \Delta e_{k-s}(T_0) \qquad (1 \le s < k). \tag{3.13}$$

Write $T_0 = Q_0 C_1$ with $Q_0 = C_e \cdots C_2$ and let $u = Q_0 \mathbf{e}_1$. Varying the missing entries changes T_0 by a matrix uv^t. Its characteristic polynomial is affine-linear in v, by the adjugate formula for a rank-one perturbation. The last fixed top-row entry of C_1 is $c = (-1)^{k-1} D_1^{-1}$, so $T_0 \mathbf{e}_k = cu$. The derivatives with respect to the first $k-1$ top-row entries are the first $k-1$ coordinates of the coefficient vectors of $\operatorname{adj}(zI - T_0)u$. These vectors are triangular linear combinations, with diagonal signs, of $u, T_0 u, \dots, T_0^{k-2} u$. Appending $\mathbf{e}_k$ gives determinant

$$\pm c^{-(k-1)} \det[\mathbf{e}_k, T_0 \mathbf{e}_k, \dots, T_0^{k-1} \mathbf{e}_k].$$

Changing the top-row entries back to the $a_{j,1}$ contributes $D_1^{-(k-1)}$ and a sign, canceling the powers of D_1.

Let K_- have columns $\mathbf{e}_k, T_0^{-1} \mathbf{e}_k, \dots, T_0^{-(k-1)} \mathbf{e}_k$. By (3.6) in the u-frame, $\det K_- = \epsilon_k Q$. Multiplying all columns by T_0^{k-1} and reversing their order gives

$$\det[\mathbf{e}_k, T_0 \mathbf{e}_k, \dots, T_0^{k-1} \mathbf{e}_k] = \epsilon_k \Delta^{-(k-1)} \det K_- = \Delta^{-(k-1)} Q.$$

The $k-1$ factors Δ in (3.13) cancel these powers. This proves (3.11) with coefficient ± 1. Since the map is affine-linear, Cramer's rule proves (3.12) over $\mathbb{Z}$. □

3.4. **The initial complex chart.** Set

$$A_{\mathbb{Z}} = \mathbb{Z}[D_h^{\pm 1}, F_1, \dots, F_{k-1}].$$

For a field K write $R_K = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ and $A_K = A_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. The next argument proves, in particular, that these displayed coefficient functions are algebraically independent.

Lemma 3.4. *One has*

$$R_{\mathbb{C}}[(x_1 \cdots x_N)^{-1}] = A_{\mathbb{C}}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]. \tag{3.14}$$

The same equality holds over $\mathbb{Q}$.

Proof. Choose the distinct scalar spectrum of (2.2) with $p = k+1$, now with period e. Multiplying all eigenvalues by a nonzero scalar ζ multiplies every determinant with index set I by $\zeta^{\sum_{i\in I}\lfloor (i-1)/e \rfloor}$. The ordinary determinant identities have the same total exponent in both full terms. The exceptional identity rescales by the corresponding determinant units. Thus the complete scalar initial and adjacent charts remain valid for this scaled spectrum.

Fix any nonzero determinant tuple (d_h) and choose $\zeta^k = \prod_h d_h$. In the scaled scalar cone impose the e consecutive frozen values

$$f_m = \prod_{r=1}^{m-1} d_r \qquad (N+1 \le m \le N+e). \tag{3.15}$$

Since $f_{r+e} = (\prod d_h) f_r$, these are equivalent to $f_1 = 1$ and the ratios $f_{r+1}/f_r = d_r$.

This scalar slice is in bijection with the entire band fiber having the chosen simple monodromy spectrum and determinant tuple, including its points with $Q = 0$. Indeed, from a band point reconstruct the vectors by (3.3) and diagonalize its monodromy in a determinant-one eigenbasis. Such a basis exists by rescaling one eigenvector. Its ambiguity is exactly H. None of its spectral rows is zero, since a consecutive frame is invertible. Conversely, the slice has every consecutive frame invertible, and (3.6) recovers its band coefficients. Periodicity of the vectors makes those coefficients periodic. These two constructions are inverse.

Quotienting the complete scalar initial chart by (3.15) gives an arbitrary mutable torus. The factors f_m in (3.6) are nonzero fixed numbers. For the fixed tuple (d_h), Lemma 3.3 identifies every prescribed-characteristic fiber on $Q \neq 0$ with the same omitted-phase open, independently of the prescribed characteristic values. It follows that

$$\operatorname{Spec} R_{0,\mathbb{C}}[(x_1 \cdots x_N)^{-1}] \longrightarrow \operatorname{Spec} \mathbb{C}[D_h^{\pm 1}, x_i^{\pm 1}]$$

is bijective on complex closed points. This finite-type map is quasi-finite: a positive-dimensional fiber would specialize to one over a closed point. It is dominant, and its generic degree is one, since in characteristic zero a larger separable degree gives several closed preimages over a nonempty open subset. The target is a normal torus, so Lemma 2.1 first makes the map an open immersion and surjectivity makes it an isomorphism. Combining with (3.12) proves (3.14). The root ζ was chosen separately on each fiber. Faithful flatness of $\mathbb{C}/\mathbb{Q}$ descends the equality to $\mathbb{Q}$. □

3.5. **Prime divisors and the integral Jacobian.**

Proposition 3.5. *The initial and initial adjacent charts satisfy*

$$R_{\mathbb{Z}}[(x_1 \cdots x_N)^{-1}] = A_{\mathbb{Z}}[x_1^{\pm 1}, \dots, x_N^{\pm 1}], \tag{3.16}$$

$$R_{\mathbb{Z}}[(y_i \prod_{j \neq i} x_j)^{-1}] = A_{\mathbb{Z}}[y_i^{\pm 1}, x_j^{\pm 1} : j \neq i], \tag{3.17}$$

where $y_1 = \phi$. *The* x_i *are distinct prime elements of* $R_{\mathbb{Z}}$, *remain prime over* $\mathbb{C}$, *and are coprime to their own neighbors.*

Proof. Assume first $N > 1$. Let m be the number of distinct irreducible factors of $x_1 \cdots x_N$ in the UFD $R_{\mathbb{C}}$. The unit ranks on the two sides of (3.14), modulo $\mathbb{C}^\times$, are $e+m$ and $e+N$. Hence $m = N$.

For each i, a complete neighboring scalar chart with the scaled simple spectrum supplies a point

$$x_i = 0, \qquad x_j \neq 0 (j \neq i), \qquad y_i \neq 0. \tag{3.18}$$

For an ordinary direction the principal column is nonzero: in the first strip x_{i-1} occurs with nonzero difference, and in the later strip x_{i-e} does. After fixing the frozen values the exchange binomial therefore has a zero in the remaining complex torus. Set $y_i = 1$. For the special direction U/V is a nonconstant mutable Laurent monomial when $N > 1$, and its polynomial has nonzero endpoints, so it has a nonzero complex root. The whole fiber correspondence in Lemma 3.4 transfers these points to the band, even for $Q = 0$.

A prime factor of x_i vanishing at (3.18) divides neither any other x_j nor y_i. These N private factors are distinct. Since there are only N factors altogether, each x_i is a unit times a power of its own factor. Equality of the exact unit groups in (3.14) forces this power to be one: the private factor becomes a unit in the localization, and its valuation one must be an integer combination of the valuations of the x_j. This proves the complex prime assertions.

Order the omitted-phase middle variables by the core diagonal:

$$t_u = a_{k-a,r+1} \quad \text{if } u = (a-1)(e-1) + r, \quad 1 \leq r \leq e-1. \tag{3.19}$$

A middle entry at core position (u, v) has index $t_{eu-(e-1)v}$. Give t_j weight $C - j^2$ for $C > N^3$, and determinant units weight zero. A determinant term with fewer middle entries loses a whole C, more than any possible gain in the square terms. For a permutation σ of a trailing block,

$$\sum_u (eu - (e-1)\sigma(u))^2 = (e^2 + (e-1)^2) \sum_u u^2 - 2e(e-1) \sum_u u\sigma(u).$$

Strict rearrangement makes the identity permutation the unique minimum. Thus

$$\operatorname{in}(x_m) = t_m \cdots t_N \quad \text{with coefficient 1.} \tag{3.20}$$

In particular all x_m are primitive over $\mathbb{Z}$, so Gauss's lemma gives the integral prime assertions.

Let $J_0 = \det(\partial x_i / \partial t_j)$, with determinant variables held fixed. Differentiating an ordinary exchange at the generic prime $x_i = 0$, for $i \geq 2$, gives a nontrivial relation among the dx_j: its nonzero principal column supplies a unit derivative. Thus $x_i \mid J_0$. The ordinary degree bound is

$$\deg J_0 \leq \sum_{m=1}^{N} (N - m) = \frac{N(N-1)}{2} = \deg \prod_{i=2}^{N} x_i.$$

Consequently $J_0 = c(D) \prod_{i=2}^{N} x_i$. Differentiating a term subtracts the weight of the column variable; every term in a full Jacobian determinant subtracts the same total. Hence its leading part is the Jacobian of the leading monomials in (3.20). This is upper triangular with coefficient-one diagonal, giving $c(D) = \pm 1$. In the next determinant, $a = (a_{j,h})_{1 \leq j < k,\, 0 \leq h < e}$ is ordered lexicographically and $D = (D_0, \ldots, D_{e-1})$. Together with (3.11), the block-triangular full Jacobian is

$$\det \frac{\partial(x_1, \ldots, x_N, F_1, \ldots, F_{k-1}, D)}{\partial(a, D)} = \pm \prod_{i=1}^{N} x_i. \tag{3.21}$$

The integral initial chart map is therefore étale over $\mathbb{Z}$ and birational by Lemma 3.4. Its source and target unit groups are both $\{\pm D^\alpha x^\beta\}$. Lemma 2.1 proves (3.16), including all vertical divisors. The neighboring full Jacobian is $\pm y_i \prod_{j \neq i} x_j$, so that map is likewise a birational étale open immersion whose image contains $D(P_i)$. In the special target P_1 is prime because it is linear in the independent polynomial variable F_1, with Laurent-unit coefficient UV^{k-1}. A first-strip ordinary binomial has determinant exponent difference one at D_i; a later-strip binomial has mutable exponent difference one at x_{i-e}. Both are prime by a

unimodular Laurent change. The old private divisor meets this exchange divisor. Lemma 2.1 proves (3.17).

If $N = 1$, then $k = e = 2$. With $a = a_{1,2}$ and $b = a_{1,1}$ one has

$$M = \begin{pmatrix} -D_1 & -bD_1 \\ a & ab - D_2 \end{pmatrix}, \quad F_1 = ab - D_1 - D_2, \quad Q = a, \quad \phi = D_2 b, \tag{3.22}$$

and $a(D_2 b) = D_2^2 + D_2 F_1 + D_1 D_2$. The initial and neighboring chart equalities follow by solving for b and a, respectively. The two elements a, b are distinct primes and the Jacobian constant is one, proving all the assertions in this case also. □

4. All mutation charts and determinant-one specialization

4.1. **A normalized auxiliary coefficient string.** Adjoin a formal unit T with $T^k = \Delta$ and put $G_s = T^{-s} F_s$. Eliminating D_1 gives polynomial/Laurent rings

$$R' = \mathbb{Z}[T^{\pm 1}, D_{h\neq 1}^{\pm 1}, a_{j,h}], \qquad A' = \mathbb{Z}[T^{\pm 1}, D_{h\neq 1}^{\pm 1}, G_1, \dots, G_{k-1}]. \tag{4.1}$$

The special exchange becomes

$$Q\phi = \sum_{s=0}^{k} G_s (TU)^s V^{k-s}, \qquad G_0 = G_k = 1. \tag{4.2}$$

The G_s are independent polynomial coefficients; their string is reversed when vertex 1 is mutated. The divided extended columns are

$$b_1 = \mathbf{e}_2 - \mathbf{e}_{e+1} - \sum_{h\neq 1} \mathbf{e}_{D_h} + \mathbf{e}_T, \tag{4.3}$$

$$b_m = \mathbf{e}_{m+1} + \mathbf{e}_{m+e-1} - \mathbf{e}_{m-1} - \mathbf{e}_{m+e} + \mathbf{e}_{D_m} \qquad (2 \le m \le \min(e, N)),$$

$$b_m = \mathbf{e}_{m-e} + \mathbf{e}_{m+1} + \mathbf{e}_{m+e-1} - \mathbf{e}_{m-e+1} - \mathbf{e}_{m-1} - \mathbf{e}_{m+e} \qquad (e < m \le N).$$

Out-of-range mutable rows are zero. The D_1 row is zero and may be omitted. As before, the actual first column is kb_1.

The T row supplies a pivot in column 1. The rows $D_2, \dots, D_{\min(e,N)}$ supply the next pivots. The remaining pivots occur in mutable row r, column $r + e$, with coefficient one for $1 \le r \le N - e$. These give a unit triangular minor. This description also covers $e = 2$: repeated arrows affect other entries but not the pivots. The extended matrix has full column rank at every seed.

At any seed, invert the mutable, determinant, and root coordinates, while retaining $G_1, \dots, G_{k-1}$ polynomial. The special exchange polynomial is prime, being linear in G_1 (or its reversed position) with a Laurent-unit coefficient. No ordinary exchange contains a G_s, so it cannot share this factor. Two ordinary exchanges have independent Newton-segment directions and are coprime by the argument used in Theorem 2.7. Endpoint coefficients one exclude coordinate factors. The finite upper bound is therefore invariant over $\mathbb{Q}$, by [13, Theorem 4.1].

Base change of Proposition 3.5 gives the initial star of charts over A'. The initial x_i remain prime: each is independent of D_1, and eliminating D_1 simply adjoins the independent Laurent variable T to its coefficient ring. Their own-neighbor coprimality also persists. The height-one argument therefore yields

$$\mathcal{U}'_{\mathbb{Q}} = R'_{\mathbb{Q}}. \tag{4.4}$$

The ground ring remains polynomial in the G_s.

4.2. **Arithmetic of arbitrary mutation words.** The integral Laurent phenomenon places every mutated variable in each ring of the initial star over A'. The intersection of that finite star is R': every initial mutable prime is covered by its neighboring chart and every other height-one prime by the initial chart. This includes integer primes, since the minors have the coefficient-one leading terms (3.20). Consequently all mutated variables are regular in R'.

Reduce the initial chart modulo any integer prime q. It is still the Laurent polynomial ring

$$A'_q[x_1^{\pm1},\dots,x_N^{\pm1}], \qquad A'_q = \mathbb{F}_q[T^{\pm1}, D_{h\neq1}^{\pm1}, G_1,\dots,G_{k-1}],$$

inside the domain R'_q. Inductively, every mutation is a nonzero birational substitution over $\operatorname{Frac} A'_q$. An ordinary exchange has two distinct extended Laurent monomials, since its column is nonzero. Exponent differences divisible by q still define different monomials in characteristic q. A special exchange is nonzero because of its independent G_1 term. The numerator is independent of the replaced variable, so the same exchange equation gives the inverse substitution. It follows that every seed is algebraically independent over $\operatorname{Frac} A'_q$, for every prime q.

Lemma 4.1. *Let $C_\mathbb{Z} = \mathbb{Z}[u_1^{\pm1},\dots,u_r^{\pm1},c_1,\dots,c_s]$ embed in a torsion-free domain $S_\mathbb{Z}$. For every integer prime q put $C_q = C_\mathbb{Z}/qC_\mathbb{Z}$ and $S_q = S_\mathbb{Z}/qS_\mathbb{Z}$. Write $C_\mathbb{Q} = C_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{Q}$. Suppose that S_q is a domain, the induced reduction map $C_q \to S_q$ is injective, and $y_1,\dots,y_N \in S_\mathbb{Z}$ remain algebraically independent over $\operatorname{Frac} C_q$ modulo every such q. Then*

$$S_\mathbb{Z} \cap C_\mathbb{Q}[y_1^{\pm1},\dots,y_N^{\pm1}] \subseteq C_\mathbb{Z}[y_1^{\pm1},\dots,y_N^{\pm1}].$$

Proof. Let a belong to the intersection on the left. Collect its finite Laurent expansion in y and expand its coefficients in the monomial basis of $C_\mathbb{Z}$. Let m be their least positive common integer denominator. If $q \mid m$, at least one coefficient of the numerator is nonzero modulo q. Clearing negative mutable powers gives $ma\,y^\beta = P(y)$, with $P \in C_\mathbb{Z}[Y_1,\dots,Y_N]$ and $\overline{P} \neq 0$ modulo q. The left side becomes zero, whereas the right side cannot vanish by algebraic independence. Thus $m = 1$. □

For a finite mutation word t, denote its mutable variables by $y_{1,t},\dots,y_{N,t}$. Equation (4.4) puts every element of R' in the corresponding rational chart. Lemma 4.1 removes its integer denominators. Since all the $y_{i,t}$ are regular, the reverse inclusion also holds, giving

$$R'[(\prod_i y_{i,t})^{-1}] = A'[y_{1,t}^{\pm1},\dots,y_{N,t}^{\pm1}] \qquad \text{for every finite } t. \tag{4.5}$$

Throughout these chart identities the variables G_s remain polynomial coefficients.

4.3. **Descent without dividing by the rank.** The *retained-determinant pattern* consists of the variables $y_{i,t}$ obtained from the normalized root seed with $G_s = T^{-s}F_s$ and $T^k = \Delta$, keeping every D_h invertible. Inside the common fraction field of R' put

$$\mathcal{U}_{\mathrm{GL},\mathbb{Z}} = \bigcap_t A_\mathbb{Z}[y_{1,t}^{\pm1},\dots,y_{N,t}^{\pm1}].$$

Theorem 4.2. *Every retained-determinant mutation word is regular in $R_\mathbb{Z}$. For every finite word t,*

$$R_\mathbb{Z}[(\prod_i y_{i,t})^{-1}] = A_\mathbb{Z}[y_{1,t}^{\pm1},\dots,y_{N,t}^{\pm1}], \qquad \mathcal{U}_{\mathrm{GL},\mathbb{Z}} = R_\mathbb{Z}. \tag{4.6}$$

Proof. Give T degree one in $\mathbb{Z}/k\mathbb{Z}$, every original retained coordinate degree zero, and G_s degree $-s$. Put $d_1 = k$ and $d_j = 1$ for $j > 1$, so the actual matrix has entries $B_{ij} = d_j b_{ij}$. Away from the incident row and column, mutation at h gives

$$b'_{ij} = b_{ij} + d_h([b_{ih}]_+[b_{hj}]_+ - [-b_{ih}]_+[-b_{hj}]_+), \tag{4.7}$$

and the incident entries change sign.

Let $\epsilon = 1$ for the original special string and $\epsilon = -1$ for its reversal. Initially, and after every mutation,

$$b_{T,1} \equiv \epsilon \pmod k, \qquad b_{T,j} \equiv 0 \pmod k \quad (j > 1). \tag{4.8}$$

Indeed, at an ordinary mutation its root-row entry is a multiple of k, so each correction in (4.7) is a multiple of k. At a special mutation the special root-row entry and the string both reverse, while corrections to ordinary columns carry the factor $d_1 = k$.

Every ordinary exchange branch has degree zero. At a special exchange write $r = b_{T,1}$. The root contribution in branch s is

$$s[r]_+ + (k-s)[-r]_+ \equiv sr \pmod{k}.$$

Its coefficient is G_s or G_{k-s}, of degree $-\epsilon s$. The full branch has degree $s(r-\epsilon) = 0$ by (4.8). Every mutable variable therefore has degree zero: it is regular in R', and multiplication by its nonzero degree-zero predecessor equals a degree-zero numerator, forcing all its other homogeneous components to vanish.

The monic relation $T^k - \Delta$ gives actual free decompositions

$$R' = \bigoplus_{j=0}^{k-1} T^j R_{\mathbb{Z}}, \qquad A' = \bigoplus_{j=0}^{k-1} T^j A_{\mathbb{Z}}. \tag{4.9}$$

Their degree-zero pieces are the original rings. Thus the mutated variables lie in $R_{\mathbb{Z}}$. Localization at degree-zero elements commutes with (4.9); taking degree zero in (4.5) proves the chart identity in (4.6). This is a direct grading argument, valid even in residue characteristics dividing k.

The chart identities put $R_{\mathbb{Z}}$ in every Laurent chart. Conversely the upper ring is contained in the initial finite star, whose height-one intersection is $R_{\mathbb{Z}}$ by Proposition 3.5. This proves the upper equality. □

4.4. **The determinant-one ring.** Put

$$B_{\mathbb{Z}} = \mathbb{Z}[a_{j,h} : 1 \le j < k, h \in \mathbb{Z}/e\mathbb{Z}], \qquad C_{\mathbb{Z}} = \mathbb{Z}[F_1, \dots, F_{k-1}],$$

where the F_s on the right are now evaluated at $D_h = 1$. The determinant-one seed has the core minors with $D_h = 1$ as its initial mutables. Its divided matrix is obtained from (4.3) by deleting the T and determinant rows; its first actual column is still multiplied by k. Its exceptional coefficient string is $(1, F_1, \dots, F_{k-1}, 1)$, reversed at every special mutation, with no relations among the interior coefficients. Its upper algebra, inside $\operatorname{Frac} B_{\mathbb{Z}}$, is

$$\mathcal{U}_{\mathrm{SL},\mathbb{Z}} = \mathcal{U}^{\mathrm{ind}}_{k,e} = \bigcap_t C_{\mathbb{Z}}[\overline{y}_{1,t}^{\pm 1}, \dots, \overline{y}_{N,t}^{\pm 1}],$$

where $\overline{y}_{i,t}$ denotes the variable at position i after the finite mutation word t.

Theorem 4.3. *For every finite mutation word t, its determinant-one variables $\overline{y}_{i,t}$ satisfy*

$$B_{\mathbb{Z}}[(\prod_i \overline{y}_{i,t})^{-1}] = C_{\mathbb{Z}}[\overline{y}_{1,t}^{\pm 1}, \dots, \overline{y}_{N,t}^{\pm 1}]. \tag{4.10}$$

All these functions are nonzero, all F_s remain independent polynomial coefficients, and

$$\mathcal{U}_{\mathrm{SL},\mathbb{Z}} = B_{\mathbb{Z}}. \tag{4.11}$$

Proof. For each individual word, quotient (4.5) by $(T-1, D_h - 1 : h \ne 1)$. Then also $D_1 = 1$, $G_s = F_s$, and the ambient quotient is $B_{\mathbb{Z}}$. Localization commutes with quotient, so this gives (4.10). Its right side is a nonzero Laurent polynomial ring. Thus none of the specialized variables is zero, and the coefficients are still independent. Specializing the recursive exchange equations identifies these functions with the determinant-one mutation pattern. In particular $B_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathrm{SL},\mathbb{Z}}$.

