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CONDITIONAL RESULTS AT $N-p$: HIGHER DIVISOR SUMS, BAND-PRIME COFACTORS, AND A CONDITIONAL REDUCTION

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ABSTRACT. Let N be a large integer and p a prime. We assume that Dirichlet L-functions have no zeros with real part greater than $7/8$, as a recent preprint asserts, together with exactly stated results of other recent preprints; none of these has been refereed. This zero-free half-plane would also follow from the generalised Riemann hypothesis for Dirichlet L-functions, of which L. Alpöge announced a proof, formalised in Lean, on 20 November 2026. Under these assumptions we prove that, for each fixed $k \geq 3$, $\sum_{p<N} \tau_k(N-p) = C_k(N)N(\log N)^{k-2}(1+o(1))$ as $N \to \infty$ through all integers, even or odd, with an explicit Euler product $C_k(N)$. We then state, as a hypothesis, a reflected form of Lü, Wang and Wu's bilinear estimates with auxiliary primes in bands. Under it, together with the zero-free half-plane and a form of Vaughan's identity from one of these preprints, every large even N is $p+aq$ with p and q prime and a a product of one prime from each of K bands, and the number of such representations with p in a dyadic interval is asymptotic to a sum of Hardy–Littlewood type over the admissible a. If the hypothesis holds for bands of polylogarithmic primes, a can be taken at most $(\log N)^E$, where E exceeds 10^{3363}. Finally, under the same assumptions with the hypothesis for a third system of bands, and for each fixed $0 < \alpha < 1/5$, the binary Goldbach asymptotic is reduced to an averaged bound, with a saving $(\log N)^{-1-\varepsilon}$, for primes in the residue classes $N \bmod d$ with moduli up to $N\exp(-(\log N)^{\alpha}/2)$, which exceeds N^{ν} for every fixed $\nu < 1$ once N is sufficiently large; we do not prove that bound.

1. INTRODUCTION

1.1. **Sums over $N-p$.** Let N be a large integer. This paper studies sums $\sum_{p<N} f(N-p)$ over primes p; with $f = \Lambda$ such a sum counts, with weights, the representations of N as a prime plus a prime power. They are the reflected counterparts of the shifted sums $\sum_{p\leq x} f(p+h)$, but a theorem for a fixed shift h does not give them: at the reflection the shift is $-N$, which grows with the range of summation, the arguments p and $N-p$ move in opposite directions, and the error terms of fixed-shift theorems may depend on h.

At fixed shifts there has been recent progress by a method that weights a shifted value by its divisors $a = p_1 \cdots p_K$ with one prime from each of K bands of auxiliary primes, and controls the resulting bilinear sums by an operator method [33]. Lü, Wang and Wu used it to obtain $\sum_{p\leq x} \mu(p+2) = o(\pi(x))$ [28]. Jiang, Lau and Lü obtained an asymptotic formula for primes of the form u^2+v^2+1 [23], and Hu, Lü, Wang and Wu, building also on [35] and [42], obtained $\sum_{x<n\leq 2x} \Lambda(n)\tau_k(n+h) = C_{k,h}\, x(\log x)^{k-1} + o_{k,h}(x(\log x)^{k-1})$ for every fixed $k \geq 2$ and every fixed $h \neq 0$ [21, Theorem 1.1]. At the reflection, Guo carried the arguments of [28] over to $\sum_{p<N} \mu(N-p)$, under hypotheses recalled in §1.4 [18]. All of these are recent preprints.

Classically, binary Goldbach with the Hardy–Littlewood asymptotic is known for almost all even N [7, 8, 10], and every large even N is the sum of a prime and a number with at most two prime factors [6]. For divisor weights of dimension two the reflected problem is classical: the case $k=2$ of the sum $\sum_{1\leq t<N} \Lambda(t)\tau_k(N-t)$ of Theorem A below is covered by Assing, Blomer and Li's uniform Titchmarsh divisor theorem [2, Theorem 1.1], and Elliott and Halberstam's asymptotic for $\sum_{p<n} 2^{\omega(n-p)}$ holds for every $n \geq 3$ [9], [16, (1.1)].

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Parity. Selberg's sequence $a_n = 1 + \lambda(n)$, with λ the Liouville function, vanishes at every prime, yet its sums over the multiples of d behave like those of the sequence $a_n = 1$, on average over d up to every level $x^{1-\varepsilon}$ [39], [14, p. 1045], [11, p. 1664]. In the dimension-one setting of Bombieri's asymptotic sieve, under its additional growth and regularity conditions, information of this kind up to every level $x^{1-\varepsilon}$ determines the weighted counts described in [15, §2], for a given number of distinct prime factors, only up to one factor $\delta(x) \in [0, 2]$. This enters as $\delta(x)$ for one parity of that number and as $2 - \delta(x)$ for the other, and every value in $[0, 2]$ occurs for some sequence [5], [15, §2]; see also [11, Theorem 1]. The counts in Theorems B and C below are restricted to integers $N - p$ with exactly $K + 1$ prime factors, and Theorem D concerns the primes themselves. Their proofs use an input that sees parity: Hypothesis RE of §2.3, reflected bilinear estimates of the kind on which the reflected theorems for μ and λ rest. Theorem A is not of this kind: its asymptotic would follow from distribution in progressions at a level above $1 - 1/k$ (§1.4).

1.2. **Results.** For even n put $\mathfrak{S}(n) = 2C_2 \prod_{p\mid n,\ p>2}(p-1)/(p-2)$, with $C_2 = \prod_{p>2}(1-(p-1)^{-2})$, and $G(N) = \sum_{1\le n<N} \Lambda(n)\Lambda(N-n)$. The premises named below are stated in §2: the zero-free half-plane QRH(θ) of Definition 2.1, the numbered statements of preprints listed in §2.2, Hypothesis RE($\mathfrak{B}, \theta$) of §2.3, and the classical inputs of §2.4. Table 1 shows which theorem uses which. The zero-free half-plane has two sources (§2.1): the preprint [34] asserts QRH(7/8), and QRH(θ) for every $\theta \in [1/2, 1)$ would follow from the generalised Riemann hypothesis for Dirichlet L-functions, of which a proof has been announced [1]. We keep it as a premise. The statements below are short forms; the full statements are in §§3–6.

Theorem A. *Assume* QRH(7/8)*, the statements of* [33]*,* [35]*,* [42] *and* [21] *listed in §2.2, and the classical inputs of §2.4. Fix an integer $k \ge 3$ and put*

$$C_k(N) = \frac{1}{(k-1)!} \prod_{p\mid N} \bigl(1-p^{-1}\bigr)^{k-1} \prod_{p\nmid N} \frac{1 - p^{-1}(1-p^{-1})^{k-1}}{1-p^{-1}}. \tag{1}$$

Then

$$\lim_{X\to\infty} \sup_{N\ge X} \left| \frac{\sum_{1\le t<N} \Lambda(t)\tau_k(N-t)}{C_k(N)\, N(\log N)^{k-1}} - 1 \right| = 0,$$

the supremum being over all integers $N \ge X$, even and odd. Moreover, $\sum_{p<N} \tau_k(N-p) = C_k(N)N(\log N)^{k-2}(1+o(1))$ as $N \to \infty$ through all integers.

Here k is fixed: the theorem does not allow k to grow with N. The order of magnitude $C_k(N)N(\log N)^{k-1}$ follows from classical sieve inputs (§3); the content of Theorem A is the ratio 1. In our proof QRH(7/8) enters only through the estimates of §3.4, and without it two arithmetic estimates that the proof needs remain unproved.

Theorems B–D assume Hypothesis RE, a pair of reflected minor- and major-arc estimates for one bilinear square, at three systems $\mathfrak{B}_{\mathrm{exp}}$, $\mathfrak{B}_{\mathrm{poly}}$, $\mathfrak{B}_{\mathrm{wide}}$ of auxiliary prime bands. Hypothesis RE is ours; it is not proved in this paper, and no proof of it is in print. Given it, each of these proofs follows the plan of [28]: a form of Vaughan's identity on integers without small prime factors (Proposition 2.2, which is [28, Proposition 2.1]) splits the sum into Type I sums, which we estimate by theorems of Bombieri–Vinogradov type, and a Type II sum, which Cauchy's inequality reduces to the square that Hypothesis RE controls. The zero-free half-plane QRH(θ) enters Theorems B–D only through the major-arc half of Hypothesis RE.

For Theorems B and C, $\delta \in (0, 1/4)$ is fixed, $\mathcal{X}_N^\delta = \{N/2^j :\ j \ge 2 \text{ an integer},\ 2^{-j} \ge \delta\}$, and $L = \log x$. For K bands $\mathcal{P}_1, \dots, \mathcal{P}_K$ of primes, put $\mathcal{P}_i^N = \{p \in \mathcal{P}_i :\ p \nmid N\}$, let $\mathcal{A}^N$ be the set of products $p_1 \cdots p_K$ with $p_i \in \mathcal{P}_i^N$, and let $P_N(x)$ be the number of primes $p \in (x, 2x]$ with $N - p = ah$ for some $a \in \mathcal{A}^N$ and some prime h.

Theorem B. *Assume* $\mathrm{QRH}(\theta)$ *and Hypothesis* $\mathrm{RE}(\mathfrak{B}_{\exp},\theta)$ *for a fixed* $\theta<1$*, together with Proposition 2.2 and the classical inputs. Fix* $A'>0$*, an integer* $K\ge K_0(A')$*, where* $K_0(A')$ *is the explicit threshold of §4, and exponents* $1/10<a_1<\cdots<a_K<1/5$*, and let* $\mathcal{P}_i=\{p:\ \exp(L^{a_i})<p\le\exp(2L^{a_i})\}$*. Then there is* x_0*, depending only on* δ*,* A'*,* K*,* (a_i) *and* θ*, such that for every even* N *and every* $x\in\mathcal{X}_N^{\delta}$ *with* $x\ge x_0$*,*

$$P_N(x)=\sum_{a\in\mathcal{A}^N}\frac{\mathfrak{S}(aN)}{a}\int_x^{2x}\frac{du}{\log u\,\log((N-u)/a)}+O\big(xL^{-A'-1}\big),$$

and the main term equals $\mathfrak{S}(N)(\log 2)^K(x/L^2)\big(1+O_{K,\delta}(L^{-0.1})\big)$*. If* $A'>1$*, every sufficiently large even* N *is* $p+ah$ *with* $p\in(N/4,N/2]$ *and* h *prime and* $a\in\mathcal{A}^N$ *for the bands at* $L=\log(N/4)$*. The implied constants depend only on* δ*,* A'*,* K*,* (a_i) *and* θ*; neither they nor* x_0 *are effective.*

Theorem C. *Assume* $\mathrm{QRH}(7/8)$ *and Hypothesis* $\mathrm{RE}(\mathfrak{B}_{\mathrm{poly}},7/8)$*, together with Proposition 2.2 and the classical inputs. Fix* $A'>1$*, and choose* K*,* c_1 *and* $c_i=5^{i-1}c_1$ *as in §5; let* $\mathcal{P}_i=\{p:\ L^{c_i}<p\le L^{2c_i}\}$ *and* $S_K=c_1+\cdots+c_K$*. Then there is* x_0*, depending only on* δ*,* A' *and the choices of §5, such that for every even* N *and every* $x\in\mathcal{X}_N^{\delta}$ *with* $x\ge x_0$*,*

$$P_N(x)=\mathfrak{S}(N)(\log 2)^K\frac{x}{L^2}\Big(1+O_K\Big(\frac{1}{c_1\log L}\Big)+O_\delta\Big(\frac{S_K\log L}{L}\Big)+O\big(L^{1-A'}\big)\Big).$$

Hence every sufficiently large even N *is* $p+aq$ *with* p *and* q *prime and* a *a product of exactly* K *distinct primes, one from each band at* $L=\log(N/4)$*, none dividing* N*, with* $(\log(N/4))^{S_K}<a\le(\log N)^{2S_K}$*. The implied constants depend only on* δ*,* A' *and these choices; neither they nor* x_0 *are effective.*

The bands of Theorem C are polylogarithmic, but their fixed exponent is enormous. Here $2S_K=c_1(5^K-1)/2$, and the choices of §5 force $K\ge 4801$ and $c_1>8K^2$, so that $2S_K>10^{3363}$; the bound $a\le(\log N)^{2S_K}$ improves on the trivial bound $a<N$ only when $\log N>7.7\cdot10^{3366}$ (§5).

For Theorem D let $L=\log N$. For a modulus d and a real y we compare the prime count $\Psi(y;d,N)$ with the local model $M_N(y;d)$, where

$$\Psi(y;d,N)=\sum_{\substack{1\le n\le\min(y,N-1)\\ n\equiv N\pmod d}}\Lambda(n),$$

$$M_N(y;d)=\frac{d}{\varphi(d)}\frac{N}{\varphi(N)}\,\#\{k\ge1:\ 1\le N-kd\le y,\ (k,N)=1\},$$

and put

$$\mathcal{E}'_N(Q)=\sum_{\substack{1\le d<Q\\ (d,N)=1}}\max_{0\le y\le N}\big|\Psi(y;d,N)-M_N(y;d)\big|. \tag{2}$$

Theorem D. *Assume* $\mathrm{QRH}(\theta)$ *and Hypothesis* $\mathrm{RE}(\mathfrak{B}_{\mathrm{wide}},\theta)$ *for a fixed* $\theta<1$*, together with Proposition 2.2 and the classical inputs. Fix* $0<a<1/5$*, and choose the number* K *of bands, their width, their exponents and the other parameters as in §6, the bands after* a*. Then, uniformly over even* N*,*

$$G(N)=\mathfrak{S}(N)N+O\big(L\,\mathcal{E}'_N(Ne^{-L^a/2})\big)+o(N).$$

If, moreover, there are fixed $\varepsilon>0$ *and* C_H *such that, for every sufficiently large even* N*,*

$$(I'_a)\qquad \mathcal{E}'_N(Ne^{-L^a/2})\le C_HNL^{-1-\varepsilon}, \tag{3}$$

then $G(N)\sim\mathfrak{S}(N)N$*, and every sufficiently large even* N *is a sum of two primes.*

The second assertion is a conditional implication: (I'_a) is beyond known techniques and is not proved here, and Theorem D is not a present route to Goldbach's conjecture. The level $Ne^{-L^a/2}$ exceeds every fixed power N^ν with $\nu < 1$. For general sequences, distribution information of this kind does not by itself detect primes: Selberg's example above satisfies an analogue of (I'_a) at the same level, with the model t/d in place of $M_N(y;d)$ (§6). In Theorem D the parity-sensitive input is a cancellation estimate for a smoothed truncation of Λ at the integers $N-p$, and it is deduced from Hypothesis RE.

1.3. **What is not claimed.** Nothing in this paper is unconditional. Each theorem rests on preprints that had not been refereed when this paper was written, and Theorems B–D also on Hypothesis RE. If QRH(7/8) or one of the preprint statements that Theorem A uses fails, Theorem A says nothing; if Hypothesis RE fails for the band system that one of Theorems B–D uses, that theorem says nothing. QRH(7/8) can fail only if both [34, Theorem 1.1] and the proof announced in [1] are wrong. No threshold in this paper is claimed to be effective, and Theorem A gives no rate. Theorems B and C are proved only for the dyadic $x \in \mathcal{X}_N^\delta$, and they are statements about the sum over $a \in \mathcal{A}^N$: they give some a for each large even N, and nothing for an individual a. Since $1 \notin \mathcal{A}^N$, neither Theorem B nor Theorem C gives a representation of N as a sum of two primes, and Theorem A, which concerns $\tau_k(N-p)$ with $k \geq 3$, gives none either. We claim no result for $\mu(N-p)$ or $\lambda(N-p)$. Hypothesis (I'_a) is not proved; we know of no result that contradicts it, and §6 compares it with the known irregularities of primes in progressions to large moduli.

1.4. **Prior work and credit.** The auxiliary band primes and the operator method behind Theorems B–D come from OpenAI's preprint [33], whose method builds on the work of Helfgott and Radziwiłł [19] and its development by Pilatte [37] and whose Theorem 3.1 is stated for every function F with $|F| \leq 1$ that is invariant under multiplication by the band primes, and from Lü, Wang and Wu [28], who adapted it to $\sum_{p \leq x} \mu(p+2)$. Guo [18] carried these arguments over to the reflection $N-p$. Assuming that the arguments of [28, version 1, §§4.2–4.7 and (A.2)–(A.4)] are correct as written and, for general N, a zero-free region for Dirichlet L-functions close to the line $\Re s = 1$, which any zero-free half-plane $\Re s > \theta$ with $\theta < 1$ implies, he proves $\sum_{p<N} \mu(N-p) = o(\pi(N))$ as $N \to \infty$ through the even integers, and the same for λ; version 2 adds the rate $O(\pi(N)/\sqrt{\log\log N})$. When $\mathrm{rad}(N) \leq (\log N)^{O(1)}$ he does not need the zero-free assumption; if the proof announced in [1] holds, it supplies that zero-free region for all large N. His proof contains reflected minor- and major-arc estimates for the cofactor square [18, version 2, Propositions 5.3 and 6.1 and Theorem 6.13], and Hypothesis RE asserts estimates of the same form. For bands of the type of Theorem B he obtains such estimates, under the same assumptions, with different parameters; for instance, in version 2 his bilinear variables are $\exp((\log N)^{3/4})$-rough, and his bands are based on $\log N$ rather than on $\log x$. His major-arc estimate is proved for the specific coefficients of his proof, not for the whole class in (RE2). The bands of Theorem C, of polylogarithmic primes, lie below the range of exponents that Guo treats, all of which exceed $1/10$; and each of his bands has reciprocal mass $\log 2 + o(1)$, while the bands of Theorem D have reciprocal mass tending to infinity.

The extraction of a band divisor $a = p_1 \cdots p_K$ from a shifted value also appears at fixed shifts, in Jiang, Lau and Lü [23] and in Hu, Lü, Wang and Wu [21]. Theorem A transfers the method of [21], which rests on [33], [35] and Walsh's work [42], from fixed shifts to the reflection; their Theorem 1.1 as stated does not imply it, since its error term and threshold may depend on the shift. The divisor model, the auxiliary marks, the graph constructions and the final assembly are those of [33], [35], [42] and [21]; the arguments written here are the transfer lemmas of §3 and their uniformity in N. For $k = 2$ the reflected problem is classical, and [2, Theorem 1.1] gives a sharper unconditional result, uniform in N; we claim nothing for $k = 2$. Elliott's Gaussian refinement of the asymptotic for $\sum_{p<n} 2^{\omega(n-p)}$ was proved by Gorodetsky and Grimmelt [16, Theorem 1.1]. Under

an Elliott–Halberstam hypothesis at a level above $1-1/k$, the asymptotic of Theorem A follows by a classical argument (§3; for fixed shifts see [29, §1.1]).

Read for the weight $\mathbf{1}_{\mathcal{A}^N}$, which depends on N and vanishes at 1 (Friedlander and Iwaniec take their weight to be 1 at 1 [15, §8]), their generalised Hardy–Littlewood conjecture [15, (12-1)–(12-2)] predicts the asymptotics of Theorems B and C. Li and Liu [26, Theorem 1.1], extending Chen [6], state that every sufficiently large even N is $p+rq$ with p and q prime and $r=1$ or a prime $r\le q^{0.9}$; in Theorems B and C the cofactor a is composite. Under zero-free half-plane hypotheses the exceptional set of binary Goldbach has been bounded further [38, 25]. J. Li and J. Liu have posted a derivation of binary Goldbach for every large even N under the generalised Riemann hypothesis for Dirichlet L-functions together with a further hypothesis of theirs [27]. Gou, Wang and Xu derive $G(N)=\mathfrak{S}(N)N+O_\varepsilon(N^{1-\omega_D+\varepsilon})$ for every even N and every $\varepsilon>0$ from the generalised Riemann hypothesis for Dirichlet L-functions and a ratios conjecture for primitive Dirichlet L-functions, which they assume uniformly over all conductors with a power saving $\omega_D>0$ in the error term [17, Theorem 1.1 and Hypothesis 1.1]. If the proof announced in [1] holds, the first of these derivations rests on the further hypothesis of Li and Liu alone, and the second on the ratios conjecture alone.

Conditional reductions of binary Goldbach of a related kind are known. Continuing work of Pan [36], Huang and Li show that the Elliott–Halberstam conjecture for the primes at level $N^{\theta_{\mathrm{HL}}}(\log N)^{12+\epsilon}$, together with its variant twisted by the Möbius function at level $N^{1-\theta_{\mathrm{HL}}}$, for some fixed $0<\theta_{\mathrm{HL}}<1$, implies that every sufficiently large even integer is a sum of two primes, and remark that their hypotheses are needed only for the residue class $N \bmod q$ [22, Theorem 1, Corollary 1 and the remark after it]. An analogous result for twin primes is due to Murty and Vatwani [31]. Theorem D differs in its levels and in the form of its inputs. Where Huang and Li fix $\theta_{\mathrm{HL}}<1$, the distribution hypothesis (I'_a) is at the level $Ne^{-L^a/2}$, which eventually exceeds N^ν for every fixed $\nu<1$, but asks only for a saving $L^{-1-\varepsilon}$ against the exact local model $M_N(y;d)$. Its Möbius-twisted input is one smoothed combination of Möbius sums at moduli below $\exp(L^a)$, deduced from Hypothesis RE rather than assumed. §6 also compares Theorem D with the asymptotic sieves of Bombieri [5] and of Friedlander and Iwaniec [14], and with [11].

We found no earlier statement of Theorems A–D in the literature we searched.

1.5. **Organisation.** §2 states the zero-free half-plane, the preprint inputs, the band systems and Hypothesis RE, and the classical inputs, with a table of which theorem uses which. §3 proves Theorem A, §4 Theorem B, and §5 Theorem C, by the changes that polylogarithmic bands require in the proof of Theorem B. §6 proves Theorem D, the conditional reduction. Appendix A restates the numbered statements of the preprints that are used, with their versions, and Appendix B records how the band scale enters estimates of the kind that Hypothesis RE asserts for polylogarithmic bands, as evidence for $\mathrm{RE}(\mathfrak{B}_{\mathrm{poly}},7/8)$; it is a census of uses, not a proof.

2. Hypotheses, inputs and notation

Every result of this paper is conditional, and this section states every premise. There are two kinds. The first are statements from recent preprints, none of which had been refereed when this paper was written: a zero-free half-plane for Dirichlet L-functions, QRH(7/8), asserted in [34] (it would also follow from the result announced in [1]; §2.1), and numbered statements of [33], [35], [42], [21] and [28] (§2.2). The second is Hypothesis $\mathrm{RE}(\mathfrak{B},\theta)$ of §2.3, which we formulate ourselves: a pair of reflected minor- and major-arc estimates for a determinant square of Lü, Wang and Wu [28], at three systems $\mathfrak{B}$ of auxiliary prime bands. Hypothesis RE is not proved in this paper, and no proof of it is in print.

If QRH(7/8) or one of the inputs of §2.2 that Theorem A uses fails, Theorem A says nothing. If Hypothesis RE fails for the band system that one of Theorems B, C and D uses, that theorem says nothing. In Theorems B–D the zero-free half-plane enters only through the major-arc half (RE2) of Hypothesis RE. Table 1 in §2.5 lists which theorem uses which input.

2.1. **The zero-free half-plane.**

Definition 2.1. For a real number $\theta < 1$, QRH(θ) is the statement: for every integer $q \geq 1$ and every Dirichlet character χ modulo q, $L(s,\chi) \neq 0$ for $\Re s > \theta$. (For $q = 1$ this includes $\zeta(s)$; the pole of $L(s,\chi)$ at $s = 1$ for principal χ is allowed.)

Theorem 1.1 of the preprint [34] asserts QRH(7/8). That preprint has not been refereed, and we use QRH(7/8), and in Theorems B and D QRH(θ) for a fixed $\theta < 1$, as a hypothesis. On 20 November 2026 L. Alpöge announced a proof, formalised in Lean, of the Riemann hypothesis and of the generalised Riemann hypothesis for Dirichlet L-functions [1]. The generalised Riemann hypothesis for Dirichlet L-functions implies QRH(θ) for every $\theta \in [1/2, 1)$, so QRH(7/8) has a second source. We have not checked the announced proof, and it has not been refereed; we keep QRH(θ) as a hypothesis, and no proof in this paper uses the announcement. If $\theta \leq \theta' < 1$, then QRH(θ) implies QRH(θ'). Each theorem is stated with the exponent its proof uses: Theorems A and C under QRH(7/8), the case asserted in [34], and Theorems B and D under QRH(θ) for an arbitrary fixed $\theta < 1$, which includes the case $\theta = 7/8$. In the proof of Theorem A, QRH(7/8) is used only through the estimates of §3.4.

2.2. **Inputs from preprints.** *Theorem A* uses the following statements; Appendix A lists each one with its quantifiers.

- From [33] (version of 24 September 2026): Lemmas 2.1–2.6; Theorem 3.1 with the part of its proof in §§3–4, namely Lemmas 3.2–3.4, Proposition 4.1 and equations (3.12)–(4.64), applied to bands from which the primes dividing N are deleted (§3 checks that [33, §§3–4] use the bands only through properties that such subsets inherit).
- From [35] (version of 24 September 2026): Lemma 2.9, Definitions 3.1–3.3, Lemmas 3.4 and 3.6–3.10, Theorem 3.5 and Theorem 4.1. Theorem 3.5 is printed for a polylogarithmic displacement; its extension is proved in §3.
- From [42] (arXiv version 1, whose numbering version 2 of 12 November 2026 keeps): Lemmas 2.1–2.3, 5.2, 6.2 and 7.1, and the definitions (7.1)–(7.14). For bands from which the primes dividing N are deleted, Lemma 2.3 is used through its proof.
- From [21] (arXiv version 1): Lemmas 2.1–2.2, the divisor progression estimate (3.2), the finite-convolution algebra of §§3 and 5, and the selection and damping-removal arguments of §§5–6, with the constants of (2.7).

The main fixed-shift theorems of these four papers are not used at the shift $-N$; where a proof is transferred to the reflection, the transfer is written out in §3. Theorem A does not use [28].

Theorems B, C and D use from [28] (arXiv version 1, whose numbering is used throughout) the following identity, a form of Vaughan's identity on Z-sifted integers, and Theorem D also uses the Bombieri–Vinogradov bound for μ in the form quoted in [28, (3.1)] (Lemma 2.14); the other arguments of [28] that §§4–6 adapt are written out there. We copy it with the notation of [28, (2.1)].

Proposition 2.2 ([28, Proposition 2.1])**.** *Let $x \geq 2$, and put $L = \log x$, $Z = \exp(L^{0.24})$, $U = V = x^{1/5}$ and $R_Z(n) = \mathbf{1}_{P^-(n)>Z}$, with the convention $P^-(1) = \infty$, so that $R_Z(1) = 1$. Put $c_V(r) = \sum_{\ell \mid r,\ \ell > V} \Lambda(\ell)$. If F is bounded on $(x, 2x]$, then*

$$\sum_{x<n\leq 2x} \Lambda(n)F(n) = T_1(F) - T_2(F) + T_3(F) + O\big(\|F\|_\infty Z\big),$$

where $\|F\|_\infty = \max_{x<n\le 2x}|F(n)|$ *and*

$$T_1(F) = \sum_{d\le U}\mu(d)R_Z(d)\sum_{x<dk\le 2x} R_Z(k)(\log k)F(dk),$$
$$T_2(F) = \sum_{\substack{d\le U\\ \ell\le V}}\mu(d)\Lambda(\ell)R_Z(d\ell)\sum_{x<d\ell k\le 2x} R_Z(k)F(d\ell k),$$
$$T_3(F) = \sum_{\substack{d>U,\ r>V\\ x<dr\le 2x}}\mu(d)R_Z(d)c_V(r)R_Z(r)F(dr).$$

The proof in [28] multiplies Vaughan's identity, valid for $n>V$, by $R_Z(n)F(n)$, uses $R_Z(dk)=R_Z(d)R_Z(k)$, and bounds by $O(\|F\|_\infty Z)$ the prime powers $p^j\in(x,2x]$ with $p\le Z$, which R_Z removes. We call T_1 and T_2 the Type I sums and T_3 the Type II sum, as in [28].

Remark 2.3 (Type I forms). Put $\gamma_q=\sum_{u\ell=q,\ u\le U,\ \ell\le V}\mu(u)\Lambda(\ell)$, so that $|\gamma_q|\le\log q$. Grouping $q=d\ell$ gives $T_2(F)=\sum_{q\le x^{2/5},\,R_Z(q)=1}\gamma_q\sum_{x<qk\le 2x}R_Z(k)F(qk)$, and q, k are odd because $Z>2$. In $T_1(F)$ write $\log k=\log(qk)-\log q$ and remove $\log(qk)$ by partial summation in the upper endpoint. Then both Type I sums are controlled, at the cost of a factor $O(L)$, by sums $\sum_{q\le x^{2/5}}\gamma'_q\sum_{x<qk\le y}R_Z(k)F(qk)$ with $y\in(x,2x]$, q odd and Z-rough, and $|\gamma'_q|\le L$; so a bound for these sums that is uniform in the upper endpoint y bounds both Type I sums [28, proof of Proposition 3.1].

Proposition 3.1 of [28] is not used as stated: it concerns $\mu(n+2)$ with a band weight, and its proof is the template for the Type I estimates of §§4.3 and 6.6, which we write out. Propositions 4.2 and 5.3 of [28] are the models for Hypothesis RE (Remark 2.11) and are not used otherwise.

2.3. **Band systems and Hypothesis RE.**

Common data. Throughout this subsection N is even and x is a real number of the form $x=2^{-j}N$ with $j\ge 1$ an integer. We put $L_x=\log x$, $Z=\exp(L_x^{0.24})$ and $R_Z(n)=\mathbf{1}_{P^-(n)>Z}$ as in Proposition 2.2; we write L_x rather than L so that the hypothesis reads the same in every section. The Type II variables run over $m\in(M,2M]$ and $n\in(\mathfrak{N},2\mathfrak{N}]$ with real $M,\mathfrak{N}\ge x^{1/5}/2$, and $X:=M\mathfrak{N}$.

Fix, once and for all, a smooth $\eta\colon(0,\infty)\to[0,1]$ supported in $[1,4]$ with $\sum_{k\in\mathbb{Z}}\eta(u/2^k)=1$ for every $u>0$ (for instance $\eta(u)=\rho(u)-\rho(u/2)$ with ρ smooth, nondecreasing, $\rho=0$ on $(0,1]$ and $\rho=1$ on $[2,\infty)$), and a real $\psi_1\in C_c^\infty(\mathbb{R})$ with $\psi_1=1$ on $[-4,4]$ [28, (4.9)]. The mark scale Y runs over $2^{\mathbb{Z}}$.

The precision chain.

Definition 2.4. Let $A>0$ and $C>0$. A pair (E_0,K) is *admissible for* (A,C) if E_0 is real with $E_0>A+4C+2$ and K is an integer with $K>200(E_0+8)$. We then put

$$G:=E_0+5,\qquad A_0:=\lceil G+4\rceil,$$

so that $K>200(G+3)$.

Here A is the precision of the estimates, C the exponent in the coefficient bound L_x^C, and A_0 the exponent of the major arcs below. The choices $E_0>A+4C+2$, $G=E_0+5$, $A_0>G+3$ and $K>200(G+3)$, made in this order and independently of the band exponents, are those of [28, §§4.4–4.7 and 5.3]; Definition 2.4 fixes $A_0=\lceil G+4\rceil$.

The three band systems.

Definition 2.5. Let $A, C > 0$ and let (E_0, K) be admissible for (A, C). A *band system* with K bands assigns to each large x sets $\mathcal{P}_1, \dots, \mathcal{P}_K$ of primes, given below in terms of L_x. Its *puncturing* at the even integer N is

$$\mathcal{P}_i^N := \{p \in \mathcal{P}_i : \ p \nmid N\}, \qquad V_i^N := \sum_{p \in \mathcal{P}_i^N} \frac{1}{p}, \qquad \mathcal{A}^N := \{p_1 \cdots p_K : \ p_i \in \mathcal{P}_i^N \ (1 \le i \le K)\}.$$

A *datum for* $\mathfrak{B}$ *at* (A, C) is (E_0, K) together with the band parameters of one of the following.

$\mathfrak{B}_{\exp}$: fixed exponents $1/10 < a_1 < \cdots < a_K < 1/5$, and $\mathcal{P}_i = \{p : \ \exp(L_x^{a_i}) < p \le \exp(2L_x^{a_i})\}$.

$\mathfrak{B}_{\mathrm{poly}}$: a fixed $\varepsilon_\sharp > 0$ and a fixed real $c_1 > c_{\mathrm{poly}}$, where, with

$$C^\sharp := 2K + 3A_0 + 4, \qquad b_\sharp := 2C^\sharp + 2E_0 + 15, \qquad h_\sharp := C^\sharp + E_0 + 4,$$
$$\varkappa := 2.02K + 2G + A + 4C + 3A_0 + 2 + \varepsilon_\sharp,$$
$$\Gamma^{\mathrm{LWW}} := A + \max\bigl(C_1^{\mathrm{LWW}}, C_3^{\mathrm{LWW}}, C_4^{\mathrm{LWW}} + 3A_0\bigr) + 1,$$
$$c_{\mathrm{poly}} := \max\bigl\{5 + h_\sharp b_\sharp,\ \varkappa,\ 4K + 6A_0 + 2G + 2,\ G + 3A_0 + 2,\ \Gamma^{\mathrm{LWW}},\ 80(A + 20A_0 + 2C + 15)\bigr\};$$

then $c_i := 5^{i-1}c_1$ and $\mathcal{P}_i = \{p : \ L_x^{c_i} < p \le L_x^{2c_i}\}$. Here C_1^{LWW} is the exponent called C_1 in the proof of [28, Lemma 4.1], which depends only on the coefficient-size exponent, and C_3^{LWW}, C_4^{LWW} are the fixed exponents called C_3, C_4 in [28, (4.74) and (4.76)], which depend on the coefficient bounds; no numerical values for them are printed there. The system $\mathfrak{B}_{\mathrm{poly}}$ is used only with $\theta = 7/8$.

