A Lean-Verified Alternative Proof of the Pressure-Flux Estimate in Navier–Stokes Lemma 10.5

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Abstract

OpenAI's written proof of (10.19) estimates a local pressure term in $L^{3/2}$. The corresponding upstream Lean development bounds the physical flux by an expression that also involves localized velocity dissipation. We give a companion Lean proof of the printed velocity-only bound. A second cutoff commutation moves two further powers of the cutoff onto the stress tensor. The resulting local pairing admits an L^2 estimate, while both commutator errors retain the required decay in the radius. The argument allows an essential-in-time energy bound for the competing velocity. Its conclusion holds at every interior time, with one constant for the whole slab. [1, 2, 8]

The physical estimate

Let u, v be the reference and competing velocities on $\mathbb{R}^3$, with respective pressures p, P. Write $w = v - u$ and $\pi = P - p$. Choose a fixed smooth compactly supported φ with $0 \le \varphi \le 1$ and $\varphi = 1$ on the unit ball. At a radius R set

$$\rho(x) = \varphi(x/R), \qquad \chi_R = \rho^8, \qquad B_R(t) = \|\rho^4 w(t)\|_6.$$

Under the slab hypotheses in Section 1, there is $C_T \ge 0$ such that for every $R \ge 1$ and every $0 < t < T$, the following holds with $B = B_R(t)$:

$$\left| \int_{\mathbb{R}^3} \pi(t,x)\, w(t,x) \cdot \nabla \chi_R(x)\, dx \right| \le \frac{C_T}{R} \left[(B+1)B^{1/2} + R^{-3/4} B^{3/4} \right]. \tag{10.19}$$

The right-hand side uses the fourth-power weighted L^6 velocity norm at the same radius as the physical cutoff. The manuscript asserts the inequality for almost every interior time. The companion yields it for each such time. Appendix A gives the estimates and an explicit admissible constant.

Proof artifacts. Lean sources, reproducibility instructions and citation metadata are available in the public repository [9].

https://github.com/TheRealDubb/navier-stokes-equation-10-19-lean

1 Formal statement and hypotheses

Fix $T > 0$. Both velocities and pressures are smooth on the closed slab $[0, T] \times \mathbb{R}^3$. Both velocities have zero divergence and the same viscosity-one Navier–Stokes residual on the interior, expressing equality of the applied force. One compact set contains the spatial supports of u at all slab times. The spatial L^2 norm of v has a finite essential bound in time. All spatial norms and integrals use Lebesgue volume.

The declaration below is `NavierStokesR3.Paper1019.equation_10_19`. Its source is `lean/Paper1019.lean`. Line wrapping is typographical.

```
theorem equation_10_19 {T : ℝ} (hT : 0 < T) {u v : VelocityField} {p P : PressureField}
    {K : Set Space} (hu : ContDiffOn ℝ ∞ u (slab 0 T)) (hv : ContDiffOn ℝ ∞ v (slab 0 T))
    (hp : ContDiffOn ℝ ∞ p (slab 0 T)) (hP : ContDiffOn ℝ ∞ P (slab 0 T)) (hK : IsCompact K)
    (hsupp : ∀ t ∈ Icc (0 : ℝ) T, tsupport (fun x => u (t,x)) ⊆ K)
    (hev : ∃ M : ℝ, 0 ≤ M ∧ ∀ᵐ t ∂volume.restrict (Icc (0 : ℝ) T),
      MemLp (fun x : Space => v (t,x)) 2 volume ∧
        comparisonLpNorm 2 (fun x : Space => v (t,x)) ≤ M)
    (hdu : ∀ t ∈ Ioo (0 : ℝ) T, ∀ x,
        NavierStokes.ProblemStatement.spatialDivergence u t x = 0)
    (hdv : ∀ t ∈ Ioo (0 : ℝ) T, ∀ x,
        NavierStokes.ProblemStatement.spatialDivergence v t x = 0)
    (hNS : ∀ t ∈ Ioo (0 : ℝ) T, ∀ x,
        navierStokesResidual 1 u p t x = navierStokesResidual 1 v P t x)
    {φ : Space → ℝ} (hφ : ContDiff ℝ ∞ φ) (hc : HasCompactSupport φ)
    (hφ0 : ∀ x, 0 ≤ φ x) (hφ1 : ∀ x, φ x ≤ 1)
    (_hunit : ∀ x : Space, ‖x‖ ≤ 1 → φ x = 1) :
    ∃ C_T : ℝ, 0 ≤ C_T ∧ ∀ R : ℝ, 1 ≤ R → ∀ t ∈ Ioo (0 : ℝ) T,
      let ρ := PaperFluxScaling.scaled φ R
      let B := comparisonLpNorm 6 (fun x => ρ x ^ 4 • (v (t,x) - u (t,x)))
      |∫ x : Space, (P (t,x) - p (t,x)) * fderiv ℝ (fun y => ρ y ^ 8) x
        (v (t,x) - u (t,x))|
          ≤ C_T / R * ((B + 1) * B ^ (1/2 : ℝ) + R ^ (-(3/4 : ℝ)) * B ^ (3/4 : ℝ))
```

