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Counterexamples to Walther's ideal equality

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We disprove the equality $\mathfrak{a}_{f,s+1} = (f, f_x, f_y)$ in Walther's Question 1.4 by an infinite family of irreducible plane curves.

1. Introduction

Let $d \geq 1$ be an integer and $R = \mathbb{C}[x_1, \ldots, x_d]$. Write $D = R\langle \partial_1, \ldots, \partial_d \rangle$ for its Weyl algebra, and fix a nonconstant $f \in R$. Here $\partial_i = \partial/\partial x_i$, $f_{x_i} = \partial_i f$, and s commutes with D. The symbol f^s generates a free rank-one $R[f^{-1}, s]$-module with action

$$\partial_i(hf^s) = \left(\partial_i h + sh\frac{f_{x_i}}{f}\right) f^s, \qquad h \in R[f^{-1}, s]. \tag{1.1}$$

We write $f^{s+j} = f^j f^s$ for $j \in \mathbb{Z}$. Functional equations over $D[s]$ are interpreted in this module.

These equations arose in the study of the analytic continuation of complex powers of a polynomial. Bernstein [1972, Theorem 1′] proved that there exist $P(s) \in D[s]$ and a nonzero $b(s) \in \mathbb{C}[s]$ satisfying

$$P(s)f^{s+1} = b(s)f^s.$$

The scalar polynomials that occur in such equations form a nonzero ideal of $\mathbb{C}[s]$. Its monic generator is the Bernstein–Sato polynomial $b_f(s)$. Kashiwara [1976] gave a holonomic proof in the analytic setting and proved that the roots are strictly negative rational numbers [Kashiwara, 1976, Corollary 5.2].

For a fixed scalar factor $q(s) \in \mathbb{C}[s]$, one can instead vary the spatial multiplier. This gives the coefficient ideal

$$\mathfrak{a}_{f,q(s)} = \{g \in R : q(s)gf^s \in D[s]f^{s+1}\}. \tag{1.2}$$

It is an ideal because $D[s]f^{s+1}$ is an R-submodule. The definition of b_f gives

$$1 \in \mathfrak{a}_{f,q(s)} \quad \Longleftrightarrow \quad b_f(s) \mid q(s).$$

Walther [2005, Definition 2.1] introduced these ideals in his study of homogeneous polynomials and generic hyperplane arrangements. For homogeneous f, his Lemma 2.2 obtains a divisibility bound for $b_f(s)$ from the containment of a power of the homogeneous maximal ideal in $\mathfrak{a}_{f,q(s)}$. The argument applies the Euler relation successively to the spatial multipliers.

Put

$$I_f = (f, f_{x_1}, \ldots, f_{x_d}),$$

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the Tjurina ideal, denoted by $\mathrm{Jac}(f)$ in Walther [2015]. The identities

$$\partial_i f^{s+1} = (s+1) f_{x_i} f^s, \qquad (s+1) f^{s+1} = (s+1) f f^s$$

show that $I_f \subseteq \mathfrak{a}_{f,s+1}$. In his survey, Walther [2015, Question 1.4, p. 393] asks whether

$$\mathfrak{a}_{f,s+1} = I_f \tag{1.3}$$

holds for every nonconstant polynomial. The motivation given there is the relation between $b_f(s)$ and the cohomology of the Milnor fibre. We give the following infinite family of counterexamples.

Theorem 1.1. *For every integer $m \geq 3$, set*

$$f_m = x^{2m+1} + y^{2m-1} + x^m y^m, \qquad g_m = x^{m-1} y^{m+1}.$$

The polynomial f_m is irreducible in $\mathbb{C}[x,y]$, and

$$g_m \in \mathfrak{a}_{f_m,s+1} \setminus (f_m, \partial_x f_m, \partial_y f_m).$$

In particular, $I_{f_m} \subsetneq \mathfrak{a}_{f_m,s+1}$.