$\mathfrak{B}_{\mathrm{wide}}$: a fixed g' with $0 < g' \le 1/(40K)$ and fixed exponents with

$$\frac{1}{10} < a_1, \qquad a_{i+1} - a_i \ge 2g' \ \ (1 \le i < K), \qquad a_K + g' < \frac{1}{5},$$

and $\mathcal{P}_i = \{p : \ \exp(L_x^{a_i}) < p \le \exp(L_x^{a_i + g'})\}$.

Remark 2.6. (i) In each system the bands are pairwise disjoint for large x, and every band prime is odd and lies below Z. Band primes exceed every fixed power of L_x for $\mathfrak{B}_{\exp}$ and $\mathfrak{B}_{\mathrm{wide}}$, and exceed $L_x^{c_1}$ for $\mathfrak{B}_{\mathrm{poly}}$. By construction no band prime divides N; we call this convention (G1).

(ii) Let N be even and $x = 2^{-j}N$ with $N \le xL_x$. For large x this holds at every triple of $\mathcal{R}_{\delta_0}(N)$, with δ_0 fixed, and of $\mathcal{R}_{\mathrm{wide}}(N)$ (Definition 2.8), at the scales $x \in \mathcal{X}_N^\delta$ of Theorems B and C, and on the blocks of §6.5. Then $\omega(N) \le \log N/\log 2 \le (L_x + \log L_x)/\log 2 \le 2L_x$ for large x, and, uniformly in such N, Mertens' theorem gives $V_i^N = \log 2 + o(1)$ for $\mathfrak{B}_{\exp}$; $V_i^N = \log 2 + O(1/(c_1 \log L_x))$ for $\mathfrak{B}_{\mathrm{poly}}$; and $V_i^N = g' \log L_x + O(L_x^{-a_i}) + O(\omega(N) e^{-L_x^{a_i}})$ for $\mathfrak{B}_{\mathrm{wide}}$. In particular $\prod_i V_i^N = L_x^{o(1)}$ in every case, and $\prod_i V_i^N \asymp_K 1$ for $\mathfrak{B}_{\exp}$ and $\mathfrak{B}_{\mathrm{poly}}$.

(iii) For $\mathfrak{B}_{\mathrm{poly}}$, $S_K := c_1 + \cdots + c_K = c_1(5^K - 1)/4$ and every $a \in \mathcal{A}^N$ satisfies $L_x^{S_K} < a \le L_x^{2S_K}$. Since $h_\sharp b_\sharp > 2(C^\sharp)^2 > 8K^2$, the condition $c_1 > c_{\mathrm{poly}}$ gives $c_1 > 8K^2$. It also gives $c_1 > G + 3A_0 + 2 > A + 7$ and $c_1 > A_0 + 1$. So the conditions $c_1 > 2$, $c_1 \ge A + 7$ and $c_{i+1} > 2c_i$ that the deduction in §5.2 uses hold for every datum of $\mathfrak{B}_{\mathrm{poly}}$. The entry $80(A + 20A_0 + 2C + 15)$ is $80(1 + 2A_{\mathrm{s}})$ at the least value $2A_{\mathrm{s}} = A + 20A_0 + 2C + 14$ of the threshold A_{s} of Appendix B, and its factor 80 is tied to $\theta = 7/8$; Appendix B explains how each entry of c_{poly} arises.

(iv) For $\mathfrak{B}_{\mathrm{wide}}$ the masses V_i^N grow like $g' \log L_x$, and the bands lie in $(\exp(L_x^{1/10}), \exp(L_x^{1/5}))$.

Coefficients and ranges.

Definition 2.7. Let $C, B > 0$. A sequence $\alpha = (\alpha_m)$ of complex numbers is in $\mathcal{C}_N(M)$ if, for some interval $I \subseteq (M, 2M]$, it is supported on the integers $m \in I$ with $R_Z(m) = 1$ and $(m, N) = 1$, and satisfies $|\alpha_m| \le L_x^C$; similarly for $\mathcal{C}_N(\mathfrak{N})$. It is in $\mathcal{C}_N^\mu(M; B)$ if

$$\alpha_m = \mu(m) R_Z(m) \mathbf{1}_{(m,N)=1} \mathbf{1}_I(m)\, m^{iv}$$

for an interval $I \subseteq (M, 2M]$ and a real v with $|v| \le L_x^{2B}$. The support condition $(m, N) = 1$ is convention (G2).

Definition 2.8. Let N be even. For a fixed $\delta_0 \in (0,1)$, the range $\mathcal{R}_{\delta_0}(N)$ is the set of triples $(x, M, \mathfrak{N})$ with $x = 2^{-j}N$ for an integer $j \ge 1$, real $M, \mathfrak{N} \ge x^{1/5}/2$, and $X = M\mathfrak{N}$ satisfying

$$x/5 < X < \min(3x, N) \qquad \text{and} \qquad X \ge \delta_0 N.$$

The range $\mathcal{R}_{\mathrm{wide}}(N)$ is defined in the same way with $X \ge \delta_0 N$ replaced by $N/X \le L_x^{1/40}$.

The two squares.

Definition 2.9. Let a datum for $\mathfrak{B}$ at (A, C) be given, let N be even, $(x, M, \mathfrak{N})$ as above, $Y \in 2^{\mathbb{Z}}$, and $\alpha = (\alpha_m)$, $\beta = (\beta_n)$ complex sequences supported on $(M, 2M]$ and $(\mathfrak{N}, 2\mathfrak{N}]$ respectively. For $h \in \mathbb{Z} \smallsetminus \{0\}$ put

$$S_Y(h) := \sum_{a \in \mathcal{A}^N} \eta(a/Y) \sum_{\substack{m,n \ge 1 \\ N - mn = ah}} \alpha_m \beta_n,$$

and define the *signed square*

$$\begin{aligned} Q_Y^{\mathrm{sgn}} &:= \Big(\prod_{i=1}^{K} V_i^N\Big)^{-2} \sum_{h \in \mathbb{Z} \smallsetminus \{0\}} |S_Y(h)|^2 \\ &= \Big(\prod_{i=1}^{K} V_i^N\Big)^{-2} \sum_{a,b \in \mathcal{A}^N} \eta\Big(\frac{a}{Y}\Big)\eta\Big(\frac{b}{Y}\Big) \sum_{\substack{m,n,r,s \ge 1 \\ (N-mn)/a = (N-rs)/b \in \mathbb{Z} \smallsetminus \{0\}}} \alpha_m \overline{\alpha_r} \beta_n \overline{\beta_s}. \end{aligned}$$

With the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\|\cdot\|$ the distance to the nearest integer and $e(t) = e^{2\pi i t}$, put

$$\mathfrak{M}_1 := \bigcup_{1 \le k \le L_x^{A_0}} \bigcup_{\substack{h \bmod k \\ (h,k)=1}} \big\{\varphi \in \mathbb{T} : \ \|\varphi - h/k\| \le 2L_x^{A_0}/Y\big\}, \qquad \mathfrak{M}_N := \{\varphi \in \mathbb{T} : \ N\varphi \in \mathfrak{M}_1\},$$

with A_0 from Definition 2.4, and for integers t

$$H_N(t; a, b) := \psi_1\Big(\frac{t}{NY}\Big) \int_{\mathfrak{M}_N} e\big(\varphi\,(t - N(b-a))\big)\, d\varphi.$$

The *reflected major-arc model* is

$$Q_Y^{(N),\mathrm{maj}} := \Big(\prod_{i=1}^{K} V_i^N\Big)^{-2} \sum_{a,b \in \mathcal{A}^N} \eta\Big(\frac{a}{Y}\Big)\eta\Big(\frac{b}{Y}\Big) \sum_{m,n,r,s \ge 1} \alpha_m \overline{\alpha_r} \beta_n \overline{\beta_s}\, H_N(bmn - ars; a, b).$$

In both squares the pairs (a, b) with a common prime factor are included.

Lemma 2.10. (i) *For every integer t,*

$$H_N(t; a, b) = \psi_1\Big(\frac{t}{NY}\Big)\, \mathbf{1}_{N \mid t} \int_{\mathfrak{M}_1} e\big(\varphi\,(t/N - b + a)\big)\, d\varphi.$$

(ii) *If $(a, b) = 1$, the integers m, n, r, s are odd and N is even, then $(N - mn)/a = (N - rs)/b \in \mathbb{Z} \smallsetminus \{0\}$ if and only if*

$$bmn - ars = -N(a - b). \tag{4}$$

Proof. (i) Let $\widetilde{\mathfrak{M}}_1 \subset [0,1)$ represent $\mathfrak{M}_1$. Then $\mathfrak{M}_N$ is represented by the disjoint union of the sets $(\widetilde{\mathfrak{M}}_1 + j)/N$, $0 \le j < N$. For an integer u, $\int_{\mathfrak{M}_N} e(\varphi u)\, d\varphi = N^{-1} \sum_{0 \le j < N} e(ju/N) \int_{\widetilde{\mathfrak{M}}_1} e(\varphi' u/N)\, d\varphi'$, which is $\mathbf{1}_{N \mid u} \int_{\mathfrak{M}_1} e(\varphi' u/N)\, d\varphi'$, since for $N \mid u$ the integrand is 1-periodic. Take $u = t - N(b-a)$; then $N \mid u$ if and only if $N \mid t$, and $u/N = t/N - b + a$.

(ii) If both quotients equal $h \neq 0$, then $b(N-mn) = a(N-rs)$, which is (4). Conversely, (4) says $b(N-mn) = a(N-rs)$; since $(a,b) = 1$, $a \mid N-mn$ and $b \mid N-rs$, and the two quotients are equal. They are nonzero because $N-mn$ is odd. □

Since the coefficients of Definition 2.7 are supported on Z-rough, hence odd, integers, Lemma 2.10(ii) applies to every pair of coprime marks in Q_Y^{sgn}.

Remark 2.11 (The model at the shift $+2$). Lü, Wang and Wu define the corresponding quantities Q_Y and Q_Y^{maj} for the shifted values $mn+2$ in place of the reflected values $N-mn$ [28, (4.5) and (4.9)], with unpunctured bands. Replacing $mn+2$ by $N-mn$ turns their determinant equation $bmn - ars = 2(a-b)$ into (4) and their kernel H_{-2} into H_N. Their Proposition 4.2 bounds $Q_Y - Q_Y^{\mathrm{maj}}$ and their Proposition 5.3 bounds Q_Y^{maj}. In Q_Y^{sgn} the common quotient h runs over all nonzero integers, since $N-mn$ need not be positive for m, n in a dyadic box.

The hypothesis.

Hypothesis RE($\mathfrak{B},\theta$)**.** *Let $\mathfrak{B}$ be one of $\mathfrak{B}_{\mathrm{exp}}$, $\mathfrak{B}_{\mathrm{poly}}$, $\mathfrak{B}_{\mathrm{wide}}$, and let $\theta < 1$, with $\theta = 7/8$ if $\mathfrak{B} = \mathfrak{B}_{\mathrm{poly}}$. Let $A, C, B > 0$ be fixed, let a datum for $\mathfrak{B}$ at (A,C) be fixed (Definitions 2.4 and 2.5), and, if $\mathfrak{B} \neq \mathfrak{B}_{\mathrm{wide}}$, let $\delta_0 \in (0,1)$ be fixed. Then there are x_0 and C_0 such that, for every even N, every $(x, M, \mathfrak{N})$ in $\mathcal{R}_{\delta_0}(N)$ (or in $\mathcal{R}_{\mathrm{wide}}(N)$ if $\mathfrak{B} = \mathfrak{B}_{\mathrm{wide}}$) with $x \geq x_0$, and every $Y \in 2^{\mathbb{Z}}$, the following hold.*

(RE1) (Reflected minor arcs.) *For all $\alpha \in \mathcal{C}_N(M)$ and $\beta \in \mathcal{C}_N(\mathfrak{N})$,*

$$\left|Q_Y^{\mathrm{sgn}} - Q_Y^{(N),\mathrm{maj}}\right| \leq C_0\, XY\, L_x^{-A}.$$

(RE2) (Reflected major arcs.) *If* QRH(θ) *holds, then for all $\alpha \in \mathcal{C}_N^{\mu}(M;B)$ and $\beta \in \mathcal{C}_N(\mathfrak{N})$,*

$$\left|Q_Y^{(N),\mathrm{maj}}\right| \leq C_0\, XY\, L_x^{-A}.$$

The numbers x_0 and C_0 may depend on A, C, B, the datum, δ_0 (if present), θ, η and ψ_1, but not on N, x, M, $\mathfrak{N}$, Y, v, the intervals or the coefficients. They need not be effective.

Remark 2.12. (i) *How RE is used.* Theorems B–D use Hypothesis RE only through the consequence $|Q_Y^{\mathrm{sgn}}| \leq 2C_0\, XY\, L_x^{-A}$ of (RE1) and (RE2), for $\alpha \in \mathcal{C}_N^{\mu}(M;B)$ and $\beta \in \mathcal{C}_N(\mathfrak{N})$. Theorems B and C apply it with $C = 2$ and $A = 2A' + 4$, where A' is their target precision (§§4.4 and 5.2), and Theorem D with $C = 1$ and $A = D_* > 18$ (§6.7).

(ii) *Where* QRH(θ) *enters.* (RE2) is formulated as a consequence of QRH(θ), and (RE1) involves no zero-free region. In the proofs of Theorems B–D, QRH(θ) is used nowhere else. If the proof announced in [1] holds, the antecedent of (RE2) is true for every $\theta \in [1/2, 1)$, so that for such θ (RE2) asserts its major-arc bound outright; Hypothesis RE remains unproved (part (v)).

(iii) *Normalisation.* $Q_Y^{\mathrm{sgn}} \geq 0$. Replacing β by β/c with $c > 0$ multiplies Q_Y^{sgn} and $Q_Y^{(N),\mathrm{maj}}$ by c^{-2}, and likewise for α. So if $|\beta_n| \leq c\, L_x^C$ for a fixed $c \geq 1$, (RE1) and (RE2) applied to $(\alpha, \beta/c)$ give the same bounds for (α,β) with $c^2 C_0$ in place of C_0. In (RE1) a sequence α with $|\alpha_m| \leq c\, L_x^C$ may be rescaled in the same way, at a further factor c^2. A sequence in $\mathcal{C}_N^{\mu}(M;B)$ satisfies $|\alpha_m| \leq 1$ and is not rescaled.

(iv) *Uniformity in N.* Hypothesis RE is stated uniformly over all even N, including those with many small prime factors. Under the puncturing (G1) and the support condition (G2), every mark and every Type II variable is a unit modulo N.

(v) *Status.* Hypothesis RE is a statement of ours. It is not proved in this paper, and no proof of it is in print; §1.4 describes the earlier work closest to it. For $\mathfrak{B}_{\mathrm{poly}}$, Appendix B records how the band scale enters estimates of this kind, as evidence for RE($\mathfrak{B}_{\mathrm{poly}}, 7/8$); that record is a census of uses, not a proof.

2.4. **Classical inputs.** We use the following results from the literature in the forms stated.

Lemma 2.13 (Bombieri–Vinogradov; [24, Theorem 18.9, display (18.11)])**.** *Fix $A \ge 0$. For $x \ge 2$ and $1 \le Q \le x^{1/2}/(\log x)^{A+3}$,*

$$\sum_{q\le Q} \max_{y\le x} \max_{(a,q)=1} \Big|\pi(y;q,a) - \frac{\mathrm{li}(y)}{\varphi(q)}\Big| \ll_A \frac{x}{(\log x)^{A+1}}, \qquad \mathrm{li}(y) = \int_2^y \frac{dt}{\log t}.$$

This is due to Bombieri [3] and A. I. Vinogradov [41]; the proof gives no effectively computable implied constant [4, §7, p. 57]. Applied with $\max(A-1,0)$ in place of A and restricted to $2 \le z \le H$, it gives, for every $A > 0$, $H \ge 2$ and $Q \le H^{1/2}(\log H)^{-\max(A,1)-2}$,

$$\sum_{m\le Q} \max_{(c,m)=1} \max_{2\le z\le H} \Big|\pi(z;m,c) - \frac{\mathrm{li}(z)}{\varphi(m)}\Big| \ll_A H(\log H)^{-A},$$

the form used in §4.3.

The next two statements are the Möbius analogue and its divisor-weighted form; with Proposition 2.2 and Lemma 2.16 below they are the inputs of the Type I estimate in §6.6, and they are not part of Hypothesis RE.

Lemma 2.14 ([28, (3.1)])**.** *For every $A > 0$ there is $B = B(A) > 0$ such that, for $x \ge 2$,*

$$\sum_{q\le x^{1/2}(\log x)^{-B}} \max_{(a,q)=1} \max_{y\le x} \Big| \sum_{\substack{n\le y\\ n\equiv a \pmod q}} \mu(n)\Big| \ll_A \frac{x}{(\log x)^A}.$$

Lemma 2.15. *For every integer $j \ge 0$ and every $P > 0$ there is $B = B(P,j)$ such that, for $y \ge 2$,*

$$\sum_{q\le y^{1/2}(\log y)^{-B}} \tau(q)^j \max_{(c,q)=1} \max_{z\le y} \Big| \sum_{\substack{n\le z\\ n\equiv c \pmod q}} \mu(n)\Big| \ll_{P,j} y(\log y)^{-P}.$$

Proof. This is the argument of [28, §3] for $j = 1$. Let $E(q)$ denote the double maximum, so that $E(q) \le y/q + 1$, and let $Q = y^{1/2}(\log y)^{-B}$. By Cauchy's inequality,

$$\Big(\sum_{q\le Q} \tau(q)^j E(q)\Big)^2 \le \Big(\sum_{q\le Q} E(q)\Big)\Big(\sum_{q\le Q} \tau(q)^{2j}\Big(\frac{y}{q}+1\Big)\Big) \ll_j \Big(\sum_{q\le Q} E(q)\Big) y(\log y)^{4^j},$$

using $\sum_{q\le Q} \tau(q)^{2j}/q \ll_j (\log Q)^{4^j}$ and $\sum_{q\le Q} \tau(q)^{2j} \ll_j Q(\log Q)^{4^j-1}$. Lemma 2.14 with $A = 2P + 4^j$ and $B = B(2P+4^j)$ bounds the first factor by $y(\log y)^{-2P-4^j}$. □

Lemma 2.16 (Rankin)**.** *Let $v_0 \ge 1$ be an integer and $0 < \sigma < 1$. For $E \ge 1$ and $y \ge 1$,*

$$\sum_{\substack{e>E\\ e\mid v_0^\infty}} \frac{1}{e} \le E^{-\sigma} \prod_{p\mid v_0} \big(1-p^{\sigma-1}\big)^{-1}, \qquad \#\{e \le y :\ e \mid v_0^\infty\} \le y^\sigma \prod_{p\mid v_0} \big(1-p^{-\sigma}\big)^{-1}.$$

Proof. Bound $\mathbf{1}_{e>E} \le (e/E)^\sigma$ and $\mathbf{1}_{e\le y} \le (y/e)^\sigma$, and sum over all $e \mid v_0^\infty$ by the Euler product. □

In §6.6 the terms are grouped by their modulus before Lemma 2.15 is applied; each modulus arises from at most a fixed power of the divisor function many terms, which is why the weighted form is needed. This is the grouping of [28, §3]. Lemma 2.16 with $\sigma = 1/2$ and $\sigma = 1/20$ is used there as in [28, (3.5)].

We also use, in the forms stated at their first use:

- the prime number theorem for μ with de la Vallée Poussin's error term, $\sum_{n\le y} \mu(n) \ll y\exp(-c_0\sqrt{\log y})$, hence $\sum_{n\le y} \mu(n) \ll_b y(\log y)^{-b}$ for every fixed b, and $\sum_{n\ge 1} \mu(n)/n = 0$;
- Abel's theorem for Dirichlet series, and the Laurent expansion of the Riemann zeta function at $s = 1$ (in the proof of Lemma 4.5);

- Mertens' estimates $\sum_{p\le y} 1/p = \log\log y + M_0 + O(1/\log y)$ and $\prod_{p\le y}(1-1/p)^{-1} \ll \log y$ [24, Theorem 3.4(b)–(c)], and Chebyshev's bound $\pi(z) \le 2z/\log z$ for $z \ge 2$;
- the divisor-moment bounds $\sum_{n\le y} \tau(n)^j \ll_j y(\log y)^{2^j-1}$ and $\tau^2 \le \tau_4$, and the inequality $\tau_4(n) \le 4\sum_{d\mid n,\ d\le n^{3/4}} \tau_3(d)$;
- the Brun–Titchmarsh inequality in reduced residue classes, also on translated intervals [24, Theorem 20.1] (§6.4);
- Selberg's upper-bound sieve; the uniform constant and remainder that §6.4 needs are proved there;
- for Theorem A: Henriot's New Theorem 5 [20, p. 377], and the local formulas (L1)–(L3) for the logarithmic derivatives of Dirichlet L-functions, uniform in the modulus [40, Proposition 19, Exercise 45(i)–(ii) and display (46)], stated in Appendix A.

TABLE 1. Inputs of Theorems A–D. "Preprint" means not refereed; "announced" means a proof announced, not refereed, and not used in our proofs (§2.1); "—" means not used; "template" means that the proof is adapted and written out (§§4.3 and 6.6). QRH(θ) enters Theorem A only through §3.4, and Theorems B–D only through (RE2). Mertens' estimates and divisor bounds are used throughout.

Input	Status	A	B	C	D
QRH(θ); [34, Thm. 1.1] asserts $\theta = 7/8$; for $\theta \in [1/2, 1)$ it would follow from [1]	preprint; announced	$\theta = 7/8$	fixed $\theta < 1$	$\theta = 7/8$	fixed $\theta < 1$
[33], [35], [42], [21] (§2.2)	preprints	used	—	—	—
[28, Prop. 2.1] (Prop. 2.2)	preprint	—	used	used	used
[28, Prop. 3.1]	preprint	—	template	template	template
Hypothesis RE($\mathfrak{B}, \theta$)	hypothesis	—	$\mathfrak{B}_{\mathrm{exp}}$	$\mathfrak{B}_{\mathrm{poly}}$	$\mathfrak{B}_{\mathrm{wide}}$
Classical (§2.4)	in print	BV, Henriot, (L1)–(L3)	BV, PNT for μ	as for B	BV for μ, Brun–Titchmarsh, Selberg
(I'_a), the hypothesis of Corollary D.1	open	—	—	—	Corollary D.1 only

2.5. **Table of inputs, and notation.** Hypothesis RE is modelled on [28, Propositions 4.2 and 5.3], whose proofs draw on [33] and, through [28, Proposition 2.2], on [30, Lemma 3.9(ii)]. None of these is used in the proofs of Theorems B–D beyond what Hypothesis RE states. Appendix A lists, for each preprint, the numbered statements used and the step that uses each one, with versions pinned.

Table 2 fixes the notation that differs between sections.

3. The reflected higher-order Titchmarsh divisor problem

Throughout this section $k \ge 3$ is a fixed integer and N is a large integer, even or odd; $L := \log N$ and $S_k(N) := \sum_{1\le t<N} \Lambda(t)\tau_k(N-t)$, where $\tau_k(n)$ is the number of ordered k-tuples of positive integers with product n. Implied constants and thresholds may depend on k and on the fixed auxiliary parameters introduced below, but never on the factorisation of N; this is what "uniformly in N" means here. We write $e(t) = e^{2\pi i t}$, put $P^-(1) = \infty$, extend Dirichlet characters by zero on non-units, and write $\widehat{f}(s) = \int_0^\infty f(t)t^{s-1}\,dt$. Numbered statements of [33], [35], [42] and [21] carry their authors' numbering (§2.2; Appendix A); for [42] this is the numbering of version 1, which version 2 (12 November 2026) keeps, and [21] is cited in its arXiv version 1.

TABLE 2. Notation.

Symbol	Meaning	Where
N	the integer whose differences $N-p$ are studied: every large integer in Theorem A, even in Theorems B–D	all
L, L_x	$L=\log N$ in §§3 and 6; $L=\log x$ in §§4 and 5; $L_x=\log x$ always, and in Hypothesis RE	all
x	a block scale $x=2^{-j}N$; Theorems B and C take $x\in\mathcal{X}_N^{\delta}$	§§2,4–6
Z, R_Z, U, V, c_V	$Z=\exp(L_x^{0.24})$; $R_Z(n)=\mathbf{1}_{P^-(n)>Z}$; $U=V=x^{1/5}$; $c_V(r)=\sum_{\ell\mid r,\,\ell>V}\Lambda(\ell)$	§2.2
K; $K_0(A')$	the number of bands; the least admissible K in Theorem B	§§2,4
E_0, G, A_0	the precision chain of Definition 2.4; $G=E_0+5$. With an argument, $G(N)$ is the Goldbach sum of §6	§§2,4–6
$\mathcal{P}_i^N$, V_i^N, $\mathcal{A}^N$	punctured bands, their reciprocal masses, the marks	§§2,4–6
a, b	marks (elements of $\mathcal{A}^N$) in §§2, 4, 5. In §6, where a is the cut exponent of Theorem D, the marks are written A, B	
Y; Y'	the mark scale, $Y\in 2^{\mathbb{Z}}$; the Bombieri–Vinogradov height in §6.6	
M, $\mathfrak{N}$, X	dyadic ranges of the Type II variables, $X=M\mathfrak{N}$	
A; D_*; C; B	the precision of RE; the precision at which §§4–6 apply it; the coefficient exponent; the Mellin height exponent, $\lvert v\rvert\le L_x^{2B}$	
η, ψ_1, $\mathfrak{M}_1$, $\mathfrak{M}_N$, H_N	dyadic cutoff, major-arc cutoff, major arcs, reflected kernel	§2.3
Q_Y^{sgn}, $Q_Y^{(N),\mathrm{maj}}$, $S_Y(h)$	the squares of Hypothesis RE; $Q_a=Ne^{-L^a/2}$ is the level in Theorem D	
θ; θ_{HL}	only the exponent in QRH(θ); the level parameter of [22]	§§1,6
c_i, S_K, $\varepsilon_\sharp$, $\varkappa$, $C^\sharp$, $b_\sharp$, $h_\sharp$	the data of $\mathfrak{B}_{\mathrm{poly}}$; $S_K=c_1+\cdots+c_K$	§§2,5, App. B
g'	the width of the bands of $\mathfrak{B}_{\mathrm{wide}}$	§§2,6
δ; δ_0	$x\ge\delta N$ in Theorems B–C, and $\delta=L^{-g'}$ in §6; the scale parameter of $\mathcal{R}_{\delta_0}(N)$	
ε, C_H	the exponent and the constant in (I'_a) only	§6
C_B	the Bonferroni depth constant of the band indicator in §6.6	§6
H; $\mathbf{1}_H$	the height in Lemma 2.13 and §4.3; in §6, the property that every band is hit, and its indicator	§§2,4,6
κ; $\varkappa$	$\kappa(s)=\sum_{b\mid s}\mu(s/b)\log b\,\psi(\log b/\ell)$ in §6; the hierarchy constant of $\mathfrak{B}_{\mathrm{poly}}$	
$\Psi(y;d,N)$	$\sum_{1\le n\le\min(y,N-1),\ n\equiv N\,(\mathrm{mod}\,d)}\Lambda(n)$	§6
k_0; D_0	the cutoff in the dual split of §6, and $D_0=N/k_0$	§6

3.1. **Statement.** For an integer $N\ge 2$ put

$$C_k(N):=\frac{1}{(k-1)!}\prod_{p\mid N}\Big(1-\frac{1}{p}\Big)^{k-1}\prod_{p\nmid N}\frac{1-p^{-1}(1-p^{-1})^{k-1}}{1-p^{-1}}. \tag{5}$$

Theorem A. *Assume* QRH(7/8) (*Definition 2.1*), *the statements of* [33], [35], [42] *and* [21] *listed in §2.2, and the classical inputs of §2.4. Fix an integer $k\ge 3$. Then*

$$\sum_{1\le t<N}\Lambda(t)\tau_k(N-t)=C_k(N)\,N(\log N)^{k-1}\big(1+o_k(1)\big) \tag{6}$$

(a) *as $N\to\infty$ through the even integers, and* (b) *as $N\to\infty$ through the odd integers. In each case the $o_k(1)$ is uniform in N: for each $\varepsilon>0$ there is $N_0(k,\varepsilon)$, independent of the factorisation of N, such that the $o_k(1)$ has modulus at most ε for every $N\ge N_0(k,\varepsilon)$ in that class. Combining*

(a) *and* (b),

$$\lim_{X\to\infty}\ \sup_{N\geq X}\left|\frac{S_k(N)}{C_k(N)N(\log N)^{k-1}}-1\right|=0.$$

Corollary A.1. *Under the assumptions of Theorem A, for each fixed integer $k\geq 3$,*

$$\sum_{p<N}\tau_k(N-p)=C_k(N)\,N(\log N)^{k-2}\big(1+o_k(1)\big)$$

as $N\to\infty$ through the even integers and through the odd integers, uniformly in N in the same sense.

Remark 3.1. No rate is asserted for the $o_k(1)$: the auxiliary estimates save arbitrary powers of L, but the last steps take limits in fixed parameters (§3.7). The integer k may not grow with N, and we give no effective $N_0(k,\varepsilon)$. Corollary A.1 is proved directly (§3.8): partial summation in N is not legitimate, since both the argument of τ_k and $C_k(N)$ move with N. By Lemma 3.5, $C_k(N)\asymp_k(\varphi(N)/N)^{k-1}\gg_k(\log\log 3N)^{-(k-1)}$, so a saving of a fixed power of L absorbs $1/C_k(N)$, whereas an error $o(NL^{k-1})$ without the factor $C_k(N)$ would not.

Remark 3.2 (Fixed shifts, $k=2$, and the order of magnitude). Theorem 1.1 of the preprint [21] asserts, for fixed integers $k\geq 2$ and $h\neq 0$, that $\sum_{x<n\leq 2x}\Lambda(n)\tau_k(n+h)=C_{k,h}x(\log x)^{k-1}+o_{k,h}(x(\log x)^{k-1})$, with $C_{k,h}$ given by (5) with h in place of N. That statement is for a fixed shift and does not apply with $h=-N$; the proof below transfers its method to the reflection, with uniformity in N. For $k=2$ the reflected problem is classical: Assing, Blomer and Li [2, Theorem 1.1], with $\sigma=-1$, $f=N$, $Y=0$, $X=N-1$, give unconditionally an asymptotic for $S_2(N)$ with leading term $C_2(N)N\log N$ (the constant $c_0(f)$ of [2, (1.6)] at $f=N$ is $C_2(N)$), an explicit secondary term and error $O_A(NL^{-A})$. We claim nothing for $k=2$. For every fixed $k\geq 2$ the order of magnitude $S_k(N)\asymp_k C_k(N)NL^{k-1}$ follows from classical inputs: the upper bound is the τ_k part of Lemma 3.8 (Henriot's theorem, with the check at $p=2$ of §3.8), and the lower bound follows from $S_k(N)\geq\frac{N}{4}\sum_{d\leq N^{1/4},\,(d,N)=1}\tau_{k-1}(d)/\varphi(d)-O_A(NL^{-A})$, by Lemma 2.13 as in Lemma 3.7. So the content of Theorem A is the ratio 1.

Remark 3.3 (Elliott–Halberstam). If Λ has a fixed level of distribution $\theta_1>1-1/k$, with maxima over reduced residues and over endpoints, then (6) follows by the classical argument: the generalised hyperbola method singles out a largest factor of an ordered k-factorisation of $N-t$, and the prime then lies in an interval in the class N modulo the remaining product $d\leq N^{1-1/k}$. For fixed shifts, Matomäki, Radziwiłł and Tao note that sieve methods give the asymptotic under a level of distribution of Λ greater than $1-1/k$ [29, §1.1]; see also [21, §1]. We do not use this hypothesis.

Remark 3.4 (Where QRH(7/8) enters, and what is transferred). QRH(7/8) is used only through Lemma 3.10, in the reflected major arcs (Propositions 3.16 and 3.22). It is asserted in [34], and the result announced in [1] (§2.1) would also give it. Without it the proof below still reduces Theorem A to the bounds (27) and (36), and these we do not prove without it: at conductor $q\asymp N$ and for primes of size $P=e^{L^a}$, $a<1$, a zero-free region of width $\asymp 1/\log q$ saves only $\exp(-O(\log P/\log q))=1-o(1)$. The proof follows [21], with their fixed-shift estimates [21, Lemma 4.1 and Proposition 4.2] replaced by Propositions 3.13 and 3.17 at the shift $-N$. Written out in full are the root count (Lemma 3.14) and its transport to index-N class lattices (Proposition 3.15), the character estimates at polynomial conductor, the displacement-N pairing (Proposition 3.20), the exact principal centring, and the local and endpoint bookkeeping uniform in N; the density model, marks, graph architecture and Boolean assembly are those of [33], [35], [42] and [21].