For the reverse inclusion use only the initial star. Suppose $N > 1$. In the polynomial UFD $B_{\mathbb{C}}$ the initial chart has unit rank N, so the product of its mutables has exactly N distinct prime factors. In each neighboring chart set the new coordinate equal to one. Every ordinary principal column is nonzero, so its binomial has a zero in the other mutable torus. The exceptional ratio U/V is nonconstant, and the special polynomial has nonzero endpoint coefficients, so it too has a nonzero root over $\mathbb{C}$. These give the private points (3.18) after determinant specialization. As in Proposition 3.5, counting private factors and comparing exact unit lattices show that the initial mutables are distinct absolute primes, coprime to

their own neighbors. The coefficient-one leading terms (3.20) persist at $D_h = 1$, proving primitivity and primality over $\mathbb{Z}$.

Every height-one prime of $B_{\mathbb{Z}}$ is therefore covered by the initial chart or the appropriate neighboring chart. Normality puts the finite intersection in $B_{\mathbb{Z}}$, proving (4.11). When $N = 1$, formula (3.22) becomes $F_1 = ab - 2$. The two chart rings are $\mathbb{Z}[F_1][a^{\pm 1}] = \mathbb{Z}[a, b][a^{-1}]$ and $\mathbb{Z}[F_1][b^{\pm 1}] = \mathbb{Z}[a, b][b^{-1}]$; their intersection is $\mathbb{Z}[a, b]$ because a, b are distinct primes. This also proves the assertion in rank one. □

5. The full cyclic Hall algebra and its bounded dual

Throughout this section, $e, k \geq 2$. The cyclic quiver has vertices $I = \mathbb{Z}/e\mathbb{Z}$ and arrows $i \longrightarrow i + 1$. In particular, when $e = 2$ there are two arrows, one in each direction. Its Euler form and symmetrization are

$$E(\alpha, \beta) = \sum_i \alpha_i \beta_i - \sum_{i \to j} \alpha_i \beta_j, \qquad C(\alpha, \beta) = E(\alpha, \beta) + E(\beta, \alpha).$$

When a vertex is an argument of E or C, it denotes the corresponding unit dimension vector. We write $|\beta| = \sum_i \beta_i$. A segment $[h; j)$ has a basis in successive colors $h, h + 1, \ldots, h + j - 1$, with the consecutive arrows acting as identities. A multisegment is a finite sum of segments. Nilpotent orbits are indexed by multisegments, including the periodic ones.

Let H be the full generic Hall algebra, with lower intersection cohomology basis b_τ. We use perverse-normalized intersection cohomology complexes; the *raw IC* of an orbit closure is $\mathrm{IC}(\overline{O})[-\dim O]$. The first Hall factor is the submodule and the second is the quotient. We use the parameter v with the finite-field convention $v = q_f^{-1}$, where the field has q_f^2 elements. These are the conventions of [21, §§1.2–1.5]. The dual below is the graded *evaluation dual*:

$$R = H^\vee, \qquad B_\tau(b_\sigma) = \delta_{\tau\sigma}.$$

We may extend scalars to $\mathbb{Z}[v^{1/2}, v^{-1/2}]$ when normalizing classes.

For a dimension vector β, write E_β for the affine arrow space and $G_\beta = \prod_i \mathrm{GL}(V_i)$. Put

$$Z_{k,\beta} = \{x \in E_\beta : x^k = 0\}.$$

Its orbits are exactly the multisegments whose segment lengths do not exceed k. Their lower classes span a coalgebra H_k: a triangular matrix satisfying $x^k = 0$ has diagonal blocks satisfying the same equation. Its evaluation dual is the based algebra quotient

$$R_k = H_k^\vee = R/J_k, \qquad J_k = \bigoplus_{\tau \text{ containing a segment of length } > k} \mathbb{Z}[v^{\pm 1/2}] B_\tau. \tag{5.1}$$

5.1. **Strict flags and the full basis.** For $\mathbf{d} = (d_1, \ldots, d_k)$, with $\sum_a d_a = \beta$ and with zero layers allowed, consider decreasing graded flags

$$V = F_0 \supset F_1 \supset \cdots \supset F_k = 0, \quad \dim(F_{a-1}/F_a) = d_a, \quad xF_{a-1} \subset F_a.$$

Their incidence variety $X_{\mathbf{d}}$ is a vector bundle over a product of partial flag varieties. It is smooth and irreducible, and the projection $\pi_{\mathbf{d}} : X_{\mathbf{d}} \to E_\beta$ is proper. If b_d denotes the lower class of the zero representation of dimension d, the corresponding Hall product is

$$L_{\mathbf{d}} = b_{d_k} \cdots b_{d_1}, \qquad \dim X_{\mathbf{d}} = \sum_{a<b} \left(\sum_i d_{a,i} d_{b,i} + \sum_{i \to j} d_{a,i} d_{b,j} \right). \tag{5.2}$$

The displayed dimension is also the sum of the induction shifts.

Lemma 5.1. *For each bounded multisegment τ, the radical flag of its representation gives a proper birational map $X_{\mathbf{d}(\tau)} \to \overline{O_\tau}$. Consequently the products (5.2) span H_k integrally in every dimension vector. All these flag fibers, and all raw orbit IC stalks, have even Tate cohomology.*

Proof. Take $F_a = x^a V$ and $d_a = \dim(x^{a-1}V/x^a V)$. For any point of this incidence variety, $x^a V \subset F_a$. The conditions that these inclusions have equal dimensions at every vertex form a nonempty open set. There the ranks of every graded power of x are prescribed. Their successive differences determine all segment multiplicities, so the image of this open set is exactly O_τ. Its flag is forced to be the radical flag. Irreducibility of $X_{\mathbf{d}}$ proves that its whole image is $\overline{O_\tau}$, and generic uniqueness proves birationality. The decomposition theorem therefore gives $L_{\mathbf{d}(\tau)} = b_\tau +$ terms on smaller orbits, with diagonal coefficient one. There are finitely many orbits in a fixed dimension vector. Finite unitriangular inversion proves the integral spanning assertion.

Here is the flag-fiber argument for parity, in the form used in [7, Theorem 6.3 and Lemma 6.4]. On the graded socle W of a nilpotent module M, put $I_j = W \cap x^j M$. The parabolic preserving this filtration lifts to $\mathrm{Aut}(M)$. Indeed, choose a chain basis and identify the vectors at distance j from their socles with I_j, with their vertex grading shifted by $-j$. A filtration-preserving map acts on that level by its restriction to I_j. These actions commute with the inclusions $I_j \subset I_{j-1}$ representing the arrows, so they define the required lift. The chain levels remain distinct even when their colors agree.

The first space of an increasing strict flag lies in W. Choose a Borel in this parabolic. Over a Schubert cell of possible first spaces, its unipotent cell representatives, lifted as above, identify the remaining fiber with the strict flag variety of the quotient module. Thus each stratum is an affine space times a smaller strict flag variety. Induction on $\dim M$, using the closed filtration by Schubert cells, gives an affine paving. It is defined over the split ground field. Its cohomology is therefore even and Tate. The raw IC of $\overline{O_\tau}$ is a zero-shift summand of its radical resolution, so the same statement holds for its stalks. The stabilizer of a representation is the group of units of its finite-dimensional endomorphism algebra. This is an irreducible open subset of an affine space and is connected, so every simple equivariant local system on an orbit is constant. □

5.2. **The Miura map and its exact leading term.** Let $\mathcal{T}_{k,e}$ be the quantum torus with letters $X_{a,i}$, $1 \le a \le k$, $i \in I$, and commutation matrix

$$\Omega((a,i),(b,j)) = \begin{cases} E(i,j) - E(j,i), & a = b, \\ -C(i,j), & a < b, \\ C(i,j), & a > b. \end{cases} \tag{5.3}$$

Here $X_u X_w = v^{\Omega(u,w)} X_w X_u$. For an exponent vector n, write $\mathsf{M}(n)$ for its Weyl-normalized monomial: if the letters are ordered by row and then color,

$$\mathsf{M}(n) = v^{-\frac{1}{2}\sum_{u<w}\Omega(u,w)n_u n_w} \prod_u X_u^{n_u}.$$

When the argument is a monomial, M applies to its exponent vector. Its row degree is

$$\deg_{\mathrm{row}} \mathsf{M}(n) = \left(\sum_c n_{1c}, \dots, \sum_c n_{kc} \right) \in \mathbb{Z}^k,$$

ordered lexicographically, starting with the first row. This is an ordered abelian group, also for Laurent monomials. Set $\eta(\beta) = |\beta| - E(\beta,\beta)$ and $\widehat{B}_\tau = v^{\eta(\dim \tau)/2} B_\tau$. The half powers here extend scalars in the Grothendieck algebra. They normalize basis symbols; all categorical assertions below use actual integer-graded objects, up to their ordinary grading shifts.

Theorem 5.2. *There is an injective algebra map from R_k into the polynomial part of $\mathcal{T}_{k,e}$ such that*

$$\Phi_k(\widehat{B}_\tau) = \sum_{\mathbf{d}} a^\tau_{\mathbf{d}}(v)\mathsf{M}(\mathbf{d}), \qquad L_{\mathbf{d}} = \sum_\tau a^\tau_{\mathbf{d}}(v) b_\tau. \tag{5.4}$$

Every $a^{\tau}_{\mathbf{d}}$ lies in $\mathbb{N}[v,v^{-1}]$ and is invariant under $v\mapsto v^{-1}$. The unique maximal row-degree term is

$$\mathsf{M}(m(\tau)),\qquad m(\tau)=\prod_{[h;j)\in\tau}\prod_{a=1}^{j}X_{a,h+a-1},\tag{5.5}$$

and its coefficient is one. Distinct full multisegments have distinct words in (5.5).

Proof. The one-row quotient is the evaluation dual of the semisimple coalgebra. Its basis y_d satisfies $y_a y_b = v^{-E(b,a)}y_{a+b}$: restriction fixes the subspace and the zero arrow tuple is the only lift of two zero tuples. Green's compatibility with Hall multiplication uses the braided tensor product with crossing v^{-C}. Dualizing the product $b_{d_k}\cdots b_{d_1}$, in reversed tensor order, therefore gives an algebra map into k copies of this one-row quotient. Its letter commutations are exactly (5.3). By Lemma 5.1, its kernel is exactly J_k.

We verify its normalization on arbitrary row dimensions. For one row put

$$r(d)=-\sum_i\binom{d_i}{2}-\sum_{i<j}E(j,i)d_id_j,\qquad t(d)=\sum_{i<j}(E(i,j)-E(j,i))d_id_j.$$

Then $\prod_i X_i^{d_i}=v^{r(d)}y_d$ and $r(d)-t(d)/2=\eta(d)/2$. For $\beta=\sum_a d_a$ set $S=\sum_{a<b}C(d_a,d_b)$. The ordered row product is $v^{\sum_a r(d_a)-S}(y_{d_1}\otimes\cdots\otimes y_{d_k})$, and its total skew exponent is $\sum_a t(d_a)-S$. Hence

$$\mathsf{M}(\mathbf{d})=v^{\frac12(\sum_a\eta(d_a)-S)}(y_{d_1}\otimes\cdots\otimes y_{d_k})=v^{\eta(\beta)/2}(y_{d_1}\otimes\cdots\otimes y_{d_k}).$$

This proves (5.4) with exactly the stated scalar, including $e=2$, where $C(0,1)=-2$.

The proper pushforward from $X_{\mathbf{d}}$ is Verdier self-dual after its middle shift. Lemma 5.1 makes its raw IC multiplicities even and Tate. If $D=\dim X_{\mathbf{d}}$ and $d_\tau=\dim O_\tau$, its coefficient is consequently

$$a^{\tau}_{\mathbf{d}}(v)=\sum_r m_{\mathbf{d},\tau,r}v^{D-d_\tau-2r},\qquad m_{\mathbf{d},\tau,r}\in\mathbb{N}.$$

Verdier duality pairs r with $D-d_\tau-r$, proving the symmetry.

For any strict flag of a representation in O_τ, the inclusions $(x^tV)_i\subset(F_t)_i$ give

$$\sum_{a\le t}d_{a,i}\le\dim V_i-\operatorname{rank}(x^t:V_{i-t}\to V_i).$$

After summing over i, the cumulative row degrees are bounded by those of $\lambda(\tau)$, where $\lambda_a(\tau)$ is the number of segments of length at least a. If the total row-degree vector equals $\lambda(\tau)$, every nonnegative vertexwise deficit has sum zero; all deficits vanish. The flag is therefore $F_t=x^tV$ at every vertex. Its full row-and-color type is unique, and its coefficient is one by the birational resolution. Finally, if $E_{a,c}$ is the exponent array of its word, then

$$\operatorname{mult}_{[h;a)}(\tau)=E_{a,h+a-1}-E_{a+1,h+a},\qquad E_{k+1,c}=0.\tag{5.6}$$

This proves full-word injectivity, including at a fixed row degree. □

For a single segment, the strict flag chooses strictly increasing rows for its successive positions. If an orbit of total dimension d degenerates to a one-chain orbit, its rank of x is at least $d-1$; it too must consist of one block. It cannot be a proper degeneration between such blocks: semicontinuity of all graded ranks, whose totals are already equal for every power, forces vertexwise equality. Thus the upper segment coordinate is also the evaluation-dual orbit coordinate. Each admissible flag is unique, and we obtain

$$\Phi_k(\widehat{B}_{[h;j)})=\sum_{1\le a_1<\cdots<a_j\le k}\mathsf{M}(X_{a_1,h}\cdots X_{a_j,h+j-1}).\tag{5.7}$$

The sum is zero for $j>k$. Ordered products of these segment classes generate R_k integrally. Indeed, the product indexed by a multisegment τ has the unique top word $m(\tau)$, with a Laurent-unit coefficient. In its expansion in the full basis, no larger row degree can occur,

since the distinct top words at that degree cannot cancel. At the equal row degree only τ occurs. Finite triangular inversion in each dimension vector proves the assertion.

At $v = 1$ the ring is therefore

$$\mathscr{R}_k^+ = \mathbb{Z}[a_{j,h} : 1 \le j \le k,\ h \in I], \qquad a_{j,h} = B_{[h;j)}\big|_{v=1}. \tag{5.8}$$

There are no relations: products indexed by different multisegments have different top words, so the same triangular argument gives a polynomial basis.

Proposition 5.3. *Let* $\mathrm{ord}(\varepsilon^2) = e$. *At* $v = 1$, *the full bounded dual IC basis identifies with the simple basis in the polynomial restricted quantum-loop* GL_k *ring on one spectral orbit. Its fundamental coordinates have the following indexing: if* $t_{j,n} = [V_{\omega_j}(\varepsilon^{2n})]$ *and* $t_{0,n} = 1$, *then*

$$a_{j,h} = t_{j,h+j-1}, \qquad \Delta t_{j,n} = \sum_{a=0}^{j} t_{a,n} \otimes t_{j-a,n-a}. \tag{5.9}$$

At finite rank r, *put* $t_{j,n} = 0$ *for* $j < 0$ *or* $j > r$. *The identification retains all periodic labels.*

Proof. The simultaneous standard/orbit and simple/IC comparison is [24, §12.3(a)]; the full standard/PBW Hopf pairing is [11, §§11.1–11.2]. They use the same standard objects and the same orbit functionals, as is also specified in the introduction of [11]. Their unitriangular transition matrices therefore identify the simple bases in the stable ring. These transition matrices are integral in each finite dimension vector, so the comparison over $\mathbb{C}$ identifies the integral bases. Substituting the coproduct in (5.9) gives the head index $h = n - j + 1$; this fixes the spectral convention.

We spell out the finite-rank passage. In a polynomial simple of large rank, take the subspace whose integer weight coordinates after row k are zero. The restricted Cartan binomials distinguish these integer weights at roots of unity [12, Corollary 4.3]. If its highest weight has a nonzero tail, the dominance inequality for the first k coordinates makes this subspace zero. Otherwise, a zero-tail weight can be obtained from the highest weight only using the internal simple roots: starting with the last coordinate and working backwards shows that the coefficients of $\alpha_k, \dots, \alpha_{n-1}$ in its lowering expression vanish. The retained subspace is therefore generated from the original highest line by internal negative divided Drinfeld generators. In any nonzero smaller-rank submodule choose a vector of maximal finite weight. Internal positive generators kill it by maximality, and external positive generators kill it because they would create a negative tail coordinate. The full triangular decomposition and simplicity upstairs force this vector to the original highest line; otherwise its generated full submodule would have strictly smaller highest finite weight. The retained subspace is consequently simple.

On characters the operation sets the tail row letters to zero. A strict flag with k rows exists precisely for labels whose segment lengths are at most k, and the radical argument retains their IC classes separately. Thus the rank passage is exactly the based quotient (5.1). This retains both periodic and aperiodic labels for all $k, e \ge 2$. □

6. An exact tensor category for the bounded full basis

We construct the exact tensor category, its subobjects, and its normalized intertwining maps.

Let $\mathcal{P}_{k,\beta}$ be the additive Karoubi category generated by all shifts of the constant-local-system equivariant orbit ICs in $Z_{k,\beta}$, with Ext computed in $D^b_{G_\beta}(E_\beta)$. A finite sum of strict k-row flag complexes generates it by Lemma 5.1. Let $\mathcal{C}_{k,\beta}$ be the category of finite-dimensional graded contravariant modules on this projective category. Equivalently, choose a basic sum of the perverse ICs and take finite-dimensional graded modules over its Ext algebra. Negative Ext between perverse generators vanishes and degree zero is a product of copies of $\mathbb{C}$. Its simple objects, up to shift, are thus indexed by the bounded multisegments. Denote them by S_τ.

We will also use an ungraded companion $\mathcal{C}^{\mathrm{ung}}_{k,\beta}$: its objects are finite-dimensional continuous modules over the completion of this Ext algebra at the positive equivariant central ideal. A finite-dimensional graded module is continuous, since sufficiently large positive central degree acts by zero, and therefore has a forgetful image in this companion. The companion contains all its ungraded subobjects; the grading-stable ones recover the graded subobjects. In each ungraded construction below, invariance under the grading torus supplies this stability.

6.1. **Right-adjoint restriction.** For $V = A \oplus B$ of dimension vectors α, β, put

$$g = \sum_i \alpha_i\beta_i, \qquad h = \sum_{i\to j} \beta_i\alpha_j, \qquad h^{\mathrm{op}} = \sum_{i\to j} \alpha_i\beta_j.$$

Let E_+ retain the cross arrows $B \to A$, and E_- retain the cross arrows $A \to B$. Their projections to $E_0 = E_\alpha \times E_\beta$ and inclusions into $E_{\alpha+\beta}$ are denoted $\pi_\pm, i_\pm$. Ambient normalized induction has shift $g + h$. Its right adjoint is

$$F_{\alpha,\beta} = \pi_{+*}i_+^![g-h] \simeq \pi_{-!}i_-^*[g-h]. \tag{6.1}$$

For the first formula, a local section over the flag variety makes a G-equivariant complex constant along a smooth chart of dimension g; exceptional fiber restriction contributes $2g$, leaving $g - h$ after adjunction. The second is Braden's natural hyperbolic comparison [3, Theorem 1]. All inclusions, units, and base-change maps exist equivariantly; the ordinary isomorphism makes their equivariant cone zero. Since diagonal blocks of x^k are x_A^k, x_B^k, its support is bounded in each factor.

Lemma 6.1. *The functor $F_{\alpha,\beta}$ sends $\mathcal{P}_{k,\alpha+\beta}$ to $\mathcal{P}_{k,\alpha} \boxtimes \mathcal{P}_{k,\beta}$, including on all Ext morphisms. Its unshifted version $F_0 = \pi_{-!}i_-^*$ preserves the category of raw flag projectives with even shifts.*

Proof. Write $Q_\gamma = \pi_{\gamma!}\mathbb{C}$ for a raw strict-flag complex. Filter $X_\gamma \times_E E_-$ by relative position of its flag with respect to B. A stratum is indexed by rowwise splits $\lambda_a + \mu_a = \gamma_a$. Over fixed flags of A, B, set

$$U = \bigoplus_i \mathrm{Hom}(A_i, B_i), \quad U_F = \{f : f(F_r^A) \subset F_r^B\}, \quad W_F = \{c : c(F_r^A) \subset F_{r+1}^B\}.$$

The entire stratum, including its cross arrow, is the vector bundle

$$(U \oplus W_F)/\{(f, x_Bf - fx_A) : f \in U_F\}. \tag{6.2}$$

Changing a lift of the flag by $f \in U_F$ changes the cross arrow by the indicated coboundary. The first component makes the graph map injective, with a vector-bundle quotient of rank

$$r(\lambda,\mu) = \sum_{b<a,i} \lambda_{a,i}\mu_{b,i} + \sum_{a<b,i\to j} \lambda_{a,i}\mu_{b,j}.$$

Indeed, in the decreasing flags used here, a map in U_F sends source layer a to target layers $b \geq a$, whereas a map in W_F sends it to layers $b > a$. Subtracting $\dim U_F$ from $\dim U + \dim W_F$ gives exactly the displayed rank. Strictness of x_A, x_B also gives $x_Bf - fx_A \in W_F$ for $f \in U_F$, so the graph in (6.2) is contained in the stated bundle with this same flag convention. Proper base change and compact Thom isomorphism give successive terms $(Q_\lambda \boxtimes Q_\mu)[-2r](-r)$.

For two raw flags, their Steinberg variety has a finite stratification over the relative-position orbits in the product of flag varieties. Its arrow fibers are vector bundles. A stabilizer of a flag pair has a product-of-general-linear-groups Levi and unipotent radical, so its rational equivariant cohomology is even. The closed filtration and the Ext/Borel–Moore identification, whose dimension shifts are even, therefore prove

$$\mathrm{Ext}^{2m+1}_{G_\beta}(Q_\lambda, Q_\mu) = 0. \tag{6.3}$$

All connecting maps in the filtration of (6.2) have odd degree between even shifts and vanish. It splits equivariantly. A raw orbit IC is a shift-zero summand of its radical flag resolution. If any odd shift of a raw IC occurred in a flag, composing its inclusion with that radical flag's projection would contradict (6.3). Hence the raw-even category is exactly the envelope of

the raw ICs and their even shifts. The splitting just proved gives the asserted closure on all its summands. Naturality of F_0 on Ext is built into its six-functor definition; the splitting proves closure on projectives.