Reading the signature

The spatial Fréchet derivative `fderiv` of ρ^8 applied to $v - u$ is the directional derivative $w \cdot \nabla \chi_R$. The premise `hev` puts L^2 membership and a uniform bound on the same full-measure set of times. The conclusion chooses C_T before quantifying over R and t. The proof uses neither an initial-data equality nor a magnitude bound on the competing pressure.

The zero case is also checked separately. If $B = 0$, the weighted velocity vanishes almost everywhere. At such points either $\rho = 0$ or $w = 0$, so the physical integrand vanishes. This follows for any pressure from `PaperFluxZero.integrand_ae_zero`.

2 Prior work and numerical comparison

Written estimates and the reported discrepancy

OpenAI states (10.19) after commuting the fourth-power cutoff through the double Riesz transform and estimating the local term in $L^{3/2}$ [1]. KokunoYumeto's pinned workbench records the same normalized expression as (E.21) [3]. Open Maths derives the quantitative bound before absorption into C_T in its re-audit [4]. Bastounis, Circelli and Hansen identify the change between the printed argument and the upstream Lean estimate, whose right-hand side also contains the localized gradient norm A_R [2, §3.2].

Related formalizations

MathGraph proves a bound for a legacy canonical weighted-pressure pairing and develops a transpose identity [5]. Its polynomial upper bound is proportional to $R^{-1}(B^{3/2} + B + 1)$. Specializing the underlying raw bound at $R = L = M = V = 1$ leaves a nonnegative contribution of 16 for every B. The pairing also requires an identification with the physical flux. The later MathGraph comparison work records (10.19) as open in that work [5].

Bourakkadi's Layer 7 verifies conditional scalar scaling, absorption and cancellation identities relevant to the comparison [6]. Hunt reports a formal estimate using enlarged cutoff radii and weighted dissipation [7]. His support example concerns radius dilations. For the original powers, the zero-case lemma in Section 1 gives a vanishing physical integrand whenever $B = 0$.

A scalar test for numerical implication

Fix the radius at one and take $B(t) = t^4$, $F(t) = t$ on $0 < t < 1$. These scalar data meet the raw upper bound above. For any proposed constant $C \geq 0$, the target right-hand side is bounded by

$$C\big[(1 + t^4)t^2 + t^3\big] \leq 3Ct^2.$$

Thus $F(t)$ exceeds the target on an interval of positive length for every fixed C. This establishes failure of numerical domination even with a null exceptional set of times. Additional arguments using the fields or their equations could still yield (10.19). The continuous-family calculation is analytic. The separate audit theorem `PressureReview.no_uniform_scalar_implication` kernel-checks the positive scalar witness for each C.