3.2. **Notation, the local factor and the density model.** Put $y=e^{L^{2/5}}$, $S=\{p\leq y\}$, $E=e^{\sqrt{L}}$, $Z=e^{L^{0.24}}$, $V(w)=\prod_{p\leq w}(1-1/p)$ and $V_y=V(y)$. For a set $\mathcal{U}$ of primes, $n_{\mathcal{U}}=\prod_{p\in\mathcal{U}}p^{v_p(n)}$ and $\mathcal{U}'$

is the complement. Let $\widetilde{M}_k(v)=\mathrm{Res}_{s=1}(\zeta(s)^k\prod_{p\in S}(1-p^{-s})^k v^{s-1})$, fix a smooth $\varsigma\colon[0,\infty)\to[0,1]$ equal to 0 on $[0,1/2]$ and to 1 on $[1,\infty)$, and let $M_k(v)=\varsigma(v/N^{1/2})\widetilde{M}_k(v)$ for $v>N^{1/2}/2$ and $M_k(v)=0$ otherwise. Put

$$D_k(u)=\tau_k(u_S)M_k(u_{S'})/V_y,\qquad \xi_k=\tau_k-D_k,\qquad \Xi_k=\tau_k+D_k.$$

For large N, $M_k\ge 0$: the leading term of the residue is $V_y^k(\log v)^{k-1}/(k-1)!$, the lower coefficients have relative size $O_k(\log y/\log v)$ by [42, (2.4)–(2.6)], and $\log v\ge L/2-O(1)$. So M_k is a smooth nonnegative model, equal to the residue for $v\ge N^{1/2}$, with $M_k(v)\ll_k V_y^kL^{k-1}$ for $v\le N^2$, as in [42, §2.2]; on scales $u=N^{1+o(1)}$, $u_S\le E$ implies $N^{1/2}<u_{S'}<N^2$. A function F is *invariant* under a set of primes if $F(pu)=F(u)$ for each of its primes and every u.

Marks. Fix K and exponents $1/10<a_1<\cdots<a_K<1/5$, and put

$$\mathfrak{B}_i:=\{p:\ e^{L^{a_i}}<p\le e^{2L^{a_i}},\ p\nmid N\},\qquad V_i:=\sum_{p\in\mathfrak{B}_i}\frac{1}{p},\qquad \omega_i(u):=\#\{p\in\mathfrak{B}_i:\ p\mid u\},\tag{7}$$

$\omega=\sum_i\omega_i$ and, for $0<\varpi<1$, $W_\varpi(u):=\prod_{i=1}^K\varpi^{\omega_i(u)-1}\omega_i(u)/V_i$. These are the bands of [33, (3.2)] based at $L=\log N$ with the primes dividing N deleted, and W_ϖ is the marked weight of [33, (3.4)] with its q written ϖ. Bank primes are odd and lie in $(e^{L^{1/10}},Z)\cap S$, and $V_i=\log 2+o(1)$ uniformly in N by Mertens' theorem and (9).

Lemma 3.5 (The local factor)**.** *Let $J_{k,p}(N)=\int_{\mathbb{Z}_p}\tau_k(p^{v_p(u)})\mathbf{1}_{p\nmid N-u}\,du$ (Haar measure). Then $J_{k,p}(N)=(1-1/p)^{1-k}-1/p$ if $p\nmid N$ and $J_{k,p}(N)=1-1/p$ if $p\mid N$, and $C_k(N)=\frac{1}{(k-1)!}\prod_p(1-1/p)^{k-2}J_{k,p}(N)$. With $g_k(p):=(1-(1-1/p)^{k-1}/p)/(1-1/p)=1+O_k(p^{-2})>0$ and $G_k:=\prod_p g_k(p)$,*

$$C_k(N)=\frac{G_k}{(k-1)!}\Big(\frac{\varphi(N)}{N}\Big)^{k-1}\prod_{p\mid N}g_k(p)^{-1}\asymp_k\Big(\frac{\varphi(N)}{N}\Big)^{k-1}\gg_k(\log\log 3N)^{-(k-1)}.$$

Proof. $\int_{\mathbb{Z}_p}\tau_k(p^{v_p(u)})\,du=(1-1/p)\sum_{a\ge0}\binom{k+a-1}{a}p^{-a}=(1-1/p)^{1-k}$. If $p\nmid N$ the excluded class $u\equiv N\pmod p$ has valuation zero and measure $1/p$; if $p\mid N$, $p\nmid N-u$ forces u to be a unit. The rest is arithmetic. □

So (5) is the local-density prediction, and Lemma 3.7 shows that it is the constant of the density model; $J_{k,p}(N)$ is the factor J_p of [21, (2.6)] with N in place of h.

Lemma 3.6 (Moments and deletions)**.** (i) *For fixed a,r and $0<\delta\le C$, uniformly for $N^\delta\le T\le N^C$, $\sum_{u\asymp T}\tau_a(u)^r\ll TL^{C_{a,r}}$ and $\sum_{u\asymp T}\tau_a(u)^r\mathbf{1}_{u_S>E}\ll TL^{C_{a,r}}e^{-c_{a,r}L^{1/10}}$; for $T\asymp N$ the same holds with $N-u$ in place of u; and, by Hölder, both hold for any fixed product of the divisor weights used below.* (ii) *$D_k(u)\ll L^{k-1}\tau_k(u_S)$, and on a fixed interior support $u\asymp N$ with $u_S\le E$,*

$$D_k(u)=B_k\tau_k(u_S)\big(1+O(L^{-1/2})\big),\qquad B_k=V_y^{k-1}L^{k-1}/(k-1)!.\tag{8}$$

(iii) *For every set $\mathfrak{B}$ of primes,*

$$\sum_{p\in\mathfrak{B},\ p\mid N}1/p\le\omega(N)/\min\mathfrak{B}\ll L/\min\mathfrak{B}.\tag{9}$$

(iv) *For $M\ge N^\delta$ and an interval of length $\asymp M$,*

$$\#\{m\asymp M:\ P^-(m)>Z,\ (m,N)>1\}\ll ML/Z+L.\tag{10}$$

Proof. (i) is [42, Lemma 2.2], whose tilted moment bound contains the ordinary one, at every scale $N^\delta\le T\le N^C$ (see also [21, Lemma 2.2]); (ii) is [42, Lemma 2.1] and [21, Lemmas 2.1–2.2], whose hypotheses concern the size of u, not a shift. In (iv) a prime factor of N dividing m exceeds Z, so the count is at most $\sum_{p\mid N,\,p>Z}(M/p+1)$; the term L must be kept, since M/p need not exceed 1. □

Hölder's inequality and (i)–(ii) turn (9)–(10) into errors $\ll NL^{-A}$ for every fixed A; the same argument with threshold y removes integers divisible by a prime $p \mid N$, $p > y$, in the progression reductions.

Lemma 3.7 (The density model)**.** *For each fixed $\Phi \in C_c^\infty((0,1))$ and $A > 0$, uniformly in N,*

$$\sum_{t<N} \Lambda(t)\Phi\Big(\frac{N-t}{N}\Big)D_k(N-t) = C_k(N)NL^{k-1}\int_0^1 \Phi + O_{k,\Phi}\big(C_k(N)NL^{k-3/2}\big) + O_{A,k,\Phi}\big(NL^{-A}\big). \tag{11}$$

Proof. By (8) it suffices to evaluate $B_k\sum_t \Lambda(t)\Phi((N-t)/N)\tau_k((N-t)_S)$ (with $|\Phi|$ for the relative error), and $\tau_k(u_S) = \sum_{d|u,\ d\ S\text{-smooth}} \tau_{k-1}(d)$. By Lemma 3.6(i), even with $\Lambda(t) \le L$, the terms with $(N-t)_S > E$ contribute $NL^{O_k(1)}e^{-cL^{1/10}}$, so $d \le E$. If $(d,N) > 1$, $t \equiv N \pmod d$ and $\Lambda(t) \ne 0$ force t to be a power of a prime dividing N: $O(L^2)$ terms. For $(d,N) = 1$ let $\mathcal{E}_d$ be the difference between $\sum_{t<N,\, t\equiv N\,(d)} \Lambda(t)\Phi((N-t)/N)$ and $\frac{N}{\varphi(d)}\int\Phi$. Partial summation in t, the passage from π to ψ (a factor L, and $O(N^{1/2}L)$ for prime powers) and Lemma 2.13 at $x = N$ (its maximum over $y \le x$ covers every height in the partial summation) give $\sum_{d\le E}|\mathcal{E}_d| \ll_{A'} NL^{-A'}$ for every A', as $E \le N^{1/2}L^{-B}$. The divisor weight is not bounded by a power of L; since $d|\mathcal{E}_d| \ll NL$,

$$\sum_{d\le E}\tau_{k-1}(d)|\mathcal{E}_d| \le \Big(\sum_{d\le E}\frac{\tau_{k-1}(d)^2}{d}\Big)^{1/2}\Big(\sum_{d\le E} d|\mathcal{E}_d|^2\Big)^{1/2} \ll L^{O_k(1)}\Big(NL\sum_{d\le E}|\mathcal{E}_d|\Big)^{1/2},$$

which is $\ll NL^{-A}$ after multiplication by B_k. Extending $\sum_{d\ S\text{-smooth},\,(d,N)=1}\tau_{k-1}(d)/\varphi(d)$ beyond E costs $NL^{O_k(1)}e^{-cL^{1/10}}$ (Rankin). Its Euler factor at $p \le y$, $p \nmid N$, times $(1-1/p)^{k-1}$, is $g_k(p)$; at $p \mid N$ the factor $(1-1/p)^{k-1}$ remains; the tail $p > y$ costs $1 + O_k(1/y)$, and the primes $p \mid N$, $p > y$, a factor $1 + O_k(L/y)$. With $B_k = V_y^{k-1}L^{k-1}/(k-1)!$ the main coefficient is $C_k(N)L^{k-1}(1 + O_k(L/y))$. □

3.3. **Positive envelopes and small moduli.** The upper-bound input is the corrected theorem [20, New Theorem 5, p. 377] (Appendix A), whose constant does not depend on the discriminant or the coefficients once its parameters are fixed, and which allows absolute values, hence negative slopes. Below it is applied to products of two primitive linear forms of height $N^{O(1)}$ on dyadic intervals of length $\ge N^{\delta_1}$, with its height exponent small, its interval exponent $1/2$, and then its growth parameter small. For a finite set $\mathcal{U}$ of primes not dividing N, $\mu_{N,k}^*$ is the product law normalising $\tau_k(p^{v_p(u)})\mathbf{1}_{p\nmid N-u}\,du$ at each $p \in \mathcal{U}$ (normaliser $J_{k,p}(N)$), and $\mu_{k,j}$ is the product law on pairs of valuations with unnormalised masses $1-2/p$ at $(0,0)$, $(1-1/p)\tau_k(p^a)p^{-a}$ at $(a,0)$ and $(1-1/p)\tau_j(p^b)p^{-b}$ at $(0,b)$, $a,b \ge 1$ (normaliser $(1-1/p)^{1-k} + (1-1/p)^{1-j} - 1$).

Lemma 3.8 (Positive envelopes)**.** *Fix $\Phi \in C_c^\infty((0,1))$, $\Phi \ge 0$. Uniformly in N, with constants depending only on fixed orders, Φ and fixed exponents:*

(i) *if $\mathcal{U} \subset \{e^{L^{1/10}} < p \le y : p \nmid N\}$ and $0 \le F \le L^C$ is a function of $u_{\mathcal{U}}$, then $\sum_{p<N}\Phi(\frac{N-p}{N})\Xi_k(N-p)F((N-p)_{\mathcal{U}}) \ll C_k(N)NL^{k-2}\,\mathbb{E}_{\mu_{N,k}^*}F + NL^{C'}e^{-cL^{1/10}}$, the first constant being independent of F and $\mathcal{U}$;*

(ii) *if $\mathcal{U} \subset \{Z < p \le y : p \nmid N\}$ and $0 \le F \le L^C$ is a function of $(u_{\mathcal{U}}, (N-u)_{\mathcal{U}})$, then*

$$\sum_u \Phi\Big(\frac{u}{N}\Big)\Xi_k(u)\tau_j(N-u)\mathbf{1}_{P^-(N-u)>Z}F \ll C_k(N)NL^{k+j-2}V(Z)^j\,\mathbb{E}_{\mu_{k,j}}F + NL^{C'}e^{-cL^{1/10}};$$

(iii) *for fixed $\gamma > 0$ and $j \ge 1$, if $b_1,\dots,b_j$ are supported on Z-rough $m_i \asymp H_i$ with $H_i \ge N^\gamma/2$, $\prod_i H_i \asymp N$ and $|b_i| \le (\gamma LV(Z))^{-1}$, then on a fixed interior product support $\sum \Xi_k(N - m_1\cdots m_j)\prod_i|b_i(m_i)| \ll C_k(N)NL^{k-j-1}$;*

(iv) *for fixed $0 < \eta_1 < 1/4$ and $\epsilon > 0$, $\sum_{t<N,\ \min(t,N-t)\le\eta_1 N}\Lambda(t)\tau_k(N-t) \ll_k C_k(N)\eta_1 NL^{k-1} + N^{1/2+\epsilon}$, with a constant independent of η_1 (the threshold for N may depend on η_1).*

Proof. The argument is that of [21, Lemmas 5.1–5.3], with the shift replaced by the reflection and the local factors of Lemma 3.5. Discard oversized smooth records by Lemma 3.6(i) and rough non-units of N by (10). In (ii) fix the records $r_1 = u_{\mathcal{U}}$ and $r_2 = (N-u)_{\mathcal{U}}$, both $\le E$; they are coprime (a common prime would divide N), and u lies in one class modulo r_1r_2. The residual linear forms, one of negative slope, have resultant $\pm N$ and height $N^{1+o(1)}$, and are primitive because $(r_1r_2, N) = 1$. In Henriot's corrected exact-valuation density [20, (0·1)–(0·2)] the local factor is $I_p + \rho(p)/p$, with I_p the exact local integral and $\rho(p)$ the number of roots modulo p; since $I_p \ge 1 - \rho(p)/p$, $(1-\rho(p)/p)(I_p + \rho(p)/p) \le I_p$ for $p > 2$, and $p = 2$ costs at most a factor 3. At $p \in \mathcal{U}$ this gives the masses of $\mu_{k,j}$; other $p \nmid N$ give $L^{k+j-2}V(Z)^j$ up to constants; at $p \mid N$, $p \le Z$, roughness forces u to be a unit, a factor $(1-1/p)^{k-1}(1+O(p^{-2}))$ relative to the generic one; primes dividing N above Z have reciprocal mass $O(L/Z)$. This extracts $C_k(N)$, with no factor involving $\tau(N)$. For D_k use (8) and the same positive calculation (the density-model part of [21, Lemma 5.3]). The argument sums positively against any $F \ge 0$. For (i) dominate the prime condition by an upper-bound sieve to a fixed power of the scale: the local density is $J_{k,p}(N)$ and the sieve gives L^{-1}. For (iii) fix $m_2, \dots, m_j$, put $a = \prod_{i \ge 2} m_i$ and apply the theorem to m and $N - am$: the inner sum is $\ll C_k(N)H_1V(Z)L^{k-1}$, and each other variable supplies $H_iV(Z)$. For (iv) bound the variable below $N^{1/2}$ pointwise, and on dyadic $[R', 2R']$, $R' \ge N^{1/2}$, the two-form bound gives $\ll C_k(N)R'L^{k-1}$. □

Lemma 3.9 (Small moduli and the sifted model)**.** *Let $c_0 = 1/(2(k+2))$.* (i) *For every function $\mathfrak{h}$ of u_S with $|\mathfrak{h}| \le L^C$ and every A,*

$$\sum_{r \le N^{c_0}} \Big| \sum_{u:\ r \mid N-u} \Phi\Big(\frac{u}{N}\Big)\mathfrak{h}(u_S)\xi_k(u)\Big| \ll_{A,k,C,\Phi} NL^{-A}. \tag{12}$$

(ii) *Let s be a large even integer, $\gamma = c_0/(8s+8)$ and $z = N^\gamma$; fix a mark system (7) and ϖ, and let $m_W = \mathbb{E}_{\mu^*_{N,k}} W_\varpi$, which lies between positive constants depending only on K, k and ϖ, uniformly in N. For every real $\Phi \in C_c^\infty((0,1))$,*

$$\limsup_{N \to \infty,\ 2 \mid N} \frac{|\sum_u \Phi(u/N)\xi_k(u)W_\varpi(u)\mathbf{1}_{P^-(N-u)>z}|}{C_k(N)NL^{k-2}m_W} \ll_{k,\Phi} \frac{e^{-s}}{\gamma}, \tag{13}$$

with an implied constant depending only on k and Φ: not on s, γ, K, the mark system or its exponents, or N.

Proof. (i) is the reflected form of [21, Lemma 3.1], from the divisor-progression estimate [21, (3.2)]. Restrict to $a = u_S \le E$, put $g = (a, r)$, which must divide N, and $q = r/g$; the rough part of $u = av$ lies in the class $v \equiv (N/g)(a/g)^{-1} \pmod q$. A class not reduced at some $p \le y$ contains no rough v; one not reduced at some $p > y$ forces $p \mid N$ and $p \mid u$, and is deleted with Lemma 3.6, multiplicity $\tau(N-u)$ included. In reduced classes the two principal terms agree up to the large-prime Euler factors of [21, (3.3)–(3.4)]: for $G_{j,q}(s) = \prod_{p \mid q,\, p > y}(1-p^{-s})^j$ the needed derivatives of $G_{k,q} - 1$ at 1 and $G_{1,q}(1) - 1$ are $O_k(e^{-L^{2/5}}L^{C_k})$ for $q \le N^{c_0}$, and $\sum_{q \le N^{c_0}} 1/\varphi(q) \ll L$. The contour errors $O(Y_0^{1/2}q^{k/2})$ of [21, (3.2)], at summation length $Y_0 \le N$, sum to $N^{1/2+(k/2+1)c_0+o(1)} = N^{3/4+o(1)}$, and the multiplicity of g, at most $\tau(a)$, costs $L^{O_k(1)}$; this replaces every bound depending on the divisors of h in [21].

(ii) For the additional sieve insertion of [21, Lemma 3.2] use $\sum_{a\ S\text{-smooth}} \tau_k(a)(a,d)/a \le V_y^{-k}(k+1)^{\omega(d)}$ for squarefree d supported on S; the positive Bonferroni remainder at degree $\lceil C_A \log L \rceil$ saves any power of L, with no condition on (d, N); restrict the rough-supported coefficients to $r \le N^{c_0/2}$; as a Bonferroni divisor satisfies $d \le y^{\lceil C_A \log L \rceil + 1} = N^{o(1)}$, $rd < N^{c_0}$ and (12) applies; the block sieve uses only $r \le z^{4s+2}$, and $(4s+2)\gamma < c_0/2$. The CRT calculation of [21, Lemma 3.3] now has the factors $J_{k,p}(N)$ and a main term with $C_k(N)$. Then [33, Lemma 2.6] (that is, [35,

Lemma 2.9]) gives (13) for $\Phi \geq 0$, the reflected form of [21, Proposition 3.4]. The term e^{-s}/γ arises only from $\sum_r(\lambda_r^+ - \lambda_r^-)/r \ll e^{-s}V(z)/V_y$, with an absolute constant ([33, Lemma 2.6] with $g(p) = 1/p$ on $y < p \leq z$, $p \nmid N$), from $V(z) \asymp 1/(\gamma L)$, and from the main term $(N/r)\int \Phi \cdot B_k m_W \prod_{p\leq y} J_{k,p}(N) = (N/r)\int \Phi \cdot C_k(N)(1+O_k(L/y))L^{k-1}V_y m_W$, in which $C_k(N)$ and m_W cancel against the normalisation. All other terms are $o(C_k(N)NL^{k-2}m_W)$ as $N \to \infty$ with s, K and the marks fixed, since $m_W \geq c_{K,k} > 0$ and $C_k(N) \gg (\log\log N)^{-(k-1)}$; [21] state the same independence for [21, (3.10)]. A real Φ is $(\Phi + \|\Phi\|_\infty\Phi_0) - \|\Phi\|_\infty\Phi_0$ with $\Phi_0 \geq 0$ smooth in $(0,1)$ and ≥ 1 on $\operatorname{supp}\Phi$. □

3.4. **Arithmetic tools under the zero-free half-plane.** This subsection is the only place where QRH(7/8) is used. We also use the local formulas [40, Proposition 19, Exercise 45(ii), (46)] (Appendix A): for fixed $C, \epsilon > 0$, uniformly in q, in non-principal $\chi \bmod q$ and in $\epsilon \leq \sigma \leq C$, $-L'/L(s,\chi) = -\sum_{|s-\rho|\leq\epsilon/2}(s-\rho)^{-1} + O(\log(q(2+|t|)))$; the same for ζ with the extra term $1/(s-1)$; and $L(s,\chi) = \zeta(s)\prod_{p|q}(1-p^{-s})$ for principal χ.

Lemma 3.10 (Bank primes at polynomial conductor)**.** *Assume* QRH(7/8) *and fix $a, B, D > 0$. Uniformly for $e^{L^a} \leq P \leq N^B$, characters χ modulo $q \leq N^B$, $|t| \leq N^D$ and smooth w supported in $[1/2, 3]$ with logarithmic derivatives bounded by fixed powers of L,*

$$\frac{1}{P}\sum_p \chi(p)w\Big(\frac{p}{P}\Big)p^{it} = \mathbf{1}_{\chi \text{ principal}}P^{it}\int_0^\infty \frac{w(u)u^{it}}{\log(Pu)}\,du + O\big(P^{-1/16}L^{C_w}\big). \tag{14}$$

For non-principal characters the harmonic prime sums over a bank $\{e^{L^a} < p \leq e^{2L^a} : p \nmid N\}$ at polylogarithmic heights are smaller than every power of L.

Proof. This is the contour proof of [42, Lemma 5.1] with its conductor range enlarged. Under QRH(7/8) no zero lies within 1/32 of a point with $\sigma \geq 15/16$, so the local formulas with $\epsilon = 1/16$, $C = 2$ have empty zero sums, and principal Euler factors add $O(\log q)$:

$$-\frac{L'}{L}(\sigma+iv,\chi) - \frac{\mathbf{1}_{\chi \text{ principal}}}{\sigma+iv-1} \ll \log\big(q(|v|+3)\big), \qquad \frac{15}{16} \leq \sigma \leq 2. \tag{15}$$

With $f_P(u) = w(u)/\log(Pu)$,

$$\frac{1}{P}\sum_n \Lambda(n)\chi(n)n^{it}f_P\Big(\frac{n}{P}\Big) = \frac{1}{2\pi i}\int_{(2)} -\frac{L'}{L}(s,\chi)P^{s-1+it}\widehat{f_P}(s+it)\,ds.$$

Shift to $\Re s = 15/16$; the only residue is the main term. With $z = \Im s + t$ the new integral is $\ll P^{-1/16}L^C\int_{\mathbb{R}}(1+\log(q(|z-t|+3)))(1+|z|)^{-2}\,dz \ll P^{-1/16}L^{C'}$, as $\log q + \log(|t|+3) = O(L)$. Prime powers cost $O(P^{-1/2}L^{C'})$, and imprimitive moduli $O(L^C \log q/P)$. For sharp bank endpoints smooth in strips of relative width L^{-B_1}, of total normalised mass $O(L^{a-B_1} + L^a/P_{\min})$ (counting integers), insert $1/p$ by partial summation and choose B_1 after the saving; deleting the primes dividing N costs $O(L/P_{\min})$. Below, these strip estimates are applied after the character mean squares, without a factor for the number of characters. □

Lemma 3.11 (Character normalisation)**.** *Let $q = q_0q_1$, every prime factor of q_0 dividing N and $(q_1, N) = 1$, and $Q = Nq_0$.* (i) *For units X_1, X_2 modulo N and every integer a_0, with $b_\chi(z) = \sum_{j \bmod q_0}\overline{\chi(1+Nj)}\,e(a_0jz/q_0)$,*

$$\mathbf{1}_{N|X_1-X_2}\,e\Big(\frac{a_0(X_1-X_2)}{Q}\Big) = \frac{1}{\varphi(Q)}\sum_{\chi \bmod Q}\chi(X_1)\overline{\chi(X_2)}\,b_\chi(X_2); \tag{16}$$

b_χ has a character expansion modulo q_0 with at most $\varphi(q_0)$ coefficients, each at most q_0. (ii) *For (c_n) supported on an interval of length $O(T)$ and every real t,*

$$\sum_{\chi \bmod Q} \Big|\sum_n c_n\chi(n)n^{it}\Big|^2 \ll \big(\varphi(Q)+T\big)\sum_n |c_n|^2. \tag{17}$$

(iii) *Hence, if $\sum_{n\asymp T}|c_n|^2 \le TL^C$, then*

$$\sum_{\chi \bmod Q} \Big|\frac{1}{T}\sum_{n\asymp T} c_n\chi(n)n^{it}\Big|^2 \ll L^C\Big(1+\frac{\varphi(Q)}{T}\Big), \tag{18}$$

and if moreover $T = X \asymp N$ and $q_0 \le L^B$,

$$\frac{X}{\varphi(Q)}\sum_{\chi \bmod Q} \Big|\frac{1}{X}\sum_{n\asymp X} c_n\chi(n)n^{it}\Big|^2 \ll L^C\Big(1+\frac{X}{\varphi(Q)}\Big) \ll L^{C+1}. \tag{19}$$

Proof. (i) X_1/X_2 lies in the kernel of reduction from units modulo Q to units modulo N exactly when it is $1+Nj$ for a unique j mod q_0, and then $X_1 - X_2 \equiv NjX_2 \pmod Q$; apply orthogonality. (ii) The left side is $\varphi(Q)$ times the sum over reduced classes of the squared class sums, each class has $O(1+T/Q)$ elements, and $\varphi(Q) \le Q$; this holds at each height separately. (iii) uses $\varphi(Nq_0) = q_0\varphi(N)$ and $N/\varphi(N) \ll \log\log 3N$. □

Lemma 3.12 (Primes minus the rough model)**.** *Assume* QRH(7/8)*. Let $N^\delta \le M \le N^{1-\delta}$, $I \subset [M,2M]$ an interval, $|u| \le L^B$ and ψ a character modulo $q \le L^B$. Then, also with $\mathbf{1}_{(m,N)=1}$ inserted,*

$$\sum_{m\in I}\psi(m)m^{iu}\Big(\mathbf{1}_{m\ prime} - \frac{\mathbf{1}_{P^-(m)>Z}}{V(Z)\log m}\Big) \ll_{A,B,\delta} ML^{-A}. \tag{20}$$

Proof. For the primes use Lemma 3.10 after smoothing the endpoints in strips of relative width L^{-B_1}: the main term is $\mathbf{1}_{\psi\ \text{principal}}\int_I t^{iu}/\log t\,dt$. For the rough part use Bonferroni truncations to products of at most r primes $\le Z$, r a large fixed multiple of $\log L$: in a reduced class modulo q the count for d coprime to q is $|I|/(qd)+O(1)$, the main term is $|I|V(Z)/\varphi(q)$, and the omitted degree has mass $\le M(\sum_{p\le Z}1/p)^{r+1}/(r+1)! + Z^{r+1}q \ll ML^{-D}$. Summing over classes gives 0 for non-principal ψ and $V(Z)|I|$ otherwise; partial summation costs $L^{O(B)}$, and the principal terms cancel. The unit restriction removes $O(ML/Z+L)$ rough integers and $O(L)$ primes. □

3.5. **The reflected prime-versus-rough estimate.**

Proposition 3.13. *Assume* QRH(7/8) *and the inputs from* [33]*. Let N be even. Fix $\delta, C, A > 0$ and $0 < \varpi < 1$. Let $M, R \ge N^\delta$ with $X = MR \asymp N$, take the banks* (7)*, let $|F| \le 1$ be invariant under all bank primes, let β_n be supported on Z-rough $n \in [R,2R]$ with $|\beta_n| \le L^C$, and let $\alpha_m = \mathbf{1}_I(m)m^{it}(\mathbf{1}_{m\ prime} - \mathbf{1}_{P^-(m)>Z}/(V(Z)\log m))$ with $I \subset [M,2M]$ an interval and $|t| \le L^C$. If K is large enough in terms of δ, C, A, ϖ, independently of the exponents a_i and of N, then*

$$\sum_{m,n}\alpha_m\beta_n F(N-mn)W_\varpi(N-mn) \ll_{A,C,\delta,\varpi} XL^{-A}, \tag{21}$$

where $F(N-mn)W_\varpi(N-mn)$ is 0 when $mn \ge N$; the threshold may depend on the a_i. The same holds after inserting a fixed smooth product cutoff supported on an interior subset of $\{0 < mn < N\}$.

This is the reflected form of [33, Theorem 3.1], whose argument is $mn-1$, and not a consequence of it. Odd N is treated in §3.8.

The bands. By (10), rough non-units of N may be deleted first, so all coefficient variables are units modulo N. Deleting the primes dividing N from the bands of [33, (3.2)] changes nothing in [33, §§3–4]: every use of the bands there is of one of four kinds, all inherited by subsets.

They are disjointness; the window $\mathfrak{B}_i \subset [e^{L^{a_i}}, e^{2L^{a_i}}]$, which gives $|\mathfrak{B}_i| \le e^{2L^{a_i}}$, atoms $\ll e^{-L^{0.1}}$, $\sum_p p^{-2} \ll e^{-L^{0.1}}$ and $p_{\min} \ge e^{L^{0.1}}$, and makes every bank prime odd and below Z; two-sided bounds on the V_i; and normalisation identities exact for any finite sets of primes, namely [33, (3.14), (3.15), (4.3)–(4.4), (4.56), (4.58)] and the list sum proving [33, (3.18)]. The one use of interval structure in §§3–4, $V_i = \log 2 + o(1)$ by partial summation, is replaced by (9); afterwards §§3–4 use only that the V_i are bounded above and below, and interval structure next enters [33, §5], which Proposition 3.16 replaces. The deletion is needed, not just tolerated, in three places handled below: the first row of $\widetilde{g}$ is non-zero modulo each mark prime, the map $g \mapsto \widetilde{g}$ is bijective modulo S_0, and $(\ell, N) = 1$ in the pad stripping.

Marks and the signed square. Expand W_ϖ by selecting one prime from each bank; let a be their product and $N - mn = ah$. Then $F(N - mn) = F(h)$, and replacing $\varpi^{\omega(N-mn)-K}$ by $\varpi^{\omega(h)}$ is exact unless the square of a selected prime divides $N - mn$, as in [33, (3.6)]. For fixed m and a bank prime p, $p \nmid m$ (m is prime or Z-rough and $p < Z$), so $mn \equiv N \pmod{p^2}$ fixes one class of n, with $O(R/p^2)$ solutions; as $\sum_i \sum_{p \in \mathfrak{B}_i} p^{-2} \ll_K e^{-L^{0.1}}$ these terms cost $O_A(XL^{-A})$, whatever the factorisation of N. Keep the full box of (m, n) and extend $h \mapsto F(h)\varpi^{\omega(h)}$ by zero to $h \le 0$. With a fixed bump η on $[1, 4]$ as in [33, (3.8)], Cauchy's inequality over h bounds the square of each dyadic marked sum by $O(X/Y)$ times

$$\mathcal{Q}_Y^{\mathrm{sgn}} := \Big(\prod_i V_i\Big)^{-2} \sum_{h \in \mathbb{Z} \smallsetminus \{0\}} \Big| \sum_a \eta\Big(\frac{a}{Y}\Big) \sum_{mn = N - ah} \alpha_m \beta_n \Big|^2,$$

a running over products of one prime from each $\mathfrak{B}_i$. This square's expansion over all $h \ne 0$ is an identity; the support condition $mn < N$ is dropped only through the Cauchy inequality.

Lemma 3.14 (Root count)**.** *Let N be even, $U_1, U_2 \ge N^\delta$, and $U_1U_2/N = N^{o(1)}$, bounded below by a fixed positive constant. Let S_0 be odd and squarefree, $(S_0, N) = 1$, $\log S_0 \le CL^{0.98}$. For coprime odd $(u, v) \in [U_1, 16U_1] \times [U_2, 2U_2]$ write uniquely*

$$g = \begin{pmatrix} u & c \\ v & d \end{pmatrix}, \quad ud - vc = 1, \quad 0 \le c < u, \qquad \mathfrak{f} = \lfloor -Nc/u \rfloor, \quad \vartheta = \{-Nc/u\}.$$

Uniformly in intervals $I_1 \subset [1, 16]$, $I_2 \subset [1, 2]$, $I_3 \subset [0, 1]$, in $g_0 \in \mathrm{SL}_2(\mathbb{Z}/2S_0\mathbb{Z})$ with odd first column modulo 2, and in $\mathfrak{f}_* \bmod S_0$, the number of these (u, v) with $g \equiv g_0 \pmod{2S_0}$, $\mathfrak{f} \equiv \mathfrak{f}_* \pmod{S_0}$ and $(u/U_1, v/U_2, \vartheta) \in I_1 \times I_2 \times I_3$ is

$$\frac{U_1U_2|I_1||I_2||I_3|}{\zeta(2)\,|\mathrm{SL}_2(\mathbb{Z}/2S_0\mathbb{Z})|\,S_0} + O\big(U_1U_2N^{-c_\delta}\big) \tag{22}$$

for some $c_\delta > 0$; C affects only the threshold. The two completions modulo 2 give total density $1/(3\zeta(2))$. The error is additive, not relative for small boxes.