Earlier functional equations already yield strict inclusion. In his study of total-symbol assertions, Yano [1978, Example 2.28, pp. 144–145] constructed a polynomial differential operator for $x^6 + tx^5 y - y^5$, with $t \neq 0$, whose spatial multiplier has nonzero class modulo the Tjurina ideal. His example therefore implies a negative answer to the later question (1.3).

The member f_3 occurs in the deformation family of $x^7 + y^5$ studied by Kato [1981]. Kato stratified a family with constant Milnor number according to its Bernstein–Sato polynomial. For

$$F_t = \frac{\xi^7}{7} + \frac{\eta^5}{5} - t\xi^3\eta^3, \qquad t \neq 0,$$

his equation (2.1.5), on p. 8, gives a second-order operator $P_2(s)$ with holomorphic coefficients near the origin such that

$$P_2(s) F_t^s = -\frac{72t^4}{35^2} \xi^5 \eta^2 s F_t^{s-1}.$$

Under

$$\xi = 7^{1/7} x, \qquad \eta = 5^{1/5} y, \qquad t = -7^{-3/7} 5^{-3/5},$$

we have $F_t = f_3$, and the multiplier becomes cx^5y^2, where

$$c = -\frac{72}{35^2} \left(7^{-3/7} 5^{-3/5}\right)^4 7^{5/7} 5^{2/5} = -\frac{72}{7^3 5^4} \neq 0.$$

Let $\widetilde{P}_2(s)$ be obtained by the above substitutions and $\partial_\xi = 7^{-1/7}\partial_x$, $\partial_\eta = 5^{-1/5}\partial_y$. Then

$$\widetilde{P}_2(s+1) f_3^{s+1} = c(s+1) x^5 y^2 f_3^s.$$

Since $x^2(f_3)_y = 3x^5y^2 + 5g_3$, it follows that

$$\frac{1}{5}\left(x^2\partial_y - \frac{3}{c}\widetilde{P}_2(s+1)\right) f_3^{s+1} = (s+1)g_3 f_3^s.$$

Thus Kato's equation already gives the local analytic membership for g_3.

For deformations of $x^p + y^q$ with constant Milnor number, where $p, q \geq 2$ are coprime integers, Cassou-Noguès [1987] described the variation of the Bernstein–Sato roots by a matrix of residue polynomials. Briançon et al. [1989] developed an algorithm for the Bernstein–Sato polynomial and a corresponding functional equation for Newton-nondegenerate singularities, with a detailed treatment of semiquasihomogeneous deformations. Kato's family also appears in the study of Jacobian relation ideals and Kashiwara operators by Àlvarez Montaner and Planas-Vilanova [2021, Section 6.1].

For reduced hypersurfaces, Lőrincz and Yang [2026, Theorem 1.15] bound $\mathfrak{a}_{f,s+1}$ by the intersection of the adjoint ideal and $\sqrt{I_f}$. Their Example 6.12 gives

$$I_{x^2+y^3} = \mathfrak{a}_{x^2+y^3,s+1} = (x, y^2),$$

whereas that upper bound is (x, y).

The proof below constructs a polynomial differential operator for every member of Theorem 1.1. The Euler operators and shifted products follow the construction for functions of simplex type in Yano [1978, Section 5, pp. 128–133]. Two monomial relations give scalar factors whose roots in $u = s+1$ have opposite signs. An explicit polynomial Bézout identity removes the factors. Sections 2 and 3 give this construction. Section 4 proves nonmembership in the Tjurina ideal and irreducibility.

2. Euler operators

Fix an integer $m \geq 3$ for the rest of the proof, and write $f = f_m$ and $g = g_m$. Set

$$u = s+1, \qquad X = x\partial_x, \qquad Y = y\partial_y.$$

We work in the free module

$$N = \mathbb{Q}[x, y, f^{-1}, u]F, \qquad F = f^u,$$

with the action (1.1), using u in place of s. For an integer j, the expression f^{u-j} means $f^{-j}F$. The coefficient ring is a domain, so multiplication by x on N is injective. Products of operators act from right to left, and $[P, Q] = PQ - QP$.