Earlier written derivations of the estimate and related Lean results appear above. The contribution offered here is the alternative formal proof for the physical, normalized estimate using a second commutator. Historical priority for this argument or an exact formalization remains unresolved.

3 Verification and research contributions

Local verification

The frozen build uses Lean 4.34.0-rc2, the OpenAI revision in [8], and the locked Mathlib revision below. A clean Windows build with the official dependency cache reproduced all 58 upstream and 14 companion module objects byte-for-byte. A second checkout built the dependencies from source with the cache disabled. All 3,133 main dependency objects matched their official counterparts by SHA-256. Using those source-built dependencies, all 72 project objects and the fresh theorem-import log again matched the frozen originals.

The native kernel checker replayed all 14 companion modules. An expanded inspection of 70 companion declarations found only subsets of `propext`, `Classical.choice` and `Quot.sound`. The project dependency trace covers 731 reachable declarations and includes pressure-gradient recovery and both commutators. Velocity uniqueness and the global blowup theorem play no role in this derivation. Both replays share one Windows host and the released compiler and standard library.

Reproducing the checks

The public companion sources are in `lean/` in the repository [9]. Its reproducibility instructions explain how to check their hashes and build them against the pinned public dependencies. The two earlier replays are documented in private verification records. Independent reproduction on a second machine and external expert mathematical review are pending.

Lean compiler	`6a10ac8c22beadecabdbb0919c2b50214762f91d`
Mathlib	`85e3a25e006c35636f0e53b0e9296caca2685bc0`
OpenAI	`f9e8bc5b38b6e212696e8a30e3e91517af887bbd`

Research direction and AI contribution

Troy Winarski selected and directed the investigation, set priorities, authorized resource allocation and required formal verification. Cody/Codex developed the alternative two-commutator argument, implemented the Lean companion, and performed the documented local verification procedures. Eidos/ChatGPT contributed to research planning, mathematical discussion, audit guidance and publication preparation. The mathematical claims rest on the proof terms, their interpretation under the stated hypotheses, and the verification records. Troy Winarski is the sole human contributor and retains responsibility for reviewing the final text and deciding whether to submit.

The verified result is the normalized pressure-flux estimate. Formalizing the manuscript's general $L^{3/2}$ argument and assessing the full blowup construction are separate mathematical tasks.

Appendix A. From the physical flux to three pairings

A.1. Pressure recovery on a time slice

Fix $0 < t < T$ and $R \geq 1$. Write $w = v - u$, $\pi = P - p$, $\rho(x) = \varphi(x/R)$, $\chi_R = \rho^8$, and $B = \|\rho^4 w\|_6$. The fixed smooth compactly supported cutoff satisfies $0 \leq \varphi \leq 1$ and equals one on the unit ball. Put $L \geq \|\nabla\varphi\|_\infty$ and $a = 8L$. The chain rule gives $\nabla\chi_R = 8\rho^7\nabla\rho$ and $\|\nabla\rho\|_\infty \leq L/R$.

For $D_{ij} = R_i R_j$, put $g_{ij} = u_i u_j - v_i v_j$ and $\psi_k = 8\rho^k \nabla\rho \cdot (u - v)$. The pairing $\langle f, h\rangle$ denotes $\int f h\, dx$. Testing the recovered pressure gradient against the compactly supported vector field $\rho^8(u - v)$ and using zero divergence gives

$$F = \int \pi\, w \cdot \nabla\chi_R\, dx = \operatorname{Re} \sum_{i,j} \langle g_{ij}, D_{ij}\psi_7\rangle.$$

Both pressure and velocity differences reverse signs in the upstream convention. A spatially constant pressure correction contributes zero because the test field is compactly supported and w is divergence-free.

Lean declarations. `ActualPressureFlux.pressure_flux_eq_canonical`, `Paper1019.physical_le_of_pair`, `PaperFluxScaling.exists_bounds`.