Proof. Case $U_1 \le U_2$. Put $q = 2S_0$. For fixed $u \equiv u_* \pmod q$, the pairs (c, v) modulo uq with $c \equiv c_*$, $v \equiv v_* \pmod q$ and $cv \equiv ud_* - 1 \pmod{uq}$ number $u\prod_{p|u,\, p\nmid q}(1 - 1/p)$, as in [33, (3.25)–(3.28)]. Use the torus coordinates $x_1 = c/(uq)$, $x_2 = v/(uq)$ and $x_3 = -2Nc/(uq) = (-Nc/u)/S_0$: the conditions are $x_1 \in [0, 1/q)$ and $x_3 \in (\mathfrak{f}_* + I_3)/S_0 \pmod 1$, and x_3 is a function of x_1. Split the range of v into periods of length uq, smooth the indicators at width $U_1^{-1/1000}$, and cut frequencies at $F_0 = U_1^{1/100}$; the tail is $O(U_1^{-1/10})$, and a retained triple (h, b, h') has frequency $a = h - 2Nh'$ on c. If $b \ne 0$, [33, (3.24)] bounds the complete sums of [33, (3.28)] by $q^{O(1)}u^{3/4+o(1)}F_0^{1/4}$, since $(a, b, u) \le |b|$, whatever the size of a; the total is $O(U_1U_2q^{O(1)}U_1^{-1/5+o(1)})$. If $b = 0$ and $(h, h') \ne (0, 0)$, then $a \ne 0$ as $N > F_0$; the sums are Ramanujan sums, $\ll q^{O(1)}(a, u)u^{o(1)}$, and summing over u first,

$$\sum_{u \asymp U_1} (a, u) = \sum_{d|a} \varphi(d) \#\{u \asymp U_1 :\ d \mid u\} \ll U_1\tau(|a|) = U_1N^{o(1)}, \tag{23}$$

which gives $O(U_1U_2q^{O(1)}U_1^{-1}F_0^2N^{o(1)})$. This step admits large (u,N) and every fixed δ: no root is discarded. The zero mode and the strips give the volume with error $O(U_1U_2q^{O(1)}U_1^{-1/1000+o(1)})$; averaging $\prod_{p\mid u,\,p\nmid q}(1-1/p)$ over $u\equiv u_*\pmod q$, as at the end of the proof of [33, Lemma 3.3], gives density $\zeta(2)^{-1}\prod_{p\mid q}(1-p^{-2})^{-1}$, and $|\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|=q^3\prod_{p\mid q}(1-p^{-2})$. All powers of q are $N^{o(1)}$, so $c_\delta=\delta/1000-o(1)$ works.

Case $U_2<U_1$. Interchange the coordinates, with the determinant sign reversed, as in [33, Lemma 3.3], using $-Nc/u=-Nd/v+N/(uv)$. Track the full circle coordinate $(-Nc/u)/S_0 \bmod 1$, whose interval $(\mathfrak{f}_*+I_3)/S_0$ encodes both $\mathfrak{f}\bmod S_0$ and ϑ; the translation $N/(uvS_0)$ and its normalised derivatives are bounded as $N/(U_1U_2)=O(1)$. Freeze it on rectangles of side $\min(U_1,U_2)^{-\varepsilon_1}$, ε_1 small after c_δ, and enlarge or shrink the torus intervals by $O(\min(U_1,U_2)^{-\varepsilon_1})$: a power saving remains. □

Line distribution. Put $\widetilde{g}=g\,\mathrm{diag}(1,N)\left(\begin{smallmatrix}1&\mathfrak{f}\\0&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}u&-u\vartheta\\v&w\end{smallmatrix}\right)$, of determinant N. For fixed $\mathfrak{f}\bmod S_0$ this right multiplication is a bijection from $\mathrm{SL}_2(\mathbb{Z}/S_0\mathbb{Z})$ onto the matrices of determinant N modulo S_0, as $(N,S_0)=1$; at $p\mid S_0$ the first row maps $(u,c)\mapsto(u,u\mathfrak{f}+Nc)$, a bijection of non-zero rows. So the kernels of the first row at the primes of S_0 are uniform on $\mathbb{P}^1(\mathbb{F}_p)$, independent over these primes and, by Lemma 3.14, of the real root data, in its additive sense; summing residue conditions costs $S_0^{O(1)}=N^{o(1)}$.

Class lattices. Let $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ be the orbits of unimodular pairs modulo N under units, and for $C=[(a',b')]$ put $\Lambda_C=\{(x,y)\in\mathbb{Z}^2:\ a'y-b'x\equiv 0\pmod N\}$, of index N since $(a',b',N)=1$. A primitive $P\in\mathbb{Z}^2$ lies in exactly one $\Lambda_{[P]}$ (a unimodular vector on its line is a unit multiple) and is primitive there. The columns of $\widetilde{g}$ lie in $\Lambda_{[P]}$ for the root $P=(u,v)$, the second being $\mathfrak{f}P$ plus N times an integral vector; equal indices give $\widetilde{g}\mathbb{Z}^2=\Lambda_{[P]}$. These facts do not use the parity of N. For the classes $C=[(1,\varrho)]$ with ϱ a unit modulo N, the lattices Λ_C, and coordinates relative to a basis $g\,\mathrm{diag}(1,N)U$ of determinant N with U unimodular, also appear in Guo's proof of his reflected theorem [18, §3].

The squares. With $\mathcal{Q}_Y^{\mathrm{sgn}}$ as above define, as in Definition 2.9 with L in place of L_x and the marks of this subsection,

$$\mathcal{Q}_Y^{\mathrm{maj}}:=\Big(\prod_i V_i\Big)^{-2}\sum_{a,b}\eta\Big(\frac{a}{Y}\Big)\eta\Big(\frac{b}{Y}\Big)\sum_{m,n,r,s}\alpha_m\overline{\alpha_r}\beta_n\overline{\beta_s}\,H_N(bmn-ars;a,b),$$

where $H_N(T;a,b)=\mathbf{1}_{N\mid T}\psi_1(T/(NY))\int_{\mathfrak{M}_1}e(\phi(T/N-b+a))\,d\phi$ (Lemma 2.10(i)), $\psi_1=1$ on $[-4,4]$ and $\mathfrak{M}_1=\bigcup_{q\le L^{A_0}}\bigcup_{(a_0,q)=1}\{\phi:\|\phi-a_0/q\|\le 2L^{A_0}/Y\}$. Both squares include pairs (a,b) sharing a prime; for coprime a,b the equations $mn=N-ah$, $rs=N-bh$ amount to $T:=bmn-ars=N(b-a)$ (Lemma 2.10(ii)).

Proposition 3.15 (Reflected determinant comparison)**.** *Assume the inputs from* [33]. *Let N be even, fix δ,C,ϖ, let $M,R\ge N^\delta$ with $X=MR\asymp N$, and let α, β be supported on Z-rough units modulo N in $[M,2M]$ and $[R,2R]$ with $|\alpha_m|,|\beta_n|\le L^C$. For every A, if A_0 and then K are large in terms of A,δ,C,ϖ, independently of the a_i and of N, then for every $Y\in 2^{\mathbb{Z}}$*

$$\big|\mathcal{Q}_Y^{\mathrm{sgn}}-\mathcal{Q}_Y^{\mathrm{maj}}\big|\ll_A XYL^{-A}. \tag{24}$$

Proof. We follow [33, Proposition 4.1] and record the changes. Its pads are $J=\lfloor L^{0.01}\rfloor$, $M_{\mathrm{pad}}=J+1$, $R_0=\lceil L^{1/2}/2\rceil$, a pad product $D\asymp d_0$ and $H=d_0Y$ [33, (3.13)]; put $U_1=HM$, $U_2=R$, with physical endpoints $P_0=(Dbm,s)$, $Q_0=(Dar,n)$. As $U_1U_2=HX\asymp HN$ and $H=N^{o(1)}$, Lemma 3.14 applies.

State space and dictionary. For each class C reducing to odd–odd modulo 2 take the lattice-primitive positions of $\Lambda_C\cap([U_1,16U_1]\times[U_2,2U_2])$, with the lists and measure $\prod_i V_i^{-M_{\mathrm{pad}}}$ of [33, (3.14)], and form the disjoint union over C; a non-primitive ambient vector may give several states,

no parity or odd-prime mask is placed on intermediate positions, and only the original ambient-primitive odd roots are averaged at the end. For a root $P_0 = (u, v)$, integral z, z' and $\tau(z) := z_1 - \vartheta z_2$,

$$\det(P_0, \widetilde{g}z) = Nz_2, \quad \det(\widetilde{g}z, \widetilde{g}z') = N\det(z, z'), \quad (\widetilde{g}z)_1 = u\tau(z), \quad (\widetilde{g}z)_2 = v\tau(z) + Nz_2/u,$$

so the box becomes the tilted region

$$s_1\tau(z) \in [1, 16], \qquad t_1\tau(z) + \frac{N}{s_1U_1U_2}z_2 \in [1, 2], \qquad s_1 = \frac{u}{U_1},\ t_1 = \frac{v}{U_2}; \tag{25}$$

thus $\tau(z) \asymp 1$ and $|z_1| + |z_2| = O(H)$. A mark prime p divides the first physical coordinate exactly when $[z]_p$ is the kernel line of the first row $(u, -u\vartheta)$, which is non-zero modulo p since $p \nmid N = \det \widetilde{g}$. Use the row operation of [33, (3.15)], the list symmetrisations, the damping and the good-state projection with $\det(P, Q)/N$ in place of the determinant; the target is again $\det(z, z')/D = b - a$. The two good tests of [33, §3.2] are applied to the *tested ratio*

$$r(z) := \frac{\tau(w_z)}{\tau(z)} = \frac{(\widetilde{g}w_z)_1}{(\widetilde{g}z)_1} \pmod 1, \qquad w_z \in \mathbb{Z}^2,\ \det(z, w_z) = 1,$$

well defined because $w_z \mapsto w_z + jz$ adds j. At the root $r(e_1) = -\vartheta$, uniform by Lemma 3.14; this is what the measure bound [33, Lemma 3.2] is integrated against. (The ratio of [33] needs a determinant-one completion, which does not exist at an ambient non-primitive lattice position.) Each part of the moment proof then uses pointwise estimates of [33] after the dictionary, not the fixed-shift proposition at a moving shift:

- [33, Lemma 3.4] and [33, (3.30)–(3.34)]: Lemma 3.14 replaces [33, Lemma 3.3]. Paths and labels have complexity $e^{O_K(L^{0.73})}$, the Bonferroni degree is $L^{0.76}$, $\log S_0 = O_K(L^{0.96})$, and there are $(2S_0)^3S_0 = e^{o(L)}$ residue matrices. The faces of (25) have transverse derivatives $\asymp 1$ ($\tau \asymp 1$, $N/(U_1U_2) = O(1/H)$) and other derivatives $N^{o(1)}$; the good tests are monotone in ϑ, with derivatives of modulus $1/\tau(z)^2 \asymp 1$. A grid of side $N^{-c_\delta/8}$ has $O(N^{3c_\delta/8})$ cubes, so discrepancy and boundary errors save a fixed power N^{-c}; and a fixed power N^{-c} is below L^{-AR_0} for every fixed A, since $R_0 \asymp L^{1/2}$.
- [33, §§4.1–4.3]: the equalities use the integral $\det(z, z') = \det(P, Q)/N$, primitive z and independent projective lines ("$p \mid D$ implies $\det(z, z') \equiv 0 \pmod p$, and primitivity implies $[z]_p = [z']_p$"), all unchanged; the tilted indicator is an endpoint multiplier at every position; initial and final positions are e_1.
- [33, (4.20)–(4.25)]: a target is $z' = jw_z + lz$ with $|l + j\tau(w_z)/\tau(z)| = O(1)$, and $\tau(z')/\tau(z) \in [1/16, 16]$ by the first face; so there are $O(1)$ targets at a fixed determinant, $O(Y)$ comparison determinants and comparison multiplier $O(L^{3A_0}/Y)$, and no physical displacement or ambient index occurs.
- [33, §4.5]: partition by $z_2/\tau(z)$, with the cell identity $z_2'\tau(z) - z_2\tau(z') = \det(z, z')$. The shared-label lattice has covolume D and region $|l| \ll Y$, $|h' + l\alpha'| \ll 1$, $\alpha' = D\tau(w_z)/\tau(z)$; the first good test gives a box $A'B' \ll Y$ with $Y^{0.1} \le A', B' \le Y^{0.9}$; unimodular changes preserve $\det/D$, and [33, (4.32)] gives [33, (4.33)]. Tilted cutoffs are endpoint diagonal multipliers, which [33] allows after its (4.29).
- [33, §§4.6–4.7]: only the ban, $p_* \nmid D$ and $p_* \ge e^{L^{a_{i_*}}}$ are used; the shared-label lattice is invertible modulo p_*, a prescribed line has $O(A'B'/p_* + \max(A', B'))$ points, and N does not appear after the dictionary.
- [33, §4.8]: the fresh atom bound is $e^{-L^{0.1}}$ up to fixed factors; low-rank terms keep signed contractions and high-rank terms one atom saving per independent equality.
- [33, §4.9]: [33, Lemma 3.2] is integrated over ϑ at the root, where $r(e_1) = -\vartheta$.

Consequently

$$\sum_{P \text{ original roots}} \big\langle u_P, (\mathcal{A}\mathcal{A}^*)^{R_0} u_P \big\rangle \ll U_1U_2L^{-2R_0E_0}, \tag{26}$$

with no factor for the number of classes. To pass to the endpoints use a partial positive trace: in each Λ_C partition by $P_2/(NP_1)$ into intervals of length H/U_1^2. An edge has $|\det(z,z')| \ll H$, so it meets $O(1)$ blocks, and a block holds $O(H)$ lattice-primitive positions ($\det(P,Q)/N$ takes $O(H)$ values against a lattice-primitive pivot, solutions with one value differ by multiples of the pivot, and $O(1)$ fit the box). The matrix of $(\mathcal{A}\mathcal{A}^*)^{R_0}$ on the root vectors of a block is positive semidefinite, so its norm is at most its trace; spectral Hölder, then Hölder over all pairs (class, block) with exponents 2, $2R_0/(R_0-1)$, $2R_0$, give the analogue of [33, (3.20)], with [33, (3.18)] on the union (endpoint vectors live on ambient-primitive positions, each in one class) and block cost bounded as $\log H = O_K(L^{0.21}) = o(R_0)$. No moments of a compressed operator are compared.

The original endpoints on a disjoint-label edge are ambient-primitive: a common prime of Dbm and s exceeds Z, divides T, is coprime to N, hence divides the non-zero integer T/N of size $O(Y) < Z$, which is impossible; and $T \neq 0$ because b has a prime dividing none of a, r, s. So the endpoint vectors lie in unique classes and the pairing counts exactly the signed difference. For the goodness deletion, pad stripping and collision restoration of [33, §4.9]: uniformly for a bank product $\ell \asymp Y$ with $(\ell, N) = 1$, $\sum_{h:\,|N-\ell h| \asymp N} \tau(|N-\ell h|)^2 \ll (N/Y)L^3$, since $\tau^2 \le \tau_4$, deleting a largest factor leaves $d \ll N^{3/4}$ of weight $\le 4\tau_3(d)$, $(d,\ell) = 1$ puts h in one class modulo d, and $N^{3/4}L^2$ is absorbed as $Y = N^{o(1)}$. For the comparison, $N \mid bk - a\ell'$ with a, b units forces $O(1)$ values of ℓ' in its interval of length $O(N)$ per k, and conversely, and $\sum_{k \asymp N} \tau(k)^2 \ll NL^3$. Mark collisions have harmonic mass $e^{-L^{0.1}+O(\log L)}$, and [33, (4.58)] is unchanged. Sum over the $O_K(L)$ pad dyads and choose E_0, then A_0, then K, as in [33, Proposition 4.1]. □

Proposition 3.16 (The reflected major term)**.** *Assume* QRH(7/8)*. For α as in Proposition 3.13, β as in Proposition 3.15 and every A, $|\mathcal{Q}_Y^{\mathrm{maj}}| \ll_A XYL^{-A}$.*

Proof. Write $\phi = a_0/q + \lambda/Y$ with $q \le L^{A_0}$ and $|\lambda| \le 2L^{A_0}$; the arcs are disjoint for large N, as reduced fractions with denominators $\le L^{A_0}$ are $\ge L^{-2A_0}$ apart while $Y \ge e^{cL^{0.1}}$. Apply (16) to $X_1 = bmn$, $X_2 = ars$; on the q_1-part split the additive phase between X_1 and X_2 with the inverse of N modulo q_1 and expand on unit classes, and expand $e(a_0(a-b)/q)$ on the bank variables (all variables are units modulo Nq, by roughness and bank deletion). This gives the exact factor $1/\varphi(Nq_0)$ and at most q^{12} fixed tuples of small characters, with coefficients bounded by q_0 before Cauchy's inequality is used. The remaining kernel is smooth in $b/Y, m/M, n/R, a/Y, r/M, s/R$ with log-power derivatives, because $T/(NY)$ has scale $X/N \asymp 1$; so its Mellin separation has L^1 mass L^C, tails L^{-A} beyond a polylogarithmic height (bounded with (19)), and no height comparable to N. After Cauchy's inequality it suffices to bound, uniformly at polylogarithmic heights and for fixed small twists,

$$\mathcal{E} := \frac{X}{\varphi(Q)} \sum_{\chi \bmod Q} \left|\mathcal{B}_\chi(t)\mathcal{M}_\chi(t)\mathcal{N}_\chi(t)\right|^2, \qquad Q = Nq_0, \tag{27}$$

the factors normalised by $1/Y$, $1/M$, $1/R$, the bank factor including $\prod_i V_i^{-1}$. Without $\mathcal{B}_\chi$, the product $\mathcal{M}_\chi\mathcal{N}_\chi$ is $X^{-1}\sum_n c_n\chi(n)n^{it}$ with $|c_n| \le L^C\tau(n)$, $\sum|c_n|^2 \ll XL^{C'}$ and c_n independent of χ, so (19) bounds its energy by $L^{C'}$. Mellin-expanding $(b/Y)\eta(b/Y)e(\pm\lambda b/Y)$, one bank supplies $V_i^{-1}\sum_{p\in\mathfrak{B}_i}(\chi\psi_b)(p)p^{iv}/p$ and the others have modulus ≤ 1; by Lemma 3.10, $\mathcal{B}_\chi$ is smaller than every power of L unless $\chi\psi_b$ is principal modulo Qq_1. For a fixed twist at most one χ is exceptional; it is induced from a character of conductor dividing q_0, so its twist on m has conductor dividing $q \le L^{A_0}$, and Lemma 3.12 (with $\mathbf{1}_{(m,N)=1}$, and $\delta/2$, as $M \le X/R \le N^{1-\delta/2}$) gives $\mathcal{M}_\chi \ll L^{-A'}$; bound $\mathcal{N}_\chi$, $\mathcal{B}_\chi$ trivially and use $X/\varphi(Q) \ll \log\log N$. Choose A' after the logarithmic costs. The strips of the bank sums are treated inside the energy bound, before summing characters, with normalised mass $O(L^{-B_1})$. □

Proof of Proposition 3.13. The square of each dyadic marked sum is $O(X/Y)\mathcal{Q}_Y^{\mathrm{sgn}}$, which is $\ll X^2L^{-A'}$ by Propositions 3.15 and 3.16. Choose A' before K and sum over the polynomially many dyads and Mellin integrals. □

The invariant test. Let $\mathfrak{B}=\bigcup_i\mathfrak{B}_i\subset S$ and $A_{\mathfrak{B}}(u)=\tau_k(u_{\mathfrak{B}'})-\tau_k(u_{S\setminus\mathfrak{B}})M_k(u_{S'})/V_y$, which is invariant under the bank primes. Off integers divisible by the square of a bank prime, $\tau_k(u)=k^{\omega(u)}\tau_k(u_{\mathfrak{B}'})$ and $\tau_k(u_S)=k^{\omega(u)}\tau_k(u_{S\setminus\mathfrak{B}})$, so

$$\xi_k(u)W_{1/(2k)}(u)=k^KA_{\mathfrak{B}}(u)W_{1/2}(u). \tag{28}$$

Truncate to $L^{-B_0}A_{\mathfrak{B}}\mathbf{1}_{|A_{\mathfrak{B}}|\le L^{B_0}}$, removing the tail by Lemma 3.6(i); the exponents are chosen before K, and k^K affects only the threshold. So Proposition 3.13 applies with this F and $\varpi=1/2$.

3.6. **The reflected centred invariant correlation.** Take the bank families of [21, (4.6)]: disjoint small banks in $[e^{L^{0.245}},e^{L^{0.249}}]$ and big banks in $[e^{L^{0.34}},e^{L^{0.39}}]$, each of reciprocal mass bounded above and below, $O(\log L)$ in each family. Select at least $c\log L$ banks from each family ($c>0$ fixed), delete the primes dividing N, and write $\mathcal{P}_g$ for the resulting sets, $V_g=\sum_{p\in\mathcal{P}_g}1/p$, $\omega_g(u)=\#\{p\in\mathcal{P}_g:p\mid u\}$, and $\mathcal{P}$ for their union; these primes exceed Z. Put $A_{\mathcal{P}}(u)=\tau_k(u_{\mathcal{P}'})-\tau_k(u_{S\setminus\mathcal{P}})M_k(u_{S'})/V_y$ and $\Omega_{\mathcal{P}}(u)=\prod_gV_g^{-1}\mathbf{1}_{\omega_g(u)=1}$.

Proposition 3.17. *Assume* QRH(7/8) *and the inputs from* [35], [42] *and* [21]. *Let* $\mathfrak{h}(u)=\mathfrak{h}(u_{S\setminus\mathcal{P}})$ *with* $|\mathfrak{h}|\le L^C$, *and let* $\mathfrak{t}(pv)=\mathfrak{t}(v)$ *for* $p\in\mathcal{P}$, $|\mathfrak{t}(v)|\le L^C\tau_j(v_{\mathcal{P}'})$, *and* $\mathfrak{t}(v)=0$ *unless* $P^-(v)>Z$. *For fixed* k,j,c,C,A *and* $\Phi\in C_c^\infty((0,1))$,

$$\sum_{0<u<N}\Phi(u/N)\,\mathfrak{h}(u)A_{\mathcal{P}}(u)\,\mathfrak{t}(N-u)\,\Omega_{\mathcal{P}}(u)\Omega_{\mathcal{P}}(N-u)\ll NL^{-A}, \tag{29}$$

uniformly in $N\ge3$, *in the selected subfamilies and in every permitted* $\mathfrak{h},\mathfrak{t}$.

The proof works for every $N\ge3$. First impose $(u,N)=(N-u,N)=1$: otherwise a prime $p\mid N$, $p>Z$, divides both endpoints (if $p\le Z$ divided $N-u$, then $\mathfrak{t}(N-u)=0$), and Lemma 3.6, Hölder and translated moments make these terms $\ll NL^{-A}$, also with the complete-list weights below, which are pointwise $L^{O(1)}$. Put $\mathfrak{a}(u)=\mathfrak{h}(u)A_{\mathcal{P}}(u)\mathbf{1}_{(u,N)=1}$ and $\mathfrak{t}_0(v)=\mathfrak{t}(v)\mathbf{1}_{(v,N)=1}$, both invariant under $\mathcal{P}$. The large-prime part of the unit masks is kept through the graph argument and removed only at the principal character in Proposition 3.22.

Lemma 3.18 (Complete lists on deleted banks)**.** *For each fixed* r *the complete-list moment bound of* [42, Lemma 2.3], *as used in* [21, (4.9)], *holds on the banks* $\mathcal{P}_g$ *for the endpoint functions* $\mathfrak{a}$, $\mathfrak{t}_0$, *with a logarithmic exponent independent of the padded list length* M_{pad} *(the threshold may depend on* M_{pad}*).*

Proof. Walsh's proof uses, besides [42, Lemma 2.2] at scale N/P_0, only three facts: every list product P_0 consists of $O_{M_{\mathrm{pad}}}(\log L)$ primes $\le y$, so $P_0=N^{o(1)}$; for $P_0\mid n$ composed of active primes, $|A_{\mathcal{P}}(n)|\ll L^{k-1}\tau_k(n/P_0)$ and $|\mathfrak{t}(n)|\le L^C\tau_j(n/P_0)$, for every $\mathcal{P}\subset S$; and $\sum_{\text{lists}}P_0^{-1}\prod_gV_g^{-M_{\mathrm{pad}}}\le\prod_g(\sum_{p\in\mathcal{P}_g}1/(V_gp))^{M_{\mathrm{pad}}}=1$ for any finite non-empty $\mathcal{P}_g$. The interval shape and the bounds $1/2\le V_g\le2$ of [42, (2.1)] are not used; $\mathfrak{h}$ and the unit masks cost a fixed power of L. □

Lemma 3.19 (Ideal comparisons after deletion)**.** *The ideal comparison family of* [35, Theorem 4.1] *exists on the deleted banks, with its coefficient exponents and probe count chosen before* M_{pad}*; its coefficients keep the separated form of* [42, Lemma 7.1] *and use only big-bank labels.*

Proof. Apply [35, Theorem 4.1] with two extra powers of ideal accuracy to the original selected banks (interval banks, with the fixed bounds $1/8,1/4,1/3,2/5$). Let μ_g be their harmonic law, $\varepsilon_g\ll Le^{-L^{0.245}}$ its mass on primes dividing N, and μ_g' its normalised restriction to the other primes. Compress the ideal Hilbert space to tuples of good primes; the identification is multiplication

by $(\prod_g(1-\varepsilon_g)^{-M_{\mathrm{pad}}})^{1/2}$, which commutes with permutation averaging. Conditioned on either endpoint, one of the $O_m(\log L)$ free auxiliary labels is bad with probability $O_m(\log L\max_g\varepsilon_g)$, and the absolute row and column bounds are L^{C_1}; by Schur's test, deleting these draws and changing the remaining laws to μ'_g perturb the norm by $O_{M_{\mathrm{pad}},m}(L^{C'}e^{-L^{0.245}})$. The raw coefficient is still 1, the kernels are those built on the original banks, and the expansion of [42, Lemma 7.1] survives restriction, with count and supremum mass L^{C_1}, C_1 independent of M_{pad}. □

Proposition 3.20 (Large displacement)**.** *In* [35, Definition 3.3] *replace the multiplier k by a nonzero integer h coprime to every bank prime, keeping the other definitions and the full integer position space. Under the ideal-norm hypothesis of* [35, Theorem 3.5]*, uniformly in h,*

$$\sum_{n\in I}\langle u_n,(\mathcal{A}\mathcal{A}^*)^{R_D}u_n\rangle\ll|I|L^{-2R_DE},\qquad R_D=\lceil L^{0.52}\rceil,\ I=[X,2X),\ N^{\gamma_1}\le X\le N^{1/\gamma_1}. \tag{30}$$

If the maximal displacement is $|h|D_{\max}$ with $\log D_{\max}=o(R_D)$, then for mark-independent f on $[X,2X)$ and g with $\|f\|_2,\|g\|_2\le X^{1/2}L^{C_3}$ and $\|f\|_r^r\ll XL^{C_r}$ for a fixed $r>2$, uniformly in h,

$$|\langle f,\mathcal{A}g\rangle|\ll XL^{-E+2C_3}. \tag{31}$$

Proof. [35, Theorem 3.5] is printed for polylogarithmic displacement; every use of the displacement in its proof survives. (1) [35, Lemma 3.4]: the absolute bounds use no property of positions beyond divisibility and translation. (2) [35, Lemma 3.6]: a path fixes offsets with $h_{i+1}-h_i=\pm hD_i$, whose sizes never enter; active congruences are incompatible or fix a class modulo Q_{act} with $\log Q_{\mathrm{act}}=O_{M_{\mathrm{pad}}}(R_DL^d\log L)=o(L)$, $d<0.47$; each CRT counting error is $O(1)$ for any residue, and the relative error $e^{-\gamma_1L+o(L)}$ is absorbed by the path mass $L^{O(R_D)}$; the return condition is the exact Fourier identity for $\sum\pm hD_i=0$. (3) [35, §3.3, (3.12)–(3.14)]: identities for residues; "every group prime exceeds $|k|$" serves only to infer $p\mid D$ from $p\mid hD$, which $p\nmid h$ supplies, keeping the entry/exit exclusion and the factor $V_g^{-r}/(pb_p)$. (4) [35, §§3.4–3.5]: residues translate by $\pm hD$, a permutation of the residue spaces; nothing else changes. (5) [35, Lemma 3.9]: a promotion requires $r'\pm hD_BD_S\equiv0\pmod p$ with $p\nmid D$, and h, D_B and the small-label product are units modulo p, so each pending particle permits at most one omitted product, as in its (3.31). (6) [35, Lemma 3.10]: Fourier transformation turns translation by $\pm hD$ into $e(\pm\Theta D_B)$, and the ideal hypothesis is a supremum over all real Θ. (7) [35, §3.8]: (3.40)–(3.43) involve prime labels, not h, and the choices of its (3.37) give (30).

For the pairing, [35, Corollary 3.11] uses blocks of size $\max|kD|$; instead, in each residue class modulo $|h|$, partition the source positions into blocks of at most $\lceil D_{\max}\rceil$ consecutive positions of the progression. Each block meets $O(1)$ neighbouring blocks of its class, with bounded overlap over all classes together. For mark-independent f with $\sup|f_n|\le e^{O(\sqrt{L})}$, spectral Hölder and positivity give for a block J

$$|\langle f_J,\mathcal{A}g_J\rangle|\le\|g_J\|\,\|f_J\|^{1-1/R_D}\Big(\sum_{n\in J}|f_n|^2\sum_{n\in J}\langle u_n,(\mathcal{A}\mathcal{A}^*)^{R_D}u_n\rangle\Big)^{1/(2R_D)},$$

where $(\sum_J|f_n|^2)^{1/(2R_D)}\le(D_{\max}e^{O(\sqrt{L})})^{1/(2R_D)}=O(1)$; Hölder over blocks with exponents 2, $2R_D/(R_D-1)$, $2R_D$ and (30) give (31), with no factor $|h|$. For unbounded f clip at $e^{\sqrt{L}}$ as in the proof of [42, Theorem 7.2], using $\|f_{\mathrm{tail}}\|_2^2\le e^{-(r-2)\sqrt{L}}\|f\|_r^r$ and the absolute Schur bound, and work with the represented, uncompressed operator, extending exact-count endpoint vectors by zero (this keeps norms and mark independence); then $\langle f,P_M\mathcal{A}P_Mg\rangle=\langle f,\mathcal{A}g\rangle$, as in [42, (7.14)]. No higher moments of $P_M\mathcal{A}P_M$ and $\mathcal{A}$ are compared. At $h=-N$, $\log D_{\max}=O_{M_{\mathrm{pad}}}(L^{0.39}\log L)=o(R_D)$. □

Lemma 3.21 (Exact comparisons)**.** *For lists of length t_g, distinct within each bank, put $W_{\boldsymbol{t}}^b(u)=\prod_g(V_g^{-t_g}\mathbf{1}_{\omega_g(u)=t_g})\sum_{\boldsymbol{p}\mid u}b(\boldsymbol{p})$, and $\Psi(t)=t\Phi(t)$. There are a fixed integer m and finitely many*

comparison terms with

$$\Sigma_0 + \sum_\nu c_\nu \Sigma_\nu = O(L^{-A}), \qquad \#\{\nu\} + \sum_\nu |c_\nu| \le L^{C_1}, \tag{32}$$

where $\Sigma_0 = \sum_u \frac{\Psi(u/N)}{u} \mathfrak{a}(u)\mathfrak{t}_0(N-u)\Omega_{\mathcal{P}}(u)\Omega_{\mathcal{P}}(N-u)$ *and*

$$\Sigma_\nu = \mathbb{E}_{\boldsymbol{d}_1,\boldsymbol{d}_2} a_\nu(\boldsymbol{d}_1) b_\nu(\boldsymbol{d}_2) \sum_w \frac{\Psi(w/(ND_1D_2))}{w}\, \mathfrak{a}(w)\mathfrak{t}_0(ND_1D_2-w) W_{\boldsymbol{t}}^{u_\nu}(w) W_{\boldsymbol{t}}^{v_\nu}(ND_1D_2-w). \tag{33}$$

Auxiliary draws use $\mu_g(p) = 1/(V_g p)$*; each big-bank block has a non-empty auxiliary product* D_i *with at most* m *draws per bank, and small banks have none;* $t_g = 1 + r_g$*,* $0 \le r_g \le m$*,* $r_g = 0$ *on small banks; the coefficients have modulus* ≤ 1 *and* u_ν, v_ν *depend only on big-bank lists. All exponents are independent of* M_{pad}*.*

Proof. Apply Lemma 3.19 and Proposition 3.20 with $h = -N$ and endpoint weights $\mathfrak{a}(n)$ and $\mathfrak{t}_0(-n')$ on negative target positions; the $r = 2, 4$ moments of the pairing are Lemma 3.18, over the positive integers $-n'$ of the size of the source dyad, and all positions have size $N^{1+o(1)}$. At $h = -N$ there is a localisation issue absent for fixed h: on $\Psi(n/(ND))$, $-n' = ND - n$ is positive and comparable to n. For each source dyad insert a target cutoff equal to 1 there, and keep it through the Mellin separation of $\Psi(n/(ND))$ (the D-phase is the operator's ζ, the n-phase is mark-independent); its norm is covered by Lemma 3.18. Fix $|\zeta| \le L$ before choosing the ideal family, and bound the omitted tails, by integration by parts, against the absolute Schur bound. With D_R the product of the shared labels, the endpoints are $n = D_R w$ and $n' = D_R(w - ND_1D_2)$; invariance removes D_R from both endpoint functions, the harmonic factor gives $1/D_R$, and the normalisation in bank g is exactly $V_g^{-M_{\mathrm{pad}}-t_g}/(D_R)_g = V_g^{-2t_g} \prod_{p \text{ shared in } g} (V_g p)^{-1}$, so the shared draws become independent harmonic laws, and [42, Lemma 7.1] gives the separated coefficients. The exclusions of [21, (4.10)–(4.11)] remain negligible: w and $v = ND_1D_2 - w$ are $N^{1+o(1)}$; bad shared choices have mass $L^{O_{M_{\mathrm{pad}}}(1)} e^{-L^{0.245}}$; for an auxiliary $p \mid D_1D_2$, $p \mid w$ is equivalent to $p \mid v$, and Hölder with fourth moments bounds these terms by $L^{C'} p^{-1/2}$; and a retained prime dividing w and v divides ND_1D_2 but not N, hence divides D_1D_2. This replaces the use of $p > |h|$ in [21]. The list factors cost $\prod_g (t_g!) V_g^{-t_g} = L^{O_m(1)}$. □

Fourier reduction. Restrict D_i to dyads $[H_i, 2H_i)$, and put $H_{\mathrm{aux}} = H_1H_2$ and $Y_* = NH_{\mathrm{aux}}$; then $H_i \ge e^{L^{0.34} - O(1)}$ and $\log H_{\mathrm{aux}} = O_m(L^{0.39} \log L)$. Mellin separation of $w/(ND_1D_2)$ reduces Σ_ν to integrals, of polylogarithmic L^1 mass, of $Y_*^{-1} \mathbb{E} a(D_1) b(D_2) \sum_u f_1(u) f_2(ND_1D_2 - u)$, with

$$f_1(u) = \Upsilon_1(u/Y_*) u^{i\sigma} \mathfrak{a}(u) W_{\boldsymbol{t}}^{b_1}(u), \qquad f_2(v) = \Upsilon_2(v/Y_*) \mathfrak{t}_0(v) W_{\boldsymbol{t}}^{b_2}(v), \tag{34}$$

supported on integers $\asymp Y_*$, $\|f_i\|_2^2 \ll Y_* L^C$, Υ_1, Υ_2 smooth cutoffs, b_1, b_2 tuple tests on big-bank lists, and $|\sigma| \le L^D$. With $\widehat{f}(\kappa) = \sum_u f(u) e(\kappa u)$ and $m(\phi) = \mathbb{E} a(D_1) b(D_2) e(-\phi D_1 D_2)$ the sum is $Y_*^{-1} \int \widehat{f_1}\widehat{f_2}(\kappa) m(N\kappa)\, d\kappa$. [42, Lemma 6.2] applies to the deleted banks (it allows any disjoint sets of primes with $V_g \asymp 1$, at most m draws per bank, and tuple tests after conditioning): with Dirichlet approximation at denominator $H_{\mathrm{aux}}/L^{B_0-1}$ it gives $m(\phi) \ll L^{-A'}$ off $\phi = a/q + \lambda/H_{\mathrm{aux}}$, $q, |\lambda| \le L^{B_0}$, and Parseval bounds that part. On these arcs, using all N inverse branches, $N^{-1} \sum_{r \bmod N} \widehat{f_1}\widehat{f_2}((r+\phi)/N) = \sum_{N \mid u+v} f_1(u) f_2(v) e(\phi(u+v)/N)$, with prefactor $1/(Y_* H_{\mathrm{aux}}) = N/Y_*^2$. So it suffices to prove, for every A,

$$\sum_{N \mid u+v} f_1(u) f_2(v)\, e\Big(\frac{a(u+v)}{Nq}\Big) e\Big(\frac{\lambda(u+v)}{Y_*}\Big) \ll \frac{Y_*^2}{N} L^{-A}; \tag{35}$$

a bound with Y_*^2 in place of Y_*^2/N would not close the argument.