The equivariant proper decompositions can be checked on finite approximations. For every GL_m factor choose $U_{m,t} = \mathrm{Inj}(\mathbb{C}^m, \mathbb{C}^t)$. The associated finite approximations are bundles over products of Grassmannians. The first positive cohomology of $U_{m,t}$ is in degree $2(t-m)+1$. If the cohomology of fixed objects K, H lies respectively below b and above c, choosing $2(t-m) > b-c$ makes pullback an isomorphism on $\mathrm{Hom}(K,H)$: the cone of $H \to Ra_*a^*H$ starts above $c+2(t-m)$, and t-structure orthogonality proves the assertion. One choice handles the finite list of perverse truncation and composition-series extension classes. Proper decomposition, or Braden's purity theorem on the smooth associated arrow bundle [3, Theorem 8], kills these classes on the finite approximation. Faithfulness on the specified Hom spaces kills them equivariantly. This also supplies a direct purity proof of the same closure assertion. □

6.2. **Ordinary tensor signs and exactness.** The tensor on ordinary graded modules is obtained by a coherent sign transport on the projectives. If P_i is a perverse IC of orbit dimension d_i, let $p_i = d_i \bmod 2$. Raw parity gives $\deg a = p_i + p_{i'} \bmod 2$ for $a : P_i \to P_{i'}$. Define

$$a \odot b = (-1)^{p_i \deg b}(a \boxtimes b). \tag{6.4}$$

The derived interchange sign cancels, since $(p_i + p_{i'} + \deg a)\deg b' = 0 \bmod 2$. The three-factor signs $p_i \deg b + (p_i + p_j)\deg c$ and $p_j \deg c + p_i(\deg b + \deg c)$ agree. This transports the ordinary product on raw-even generators coherently.

The unshifted restriction gives an ordinary module tensor $\otimes_0$ by precomposition. The Hall tensor is

$$M \star N = q^{s(\alpha,\beta)}(M \otimes_0 N), \qquad s(\alpha,\beta) = g - h, \tag{6.5}$$

where q raises the internal module grading. Bilinearity of s gives the cocycle equality for three factors and transports the associator. The associator therefore acts on ordinary graded modules. One can equivalently unshift every IC to a raw-even generator, perform every correspondence there, and shift back by (6.5).

Take the sum of all strict k-row flag types, retaining placements of zero layers. The splitting in Lemma 6.1 has one term for every pair of flag types. Evaluation on this finite generator shows that the tensor is finite, exact in both arguments, and nonzero on every pair of nonzero objects; its ungraded generator dimension is multiplicative. On Grothendieck groups, its coefficients are the full Green restriction coefficients, with

$$v = q^{-1}. \tag{6.6}$$

Indeed positive Green restriction has shift $h-g$, while (6.1) is its Verdier conjugate. A projective summand shift is reversed by duality, and contravariant evaluation of $P[r]$ raises module degree by r. This gives precisely (6.6). In particular all tensor composition coefficients lie in $\mathbb{N}[q, q^{-1}]$; their specialization at one is multiplication in (5.8).

6.3. **The Schur realization on all morphisms.** The completed Schur comparison gives a faithful tensor realization preserving subobjects. In this comparison we forget the internal grading; the graded tensor has already been constructed above.

Let $H_d(\zeta)$ be the affine Hecke algebra in Bernstein presentation, where $\zeta = \varepsilon^2$ has order e. Put $P = \mathbb{C}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$. For a parabolic subgroup $W_J \subset S_d$, let $v_J = \sum_{w \in W_J} T_w$ and $M_J = v_J H_d$. The PBW basis identifies M_J with $\mathbb{C}_\zeta \otimes_{H_J^{\mathrm{fin}}} H_d$, even when $v_J^2 = 0$. Thus

$$\mathrm{Hom}_{H_d}(M_J, M_K) = \{m \in M_K : mT_j = \zeta m \ (j \in J)\}. \tag{6.7}$$

For a minimal representative $w \in W_K \backslash S_d / W_J$, put $W_L = W_J \cap w^{-1} W_K w$. Its PBW double-coset block is the image of $v_L H_J^{\mathrm{aff}}$ under $v_L T_a f \mapsto v_K T_w T_a f$. The anti-involution fixing

T_i, X_j turns the right-J eigencondition into a vector $v_J f$. The remaining right-L condition reads

$$v_J(s_i(f) - f)\frac{X_{i+1} - \zeta X_i}{X_{i+1} - X_i} = 0 \qquad (i \in L).$$

Clearing the denominator in the Laurent domain and using PBW injectivity of $P \to v_J P$ gives $f \in P^{W_L}$. Hence the whole Hom space, at this value of ζ, is

$$\mathrm{Hom}_{H_d}(M_J, M_K) = \bigoplus_{w \in {}^K W^J} P^{W_J \cap w^{-1} W_K w}, \tag{6.8}$$

Here ${}^K W^J$ denotes the minimal-length double-coset representatives, and ${}^L W_J$ the minimal-length representatives for $W_L \backslash W_J$. The summand is represented by $v_J \mapsto v_K T_w f \sum_{a \in {}^L W_J} T_a$.

These operators are generated by polynomial multiplication, splits, merges, and block permutations. There is a denominator-free proof for block permutations. Merge adjacent blocks of sizes a, b, then split into sizes b, a. Its finite double-coset terms have coefficient one and intersection matrices

$$\begin{pmatrix} t & b-t \\ a-t & t \end{pmatrix}, \qquad 0 \le t \le \min(a, b).$$

For $t > 0$, refine to $(t, a-t, b-t, t)$, interchange the two middle blocks, and merge. Length-additive coset sums give exactly the corresponding term, once. Its middle size is $a+b-2t < a+b$. Subtracting these terms and inducting constructs the full block permutation. The construction uses coset sums and subtraction.

All modules in (6.8) are finite over $Z = P^{S_d}$. Flat central completion commutes with these Hom spaces: the relevant modules are finitely presented over the noetherian finite Z-algebra H_d. Artin–Rees identifies the induced image topology with the central-adic topology. The finite direct sum in (6.8) remains finite, and an arbitrary formal coefficient can be inserted in the same finite operator factorization. Surjectivity after completion is therefore an actual equality of operator algebras.

We first describe the geometric polynomial representation. Read each strict flag from its submodule end. Its corner is

$$\mathcal{L}_{\mathbf{d}} = \mathbb{C}[u_{i,p} : i \in I,\ 1 \le p \le \beta_i]^{\prod_{a,i} S_{d_{a,i}}}.$$

The vector-bundle description of the incidence variety identifies this ring with its equivariant cohomology. For a split of adjacent layers $a + b$ into $a \mid b$, convolution acts by multiplication by

$$\mathcal{E}_{a,b} = \prod_{i \in I} \prod_{p=1}^{a_{i+1}} \prod_{r=1}^{b_i} (u^a_{i+1,p} - u^b_{i,r}).$$

The reverse merge acts by the same-color divided-difference operator

$$\partial_{a,b} f = \sum_{\sigma \in \prod_i S_{a_i+b_i}/(S_{a_i} \times S_{b_i})} \sigma\left(\frac{f}{\prod_i \prod_{p=1}^{a_i} \prod_{r=1}^{b_i} (u^a_{i,p} - u^b_{i,r})}\right).$$

The summands are indexed by shuffles within each color alphabet. Multiplication by the full same-color Vandermonde makes their sum alternating, so it is divisible by that Vandermonde. Thus the displayed operator is polynomial and takes values in the merged invariants. The split formula is the zero-section Euler class; the merge formula is integration along the product of partial flag varieties. Relative-position strata in the Steinberg variety give polynomially decorated split–permutation–merge operators. Their distinct leading relative positions prove that they span the Ext algebra and that this polynomial representation is faithful [23, Propositions 2.6, 2.7, 3.4 and Theorem 3.11]. These are the operators of the geometric quiver Schur algebra in [20, §8.2].

For a flag type $\mathbf{d}$, let $\mathcal{E}_{\mathbf{d}} = \prod_{a<b} \mathcal{E}_{d_a,d_b}$. Conjugation by multiplication with $\mathcal{E}_{\mathbf{d}}$ transfers the Euler factor from the split to the merge. The next comparison reverses all layers and changes the root coordinates by

$$\theta(y_{i,r}) = -u_{i,\beta_i+1-r}.$$

Thus a geometric pair $a \mid b$ becomes the modified pair $b \mid a$. After the indicated reversal of positions within each layer,

$$\theta\left(\prod_i \prod_{p=1}^{b_i} \prod_{r=1}^{a_{i+1}} (y^b_{i,p} - y^a_{i+1,r})\right) = \mathcal{E}_{a,b},$$
$$\theta\left(\prod_i \prod_{p=1}^{b_i} \prod_{r=1}^{a_i} (y^b_{i,p} - y^a_{i,r})\right) = \prod_i \prod_{r=1}^{a_i} \prod_{p=1}^{b_i} (u^a_{i,r} - u^b_{i,p}).$$

The first identity transports the Euler numerator and the second transports the divided-difference denominator. Consequently merges agree exactly, and splits are inclusions on both sides. This gives the modified quiver Schur presentation of [20, §§9.2–9.3]. We fix this layer and root reversal throughout. On three concatenated blocks both parenthesizations reverse the block order and change the sign of each root once.

Complete at the Bernstein multiset with β_i copies of ζ^i. The explicit comparison of [20, Proposition 9.1] uses $X = \zeta^i(1-y)$ on the color-i summand. Polynomial multiplication and splits agree under this change. A Schur merge is a quiver merge composed with multiplication by a unit in the *refined source* corner. In the following formula, a, b are the adjacent layers in the modified refined source. The constant term of the unit is

$$\prod_i (1-\zeta)^{a_i b_i} \prod_i \left(\frac{-\zeta}{1-\zeta}\right)^{a_i b_{i+1}} \prod_{i,s\neq i,i+1} \left(\frac{\zeta^{i+1}-\zeta^s}{\zeta^i-\zeta^s}\right)^{a_i b_s}.$$

Every factor is nonzero for $e \geq 2$. The source-corner invariance is the invariance under permutations inside the split color layers. The inverse unit therefore acts in the refined source corner.

The Euler conjugation in that comparison is faithful on its prescribed polynomial representation. To see this directly, for fixed color dimensions take the common symmetric polynomial

$$C_\beta = \prod_i \prod_{p=1}^{\beta_i} \prod_{r=1}^{\beta_{i+1}} (u_{i+1,r} - u_{i,p}).$$

Every corner Euler factor divides C_β. Multiplication by C_β is injective on every polynomial corner, and it commutes with splits and the same-color Demazure merges. An operator zero on all corner Euler multiples is consequently zero on C_β times the whole polynomial representation, and hence is zero. For two concatenated blocks with geometric root alphabets u, w, the additional factor is

$$\mathcal{E}_{\alpha,\beta} = \prod_{i\in I} \prod_{p=1}^{\alpha_{i+1}} \prod_{r=1}^{\beta_i} (u_{i+1,p} - w_{i,r}), \qquad \mathcal{E}_{\mathbf{d}|\mathbf{f}} = \mathcal{E}_{\mathbf{d}}\mathcal{E}_{\mathbf{f}}\mathcal{E}_{\alpha,\beta}.$$

This cross factor is independent of the internal flag types and symmetric in each color alphabet of each block. It commutes with polynomial multiplication and internal splits. For every internal merge, the shuffle formula gives

$$\partial_{a,b}(\mathcal{E}_{\alpha,\beta} f) = \mathcal{E}_{\alpha,\beta}\partial_{a,b}(f).$$

Each source-corner unit above depends only on the roots of the two layers being merged. Adjoining another block leaves this series unchanged, so both it and its inverse commute with horizontal extension. Generation by these operators and (6.8) prove compatibility on every

flag morphism. For three blocks, both parenthesizations have cross factor $\mathcal{E}_{\alpha,\beta}\mathcal{E}_{\alpha,\gamma}\mathcal{E}_{\beta,\gamma}$, which gives coherence.

Here is why this horizontal comparison identifies the particular right adjoint (6.1). Let C_β^{full} be the ambient sum of complete colored flag complexes, with the normalized induction shifts $g+h$ used above, and set $J(P) = \widehat{\mathrm{Hom}}^\bullet(C_\beta^{\mathrm{full}}, P)$. In the completed endomorphism description just proved, the complete sum is the right regular Hecke module. For selected flag projectives P, Q, let T_P, T_Q be their corresponding completed permutation-module summands. The map on Homs is the identity

$$\mathrm{Hom}_H(T_P, T_Q) \longrightarrow \mathrm{Hom}_H(\mathrm{Hom}_H(H, T_P), \mathrm{Hom}_H(H, T_Q))$$

under evaluation at 1. Thus J is fully faithful on the selected projective category, including bounded corners for which the complete flags are ambient objects.

Let T_β be the sum of the completed k-row permutation modules and $S_\beta = \mathrm{End}_H(T_\beta)$. Put $\gamma = \alpha + \beta$ and let $\widehat{H}_\nu$ be the affine Hecke algebra completed at the multiset ν. Under the fixed layer reversal, $e_{\alpha,\beta}$ selects words whose first $|\beta|$ positions have multiset β and whose last $|\alpha|$ positions have multiset α. Its right action of $\widehat{H}_\alpha \widehat{\otimes} \widehat{H}_\beta$ puts the α factor in the last positions and the β factor in the first positions. Concatenation of complete flags gives

$$\mathrm{Ind}_{\alpha,\beta}(C_\alpha^{\mathrm{full}} \boxtimes C_\beta^{\mathrm{full}}) \simeq C_\gamma^{\mathrm{full}} e_{\alpha,\beta}.$$

The horizontal comparison identifies the internal endomorphism actions on these objects. For every selected projective P, adjunction therefore gives natural isomorphisms of right $\widehat{H}_\alpha \widehat{\otimes} \widehat{H}_\beta$-modules:

$$\begin{aligned} J_\gamma(P) e_{\alpha,\beta} &\simeq \widehat{\mathrm{Hom}}^\bullet(C_\gamma^{\mathrm{full}} e_{\alpha,\beta}, P) \\ &\simeq \widehat{\mathrm{Hom}}^\bullet(\mathrm{Ind}_{\alpha,\beta}(C_\alpha^{\mathrm{full}} \boxtimes C_\beta^{\mathrm{full}}), P) \\ &\simeq \widehat{\mathrm{Hom}}^\bullet(C_\alpha^{\mathrm{full}} \boxtimes C_\beta^{\mathrm{full}}, F_{\alpha,\beta} P) = J_{\alpha,\beta}(F_{\alpha,\beta} P). \end{aligned} \tag{6.9}$$

The right action in every line is precomposition by the internal flag endomorphisms. In tensor space, restriction to these groups of positions gives $T_\gamma e_{\alpha,\beta} \simeq T_\beta \widehat{\otimes} T_\alpha$. Flipping the displayed factors identifies this with $T_\alpha \widehat{\otimes} T_\beta$, carrying the opposite Schur coproduct action. The permutation-module realization [5, Proposition 3.3.1] makes this an isomorphism of S_γ–$(\widehat{H}_\alpha \widehat{\otimes} \widehat{H}_\beta)$ bimodules. Row weights split row by row; the row indices modulo k and spectral colors modulo e are separate data. Consequently

$$\begin{aligned} J_{\alpha,\beta}(F_{\alpha,\beta} P) &\simeq (P \otimes_{S_\gamma} T_\gamma) e_{\alpha,\beta} \\ &\simeq J_{\alpha,\beta}(P \otimes_{S_\gamma} (S_\alpha \widehat{\otimes} S_\beta)). \end{aligned}$$

Full faithfulness lifts this to a unique natural isomorphism of projective functors. On three blocks both composites restrict to the same three groups of positions in reversed order, so faithfulness gives their equality. The empty block gives the unit comparison.

6.4. **Finite jets and the Hopf realization.** Let $\mathscr{D}_\varepsilon$ be the specialized double-Hall integral quantum loop algebra of Du–Fu. Their Theorems 5.6, 5.8 and Corollary 5.9 in [6] give a Hopf algebra and surjections

$$\zeta_d : \mathscr{D}_\varepsilon \twoheadrightarrow S_\Delta(k, d)_\varepsilon \qquad (d \geq 0).$$

This integral form is kept distinct from the restricted GL_k form used for the character comparison in (5.9). Write $\Omega_k = \mathbb{C}^k \otimes \mathbb{C}[t, t^{-1}]$ for affine tensor space. The action of the coproduct is the positionwise action of [5, §§3.5–3.6]. It descends to a Schur coproduct: if $\zeta_{a+b}(u) = 0$, then $(\zeta_a \otimes \zeta_b)\Delta(u)$ is zero on $\Omega_k^{\otimes a} \otimes \Omega_k^{\otimes b}$ and is therefore zero in $S_a \otimes S_b$. The latter tensor representation is faithful, even though tensor space is infinite-dimensional. Indeed, finitely many linearly independent operators in one factor can be separated by finitely many matrix coefficients. The same argument on three factors proves coassociativity of the descended map. The antipode is bijective: the triangular coproducts of positive and

negative semisimple Hall generators recursively express both the antipode and its inverse in the integral form, with invertible torus terms; the toral binomials are preserved under inversion. The same identities specialize. The layer convention fixed above uses $\mathscr{D}_\varepsilon^{\mathrm{cop}}$ for the tensor realization; its antipode is the inverse of the antipode of $\mathscr{D}_\varepsilon$.

For a fixed color block, let $\mathfrak{m}$ be its symmetric Bernstein maximal ideal. Every finite continuous module factors through a finite jet, and for a common jet of any finite family of modules,

$$\mathscr{D}_\varepsilon \twoheadrightarrow S_d \twoheadrightarrow S_d/\mathfrak{m}^r S_d = \widehat{S}_d/\mathfrak{m}^r \widehat{S}_d. \tag{6.10}$$

The Hopf algebra and the completed Schur algebra have exactly the same image on that family. Thus they have exactly the same subobjects and Hom spaces. Each comparison factors through the displayed finite jet.

The completion is compatible with tensor. The symmetric polynomials of the union of two alphabets satisfy $e_r(A \cup B) = \sum_{p+q=r} e_p(A)e_q(B)$. If the factor augmentation ideals have nilpotence exponents r, s, their total ideal has exponent at most $r + s - 1$ on the tensor. The extension of symmetric rings splitting an alphabet is finite, and in its zero fiber two monic polynomials whose product is a power of the variable must themselves be powers. The radical of the total augmentation is therefore the joint augmentation, so the two completion topologies are cofinal. Every finite-module comparison above takes place in an ordinary finite quotient of the completed tensor product, including when spectra coincide.

Theorem 6.2. *The category $\mathcal{C}_k = \bigoplus_\beta \mathcal{C}_{k,\beta}$ has the exact tensor (6.5). Its graded Grothendieck algebra is R_k with $v = q^{-1}$ and its full simple basis. The corresponding ungraded companion has a faithful exact strong tensor realization in finite modules of one Hopf algebra with bijective antipode. In each prescribed Hall degree and color block, this ungraded realization preserves all subobjects and all morphisms. The realization of the graded category is its composite with forgetting grading; graded subobjects are the grading-stable subobjects in the ungraded companion.*

Proof. The projective construction, its closure, and the coherent ordinary transport give the tensor and its full Green multiplicities. The Schur comparison identifies precisely that tensor on all morphisms. Inflating its finite completed-block modules along (6.10) proves the last assertions. Distinct integer row weights are retained by the restricted toral binomials, including at even root orders [12, Corollary 4.3]; hence the realization retains the degree blocks. Alternatively each of the subsequent uses may be made within its single prescribed block, where (6.10) alone proves fullness. The PBW description and the projective comparison make the basic perverse-IC Ext algebra finite over its equivariant center. Its finite central jets have a finite nonnegative grading and semisimple degree zero; the positive-degree ideal is nilpotent. Thus their ungraded simple labels are exactly the same IC labels, with the grading shift forgotten; grading-stable subobjects recover the graded ones. □

7. Polynomial deformations and actual R-matrices

We suppress $\star$ in tensor products. The following construction will be used on graded objects and on finite ungraded zero-block objects. The deformation is constructed in the intrinsic Ext category; the Hopf realization is used on its fiber at zero. In this construction $\mathrm{wt}\, M$ denotes the Hall dimension vector of M, and later its value in the group obtained by determinant localization.

7.1. **Coordinate deletion and partial three-block identities.** For a Levi representation $E = E_0 \oplus \bigoplus_{r\in I} W_r$, let E_U retain the coordinates indexed by $U \subset I$, and put $T_U = p_{U!}i_U^*$. If $V \subset U$ and $j : E_V \hookrightarrow E_U$, the unit for this closed inclusion defines the natural transformation

$$\delta_{U,V} = p_{U!}(\mathrm{id} \longrightarrow j_*j^*)i_U^* : T_U \longrightarrow T_V. \tag{7.1}$$

Its cone is compact pushforward from $E_U \setminus E_V$, shifted once. On this complement the omitted coordinate is a nowhere-zero section of the equivariant bundle $W_{U\setminus V}$. Its top Chern class

therefore acts as zero on the cone, as a derived morphism. Localizing graded Homs at this central Euler class makes the cone zero, so (7.1) is invertible. The units of composite closed inclusions give the literal transitivity identity

$$\delta_{V,W}\delta_{U,V} = \delta_{U,W}. \tag{7.2}$$

For two blocks the ratio $\delta_{BA,0}^{-1}\delta_{AB,0}$ compares the two chamber restrictions. Its inverse, evaluated contravariantly, gives a raw module map moving the first block past the second. The raw degree after the Hall shifts is

$$d(\alpha,\beta) = h - h^{\mathrm{op}}, \tag{7.3}$$

which is bilinear and antisymmetric.

For three blocks, the factorization holds with arbitrary stationary spectra. Iterated restriction for the order ABC is the single correspondence retaining AB, AC, BC: pull back the inner triangular inclusion along the outer projection and apply base change. The same Cartesian square carries its closed-inclusion unit to the unit deleting the indicated cross coordinate. For $ABC \to BAC \to BCA$, compare all three spaces to the space retaining only BC. Equation (7.2) makes their ratios telescope. Only Euler classes involving A are inverted. The direct comparison $A \mid (B \oplus C)$ is that same ratio. Keeping AB proves the other factorization. Pasting the Cartesian squares gives the associator compatibility. On raw-even projectives these entire diagrams have even degree and ordinary interchange; the bilinear Hall twist transports them coherently.

7.2. **Polynomial lattices and their fixed quotients.** Twist all vertex bundles by one universal line. The group automorphism

$$G_\beta \times \mathbb{G}_m \longrightarrow G_\beta \times \mathbb{G}_m, \qquad ((g_i),t) \longmapsto ((tg_i),t)$$

fixes arrows and flags, since the scalars cancel. On the uncompleted flag Ext algebra it induces $T_z : A \to A[z]$, sending every equivariant root to its translate by z, with $\deg z = 2$. Each entry is polynomial. In a fixed matrix entry between shifted flag complexes, Ext has a finite lower bound d_0; Künneth in degree n reads

$$\mathrm{Ext}^n_{G\times\mathbb{G}_m}(K,L) = \bigoplus_{0\le j\le (n-d_0)/2} \mathrm{Ext}^{n-2j}_G(K,L)z^j.$$

Every homogeneous entry, including every entry of an IC idempotent, is therefore a finite polynomial. Under the Schur comparison, the translation $u \mapsto u+z$ becomes $y \mapsto y-z$. Thus the modified coordinates use the parameter $-z$; its valuation and the nonvanishing of the specialized normalized maps are unchanged.