2.1. Three commuting operators.

Define

$$\begin{aligned} A &= -(m-1)X - mY + m(2m-1)u, \\ B &= -mX - (m+1)Y + m(2m+1)u, \\ C &= (2m-1)X + (2m+1)Y - (4m^2-1)u. \end{aligned} \tag{2.1}$$

The operators X, Y, u commute, so A, B, C commute. Collecting the coefficients of X, Y, u gives

$$A + B + C = u, \qquad X = (2m+1)A + mC, \qquad Y = (2m-1)B + mC. \tag{2.2}$$

Write

$$z_1 = x^{2m+1}, \qquad z_2 = y^{2m-1}, \qquad z_3 = x^m y^m, \qquad f = z_1 + z_2 + z_3.$$

The derivatives

$$f_x = (2m+1)x^{2m} + mx^{m-1}y^m, \qquad f_y = (2m-1)y^{2m-2} + mx^m y^{m-1}$$

give the polynomial identities

$$\begin{aligned} -(m-1)xf_x - myf_y + m(2m-1)f &= z_1, \\ -mxf_x - (m+1)yf_y + m(2m+1)f &= z_2, \\ (2m-1)xf_x + (2m+1)yf_y - (4m^2-1)f &= z_3. \end{aligned} \tag{2.3}$$

Applying (1.1) therefore gives

$$AF = uz_1 f^{u-1}, \qquad BF = uz_2 f^{u-1}, \qquad CF = uz_3 f^{u-1}. \tag{2.4}$$

2.2. Shifted products. For an operator L and an integer $r \geq 0$, put

$$(L)_{\underline{r}} = L(L-1)\cdots(L-r+1), \qquad (L)_{\underline{0}} = 1.$$

All empty products below are 1. We claim that, for nonnegative integers i, j, k and $\ell = i + j + k$,

$$(A)_{\underline{i}}(B)_{\underline{j}}(C)_{\underline{k}}F = (u)_{\underline{\ell}} z_1^i z_2^j z_3^k f^{u-\ell}. \tag{2.5}$$

To prove it, let $h = z_1^i z_2^j z_3^k$. Its exponents give

$$Xh = ((2m+1)i + mk)h, \qquad Yh = ((2m-1)j + mk)h.$$

Consequently

$$\begin{aligned} [A,h] &= (i - m(2m-1)\ell)h, \\ [B,h] &= (j - m(2m+1)\ell)h, \\ [C,h] &= (k + (4m^2-1)\ell)h. \end{aligned} \tag{2.6}$$

For example, the coefficient in the first line is

$$-(m-1)((2m+1)i + mk) - m((2m-1)j + mk) = i - m(2m-1)(i+j+k).$$

The product rule and (2.3) now give

$$\begin{aligned} (A-i)(hf^{u-\ell}) &= m(2m-1)(u-\ell)hf^{u-\ell} \\ &\quad + (u-\ell)h(z_1 - m(2m-1)f)f^{u-\ell-1} \\ &= (u-\ell)z_1 hf^{u-\ell-1}, \\ (B-j)(hf^{u-\ell}) &= m(2m+1)(u-\ell)hf^{u-\ell} \\ &\quad + (u-\ell)h(z_2 - m(2m+1)f)f^{u-\ell-1} \\ &= (u-\ell)z_2 hf^{u-\ell-1}, \\ (C-k)(hf^{u-\ell}) &= -(4m^2-1)(u-\ell)hf^{u-\ell} \\ &\quad + (u-\ell)h(z_3 + (4m^2-1)f)f^{u-\ell-1} \\ &= (u-\ell)z_3 hf^{u-\ell-1}. \end{aligned}$$

The case $i=j=k=0$ of (2.5) is immediate. Each of the three displayed calculations increments the corresponding index by one. Since A,B,C commute and u is central, induction on ℓ proves (2.5).