A.2. Two exact decompositions and the local estimate

Suppress i, j. Define $E(h, f, \psi) = \langle f, D(h\psi) - hD\psi\rangle$. Using $\psi_7 = \rho^4\psi_3$ and $\psi_3 = \rho^2\psi_1$ gives

$$\langle g, D\psi_7\rangle = \langle \rho^4 g, D\psi_3\rangle + E(\rho^4, g, \psi_3),$$
$$\langle \rho^4 g, D\psi_3\rangle = \langle \rho^6 g, D\psi_1\rangle + E(\rho^2, \rho^4 g, \psi_1).$$

Energy gives $g \in L^1$. The localized tensors are continuous with compact support, hence lie in $L^{6/5}$. Their products with transformed Schwartz tests are integrable by Hölder's inequality. For the local term, take $M \geq \|w\|_2$ and $U_q \geq \|u\|_q$. Since $\rho^6|g| \leq (\rho^2|w|)(\rho^4|w| + 2|u|)$, Hölder's inequality with exponents 3 and 6 and the interpolation in A.3 give

$$\|\rho^6 g\|_2 \leq M^{1/2} B^{1/2}(B + 2U_6), \qquad \|\psi_1\|_2 \leq \frac{aM}{R}.$$

The double-Riesz multiplier has L^2 operator norm at most one. Cauchy–Schwarz bounds the local pairing by the product of these two quantities.

Lean declarations. `PressureFlux.pressurePair_decomposition`, `PaperFluxDecomposition.split_twice`, `PaperFluxDecomposition.local_l2_bound`, `LocalizedFluxTensor.tensorDiff_bounds`.

Appendix A. Commutator bounds and uniformity

A.3. The two commutator bounds

Weighted interpolation uses the endpoints $\|w\|_2 \le M$ and $\|\rho^4 w\|_6 = B$. Pointwise, $\rho^2|w| = |w|^{1/2}(\rho^4|w|)^{1/2}$ and $\rho^3|w| = |w|^{1/4}(\rho^4|w|)^{3/4}$. Hölder's inequality and $0 \le \rho \le 1$ give

$$\|\rho^3 w\|_3 \le \|\rho^2 w\|_3 \le M^{1/2}B^{1/2},$$
$$\|\rho^3 w\|_4 \le M^{1/4}B^{3/4}.$$

Thus $\|\psi_3\|_4 \le (a/R)M^{1/4}B^{3/4}$. Applying Hölder's inequality to $\rho^3|w|$ and $\rho(|w| + 2|u|)$ gives

$$\|\rho^4 g\|_{4/3} \le M^{1/4}B^{3/4}(M + 2U_2).$$

The heat representation bounds the absolute commutator kernel by a cutoff-dependent multiple of $H_R(z) = |z|^{-3}\min\{|z|/R, 1\}$. Scaling in three dimensions gives $\|H_R\|_{4/3} = R^{-3/4}\|H_1\|_{4/3}$.

Set $K_1 = k\max\{4L, 1\}$ and $K_2 = k\max\{2L, 1\}$, where k is the universal Riesz commutator constant. For the measurable cutoffs ρ^2 and ρ, the Lipschitz bounds are $2L/R$ and L/R. The squared-cutoff estimates therefore give

$$|E(\rho^4, g, \psi_3)| \le K_1 R^{-3/4}\|g\|_1\|\psi_3\|_4,$$
$$|E(\rho^2, \rho^4 g, \psi_1)| \le K_2 R^{-3/4}\|\rho^4 g\|_{4/3}\|\psi_1\|_2.$$

The first convolution maps L^1 into $L^{4/3}$ with kernel H_R. In the second, two measurable $L^{4/3}$ factors convolve into L^2. The smooth compactly supported tests also belong to L^4 as required in the heat representation. Both remainders acquire the additional R^{-1} from $\nabla\rho$ in their test.