For a finite module M, let $M_z = \mathbb{C}[z]\otimes M$ with action pulled back by T_z. Evaluation at a translated idempotent is finite projective over $\mathbb{C}[z]$, hence free. Restriction of a fixed projective is a finite sum of shifted projectives, so the evaluated product lattices are free as well. If $0 \to K \to E \to Q \to 0$ is a *fixed* exact sequence, its translated, tensored, and idempotent-evaluated sequence is exact, with free quotient. Thus

$$L_K \cap z^a L_E = z^a L_K \qquad (a \ge 0). \tag{7.4}$$

This applies to the translated families of fixed kernels, images, and composition quotients.

The invariant cross Euler product after translating the moving block is a signed monic polynomial in z, with lower coefficients in a positive-degree commutative central ideal. That ideal is nilpotent on a finite graded module, and on a finite ungraded zero-block module by continuity. Factoring out its leading power of z and expanding the nilpotent remainder gives a finite Laurent inverse. Hence the coordinate comparison evaluates to actual invertible Laurent maps

$$R_{M_z,N} : M_zN \longrightarrow NM_z, \qquad R_{N_w,M_z}R_{M_z,N_w} = \mathrm{id}\,.$$

Common line twisting commutes with every coordinate unit, so

$$R_{M_z,N_w} = (R_{M_{z-w},N})_w. \tag{7.5}$$

These are maps of intrinsic Ext modules. The Schur and Hopf realizations are applied after specialization at $z = 0$.

For a nonzero pair define the signed integer

$$\begin{aligned} a(M,N) &= \max\{a \in \mathbb{Z} : R_{M_z,N}(M_z N) \subset z^a N M_z\}, \\ \widehat{R}_{M_z,N} &= z^{-a(M,N)} R_{M_z,N}, \qquad r_{M,N} = \widehat{R}_{M_z,N}|_{z=0}, \\ \Lambda(M,N) &= d(\operatorname{wt} M, \operatorname{wt} N) - 2a(M,N). \end{aligned} \tag{7.6}$$

Finitely many projective evaluations compute $a(M,N)$, and maximality makes $r_{M,N}$ nonzero. The integer a is allowed to be negative. Equation (7.5) identifies moving the second factor with moving the first at $-z$, up to the resulting nonzero scalar. For a fixed subobject or quotient we use raw naturality with one common normalization, retaining the difference of the two orders as a power of z. For an inclusion $i : S \hookrightarrow E$, the exact identity is

$$\widehat{R}_{A_z,E}(A_z i) = z^{a(A,S)-a(A,E)}(iA_z)\widehat{R}_{A_z,S}.$$

Saturation controls the restriction on its free sublattice. In particular,

$$E \twoheadrightarrow Q \implies \Lambda(A,Q) \le \Lambda(A,E), \qquad K \hookrightarrow E \implies \Lambda(A,K) \le \Lambda(A,E), \tag{7.7}$$

and similarly with A second. To verify either inequality, insert $R = z^a \widehat{R}$ in the naturality square, divide by the lesser power, and specialize. The contrary inequality would force a nonzero normalized map to vanish after an epimorphism or before a monomorphism. Raw degrees are equal because all compared objects have the same Hall weight.

7.3. **Coefficient extraction and finite adjoints.** Theorem 6.2 supplies two useful elementary consequences. For subobjects $X \subset M_1 M_2$ and $Y \subset M_2 M_3$,

$$XM_3 \subset M_1 Y \implies \exists K \subset M_2 : \quad X \subset M_1 K, \quad KM_3 \subset Y, \tag{7.8}$$

$$M_1 Y \subset XM_3 \implies \exists K \subset M_2 : \quad M_1 K \subset X, \quad Y \subset KM_3. \tag{7.9}$$

For the first, take the image of the Hopf evaluation $M_1^* \otimes X \to M_2$, equivalently the span of the first-factor coefficients of X. Extracting coefficients from the assumed containment proves the second inclusion. Use last-factor coefficients of Y and the other dual for the second implication. These images are subobjects of M_2, so finite-jet fullness puts them back in the required Schur block. For graded data the first image is intrinsically the smallest K with $X \subset M_1 K$. Exact tensor preserves intersections, and the monoidal grading torus preserves this smallest subobject. Thus K is graded. The same argument applies to the second implication.

For fixed A, L and one specified input Hall block, the ambient dual tensor representing $\operatorname{Hom}(AZ, L)$ has a largest submodule annihilated by the Schur-kernel ideal. Taking its prescribed generalized central component gives a finite object X with

$$\operatorname{Hom}(AZ, L) = \operatorname{Hom}(Z, X). \tag{7.10}$$

Using a largest quotient of the other dual tensor gives a finite Y with $\operatorname{Hom}(L, ZA) = \operatorname{Hom}(Y, Z)$. These finite adjoints are constructed in one specified block at a time and may be used ungraded.

If B is simple and $u : L \to M_1 B$, $v : BM_3 \to L'$ are nonzero, then $(M_1 v)(uM_3) \neq 0$. Otherwise $\operatorname{im}(u)M_3 \subset M_1 \ker(v)$; (7.8) gives a nonzero $K \subset B$ with $KM_3 \subset \ker(v)$, contradicting simplicity. The opposite orientation follows from (7.9). Applying this to the two adjacent specialized R-maps with simple moving factor A proves that their composite gains no additional factor of z. The partial three-block identity consequently gives

$$\Lambda(A, EF) = \Lambda(A,E) + \Lambda(A,F), \qquad \Lambda(EF, A) = \Lambda(E,A) + \Lambda(F,A) \tag{7.11}$$

for arbitrary nonzero stationary E, F, and gives the corresponding factorizations of normalized maps.

8. Real factors, strict extrema, and signed division

A simple object is *real* if its tensor square is simple. Two simples strongly commute if their product is simple; when one is real, the next theorem also proves that the two orders are isomorphic up to shift.

Theorem 8.1. *Let A be real simple and B simple. Both AB and BA have simple heads and socles. Up to their indicated grading shifts,*

$$H = \mathrm{hd}(AB) = \mathrm{soc}(BA) = \operatorname{im} r_{A,B}, \qquad J = \mathrm{hd}(BA) = \mathrm{soc}(AB) = \operatorname{im} r_{B,A}.$$

Each displayed label has total ungraded multiplicity one. Moreover

$$\Lambda(A,H) = \Lambda(A,B), \qquad \Lambda(A,T) \le \Lambda(A,B) - 2 \quad (T \text{ in } \ker(AB \twoheadrightarrow H)), \tag{8.1}$$

$$\Lambda(J,A) = \Lambda(B,A), \qquad \Lambda(T,A) \le \Lambda(B,A) - 2 \quad (T \text{ in } \ker(BA \twoheadrightarrow J)). \tag{8.2}$$

The same strict inequalities hold outside the corresponding opposite socles. All four head and socle label maps, with A fixed, are injective modulo grading shift. Finally

$$\Lambda(A,B) + \Lambda(B,A) = 2\ell \ge 0, \qquad \ell = 0 \iff AB \text{ is simple.} \tag{8.3}$$

Proof. We first prove the simple-image assertion for a *simple* M satisfying $r_{M,M} = c\,\mathrm{id}$, $c \neq 0$, and a simple N. Thus the moving factor in (7.11) is already simple. For a nonzero fixed subobject $S \subset MN$, move M_z from the right through MN. Equation (7.5) transfers the varying parameter to the first argument when computing orders; the substitution $z \mapsto -z$ preserves those orders. Simple-moving factorization gives, up to a nonzero scalar,

$$r_{MN,M} = (r_{M,M}N)(Mr_{N,M}) = c\,Mr_{N,M}.$$

Raw naturality on SM with the normalization of MN has factor $z^{a(S,M)-a(MN,M)}$. Its exponent is nonnegative by (7.7). Saturation in (7.4) therefore places the specialized image in MS. Consequently $(Mr_{N,M})(SM) \subset MS$. Exactness identifies the inverse image of MS under $Mr_{N,M}$ with $Mr_{N,M}^{-1}(S)$, giving

$$SM \subset Mr_{N,M}^{-1}(S).$$

By (7.8), there is $K \subset N$ with $S \subset MK$ and $KM \subset r_{N,M}^{-1}(S)$. Necessarily $K = N$, so $\operatorname{im} r_{N,M}$ lies in every nonzero subobject of MN. Choosing a simple subobject of that image proves it is the unique simple socle.

For a nonzero proper fixed subobject $K \subset MN$, moving M_z from the left gives

$$r_{M,MN} = (Mr_{M,N})(r_{M,M}N) = c\,Mr_{M,N}.$$

Raw naturality on MK has the nonnegative order difference $a(M,K) - a(M,MN)$. Saturation of the fixed target lattice places the specialized image in KM, hence $Mr_{M,N}(K) \subset KM$. Implication (7.9) supplies $K_0 \subset N$ with $MK_0 \subset K$ and $r_{M,N}(K) \subset K_0M$. If the latter image were nonzero, $K_0 = N$, contradicting properness. Thus $r_{M,N}$ kills every proper subobject. Its kernel is the unique maximal subobject and its image is the simple head. Reversing the monoidal category proves the two opposite statements. Taking $N = M$, the image of the scalar self-map is all of M^2, which is therefore simple. Thus scalar self-R implies reality for a simple M. Conversely, for real simple A, the nonzero map $r_{A,A}$ is a scalar endomorphism of the underlying simple A^2, and has degree zero since a finite graded object cannot be isomorphic to a nonzero shift of itself. The preceding argument applies.

We prove head-label injectivity using the finite adjoint. Fix simple L and use the finite adjoint X in (7.10). If $X \neq 0$, its universal evaluation $p_X : AX \twoheadrightarrow L$ is nonzero on AZ for every nonzero subobject $Z \subset X$: under adjunction the restriction is exactly $Z \hookrightarrow X$. Let $K = \ker p_X$. The three-block calculation with scalar $r_{A,A}$ gives $Ar_{A,X}(K) \subset KA$. By (7.9) there is $Z \subset X$ with $AZ \subset K$ and $r_{A,X}(K) \subset ZA$. The universal property forces $Z = 0$. Hence

$$r_{A,X} = \xi p_X, \qquad 0 \neq \xi : L \hookrightarrow XA.$$

For every simple subobject $i_S : S \hookrightarrow X$, its evaluation $p_X(Ai_S)$ is nonzero, and therefore surjective onto L. The subobject order inequality gives $a(A,S) \geq a(A,X)$. If it were strict, the common-normalization identity above would give $r_{A,X}(Ai_S) = 0$ at $z = 0$. This contradicts $r_{A,X}(Ai_S) = \xi p_X(Ai_S) \neq 0$. Thus the orders are equal. Specializing that identity now places the entire image $\xi(L)$ inside SA. Two distinct simple subobjects would have $(S_1A) \cap (S_2A) = (S_1 \cap S_2)A = 0$, a contradiction. Thus X has a unique simple socle. If $L = \mathrm{hd}(AB)$, its quotient map corresponds to an injection $B \hookrightarrow X$, determining B uniquely. Reversal gives the other head, and the head–socle identifications give both socle maps. Equal head labels force equal Hall weights of B, so this argument needs adjoints only in a single fixed block. Underlying simple labels have unique graded lifts up to shift by the degree-zero description of the basic IC algebra.

We prove equality at the extrema separately from the strictness on their kernels. Factor $r_{A,B} = ip$ and $r_{B,A} = j\rho$, where $p : AB \twoheadrightarrow H$, $i : H \hookrightarrow BA$, $\rho : BA \twoheadrightarrow J$, and $j : J \hookrightarrow AB$. The maps

$$g = (pA)(Ai) : AH \to HA, \qquad h = (A\rho)(jA) : JA \to AJ$$

are nonzero. If $g = 0$, then $Ai(H) \subset (\ker p)A$. Implication (7.9), with middle factor B, gives $K \subset B$ with $AK \subset \ker p$ and $i(H) \subset KA$. The second inclusion forces $K = B$, contradicting $p \neq 0$. The proof for h uses (7.8) and ρ.

By (7.11), the normalized map moving A_z past AB has degree $\Lambda(A,B)$ and specializes, up to scalar, to $Ar_{A,B}$. It preserves the fixed kernel $K = \ker p$. Its specialization is zero there, so (7.4) divides it by one further z inside the kernel lattice. Its normalized degree on K is at most $\Lambda(A,B) - 2$. Equation (7.7) gives the same bound on every constituent of K. On the quotient H, its specialization is the nonzero map g, since $(pA)(Ar_{A,B}) = g(Ap)$. There is no additional z factor there, proving (8.1).

The map moving A_z past BA specializes to $r_{A,B}A$. Its image lies in $i(H)A$, so on $BA/i(H)$ it has one further factor of z; on $Ai(H)$ its specialization is nonzero because $(r_{A,B}A)(Ai) = (iA)g$. This proves the opposite-socle version. Similarly the maps moving A_z in the other orientation specialize to $r_{B,A}A$ and $Ar_{B,A}$. Their kernel and quotient calculations use the nonzero h, and prove (8.2) and its opposite-socle version. A second occurrence of the same head label, in any shift, would lie in its kernel and contradict the strict inequality. This proves total ungraded multiplicity one at all four ends.

Finally normalize the literal inverse Laurent maps at spectra $(z,0)$ as $F = z^{-a}R_{A_z,B}$ and $G = z^{-b}R_{B,A_z}$. Then

$$GF = z^{\ell}\,\mathrm{id}, \qquad FG = z^{\ell}\,\mathrm{id}, \qquad \ell = -(a+b).$$

Both preserve nonzero free lattices, so $\ell \geq 0$. Antisymmetry of (7.3) and (7.5) give (8.3). If $\ell = 0$, their specialized maps are inverse isomorphisms and have simple images, so AB is simple. Conversely, if AB is simple, its nonzero map to BA is injective. The raw Laurent isomorphism makes the two free lattices have the same rank at each finite projective evaluation; their specializations therefore have equal dimensions. The injection is an isomorphism, and the reverse map is also one. Their nonzero specialized composite forces $\ell = 0$. □

8.1. **Localization at determinant objects.** Put $D_h = a_{k,h}$ and write the corresponding simple also as D_h. Its Miura formula (5.7) has exactly one term: a chain of length k must use every one of the k flag rows. The fiber functor of Theorem 6.2 evaluates a simple on the sum of these flag projectives, and its dimension is the sum of their IC multiplicities at $v = 1$. Hence the fiber of D_h is a line. Tensoring by this Hopf line is an equivalence, and finite-jet fullness gives

$$\mathrm{Hom}(M,N) \simeq \mathrm{Hom}(D_hM, D_hN) \tag{8.4}$$

inside any fixed Hall block. The graded assertion follows by taking homogeneous components of the unique ungraded preimage. In particular D_hS is simple whenever S is simple.

Its action on labels is addition of $[h;k)$. Indeed D_hS_τ is simple by the preceding paragraph, and its unique highest Miura word is $m(D_h)m(\tau)$ by Theorem 5.2. Formula (5.6) identifies

the resulting label as $\tau + [h; k)$. The coefficient-one top terms fix its Laurent unit in Weyl normalization. Thus determinant multiplication is a based label translation.

Telescoping (5.3) gives

$$\Omega(m(D_h), (a, t)) = \rho_h(t) := \delta_{t,h-1} - \delta_{t,h} + \delta_{t,h+k-1} - \delta_{t,h+k}, \tag{8.5}$$

independent of a. Every word in a homogeneous simple character has the same color degree, so $\widehat{D}_h\widehat{S} = v^{\rho_h(\mathrm{wt}\, S)}\widehat{S}\widehat{D}_h$. The simple-product statement and (8.3) then identify the normalized determinant crossings and their degrees.

To localize as an actual category, use the full subcategory $\mathcal{C}_k^{\mathrm{ten}}$ of subquotients of finite sums of tensor words in simples. It is abelian, finite length, exact in the ambient category, and tensor closed. On a word W, all successive $r_{D_h,S}$ are isomorphisms, so their product is an isomorphism $D_hW \to WD_h$ of degree $\rho_h(\mathrm{wt}\, W)$. Its normalizing power depends only on that Hall weight. Raw naturality and (7.4) descend this map and its inverse to every fixed subquotient of a sum of such words. Thus the determinants form a natural coherent family of invertible crossings on $\mathcal{C}_k^{\mathrm{ten}}$.

Choose an order on the determinants. A localized object is represented by $D^{-a}M$, $a \in \mathbb{N}^e$. For two objects of equal localized Hall degree, choose $c \geq a, b$ and compute their Homs between $D^{c-a}M$ and $D^{c-b}N$ in one positive block. The transitions are isomorphisms by (8.4). The coherent crossings define tensor on representatives. Exactness, subquotients, heads, and multiplicities are computed after the same positive translation. This constructs an actual localization with based ring

$$\mathscr{R}_k = \mathbb{Z}[a_{j,h}\ (j < k), D_h^{\pm 1} : h \in I]. \tag{8.6}$$

Its simple labels allow integral multiplicities of length-k segments and nonnegative multiplicities of shorter segments.

All real-factor assertions persist here. To check the strict inequalities explicitly, let T be a constituent of AB and let P, Q be determinant monomials. Bilinearity of their degrees gives

$$\Lambda(PA, PQT) - \Lambda(PA, QB) = \Lambda(A, T) - \Lambda(A, B) + \Lambda(P, T) - \Lambda(P, B) + \Lambda(A, P).$$

Since $\mathrm{wt}\, T = \mathrm{wt}\, A + \mathrm{wt}\, B$, the last three terms sum to zero by (8.5). The other extrema are identical. One common positive determinant translation also proves the leading-word assertion for a finite set of localized labels: it shifts every row degree and full word by the same monomial. Equation (5.6) continues to recover the integral determinant multiplicities.

8.2. **The semiclassical bracket and signed support.** The anti-involution fixing every normalized full IC class is induced by Verdier duality and the scalar $v^{\eta(\beta)}$. Indeed raw restriction coefficients satisfy $c^T_{A,B}(v^{-1}) = v^{C(\alpha,\beta)}c^T_{B,A}(v)$, and polarization of η removes this defect. Equivalently Theorem 5.2 intertwines it with the torus anti-involution fixing Weyl monomials. At $v = 1$ the algebra is commutative. Define

$$\{f, g\} = \left.\frac{\widehat{f}\widehat{g} - \widehat{g}\widehat{f}}{v - 1}\right|_{v=1}.$$

The definition is independent of lifts. It gives the log-canonical letter bracket with matrix (5.3). If A, B strongly commute and A is real, the grading convention is

$$AB = q^{-\Lambda(A,B)}BA = v^{\Lambda(A,B)}BA, \qquad \{A, B\} = \Lambda(A, B)AB. \tag{8.7}$$

Lemma 8.2 (Signed support)**.** *Let A be a real simple class and z any finite signed or complex linear combination of localized simple classes. If*

$$\{A, z\} = \lambda Az,$$

then every simple B in its support strongly commutes with A, and $\Lambda(A, B) = \lambda$, $\Lambda(B, A) = -\lambda$.

Proof. For $z = 0$ the support is empty. Assume $z \neq 0$. The Hall grading is cancellative, so the equation splits by the Hall degree of z; work in one degree. Write $z = \sum_B c_B B$ and use the constant-coefficient lift in the normalized simple basis. Set $u = \max_{c_B \neq 0} \Lambda(A, B)$, choose a maximizing B, and let $T = \mathrm{hd}(AB)$. By (8.1), this label cannot occur as a nonhead constituent of any other supported AB': all such constituents have smaller interaction than u. Head-label injectivity excludes it as the head of another supported product. This isolation is true for the entire Laurent coefficient polynomial. It is true in reverse products as well, either by the opposite-socle extremum or because a coefficient in $\mathbb{N}[v, v^{-1}]$ vanishing at one must be zero.

Total ungraded multiplicity one gives, for one integer or half-integer b, coefficient $c_B v^b$ in $\widehat{A}\widehat{z}$ and $c_B v^{b-u}$ in $\widehat{z}\widehat{A}$. Differentiating their difference at one gives $c_B u$, while the coefficient of T in Az is c_B. Thus $u = \lambda$. Repeating with the maximizing right head gives $w = \max \Lambda(B, A) = -\lambda$. For every supported B,

$$0 \leq \Lambda(A, B) + \Lambda(B, A) \leq u + w = 0.$$

Both upper bounds are equalities, and the distance criterion proves strong commutation. The isolated head coefficient determines each extremum before differentiation. □

Corollary 8.3 (Saturated division)**.** *Let A, H be actual simple classes with A real. Suppose a regular element $z \in \mathscr{R}_k \otimes \mathbb{C}$ satisfies*

$$Az = H, \qquad \{A, z\} = \lambda Az.$$

Then z is one actual simple class, with coefficient one, and it strongly commutes with A.

Proof. Regularity supplies a finite signed expansion in the full simple basis. By Lemma 8.2, each supported B strongly commutes with A, so AB is one simple at $v = 1$. Their labels are distinct by head-label injectivity. Comparison with the single basis element H leaves exactly one coefficient, equal to one. □

Lemma 8.4. *A product of pairwise strongly commuting real simples is real simple. It strongly commutes with every simple commuting with all its factors.*

Proof. For two such objects A, B, Theorem 8.1 first makes $M = AB$ simple. Expand its self-R using the two factorizations:

$$r_{M,M} = (Ar_{M,B})(r_{M,A}B), \quad r_{M,A} = (r_{A,A}B)(Ar_{B,A}), \quad r_{M,B} = (r_{A,B}B)(Ar_{B,B}).$$

All factors are invertible, so no further deformation factor appears. The self-maps of A, B are scalar, and the distance-zero inverse identity makes $r_{A,B}r_{B,A}$ scalar. Thus $r_{M,M}$ is a nonzero scalar identity. The scalar-self-R argument in Theorem 8.1 makes M real. If C commutes with A, B, the normalized factorization of $r_{C,AB}$ is a composite of isomorphisms. The now real factor AB gives simplicity of ABC. Induction proves the claim, including every positive power of a real simple. Determinant twists follow by the actual localization above. □

Lemma 8.5 (Reality of a commuting head)**.** *Let R, N be real simples and $r \geq 0$. If $J = \mathrm{hd}(R^r N)$ strongly commutes with R, then J is real.*

Proof. Let $p : R^r N \twoheadrightarrow J$ and let $i : J \hookrightarrow NR^r$ be its opposite-socle inclusion. The commuting isomorphisms give respectively an epimorphism and a monomorphism

$$R^{2r}N^2 \twoheadrightarrow R^r JN \simeq JR^r N \twoheadrightarrow J^2, \qquad J^2 \hookrightarrow NR^r J \simeq NJR^r \hookrightarrow N^2 R^{2r}.$$

Lemma 8.4 makes R^{2r} real and N^2 simple. The two outer products have a common simple head and opposite socle K, with total ungraded multiplicity one. Every nonzero quotient of an object with simple head has that head, and every nonzero subobject of an object with simple socle has that socle. Hence J^2 has head and socle K and at most one occurrence of its label. The socle-to-head map cannot be zero, since otherwise its kernel and quotient give two occurrences of K. It is an isomorphism between these simple ends and splits them. A nonzero complement would have a head, contradicting uniqueness of the head. Thus $J^2 = K$ is simple. □

We will also use the elementary cancellation consequence: if A is real, B is simple, AB is real, and A strongly commutes with B, then B is real. Indeed $(AB)^2 \simeq A^2B^2$ up to shift. If B^2 had two nonzero successive quotients, exact tensor with nonzero A^2 would preserve both, contradicting simplicity of $(AB)^2$.