The two monomial relations needed below are

$$xz_3^2 = yz_1z_2, \qquad z_3^{2m-1} = xz_1^{m-1}z_2^m. \tag{2.7}$$

In the first relation both sides are $x^{2m+1}y^{2m}$. In the second, the x-exponents agree because $1+(2m+1)(m-1) = m(2m-1)$, and the y-exponents are both $m(2m-1)$. The total shifted-product lengths are equal on the two sides of each relation. Thus (2.5) yields

$$\begin{aligned} [xC(C-1) - yAB]F &= 0, \\ [(C)_{\underline{2m-1}} - x(A)_{\underline{m-1}}(B)_{\underline{m}}]F &= 0. \end{aligned} \tag{2.8}$$

3. The functional equation

Set

$$w = ugf^{u-1} \in N.$$

We construct an operator P with coefficients in $\mathbb{Q}[x,y,u]$ such that $PF = w$.

3.1. Two equations for the target. Equation (2.4) gives

$$xw = yCF, \qquad y^{m-2}w = x^{m-1}BF. \tag{3.1}$$

Indeed, the respective spatial multipliers on both sides are $x^m y^{m+1}$ and $x^{m-1}y^{2m-1}$.

From (2.1),

$$[B,x] = -mx, \qquad [B,y] = -(m+1)y.$$

Hence

$$x(B+1)w = (B+m+1)xw = (B+m+1)yCF = yBCF.$$

Using $mC = X-(2m+1)A$, $BX = XB$, and the first identity in (2.8), we obtain

$$\begin{aligned} mx(B+1)w &= yB\big(X-(2m+1)A\big)F \\ &= xy\partial_x BF - (2m+1)yABF \\ &= x\big[y\partial_x B - (2m+1)C(C-1)\big]F. \end{aligned}$$

Multiplication by x is injective on N, so

$$m(B+1)w = \big[y\partial_x B - (2m+1)C(C-1)\big]F. \tag{3.2}$$

Applying ∂_x to $xw = yCF$ also gives

$$(X+1)w = y\partial_x CF. \tag{3.3}$$

Define the central polynomial and the two operators

$$\begin{aligned}\lambda &= \frac{2m+1}{m+1}u,\\ H &= \frac{(2m+1)C(C-1)-y\partial_x B-m^2 y\partial_x C}{m(m+1)},\\ G &= (2m-1)y\partial_x C+(2m+1)H.\end{aligned} \tag{3.4}$$

The denominators in this display are nonzero integers. The definitions give

$$\begin{aligned}B+1 &= -m(X+1)-(m+1)(Y-1-m\lambda),\\ C-2-\lambda &= (2m-1)(X+1)+(2m+1)(Y-1-m\lambda).\end{aligned}$$

Multiply the first identity by m, apply it to w, and use (3.2) and (3.3). Division by $-m(m+1)$ gives the first equation below; the second follows from the second identity:

$$(Y-1-m\lambda)w = HF, \qquad (C-2-\lambda)w = GF. \tag{3.5}$$

3.2. Scalar factors. Put

$$E(z)=\prod_{k=1}^{m-2}(z+k), \qquad T(z)=\prod_{j=1}^{2m-2}(z-j).$$

For an integer $r\geq 1$ and $a_1,\dots,a_r\in\mathbb{Q}$, put $\Phi(z)=\prod_{i=1}^{r}(z-a_i)$ and use the polynomial divided difference

$$\begin{aligned}\Delta_\Phi(z,t) &= \frac{\Phi(z)-\Phi(t)}{z-t}\\ &= \sum_{j=1}^{r}\left(\prod_{1\leq i<j}(z-a_i)\right)\left(\prod_{j<i\leq r}(t-a_i)\right).\end{aligned} \tag{3.6}$$

Here z,t are commuting indeterminates. Multiplication of the last line by $z-t$ telescopes to $\Phi(z)-\Phi(t)$. Thus the displayed sum defines $\Delta_\Phi\in\mathbb{Q}[z,t]$. We apply this identity to E,T with a central second argument.