Proposition 3.22 (Large-character comparison)**.** *Assume* QRH(7/8)*. Then* (35) *holds, uniformly in all the data above.*

Proof. Absorb $e(\lambda u/Y_*)$, $e(\lambda v/Y_*)$ into the cutoffs, split $q = q_0q_1$ and apply (16) with $Q = Nq_0$, $X_1 = u$, $X_2 = -v$. Keep $b_\chi(-v)$ on the target side, expanded in small characters modulo q_0 with χ-dependent coefficients bounded by q_0. On the q_1-part the phase splits into bounded periodic functions of u and of v; the target one is left unexpanded. The source need not be a unit modulo q_1: with $d = (u, q_1)$, write $u = dz$ and expand on the units modulo q_1/d, so the source twist is $\chi(u)\psi(u/d)$ with $d \le L^{B_0}$ and $(d, N) = 1$. For each of the polylogarithmically many terms put $\mathcal{F}_\chi = Y_*^{-1}\sum_u f_{1,d}(u)\chi(u)\psi(u/d)$ and $\mathcal{G}_\chi = Y_*^{-1}\sum_v f_{2,*}(v)\overline{\chi(-v)}$, with coefficients independent of χ; it is enough to prove

$$\sum_{\chi \bmod Q} |\mathcal{F}_\chi\mathcal{G}_\chi| \ll_A L^{-A}, \tag{36}$$

since (16) then gives $Y_*^2L^{-A}/\varphi(Q)$ and $N/\varphi(Q) \ll \log\log N$.

Choose one selected small bank $\mathcal{P}_g$; since $t_g = 1$ there and b_1 uses only big lists, each u counted in $\mathcal{F}$ has a unique prime divisor $p \in \mathcal{P}_g$. The $u \asymp Y_*$ with $p^2 \mid u$ number $\le Y_*e^{-cL^{0.245}}$ and have $\sum|f_1|^2 \ll Y_*L^Ce^{-cL^{0.245}}$; this error is handled after the character expansion, by (17) with $Q \ll Y_*$ and Cauchy's inequality against $\mathcal{G}$ (not by a bound per character summed over Q characters). Now $u = dpm$, with m free of primes of $\mathcal{P}_g$; the other conditions do not depend on p, since $\mathfrak{h}$ and $A_\mathcal{P}$ are invariant, the other lists are from disjoint banks and $p > q_1$, and the unit mask becomes $(m, N) = 1$. For p in a dyad $[P, 2P]$, $e^{L^{0.245}}/2 \le P \le e^{L^{0.249}}$, the scale $T = Y_*/(dP)$ has $T/Q = H_{\mathrm{aux}}/(dPq_0)$ larger than every power of L, and Mellin separation writes $\mathcal{F}_\chi$ through integrals of logarithmic L^1 mass of

$$\frac{\chi(d)}{dV_g}\Big(\frac{1}{P}\sum_{p\in\mathcal{P}_g\cap[P,2P]}(\chi\psi)(p)p^{it}w\Big(\frac{p}{P}\Big)\Big)\Big(\frac{1}{T}\sum_{m\asymp T}c_m(\chi\psi)(m)m^{is}\Big), \qquad \sum|c_m|^2 \ll TL^C,$$

with exponents depending on m, k, j, C but not on M_{pad}. By (18) the second factor and $\mathcal{G}_\chi$ have character mean squares $\ll L^C$ (as $T, Y_* \gg Q$), so Cauchy's inequality lets the supremum of the prime factor be taken outside, and by Lemma 3.10 it is $\ll L^{-A'}$ whenever $\chi\psi$ is non-principal modulo $Q(q_1/d)$. Strips and Mellin tails are bounded at each height by (18), with normalised mass $O(L^{-B_1} + 1/P)$, before the sum over χ.

Since $(Q, q_1/d) = 1$, $\chi\psi$ is principal only if χ and ψ both are: one exception per stratum, treated by exact centring. Remove the part of the unit mask from primes $p \mid N$, $p > y$, at cost $\ll L^Ce^{-cL^{0.4}}$ (Hölder, fourth moments, $\sum_{p|N,\,p>y} 1/p \ll L/y$). What remains depends on u only through the S-record $A = u_S$, and for $u = Av$, $(v, S) = 1$, $A_\mathcal{P}(Av) = \tau_k(A_{S\setminus\mathcal{P}})\{\tau_k(v) - M_k(v)/V_y\}$, with record coefficient $\ll L^C\tau_k(A)$. [42, Lemma 5.2] with $R = Y_* = N^{1+o(1)}$, $q = 1$, $j = K = k$, $b_A = 1$ (absolute values in its (5.4); fixed compact interior support, log-power derivatives, heights $\le L^D$) gives $|\mathcal{F}_\chi| \ll L^{C'}e^{-cL^{1/10}}$, and the target is $O(L^C)$. □

Proof of Proposition 3.17. By Proposition 3.22 every comparison in (33) is smaller than every power of L; with accuracy above C_1 in (32), $\Sigma_0 \ll L^{-A}$, which is (29) as $\Psi(u/N)/u = \Phi(u/N)/N$. The parameters are chosen in this order: the target graph accuracy from the exponents of Lemma 3.18; the ideal accuracy; the probes m and coefficient exponents; then M_{pad}, large for the rare transfers. The accuracy in Proposition 3.22 comes after the coefficient exponent. No estimate whose exponent grows with M_{pad} feeds back into its choice; exponentially small errors may have constants depending on M_{pad}. □

3.7. **Proof of Theorem A for even N.** We use the finite-convolution and Boolean identities of [21, §§3, 5–6] with Lemma 3.8; no other shifted-convolution estimate enters. Let N be even, fix $\Phi \geq 0$ in $C_c^\infty((0,1))$ and one mark system, and write $W = W_{1/(2k)}$.

Steps 1–2: sieve, and primes to the rough model. Fix a large even s and $\gamma = c_0/(8s+8)$. Lemma 3.9(ii) controls the centred sum over N^γ-rough integers at the prime scale; the complement is a finite convolution expansion with between 2 and $1/\gamma$ prime factors $> N^\gamma$, repeated factors contributing $N^{1-\gamma+o(1)}$. With $a(m) = \mathbf{1}_{P^-(m)>Z}/(V(Z)\log m)$, $m > N^\gamma$, telescope one prime coefficient at a time on each dyadic box: each difference is a sum (21), with the other variables as β (truncated to $|\beta| \leq L^C$ by Lemma 3.6(i) and Hölder) and the test of (28); apply Proposition 3.13 at the box's δ and sum over the polynomially many boxes.

Step 3: invariance. With $\mathcal{C}$ the union of the bank families of §3.6, replace a by $\widetilde{a}(m) = \mathbf{1}_{P^-(m)>Z}\mathbf{1}_{m_{\mathcal{C}'}>N^\gamma}/(V(Z)\log m_{\mathcal{C}'})$, invariant under $\mathcal{C}$. Removing $m_{\mathcal{C}}$ changes the logarithm by $O(\sqrt{L})$ outside small-record tails, a relative change $O(L^{-1/2})$, controlled by Lemma 3.8(iii) over $O(L^{j-1})$ product boxes; for $j \geq 2$ the threshold crossing confines one logarithmic coordinate to width $O(\sqrt{L})$, so $O(L^{j-3/2})$ boxes give $O(C_k(N)NL^{k-5/2})$. This is the reflected form of [21, §5.2].

Step 4: the marked centred estimate. By Lemma 3.8(ii) restrict to pairs in which enough banks contain exactly one prime at both endpoints: at $p \nmid N$ the pair law is $\mu_{k,j}$, each bank succeeds with probability bounded below, independently, and Chernoff bounds give $O(C_k(N)NL^{k-2-\eta'})$, $\eta' > 0$ fixed. Expand by the Boolean identity [21, (5.18)], of mass $3^{O(\log L)} = L^{O(1)}$. After removing selected squares, with r selected banks and $\mathfrak{t}(v) = \widetilde{a}^{*j}(v_{\mathcal{P}'})$ (invariant, zero unless $P^-(v) > Z$, $|\mathfrak{t}(v)| \leq (\gamma LV(Z))^{-j}\tau_j(v_{\mathcal{P}'})$), [21, (5.19)] gives on the selected single-prime records

$$\widetilde{a}^{*j}(v) = j^r\mathfrak{t}(v), \qquad \xi_k(u) = k^r A_{\mathcal{P}}(u), \qquad \prod_g I_g = \Big(\prod_g V_g^2\Big)\Omega_{\mathcal{P}}(u)\Omega_{\mathcal{P}}(v);$$

each selected prime may go to any of the j ordered factors, all j^r assignments having the same weight by invariance (a stripped convolution, not a claim that the original one is invariant). The scalar $(kj)^r\prod_g V_g^2$ is $L^{O(1)}$; absorb small-record factors into $\mathfrak{h}$ and choose the accuracy of Proposition 3.17 after this exponent and the Boolean mass. Restoring the square terms by Lemma 3.6(i), Hölder and $\sum_{p\in\mathcal{P}} p^{-2} \ll e^{-L^{0.245}}$, we get

$$\limsup_{N\to\infty,\ 2|N} \frac{|\sum_{p<N}\Phi((N-p)/N)\xi_k(N-p)W(N-p)|}{C_k(N)NL^{k-2}m_W} \ll_{k,\Phi} \frac{e^{-s}}{\gamma} \tag{37}$$

with the constant of (13): apart from (13) the four steps contribute $o(C_k(N)NL^{k-2})$ as $N\to\infty$ with s, K, marks fixed. So the constant does not depend on s, K (chosen after s), the number of mark systems or the mark exponents.

Step 5: removing the damping. Average W/m_W over b' disjoint fixed mark systems. Under the law of Lemma 3.8(i) the average has mean 1 and variance $O_{K,k}(1/b')$, uniformly in N and the bank exponents, by the constants of [21, (2.7)] (bank primes do not divide N, and at such primes $\mu^*_{N,k}$ coincides with the local law of [21]; deleted primes have negligible mass). Lemma 3.8(i) with $|\xi_k| \leq \Xi_k$ then bounds the difference between the unweighted centred sum and the averaged weighted sums by $O_{K,k}(C_k(N)NL^{k-2}b'^{-1/2}) + O_A(NL^{-A})$. Let $N\to\infty$ with s, K, b' fixed, then $b'\to\infty$, then $s\to\infty$; as $e^{-s}/\gamma \to 0$,

$$\sum_{p<N}\Phi\Big(\frac{N-p}{N}\Big)\xi_k(N-p) = o\big(C_k(N)NL^{k-2}\big) \tag{38}$$

uniformly over even N, for $\Phi \geq 0$ and then, by linearity, for every real $\Phi \in C_c^\infty((0,1))$. The choices before the limit in N follow [21, Remark 6.3]: truncation and error exponents precede the number of marks, and the band exponents come last.

Step 6: Λ and the model. On the interior support $\log p = L + \log(1-v)$, $v = (N-p)/N$; apply (38) to Φ and to $-\Phi(v)\log(1-v) \geq 0$. Prime powers contribute $N^{1/2+\epsilon}L^{O(1)}$. So $\sum_{t<N}\Lambda(t)\Phi((N-t)/N)\xi_k(N-t) = o(C_k(N)NL^{k-1})$, and Lemma 3.7 gives the smoothed form of (6). Finally take $\Phi \geq 0$ equal to 1 on $[2\eta_1, 1-2\eta_1]$ and supported in $[\eta_1, 1-\eta_1]$, so that $\int_0^1 \Phi = 1+O(\eta_1)$; by nonnegativity and Lemma 3.8(iv), $S_k(N)$ differs from the smoothed sum by $O_k(\eta_1 C_k(N)NL^{k-1}) + N^{1/2+\epsilon}$. Let $N \to \infty$, then $\eta_1 \to 0$; Lemma 3.5 absorbs $N^{1/2+\epsilon}$. This proves Theorem A(a).

3.8. **Odd N, and the prime-only form.** *Class lattices and root count.* For odd N take *every* class $C \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ in the state space: a class modulo odd N has no reduction modulo 2, so "reducing to odd–odd modulo 2" is not used. Keep all lattice-primitive intermediate positions with the lists and measure of Proposition 3.15; only the original roots in the final trace must be ambient-primitive and odd–odd, which the rough coefficients and marks ensure; copies of a non-primitive ambient vector in different classes are separate states. Lemma 3.14 holds for odd N with the same main term and error: $q = 2S_0$ forces odd–odd roots whatever the parity of N; the congruences and their number depend on u and q only; the third coordinate is $(-Nc/u)/S_0$ with frequency $a = h - 2Nh'$; for $b \neq 0$, $(a,b,u) \leq |b| \leq F_0$, and for $b = 0$, $(h,h') \neq (0,0)$ we have $a \neq 0$ and (23), roots with $(u,N) > 1$ included; for $U_2 < U_1$ the relation $(-Nc/u)/S_0 = (-Nd/v)/S_0 + N/(uvS_0)$ (mod 1) keeps $\mathfrak{f} \bmod S_0$ and ϑ; the two completions modulo 2 give density $1/(3\zeta(2))$; at $p \mid S_0$, N is a unit, so the line distribution holds; no projective line at 2 is used; and c_δ, decreased if needed, does not depend on the parity of N.

Determinant comparison and major arcs. Every item in the proof of Proposition 3.15 applies with its bounds: the path modulus has only odd mark primes, $\det/N$ becomes $\det(z,z')$, the endpoint restrictions are the same tilted diagonals, and the parity coordinate is a root restriction, not a memory label. So (26) holds over the odd–odd roots in all classes. A lattice-primitive pivot completes to a lattice basis, so blocks have $O(H)$ positions in every class, and endpoint supports lie in unique classes: no class-count factor. The endpoints are ambient-primitive as before; for odd N, $\det/N$ is even at these odd endpoints and so is the raw target $D(b-a)$, which restricts nothing. Lemma 2.10(ii) holds for odd N with the same proof, except that the two quotients are non-zero because $m, r > 1$ are units modulo N, so that $mn \neq N$ and $rs \neq N$. The kernels, the signed-square identity and the pad-stripping bound (which needs only $(\ell, N) = 1$) are unchanged, so (24) holds. In Proposition 3.16, (16) and $\varphi(Nq_0) = q_0\varphi(N)$ hold for every N; if q is even and N odd, the 2-part of q belongs to q_1; all of m, n, r, s, a, b are odd, so the q_1 expansion is on units; small characters modulo powers of 2 are allowed in Lemmas 3.10 and 3.12. So Proposition 3.13 holds for odd N, with the same dependence of K, and (28) is unchanged. Proposition 3.17 was proved for every $N \geq 3$.

Local factors and assembly. Lemmas 3.5, 3.7 and 3.9 hold for either parity (their residue is $N \bmod d$, their unit condition $p \mid N$, their multiplicities bounded in the smooth record), and (13) holds with lim sup over odd N. In Lemma 3.8 check $p = 2$: for odd N a rough or odd-prime target $N - u$ forces u even, and

$$\int_{\mathbb{Z}_2} \tau_k(2^{v_2(u)})\mathbf{1}_{2\nmid N-u}\,du = \frac{1}{2}\sum_{a\geq 1}\binom{k+a-1}{a}2^{-a} = 2^{k-1} - \frac{1}{2},$$

which times 2^{2-k} is $g_k(2) = 2-2^{1-k}$, the generic factor of (5); for even N the integral is $1/2$ and the factor 2^{1-k}. In a rough pair the second endpoint is odd, so τ_j contributes 1 there, and the record banks omit 2. Henriot's sieve product runs over $p > \deg = 2$ ([20, p. 377]: $\prod_{g<p\leq x}(1-\rho(p)/p)$, g the degree), so no factor $1-2/2$ arises, and the 2-adic valuation contribution is bounded by the series displayed. So Lemma 3.8 keeps $C_k(N)$ for odd N, with no loss depending on N. In §3.7, all convolution factors are rough, hence odd, so N minus their product is even, which the centred divisor test allows; primes and prime powers based at 2 contribute $N^{o(1)}L^{O(1)}$; the Boolean

probabilities involve only odd primes; and Lemma 3.7 supplies the factor at 2. The limits are taken in the same order. This proves Theorem A(b).

Proof of Corollary A.1. For either parity remove the prime powers from the smoothed form of (6) (Step 6 of §3.7) and use $\log p = L + O_{\eta_1}(1)$ on the interior support; positivity and Lemma 3.8(i) bound the conversion error by $O_{\eta_1}(C_k(N)NL^{k-3})$. (Equivalently, combine (38) with the same conversion applied to (11).) At the endpoints, $p \le \sqrt{N}$ costs $N^{1/2+\epsilon}$, and for $p > \sqrt{N}$, $1 \le 2\log p/L$ and Lemma 3.8(iv) give $O_k(\eta_1 C_k(N)NL^{k-2}) + N^{1/2+\epsilon}$. Let $N \to \infty$, then $\eta_1 \to 0$. □

4. Band-prime cofactors: Theorem B

This section states and proves Theorem B. Throughout, N is even, $\delta \in (0, 1/4)$ is fixed, and $\mathcal{X}_N^\delta := \{N/2^j : \ j \ge 2 \text{ an integer}, \ 2^{-j} \ge \delta\}$. For real x we put $L := L_x = \log x$, and $Z = \exp(L^{0.24})$, $U = V = x^{1/5}$, R_Z and c_V are those of Proposition 2.2; $P_Z := \prod_{p \le Z} p$. We fix an integer $K \ge 1$ and exponents $1/10 < a_1 < \cdots < a_K < 1/5$, and $\mathcal{P}_i$, $\mathcal{P}_i^N$, V_i^N and $\mathcal{A}^N$ are the bands of $\mathfrak{B}_{\exp}$ formed with $L_x = \log x$, their puncturings at N, their reciprocal masses and the set of marks (Definition 2.5). For an even integer n,

$$\mathfrak{S}(n) := 2C_2 \prod_{p \mid n, \ p>2} \frac{p-1}{p-2}, \qquad C_2 := \prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big) = 0.6601618158\ldots,$$

and for integers k we put $G_N(k) := \sum_{a \in \mathcal{A}^N} \mathbf{1}_{a \mid N-k}\, \mathbf{1}_{\text{prime}}((N-k)/a)$, the number of ways of writing $N - k = ah$ with $a \in \mathcal{A}^N$ and h prime.

4.1. **Statement.** The proof follows the structure of the proof by Lü, Wang and Wu [28] at the shift $+2$, with $N - k$ in place of $k + 2$ and the weight G_N in place of theirs: Vaughan's identity on Z-sifted integers (Proposition 2.2), a Type I estimate modelled on [28, proof of Proposition 3.1] (§4.3), and a reduction of the Type II sum to the signed square of Definition 2.9 (§4.4), which Hypothesis RE bounds. Prior work is described in §1.4.

Theorem B. *Let $\theta < 1$ be fixed, and assume* $\mathrm{QRH}(\theta)$ *(Definition 2.1) and Hypothesis* $\mathrm{RE}(\mathfrak{B}_{\exp}, \theta)$ *(§2.3); the proof uses in addition Proposition 2.2, which is* [28, Proposition 2.1]*, and the classical inputs of §2.4. Fix $\delta \in (0, 1/4)$ and $A' > 0$. Let $K \ge K_0(A')$ be an integer, where $K_0(A') := \lfloor 200(E_0 + 8)\rfloor + 1$ with $E_0 := 2A' + 15$, and fix exponents $1/10 < a_1 < \cdots < a_K < 1/5$. There is $x_0 = x_0(\delta, A', K, (a_i), \theta)$ such that for every even N, every $x \in \mathcal{X}_N^\delta$ with $x \ge x_0$, and every $y \in (x, 2x]$,*

$$T_N(x; y) := \sum_{x < k \le y} \Lambda(k) G_N(k) = M_N(x; y) + O\big(xL^{-A'}\big), \tag{39}$$

$$M_N(x; y) := \sum_{a \in \mathcal{A}^N} \frac{\mathfrak{S}(aN)}{a} \int_x^y \frac{du}{\log((N-u)/a)}, \qquad \mathfrak{S}(aN) = \mathfrak{S}(N) \prod_{p \mid a} \frac{p-1}{p-2}. \tag{40}$$

The bands, hence $\mathcal{A}^N$, are those of $L = \log x$, the same for every y. The implied constant depends only on δ, A', K, (a_i) and θ. It is not effective.

Hypothesis $\mathrm{RE}(\mathfrak{B}_{\exp}, \theta)$ is used at one instance only (§4.4): precision $A = 2A' + 4$, coefficient exponent $C = 2$, Mellin exponent $B = A' + 4$, $\delta_0 = \delta/5$, and the datum (E_0, K) with $E_0 = 2A' + 15$, together with the exponents (a_i). Since $E_0 > A + 4C + 2 = 2A' + 14$ and $K > 200(E_0 + 8)$, this pair is admissible for (A, C) (Definition 2.4). The zero-free half-plane $\mathrm{QRH}(\theta)$ enters only through (RE2). As stated in Theorem B, the proof also uses Proposition 2.2, which is [28, Proposition 2.1], and the classical results of §2.4. The implied constant in (39) is not effective because the constant in Lemma 2.13, as proved, is not effectively computable and Hypothesis RE does not assert effective constants.

Corollary B.1. *In the setting of Theorem B, put*

$$P_N(x) := \#\{p \in (x, 2x] \text{ prime}: \ N - p = a \cdot h \text{ with } a \in \mathcal{A}^N \text{ and } h \text{ prime}\}.$$

Then

$$P_N(x) = \sum_{a \in \mathcal{A}^N} \frac{\mathfrak{S}(aN)}{a} \int_x^{2x} \frac{du}{\log u \cdot \log((N-u)/a)} + O\big(xL^{-A'-1}\big), \tag{41}$$

and the main term equals $\mathfrak{S}(N)\big(\prod_i V_i^N\big)(x/L^2)\big(1 + O_{K,\delta}(L^{-0.8})\big) = \mathfrak{S}(N)(\log 2)^K(x/L^2)\big(1 + O_{K,\delta}(L^{-0.1})\big)$. *Uniformly in even* N, *the main term is at least* $(\log 2)^K x/(2L^2)$ *for large* x, *since* $\mathfrak{S}(N) \geq 2C_2 > 1.32$. *Hence* $P_N(x) = (1 + O(L^{1-A'})) \cdot (\text{main term})$. *At the same* K, *the sharper form* (48) *of* (39) *proved in §4.5 gives the error* $O(xL^{-A'-2})$ *in* (41), *that is,* $P_N(x) = (1 + O(L^{-A'})) \cdot (\text{main term})$. *In particular, for every* $A' > 0$ *and every* $K \geq K_0(A')$, *every sufficiently large even* N *is* $p + a \cdot h$, *with* $p \in (N/4, N/2]$ *prime,* h *prime, and* a *a product of one prime from each band* $\mathcal{P}_i$ (*formed with* $L = \log(N/4)$), *none dividing* N. *If only* (39) *as stated is used, this needs* $A' > 1$.

Remark 4.1. (i) *What Theorem B is not.* Every mark is a product of $K \geq K_0(A')$ primes, each larger than $\exp(L^{1/10})$; $a = 1$ is not a mark, and Theorem B gives no representation $N = p + h$ with p and h prime. It is an average over the marks: for each large N it gives some admissible a, not a statement for each individual a. No effective numerical threshold is obtained.

(ii) *Marks sharing a prime with* N *contribute nothing to* P_N. If $p \mid a$, $p \mid N$ and $N - p' = a \cdot h$ with p' prime, then $p \mid p'$, which is impossible for a prime $p' > x > p$. So $P_N(x)$ is unchanged if $\mathcal{A}^N$ is replaced by the set of all products of one prime from each $\mathcal{P}_i$.

(iii) *Dyadic* x. Theorem B is stated for $x \in \mathcal{X}_N^\delta$ because Hypothesis RE is formulated for $x = 2^{-j}N$; the Type I estimate (Proposition 4.3) holds for every real $x \in [\delta N, N/4]$. The existence statement uses only $x = N/4$, which lies in $\mathcal{X}_N^\delta$ for every $\delta < 1/4$.

(iv) *Sizes.* $K_0(A') > 4600$ for every $A' > 0$, and $K_0(2) = 5401$. The defining half-open intervals of the bands are pairwise disjoint whenever $L^{a_{i+1}-a_i} \geq 2$ for every i. Since the K exponents lie in an open interval of length $1/10$, this sufficient separation condition forces $L > 2^{10(K-1)}$; the threshold obtained by this separation argument is astronomically large.

4.2. **Elementary facts and Vaughan's identity.**

Lemma 4.2. *Let* N *be even and let* x *be real with* $\delta N \leq x \leq N/4$, *large in terms of* δ, K *and* (a_i).

(a) *Every band prime lies in* $(\exp(L^{1/10}), \exp(2L^{1/5}))$, *so it is odd, smaller than* Z, *and larger than* L^C *for every fixed* C, *and the bands are pairwise disjoint. Every* $a \in \mathcal{A}^N$ *is odd and squarefree, with* $\exp(KL^{1/10}) < a \leq \exp(2KL^{1/5}) = x^{o(1)}$; *also* $\#\mathcal{A}^N \leq \exp(2KL^{1/5})$ *and* $\sum_{a \in \mathcal{A}^N} 1/a = \prod_i V_i^N \leq 1$.

(b) $\omega(N) \leq 2L$. *With* $V_i := \sum_{p \in \mathcal{P}_i} 1/p = \log 2 + O(L^{-a_i})$, *one has* $0 \leq V_i - V_i^N \leq 2L\exp(-L^{1/10})$, *so* $\prod_i V_i^N = (\log 2)^K\big(1 + O_K(L^{-1/10})\big)$ *uniformly in even* N.

(c) $(a, N) = 1$ *for every* $a \in \mathcal{A}^N$, *and* $(a, q) = 1$ *for every* Z*-rough* q.

(d) $G_N(k) = 0$ *for* $k \geq N$.

(e) *If* $N - k \geq N/3$, *then* $G_N(k) \in \{0, 1\}$. *In particular* $G_N(k) \leq 1$ *for* $k \leq 2x(1 + L^{-B})$, *for every fixed* $B > 0$.

Proof. (a) $2L^{1/5} < L^{0.24}$ for large L, and the bands are disjoint once $L^{a_{i+1}-a_i} > 2$; so each $a \in \mathcal{A}^N$ comes from exactly one choice $(p_1, \dots, p_K)$, and $V_i^N \leq V_i < 1$ by (b). (b) $\omega(N) \leq \log N/\log 2 \leq (L + \log(1/\delta))/\log 2 \leq 2L$; Mertens' estimate gives V_i, and at most $\omega(N)$ primes of $\mathcal{P}_i$, each larger than $\exp(L^{a_i})$, divide N; and $a_i > 1/10$. (c) is the definition and (a). (d) $(N-k)/a \leq 0$ is not a prime. (e) If $N - k = ah$ with h prime, then $h \geq N/(3a) > \exp(2L^{1/5})$, so h is not a band prime

and a is the product of all band-prime factors of $N-k$; so a and h are determined by k. For $k \le 2x(1+L^{-B})$, $N-k \ge N/2 - NL^{-B}/2 > N/3$. □

Fix $y \in (x, 2x]$, put $I := (x, y]$ and $F := G_N \mathbf{1}_I$. Then $\|F\|_\infty \le 1$ by Lemma 4.2(e), and Proposition 2.2 gives

$$T_N(x;y) = T_1(F) - T_2(F) + T_3(F) + O(Z). \tag{42}$$

4.3. **The Type I sums.**

Proposition 4.3. *For every $A_{\mathrm{I}} > 0$,*

$$(T_1 - T_2)(F) = M_N(x;y) + O_{A_{\mathrm{I}},\delta,K}\big(xL^{-A_{\mathrm{I}}}\big),$$

uniformly in even N, in real $x \in [\delta N, N/4]$ with $x \ge x_1(A_{\mathrm{I}}, \delta, K, (a_i))$, and in $y \in (x, 2x]$.

This uses no hypothesis: its inputs are Proposition 2.2, Lemma 2.13 and the classical results of §2.4. The indicator $R_Z(k)$ of the variable k is replaced by a truncated inclusion–exclusion, as in [28, proof of Proposition 3.1], and the resulting sums over primes in progressions are handled by Lemma 2.13.

Lemma 4.4. *For every integer $n \ge 1$ and every integer $R \ge 0$,*

$$\Big| R_Z(n) - \sum_{\substack{r \mid (n, P_Z) \\ \omega(r) \le R}} \mu(r) \Big| \le \#\{r \mid (n, P_Z) :\ \omega(r) = R+1\}.$$

Proof. Let $s := \omega((n, P_Z))$. If $s = 0$, both sides are 0. If $s \ge 1$, then $R_Z(n) = 0$, and the truncated sum is $\sum_{j \le R} (-1)^j \binom{s}{j} = (-1)^R \binom{s-1}{R}$, while $\binom{s-1}{R} \le \binom{s}{R+1}$. □

Lemma 4.5. *Put $\mathfrak{m}_0(y) := \sum_{n \le y} \mu(n)/n$ and $\mathfrak{m}_1(y) := \sum_{n \le y} \mu(n) \log n/n$. There is an absolute $c_1 > 0$ such that, for $y \ge 2$, $\mathfrak{m}_0(y) \ll \exp(-c_1\sqrt{\log y})$ and $\mathfrak{m}_1(y) = -1 + O(\exp(-c_1\sqrt{\log y}))$. For all $y \ge 1$, $|\mathfrak{m}_0(y)| \le 1 + \log y$ and $|\mathfrak{m}_1(y)| \le (1+\log y)^2$.*

Proof. Let $M(y) := \sum_{n \le y} \mu(n) \ll y \exp(-c_0\sqrt{\log y})$ (§2.4). Since $\sum_n \mu(n)/n = 0$, partial summation gives $\mathfrak{m}_0(y) = M(y)/y - \int_y^\infty M(t) t^{-2}\, dt \ll \exp(-c_0\sqrt{\log y})(1 + \sqrt{\log y})$. Next, $\mathfrak{m}_1(y) = \mathfrak{m}_0(y)\log y - \int_1^y \mathfrak{m}_0(t)\, dt/t$; the integral converges, so $\mathfrak{m}_1(y)$ tends to $\mathfrak{m}_1(\infty) := -\int_1^\infty \mathfrak{m}_0(t)\, dt/t$ with error $\ll \exp(-c_1\sqrt{\log y})$ for any $c_1 < c_0$. By Abel's theorem for Dirichlet series, $\mathfrak{m}_1(\infty) = \lim_{s \to 1^+} \sum_n \mu(n)(\log n) n^{-s} = \lim_{s \to 1^+} \zeta'(s)/\zeta(s)^2 = -1$, since $\zeta(s) = 1/(s-1) + O(1)$ and $\zeta'(s) = -1/(s-1)^2 + O(1)$ as $s \to 1^+$. The trivial bounds hold term by term. □

For even $n \ge 2$ put $V_Z(n) := \prod_{p \le Z,\ p \nmid n} \big(1 - 1/(p-1)\big)$; the prime $p = 2$ never occurs.