9. The initial simple objects

Throughout the representation-theoretic argument, $e, k \geq 2$, $d = e - 1$, and $N = (k-1)d$. We work first in the determinant-localized bounded Hall ring and its actual graded tensor category from Theorem 6.2. Its classical ring is

$$R = \mathbb{C}[a_{j,h} : 1 \leq j < k,\ h \in \mathbb{Z}/e\mathbb{Z}][D_h^{\pm 1} : h \in \mathbb{Z}/e\mathbb{Z}], \qquad D_h = a_{k,h}.$$

We write $C_\tau = B_\tau|_{v=1}$ for its localized simple class, with the normalization of Theorem 5.2. The Hall parameter is $v = q^{-1}$, where q denotes a grading shift in the category. Thus, for strongly commuting simple objects, the exponent of v in their commutation relation is $\Lambda(A, B)$. We use this convention for the Poisson bracket. The quantum-loop parameter ε is fixed throughout, with $\operatorname{ord}(\varepsilon^2) = e$.

9.1. **Evaluation and skew characters.** Write $t_{j,n}$ for the polynomial fundamental class with evaluation parameter ε^{2n}. The full Hall convention is

$$a_{j,h} = t_{j,h+j-1}, \qquad \Delta(t_{j,n}) = \sum_{a+b=j} t_{a,n} \otimes t_{b,n-a}. \tag{9.1}$$

At a generic quantum parameter, its character is

$$\chi(t_{j,n}) = \sum_{a_1 < \cdots < a_j} \Lambda_{a_1,n}\Lambda_{a_2,n-1} \cdots \Lambda_{a_j,n-j+1}.$$

Here $\Lambda_{a,n}$ means the letter with spectral parameter ζ^{2n} for a generic quantum-loop parameter ζ, and then with parameter ε^{2n} after $\zeta = \varepsilon$. The parameter ζ is unrelated to the categorical shift q. The Λ row order is opposite to the row order in the Hall Miura map: $X_{a,c} = \Lambda_{k+1-a,c}$. This equality calibrates the entire fundamental character in (9.1), including its determinant.

For a partition λ and an integer a, let $\mathsf{E}_\lambda(a)$ be the specialized polynomial evaluation character. In each finite rank, $\mathrm{SSYT}(\lambda)$ is the set of tableaux with entries in the ordered row alphabet, weakly increasing along rows and strictly increasing down columns. The stable expressions agree under truncation of this alphabet. The tableau expression for $\mathsf{E}_\lambda(a)$ is

$$\sum_{T \in \mathrm{SSYT}(\lambda)} \prod_{(r,c) \in \lambda} \Lambda_{T(r,c),\, a+c-r}. \tag{9.2}$$

This is the class of the specialized evaluation module. We also use the corresponding skew expression $\mathsf{S}_{\lambda/\mu}(a)$, with the contents inherited from the outer diagram.

Lemma 9.1. *The skew expression $\mathsf{S}_{\lambda/\mu}(a)$ is an actual positive class in the stable polynomial root-of-unity ring. Its coproduct is the alphabet-splitting coproduct, and*

$$\Delta \mathsf{E}_\lambda(a) = \sum_{\mu \subseteq \lambda} \mathsf{E}_\mu(a) \otimes \mathsf{S}_{\lambda/\mu}(a). \tag{9.3}$$

Proof. Use the two commuting Drinfeld blocks in the generic polynomial evaluation module; evaluation commutes with restriction to the upper-left block [11, Sections 4.4–4.5 and 5.4]. The first block acts through a finite-type quantum group at a generic parameter, so its restriction is semisimple. Consequently

$$V_\lambda(a)\big|_{\text{two blocks}} = \bigoplus_\mu V_\mu(a) \otimes \mathrm{Hom}_{\text{first block}}(V_\mu(a), V_\lambda(a)).$$

The second block acts on each displayed multiplicity space because the two blocks commute. Fixing the first-block tableau leaves a skew tableau of shape λ/μ, with precisely the original

contents. Thus its multiplicity space has the indicated skew character. This constructs a module with the stated positive class.

Choose restricted free lattices in the finitely many generic constituents and specialize them [11, Lemmas 8.5–8.7]. The character-specialization theorem [10, Proposition 2.5 and Theorem 3.2] gives actual positive classes with these specialized characters. The evaluation classes $\mathsf{E}_\mu(a)$ remain linearly independent in each finite degree: their distinct finite highest weights give a unitriangular Weyl-to-simple transition. Hence specializing the generic alphabet identity yields (9.3), with the same uniquely determined skew coefficients. □

Positive multiplication in the stable full polynomial ring follows from restriction to SL. For a calculation of total polynomial degree D, choose rank $n > D$. Each simple involved has last Drinfeld polynomial $Q_n = 1$. Restriction to restricted SL_n preserves simplicity, and the SL_n polynomial ratios determine the GL_n tuple uniquely when $Q_n = 1$. The coefficients of a product of full simple characters are therefore the nonnegative multiplicities in the actual SL_n tensor product. We shall use this stable positivity before returning to rank k.

9.2. **The extremal partitions and the source minors.** For $m \geq 0$ put

$$\lambda_m = (m, m-d, m-2d, \ldots > 0), \qquad \lambda_0 = \varnothing, \qquad \mathsf{E}_m = \mathsf{E}_{\lambda_m}(1-m).$$

Set $m^- = \max(m-e, 0)$. If $m^- > 0$, put $\mathsf{S}_m = \mathsf{S}_{\lambda_m/\lambda_{m^-}}(1-m)$; otherwise put $\mathsf{S}_m = \mathsf{E}_m$. The two spectral exponents in the former expression differ by $2e$ and give the same parameter after root-of-unity specialization.

We spell out the level-one matrix-coefficient interpretation used below. A partition of charge a has occupied positions $p_r = a + \lambda_r - r + 1$ in $\mathbb{Z}$. Adding a box moves an occupied position p to a vacant position $p+1$, and has color $p \bmod e$. In any finite calculation choose a window $[L, U]$ containing every move, with L occupied and U vacant on all states in question. On the exterior power of the finite natural space define

$$f_i = \sum_{\substack{L \leq p < U \\ p \equiv i \pmod e}} E_{p+1,p}, \qquad e_i = \sum_{\substack{L \leq p < U \\ p \equiv i \pmod e}} E_{p,p+1}.$$

Adjacent moves have positive exterior sign. Increasing the window preserves each fixed calculation. Write Λ_i for the level-one affine fundamental weight with simple-coroot pairings $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ and derivation value zero, and s_i for the corresponding simple reflection. Their indices are taken modulo e.

Lemma 9.2. *The vacuum-generated space is the irreducible integrable highest-weight module $L(\Lambda_a)$ for the symmetric affine Cartan datum of the e-cycle. The divided powers add or remove sets of corners with coefficient one. For charge $a = 1 - m$, $|\lambda_m\rangle$ is the normalized extremal vector of weight*

$$u_m \Lambda_{1-m}, \qquad u_m = s_0 s_{-1} \cdots s_{1-m}.$$

The word u_m is reduced.

Proof. On the natural space, $f_i^2 = e_i^2 = 0$. For $e \geq 3$, $f_i f_j f_i = 0$ for adjacent distinct colors, which gives the double-commutator Serre relation. For $e = 2$ the two directed arrows remain present; the valid relation is $(\operatorname{ad} f_i)^3 f_j = 0$, and the same holds for e_i. The mixed commutator is the diagonal operator whose eigenvalue on a partition is its number of addable i-corners minus its number of removable i-corners. The boundary contribution is

$$\sum_i [e_i, f_i] = E_{L,L} - E_{U,U}.$$

It acts as 1. Thus the level is one, and on the vacuum the Cartan eigenvalues are $\delta_{i,a}$. The derivation with eigenvalue minus the number of color-0 boxes retains the full affine weight. These formulas give the claimed affine action, including the two-vertex case.

Adding an i-corner creates possible new corners only of adjacent colors. Hence distinct currently addable i-corners can be added in every order, and no new i-corner appears during these moves. It follows that

$$f_i^{(r)}|\lambda\rangle = \sum_{\substack{S \text{ a set of} \\ r \text{ addable } i\text{-corners}}} |\lambda + S\rangle. \tag{9.4}$$

Removal is the transpose formula. This proves integrability and the integral coefficient-one normalization.

Give the partition basis its positive definite orthonormal form. Then $f_i^* = e_i$. If the vacuum-generated space had a proper nonzero submodule, choose a weight vector w of its smallest box degree. All $e_i w$ vanish. The degree cannot be zero, since the vacuum generates the space. For every nonempty word in the f_i, adjointness gives

$$(w, f_{i_1} \cdots f_{i_t}|\varnothing\rangle) = (e_{i_t} \cdots e_{i_1} w, |\varnothing\rangle) = 0.$$

This contradicts $(w, w) > 0$, since these words span the vacuum-generated space. Thus it is $L(\Lambda_a)$.

At step r, the color $a + r - 1$ has $L_r = \lceil r/d \rceil$ addable corners of λ_{r-1} and no removable corner. To check the possible bottom corner, write $r - 1 = qd + s$, $0 \le s < d$. If $s = 0$, its residue difference from the row ends is qe; if $s > 0$, it is $qe+s+1$, with $2 \le s+1 \le e-1$. Thus it is added exactly when $s = 0$, which is exactly when λ_r acquires another row. Equation (9.4) gives

$$f_{a+r-1}^{(L_r)}|\lambda_{r-1}\rangle = |\lambda_r\rangle.$$

This is the full $\mathfrak{sl}_2$ string reflection, with positive coroot pairing L_r and coefficient one. Taking $a = 1 - m$ proves the extremal-vector assertion. Inductively the word is reduced: if the last left reflection shortened it, the pullback of that simple coroot would be negative and would pair nonpositively with the original dominant fundamental weight, contrary to the displayed positive pairing. □

In the source comparison and its based section below, specialize the grading parameter to $v = 1$. Let $H_{\mathrm{comp}} \subset H$ be the composition subalgebra generated by the one-vertex simple classes. Restriction of linear functionals gives $\pi : H^\vee \to H_{\mathrm{comp}}^\vee$. Let $*$ fix the Chevalley generators and reverse their products, and use the same symbol for its dual action. Write $\mathcal{F} = *\pi$. Fix a positive integer R_0 at least as large as every index m under consideration. In the convention of [18, §11], use the ordinary unipotent coordinate algebra associated with $w = u_{R_0}^{-1}$, where $u_{R_0} = s_0 s_{-1} \cdots s_{1-R_0}$, in its positive coordinate realization. Its determinantial minors $D(m, j)$, for $j = 0, m^-$, are the matrix coefficients

$$\langle g, D(m, j)\rangle = (g|\lambda_j\rangle, |\lambda_m\rangle),$$

where both extremal vectors have charge $1 - m$, the normalization is that of Lemma 9.2, and g is a Chevalley word. Then

$$\mathcal{F}(\mathsf{E}_m) = D(m, 0), \qquad \mathcal{F}(\mathsf{S}_m) = D(m, m^-). \tag{9.5}$$

Indeed, pairing the second factor of (9.3) with a lower semisimple orbit class removes a horizontal strip with the prescribed residues. Pairing with a Chevalley word removes its boxes in reverse order. Consequently

$$\langle g, \mathsf{S}_{\lambda/\mu}(a)\rangle = (g^*|\mu\rangle, |\lambda\rangle).$$

Precomposing with $*$ and using Lemma 9.2 gives the normalized matrix coefficients in (9.5). The charge $1 - m$ is the fundamental-weight index at position m of the fixed word u_{R_0}.

9.3. **A based section on the source category.**

Lemma 9.3. *Every simple constituent of a finite product of the* E_m *and* S_m *is aperiodic.*

Proof. Choose stable rank n larger than the total degree of the product. For an SL_n character monomial define

$$H_i = \text{exponent of } Y_{i,\varepsilon^{i+1}}, \qquad H = (H_{n-1}, \dots, H_1),$$

ordered lexicographically. Put $Y_{0,a} = Y_{n,a} = 1$ for every spectral parameter a in the SL_n character ring. For $1 \le j \le n$, the letter substitution is

$$\Lambda_{j,b} \longmapsto Y_{j,\varepsilon^{2b+j-1}} / Y_{j-1,\varepsilon^{2b+j}}.$$

For the skew tableau with outer shape λ_m, therefore, $H_i = A_i - B_{i+1}$: A_i counts entries i in boxes of content m mod e, and B_{i+1} counts entries $i+1$ in boxes of content $m-1$ mod e.

Every box counted by an A_i has a box immediately below it. In fact the boxes of a row without lower neighbors have contents $m-1, m-2, \dots, m-d$ modulo e, or an initial subinterval of this list, and none has content m. Removing an inner Young diagram preserves the assertion: if the lower box were removed, the upper one would also have been removed. Column strictness injects A-boxes with entry i into B-boxes with strictly larger entry. Hence $A_{n-1} \le B_n$. If equality holds, all the available B_n have already been used, so $A_{n-2} \le B_{n-1}$; continuing in this way proves $H \le 0$. The same conclusion holds for each monomial in a product, since the lexicographically nonpositive cone is additively closed.

By Lemma 9.1 and stable positivity, every constituent has a dominant highest monomial occurring with positive multiplicity in that product character. Its H_i are individually nonnegative. The preceding inequality therefore forces all $H_i = 0$. Under (9.1), $Y_{i,\varepsilon^{i+1}}$ is the highest letter of the segment $[1; i)$. Thus every constituent omits head 1 at every possible length below n. A length-n determinant cannot occur, because n exceeds the total degree. At every length at least one head is consequently missing, which is aperiodicity. □

The lower canonical basis of the composition algebra is exactly the aperiodic subset of the full lower IC basis [21, Section 1.8]. On evaluation-dual bases, π kills periodic labels and bijects the aperiodic labels with the composition upper basis; $*$ permutes that basis. The projection therefore retains every constituent of these products.

Proposition 9.4. *The map* $\mathcal{F}$ *has an integral based section on the ordinary source algebra for* $w = u_{R_0}^{-1}$. *It sends* $D(m, 0)$ *to* E_m *and* $D(m, m^-)$ *to* S_m.

Proof. The S_m are positive and have aperiodic support. Their images in (9.5) are single upper canonical elements with coefficient one. Since $\mathcal{F}$ keeps different aperiodic labels distinct, each S_m is itself a single full simple.

The source category $\mathcal{C}_w$ is the smallest monoidal Serre category generated by these determinantial root objects, with grading shifts [18, Definition 11.2.1]. Every one of its simple objects occurs in a finite product of the root objects. Let Σ be the set of labels occurring in such products. Exactness of tensoring makes Σ closed under constituents of products of its members: the product of two such constituents is a subquotient of the concatenated root product. Objects with constituents in Σ thus form a monoidal Serre category containing the generators, so minimality gives the assertion.

Lift a root product to the corresponding S-product. Its positive full support is aperiodic by Lemma 9.3. Each source simple in it therefore has one full simple lift, independent of the chosen product, because the aperiodic-basis bijection is global. Every monomial of that lift still has $H \le 0$, since it is a positive constituent of the product. Products of lifts again have entirely aperiodic support. Applying $\mathcal{F}$ then identifies every multiplication coefficient separately. This proves the based section and its integrality. The same argument applied to E_m and (9.5) identifies its lift. Each finite calculation is made in a rank larger than its total degree. □

9.4. **From the source to the bounded initial seed.** Write T_k for the based rank-k quotient of stable full classes, specialized at $v = 1$, described in (5.1) and Proposition 5.3. Put $P_p = T_k(\mathsf{E}_p)$ for $0 \le p \le N$, so $P_0 = 1$. The core determinant in the seed of Section 4.1 is exactly the specialization of E_{N+1-m}. Here are the indices in that identity. If $p = N + 1 - m$, antidiagonal transposition of the core minor, followed by reversing both orders, gives

$$x_m = \det_{1 \le u,v \le p} t_{\lambda'_{p,v} - v + u,\, u - p}, \tag{9.6}$$

where $\lambda'_{p,v}$ is column v of λ_p. To check this from the original core, write $b_v = 1 + \lfloor (v-1)/d \rfloor$. Its transposed entry has height $b_v - u + v$ and head $(k-1)e + 2 - v - b_v$; its fundamental index is $1 - u$ modulo e. Reversing the two orders gives (9.6). Heights 0 and k retain respectively the unit and the actual determinant.

The column path expansion of (9.6) is (9.2) for λ_p and $a = 1 - p$. Indeed an entry in row u and column v is a strict column whose last spectral index is $v - p - \lambda'_{p,v} + 1$, independent of u. Switching tails at a first intersection therefore preserves the letter weights and cancels intersecting path families in the determinant. The surviving nonintersecting families are precisely the semistandard tableaux, with index $1 - p + c - r$ at box (r, c). This argument is performed with unfolded integer indices before reducing modulo e, so repeated residues create no exception.

The full preferred label of this prefix is

$$\mathfrak{p}_p = \sum_{i=1}^{p} [1 - r_i; j_i), \qquad i = (j_i - 1)d + r_i, \quad 1 \le r_i \le d. \tag{9.7}$$

Set $\mathfrak{p}_p = 0$ for $p \le 0$. A highest tableau column of height j has head $c - p - j + 1$; reading the columns from the other end gives $1 - r_i$ modulo e, proving (9.7).

Proposition 9.5. *Every product*

$$\prod_{m=1}^{N} x_m^{c_m} \prod_h D_h^{t_h}, \qquad c_m \in \mathbb{N}, \quad t_h \in \mathbb{Z},$$

is the classical class of a single real simple object. In particular the initial variables are pairwise strongly commuting real simples.

Proof. In one source seed all $D(m, 0)$, including its frozen members, have simple mixed tensor products [18, Theorems 11.2.2–11.2.3]. At the source grading parameter 1, the based section therefore makes every product of the E_m one stable full simple with coefficient one. For $m \le N$, λ_m has at most $k - 1$ rows. The highest finite weight of any such product still has zero tail after row k. Proposition 5.3 therefore makes its based rank-k quotient nonzero and simple.

By (9.6), these retained classes are precisely the products of the x_m. In the actual bounded graded category, tensor multiplicities are nonnegative Laurent polynomials in q. Their value at $q = 1$ is zero for every unwanted label and one for the surviving label. Hence the former polynomials are zero and the latter is one grading monomial. The actual tensor product is simple up to shift. Applying the statement to doubled exponents proves reality. Based invertible determinant translation extends the assertion to all $t_h \in \mathbb{Z}$. □

10. The initial Poisson frame and ordinary neighbors

10.1. **A uniform compatibility calculation.** Let $\ell_{a,c}$ be the unit exponent vector of $X_{a,c}$, where $1 \le a \le k$ and $c \in \mathbb{Z}/e\mathbb{Z}$ is its cyclic color. Put

$$E(c, t) = \delta_{c,t} - \delta_{c+1,t}, \qquad C(c, t) = E(c, t) + E(t, c).$$

The Miura form in Theorem 5.2 is

$$\Omega(\ell_{a,c},\ell_{b,t}) = \begin{cases} E(c,t) - E(t,c), & a = b, \\ -C(c,t), & a < b, \\ C(c,t), & a > b. \end{cases} \tag{10.1}$$

All symbols are cyclic; in particular, both arrows are counted when $e = 2$. For a head h define $s_{j,h} = \sum_{a=1}^{j} \ell_{a,h+a-1}$. If $i = (j-1)d + r$, $1 \le r \le d$, set $w_i = s_{j,1-r}$ and $v_p = \sum_{i=1}^{p} w_i$. Set $s_{0,h} = 0$ and $w_i = 0$ for $i \le 0$, and use the same formula for w_i through $i = kd$. Empty prefixes are zero. Equation (9.7) says that v_p is the preferred exponent of the prefix P_p, and $x_m = P_{N+1-m}$. Pull back Ω along these exponent vectors to the initial frame, using the same symbol for the resulting form. For the frame basis vectors e_{x_m}, e_{D_h}, define the Hall-degree map by

$$\Gamma(e_{x_m}) = \dim \mathfrak{p}_{N+1-m}, \qquad \Gamma(e_{D_h}) = \dim[h;k).$$

Here the dimension vector of $[h;j)$ has one contribution at each of its colors $h, h+1, \dots, h+j-1$, counted with multiplicity. For the extended frame $\mathbf{x}$, write $\mathbf{x}^{\alpha} = \prod_i x_i^{\alpha_i}$.

Proposition 10.1. *In the actual initial frame $(x_1, \dots, x_N, D_0, \dots, D_{e-1})$, with the divided extended matrix $\widetilde{B}$ on the actual frame, namely the columns of Section 4.1 with the formal T row deleted and the zero D_1 row restored,*

$$\Omega\widetilde{B} = \begin{pmatrix} -2I_N \\ 0 \end{pmatrix}. \tag{10.2}$$

The formal root has a zero Poisson row, so reattaching its row gives the equivalent compatibility identity for the auxiliary monic seed. Its special column b satisfies

$$b = e_{x_2} - e_{x_{e+1}} - \sum_{h \ne 1} e_{D_h}, \qquad \Gamma b = -\delta, \qquad b_{D_1} = 0,$$

where out-of-range mutable coordinates are omitted, $\delta = (1, \dots, 1)$.

Proof. For two rows $a < b$ and one color c, subtraction in (10.1) gives

$$\Omega(\ell_{r,t}, \ell_{b,c} - \ell_{a,c}) = \begin{cases} -2E(t,c), & r = a, \\ -2C(t,c), & a < r < b, \\ -2E(c,t), & r = b, \\ 0, & r < a \text{ or } r > b. \end{cases} \tag{10.3}$$

We apply this local formula to each column.

For an ordinary direction m, put $p = N + 1 - m$. Using the two ordinary exchange monomials, the letter exponent of its divided column is

$$\beta_m = w_{p+e} + w_{p-e+1} - w_p - w_{p+1}. \tag{10.4}$$

For $m > e$ this follows by successive differences of prefixes. For $2 \le m \le e$, the term formerly beyond the prefix range is $w_{p+e} = s_{k,m} = D_m$, exactly the first-strip determinant factor in the exchange.