The Weyl relation gives

$$\partial_y^r y^r = \prod_{k=1}^{r}(Y+k) \qquad (r\geq 0). \tag{3.7}$$

The product rule gives $(Y+1)y^r = y^r(Y+r+1)$. Therefore

$$\partial_y^{r+1}y^{r+1} = \partial_y^r(Y+1)y^r = \partial_y^r y^r(Y+r+1),$$

which proves (3.7) by induction from $r=0$. Apply ∂_y^{m-2} to the second equation in (3.1). Since ∂_y commutes with x,

$$E(Y)w = x^{m-1}\partial_y^{m-2}BF.$$

Also, by the first equation in (3.5),

$$[E(Y)-E(1+m\lambda)]w=\Delta_E(Y,1+m\lambda)HF.$$

Consequently the operator and scalar polynomial

$$\begin{aligned} P_- &= x^{m-1}\partial_y^{m-2}B-\Delta_E(Y,1+m\lambda)H,\\ Q_-(u) &= E(1+m\lambda)=\prod_{k=1}^{m-2}\left(\frac{m(2m+1)}{m+1}u+k+1\right) \end{aligned} \tag{3.8}$$

satisfy $P_-F=Q_-(u)w$. The roots of Q_- are the distinct negative rational numbers

$$\rho_k=-\frac{(m+1)(k+1)}{m(2m+1)},\qquad 1\le k\le m-2. \tag{3.9}$$

For a second equation, the commutators

$$[C,x]=(2m-1)x,\qquad [C,y]=(2m+1)y$$

give, for every integer j,

$$x(C-2-j)=(C-(2m+1)-j)x,\qquad (C-(2m+1)-j)y=y(C-j).$$

Applying these commutation identities successively to the factors of T gives

$$xT(C-2)=T(C-(2m+1))x,\qquad T(C-(2m+1))y=yT(C).$$

The first equation in (3.1) and the second annihilator in (2.8) now give

$$\begin{aligned} xT(C-2)w &= T(C-(2m+1))xw\\ &= T(C-(2m+1))yCF\\ &= yT(C)CF\\ &= y(C)_{\underline{2m-1}}F\\ &= xy(A)_{\underline{m-1}}(B)_{\underline{m}}F. \end{aligned}$$

Cancel x in the free module N to obtain

$$T(C-2)w=y(A)_{\underline{m-1}}(B)_{\underline{m}}F.$$

The second equation in (3.5) gives

$$[T(C-2)-T(\lambda)]w=\Delta_T(C-2,\lambda)GF.$$

Thus

$$\begin{aligned} P_+ &= y(A)_{\underline{m-1}}(B)_{\underline{m}}-\Delta_T(C-2,\lambda)G,\\ Q_+(u) &= T(\lambda)=\prod_{j=1}^{2m-2}\left(\frac{2m+1}{m+1}u-j\right) \end{aligned} \tag{3.10}$$

satisfy $P_+F=Q_+(u)w$. The roots of Q_+ are

$$\frac{(m+1)j}{2m+1},\qquad 1\le j\le 2m-2,$$

and are strictly positive.