Lean declarations. `WeightedInterpolation.cutoff_interpolation_three`, `WeightedInterpolation.cutoff_interpolation_four`, `Lemma105PaperFlux.residual_three_four`, `Comparison.radialCommutatorKernel_lpNorm_scale`, `Comparison.riesz_commutator_pair_bound`, `MixedRieszCommutator.riesz_commutator_pair_bound`.

A.4. One constant for the whole slab

For each $t \in [0, T]$, choose good times tending to t where $\|v\|_2 \le M_v$. Slab continuity gives spatial pointwise convergence. Fatou's lemma applied to squared velocities yields $\int |v(t,x)|^2\,dx \le M_v^2$, including at the endpoints. Continuity and common compact support bound U_2, U_6 uniformly. Energy then supplies uniform M and $G \ge \max_{i,j}\|g_{ij}\|_1$.

Sum the three bounds and the nine tensor components. Using $B + 2U_6 \le (1 + 2U_6)(B + 1)$, we obtain (10.19) for every $R \ge 1$ and every interior t with

$$C_T = 9\max\Big\{aM^{3/2}(1 + 2U_6),\ aM^{1/4}\big[K_1 G + K_2 M(M + 2U_2)\big]\Big\}.$$

Lean declarations. `EssentialEnergy.uniformFiniteEnergy_of_ae_bound`, `FluxEnvelope.exists_uniform_slice_bounds`, `PaperFluxSlice.pair_three_term`, `FluxEnvelope.exists_scalar_envelope`, `Paper1019.exists_uniform_physical_bound`.

The supplemental quantitative corollary `Paper1019.physical_informative` collects both error coefficients into a single tensor bound G by also requiring $G \ge M(M + 2U_2)$. In the normalized theorem just proved, C_T absorbs the two coefficients separately. The main theorem has no hypothesis involving G.

Sources and proof locations

[1] OpenAI. *Finite Time Blowup for Navier–Stokes*. The 166-page manuscript, Lemma 10.5, pp. 121–123, equations (10.18)–(10.19).

[2] A. Bastounis, F. Circelli and A. C. Hansen. *Navier–Stokes lost in translation: Why Lean verification of AI autoformalisation does not guarantee correct natural language proofs*. arXiv:2610.08144v1, §3.2 (6 October 2026).

[3] KokunoYumeto. *Navier–Stokes workbench*, equation (E.21). TeX source at commit `e0a04c0`, dated 9 September 2026. The cited lines display the weighted interpolation and normalized inequality.

[4] Open Maths. *Whole-space comparison and quantitative bounds*. Lemma 10.5 re-audit, especially equations (18)–(33).

[5] MathGraph. *Legacy canonical pressure estimate*, commit `c602035`. See PR 142 for the transpose identity, PR 143 for the checked estimate, and PR 146 for later comparison work. The numerical specialization in Section 2 uses the pinned `rawBound` definition.

[6] I. Bourakkadi. *NavierStokesRefereeAudit*, Layer 7. `Layer7_UniquenessAndBenchmarks.lean`, commit `12051b5` (November 2026).

[7] B. Hunt. *The Lean Code Is Verified; the Paper's Argument Is Not: A Counterexample to the Pressure-Flux Estimate in "Finite Time Blowup for Navier–Stokes"*. FalseGreen paper A, §§4–5, 8.

[8] OpenAI. *NavierStokesAndEuler*, frozen source repository. The original pressure-recovery and comparison developments are under `NavierStokes/R3/`.

[9] T. Winarski. *A Lean-Verified Alternative Proof of the Pressure-Flux Estimate in Navier–Stokes Lemma 10.5*. Public companion repository, original commit `6ad63c5fa8fb01f6cd88cbbea0fb8dc67ed4de3a`. Includes the Lean sources, reproducibility instructions and citation metadata.

How to locate the proof terms

Declaration names in the text and Appendix A omit the common namespace `NavierStokesR3`. The companion sources are in `lean/` in [9]. Upstream declarations are in `NavierStokes/R3/` in [8]. The separate scalar comparison in Section 2 was checked in the private audit. Its source and historical build records are outside this public distribution.