Write $p = (j-1)d + r$. If $r < d$, put $h = 1 - r$. Then

$$\beta_m = s_{j+1,h-1} + s_{j-1,h} - s_{j,h} - s_{j,h-1} = \ell_{j+1,h+j-1} - \ell_{j,h+j-1}.$$

Formula (10.3) vanishes against $s_{l,g}$ if $l < j$ or $l > j$. At $l = j$ it is $-2(\delta_{g,h} - \delta_{g,h-1})$. These two allowed heads occur exactly at increments p and $p+1$. Thus

$$\Omega(w_q, \beta_m) = -2(\delta_{q,p} - \delta_{q,p+1}).$$

If $r = d$, then $h = 2$ modulo e, and

$$\beta_m = s_{j+2,0} + s_{j-1,2} - s_{j,2} - s_{j+1,0} = \ell_{j+2,j+1} - \ell_{j,j+1}.$$

In (10.3) the only possible nonzero segment lengths are j and $j+1$: after the factor -2 their values are respectively $\delta_{g,2} - \delta_{g,1}$ and $\delta_{g,1} - \delta_{g,0}$. The increment list omits head 1, so

the same formula follows. At $e = 2$ the equal residues 2 and 0 occur at different lengths, and both contributions are retained. Summing increments gives $\Omega(v_s, \beta_m) = -2\delta_{s,p}$. Every determinant has length k, above the relevant cancellation threshold, so its pairing is zero.

For the special column, the last prefix block gives

$$\beta_1 = v_{N-1} - v_{N-e} - \sum_{h \neq 1} s_{k,h} = -\ell_{k-1,k} - \sum_{c \neq k} \ell_{k,c}.$$

The complete last-row holonomy $\sum_c \ell_{k,c}$ is in the radical of Ω. Modulo this radical, β_1 is therefore $\ell_{k,k} - \ell_{k-1,k}$. The adjacent-row calculation above has just one nonzero allowed increment: its pairing with w_N is -2. The positive endpoint at the omitted head 1 is absent. Thus $\Omega(v_s, \beta_1) = -2\delta_{s,N}$, and all determinant pairings are zero. The explicit expression for β_1 has one negative letter of every color, proving $\Gamma b = -\delta$ and the stated determinant entries.

All frame pairings used here are actual pairings: Proposition 9.5 gives their strong commutation, and the coefficient-one preferred words in Theorem 5.2 determine the commutation exponents. This completes the calculation of (10.2). □

10.2. **The first ordinary replacements.**

Proposition 10.2. *For each $2 \leq m \leq N$, the ordinary initial neighbor y_m is a real simple. Its arbitrary nonnegative powers and its products with the unchanged initial variables are single real simples.*

Proof. Take source length $R_0 = kd = N + d$. Its mutable positions are $1, \ldots, N-1$, and positions $N, \ldots, N+d$ are frozen. Set

$$Q = x_1, \qquad \kappa_r = \prod_{t=1}^{r} D_{1-t}, \qquad \kappa_0 = 1.$$

The exact rank-k images of the source initial variables are

$$T_k(\mathsf{E}_j) = x_{N+1-j} \ (j \leq N), \qquad T_k(\mathsf{E}_{N+r}) = Q\kappa_r \ (0 \leq r \leq d). \tag{10.5}$$

Indeed, λ_{N+r} has r forced full-height columns. Their heads are $1-r, \ldots, 0$. Removing them and shifting the remaining column indices cancels the change in evaluation parameter, leaving the tableau of Q.

The exchange at source position j is

$$\mathsf{E}_j \mathsf{E}'_j = \mathsf{E}_{j+e}\mathsf{E}_{j-1}\mathsf{E}_{j-d} + \mathsf{E}_{j+d}\mathsf{E}_{j+1}\mathsf{E}_{j-e}. \tag{10.6}$$

Nonpositive indices are replaced by 1. This follows directly from the determinantial seed for $i_j = 1 - j$: the equal-color successor and predecessor are $j+e, j-e$, while the two interlacing adjacent-color intervals have endpoints $j-1, j-d$ and $j+1, j+d$. There are no other intervals meeting $[j, j+e]$ with the required interlacing. At $e = 2$ the coincident adjacent positions contribute two factors. This is the ordinary source seed of [18, Sections 8.2 and 11.2].

Put $j = N+1-m$. If $m \leq e$, the two high indices crossing N in (10.6) give $Q\kappa_{e-m+1}$ and $Q\kappa_{e-m}$. Their ratio is D_m. Hence, if H_m denotes the rank-k image of the source partner,

$$x_m H_m = Q\kappa_{e-m}(D_m x_{m+1} x_{m+e-1} + x_{m-1} x_{m+e}).$$

If $m > e$, the same substitution gives

$$x_m H_m = x_{m-e} x_{m+1} x_{m+e-1} + x_{m-e+1} x_{m-1} x_{m+e}.$$

By the geometric ordinary identities, therefore,

$$H_m = \begin{cases} Q\kappa_{e-m} y_m, & 2 \leq m \leq \min(e, N), \\ y_m, & e < m \leq N. \end{cases} \tag{10.7}$$

The extra factor in the first strip is retained before any determinant-one specialization.

The based source section and based rank quotient send each once-mutated source monomial to one bounded simple or to zero. It is nonzero here: evaluate the displayed exchange at

positive Miura letters. Every initial image and the numerator are strictly positive. Localizing this nonvanishing denominator therefore defines the required specialization.

For arbitrary nonnegative a, b_i, the image of the corresponding once-mutated source monomial, using its frozen E_N for powers of Q, is consequently a simple H of the form

$$H = A\,C, \qquad C = y_m^a \prod_{i\neq m} x_i^{b_i}, \qquad A = (Q\kappa_{e-m})^a$$

in the first strip, and $A = 1$ otherwise. The factor A is already real by Proposition 9.5. Geometric regularity makes C a finite signed class. For every unchanged initial coordinate, the two terms in the ordinary exchange have equal logarithmic weights, by (10.2). Hence y_m, and then C, is a logarithmic eigenfunction against every factor of A. The saturated division statement, Corollary 8.3, shows that C itself is one simple. Applying this assertion to doubled exponents proves reality and all required compatibilities. □

11. The first higher-degree exchange

The functions F_s in this section are initially the geometric characteristic coefficients of the monodromy in (3.13). Put

$$Q = x_1, \quad U = x_2, \quad B = x_{e+1}, \quad D_* = \prod_{h\neq 1} D_h, \quad \Delta = \prod_h D_h, \quad V = BD_*,$$

where $x_j = 1$ if $j > N$. Define

$$t = U/V, \qquad P(t) = \sum_{s=0}^{k} F_s t^s, \qquad F_0 = 1, \quad F_k = \Delta, \qquad \phi = \frac{V^k}{Q} P(t). \tag{11.1}$$

The geometric identities and charts in Theorem 4.2 prove that ϕ is in R. We retain the two coefficient rings

$$A = \mathbb{C}[F_1, \dots, F_{k-1}, D_h^{\pm 1}], \qquad Z_c = \mathbb{C}[F_1, \dots, F_{k-1}, \Delta^{\pm 1}].$$

The initial chart is exactly $R[(x_1 \cdots x_N)^{-1}] = A[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$. The interior F_s remain polynomial coefficients.

11.1. **Holonomies and the joint integral kernel.** Let $M_a = \prod_{c\in\mathbb{Z}/e\mathbb{Z}} X_{a,c}$ be the ath row holonomy. In the Ore algebra with $\tau f_r = f_{r+1}\tau$, the fundamental Miura formula gives

$$\prod_{a=1}^{k} (1 - X_{a,r}\tau) = \sum_{j=0}^{k} (-1)^j a_{j,r} \tau^j.$$

A first-order factor has period multiplier M_a^{-1}. Successively taking the kernels of the ordered factors gives an invariant filtration of the solution space, so these are the period multipliers, counted with multiplicity. The vector monodromy is the inverse scalar companion monodromy, as shown in (3.13). Consequently

$$F_s = e_s(M_1, \dots, M_k), \qquad \Delta = M_1 \cdots M_k. \tag{11.2}$$

The cyclic row and column sums of E and C in (10.1) vanish. Every M_a is therefore Poisson central. In particular the F_s are Poisson central and have Hall degree $s\delta$.

Write $\mathbf{d}$ for the exponent vector having one at every determinant coordinate and zero at every mutable coordinate. The vector b is the special column in Proposition 10.1.

Lemma 11.1. *Over $\mathbb{Q}$, the solutions of*

$$\Omega a = 0, \qquad \Gamma a + w\delta = 0$$

are $(a, w) = \ell(\mathbf{d}, -k)$. More generally, the solutions of

$$\Omega a = c\, e_Q, \qquad \Gamma a + w\delta = 0 \tag{11.3}$$

are

$$(a, w) = u(b, 1) + \ell(\mathbf{d}, -k), \qquad c = -2u. \tag{11.4}$$

If a, w are integral, then u, ℓ are integral. These assertions include $N = 1$.

Proof. Transposing (10.2) gives $\widetilde{B}^t\Omega = (2I_N, 0)$. In the central case it forces all mutable components of a to vanish. Write $z = (z_h)$ for the remaining determinant components. Telescoping (10.1) along a length-k segment gives

$$\Omega(D_h, X_{a,j}) = \delta_{j,h-1} + \delta_{j,h+k-1} - \delta_{j,h} - \delta_{j,h+k}, \tag{11.5}$$

independent of the row a. The first $e-1$ length-one prefix increments span all color vectors except color 1. The degree of D_1 has a nonzero color-1 coordinate. Thus the Hall degrees of the full frame span $\mathbb{Q}^e$, and vanishing against that frame in (11.5) is vanishing against every color.

Let $(Sz)_j = z_{j+1}$. The determinant bracket and its Hall-degree operator are respectively

$$(S-1)(1+S^{-k})z, \qquad B_k z = (1 + S^{-1} + \cdots + S^{-(k-1)})z.$$

The degree equation says $B_k z = -w\delta$. Applying $1 - S^{-1}$ gives $(1 - S^{-k})z = 0$. On this subspace the bracket equation becomes $2(S-1)z = 0$. Hence z is a constant vector $\ell\delta$, and $w = -k\ell$.

For (11.3), skew-symmetry of the mutable principal block gives $2a_{\mathrm{mut}} = -c\, b_{\mathrm{mut}}$. Put $u = -c/2$. Then $a - ub$ has only determinant components, has zero bracket, and has Hall degree $(u-w)\delta$. The central case gives (11.4). This is also the exact integral lattice: on its two generators, the coordinates (a_{D_1}, w) form the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix},$$

whose determinant is -1. Explicitly, $\ell = a_{D_1}$ and $u = w + ka_{D_1}$. The displayed unimodular minor proves integrality for all $k, e \geq 2$. □

11.2. **The coefficient algebra is based.** For $0 \leq s \leq k$ put $\omega_s = (\underbrace{1, \ldots, 1}_{s}, \underbrace{0, \ldots, 0}_{k-s}) \in \mathbb{Z}^k$.

Proposition 11.2. *The algebra Z_c is spanned by a subset of the localized simple basis. For $0 \leq s \leq k$,*

$$F_s = C_{\sum_{h \in \mathbb{Z}/e\mathbb{Z}} [h;s)}, \tag{11.6}$$

where length-zero segments are omitted. Every coefficient simple strongly commutes with every real simple, with zero one-sided R degrees.

Proof. Let $f \in Z_c$ be homogeneous of Hall degree $r\delta$, and expand it as a finite signed linear combination of actual simple classes. Apply Lemma 8.2 to each initial mutable and determinant, all of which are real by Proposition 9.5. Every supported simple C separately Poisson-commutes with the whole initial frame and has the same Hall degree.

Expand C in the exact initial chart. Its independent monomials have the form $F^\nu \mathbf{x}^\alpha$, with $\nu_s \geq 0$ and integral frame exponents α. Each surviving monomial obeys

$$\Omega\alpha = 0, \qquad \Gamma\alpha + \left(\sum_s s\nu_s - r\right)\delta = 0.$$

Lemma 11.1 implies $\alpha = \ell\mathbf{d}$. Hence $C \in Z_c$. Decomposing by Hall degree proves basedness. This argument only removes forbidden monomials in the integral initial chart, so it is valid over $\mathbb{Z}$ as well.

By (11.2), Z_c is the symmetric Laurent ring in the independent variables $M_1, \ldots, M_k$. A nonzero homogeneous element has a largest exponent tuple $\mu_1 \geq \cdots \geq \mu_k$ with the prescribed total degree. For total degree s, $0 \leq s \leq k$, the lexicographically least dominant integer tuple is ω_s. Indeed, at a first smaller entry in position $j \leq s$, that entry and all later entries are nonpositive, making the total at most $j - 1 < s$. At a first smaller entry after s, all remaining entries are negative, again making the total less than s. Negative determinant powers do not change this argument.

The full Miura leading component of every localized simple is one word of coefficient one, and different labels have different such words (Theorem 5.2). Thus the largest row of a finite signed simple expansion cannot cancel. The function $F_s = e_s(M)$ has largest word M^{ω_s} with coefficient one. Every simple in its support lies in Z_c, so the preceding minimality leaves only that dominant tuple. Full-word injectivity then leaves just the label $\sum_h [h; s)$, with coefficient one. This proves (11.6).

Finally, every supported coefficient simple belongs to Z_c, whose generators are Poisson central by (11.2). It therefore Poisson-commutes with every element of $\operatorname{Frac} R$. Against any real simple it therefore satisfies the zero eigenrelation. The singleton case of Lemma 8.2 proves strong commutation and both zero degrees. Products of coefficient simples are positive and stay in the based algebra Z_c, but may have several constituents. □

The coefficient normalization is also compatible with the quantum Miura map. A coefficient simple has, at $v = 1$, only holonomy words. Positivity of the Miura coefficients excludes any other quantum word, and every holonomy word is central. Its normalized quantum lift is therefore central. For the elementary F_s, each classical coefficient is one; positivity and invariance under $v \mapsto v^{-1}$ make each quantum coefficient exactly one.

Remark 11.3. The Hall-degree equation in Lemma 11.1 is essential. For $e = 2$ and odd k, (11.5) vanishes for each individual determinant. In particular D_0/D_1 is Poisson central, but its Hall degree is not a multiple of δ and it does not belong to Z_c. The needed kernel is the joint Poisson and Hall-degree kernel.

11.3. **The leading constituent of a regular power.** Let $n = k - 1$. For a full label τ, write $\lambda(\tau)$ for its row partition and $m(\tau)$ for its complete preferred word. For a simple class Z, write λ_Z for the row partition of its label. The unit class has row partition zero. The prefix labels satisfy

$$\begin{aligned} \mathfrak{p}_N - \mathfrak{p}_{N-1} &= [2; n), \\ \mathfrak{p}_N - \mathfrak{p}_{N-e} &= \sum_{h \neq 1} [h; n) + [2; n-1), \end{aligned}$$

with empty prefixes and length-zero segments omitted. These equalities follow by reading the final e increments in (9.7); they hold also when $B = 1$. Consequently

$$\begin{aligned} \lambda_Q &= d(n, n-1, \dots, 1, 0), \\ \lambda_U &= \lambda_Q - \omega_n, \\ \lambda_V &= \lambda_Q - d\omega_n - \omega_{n-1} + d\omega_k, \\ \gamma := \lambda_U - \lambda_V &= (0, \dots, 0, -1, -d). \end{aligned} \tag{11.7}$$

Lemma 11.4. *For every $r \geq 1$, the finite signed simple expansion of ϕ^r contains*

$$H_r = C_{r\tau_H}, \qquad \tau_H = \sum_{h \neq 1} [h; k) + n\mathfrak{p}_{N-1} - [2; n-1) + [1; n),$$

with coefficient one. This is its unique constituent at the largest row. Moreover

$$H_r = (V^k/Q)^r f_r(t), \qquad f_r(t) \in Z_c[t^{\pm 1}], \tag{11.8}$$

where the coefficient of t^m has weighted degree m, for $\deg F_s = s$, $\deg \Delta = k$ and $\deg t = -1$.

Proof. The row of branch s in the numerator of (11.1) is

$$T_s = e\omega_s + s\lambda_U + (k - s)\lambda_V.$$

Its difference from T_n is $e(\omega_s - \omega_n) + (s - n)\gamma$. For $s \leq n - 2$ the first nonzero entry is $-e$; for $s = n - 1$ it is $-d$ in row n; and for $s = k$ it is -1 in row n. Thus $s = n$ is uniquely largest. The quotient of its full word by $m(Q)$ is precisely $m(\tau_H)$, with coefficient one. The displayed label is nonnegative: when $n \geq 2$, the subtracted segment already occurs in $\mathfrak{p}_{N-1}$; when $n = 1$ it is absent.

Row degree is an ordered-group degree on the Miura Laurent ring and extends to its fraction field. Its leading components multiply. Therefore ϕ^r has the single leading word $m(r\tau_H)$, with coefficient one. The largest row in a finite signed simple expansion cannot cancel. For localized labels this last fact can be reduced literally to the nonlocalized statement: multiply the whole finite expansion by one common power Δ^L clearing all negative determinant multiplicities. Based determinant translation shifts every row and full word by the same monomial. Theorem 5.2 then applies. This proves the assertion about H_r.

Let a_0 be the initial-frame exponent vector of V^k/Q. Since $\Omega b = -2e_Q$, the function $t = U/V$ commutes with every initial coordinate except Q. Thus, for every frame index $i \neq Q$,

$$\{x_i, \phi^r\} = r\Omega(e_i, a_0)x_i\phi^r.$$

Apply signed support with the already real factor x_i. The constituent H_r inherits exactly those eigenvalues. It has Hall degree $r\Gamma a_0$, because $\Gamma b = -\delta$ and every $F_s t^s$ has degree zero.

For a monomial $F^\nu \mathbf{x}^\alpha$ in the initial-chart expansion of H_r, put $a = \alpha - ra_0$, $w = \sum_s s\nu_s$. Its equations are (11.3). The integral conclusion (11.4) then gives $\mathbf{x}^\alpha = \mathbf{x}^{ra_0}t^u\Delta^\ell$ and $w = u - k\ell$. Grouping equal powers of t proves (11.8), including its coefficient degrees. All coefficients remain polynomial in the interior F_s. □

11.4. **Division in the polynomial coefficient ring.**

Lemma 11.5. *For every $r \geq 1$ there is $g_r \in Z_c[t^{\pm 1}]$ of total weighted degree zero such that*

$$H_r = \phi^r g_r(t). \tag{11.9}$$

If $N > 1$, then $g_r \in Z_c[t]$.

Proof. Let $S = Z_c[t^{\pm 1}]$ and $L = A[x_i^{\pm 1} : i \neq 1]$. If $N > 1$, the changes $U = tBD_*$ and $D_1 = \Delta D_*^{-1}$ express L as a Laurent-polynomial extension of S in independent extra variables. If $N = 1$, necessarily $k = e = 2$, and $U = B = 1$, $t = D_0^{-1}$; then $L = S$ via $D_0 = t^{-1}$, $D_1 = \Delta t$. In both cases

$$L \cap \mathrm{Frac}(S) = S. \tag{11.10}$$

Indeed, compare coefficients of the extra independent Laurent variables after extending to $\mathrm{Frac}(S)$. An element independent of those variables has only its constant coefficient, which is in S.

The exact special-neighbor chart is $L[\phi^{\pm 1}]$, and ϕ is algebraically independent over $\mathrm{Frac}(L)$. Rewriting (11.8) gives $H_r = \phi^r f_r/P(t)^r$. Since H_r is regular, uniqueness of its Laurent expansion in ϕ forces $f_r/P(t)^r$ to belong to L. It also belongs to $\mathrm{Frac}(S)$, so (11.10) places it in S. This proves (11.9).

Suppose $N > 1$. In the ordinary $U = x_2$ neighbor chart the exact exchange is

$$Uy_2 = D_2x_3x_{e+1} + Qx_{e+2} = b + aQ. \tag{11.11}$$

Put

$$S_0 = A[x_j^{\pm 1} : j \neq 1, 2][y_2^{\pm 1}], \qquad S_U = S_0[Q^{\pm 1}].$$

The two elements $a = x_{e+2}$ and $b = D_2x_3x_{e+1}$ are units of S_0, with out-of-range variables equal to 1. Hence

$$S_U/(aQ + b) \simeq S_0, \qquad Q = -b/a.$$

The principal ideal is a height-one prime of the Laurent-polynomial UFD S_U, and its contraction to A is zero. Its discrete valuation ν therefore satisfies

$$\nu(c) = 0 \quad (0 \neq c \in Z_c), \qquad \nu(U) = 1, \quad \nu(Q) = \nu(V) = 0.$$

Modulo this prime the special exchange becomes $Q\phi = V^k$, so $\nu(\phi) = 0$. If $g_r = \sum_{m=p}^{q} c_m t^m$ with $c_p \neq 0$, the term of degree p has uniquely least valuation p. Since the actual class H_r is regular in this neighbor chart, (11.9) gives $p = \nu(H_r) \geq 0$. □

11.5. **Eliminating the multiplier.**

Theorem 11.6. *For every integer $r \geq 1$,*

$$\phi^r = C_{r\tau_H}. \tag{11.12}$$

In particular the first special replacement is a real simple, strongly commuting with every unchanged initial variable.

Proof. Subtracting the preferred words in the prefix identities gives

$$m_t = X_{k-1,k}^{-1} \prod_{c \neq k} X_{k,c}^{-1}. \tag{11.13}$$

There is exactly one inverse letter of each cyclic color: color k is in row $k-1$, and the others are in row k. This formula remains valid when colors repeat in long segments.

Write $g_r = \sum_m c_m t^m$. For $c_m \neq 0$, let $\mu(m)$ be its largest exponent tuple in the M_a. It is dominant and has sum m. The leading word of that summand is a nonzero scalar times $M^{\mu(m)} m_t^m$, with row $e\mu(m) + m\gamma$. The map

$$(m, \mu) \longmapsto M^{\mu} m_t^m \tag{11.14}$$

is injective. The difference between color k and any other color in row $k-1$ recovers $-m$; another color exists because $e \geq 2$. The remaining row-constant exponents recover every μ_a. Thus the words at the largest row among the summands of g_r cannot cancel, even when its coefficients have signs.

This comparison is valid for rational expressions as well: choose one L clearing all negative determinant powers in the finitely many c_m, and choose $p, q \geq 0$ so that $p + m, q - m \geq 0$ for every occurring m. Multiplication by $\Delta^L U^p V^q$ turns all summands into finite Miura Laurent polynomials. Its leading component is a single word, so it makes one common translation of all the words in (11.14). Cancelling that translation proves the stated noncancellation.