3.3. Removing the scalar factors. Define

$$\begin{aligned}\beta(u) &= \sum_{k=1}^{m-2} \frac{1}{Q_+(\rho_k)} \prod_{\substack{1\le j\le m-2\\ j\neq k}} \frac{u-\rho_j}{\rho_k-\rho_j},\\ \alpha(u) &= \frac{1-\beta(u)Q_+(u)}{Q_-(u)}.\end{aligned} \tag{3.11}$$

The ρ_k are distinct and negative, whereas the roots of Q_+ are positive. Every denominator in the expression for β is therefore a nonzero rational number, so $\beta \in \mathbb{Q}[u]$. Substitution of $u=\rho_k$ leaves just the k-th summand and gives $\beta(\rho_k) = 1/Q_+(\rho_k)$. The numerator defining α vanishes at every root of Q_-. Since these roots are simple, the factor theorem shows that Q_- divides that numerator in $\mathbb{Q}[u]$. Hence $\alpha \in \mathbb{Q}[u]$ and

$$\alpha Q_- + \beta Q_+ = 1.$$

Because these coefficients are central, the polynomial differential operator

$$P = \alpha(u)P_- + \beta(u)P_+ \tag{3.12}$$

satisfies

$$PF = \alpha Q_- w + \beta Q_+ w = w = ugf^{u-1}.$$

Substituting $u = s+1$ gives an operator in $\mathbb{Q}[x,y]\langle\partial_x,\partial_y\rangle[s]$ with

$$P\big|_{u=s+1} f^{s+1} = (s+1)gf^s.$$

This proves $g \in \mathfrak{a}_{f,s+1}$.

4. The Tjurina ideal and irreducibility

4.1. Nonmembership. The second identity in (2.3) gives $y^{2m-1} \in I_f$. Solving that identity for f also gives $f \in (f_x, f_y, y^{2m-1})$, since $m(2m+1) \neq 0$. Therefore

$$I_f = (f_x, f_y, y^{2m-1}).$$

Consider the following monic generators:

$$\begin{aligned}G_1 &= x^{2m} + \frac{m}{2m+1}x^{m-1}y^m,\\ G_2 &= x^m y^{m-1} + \frac{2m-1}{m}y^{2m-2},\\ G_3 &= y^{2m-1}.\end{aligned} \tag{4.1}$$

They generate I_f because $f_x = (2m+1)G_1$ and $f_y = mG_2$. Use lexicographic order with $x > y$: exponents of x are compared first, and exponents of y break ties. For a nonzero polynomial h, let $\mathrm{LM}(h)$ denote its largest monomial in this order, without its coefficient. The leading monomials of (4.1) are

$$x^{2m}, \qquad x^m y^{m-1}, \qquad y^{2m-1}.$$

For monic G_i, G_j, set

$$L_{ij} = \operatorname{lcm}(\operatorname{LM}(G_i), \operatorname{LM}(G_j)),$$
$$S(G_i, G_j) = \frac{L_{ij}}{\operatorname{LM}(G_i)} G_i - \frac{L_{ij}}{\operatorname{LM}(G_j)} G_j.$$

The three S-polynomials have the following expressions:

$$\begin{aligned} y^{m-1}G_1 - x^m G_2 &= -\frac{2m-1}{m} y^{m-1} G_2 \\ &\quad + \left(\frac{m}{2m+1} x^{m-1} + \frac{(2m-1)^2}{m^2} y^{m-2} \right) G_3, \\ y^{2m-1}G_1 - x^{2m} G_3 &= \frac{m}{2m+1} x^{m-1} y^m G_3, \\ y^m G_2 - x^m G_3 &= \frac{2m-1}{m} y^{m-1} G_3. \end{aligned} \tag{4.2}$$

For the first line, expansion of the left side gives

$$\frac{m}{2m+1} x^{m-1} y^{2m-1} - \frac{2m-1}{m} x^m y^{2m-2}.$$

Subtracting $-((2m-1)/m) y^{m-1} G_2$ leaves

$$\frac{m}{2m+1} x^{m-1} y^{2m-1} + \frac{(2m-1)^2}{m^2} y^{3m-3},$$

which is the stated multiple of G_3. The other two lines follow by cancelling the leading terms. Every product on the right side of an S-expression has leading monomial strictly smaller than the corresponding L_{ij}. For the first expression the x-exponents are m, $m-1$, and 0, all smaller than $2m$; for the second they are $m-1 < 2m$, and for the third they are $0 < m$.