Lemma 4.6. *Put $S_0 := \sum \mu(d) R_Z(d)/\varphi(d)$ and $S_1 := \sum \mu(d) R_Z(d) \log d/\varphi(d)$, both over $d \le U$ with $(d, N) = 1$. There is an absolute $c > 0$ such that, for large x and uniformly in even N with $\omega(N) \le 2L$,*

$$S_0 \ll \exp(-c\sqrt{L}), \qquad S_1 = -\mathfrak{S}(N)/V_Z(N) + O(\exp(-c\sqrt{L})).$$

Proof. Put $f(d) := \mu(d) R_Z(d) \mathbf{1}_{(d,N)=1}\, d/\varphi(d)$. Then $S_0 = \sum_{d \le U} f(d)/d$, $S_1 = \sum_{d \le U} f(d) \log d/d$, and f is multiplicative and supported on squarefree d, with $f(p) = -p/(p-1)$ if $p > Z$ and $p \nmid N$, and $f(p) = 0$ otherwise. Put $g := f * 1$, so $f = g * \mu$; g is multiplicative with $g(p^j) = 1 + f(p)$ for $j \ge 1$, that is, $g(p^j) = 1$ if $p \le Z$ or $p \mid N$, and $g(p^j) = -1/(p-1)$ if $p > Z$ and $p \nmid N$. Then

$$S_0 = \sum_{e \le U} \frac{g(e)}{e}\, \mathfrak{m}_0(U/e), \qquad S_1 = \sum_{e \le U} \frac{g(e)}{e} \big(\log e \cdot \mathfrak{m}_0(U/e) + \mathfrak{m}_1(U/e)\big).$$

The series $\Sigma_g := \sum_{e\ge1} g(e)/e$ converges absolutely. Its Euler factor is $p/(p-1)$ at $p \le Z$ and at $p \mid N$, and $1-(p-1)^{-1}\sum_{j\ge1} p^{-j} = p(p-2)/(p-1)^2$ at $p > Z$, $p \nmid N$. Since N is even, $\mathfrak{S}(N) = \prod_{p\nmid N} p(p-2)(p-1)^{-2}\prod_{p\mid N} p(p-1)^{-1}$, and comparing factor by factor,

$$\Sigma_g = \prod_{p\le Z}\frac{p}{p-1}\prod_{\substack{p>Z\\ p\mid N}}\frac{p}{p-1}\prod_{\substack{p>Z\\ p\nmid N}}\frac{p(p-2)}{(p-1)^2} = \frac{\mathfrak{S}(N)}{V_Z(N)}.$$

Rankin. With $\sigma := 1/\log Z$,

$$\sum_e \frac{|g(e)|}{e^{1-\sigma}} = \prod_{p\le Z}\bigl(1-p^{-1+\sigma}\bigr)^{-1}\prod_{\substack{p>Z\\ p\mid N}}\bigl(1-p^{-1+\sigma}\bigr)^{-1}\prod_{\substack{p>Z\\ p\nmid N}}\Bigl(1+\frac{p^{-1+\sigma}}{(p-1)(1-p^{-1+\sigma})}\Bigr) \ll \log Z.$$

For $p \le Z$, $p^\sigma - 1 \le (e-1)\sigma\log p$, so the first product differs from $\prod_{p\le Z}(1-1/p)^{-1} \ll \log Z$ by a factor $\exp(O(\sigma\sum_{p\le Z}\log p/p)) = O(1)$. In the second, $p^{-1+\sigma} \le p^{-1/2} \le Z^{-1/2}$, so it is at most $\exp(4\omega(N)Z^{-1/2}) \le 2$; this is where $\omega(N) \le 2L$, and with it the uniformity in N, enters. The third is at most $\exp(4\sum_{p>Z} p^{-3/2}) \le 2$. Hence $\sum_{e>\sqrt{U}} |g(e)|/e \le U^{-\sigma/2}O(\log Z) = O(L^{0.24}\exp(-L^{0.76}/10))$, and $\sum_e |g(e)|/e \ll L^{0.24}$.

S_0. For $e \le \sqrt{U}$, $\log(U/e) \ge L/10$ and $|\mathfrak{m}_0(U/e)| \ll \exp(-c_1\sqrt{L/10})$; for $e > \sqrt{U}$, $|\mathfrak{m}_0(U/e)| \le 1+L$ and the Rankin tail applies. So $S_0 \ll L^{0.24}\exp(-c_1\sqrt{L/10}) + L^{1.24}\exp(-L^{0.76}/10)$.

S_1. The part with $\log e\cdot\mathfrak{m}_0$ is at most L times the bound for S_0. The part with $\mathfrak{m}_1$ is $-\sum_{e\le\sqrt{U}} g(e)/e + O(L^{0.24}\exp(-c_1\sqrt{L/10})) + O(L^{2.24}\exp(-L^{0.76}/10))$, and $\sum_{e\le\sqrt{U}} g(e)/e = \Sigma_g + O(L^{0.24}\exp(-L^{0.76}/10))$. All errors are $\ll \exp(-c\sqrt{L})$ with $c := c_1/4$. □

Proof of Proposition 4.3. Shape. Grouping $q = d\ell$ in T_2 as in Remark 2.3,

$$T_1(F) = \sum_{q\le U}\mu(q)R_Z(q)\,I_q(\log), \qquad T_2(F) = \sum_{q\le x^{2/5}}\gamma_q R_Z(q)\,I_q(1),$$

where $I_q(w) := \sum_{k:\ qk\in I} R_Z(k)\,w(k)\,G_N(qk)$ for $w = \log$ and $w = 1$. We write ϱ_q for the coefficient $\mu(q)\mathbf{1}_{q\le U}$ or γ_q, so that $|\varrho_q| \le L$. Every q with $R_Z(q) = 1$ is Z-rough, so $(a,q) = 1$ for $a \in \mathcal{A}^N$ (Lemma 4.2(c)). Writing $N - qk = ah$,

$$I_q(w) = \sum_{a\in\mathcal{A}^N}\sum_{\substack{h\in J_a \text{ prime}\\ q\mid N-ah}} R_Z\Bigl(\frac{N-ah}{q}\Bigr)\,w\Bigl(\frac{N-ah}{q}\Bigr), \qquad J_a := \Bigl[\frac{N-y}{a},\frac{N-x}{a}\Bigr), \tag{43}$$

a bijection between the pairs (k,a) counted by $I_q(w)$ and the pairs (a,h) with $h \in J_a$ prime and $q \mid N-ah$. For $h \in J_a$: $N-ah \in (x,y]$; $h \ge (N-2x)/a \ge N/(2a) > \max(Z,\exp(2L^{1/5}))$; $|J_a| \le x/a$; and $H_a := (N-x)/a \ge 3x/a \ge x^{1-o(1)}$.

Moduli not coprime to N. If a prime p divides (q,N), then $q \mid N-ah$ gives $p \mid ah$, and since $p \nmid a$, $p \mid h$, so $h = p \le q \le x^{2/5}$. But $h \ge N/(2a) \ge 2x\exp(-2KL^{1/5}) > x^{2/5}$. So $I_q(w) = 0$ when $(q,N) > 1$. From now on $(q,N) = 1$, and $q \mid N-ah$ means $h \equiv N\bar{a} \pmod q$, a reduced class ($\bar{a}$ is the inverse of a modulo q).

The sieve on k. Fix $C_{\mathrm{tr}} > e$ with $C_{\mathrm{tr}}\log(C_{\mathrm{tr}}/e) \ge A_{\mathrm{I}} + 6$, and put $R_{\mathrm{I}} := \lceil C_{\mathrm{tr}}\log L\rceil$. Apply Lemma 4.4 with $n = (N-ah)/q$ and $R = R_{\mathrm{I}}$. Since q is Z-rough and $r \mid P_Z$, $r \mid n$ if and only if $qr \mid N-ah$. Let $p \mid r$. If $p \mid a$, then $p \mid N-ah$ would force $p \mid N$, which the puncturing excludes. If $p \mid N$ and $p \nmid a$, then $p \mid ah$ gives $p \mid h$, so $h = p \le Z$, which is impossible. So only $r \in \mathcal{R}(a) := \{r \mid P_Z :\ (r, aN) = 1,\ \omega(r) \le R_{\mathrm{I}}\}$ contribute to the truncated sum, and for these

$qr \mid N - ah$ if and only if $h \equiv N\bar{a} \pmod{qr}$, a reduced class ($qr$ is odd because $2 \mid N$). Hence

$$I_q(w) = \sum_{a \in \mathcal{A}^N} \sum_{r \in \mathcal{R}(a)} \mu(r)\, \Pi_a(q, r; w) + E_q^{\mathrm{tr}}, \qquad \Pi_a(q, r; w) := \sum_{\substack{h \in J_a \text{ prime} \\ h \equiv N\bar{a} \pmod{qr}}} w\Big(\frac{N - ah}{q}\Big),$$

$$|E_q^{\mathrm{tr}}| \le (L+1) \sum_{a \in \mathcal{A}^N} \sum_{\substack{r \mid P_Z,\ \omega(r) = R_{\mathrm{I}}+1 \\ (r,a)=1}} \Big(\frac{x}{aqr} + 1\Big);$$

in the last bound the primality of h is dropped, $|w| \le L+1$, and $qr \mid N - ah$ is one class of h modulo qr in an interval of length at most x/a. Using $\sum_{r \mid P_Z,\, \omega(r)=j} 1/r \le (\sum_{p \le Z} 1/p)^j/j! \le (\log L)^j/j!$, $\#\{r \mid P_Z :\ \omega(r) = j\} \le Z^j$, $(R_{\mathrm{I}}+1)! \ge ((R_{\mathrm{I}}+1)/e)^{R_{\mathrm{I}}+1}$, $\sum_{q \le x^{2/5}} 1/q \le L$ and $\sum_a 1/a \le 1$,

$$\sum_q |\varrho_q|\, |E_q^{\mathrm{tr}}| \le 2L^2 \Big[xL \Big(\frac{e \log L}{R_{\mathrm{I}}+1}\Big)^{R_{\mathrm{I}}+1} + x^{\frac{2}{5}}\, \#\mathcal{A}^N\, Z^{R_{\mathrm{I}}+1} \Big] \ll xL^{3 - C_{\mathrm{tr}} \log \frac{C_{\mathrm{tr}}}{e}} + x^{\frac{2}{5}+o(1)} \ll xL^{-A_{\mathrm{I}}-3}.$$

Primes in progressions. Every modulus is $m = qr \le Q_0 := x^{2/5} Z^{R_{\mathrm{I}}} = x^{2/5+o(1)}$, and m determines q (its Z-rough part) and r (its Z-smooth part); so, for fixed a, each m occurs at most once in T_1 and at most once in T_2. The weight $W(h) := w((N - ah)/q)$ is monotone on J_a, with $\sup |W| \le L + 1$ and total variation at most $\log(y/x) \le \log 2$. Put $\Delta(H; m) := \max_{(c,m)=1} \max_{2 \le z \le H} |\pi(z; m, c) - \mathrm{li}(z)/\varphi(m)|$. Partial summation against $\pi(\cdot; m, c) - \mathrm{li}(\cdot)/\varphi(m)$ on $J_a \subset [2, H_a]$ (with left limits at the endpoints, to which the maximum passes since li is continuous) gives

$$\Pi_a(q, r; w) = \frac{1}{\varphi(qr)} \int_{J_a} w\Big(\frac{N - at}{q}\Big) \frac{dt}{\log t} + E_a^{\mathrm{BV}}(q, r),$$

with $|E_a^{\mathrm{BV}}(q, r)| \le (2 \sup |W| + \operatorname{Var} W)\, \Delta(H_a; qr) \le 3L\, \Delta(H_a; qr)$, because the class $N\bar{a}$ modulo qr is reduced. Since $Q_0 \le H_a^{1/2} (\log H_a)^{-A_{\mathrm{I}}-5}$ for large x, the form of Lemma 2.13 displayed after it, at precision $A_{\mathrm{I}} + 3$, gives, for each of T_1 and T_2,

$$\sum_{q,r} |\varrho_q \mu(r)|\, |E_a^{\mathrm{BV}}(q, r)| \le 3L^2 \sum_{m \le Q_0} \Delta(H_a; m) \ll L^2 H_a (\log H_a)^{-A_{\mathrm{I}}-3} \ll \frac{N}{a} L^{-A_{\mathrm{I}}-1},$$

and $\sum_a (N/a) L^{-A_{\mathrm{I}}-1} \le (x/\delta) L^{-A_{\mathrm{I}}-1}$. The integer N enters only through the residue class, which the maximum covers; so the bound is uniform in N and in a.

Completing the sum over r. For $(q, r) = 1$, $\varphi(qr) = \varphi(q)\varphi(r)$, and $\sum_{r \mid P_Z,\, (r, aN)=1} \mu(r)/\varphi(r) = V_Z(aN)$. With $s := \sum_{2 < p \le Z} 1/(p-1) \le \log L$, the terms with $\omega(r) > R_{\mathrm{I}}$ total at most $\sum_{j > R_{\mathrm{I}}} s^j/j! \le e^s (es/(R_{\mathrm{I}}+1))^{R_{\mathrm{I}}+1} \le L^{1 - C_{\mathrm{tr}} \log(C_{\mathrm{tr}}/e)}$. Since $|\int_{J_a} w((N - at)/q)\, dt/\log t| \le (L+1)(x/a)(2/L)$ and $\sum_{q \le x^{2/5}} 1/\varphi(q) \ll L$, collecting the three steps gives

$$I_q(w) = \frac{1}{\varphi(q)} \sum_{a \in \mathcal{A}^N} V_Z(aN) \int_{J_a} w\Big(\frac{N - at}{q}\Big) \frac{dt}{\log t} + E_q, \qquad \sum_q |\varrho_q|\, |E_q| \ll_\delta xL^{-A_{\mathrm{I}}-1}. \tag{44}$$

Summing over the Vaughan coefficients. By (44) and the discard of $(q, N) > 1$, $(T_1 - T_2)(F) = MT_1 - MT_2 + O(xL^{-A_{\mathrm{I}}-1})$, where

$$MT_1 := \sum_{d \le U,\, (d,N)=1} \frac{\mu(d) R_Z(d)}{\varphi(d)} \sum_{a \in \mathcal{A}^N} V_Z(aN) \int_{J_a} \big(\log(N - at) - \log d\big) \frac{dt}{\log t},$$

$$MT_2 := \sum_{\substack{d \le U,\ \ell \le V \\ (d\ell, N)=1}} \frac{\mu(d) \Lambda(\ell) R_Z(d\ell)}{\varphi(d\ell)} \sum_{a \in \mathcal{A}^N} V_Z(aN) \int_{J_a} \frac{dt}{\log t}.$$

In MT_2, $R_Z(d\ell)=R_Z(d)R_Z(\ell)$ and $\mathbf{1}_{(d\ell,N)=1}=\mathbf{1}_{(d,N)=1}\mathbf{1}_{(\ell,N)=1}$; for squarefree d and $\ell=p^j$, $\varphi(d)/\varphi(d\ell)$ is $1/\varphi(\ell)$ if $p\nmid d$ and $1/\ell$ if $p\mid d$. Hence

$$\sum_{\ell\le V,\,(\ell,N)=1}\Lambda(\ell)R_Z(\ell)\frac{\varphi(d)}{\varphi(d\ell)}=A_V(N)-\varpi(d),\qquad A_V(N):=\sum_{\ell\le V,\,(\ell,N)=1}\frac{\Lambda(\ell)R_Z(\ell)}{\varphi(\ell)},$$

where, for d with $\mu(d)R_Z(d)\neq0$, $0\le\varpi(d)=\sum_{p\mid d}\sum_{p^j\le V}\log p/(p^j(p-1))\le\sum_{p\mid d}\log p/(p-1)^2\le 2L\log Z/Z^2$. Only $0\le A_V(N)\le 2\sum_{n\le V}\Lambda(n)/n\le 2L$ is needed. With S_0, S_1 as in Lemma 4.6 and $S_\varpi:=\sum_{d\le U,\,(d,N)=1}\mu(d)R_Z(d)\varpi(d)/\varphi(d)\ll L^2\log Z/Z^2$,

$$MT_1-MT_2=\sum_{a\in\mathcal{A}^N}V_Z(aN)\int_{J_a}\Big[\big(\log(N-at)-A_V(N)\big)S_0-S_1+S_\varpi\Big]\frac{dt}{\log t}. \tag{45}$$

The singular series. In (45), $|\log(N-at)-A_V(N)|\le 3L$ and $\sum_a V_Z(aN)\int_{J_a}dt/\log t\le\sum_a(x/a)(2/L)\le 2x/L$. Since $\omega(N)\le 2L$ (Lemma 4.2(b)), Lemma 4.6 gives

$$MT_1-MT_2=\frac{\mathfrak{S}(N)}{V_Z(N)}\sum_{a\in\mathcal{A}^N}V_Z(aN)\int_{J_a}\frac{dt}{\log t}+O\big(x\exp(-c\sqrt{L})\big)+O\big(xL^2\log Z/Z^2\big).$$

Every prime $p\mid a$ satisfies $2<p<Z$ and $p\nmid N$, so $V_Z(aN)/V_Z(N)=\prod_{p\mid a}(p-1)/(p-2)$ and $\mathfrak{S}(N)V_Z(aN)/V_Z(N)=\mathfrak{S}(aN)$, by the definition of $\mathfrak{S}$ at the even number aN. The substitution $t=(N-u)/a$ maps J_a onto $u\in(x,y]$, with $dt=du/a$ up to orientation, so $\sum_a\mathfrak{S}(aN)\int_{J_a}dt/\log t=M_N(x;y)$. Collecting the errors of all steps proves the proposition. □

4.4. **The Type II sum and the hand-off to Hypothesis RE.**

Proposition 4.7. *Assume* QRH(θ) *and Hypothesis* RE($\mathfrak{B}_{\exp},\theta$) *for a fixed* $\theta<1$. *Let* $A'>0$, *let* $K\ge K_0(A')$, *and fix the exponents* a_i. *Then, for* $x\in\mathcal{X}_N^\delta$ *with* $x\ge x_2(\delta,A',K,(a_i),\theta)$, *uniformly in* $y\in(x,2x]$ *and in even* N,

$$T_3(F)\ll_{\delta,A',K,(a_i),\theta}xL^{-A'-1}.$$

Proof. Put $B:=A'+4$ and $D_*:=2A'+4$.

Variables not coprime to N. Delete from $T_3(F)$ the terms in which some prime $P\mid N$ divides mn. Since m and n are Z-rough, $P>Z$. By Lemma 4.2(e), $|\mu(m)c_V(n)|\le\log(2x)$, $\tau(Pj)\le 2\tau(j)$ and $\sum_{j\le t}\tau(j)\le t(\log t+1)$, the deleted terms total at most $\mathcal{E}_{\mathrm{del}}:=\sum_{P\mid N,\,P>Z}\sum_{k\le 2x,\,P\mid k}\tau(k)\log(2x)\le\sum_{P\mid N,\,P>Z}4(x/P)(L+2)^2\le 8x(L+2)^3/Z$.

Smoothing, boxes, Mellin separation (as in [28, §4.1]). Choose a smooth $\psi_I\colon(0,\infty)\to[0,1]$ equal to $\mathbf{1}_{I/x}$ outside the L^{-B}-neighbourhoods of 1 and y/x, with $\psi_I^{(j)}\ll_j L^{Bj}$ uniformly in I and total variation at most 2. Replacing $\mathbf{1}_I(mn)$ by $\psi_I(mn/x)$ changes only terms with mn in two intervals $\mathcal{J}_1,\mathcal{J}_2\subset[x/2,3x]$ of length $2xL^{-B}$, where $G_N\le 1$ by Lemma 4.2(e). Since $\sum_{k\in\mathcal{J}}\tau(k)\le 2\sum_{d\le\sqrt{3x}}(|\mathcal{J}|/d+1)\le 2|\mathcal{J}|(\log 3x+1)+4\sqrt{x}$ for intervals $\mathcal{J}\subset[x/2,3x]$, this costs $\mathcal{E}_{\mathrm{sm}}\le L^2\sum_{k\in\mathcal{J}_1\cup\mathcal{J}_2}\tau(k)\ll xL^{3-B}+x^{1/2}L^2$. Split $m>U$ and $n>V$ into dyadic intervals $(M,2M]$ and $(\mathfrak{N},2\mathfrak{N}]$ with $M\in 2^{\mathbb{Z}_{\ge0}}U$ and $\mathfrak{N}\in 2^{\mathbb{Z}_{\ge0}}V$, and keep the boxes that meet the support of $\psi_I(mn/x)$, which lies in $[x(1-L^{-B}),x(2+L^{-B})]$. With $X:=M\mathfrak{N}$, so that $mn\in(X,4X]$, a kept box has $x/5<X<3x$; hence $N/X<5/\delta$ and $X/N<3/4$, and there are at most $8L$ kept boxes. Mellin inversion gives

$$\psi_I(u)=\int_{|v|\le L^{2B}}\widehat{\psi_I}(v)u^{iv}\,dv+O_B(L^{-B}),\qquad\int_{|v|\le L^{2B}}|\widehat{\psi_I}(v)|\,dv\ll_B\log L,$$

uniformly in I, with $\widehat{\psi_I}(v)=(2\pi)^{-1}\int\psi_I(e^w)e^{-ivw}\,dw$ (integrate by parts once, using the total variation, and twice). By Lemma 4.2(d), the truncation error over the kept boxes is at most $\mathcal{E}_{\mathrm{Mel}}:=C_BL^{2-B}\sum_{k\le 12x,\,k<N}\tau(k)G_N(k)$. For $k<N$, $G_N(k)\le\#\{a\in\mathcal{A}^N:\ k\equiv N\pmod a\}$. If $(d,a)>1$,

no $k \equiv N \pmod a$ is divisible by d, since a common prime would divide N. With $\tau(k) \le 2\#\{d \mid k : d \le \sqrt{t}\}$ for $k \le t$, this gives $\sum_{k\le t,\ k\equiv N\,(a)} \tau(k) \le 2\sum_{d\le\sqrt{t},\,(d,a)=1}(t/(da)+1) \le (t/a)(2+\log t)+2\sqrt{t}$; at $t = 12x$, by Lemma 4.2(a), $\sum_{k\le 12x,\,k<N} \tau(k)G_N(k) \le 12x(L+5)\prod_i V_i^N + 2\sqrt{12x}\,\#\mathcal{A}^N \ll xL$. So $\mathcal{E}_{\mathrm{Mel}} \ll_B xL^{3-B}$. It remains to bound, uniformly in the kept boxes and in $|v| \le L^{2B}$,

$$\mathcal{T} := \sum_{\substack{M<m\le 2M\\ \mathfrak{N}<n\le 2\mathfrak{N}}} \alpha_m\beta_n\, G_N(mn),$$

where $\alpha_m := \mu(m)R_Z(m)\mathbf{1}_{(m,N)=1}m^{iv}$ and $\beta_n := c_V(n)R_Z(n)\mathbf{1}_{(n,N)=1}n^{iv}$, with $|\alpha_m| \le 1$ and $|\beta_n| \le \log(12x) \le L^2$. A box may contain products $mn \ge N$; there $G_N = 0$, and no sign convention is needed. Then

$$|T_3(F)| \le 8L \cdot C_B \log L \cdot \max|\mathcal{T}| + \mathcal{E}_{\mathrm{del}} + \mathcal{E}_{\mathrm{sm}} + \mathcal{E}_{\mathrm{Mel}}. \tag{46}$$

The cofactor. For every integer $k \ge 1$, $G_N(k) = \sum_{a\in\mathcal{A}^N}\sum_{h\ \mathrm{prime}} \mathbf{1}_{N-k=ah}$ exactly. With the dyadic partition $1 = \sum_{Y\in 2^{\mathbb{Z}}}\eta(a/Y)$ of §2.3, and $S_Y(h)$ as in Definition 2.9 for these α, β, we get $\mathcal{T} = \sum_Y \mathcal{T}_Y$ with $\mathcal{T}_Y := \sum_{h\ \mathrm{prime}} S_Y(h)$. Since η is supported in $[1,4]$, $\eta(a/Y) \ne 0$ forces $a/4 \le Y \le a$; so by Lemma 4.2(a) at most $3KL^{1/5}$ values of Y occur, all with $KL^{1/10} - \log 4 \le \log Y \le 2KL^{1/5}$.

Cauchy–Schwarz over the prime cofactor. If $S_Y(h) \ne 0$ with $h \ge 1$, then $h = (N-mn)/a < N/Y$. With $\widetilde{Q}_Y := \sum_{h\ne 0}|S_Y(h)|^2$,

$$|\mathcal{T}_Y|^2 \le \pi(N/Y)\sum_{h\ge 1}|S_Y(h)|^2 \le \frac{4N}{YL}\,\widetilde{Q}_Y \le \frac{20}{\delta}\cdot\frac{X}{YL}\,\widetilde{Q}_Y, \tag{47}$$

by Chebyshev's bound, $\log(N/Y) \ge L/2$ and $N < 5X/\delta$. By Definition 2.9, $\widetilde{Q}_Y = (\prod_i V_i^N)^2\, Q_Y^{\mathrm{sgn}}$ for the same α, β, η, Y and $\mathcal{A}^N$. The primality of h is used only through $\pi(N/Y)$; the square does not see the cofactor.

The hand-off. We apply Hypothesis $\mathrm{RE}(\mathfrak{B}_{\mathrm{exp}}, \theta)$ with $A := D_* = 2A'+4$, $C := 2$, the B above, $\delta_0 := \delta/5$, and the datum (E_0, K) with $E_0 = 2A'+15$ and the exponents (a_i), which is admissible for (A, C) because $E_0 > 2A'+14$ and $K \ge K_0(A')$. Its hypotheses hold. First, $x = 2^{-j}N$ with $j \ge 2$, $M, \mathfrak{N} \ge x^{1/5}$, $x/5 < X < 3x \le \min(3x, N)$ and $X > x/5 \ge \delta N/5 = \delta_0 N$, so $(x, M, \mathfrak{N}) \in \mathcal{R}_{\delta_0}(N)$ (Definition 2.8). Second, by the deletion above, $\alpha \in \mathcal{C}_N^{\mu}(M; B)$ (with the interval $(M, 2M]$ and $|v| \le L^{2B}$), $\alpha \in \mathcal{C}_N(M)$ and $\beta \in \mathcal{C}_N(\mathfrak{N})$ with $C = 2$ (Definition 2.7). Third, the marks are those of $\mathfrak{B}_{\mathrm{exp}}$, punctured at N. Then (RE1) and (RE2) give $Q_Y^{\mathrm{sgn}} \le |Q_Y^{\mathrm{sgn}} - Q_Y^{(N),\mathrm{maj}}| + |Q_Y^{(N),\mathrm{maj}}| \le 2C_0XYL^{-D_*}$ for every $Y \in 2^{\mathbb{Z}}$ and every $x \ge x_0^{\mathrm{RE}}$, where C_0 and x_0^{RE} are the constants of Hypothesis RE for this instance (Remark 2.12(i)). Since $\prod_i V_i^N \le 1$, (47) gives $|\mathcal{T}_Y| \ll_\delta C_0^{1/2}XL^{-(D_*+1)/2}$.

Summing up. $|\mathcal{T}| \le 3KL^{1/5}\max_Y|\mathcal{T}_Y| \ll_{\delta,K} L^{1/5}\,x\,L^{-(D_*+1)/2}$, and by (46)

$$|T_3(F)| \ll_{\delta,K,B} L^{1.2}\log L \cdot xL^{-(D_*+1)/2} + xL^{3-B} + x^{1/2}L^2 + xL^3/Z.$$

With $D_* = 2A'+4$ and $B = A'+4$, every term is $\ll xL^{-A'-1}$. □

The conventions (G1) and (G2) make every mark and every Type II variable a unit modulo N; with them the reduction runs over all even N, including those with many small prime factors. Hypothesis RE includes in Q_Y^{sgn} the pairs of marks (a, b) with a common prime factor; for coprime marks, Lemma 2.10(ii) turns the condition on (m, n, r, s) into the determinant equation (4). The next lemma, the reflected form of [28, Lemma 4.1], bounds the pairs with a common prime factor. Neither fact is used in the proof of Theorem B.

Lemma 4.8. *Let $\mathcal{M}$ be a set of odd squarefree integers coprime to N whose prime factors are all at least $p_{\min} \ge 3$. Let $|\alpha_m| \le 1$ and $|\beta_n| \le L^2$ be supported on $(M, 2M]$ and $(\mathfrak{N}, 2\mathfrak{N}]$, with $X = M\mathfrak{N} \le 3x$, and put $S_\ell(h) := \sum_{N-mn=\ell h}\alpha_m\beta_n$. If $Y \in 2^{\mathbb{Z}}$ and $Y \le X^{1/4}$, then*

$\sum_{a,b\in\mathcal{M},\,(a,b)>1}\eta(a/Y)\eta(b/Y)\sum_{h\neq0}|S_a(h)|\,|S_b(h)|\ll XYL^7/p_{\min}$, *with an absolute implied constant. For* $\mathcal{M}=\mathcal{A}^N$ *with the bands of* $\mathfrak{B}_{\exp}$, *the right-hand side is* $\ll XYL^7\exp(-L^{1/10})$.

Proof. Let $\ell\in\mathcal{M}$ with $Y\le\ell\le 4Y$. Then $|S_\ell(h)|\le L^2\tau(N-\ell h)$, and $S_\ell(h)=0$ unless $X<N-\ell h\le 4X$. By $\tau^2\le\tau_4$ and $\tau_4(n)\le 4\sum_{d|n,\,d\le n^{3/4}}\tau_3(d)$, $\sum_h\tau(N-\ell h)^2$ over these h is at most $4\sum_{d\le(4X)^{3/4}}\tau_3(d)\,\#\{h:\ X<N-\ell h\le 4X,\ d\mid N-\ell h\}$. If $(d,\ell)>1$, the count is 0, since a common prime would divide N; otherwise it is at most $3X/(\ell d)+1$. Hence the sum is $\ll(X/Y)L^3+X^{3/4}L^2\ll(X/Y)L^3$, because $Y\le X^{1/4}$. By Cauchy's inequality, each pair (a,b) contributes $\ll L^4(X/Y)L^3$. The pairs in $[Y,4Y]^2$ sharing a prime number at most $\sum_{p\ge p_{\min}}(4Y/p)^2\le 32Y^2/p_{\min}$. For $\mathfrak{B}_{\exp}$, $p_{\min}>\exp(L^{1/10})$, and $Y\le\exp(2KL^{1/5})\le X^{1/4}$ for large x, by Lemma 4.2(a). □

4.5. **Proof of Theorem B and of Corollary B.1; the order of choices.**

Proof of Theorem B. Insert Proposition 4.3 with $A_{\mathrm{I}}:=A'+1$ and Proposition 4.7 into (42):

$$T_N(x;y)=M_N(x;y)+O\big(xL^{-A'-1}\big)+O\big(xL^{-A'-1}\big)+O(Z)=M_N(x;y)+O\big(xL^{-A'-1}\big), \tag{48}$$

uniformly in even N, in $y\in(x,2x]$ and in $x\in\mathcal{X}_N^\delta$ with $x\ge x_0:=\max(x_1,x_2)$, with the bands fixed at $L=\log x$. This is (39), with the error $O(xL^{-A'-1})$, better than stated. □

Proof of Corollary B.1. G_N on primes. By Lemma 4.2(e), $G_N(p)\in\{0,1\}$ for $p\in(x,2x]$, and $G_N(p)=1$ if and only if $N-p=ah$ with $a\in\mathcal{A}^N$ and h prime. So $P_N(x)=\sum_{x<p\le 2x}G_N(p)$.

Prime powers. Put $\vartheta_N(u):=\sum_{x<p\le u}(\log p)G_N(p)$. The prime powers p^j, $j\ge2$, in $(x,2x]$ carry Λ-weight at most $\sum_{p^j\le 2x,\,j\ge2}\log p\ll x^{1/2}$, and $G_N\le1$ there. So by (48), $\vartheta_N(u)=M_N(x;u)+\mathcal{E}(u)$ with $|\mathcal{E}(u)|\le C'xL^{-A'-1}$, uniformly in $u\in(x,2x]$.