For $N > 1$, Lemma 11.5 gives $m \geq 0$. If $m > 0$, dominance implies $\mu_1 \geq \lceil m/k \rceil > 0$. For $k \geq 3$ the first row of $e\mu + m\gamma$ is $e\mu_1 > 0$. For $k = 2$, $N > 1$ implies $e > 2$, and the first row is

$$e\mu_1 - m \geq e\lceil m/2 \rceil - m > 0.$$

If $m = 0$ and c_0 is nonconstant, its largest dominant tuple of sum zero has $\mu_1 > 0$; otherwise all entries would be zero and symmetry would leave only a constant. Consequently any nonconstant g_r has strictly positive largest row. Equation (11.9) would then give H_r a row strictly larger than that of ϕ^r, contrary to Lemma 11.4. Thus g_r is a scalar, and its coefficient-one leading word forces $g_r = 1$.

It remains to consider $N = 1$, or $k = e = 2$. Negative m cannot be excluded by a nonexistent ordinary neighbor. For a dominant tuple of sum m, however, the first row of its multiplier is $2\mu_1 - m = \mu_1 - \mu_2 \geq 0$. Noncancellation and equality of the prescribed largest rows force $\mu_1 = \mu_2 = m/2$ for every surviving coefficient. Symmetry and total degree then give

$$c_{2j} = a_j \Delta^j, \qquad g_r = \sum_j a_j (\Delta t^2)^j = \sum_j a_j (D_1/D_0)^j.$$

These terms have row zero, but their complete words are distinct, because

$$D_1/D_0 = \frac{X_{1,1} X_{2,0}}{X_{1,0} X_{2,1}}.$$

Multiplication by the sole top word of ϕ^r gives one distinct top word for each nonzero a_j. The simple H_r has exactly the same single top word as ϕ^r, with coefficient one. Hence $a_0 = 1$ and all other $a_j = 0$. This proves (11.12) in every case.

The case $r = 1$ now makes ϕ an actual simple. For $r = 2$, positivity of the graded composition multiplicities and their coefficient-one specialization imply that its tensor square is simple. Thus it is real. Finally the logarithmic eigenrelations against all unchanged initial real objects, already used in Lemma 11.4, and singleton signed support give the required strong commutations. □

12. Propagation through every mutation

12.1. **Weighted mutation and the determinant endpoint.** We first specify the exponent calculations used in the proof. In this section $\widetilde{B}$ denotes the augmented monic matrix of Section 4.1, with its formal root row and, when convenient, the redundant zero D_1 row. There is one vertex of degree k; all other mutable degrees are 1. Write d_i for these degrees and $\widetilde{B} = (b_{ij})$ for the divided extended matrix, so its mutable principal block is skew-symmetric. For the current extended seed $\mathbf{x} = (x_j)_j$, put

$$U_i = \prod_j x_j^{[b_{ji}]_+}, \qquad V_i = \prod_j x_j^{[-b_{ji}]_+}, \qquad \mathbf{x}^u = \prod_j x_j^{u_j}.$$

These products include the retained determinant and formal-root rows. Mutating at the vertex x, with $p = d_x$, changes its entries by

$$b'_{ij} = \begin{cases} -b_{ij}, & i = x \text{ or } j = x, \\ b_{ij} + p([b_{ix}]_+[b_{xj}]_+ - [-b_{ix}]_+[-b_{xj}]_+), & i, j \neq x. \end{cases} \tag{12.1}$$

This includes every frozen row. If E is the identity on the extended row lattice except $E_{xx} = -1$ and $E_{ix} = p[b_{ix}]_+$ for $i \neq x$, and F is the transpose of its mutable principal block, then

$$\widetilde{B}' = E\widetilde{B}F, \qquad \Omega' = E^t\Omega E. \tag{12.2}$$

Indeed, multiplication gives (12.1). Also $E^2 = F^2 = I$ and $E^t(I_N, 0)^t = (I_N, 0)^t F$, so (10.2) is preserved. Each summand defining the new variable has the same bracket against every unchanged coordinate; the difference of two successive exponent vectors is the old column, whose bracket is supported only at x. Thus (12.2) is the actual classical Poisson mutation formula.

The physical determinant endpoint is $F_k = \Delta$. For uniform monic exponent calculations, adjoin a formal central T with $T^k = \Delta$ and put

$$G_s = T^{-s}F_s, \qquad G_0 = G_k = 1.$$

In the initial special column replace U by TU, or equivalently append a frozen T row of entry 1 in that column and 0 elsewhere. Then

$$\sum_{s=0}^{k} F_s U^s V^{k-s} = \sum_{s=0}^{k} G_s (TU)^s V^{k-s}.$$

The coefficient string reverses on special mutation. The row and column of T in Ω are zero. The single relation $ke_T - \mathbf{d}$ is central and fixed under every change of frame, so all formulas descend from the free exponent lattice with this relation.

Lemma 12.1. *Suppose the current seed variables are pairwise strongly commuting real simples. Every complete exchange branch and every common factor used below is an object of the original determinant-localized category, with integral determinant twists.*

Proof. Give the original ring degree 0 in $\mathbb{Z}/k\mathbb{Z}$ and T degree 1. Let $\sigma = 1$ for the string (G_s) and $\sigma = -1$ for the reversed string (G_{k-s}). At every seed,

$$b_{T,i} \equiv 0 \pmod{k} \quad (d_i = 1), \qquad b_{T,y} \equiv \sigma \pmod{k} \quad (d_y = k). \tag{12.3}$$

This is true initially. An ordinary mutation has $b_{T,x}$ divisible by k, so all its updates to the T row are divisible by k. A special mutation has multiplier k in (12.1), while its own entry changes sign and its string reverses. This proves (12.3) inductively.

For a special branch, its monomial T degree is $s[b_{T,y}]_+ + (k-s)[-b_{T,y}]_+ \equiv sb_{T,y}$. It cancels the degree $-s$ of G_s, or the degree s of G_{k-s} in the reversed string. Thus the whole branch is an integral power of Δ, an elementary coefficient F_s or F_{k-s}, and an original compatible seed monomial. An ordinary branch has all its T powers divisible by k directly. The real monomial and the coefficient simple strongly commute by Proposition 11.2, so each whole

branch is an actual simple. Partial monomials containing a nonintegral power of Δ are used only as exponent vectors.

We also verify the particular greatest common divisor needed below. Suppose $b = b_{xy} > 0$, $p = d_x$, $q = d_y$. Its T exponent is

$$q \min\bigl(pb[b_{T,x}]_+ + [b_{T,y}]_+,\ [-b_{T,y}]_+\bigr).$$

If $q = k$, this is divisible by k. If $q = 1$, then $b_{T,y}$ is divisible by k, and so is $pb[b_{T,x}]_+$: either $p = k$, or x is ordinary and (12.3) applies. The exponent is again divisible by k. No coefficient generator enters this greatest common divisor. Its other exponents are nonnegative exponents of unchanged seed factors. It is consequently an actual real monomial with integral determinant twists. □

Lemma 8.4 applies to the resulting real monomials and their integral determinant twists.

12.2. **A filtered-map lemma.** The following assertion separates the order of a meromorphic map from the filtration of its image. Both are needed in mutation.

Lemma 12.2. *Let $\mathcal{R}(z) : V(z) \to W(z)$ be a nonzero homogeneous meromorphic map between the fixed polynomial families of the actual R construction. Suppose it preserves fixed finite filtrations, all of whose successive quotients are flat. For the special fibers assume*

$$\operatorname{supp}(V_s/V_{s-1}) \cap \operatorname{supp}(W_t/W_{t-1}) = \varnothing \qquad (t < s),$$

where support means simple labels with all grading shifts forgotten. If the induced nonzero layer maps have orders m_s, then the whole map has order $\min_s m_s$. If all m_s are equal, its normalized specialization has an image filtration whose successive quotients are exactly the images of the normalized layer maps.

Proof. Let a be the whole order. Fixed-quotient flatness and saturation show that $z^{-a}\mathcal{R}$ is regular on every layer, so $a \le m_s$. If $a < \min_s m_s$, its nonzero normalized specialization f has every associated graded map zero. Thus $f(V_s) \subseteq W_{s-1}$. Induct on s. It is zero on the first layer. If it is zero on V_{s-1}, its restriction to V_s factors as

$$V_s/V_{s-1} \longrightarrow W_{s-1}.$$

A nonzero image would have a simple constituent in both supports, which the hypothesis excludes. Hence $f(V_s) = 0$, and induction contradicts $f \neq 0$. This proves the order formula.

If all orders agree, the induced maps of the normalized specialization are precisely the nonzero normalized layer maps. At stage s, the snake-lemma obstruction to exactness of the image sequence maps from the kernel of the new layer map to the earlier cokernel. Their supports lie respectively in the new source layer and the earlier target layers, so this map vanishes. Therefore

$$0 \longrightarrow \operatorname{im} f_{\le s-1} \longrightarrow \operatorname{im} f_{\le s} \longrightarrow \operatorname{im} f_s \longrightarrow 0$$

is exact. Induction proves the image-filtration assertion. Disjoint ungraded supports exclude these Hom spaces in every degree. □

For applications in the determinant-localized category, first choose common positive determinant translations of all finitely many filtration objects. Naturality and full faithfulness of determinant translation preserve kernels, images and supports. The determinant correction to the raw order depends only on the stationary Hall degree, which is common to all layers. Thus the order comparison and the resulting image filtration transport back unchanged. This uses only positive polynomial families and actual determinant objects.

12.3. **Updating a neighboring replacement.** We maintain the following induction data: every variable in the current seed is real, the seed is pairwise strongly commuting, and each one-step replacement is real and strongly commutes with the other variables of that seed. Propositions 9.5 and 10.2, together with Theorem 11.6, establish these data initially.

Mutate x to X, and update the replacement of another variable y. Assume first $b = b_{xy} > 0$, and put $p = d_x$, $q = d_y$. At most one of p, q is greater than 1. Use the current monic notation and write the old exchange at y as

$$yy' = \sum_{s=0}^{q} E_s, \qquad E_s = F_s^y U_y^s V_y^{q-s}, \qquad U_y = x^b P. \tag{12.4}$$

Here P, V_y contain neither x nor y, and F_s^y denotes either the appropriate formal G string or the ordinary endpoint string $(1,1)$. Each whole E_s is rewritten as an actual object by Lemma 12.1. It is simple. The layers are distinct: interior coefficients are independent, while equality of the two endpoint monomials would make their exponent difference zero modulo a central relation, contradicting $\Omega\widetilde{B} = (-2I_N, 0)^t$. Thus (12.4) is the ungraded composition series of yy', with each label occurring once.

Let u, v be the positive and negative parts of the divided column at x. We claim that

$$C = \mathrm{hd}(Xx) = U_x^p, \qquad U_x = \mathbf{x}^u. \tag{12.5}$$

The simple object X strongly commutes with every common seed factor. Its bracket against these factors is given by the formal exponent $\gamma_x = -e_x + pu$. The successive simple branches in its old exchange have $\Lambda(X, -)$ differing by

$$\Omega(-e_x + pu, u - v) = 2,$$

because $\Omega(e_x, b_x) = -2$ and $u_x = 0$. The strict head extremum in Theorem 8.1 therefore selects the positive endpoint U_x^p. This proves (12.5). It also gives $\Lambda(X, x) = \Lambda(X, C) = \Omega(\gamma_x, e_x)$. Together with the common-variable relations, the same linear functional $\Omega(\gamma_x, -)$ therefore gives the interaction with every factor of the old frame, including x.

Put $L = X^{qb}$. Simple-moving factorization of actual normalized R maps, as in (7.11), applies to arbitrary stationary products. Since $b_{yx} = -b$, the y component of u is zero, and off-diagonal compatibility gives

$$\Lambda(X, U_y) - \Lambda(X, V_y) = \Omega(-e_x + pu, b_y) = 0. \tag{12.6}$$

Coefficient simples have zero interaction with real objects. If a partial monomial in this calculation contains T, its displayed Λ means the linear exponent functional with $\Lambda(X, T) = 0$. For each complete branch, Lemma 12.1 replaces this formal expression by an actual F_j, an integral Δ power and original seed factors. The first two have zero interaction with X, so the formal calculation is exactly the actual degree of that complete branch. All the $\Lambda(L, E_s)$ are therefore equal. All E_s also have the same Hall degree, namely $\deg y + \deg y'$. Thus their raw R maps against L have the same normalizing order.

We next prove disjointness of the *full* simple supports of different products LE_s. If y is special, then x is ordinary and $Xx = C + D$ has two compatible monomial branches. In the classical ring,

$$LE_s = F_s^y X^{(k-s)b} (C + D)^{sb} P^s V_y^{k-s}. \tag{12.7}$$

Expand the binomial. Each complete summand, after its integral determinant rewriting, is one coefficient simple times a compatible real monomial of the new seed, and is therefore a simple. Positivity makes these exactly the full constituents. Their characters have exact Laurent X exponent $(k-s)b$. Different s have disjoint supports by independence of X in the new chart.

If y is ordinary, the two products are $X^b V_y$, of exact X exponent b, and $(Xx)^b P$, of exact X exponent 0. Expand the old x exchange in the latter. If x is special, decompose the resulting positive coefficient products in the based algebra Z_c. Every constituent is again a coefficient simple times a common-variable compatible real monomial. Thus the entire two

supports are disjoint. This proof also covers ordinary x. Reversed products have the same ungraded supports, because their positive graded multiplicities have the same specialization at $v = 1$.

Take any graded composition filtration of yy'. Tensoring it on the left and right by L gives the fixed source and target filtrations of the raw R map. Their layer supports have just been proved disjoint. Equation (12.6) and common Hall degree make all layer orders equal. Lemma 12.2 first identifies the whole normalizing order and then the image filtration.

The layer images are

$$T_s = F_s^y X^{(q-s)b} C^{sb} P^s V_y^{q-s}. \tag{12.8}$$

To construct the required quotient, start with $Xx \twoheadrightarrow C$. Recursively,

$$X^m x^m = X^{m-1}(Xx)x^{m-1} \twoheadrightarrow X^{m-1}Cx^{m-1} \simeq CX^{m-1}x^{m-1} \twoheadrightarrow C^m.$$

The displayed isomorphism moves the compatible factor C. Tensor with the remaining compatible factors to obtain a simple quotient of LE_s equal to (12.8). Since L is real and E_s is simple, the real-factor theorem identifies it with the layer R image.

Factorization through the isomorphism $r_{L,y}$ now identifies the whole image with yH, where $H = \mathrm{hd}(Ly')$ is an actual simple. Taking classes of its image filtration gives

$$yH = \sum_{s=0}^{q} T_s. \tag{12.9}$$

12.4. **Division and reality of the new replacement.** Let ϕ_y be the replacement of y in the new seed. Define the formal common monomial and its actual power by

$$A_1 = \gcd(C^b P, V_y), \qquad A = A_1^q, \qquad U_{\mathrm{new}} = C^b P/A_1, \qquad V_{\mathrm{new}} = X^b V_y/A_1.$$

These are the two new column monomials at y. Indeed, for each unchanged row put $a = pb[b_{ix}]_+$ and $t = b_{iy}$. Subtracting $g = \min(a + [t]_+, [-t]_+)$ from the two exponents gives

$$a + [t]_+ - g = [a + t]_+, \qquad [-t]_+ - g = [-a - t]_+,$$

which is exactly (12.1) when $b_{xy} > 0$. At the X row the negative monomial has exponent b. Thus (12.9) and the new exchange give

$$yH = A \sum_{s=0}^{q} F_s^y U_{\mathrm{new}}^s V_{\mathrm{new}}^{q-s} = A\, y\phi_y.$$

Cancellation in the classical domain yields

$$H = A\phi_y. \tag{12.10}$$

Lemma 12.1 proves that A is an actual real monomial, with no y factor, even when A_1 itself contains a formal root.

The function ϕ_y is regular by the all-word chart theorem. The formal Poisson rule in the new seed makes $A\phi_y$ a logarithmic eigenfunction against X. Since H is already an actual simple, singleton signed support proves that H strongly commutes with X. Apply Lemma 8.5 with $R = X$, $N = y'$, and $r = qb$. Both X and y' are real, and $H = \mathrm{hd}(X^{qb}y')$ strongly commutes with X, so H is real.

Since A uses unchanged new-seed factors, formal Poisson compatibility gives $\{A, \phi_y\} = \lambda A\phi_y$. The class ϕ_y is a finite signed regular class. Apply saturated division to (12.10): ϕ_y is one simple B, and A strongly commutes with B. Singleton signed support against each other unchanged new-seed variable gives every required compatibility of B. Finally, up to shift,

$$H^2 = (AB)^2 \simeq A^2 B^2.$$

If B^2 had at least two composition factors, exact tensoring by the nonzero A^2 would retain at least two nonzero successive quotients, contradicting the simplicity of H^2. Thus B is real.

If $b_{xy} < 0$, use the opposite tensor product, negate both the divided matrix and the Poisson form, and interchange the positive and negative column monomials. The special string is reversed at the same time, so the exchange polynomial is unchanged. The preceding proof

applies with $-b_{xy} > 0$ and gives the same actual replacement. If $b_{xy} = 0$, the old y' is the new replacement as a function; formal Poisson compatibility and singleton signed support give its compatibility with X. At the mutated vertex itself the new replacement is the old x, already real and compatible with all other seed members. All induction data are therefore preserved.

Theorem 12.3. *For every $k, e \geq 2$ and every finite mutation word in the retained generalized seed, each mutable variable is the class of a real simple in the determinant-localized bounded category. Any nonnegative product of mutable variables in one seed, multiplied by arbitrary integral determinant powers, is one simple class. In particular, every mutable cluster monomial belongs to the full localized dual IC basis at parameter* 1.

Proof. The initial induction data were proved above. The update constructs every new partner from those data in the order H simple, H strongly commuting with X, H real, B simple, B real. It applies to each adjacent direction, so induction on the word length gives every seed, including words with repeated vertices. The closure of compatible real products then gives the monomial statement. Based determinant translation gives its integral determinant twists. The support separation takes place in the full basis with determinants retained. □

The exact quantum exchange phases follow from the simple objects just constructed. Let Z_s be the distinct normalized simple branches of an exchange $yB = \sum_s Z_s$ at parameter 1. Positivity and the anti-involution give

$$\widehat{y}\,\widehat{B} = \sum_s v^{a_s}\,\widehat{Z}_s, \qquad \widehat{B}\,\widehat{y} = \sum_s v^{-a_s}\,\widehat{Z}_s.$$

The already real y strongly commutes with each branch, say $\widehat{y}\,\widehat{Z}_s = v^{\lambda_s}\,\widehat{Z}_s\,\widehat{y}$. Associativity of $y(By) = (yB)y$, followed by cancellation and basis independence, gives $2a_s = \lambda_s$ for each s. These are precisely the Weyl monomial phases from Ω. Thus the formal quantum exchange agrees with the actual normalized simple.

13. The restricted target and the original Gleitz seed

13.1. **The based restriction to SL_k.** We first identify the target of the preceding construction. The polynomial GL_k form used in the full character comparison is the restricted form of Frenkel–Mukhin. It is distinct from the double-Hall Hopf form used to construct the tensor category. Only a based ring comparison with the former is needed here.

Write $t_{j,n} = [V_{\omega_j}(\varepsilon^{2n})]$ and put $t_{0,n} = 1$. The generator coproduct in the stable polynomial ring is

$$\Delta(t_{j,n}) = \sum_{a=0}^{j} t_{a,n} \otimes t_{j-a,n-a}.$$

Thus the coordinate of the segment $[h; j)$ is $a_{j,h} = t_{j,h+j-1}$: substituting this index gives exactly the head and quotient convention of the cyclic Hall coproduct. Frenkel–Mukhin's fundamental restriction formula then gives

$$a_{j,h} \longmapsto [L(Y_{j,\varepsilon^{\,2h+j-1}})] \quad (1 \leq j < k), \qquad D_h = a_{k,h} \longmapsto 1. \tag{13.1}$$

This calibration fixes both the spectral shift and the determinant specialization.

Proposition 13.1. *There is a based ring isomorphism*

$$\mathbb{Z}[a_{j,h} : 1 \leq j \leq k,\ h \in \mathbb{Z}/e\mathbb{Z}][D_h^{-1} : h \in \mathbb{Z}/e\mathbb{Z}]/(D_h - 1 : h \in \mathbb{Z}/e\mathbb{Z}) \ \simeq\ K_0(\mathcal{C}_{k,e}).$$

Every localized full simple class maps to one restricted SL_k simple class. Two such classes have the same image exactly when their labels differ by integer determinant twists.

Proof. The fundamental-generator theorem and the full character pairing give the polynomial ring on the left before localization [11, Sections 8.2–8.4 and 11.1]. Proposition 5.3 identifies the full IC basis with the simple basis of this ring.

We recall why restriction preserves simplicity, including even root orders. Put

$$A = U_\varepsilon^{\mathrm{res}}(L\mathfrak{gl}_k), \qquad B = U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_k),$$

where B denotes its image in A. For a polynomial simple $V = L_{\mathrm{GL}}(Q_1, \dots, Q_k)$, let v be its highest vector and $\lambda_i = \deg Q_i$. The integral triangular decomposition [11, Section 7.2, Lemmas 7.4 and 8.1] gives

$$A = A^- A^0 A^+, \qquad A^- = B^-, \qquad V = A^- v.$$

All nonconstant negative divided Drinfeld generators lower the finite integer weight by a positive multiple of a simple root. Consequently

$$\mathrm{wt}(V) \subseteq \lambda - \sum_i \mathbb{N}\alpha_i, \qquad V_\lambda = \mathbb{C}v.$$

These are integer weights: the restricted torus retains its divided binomials. More explicitly, on a weight μ, the operators $k_i = t_i/t_{i+1}$ and $\left[\begin{smallmatrix} k_i;0 \\ e \end{smallmatrix}\right]$ have eigenvalues

$$\left(\varepsilon^{\mu_i - \mu_{i+1}}, \begin{bmatrix} \mu_i - \mu_{i+1} \\ e \end{bmatrix}_\varepsilon\right).$$

This pair distinguishes every integer difference, also for negative integers and even root order [12, Corollary 4.3]. Since all weights have the same total polynomial degree, their differences determine the entire GL_k weight. Interpolation in these commuting operators therefore shows that any B-submodule is a sum of intersections with the GL_k weight spaces.

Let $0 \neq W \subseteq V$ be a B-submodule and choose a nonzero vector w of maximal weight μ in W. Every positive divided Drinfeld generator kills w. The full Cartan A^0 preserves finite weight, so the triangular decomposition implies that Aw has only weights at most μ. If $\mu \neq \lambda$, it misses the highest line, contradicting the simplicity of V. Hence W contains v, and $B^- v = A^- v = V$, proving simplicity on restriction.

The highest vector has restricted Drinfeld polynomials

$$P_i(u) = \frac{Q_i(\varepsilon^{i-1}u)}{Q_{i+1}(\varepsilon^{i+1}u)} \qquad (1 \le i < k). \tag{13.2}$$

This is also the full character restriction in [11, Section 8.2]; multiplication of characters makes restriction a ring map.