Given $0 \neq h \in I_f$, choose a representation $h = \sum_{i=1}^3 a_i G_i$, with $a_i \in \mathbb{C}[x,y]$, for which

$$M = \max_{a_i \neq 0} \operatorname{LM}(a_i) \operatorname{LM}(G_i)$$

is as small as possible. Such a choice exists because lexicographic order on monomials is a well-order. If $\operatorname{LM}(h) < M$, the terms of size M cancel. Let J be the set of indices contributing to those terms, and let c_i be the leading coefficient of a_i for $i \in J$. Then $\sum_{i \in J} c_i = 0$ and $|J| \geq 2$. For a fixed $i_0 \in J$, the entire group of top products equals

$$\sum_{i \in J} c_i \frac{M}{\operatorname{LM}(G_i)} G_i = \sum_{\substack{i \in J \\ i \neq i_0}} c_i \frac{M}{L_{ii_0}} S(G_i, G_{i_0}).$$

Replace each S-polynomial by (4.2), changing signs if the indices are reversed. The monomial order is compatible with multiplication, so every replacement product has leading monomial strictly smaller than M. All remaining terms of the original $a_i G_i$ already have this property. We have obtained a representation with smaller maximum, a contradiction. Thus $\operatorname{LM}(h) = M$, and some $\operatorname{LM}(G_i)$ divides $\operatorname{LM}(h)$.

For $g = x^{m-1} y^{m+1}$, neither x^{2m} nor $x^m y^{m-1}$ divides g, because $m-1 < m$. Also y^{2m-1} does not divide g, because $m+1 < 2m-1$ for $m \geq 3$. The leading-term property therefore proves $g \notin I_f$.

4.2. Irreducibility. In the Laurent polynomial ring $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, put

$$\xi = x^{m+1}y^{-m}, \qquad \eta = x^{-m}y^{m-1}.$$

The exponent matrix has determinant

$$(m+1)(m-1) - m^2 = -1,$$

and its inverse gives

$$x = \xi^{1-m}\eta^{-m}, \qquad y = \xi^{-m}\eta^{-m-1}.$$

Thus these monomials define an invertible change of Laurent coordinates, and

$$\frac{f}{x^m y^m} = 1 + \xi + \eta.$$

Substitution of $\eta = -1 - \xi$ induces an isomorphism

$$\mathbb{C}[\xi^{\pm 1}, \eta^{\pm 1}]/(1+\xi+\eta) \;\cong\; \mathbb{C}[\xi, \xi^{-1}, (1+\xi)^{-1}].$$

Indeed, the inverse sends ξ to its residue class and $(1+\xi)^{-1}$ to $-\eta^{-1}$. The right side is a nonzero subring of $\mathbb{C}(\xi)$, hence a domain. It follows that $1+\xi+\eta$ generates a proper prime ideal and is irreducible. Multiplication by the Laurent unit $x^m y^m$ shows that f is irreducible in $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$.

If $f = ab$ with $a, b \in \mathbb{C}[x,y]$, one factor, say a, becomes a unit in this Laurent ring. Such a polynomial is a nonzero constant times a monomial: if $a^{-1} = h/(x^r y^t)$ with $h \in \mathbb{C}[x,y]$ and $r,t \geq 0$, then $ah = x^r y^t$, and unique factorization in $\mathbb{C}[x,y]$ gives this assertion. Neither x nor y divides f, because $f(0,y) = y^{2m-1}$ and $f(x,0) = x^{2m+1}$. The monomial factor is therefore constant, proving irreducibility in $\mathbb{C}[x,y]$.

Together with the functional equation and nonmembership proved above, this proves Theorem 1.1. The degrees $\deg f_m = 2m+1$ are distinct, so the family contains infinitely many counterexamples.

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