Partial summation. Since $\vartheta_N(x)=0$ and $M_N(x;x)=0$,

$$\begin{aligned}P_N(x)&=\int_{(x,2x]}\frac{d\vartheta_N(u)}{\log u}=\int_x^{2x}\frac{dM_N(x;u)}{\log u}+\frac{\mathcal{E}(2x)}{\log 2x}+\int_x^{2x}\frac{\mathcal{E}(u)\,du}{u\log^2u}\\&=\sum_{a\in\mathcal{A}^N}\frac{\mathfrak{S}(aN)}{a}\int_x^{2x}\frac{du}{\log u\cdot\log((N-u)/a)}+O\big(xL^{-A'-2}\big).\end{aligned}$$

With (39) as stated, $|\mathcal{E}(u)|\le C'xL^{-A'}$, and the same computation gives the error $O(xL^{-A'-1})$ of (41); the sharper form gives $O(xL^{-A'-2})$ at the same K (from (39) with $A'+1$ in place of A' it would need $K\ge K_0(A'+1)$). The same computation on any $(x,y']\subseteq(x,2x]$ gives the count on sub-intervals.

Size of the main term. For $u\in(x,2x]$ and $a\in\mathcal{A}^N$: $\log u=L+O(1)$; $\log((N-u)/a)=L+O_\delta(1)-\log a$ with $\log a\in[KL^{1/10},2KL^{1/5}]$, so $1/\log((N-u)/a)=L^{-1}(1+O_{K,\delta}(L^{-0.8}))$; and $\mathfrak{S}(aN)=\mathfrak{S}(N)\prod_{p\mid a}(1+1/(p-2))=\mathfrak{S}(N)(1+O(Ke^{-L^{1/10}}))$. So the main term is $\mathfrak{S}(N)\big(\sum_a1/a\big)(x/L^2)(1+O_{K,\delta}(L^{-0.8}))=\mathfrak{S}(N)\big(\prod_iV_i^N\big)(x/L^2)(1+O_{K,\delta}(L^{-0.8}))$, and $\prod_iV_i^N=(\log2)^K(1+O_K(L^{-1/10}))$ uniformly in N by Lemma 4.2(b). With $\mathfrak{S}(N)\ge2C_2=1.3203\ldots$, the main term is at least $(\log2)^Kx/(2L^2)$ for large x, uniformly in N. For the existence statement fix, say, $\delta=1/8$; then $x=N/4\in\mathcal{X}_N^\delta$, and for $A'>0$ the sharper form gives $P_N(N/4)>0$ for every even $N\ge N_0(A',K,(a_i),\theta)$; with only (39) as stated this needs $A'>1$. □

The order of choices. (1) Fix $\theta<1$, $\delta\in(0,1/4)$ and the target $A'>0$. (2) Put $D_*:=2A'+4$. This is legitimate because the loss in (46) and in the sum over Y (at most $8L$ boxes, $\int|\widehat{\psi_I}|\ll\log L$, at most $3KL^{1/5}$ values of Y) is $L^{1.2+o(1)}$, with an exponent independent of K and of the a_i; K enters only the implied constant. (3) Hypothesis RE at $A=D_*$, $C=2$: $E_0:=D_*+4C+3=2A'+15$, then $A_0=\lceil E_0+9\rceil$ (Definition 2.4), then $K_0(A'):=\lfloor200(E_0+8)\rfloor+1$; every $K\ge K_0(A')$ is

admissible, independently of the a_i, of N and of δ (for $A'=2$: $D_*=8$, $E_0=19$, $K_0=5401$). (4) Fix $K\ge K_0(A')$ and the exponents a_i. (5) $B:=A'+4$; it depends on A' only, since the Mellin truncation is bounded without a bound for G_N in terms of K, so it may equally be fixed before K; the constants of Hypothesis RE may depend on B. (6) Type I: $A_{\mathrm{I}}:=A'+1$, C_{tr} as in the proof of Proposition 4.3, and Lemma 2.13 at precision $A_{\mathrm{I}}+3$; these are free. (7) Last, $x\ge x_0(\delta,A',K,(a_i),\theta)$, equivalently N large; the constants are not effective. There is no circularity: D_* and B depend on A' only, and K is chosen after $E_0>D_*+4C+2$. The K chosen serves the one precision A'; the argument does not give every precision at a fixed K.

5. Theorem C: polylogarithmic bands

In Theorem B the auxiliary primes have size $\exp(L^{a_i})$. Theorem C replaces them by primes of size a fixed power of L, so that the cofactor a in $N=p+aq$ is at most a fixed power of $\log N$. The proof is that of Theorem B; §5.2 lists the places where the size of the band primes enters, and checks each one. The hypothesis changes: Theorem C assumes Hypothesis RE for the band system $\mathfrak{B}_{\mathrm{poly}}$, whose first band starts at L^{c_1} with $c_1>c_{\mathrm{poly}}$, an enormous exponent (Remark 5.2). In [28], [23] and [21] the auxiliary primes have size $\exp(L^{\vartheta})$ for fixed exponents ϑ: with $1/10<\vartheta<1/5$ for the band weights [28, (2.12)], [23, (2.3)–(2.4)], [21, §2.2], and with $0.245<\vartheta<0.39$ for the further prime sets of [21, (4.6)]. We found no earlier use of polylogarithmic bands in this method.

5.1. **Statement.** As in §4, for even n put $\mathfrak{S}(n):=2C_2\prod_{p\mid n,\,p>2}(p-1)/(p-2)$ with $C_2:=\prod_{p>2}(1-(p-1)^{-2})$, and $\mathcal{X}_N^{\delta}:=\{N/2^j:\ j\ge 2\text{ an integer},\ 2^{-j}\ge\delta\}$. The bands, the punctured bands $\mathcal{P}_i^N$, the masses V_i^N, the marks $\mathcal{A}^N$ and $S_K=c_1+\cdots+c_K$ are those of Definition 2.5 and Remark 2.6 for $\mathfrak{B}_{\mathrm{poly}}$, with $L=L_x=\log x$.

Theorem C. *Assume* QRH(7/8) (Definition 2.1) *and Hypothesis* RE($\mathfrak{B}_{\mathrm{poly}}$, 7/8) *of §2.3; the proof uses in addition Proposition 2.2, which is* [28, Proposition 2.1]*, and the classical inputs of §2.4. Fix $\delta\in(0,1/4)$ and $A'>1$, put $A:=2A'+4$ and $C:=2$, and fix a datum for $\mathfrak{B}_{\mathrm{poly}}$ at (A,C)* (Definitions 2.4 and 2.5)*: a real $E_0>A+4C+2$, an integer $K>200(E_0+8)$, a real $\varepsilon_\sharp>0$ and a real $c_1>c_{\mathrm{poly}}$, with $c_i=5^{i-1}c_1$. There is x_0 such that for every even N and every $x\in\mathcal{X}_N^{\delta}$ with $x\ge x_0$,*

$$\sum_{x<k\le 2x}\Lambda(k)\,G_N(k)=\sum_{a\in\mathcal{A}^N}\frac{\mathfrak{S}(aN)}{a}\int_x^{2x}\frac{dt}{\log((N-t)/a)}+O\big(xL^{-A'}\big),\qquad L=\log x,\tag{49}$$

where $G_N(k)=\#\{a\in\mathcal{A}^N:\ N-k=ah,\ h\text{ prime}\}$ and $\mathcal{A}^N$ is the set of products of one prime from each $\mathcal{P}_i^N=\{p\nmid N:\ L^{c_i}<p\le L^{2c_i}\}$, $1\le i\le K$. The implied constant and x_0 depend only on δ, A' and the datum $(E_0,K,\varepsilon_\sharp,c_1)$, not on N or x.

Corollary C.1. *In the setting of Theorem C:*

(a) *For every even N and every $x\in\mathcal{X}_N^{\delta}$ with $x\ge x_0$, with the bands taken at $L=\log x$, the number $P_N(x)$ of primes $p\in(x,2x]$ such that $N-p=aq$ with $a\in\mathcal{A}^N$ and q prime satisfies*

$$P_N(x)=\mathfrak{S}(N)(\log 2)^K\frac{x}{L^2}\Big(1+O_K\Big(\frac{1}{c_1\log L}\Big)+O_\delta\Big(\frac{S_K\log L}{L}\Big)+O\big(L^{1-A'}\big)\Big),$$

where the last implied constant depends only on δ, A' and the datum. In particular $P_N(x)\asymp\mathfrak{S}(N)(\log 2)^K x/(\log N)^2$, uniformly in N.

(b) *Every sufficiently large even N is $p+aq$ with p and q prime and $a\in\mathcal{A}^N$, the bands taken at $L=\log(N/4)$. Here a is a product of exactly K distinct primes, one in each band, and*

$$(\log(N/4))^{S_K}<a\le(\log(N/4))^{2S_K}\le(\log N)^{2S_K};$$

the lower bound is $(1-o(1))(\log N)^{S_K}$ as $N\to\infty$.

Remark 5.1. (i) Neither the implied constant in (49) nor x_0 is effective: the Bombieri–Vinogradov theorem enters the Type I estimate with the constant of Lemma 2.13, which as proved is not effectively computable, and the constants of Hypothesis RE are not claimed to be effective.

(ii) Theorem C is a statement about the sum over $\mathcal{A}^N$. Corollary C.1(b) gives some $a \in \mathcal{A}^N$ for each large even N; it says nothing about any individual a. Since $1 \notin \mathcal{A}^N$, it gives no representation of N as a sum of two primes.

(iii) The relative error in Corollary C.1(a) is $O_K(1/(c_1 \log L))$, where Corollary B.1 has a negative power of L: here the masses V_i^N are compared with $\log 2$ by Mertens' theorem at the scale L^{c_i} (Remark 2.6(ii)). No sharper form is claimed.

Remark 5.2 (The size of the marks). The bands are polylogarithmic, but the exponent is enormous. By Remark 2.6(iii), $S_K = c_1(5^K - 1)/4$ and $c_1 > 8K^2$, so $2S_K > 4K^2(5^K - 1) \geq 2K^2\, 5^K$. Since $A' > 1$ we have $A > 6$, hence $E_0 > A + 4C + 2 > 16$ and $K > 200(E_0 + 8) > 4800$, that is, $K \geq 4801$. As $\log_{10} 2 < 0.30103$, we have $2^{4801} < 10^{1445.25}$, so $5^{4801} = 10^{4801}/2^{4801} > 10^{3355.75} > 5.6 \cdot 10^{3355}$; with $4801^2 > 2.3 \cdot 10^7$ this gives

$$2S_K > 2 \cdot 4801^2 \cdot 5^{4801} > 25 \cdot 10^{3362} > 10^{3363}.$$

This bound does not involve the exponents C_i^{LWW} of Definition 2.5. For instance, for $A' = 2$ and $E_0 = 19$ the least admissible K is 5401; then $A_0 = 28$, $C^\sharp = 10890$, $b_\sharp = 21833$, $h_\sharp = 10913$ and $5 + h_\sharp b_\sharp = 238{,}263{,}534$, and $2S_K > 4K^2(5^K - 1)$, with $5^{5401} > 10^{3775.13}$ and $4 \cdot 5401^2 > 1.16 \cdot 10^8$, gives $2S_K > 10^{3783}$.

Consequently the bound $a \leq (\log N)^{2S_K}$ of Corollary C.1(b) improves on the trivial bound $a < N$ only when $(\log N)^{2S_K} < N$, that is, when $y/\log y > 2S_K$ for $y = \log N$. The function $y/\log y$ increases for $y > e$, and $y_0 = 7.7 \cdot 10^{3366}$ has $\log y_0 > 3366 \log 10 > 7750$, so $y_0/\log y_0 < 7.7 \cdot 10^{3366}/7750 < 10^{3363}$. Hence this needs $\log N > 7.7 \cdot 10^{3366}$, and since x_0 is not effective, no explicit range of N is given at all. "Polylogarithmic" is an asymptotic statement only.

The size comes from the condition $c_1 > c_{\mathrm{poly}}$ in the datum for $\mathfrak{B}_{\mathrm{poly}}$. That condition is where Appendix B places the evidence for $\mathrm{RE}(\mathfrak{B}_{\mathrm{poly}}, 7/8)$, a record of how the band scale enters estimates of the kind that Hypothesis RE asserts; it is a census of uses, not a proof. The deduction in §5.2 itself uses only $c_1 > 2$, $c_1 \geq A + 7$ and $c_{i+1} > 2c_i$, which hold for every datum (Remark 2.6(iii)).

5.2. **Proof of Theorem C and Corollary C.1.** We run the proof of Theorem B in §4, at the precision $A = 2A' + 4$ and coefficient exponent $C = 2$ used there, with the bands of $\mathfrak{B}_{\mathrm{poly}}$ in place of those of $\mathfrak{B}_{\mathrm{exp}}$. That proof sees the band system only at the places in Steps 1–7 below; everything else, in particular the Z-rough Möbius sums of the Type I estimate, involves no bands and is unchanged. Throughout, $x \in \mathcal{X}_N^\delta$, $L = \log x$, and "for large x" allows thresholds that depend on δ, A' and the datum.

Step 1 (sizes; Lemma 4.2(a) and (c)). By Remark 2.6(i) and (iii), every band prime is odd, lies in $(L^{c_1}, L^{2c_K}]$, is below $Z = \exp(L^{0.24})$ for large x and does not divide N, and the bands are disjoint since $c_{i+1} = 5c_i > 2c_i$. So every $a \in \mathcal{A}^N$ is a product of K distinct odd primes, squarefree, coprime to N and to every Z-rough integer, with $L^{S_K} < a \leq L^{2S_K} = x^{o(1)}$, and $\#\mathcal{A}^N \leq \prod_i L^{2c_i} = L^{2S_K} = x^{o(1)}$. Where §4 uses that band primes exceed every fixed power of L, the lower bound L^{c_1} is used instead: for the pairs sharing a prime (Step 5) and in the main term (Step 8).

Step 2 (masses; Lemma 4.2(b)). By Remark 2.6(ii), uniformly in even N, $V_i^N = \log 2 + O(1/(c_1 \log L))$, which includes the deficit $V_i - V_i^N \leq \omega(N) L^{-c_1} \leq 2L^{1-c_1}$ from the puncturing. So for large x, $\sum_{a \in \mathcal{A}^N} 1/a = \prod_i V_i^N \leq 1$, and $\prod_i V_i^N = (\log 2)^K \big(1 + O_K(1/(c_1 \log L))\big)$.

Step 3 (one mark per value; Lemma 4.2(e)). If $N - k \geq N/3$ and $N - k = ah$ with $a \in \mathcal{A}^N$ and h prime, then $h \geq N/(3a) \geq N/(3L^{2S_K}) > L^{2c_K}$ for large x, so h is not a band prime. Hence a is the

product of all band-prime factors of $N-k$, and $G_N(k)\in\{0,1\}$. Since $2x\le N/2$, this covers every $k\le 2x(1+o(1))$, in particular every $k\in(x,2x]$.

Step 4 (Type I; Proposition 4.3, §4.3). The marks enter the Type I estimate only through Steps 1–3 and through the size of the prime cofactor $h=(N-k)/a$ for $k\le 2x$: $h\ge N/(2a)>\max(Z,L^{2c_K})$, and $h\ge 2xL^{-2S_K}>x^{2/5}$ for large x, which is what removes the moduli not coprime to N. So the Type I estimate of §4.3 holds as stated, with the same main term and singular series $\mathfrak{S}(aN)$.

Step 5 (Type II; Proposition 4.7, §4.4).

- The truncation error of the smoothing ($\mathcal{E}_{\mathrm{Mel}}$ in the proof of Proposition 4.7) is bounded through $\sum_{k\le 12x,\,k<N}\tau(k)G_N(k)\le 12x(L+5)\prod_i V_i^N+2\sqrt{12x}\,\#\mathcal{A}^N\ll xL$, by Steps 1 and 2.
- Since $\eta(a/Y)\ne 0$ forces $a/4<Y<a$, only the scales $Y\in 2^{\mathbb{Z}}$ with $L^{S_K}/4<Y<L^{2S_K}$ occur, and there are $O(S_K\log L)$ of them. In particular $\log(N/Y)\ge L/2$, as the Cauchy–Schwarz step (47) over the prime cofactor requires.
- Pairs of marks sharing a prime (Lemma 4.8): the per-mark bound $\sum_h\tau(N-\ell h)^2\ll(X/Y)L^3$ used in its proof needs $Y\ll X^{1/4}L$, which holds since $Y<L^{2S_K}\le X^{1/4}$ for large x (as $X>x/5$). The pairs (a,b) in $[Y,4Y]^2$ with $(a,b)>1$ number at most $16Y^2\sum_{p>L^{c_1}}p^{-2}\le 16Y^2L^{-c_1}$, so they contribute $\ll XY\,L^{7-c_1}\le XY\,L^{-A}$, since $c_1\ge A+7$.
- The hand-off. The coefficients α, β of §4.4 lie in $\mathcal{C}_N^{\mu}(M;B)$ and $\mathcal{C}_N(\mathfrak{N})$ with $C=2$ and the smoothing exponent $B=A'+4$ of §4; every kept box $(x,M,\mathfrak{N})$ lies in $\mathcal{R}_{\delta/5}(N)$ (Definition 2.8); and the marks are those of the datum at $L_x=\log x$, punctured at N, by Definition 2.5. So Hypothesis RE$(\mathfrak{B}_{\mathrm{poly}},7/8)$ at $(A,C,B)=(2A'+4,2,A'+4)$ with $\delta_0=\delta/5$, together with QRH(7/8), gives $|Q_Y^{\mathrm{sgn}}|\le 2C_0\,XY\,L^{-A}$ (Remark 2.12(i)), and the Cauchy–Schwarz step gives $|T_Y|\ll_\delta XL^{-(A+1)/2}$. Only this instance of Hypothesis RE is used.
- Summing, as in (46): the $O(S_K\log L)$ mark scales, the $O(L)$ boxes and the Mellin factor $O(\log L)$ together cost $L^{1+o(1)}$, so the Type II sum is $\ll xL^{1+o(1)-(A+1)/2}=xL^{-A'-3/2+o(1)}$; with the smoothing errors of §4.4 at $B=A'+4$, it is $\ll xL^{-A'-1}$.

Step 6 (assembly; (42) and (48)). Apply Proposition 2.2 with $F=G_N\mathbf{1}_{(x,2x]}$, so that $\|F\|_\infty\le 1$ by Step 3. The Type I estimate at precision $A'+1$ and the Type II bound give (49), with error $O(xL^{-A'-1})+O(Z)\ll xL^{-A'}$.

Step 7 (the order of choices; §4.5). Fix δ and $A'>1$; then $A=2A'+4$, $C=2$ and $B=A'+4$; then E_0, $G=E_0+5$, $A_0=\lceil G+4\rceil$ and K; then $\varepsilon_\sharp$ and $c_1>c_{\mathrm{poly}}$, where c_{poly} is computed from A, C, E_0, K, $\varepsilon_\sharp$ and the exponents C_i^{LWW} of Definition 2.5; then $c_i=5^{i-1}c_1$; then the parameters of the Type I estimate, which are free (its precision $A'+1$, its sieve depth and the Bombieri–Vinogradov level, as in §4.3); last, $x\ge x_0$. No threshold depends on a later choice: K is fixed before the bands, and c_1, although it depends on K, is fixed after it. Hence x_0 and the implied constant in (49) depend only on δ, A' and the datum. This proves Theorem C.

Step 8 (Corollary C.1; the proof of Corollary B.1 in §4.5). For $x<t\le 2x$ and $a\in\mathcal{A}^N$ we have $N-t\in[N/2,N)$, $0\le\log N-L\le\log(1/\delta)$ and $0<\log a\le 2S_K\log L$, so $1/\log((N-t)/a)=L^{-1}\big(1+O_\delta(S_K\log L/L)\big)$. Every prime factor of a exceeds L^{c_1} and does not divide N, so $\mathfrak{S}(aN)=\mathfrak{S}(N)\prod_{p\mid a}(1+1/(p-2))=\mathfrak{S}(N)(1+O(KL^{-c_1}))$. With Step 2, the main term of (49) is

$$\mathfrak{S}(N)(\log 2)^K\frac{x}{L}\Big(1+O_K\Big(\frac{1}{c_1\log L}\Big)+O_\delta\Big(\frac{S_K\log L}{L}\Big)\Big).$$

By Step 3, $G_N(p)=1$ exactly when $p\in(x,2x]$ is counted by $P_N(x)$, so $P_N(x)=\sum_{x<p\le 2x}G_N(p)$. The prime powers in $(x,2x]$ contribute $O(\sqrt{x}\,L)$ to the left side of (49), and $L<\log p\le L+\log 2$ for $p\in(x,2x]$. So dividing by L costs a relative $O(1/L)$, absorbed in $O_\delta(S_K\log L/L)$, and the

error $O(xL^{-A'})$ becomes a relative $O(L^{1-A'})$, since $\mathfrak{S}(N)\ge 2C_2>1$. This is (a). Since $A'>1$, all three relative errors tend to 0, and $L\le \log N\le L+\log(1/\delta)$; this gives the $\asymp$ statement. For (b), take $x=N/4\in\mathcal{X}_N^\delta$: by (a), $P_N(N/4)>0$ for every large even N; by Step 1, each $a\in\mathcal{A}^N$ is a product of exactly K distinct primes, one in each band, with $L^{S_K}<a\le L^{2S_K}$ for $L=\log(N/4)$; and $(\log(N/4))^{S_K}=(\log N)^{S_K}(1-\log 4/\log N)^{S_K}=(1-o(1))(\log N)^{S_K}$, since S_K is fixed. □

6. A conditional reduction

This section proves a conditional reduction of the binary Goldbach asymptotic. It is not a route to Goldbach's conjecture. Theorem D splits the Goldbach sum at a smooth cut. One part is evaluated up to the discrepancy of the primes in the progressions $n\equiv N \pmod d$ with $d<Ne^{-(\log N)^a/2}$; the other is weighted by the short-divisor part $\Lambda^\flat$ of Λ, and it is $o(N)$ under QRH(θ) and Hypothesis RE($\mathfrak{B}_{\text{wide}},\theta$). Corollary D.1 deduces $G(N)\sim\mathfrak{S}(N)N$ from a hypothesis (I_a') on that discrepancy. Hypothesis (I_a') concerns moduli up to a level that eventually exceeds N^ν for every fixed $\nu<1$; it is beyond known techniques and is not proved here, and Hypothesis RE is not proved either (Remark 2.12(v)). Section 6.9 describes what (I_a') asks for.

In the proof, Hypothesis RE is used once, in §6.7, and QRH(θ) enters only through (RE2) there. Everything else is proved in §§6.2–6.8 from Proposition 2.2 and the classical inputs of §2.4.

6.1. **Statement.** Throughout this section N is an even integer, $N\to\infty$, and $L=\log N$. Put

$$G(N)=\sum_{1\le n<N}\Lambda(n)\Lambda(N-n),\qquad \mathfrak{S}(N)=2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p\mid N\\ p>2}}\frac{p-1}{p-2}. \tag{50}$$

Here $\Lambda(1)=0$ and $\mu(1)=1$, and no arithmetic function is evaluated at a nonpositive integer. For integers $d\ge 1$ and real $y\ge 0$ put

$$\Psi(y;d,N)=\sum_{\substack{1\le n\le\min(y,N-1)\\ n\equiv N \pmod d}}\Lambda(n),$$
$$M_N(y;d)=\frac{d}{\varphi(d)}\,\frac{N}{\varphi(N)}\,\#\{k\ge 1:\ 1\le N-kd\le y,\ (k,N)=1\}, \tag{51}$$

$$\mathcal{E}_N'(Q)=\sum_{\substack{1\le d<Q\\ (d,N)=1}}\max_{0\le y\le N}\big|\Psi(y;d,N)-M_N(y;d)\big|. \tag{52}$$

Both Ψ and M_N count integers $N-kd$ with $k\ge 1$. The maximum is over real y; it exists because both functions are step functions of y with finitely many jumps. $\mathcal{E}_N'(Q)$ is nondecreasing in Q.

Parameters. Fix $\theta<1$, and choose the following in this order.

- (D1) Real numbers $D_*>18$ and $E_0>D_*+6$; put $G=E_0+5$ and $A_0=\lceil G+4\rceil$; choose an integer $K>200(G+3)$.
- (D2) A real g' with $0<g'\le 1/(40K)$, and exponents with $1/10<a_1$, $a_{i+1}-a_i\ge 2g'$ $(1\le i<K)$ and $a_K+g'<1/5$.
- (D3) A real a with $0<a<a_1$, and a real function $\psi\in C^{1,1}[0,\infty)$, nonincreasing, with $0\le\psi\le 1$, $\psi=1$ on $[0,1/2]$ and $\psi=0$ on $[1,\infty)$.
- (D4) The remaining precisions of the proof (Type I, Bonferroni, Bombieri–Vinogradov and smoothing, §§6.6–6.7), sufficiently large in terms of the choices above; they do not change K. Finally N is taken sufficiently large.

Here $C^{1,1}$ means continuously differentiable with Lipschitz derivative, and norms of ψ'' are essential suprema. By (D1), (E_0,K) is admissible for $(D_*,1)$ in the sense of Definition 2.4, and with (D2) it is a datum for $\mathfrak{B}_{\text{wide}}$ at $(D_*,1)$ (Definition 2.5). An admissible example is $D_*=19$, $E_0=26$,

$G = 31$, $A_0 = 35$, $K = 6801$, $g' = 1/272040$ and $a_i = 1/10 + (2i-1)g'$, for which $a_K + g' = 3/20$. Put

(53)
$$S = \exp(L^a), \quad \ell = \log S = L^a, \quad Q_a = NS^{-1/2} = Ne^{-L^a/2}, \quad w^\flat(b) = \log b\, \psi\Big(\frac{\log b}{\ell}\Big)\ (b \ge 1),$$

and $\Lambda^\flat = \mu * w^\flat$, $\Lambda^\natural = \Lambda - \Lambda^\flat$. Since $w^\flat(b) = 0$ for $b \ge S$, $\Lambda^\flat(m) = \sum_{b|m,\, b<S} \mu(m/b) w^\flat(b)$.

Theorem D. *Assume* QRH(θ) *for a fixed* $\theta < 1$ *and Hypothesis* RE($\mathfrak{B}_{\rm wide}, \theta$) *of §2.3; Proposition 2.2 and the classical inputs of §2.4 are also used. Fix the parameters as in* (D1)–(D4).

(i) *Then*

$$\Big|\sum_{p<N} \log p\, \Lambda^\flat(N-p)\Big| \ll KNL^{-g'}(\log L)^2 = o(N), \tag{54}$$

$$G(N) = \mathfrak{S}(N)N + O\big(L\,\mathcal{E}'_N(Ne^{-L^a/2})\big) + o(N). \tag{55}$$

Both bounds are uniform over even N, including integers with many small prime factors. The constants and the function implicit in $o(N)$ may depend on θ, D_*, E_0, K, g', (a_i), a, ψ, the precisions of (D4) *and the constants of Hypothesis RE. Thresholds are ineffective. No uniformity is claimed as $a \downarrow 0$, as $a \uparrow a_1$, or as the bands or the cut vary with N.*

(ii) *The same holds for every fixed $0 < a < 1/5$ if the bands are chosen after a: keep* (D1), *choose $a_1 \in (\max(a, 1/10), 1/5)$, then $0 < g' \le 1/(40K)$ with $(2K-1)g' < 1/5 - a_1$, and put $a_i = a_1 + 2(i-1)g'$. These exponents satisfy* (D2), *so K does not depend on a, while g', the bands and the threshold may. There is no assertion at $a = 1/5$.*

Corollary D.1. *Under the hypotheses of Theorem D, if, for a fixed a as in Theorem D, there exist fixed $\varepsilon > 0$, $C_H < \infty$ and N_H such that for every even $N \ge N_H$*

$$(I'_a) \qquad \mathcal{E}'_N\big(Ne^{-L^a/2}\big) \le C_H\, N\, L^{-1-\varepsilon}, \tag{56}$$

then $G(N) = \mathfrak{S}(N)N + o(N)$, hence $G(N) \sim \mathfrak{S}(N)N$.

Indeed (55) and (56) give an error $O(C_H NL^{-\varepsilon})$, and $\mathfrak{S}(N) \ge 2\prod_{p>2}(1-(p-1)^{-2}) > 0$. The terms of $G(N)$ in which n or $N-n$ is a proper prime power contribute $O(N^{1/2}L^3) = o(N)$, so under the hypotheses of Corollary D.1 also $\sum_{p+p'=N} \log p \log p' \sim \mathfrak{S}(N)N$, and every sufficiently large even N is a sum of two primes. This is a conditional implication. Hypothesis (I'_a) is beyond known techniques and is not proved here, and Hypothesis RE is not proved either. Corollary D.1 is not a present route to Goldbach's conjecture.

Remark 6.1. (i) The level $Q_a = Ne^{-L^a/2}$ exceeds N^ν for every fixed $\nu < 1$, so (I'_a) lies outside the range of the Elliott–Halberstam conjecture as formulated at fixed levels N^ν, $\nu < 1$. It asks for one saving $L^{-1-\varepsilon}$, for the single residue class N, against the exact local model M_N.

(ii) If for some fixed $C > 0$ and $\varepsilon > 0$ one has $\mathcal{E}'_N(NL^{-C}) \ll NL^{-1-\varepsilon}$ uniformly over even N, then (I'_a) holds with the same ε, since $\mathcal{E}'_N(Q)$ is nondecreasing in Q and $e^{L^a/2} > L^C$ for large N.

6.2. **The exact-coprimality model.** Throughout the proof P denotes an arbitrary fixed positive number, a power of log to be saved, and implied constants may depend on it. Put

$$r_N = \frac{N}{\varphi(N)}, \qquad f_N(d) = \mu(d)\,\frac{d}{\varphi(d)}\,\mathbf{1}_{(d,N)=1}, \qquad F_N(y) = \sum_{1\le d<y} f_N(d).$$

Lemma 6.2. *Uniformly in even N:*

(i) *for fixed $0 < \eta_1 < 1$, $F_N(y) \ll yL^{-P}$ for $N^{1-\eta_1} \le y \le N$;*

(ii) *for fixed* $0<\epsilon_0<1$ *and* $\log D\ge L^{\epsilon_0}$,

$$\sum_{d<D}\frac{f_N(d)}{d}\ll L^{-P},\qquad \sum_{d<D}\frac{f_N(d)\log d}{d}=-\mathfrak{S}(N)+O(L^{-P}),\qquad F_N(D)\ll DL^{-P}.$$

Proof. Write $f_N=\mu*\vartheta_N$. Comparing local factors gives, for $j\ge1$, $\vartheta_N(p^j)=1$ if $p\mid N$ and $\vartheta_N(p^j)=-1/(p-1)$ if $p\nmid N$. Let $\sigma=1/\log(3L)$, so that $0<\sigma\le1/4$. Then

$$\mathcal{H}_N:=\sum_{e\ge1}\frac{|\vartheta_N(e)|}{e^{1-\sigma}}=\prod_{p\mid N}\bigl(1-p^{\sigma-1}\bigr)^{-1}\prod_{p\nmid N}\Bigl(1+\frac{1}{(p-1)(p^{1-\sigma}-1)}\Bigr)\ll(\log 3L)^{B_0}$$

with an absolute B_0. The second product converges uniformly. In the first, the primes $p\le L^2$ have $p^\sigma\le e^2$, so their factors have logarithm $O(1+\sum_{p\le L^2}1/p)$; and there are at most $L/(2\log L)$ prime divisors of N above L^2, each contributing $O(L^{-2+2\sigma})$. This holds for every even N, primorial-type N included.