For the fibers, suppose Q and Q' give the same tuple P and put $R_i = Q'_i/Q_i$. Equation (13.2) gives

$$R_i(u) = R_{i+1}(\varepsilon^2 u) = R_k(\varepsilon^{2(k-i)}u).$$

All zeros and poles of R_k lie on the chosen finite orbit. Thus

$$R_k(u) = \prod_{h \bmod e} (1 - \varepsilon^{2h}u)^{m_h}, \qquad m_h \in \mathbb{Z}.$$

The tuple for D_h is $Q_i(u) = 1 - \varepsilon^{2(h+k-i)}u$, so these ratios are exactly the integer determinant twists. Conversely every such twist is trivial on restriction. Every allowed P-tuple lifts by setting $Q_k = 1$ and solving (13.2) successively.

Finally the fundamental SL_k classes in (13.1) are algebraically independent and generate the integral Grothendieck ring [10, Section 3.1]. Generation follows from the unitriangular highest-monomial argument for standard fundamental products. In a proposed polynomial relation, choose a maximal ordinary highest weight; distinct standard monomials of that weight have distinct highest loop monomials, and lower weights cannot cancel them. This proves independence. The displayed kernel is therefore exactly the ideal of the individual $D_h - 1$, completing the ring and basis assertions. □

The full characters of the coefficient simples are

$$\chi_\varepsilon(F_s) = e_s(M_1, \dots, M_k), \qquad M_a = \mathsf{Y}_{k+1-a}/\mathsf{Y}_{k-a}, \qquad \mathsf{Y}_i = \prod_{h \bmod e} Y_{i,\varepsilon^{2h+i-1}}, \tag{13.3}$$

where $\mathsf{Y}_0 = \mathsf{Y}_k = 1$. The reversed row order agrees with $X_{a,c} = \Lambda_{k+1-a,c}$; elementary symmetric functions are unchanged by this reversal. The quantum Frobenius character formula [10, Theorems 5.4 and 5.7, Lemmas 5.8–5.9] identifies this with the fundamental Frobenius simple of highest monomial Y_s. Formula (13.3) also follows from the monodromy and Miura calculation of the first exceptional mutation. In even period the bipartite spectral shift in the Frobenius formula is retained; its full orbit is precisely Y_i, and no sign changes occur in the elementary symmetric character. If e is odd and ε has order $2e$, then $\varepsilon^{e^2} = -1$. The bipartite support $\psi(i) = (-1)^{i-1}$ in [10, Lemma 5.9] gives the same full orbit Y_i and hence the same formula.

Proof of Theorem D. The integral chart theorem, Theorem 4.3, identifies $\mathcal{U}^{\mathrm{ind}}_{k,e}$ with $\mathbb{Z}[a_{j,h} : 1 \le j < k,\ h \bmod e]$. Proposition 13.1 identifies this ring with the stated restricted Grothendieck ring. Theorem 12.3 proves that each mutable cluster monomial in the retained determinant model is one full simple class. Setting every $D_h = 1$ sends this class to one simple class by the same proposition. The assertion for the coefficients follows from (13.3). □

13.2. **Continuants and Gleitz's exact seed.** We now set $k = 3$, $d = e - 1$, and $N = 2d$. Use periodic polynomial variables

$$a_j = [L(Y_{1,\varepsilon^{2j}})], \qquad b_j = [L(Y_{2,\varepsilon^{2j+1}})], \qquad j \in \mathbb{Z}/e\mathbb{Z}.$$

For the following polynomial computations set

$$M(a,b) = \begin{pmatrix} a & -b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_j = M(a_j, b_j).$$

Let $A_s(j)$ be the $(1,1)$ entry of $M_j \cdots M_{j+s-1}$, and $B_s(j)$ its northwest 2 by 2 minor. Empty products give $A_0 = B_0 = 1$, and negative suffixes mean zero. Expanding the first row, and then the exterior square, gives

$$A_s(j) = a_j A_{s-1}(j+1) - b_j A_{s-2}(j+2) + A_{s-3}(j+3), \tag{13.4}$$
$$B_s(j) = b_j B_{s-1}(j+1) - a_{j+1} B_{s-2}(j+2) + B_{s-3}(j+3). \tag{13.5}$$

Put

$$P_s = A_s(2), \quad E_s = B_s(1), \quad C = P_d, \quad D = P_{d-1}, \qquad S_r = CE_r - DB_{r-1}(2) \quad (0 \le r \le d).$$

Here P_s, D are continuants. All retained determinant coordinates and the auxiliary normalization root have been set to 1. In this section T denotes the boundary function

$$T = S_d = A_d(2)B_d(1) - A_{d-1}(2)B_{d-1}(2). \tag{13.6}$$

To compare with the band model, first substitute $a_{1,h} = a_{2-h}$, $a_{2,h} = b_{1-h}$. An antidiagonal transposition of a trailing core minor gives columns of heights 1^p for $p \le d$ and $1^d 2^r$ for $p = d + r$. In the first case its first-row expansion is (13.4), hence it equals P_p. In the second case expansion along its first d columns has only two possible row sets, $1, \dots, d$ and $1, \dots, d-1, d+1$. They contribute

$$A_d(2)B_r(d+2) - A_{d-1}(2)B_{r-1}(d+3) = S_r$$

by periodicity. Therefore the core seed consists of $P_1, \dots, P_{d-1}, S_0, \dots, S_d$, in reverse prefix order, with $S_d = T$ exceptional.

Lemma 13.2. *The ordinary mutation word*

$$\mu_e, \mu_{e-1}, \dots, \mu_2$$

replaces $S_0, \dots, S_{d-1}$ by $E_1, \dots, E_d$. The resulting divided quiver has arrows

$$T \longrightarrow E_d, \qquad P_{d-1} \longrightarrow T, \qquad E_d \rightrightarrows P_{d-1},$$
$$E_{s-1} \longrightarrow E_s \ (2 \le s \le d), \qquad P_{s-1} \longrightarrow P_s \ (2 \le s \le d-1),$$
$$P_s \longrightarrow E_s \ (1 \le s \le d-1), \qquad E_s \longrightarrow P_{s-1} \ (2 \le s \le d-1).$$

For $d = 1$, only $T \longrightarrow E_1$ remains.

Proof. The initial divided quiver is

$$S_0 \longrightarrow S_1 \longrightarrow \cdots \longrightarrow S_d, \qquad P_1 \longrightarrow \cdots \longrightarrow P_{d-1} \longrightarrow S_0,$$
$$S_{r+1} \longrightarrow P_r \longrightarrow S_r \quad (1 \le r \le d-1).$$

The required complementary-minor identity is

$$B_r(1)B_r(2) - B_{r-1}(2)B_{r+1}(1) = A_r(2).$$

For $r \ge 1$, put $N_j = \bigwedge^2 M_j$, in the ordered basis $12, 13, 23$, and $H = N_2 \cdots N_r$, with $H = I$ for $r = 1$. Then $B_r(1) = e_1^t N_1 H e_1$, $B_r(2) = e_1^t H N_{r+1} e_1$, $B_{r-1}(2) = e_1^t H e_1$, and $B_{r+1}(1) = e_1^t N_1 H N_{r+1} e_1$. Expanding the first row of N_1 and the first column of N_{r+1} shows that their displayed difference is

$$a_{r+1} \det H_{\{1,2\},\{1,2\}} + \det H_{\{1,2\},\{1,3\}}.$$

Since $\bigwedge^2 N_j = M_j$, this is the $(1,1)$ entry of $M_2 \cdots M_{r+1}$, namely $A_r(2)$. For $r = 0$ the identity reads $1 = 1$. Substitution into S_r yields

$$S_r E_{r+1} = E_r S_{r+1} + D P_r \qquad (0 \le r < d), \tag{13.7}$$

where $P_0 = E_0 = 1$.

It remains to check that these are the actual quiver exchanges. The first vertex S_0 has incoming D and outgoing S_1. Immediately before a later S_r is mutated, its complete incident list is

$$P_r \longrightarrow S_r, \quad D \longrightarrow S_r, \quad S_r \longrightarrow E_r, \quad S_r \longrightarrow S_{r+1}.$$

The path $P_r \to S_r \to S_{r+1}$ cancels the previous $S_{r+1} \to P_r$, and $D \to S_r \to E_r$ cancels $E_r \to D$. The other paths create $P_r \to E_r$ and $D \to S_{r+1}$; reversing the incident arrows gives the next displayed list. At $r = d-1$, the two incoming neighbors coincide, so the monomial in (13.7) is D^2. The same cancellation gives the stated double arrow and the final arrows at T. For $d = 1$ there is one ordinary mutation and a single reversed arrow. This proves both the rational identities and the complete matrix update. □

Define the polynomial involution $\Phi(a_j) = b_{1-j}$, $\Phi(b_j) = a_{1-j}$. If J reverses the three coordinate axes, then $JM_j^{-1}J = M(b_j, a_j)$. Complementary minors and reversal of the product therefore give

$$\Phi(A_s(j)) = B_s(2-j-s), \qquad \Phi(B_s(j)) = A_s(2-j-s). \tag{13.8}$$

Let

$$z_1 = \operatorname{tr}(M_0 \cdots M_{e-1}), \qquad z_2 = \operatorname{tr}((M_0 \cdots M_{e-1})^{-1}).$$

The same identity gives $\Phi(z_1) = z_2$, $\Phi(z_2) = z_1$, and periodicity gives $\Phi(T) = T$. In the transformed seed put

$$g_{2s-1} = A_s(1-s) \ (1 \le s \le d), \qquad g_{2s} = B_s(-s) \ (1 \le s < d), \qquad g_{2d} = T. \tag{13.9}$$

For $N \ge 4$, its full exchange matrix is specified by

$$B_{j,j+1} = -1, \quad B_{j,j+2} = 1 \quad (1 \le j \le N-3), \tag{13.10}$$
$$B_{N-2,N-1} = -2, \quad B_{N-2,N} = 3, \quad B_{N-1,N} = -3, \qquad B_{N,N-2} = -1, \quad B_{N,N-1} = 1.$$

Entries away from the last column have their skew opposites; all unlisted entries are zero. For $e = 2$, the matrix is $\left(\begin{smallmatrix} 0 & -3 \\ 1 & 0 \end{smallmatrix}\right)$. The last column has degree 3, all other columns degree 1.

The exceptional exchange before applying Φ is

$$TT' = D^3 + z_2 D^2 E_d + z_1 D E_d^2 + E_d^3.$$

After Φ, with $u = \Phi(D)$, $v = \Phi(E_d)$, it is

$$TT'' = u^3 + z_1 u^2 v + z_2 u v^2 + v^3, \tag{13.11}$$

where $u = 1$ if $d = 1$. For $e \geq 3$, (13.10) is Gleitz's matrix (5.11), with the two independent coefficients in (13.11).

Remark 13.3 (The rank-two orientation)**.** For $e = 2$ we use the uniform direction of [16, Conjecture 5.5]. It is the negative of the matrix printed in equation (5.6) there; reversing that matrix also reverses the coefficient order. The present assignment is checked directly by

$$g_1 = a_0, \qquad T = a_0 b_1 - 1,$$
$$z_1 = a_0 a_1 - b_0 - b_1, \qquad z_2 = b_0 b_1 - a_0 - a_1,$$
$$T(a_0 b_0 - 1) = 1 + z_1 a_0 + z_2 a_0^2 + a_0^3.$$

This is (13.11) with $u = 1$, $v = g_1$, and agrees with the character identity in [16, Proposition 5.3].

We check the representation labels, including the effect of the coordinate substitutions. Their composite on the original band generators is

$$a_{1,h} \longmapsto b_{h-1}, \qquad a_{2,h} \longmapsto a_h.$$

By (13.1) this is induced on the full character ring by

$$Y_{i,\varepsilon^p} \longmapsto Y_{3-i,\varepsilon^{p-1}}. \tag{13.12}$$

This is the Dynkin involution followed by a loop rotation; both preserve the restricted integral form, so it is a based automorphism at the root. It suffices to check the formula on fundamental generators because the ring is polynomial in them.

The continuant tableau formula, or induction in (13.4)–(13.5), gives the highest monomials

$$\mathrm{hw}(g_{2s-1}) = \prod_{t=0}^{s-1} Y_{1,\varepsilon^{-2t}}, \qquad \mathrm{hw}(g_{2s}) = \prod_{t=0}^{s-1} Y_{2,\varepsilon^{-1-2t}}, \tag{13.13}$$
$$\mathrm{hw}(T) = \left(\prod_{j=2}^{e} Y_{1,\varepsilon^{2j}}\right)\left(\prod_{j=1}^{e-1} Y_{2,\varepsilon^{2j+1}}\right).$$

Here is also a direct check of the highest terms. For $s < e$, the tableau for A_s is a weakly increasing row of the three fundamental letters, with successive spectral positions; the row of first letters has its highest monomial once. A terminal nonfirst letter leaves an uncancelled negative residue, since the interval has length less than e. The same argument applies to B_s. In (13.6) the first product has highest finite weight $d\omega_1 + d\omega_2$; the subtracted product has strictly smaller finite highest weight, and so cannot cancel that monomial. Theorem 12.3 and Lemma 13.2 already establish that these functions are simple classes. Thus their displayed highest monomials identify them uniquely, giving precisely Gleitz's two descending Kirillov–Reshetikhin chains and boundary class.

The vector monodromy can be checked directly in the reflected indices used in this subsection. Substituting $a_{1,h} = a_{2-h}$, $a_{2,h} = b_{1-h}$ in the vector recurrence gives

$$w_c = a_{3-c} w_{c-1} - b_{3-c} w_{c-2} + w_{c-3}.$$

For $W_c = [w_c, w_{c-1}, w_{c-2}]$, this says $W_c = W_{c-1} M_{3-c}^t$. Consequently, if $M_{\mathrm{phys}} w_c = w_{c+e}$, then

$$W_3^{-1} M_{\mathrm{phys}} W_3 = M_{-1}^t \cdots M_{-e}^t = (M_0 \cdots M_{e-1})^t.$$

Its two characteristic coefficients are z_1, z_2, in that order.

Here is an explicit character calculation of those coefficients. Put $U_j = Y_{1,\varepsilon^{2j}}$, $V_j = Y_{2,\varepsilon^{2j+1}}$, $u_j = V_j/U_{j+1}$, and $w_j = V_{j+1}^{-1}$. The fundamental characters are

$$a_j = U_j + u_j + w_j, \qquad b_j = V_j + U_{j+1}/V_{j+1} + U_{j+2}^{-1}.$$

The periodic matrices

$$G_j = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -U_j & 0 \\ 1 & -(U_j+u_j) & u_jU_{j+1} \end{pmatrix}, \qquad H_j = \begin{pmatrix} U_j & u_j & w_j \\ 0 & u_j & w_j \\ 0 & 0 & w_j \end{pmatrix}$$

satisfy $G_jM_j = H_jG_{j+1}$, by substitution of these two fundamental characters. Each G_j has nonzero monomial determinant, so multiplying around the period conjugates the character of the monodromy to an upper triangular matrix with eigenvalues

$$X, \quad Y/X, \quad Y^{-1}, \qquad X = \prod_{j \bmod e} Y_{1,\varepsilon^{2j}}, \quad Y = \prod_{j \bmod e} Y_{2,\varepsilon^{2j+1}}.$$

Hence

$$\chi_\varepsilon(z_1) = X + Y/X + Y^{-1}, \qquad \chi_\varepsilon(z_2) = X^{-1} + X/Y + Y.$$

By (13.3), these are the two Frobenius fundamental simples. The coefficient reversal under (13.12) agrees with (13.8); the coefficient order in (13.11) is therefore fixed as well.

13.3. **All rank-three fundamental generators belong to the lower algebra.** We prove that all fundamental classes belong to Gleitz's lower algebra. Write $r = e - 2$. At a base point j the chain seed is

$$T_j, \quad B_{r+1}(j), B_r(j), \dots, B_1(j), \quad A_r(j+1), \dots, A_1(j+1),$$

where

$$T_j = A_{r+1}(j+1)B_{r+1}(j) - A_r(j+1)B_r(j+1).$$

Name its vertices

$$U = T_j, \qquad V_s = B_s(j)\ (1 \le s \le r+1), \qquad W_s = A_s(j+1)\ (1 \le s \le r).$$

Lemma 13.4. *The successive mutation blocks*

$$\alpha = (U, V_{r+1}, W_r, W_{r-1}, \dots, W_1), \qquad \beta = (U, V_{r+1}, V_r, V_{r-1}, \dots, V_1)$$

carry the chain seed at j to the chain seed at $j-1$. Between the two blocks the coordinates are $\widetilde{a}_t = b_{t-1}$, $\widetilde{b}_t = a_t$, and the two exceptional coefficients are reversed.

Proof. The exceptional mutation first replaces T_j by

$$T_j^{\mathrm{rev}} = B_{r+1}(j)A_{r+1}(j) - B_r(j)A_r(j+1).$$

The cubic identity follows from a determinant calculation. At base 1, abbreviate

$$C = A_{r+1}(2), \quad D = A_r(2), \quad E = B_{r+1}(1), \quad F = B_r(2), \quad G = B_r(1).$$

For $M = M_1 \cdots M_{e-1}M_0$, its entries and complementary minors give

$$M_{23} = D, \quad (M^{-1})_{23} = E, \quad M_{21} = C, \quad (M^{-1})_{21} = F, \quad M_{13} = A_{r+1}(1), \quad (M^{-1})_{13} = G.$$

Let L have rows $e_2^t, e_2^tM, e_2^tM^{-1}$, and let K have columns $e_3, Me_3, M^{-1}e_3$. Their determinants are $-T_1$ and T_1^{rev}. Cayley–Hamilton gives

$$LK = \begin{pmatrix} 0 & D & E \\ D & z_1D + E & 0 \\ E & 0 & z_2E + D \end{pmatrix}.$$

Its determinant is $-D^3 - z_2D^2E - z_1DE^2 - E^3$, proving the exceptional exchange with the required signs.

The next exchange replaces $B_{r+1}(j)$ by $A_{r+1}(j)$, since their product is

$$T_j^{\mathrm{rev}} + B_r(j)A_r(j+1).$$

The ordinary identities used in the descending part are

$$A_s(j)A_s(j+1) = A_{s-1}(j+1)A_{s+1}(j) + B_s(j).$$

They follow by applying the same two-by-two complementary-minor calculation above to the products of the matrices M_j. In descending order $s = r, \dots, 1$, the member $A_{s+1}(j)$ has already been produced and $A_{s-1}(j+1), B_s(j)$ are unchanged. Thus these exchanges replace every $A_s(j+1)$ by $A_s(j)$. To verify these are all the quiver mutations, put $v_i = V_{r+1-i}$ for $0 \le i \le r$ and $w_i = W_{r+1-i}$ for $1 \le i \le r$, with divided entry $b_{ab} = 1$ for $a \to b$. After the first two mutations the changed entries at the top are

$$b_{U,v_0} = 1, \quad b_{U,v_1} = -1, \quad b_{U,w_1} = 0, \quad b_{v_0,v_1} = 1, \quad b_{v_0,w_1} = 1.$$

After w_1, the permanent top entries become

$$b_{U,v_0} = 1, \quad b_{U,v_1} = -1, \quad b_{v_0,v_1} = 2, \quad b_{v_0,w_1} = -1.$$

After $w_1, \dots, w_t$, the remaining entries are exactly

$$\begin{aligned}
b_{v_i,v_{i+1}} &= -1 && (1 \le i < r),\\
b_{w_i,w_{i+1}} &= -1 && \text{except } b_{w_t,w_{t+1}} = 1 \text{ if } t < r,\\
b_{v_i,w_i} &= \begin{cases} 1 & i \le t,\\ -1 & i > t, \end{cases}\\
b_{v_i,w_{i+1}} &= \begin{cases} 0 & i \le t,\\ 1 & i > t, \end{cases}\\
b_{w_i,v_{i+1}} &= \begin{cases} 1 & i < t,\\ 0 & i \ge t. \end{cases}
\end{aligned}$$

The ranges are those for which both indices exist. All other entries vanish except skew opposites and the displayed top entries. At the next frontier, the two-path rule creates $w_t \to v_{t+1}$ and cancels $v_{t+1} \to w_{t+2}$; reversing the incident arrows gives the same list with $t+1$. At $t = r$, it is the original chain quiver with its two short chains exchanged. For $r = 0$, the two rank-two mutations restore its single divided edge.

The resulting continuants are those of

$$\widetilde{a}_t = b_{t-1}, \qquad \widetilde{b}_t = a_t, \qquad \widetilde{A}_s(t) = B_s(t-1), \qquad \widetilde{B}_s(t) = A_s(t).$$

Put $\widetilde{M}_j = M(\widetilde{a}_j, \widetilde{b}_j)$. There has been one exceptional mutation, so its string reverses. This is also the actual coefficient change of the new coordinates. In the exterior-square basis $12, 13, 23$, put

$$N_j = \bigwedge\nolimits^2 M_j = \begin{pmatrix} b_j & -1 & 0\\ a_j & 0 & -1\\ 1 & 0 & 0 \end{pmatrix}, \qquad Q_j = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & a_j \end{pmatrix}.$$

Multiplication of these matrices gives

$$Q_j N_j = \widetilde{M}_{j+1} Q_{j+1}.$$

Periodicity cancels the intermediate Q_j's, so the new monodromy is conjugate to the exterior square of the old one. Its trace and inverse trace are z_2, z_1, respectively.

The identical block in these dual coordinates is β in the original vertex names. The twice dual coordinates are $\widetilde{\widetilde{a}}_t = a_{t-1}$ and $\widetilde{\widetilde{b}}_t = b_{t-1}$. Thus the two blocks translate the base point from j to $j-1$, as asserted. □

Proof of Theorem A. Theorem 4.3 identifies the rank-three upper algebra with

$$R = \mathbb{Z}[a_0, \dots, a_{e-1}, b_0, \dots, b_{e-1}].$$

Lemma 13.4, repeated around the period, places every a_j and b_j among the mutable cluster variables: for $e > 2$, the short and long chains contain their suffix-one continuants, and for

$e = 2$ the two blocks alternate the two fundamental types. Hence $R \subseteq \mathcal{A}$. The polynomial-coefficient Laurent property gives $\mathcal{A} \subseteq \mathcal{U} = R$, so $\mathcal{A} = \mathcal{U} = R$.

Lemma 13.2, (13.9)–(13.11), and the based automorphism (13.12) identify this seed with the original labeled seed of [16, Conjecture 5.5]. Theorem 12.3 gives one full simple for every mutable cluster monomial; Proposition 13.1 and the based automorphism preserve that assertion. Equations (13.13) and (13.3) verify the specified initial variable and coefficient assignments. This proves the required isomorphism and simplicity statement for all $e \geq 2$. □

Proofs of Theorems B and C. These are the scalar conclusion of Theorem 2.7. Its hypotheses are exactly $L = \gcd(n, \ell) \geq 2$ and $n/L \geq k \geq 2$, and its coefficient ring retains every ordinary frozen variable, proving both inclusions and the equality. □

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