In $F_N(y)=\sum_{e<y}\vartheta_N(e)\sum_{m<y/e}\mu(m)$, split at $e=\sqrt{y}$. The prime number theorem for μ on $e\le\sqrt{y}$, and $\sum_{e>\sqrt{y}}|\vartheta_N(e)|/e\le y^{-\sigma/2}\mathcal{H}_N$ on the rest, give for $y\ge4$ and every fixed b

$$|F_N(y)|\ll_b y\mathcal{H}_N\bigl((\log y)^{-b}+y^{-\sigma/2}\bigr).\tag{57}$$

For $N^{1-\eta_1}\le y\le N$ we have $y^{-\sigma/2}\le\exp(-(1-\eta_1)L/(2\log 3L))$, and (i) follows. For (ii), partial summation from (57) bounds the tails beyond D of $\sum_d f_N(d)/d$ and $\sum_d f_N(d)(\log d)/d$ by $\mathcal{H}_N\{(\log D)^{2-b}+D^{-\sigma/2}((1+\log D)/\sigma+\sigma^{-2})\}$, which is $O(L^{-P})$ when $\log D\ge L^{\epsilon_0}$ and b is large; the same holds with d^{-1-s} for $0\le s\le1$, so both series converge uniformly there. For $s>0$,

$$\mathcal{D}_N(s):=\sum_{d\ge1}\frac{f_N(d)}{d^{1+s}}=\prod_{p\nmid N}\Bigl(1-\frac{p^{-s}}{p-1}\Bigr)=\zeta(1+s)^{-1}\prod_{p\mid N}\bigl(1-p^{-1-s}\bigr)^{-1}\prod_{p\nmid N}\frac{1-p^{-s}/(p-1)}{1-p^{-1-s}},$$

and the last product converges uniformly near $s=0$, because $2\mid N$. By the uniform convergence, $\mathcal{D}_N(0)=0$ and $-\sum_d f_N(d)(\log d)/d=\mathcal{D}_N'(0)=r_N\prod_{p\nmid N}(1-(p-1)^{-2})=\mathfrak{S}(N)$. With the tail bounds this gives the first two estimates of (ii); the third is (57) at $y=D$. □

Lemma 6.3. *Let* $\Delta_N(k)=r_N\mathbf{1}_{(k,N)=1}-1$. *Let* $1\le k_1\le N$, $D_1=N/k_1$, *and let* w *be nondecreasing on* $[1,\infty)$ *with* $0\le w(k)\le\log k$ *and* $w(k)=0$ *for* $k<k_1$. *Then, uniformly in even* N, k_1 *and* w,

$$\sum_{d<D_1}f_N(d)\sum_{k<N/d}w(k)\Delta_N(k)=o(N).$$

Proof. Inclusion–exclusion over $e\mid N$ gives $\sum_{k\le t}\Delta_N(k)=-(r_N-1)\{t\}-r_N\sum_{e\mid N,\,e>1}\mu(e)\{t/e\}$, so

$$\sup_{t\ge0}\Bigl|\sum_{1\le k\le t}\Delta_N(k)\Bigr|\le r_N2^{\omega(N)}.\tag{58}$$

Put $c_w(X)=\sum_{k<X}w(k)\Delta_N(k)$. By partial summation and (58), $|c_w(X)|\le2r_N2^{\omega(N)}\log X$, since $w\le\log X$ on $[1,X)$ and its variation there is at most $\log X$. Fix $0<\eta_1<1$ and put $Y_1=N^{1-\eta_1}$. As $\sum_{d<Y}|f_N(d)|\ll Y$, $r_N\ll\log\log N$ and $2^{\omega(N)}=N^{o(1)}$, the terms with $d<\min(Y_1,D_1)$ contribute $N^{1-\eta_1+o(1)}$. If $D_1>Y_1$, interchanging the two finite sums gives exactly

$$\sum_{Y_1\le d<D_1}f_N(d)\,c_w(N/d)=\sum_{k_1\le k<N/Y_1}w(k)\Delta_N(k)\bigl[F_N(N/k)-F_N(Y_1)\bigr],$$

since $w(k)\ne0$ forces $k\ge k_1$ and hence $N/k\le D_1$; for each k the d-range is $[Y_1,N/k)$, and both endpoint conventions are those of F_N. Both arguments of F_N lie in $[Y_1,N]$, so Lemma 6.2(i) and $|\Delta_N(k)|\le r_N$ bound this by $\ll Nr_NL^{-P}\sum_{k<N^{\eta_1}}(\log k)/k\ll Nr_NL^{2-P}$. No short-interval estimate for μ is used. □

6.3. **The smooth split and its Type I half.** Put $w(k) = \log k\,[1 - \psi(\log k/\ell)]$ for $k \ge 1$, and $w(t) = 0$ for $t < 1$. By (D3), w is nonnegative and nondecreasing on $[1,\infty)$, $w(k) \le \log k$, $w(k) = 0$ for $k \le S^{1/2}$, and $w(k) = \log k$ for $k \ge S$. Since $\Lambda = \mu * \log$ and $\log = w + w^\flat$, we have $\Lambda^\natural = \mu * w$, and

$$G(N) = T^\sharp + T^\flat, \qquad T^\flat = \sum_{n<N} \Lambda(n)\Lambda^\flat(N-n),$$
$$T^\sharp = \sum_{n<N} \Lambda(n)\,\natural\,(N-n) = \sum_{d<Q_a} \mu(d) \sum_{k<N/d} w(k)\Lambda(N-dk), \tag{59}$$

exactly, all sums over n starting at 1: writing $N - n = dk$, $w(k) \ne 0$ forces $k > S^{1/2}$, hence $d < N/S^{1/2} = Q_a$.

Proposition 6.4. *Uniformly in even N, $T^\sharp = \mathfrak{S}(N)N + O(L\,\mathcal{E}'_N(Q_a)) + o(N)$. The proof uses only the monotonicity and the support of ψ.*

Proof. If $(d,N) > 1$, a nonzero $\Lambda(N-dk)$ requires $N-dk$ to be a power of a prime dividing N; there are $O(\omega(N)L)$ such k for each d, of weight at most L^2, so these d contribute $O(Q_a\omega(N)L^3) = o(N)$.

For $(d,N) = 1$, $\sum_{k<N/d} w(k)\Lambda(N-dk) = \int_{[0,N]} w((N-y)/d)\,d\Psi(y;d,N)$, and the corresponding model sum $\frac{d}{\varphi(d)} r_N \sum_{k<N/d,\,(k,N)=1} w(k)$ is the same integral against $M_N(y;d)$. The integrand is nonincreasing in y, bounded by L and zero at $y = N$, and $\Psi - M_N$ vanishes at $y = 0$; integrating by parts, the two differ by at most $L \max_{y\le N} |\Psi(y;d,N) - M_N(y;d)|$. Hence

$$T^\sharp = \sum_{d<Q_a} f_N(d)\, r_N \sum_{\substack{k<N/d\\(k,N)=1}} w(k) + O\big(L\,\mathcal{E}'_N(Q_a)\big) + o(N).$$

By Lemma 6.3 with $k_1 = S^{1/2}$ and $D_1 = Q_a$, replacing $r_N \mathbf{1}_{(k,N)=1}$ by 1 costs $o(N)$. Since w is monotone and at most L, $\sum_{k<X} w(k) = \int_1^X w(u)\,du + O(L)$, at a total cost $O(Q_aL\log\log N) = o(N)$. With $\psi(t) = 1$ for $t < 0$ and

$$c_S(X) = \int_0^{\min(X,S)} \psi\Big(\frac{\log u}{\ell}\Big)\log u\,du, \qquad \int_1^X w(u)\,du = X(\log X - 1) - c_S(X) \quad (X \ge 1),$$

using $\int_0^1 \log u\,du = -1$. Therefore

$$T^\sharp = N\Big[(L-1)\sum_{d<Q_a}\frac{f_N(d)}{d} - \sum_{d<Q_a}\frac{f_N(d)\log d}{d}\Big] - \sum_{d<Q_a} f_N(d)\,c_S(N/d) + O\big(L\,\mathcal{E}'_N(Q_a)\big) + o(N).$$

Since $\log Q_a \sim L$, Lemma 6.2(ii) makes the first term $\mathfrak{S}(N)N + O(NL^{1-P})$. In the second, for $d \le N/S$ we have $c_S(N/d) = c_S(S) = O(S\ell)$ and Lemma 6.2(i) gives $O(N\ell L^{-P})$. For $N/S < d < Q_a$, the function $h(t) = c_S(N/t)$ satisfies $|h(t)| \le (N/t)\ell + 1$ and $|h'(t)| \le N\ell/t^2$, and partial summation against $F_N(t) \ll tL^{-P}$ gives $O(N\ell L^{-P}(1 + \int_{N/S}^{Q_a} dt/t)) = O(N\ell^2 L^{-P})$. □

Remark 6.5 (A sharp cut). The same argument with $w(k) = \log k\,\mathbf{1}_{k\ge k_0}$ proves: for fixed $0 < \epsilon_0 < 1$, uniformly for $L^4 \le k_0 \le Ne^{-L^{\epsilon_0}}$ and with $D_0 = N/k_0$ and $F(k) = \sum_{d<N/k}\mu(d)\Lambda(N-dk)$,

$$G(N) = \mathfrak{S}(N)N + O\big(L\,\mathcal{E}'_N(D_0)\big) + O\Big(\sum_{2\le k<k_0} \log k\,|F(k)|\Big) + o(N).$$

Here $G(N) = \sum_{k\ge k_0}\log k\,F(k) + \sum_{2\le k<k_0}\log k\,F(k)$; the $d < D_0$ not coprime to N cost $O(D_0\omega(N)L^3) = o(N)$, because $k_0 \ge L^4$ and $\omega(N) \ll L/\log L$; the variation of w, including its jump at k_0, is $O(L)$; and the main term is $N[(L-1)\sum_{d<D_0} f_N(d)/d - \sum_{d<D_0} f_N(d)(\log d)/d] - k_0(\log k_0 - 1)F_N(D_0) + o(N) = \mathfrak{S}(N)N + o(N)$ by Lemma 6.2(ii). This sharp form is not used below.

6.4. **The flat weight and a uniform fibre sieve.** Write $m = m_s m_r$, where m_s is the product of the full powers of the primes below S that divide m, and m_r has no prime factor below S. A divisor $b < S$ of m divides m_s, and $m/b = m_r \cdot (m_s/b)$ with coprime factors, so

$$\Lambda^\flat(m) = \mu(m_r)\,\kappa(m_s), \qquad \kappa(s) = \sum_{b\mid s} \mu(s/b)\log b\;\psi\Big(\frac{\log b}{\ell}\Big), \tag{60}$$

with no squarefreeness assumption. Thus $\kappa(1) = 0$, and $\kappa(s) = \Lambda(s)$ for $s \le S^{1/2}$. Put $W_S(m) = |\kappa(m_s)|$ and $\tau_S(m) = \#\{b < S :\ b \mid m\}$. Then

$$|\Lambda^\flat(m)| \le W_S(m) \le \ell\,\tau_S(m) \le \ell\,\tau(m). \tag{61}$$

The first inequality can be strict when m_r is not squarefree; for this reason the bounds below are proved for W_S.

Lemma 6.6. $\displaystyle\sum_{s\ S\text{-smooth}} \frac{|\kappa(s)|}{\varphi(s)} \ll_\psi \ell\log(2+\ell)$, *the sum running over all S-smooth $s \ge 1$, prime powers included.*

Proof. Let $\varrho(t) = t\psi(t/\ell)$ for $t \ge 0$. Then $\|\varrho'\|_\infty \le 1 + \|\psi'\|_\infty =: \Gamma_1$ and $\|\varrho''\|_\infty \le (2\|\psi'\|_\infty + \|\psi''\|_\infty)/\ell =: \Gamma_2$, the latter almost everywhere. If s has distinct prime factors $p_1 < \cdots < p_j$, then with $t = \log s$ and $h_i = \log p_i$,

$$\kappa(s) = \sum_{I\subseteq\{1,\dots,j\}} (-1)^{|I|}\varrho\Big(t - \sum_{i\in I} h_i\Big).$$

For $j = 1$ this is a first difference, and $|\kappa(p^e)| \le \Gamma_1 \log p$. For $j \ge 2$, group the terms by $I \cap \{3,\dots,j\}$; each group is a second difference in the first two coordinates at nonnegative arguments, which for $\varrho \in C^{1,1}$ is a double integral of ϱ'', so $|\kappa(s)| \le 2^{j-2}\Gamma_2 \log p_1 \log p_2$. Sum over the exponents with $\sum_{e\ge1} 1/\varphi(p^e) = p/(p-1)^2$, over the primes above p_2 with the factor $\prod_{p_2<p<S}(1+2p/(p-1)^2) \ll (\ell/\log p_2)^2$, and over $p_1 < p_2$ with $\sum_{p<y}(\log p)p/(p-1)^2 \ll \log y$. The part with $j \ge 2$ is then $\ll \Gamma_2\ell^2 \sum_{p<S} p/(p-1)^2 \ll_\psi \ell\log(2+\ell)$, and the part with $j = 1$ is $\ll \Gamma_1 \sum_{p<S}(\log p)/p \ll_\psi \ell$. □

Lemma 6.7 (Selberg's sieve with a uniform constant)**.** *Let J be an interval of real length $|J|$ and $z \ge 2$. For each prime $\varpi < z$ let $\nu(\varpi)$ residue classes modulo ϖ be forbidden, where $0 \le \nu(\varpi) \le 2$ and $\nu(\varpi) < \varpi$. Then the number of integers in J that lie in no forbidden class is*

$$\ll |J| \prod_{\varpi<z}\Big(1 - \frac{\nu(\varpi)}{\varpi}\Big) + z^2(\log 2z)^{10},$$

with an absolute implied constant, independent of the classes and of the position of J.

Proof. Selberg's upper-bound sieve [24, (21.2), Theorem 21.1 and (21.8)] at level z, for the density $\nu(d)/d$ on squarefree d composed of primes $\varpi < z$ with $\nu(\varpi) \ge 1$, bounds the count by $|J|/\mathcal{J}(z)$ plus $\sum_{d,e<z} |\rho_d\rho_e|\,|r_{[d,e]}|$, where $\mathcal{J}(z) = \sum_{d<z} \mu^2(d) \prod_{\varpi\mid d} \nu(\varpi)/(\varpi - \nu(\varpi))$ over such d, the weights satisfy $|\rho_d| \le 1$, and $|r_d| \le \nu(d) \le 2^{\omega(d)}$ because each of the $\nu(d)$ classes modulo d meets J in $|J|/d + O(1)$ integers. The remainder is at most $(\sum_{d<z} 2^{\omega(d)})^2 \ll z^2(\log 2z)^2$. For the denominator, fix $\xi = 1/8$ and consider the probability measure on squarefree d composed of primes $\varpi < z^\xi$ that selects each ϖ independently with probability $\nu(\varpi)/\varpi$. Its total mass before normalisation is $\prod_{\varpi<z^\xi}(1-\nu(\varpi)/\varpi)^{-1}$, and $\mathbb{E}(\log d) = \sum_{\varpi<z^\xi} \nu(\varpi)(\log\varpi)/\varpi \le 2\xi\log z + O(1)$. By Markov's inequality the event $d < z$ has probability at least $1/2$ for large z, and all these d occur in $\mathcal{J}(z)$. By Mertens' theorem and $\nu \le 2$, $\prod_{z^\xi\le\varpi<z}(1-\nu(\varpi)/\varpi)^{-1}$ is bounded by a constant depending only on ξ. Hence $\mathcal{J}(z) \gg \prod_{\varpi<z}(1-\nu(\varpi)/\varpi)^{-1}$ with an absolute constant. □

Let $T_0 = N^{1/4}$, $z = N^{1/8}$ and $\mathfrak{S}^\circ(N) = \prod_{p\mid N,\, 2<p<S}(p-1)/(p-2)$. For a prime $p \nmid N$ write $N - p = sm$ with $s = (N-p)_s$. Then $(s, N) = 1$ (a common prime would divide p), s is odd, and m has no prime factor below S.

Lemma 6.8 (Fibre sieve)**.** *Let $J \subset (0, N)$ be an interval and $\mathcal{Q}$ a set of primes in $[S, z)$ not dividing N. Then, uniformly in N, J and $\mathcal{Q}$,*

$$\sum_{\substack{p\in J,\ p\nmid N,\ (N-p)_s\le T_0\\ \varpi\nmid N-p\ (\varpi\in\mathcal{Q})}} \log p\, W_S(N-p) \ll_\psi |J|\,\mathfrak{S}^\circ(N)\log(2+\ell)\prod_{\varpi\in\mathcal{Q}}\Big(1-\frac{1}{\varpi-1}\Big) + N^{1/2+o(1)}.$$

Proof. Fix $s \le T_0$, S-smooth and coprime to N; then $W_S(N-p) = |\kappa(s)|$ on the fibre. The primes $p = N - sm \in J$ with $p \ge z$ correspond to integers m in an interval of length $|J|/s$ that avoid the following classes modulo the primes $\varpi < z$: for $\varpi < S$ with $\varpi \nmid sN$, the two distinct classes 0 and Ns^{-1}; for $\varpi < S$ with $\varpi \mid s$, the class 0 (the prime condition then holds automatically); for $\varpi < S$ with $\varpi \mid N$, the class 0 (the two conditions coincide; this includes $\varpi = 2$); for $S \le \varpi < z$, the class Ns^{-1} (which is 0 if $\varpi \mid N$); and for $\varpi \in \mathcal{Q}$ also the class 0, distinct from Ns^{-1} because $\varpi \nmid N$. So $\nu(\varpi) \le 2$ throughout, and

$$\prod_{\varpi<z}\Big(1-\frac{\nu(\varpi)}{\varpi}\Big) \ll \frac{\mathfrak{S}^\circ(N)}{\ell\log z}\prod_{\varpi\mid s}\frac{\varpi-1}{\varpi-2}\prod_{\varpi\in\mathcal{Q}}\Big(1-\frac{1}{\varpi-1}\Big), \qquad \frac{1}{s}\prod_{\varpi\mid s}\frac{\varpi-1}{\varpi-2} \le \frac{C_\circ}{\varphi(s)},$$

with $C_\circ = \prod_{\varpi\ge 3}(1 + 1/(\varpi(\varpi-2)))$; the first by Mertens' theorem, since a factor $1 - 2/\varpi$ is $(1 - 1/\varpi)(1-1/(\varpi-1))$. Apply Lemma 6.7, multiply by $\log p \le L$ (note $L/\log z = 8$), and add at most zL for the primes $p < z$. Summing over $s \le T_0$ with weight $|\kappa(s)| \le \ell\tau(s)$, the sieve remainders and the primes below z cost $N^{1/2+o(1)}$, and the main terms give $|J|\mathfrak{S}^\circ(N)\ell^{-1}\prod_{\mathcal{Q}}(\cdots)\sum_s |\kappa(s)|/\varphi(s)$, which Lemma 6.6 bounds. No asymptotic for primes in progressions is used. □

Lemma 6.9 (The tail)**.** $\displaystyle\sum_{\substack{p<N\\ (N-p)_s>T_0}} \log p\, W_S(N-p) \ll N\ell^4 e^{-L^{1-a}/8} + N^{o(1)} \ll NL^{-P}.$

Proof. The primes $p \mid N$ contribute $N^{o(1)}$ by (61). For $p \nmid N$ with $(N-p)_s > T_0$, multiply the prime factors of $(N-p)_s$, counted with multiplicity, one at a time; each is below S, so the first partial product exceeding T_0 is an S-smooth divisor $d \in (T_0, T_0S]$ of $N - p$, coprime to N. Since $T_0S \le N^{1/3}$, the Brun–Titchmarsh inequality [24, Theorem 20.1] in the reduced class $p \equiv N \pmod d$ and Rankin's bound at exponent $1/\ell$ give

$$\sum_{\substack{p<N,\ p\nmid N\\ (N-p)_s>T_0}} \log p \ll N \sum_{\substack{d>T_0\\ d\ S\text{-smooth}}} \frac{1}{\varphi(d)} \le NT_0^{-1/\ell}\prod_{q<S}\Big(1+\sum_{j\ge1}\frac{q^{j/\ell}}{\varphi(q^j)}\Big) \ll N\ell^{2.72}e^{-L^{1-a}/4},$$

because $q^{1/\ell} \le e$ for $q < S$, so the logarithm of the product is at most $e\sum_{q<S} 1/q + O(1)$. By (61) and Brun–Titchmarsh at the moduli $[b_1, b_2] < S^2 = N^{o(1)}$, which are coprime to N whenever the class contains a prime $p \nmid N$,

$$\sum_{p<N,\ p\nmid N} \log p\, W_S(N-p)^2 \ll N\ell^2 \sum_{b_1,b_2<S}\frac{1}{\varphi([b_1,b_2])} \ll N\ell^5;$$

for the last step write $b_1 = du$, $b_2 = dv$ with $(u, v) = 1$ and use $\varphi(duv) \ge \varphi(d)\varphi(u)\varphi(v)$ and $\sum_{n\le t} 1/\varphi(n) \ll \log 2t$. Cauchy–Schwarz with the measure $\log p$ proves the lemma, since $a < 1$. □

Lemmas 6.8 (with $\mathcal{Q} = \emptyset$) and 6.9 give, for every interval $J \subset (0, N)$,

$$\sum_{p\in J}\log p\, W_S(N-p) \ll_\psi |J|\,\mathfrak{S}^\circ(N)\log(2+\ell) + O(NL^{-P}), \qquad \mathfrak{S}^\circ(N) \ll \frac{N}{\varphi(N)} \ll \log\log 3N. \tag{62}$$

In particular the left side is $O_\psi(N(\log L)^2)$ for $J=(0,N)$. We do not claim that the factor $\mathfrak{S}^\circ(N)$ is attained.

6.5. **Blocks, marks and exceptional sets.** Put $\delta = L^{-g'}$. By (62), the primes $p<\delta N$ and $p>(1-\delta)N$ contribute

$$\ll_\psi \delta N\,\mathfrak{S}^\circ(N)\log(2+\ell)+O(NL^{-P}) \ll NL^{-g'}(\log L)^2 \tag{63}$$

to the sum in (54). The rest is covered by the blocks $I=(x,2x]\cap[\delta N,(1-\delta)N]$ with $x=2^{-j}N$, $j\ge 1$ an integer; there are $O(1+\log L)$ of them, $\sum_I x\le N$, and $N/x\le 2L^{g'}$. On each block put $L_x=\log x$, so that $L_x=L+O(\log L)$, and use the bands of $\mathfrak{B}_{\text{wide}}$ at L_x punctured at N (Definition 2.5): $\mathcal{P}_i^N=\{p\nmid N:\ e^{L_x^{a_i}}<p\le e^{L_x^{a_i+g'}}\}$, with masses V_i^N and marks $\mathcal{A}^N$. By Remark 2.6(ii), $V_i^N=g'\log L_x+O(L_x^{-a_i})+O(\omega(N)e^{-L_x^{a_i}})$, so $\prod_i V_i^N=L_x^{o(1)}$. The bands stay fixed as an endpoint of the block varies. Since $a<a_1$ and $a_K+g'<1/5$, for large N

$$S<e^{L_x^{a_1}}\le \min\bigcup_i \mathcal{P}_i^N,\qquad \max\bigcup_i \mathcal{P}_i^N<e^{L_x^{1/5}}<Z=e^{L_x^{0.24}}. \tag{64}$$

So band primes are coprime to N, odd, above S and below Z.

Let $\omega_i(m)$ be the number of distinct prime factors of m in $\mathcal{P}_i^N$, let H be the property that $\omega_i(m)\ge 1$ for every i, put $\varsigma(h)=\prod_{i=1}^K(\omega_i(h)+1)^{-1}$, and let $\mathcal{B}_{\text{sq}}$ be the set of m divisible by p^2 for some $p\in\bigcup_i\mathcal{P}_i^N$.

Lemma 6.10 (Mark identity)**.** *For $m\notin\mathcal{B}_{\text{sq}}$,*

$$\Lambda^\flat(m)\mathbf{1}_H(m)=\sum_{\substack{A\mid m\\ A\in\mathcal{A}^N}}\mu(A)\,v_S(m/A),\qquad v_S(h):=\mu(h_r)\,\kappa(h_s)\,\varsigma(h).$$

For $m\in\mathcal{B}_{\text{sq}}$ the two sides differ by at most $2\ell\tau(m)$.

Proof. By (64) a mark divides m_r. If $m\notin\mathcal{B}_{\text{sq}}$ and $A\mid m$, $A\in\mathcal{A}^N$, then $(A,m/A)=1$, $(m/A)_s=m_s$ and $\mu(m_r)=\mu(A)\mu((m/A)_r)$, so each term equals $\Lambda^\flat(m)\varsigma(m/A)$. If $t_i=\omega_i(m)\ge 1$ for all i, there are $\prod_i t_i$ marks $A\mid m$, each with $\varsigma(m/A)=\prod_i t_i^{-1}$; if some $t_i=0$, both sides vanish. On $\mathcal{B}_{\text{sq}}$, if v_i of the t_i band primes of band i divide m to a power at least 2, the cofactor weights of band i add up to $(t_i-v_i)/t_i+v_i/(t_i+1)\le 1$, so the right side is at most $\ell\tau(m)$ in absolute value, as is the left side by (61). □

Lemma 6.11 (Exceptional sets)**.** *For every set $\mathcal{B}\subseteq\{1,\dots,N-1\}$, $\sum_{k\in\mathcal{B}}\tau(k)\tau(N-k)\ll(|\mathcal{B}|N)^{1/2}L^{15/2}$.*

Proof. Cauchy's inequality twice, and $\sum_{n\le N}\tau(n)^4\ll NL^{15}$ (§2.4). □

With (61), Lemma 6.11 replaces every bound for the size of the target $\Lambda^\flat(N-\cdot)$ that the proof needs before Cauchy's inequality; $\max_{m\le N}\tau(m)=N^{o(1)}$ is used only against power savings. The cases are these. *Prime powers* on the prime side: there are $O(N^{1/2})$ of them below N, so they cost $N^{1/2+o(1)}$. *The remainder* $O(\|F\|_\infty Z)$ in Proposition 2.2: it is $ZN^{o(1)}=N^{o(1)}$. *Variables sharing a prime with N*: in the Type I and Type II sums the variables are Z-rough, so such a prime P exceeds Z; the products $n\le 2x$ divisible by such a P form a set of size $O(x\omega(N)/Z)$, each n carries Vaughan coefficients of total size at most $L\tau(n)$, and Lemma 6.11 gives $xL^{O(1)}Z^{-1/2}$, using $N/x\ll L^{g'}$. *Band squares*: the set $\{k<N:\ N-k\in\mathcal{B}_{\text{sq}}\}$ has size $O(N\sum_p p^{-2})=O(Ne^{-L_x^{1/10}})$, the sum over band primes, so the error $2\ell\tau(N-k)$ of Lemma 6.10 costs $O(Ne^{-L_x^{1/10}/3})$. *Endpoint layers and the Mellin truncation* in §6.7: see there. *The escape from H and the end intervals*: Lemmas 6.8 and 6.9, which are proved for W_S.

6.6. **Type I.** Fix a block I and put $F_0(n) = \Lambda^\flat(N-n)\mathbf{1}_H(N-n)\mathbf{1}_I(n)$; it is a bounded function on $(x, 2x]$, and we apply Proposition 2.2 to it with $U = V = x^{1/5}$ and $Z = e^{L_x^{0.24}}$. By Remark 2.3, both Type I sums are bounded, up to a factor $O(L_x)$, by sums of the form treated in the next proposition, uniformly in their upper endpoint; the condition $n \in I$ only moves the endpoints.

Proposition 6.12. *For every fixed P, uniformly in even N, the blocks, $x \le x' \le y' \le 2x$ and coefficients $|\gamma'_q| \le L_x$,*

$$\sum_{\substack{q\le x^{2/5},\ q\ odd\\ R_Z(q)=1}} \gamma'_q \sum_{x'<qk\le y'} R_Z(k)\,\Lambda^\flat(N-qk)\,\mathbf{1}_H(N-qk) \ll xL_x^{-P}.$$

Proof. Reductions. The terms with $(q,N)>1$ cost $xL^{O(1)}Z^{-1/2}$ (§6.5). For odd k, the Bonferroni inequalities give $R_Z(k) = \sum \mu(r) + O(\sum 1)$, the first sum over squarefree $r \mid k$ composed of primes in $(2, Z]$ with $\omega(r) \le R_I$, the second over those with $\omega(r) = R_I + 1$, where $R_I = \lceil C_I \log L_x \rceil$; so $r \le Z^{R_I+1} = x^{o(1)}$. Write $k = rj$ with j odd. Expand the whole indicator $\mathbf{1}_H(N-qk) = \prod_i (1 - \mathbf{1}_{\omega_i(N-qk)=0})$: each factor is a signed sum over squarefree divisors $D_i \ne 1$ of $N-qk$ composed of primes of $\mathcal{P}_i^N$; truncate each at $\omega(D_i) \le R_B = \lceil C_B \log L_x \rceil$ and telescope the product. This gives $\mathbf{1}_H(n) = \sum_{D\mid n} c_D + O(\sum_{D\mid n} c_D^{\mathrm{err}})$, with $c_D^{\mathrm{err}} \ge 0$, both supported on squarefree products D of band primes, $D \le x^{o(1)}$, total masses $x^{o(1)}$, and, by $\sum_{\omega(D_i)=t} 1/D_i \le (V_i^N)^t/t!$,

$$\sum_D \frac{|c_D|}{D} \le L_x^{Kg'+o(1)}, \qquad \sum_D \frac{c_D^{\mathrm{err}}}{D} \le L_x^{Kg'-C_B\log(C_B/(eg'))+o(1)}. \tag{65}$$

A compatible term has $(D, 2qrN) = 1$: band primes are odd, prime to N and below Z (so prime to q), and a prime of D and r would divide both qk and $N-qk$, hence N.

The common-factor calculation. In the main terms insert

$$\Lambda^\flat(N-qrj) = \sum_{b<S,\ b\mid N-qrj} w^\flat(b)\,\mu\big((N-qrj)/b\big).$$

Here b is odd and $(b, qD) = 1$ by (64), so $bD \mid N-qrj$; but b may share primes with r and with N. Put $c = (b, r)$ and $r = cr_1$. The congruence is empty unless $c \mid N$. Then, r being squarefree, $(b/c, 2qr_1) = 1$, and $t := (N-qrj)/(bD)$ satisfies $(b/c)Dt = N/c - qr_1 j$; as j runs over odd integers, t runs over one class t_0 modulo $2qr_1$, and

$$g := (N/c - qr_1,\ 2qr_1) = (N, r_1), \qquad (g, 2qr_1/g) = 1, \tag{66}$$

because N/c is even, q and r_1 are odd, $(q,N) = 1$ and $(c, r_1) = 1$. Write $t = gv$; then Dg is squarefree, since $(D,N) = 1$ and $g \mid N$. For squarefree v_0 and $u \ge 1$, $\mu(v_0u) = \mu(v_0)\sum_{e\mid u,\ e\mid v_0^\infty} \mu(u/e)$ (prime by prime, the sum is $\mu(u)\mathbf{1}_{(u,v_0)=1}$). With $v_0 = Dg$,

$$\mu\big((N-qrj)/b\big) = \mu(Dg) \sum_{\substack{e\mid v\\ e\mid (Dg)^\infty}} \mu(v/e).$$

Factor $e = e_D e_g$ by its prime supports in D and in g. For fixed e the Möbius variable $u = v/e$ runs over a *reduced* class modulo

$$q' = \frac{2qr}{cg}, \qquad \text{in an interval inside } [0, Y'), \qquad Y' = \frac{N-x}{bDge}\,: \tag{67}$$

indeed $eu \equiv t_0/g \pmod{q'}$, $(t_0/g, q') = 1$ by (66), and e is a unit modulo q'. No coprimality of b with N, with r or with u has been imposed. The height Y' is a common upper bound for the initial segments that express the reflected interval, not the length of that interval.

Main terms. Retain $De_D \le x^{1/20}$ and $ge_g \le x^{1/40}$. Since $N - x \ge x$ and $b < S = x^{o(1)}$,

$$Y' \ge x^{37/40-o(1)}, \qquad q' \le 2qr \le x^{2/5+o(1)} < Y'^{1/2}(\log Y')^{-B} \tag{68}$$

for every fixed B, eventually; the exponent margin is $37/80 - 2/5 = 1/16$. For fixed (b, c, D, g, e), the modulus $q' = 2q\cdot(r_1/g)$ determines q (its Z-rough part) and r_1/g (its Z-smooth part), hence $r = cg(r_1/g)$; so distinct pairs (q, r) give distinct moduli, with coefficients of size at most $L_x\ell|c_D|$. Each inner sum is at most $2E(Y'; q')$, where $E(Y'; q') = \max_{(u_0,q')=1} \max_{z_0\le Y'} |\sum_{n\le z_0,\, n\equiv u_0\ (q')} \mu(n)|$, and Lemma 2.15 (with $j = 0$) gives $\sum_{q'} E(Y'; q') \ll Y'(\log Y')^{-P'}$ for every P'. Now sum over (b, c, D, g, e) with the weight $Y' \le N/(bDge)$. For each b there are at most $\tau(b)$ choices of c; the g and e_g sums give

$$\sum_{g|\mathrm{rad}N} \frac{1}{g} \prod_{p|g} \Big(1 - \frac{1}{p}\Big)^{-1} = \prod_{p|N} \Big(1 + \frac{1}{p-1}\Big) = r_N \ll \log\log N, \tag{69}$$

without counting the divisors of N; the D and e_D sums give $L_x^{Kg'+o(1)}$ by (65); and $\sum_{b<S} \tau(b)/b \ll \ell^2$. With $N \le 2xL^{g'}$ and $\log Y' \asymp L_x$, the main terms are $\ll xL_x^{C_{\mathrm{loss}}+Kg'+g'+3a-P'}$ for a fixed C_{loss}, which is $\ll xL_x^{-P}$ once P' is chosen.

Tails. If $ge_g > x^{1/40}$ then $e_g > x^{1/40-o(1)}$, as $g \le r = x^{o(1)}$; Lemma 2.16 with $\sigma = 1/2$ bounds the reciprocal tail by $x^{-1/80+o(1)} \prod_{p|g}(1 - p^{-1/2})^{-1} = x^{-1/80+o(1)}$, since $\omega(g) \le R_I + 1$. The tail $De_D > x^{1/20}$ is $x^{-1/40+o(1)}$ in the same way. Here bound each progression by its number of terms. The u-interval has length at most $x/(bDge)$ and the modulus is $q' = 2qr/(cg)$, so a progression has at most $xc/(2qrbDe) + 1$ terms: g cancels, and c appears in the numerator. Writing $r = cr_1$, $\sum_{r:\, c|r} c/r = \sum_{r_1} 1/r_1 \ll \log Z$, and $\sum_{q\le x^{2/5}} |\gamma'_q|/q \ll L_x^2$; for each r the e-tail is at most $x^{-1/80+o(1)}$, uniformly in $g = (N, r_1)$. With at most $\tau(b)$ choices of c, $\sum_{b<S} \ell\tau(b)/b \ll \ell^3$ and (65), the length terms of the tails total $x^{1-1/80+o(1)}$. For the terms $+1$, Lemma 2.16 with $\sigma = 1/20$ gives $\#\{e \le 3N :\ e \mid (Dg)^\infty\} \le (3N)^{1/20} \prod_{p|Dg}(1 - p^{-1/20})^{-1} = x^{1/20+o(1)}$, and there are at most $x^{2/5+o(1)}$ choices of (q, r, b, c, D), so these terms total $x^{9/20+o(1)}$.

Truncation errors. Use $|\Lambda^\flat(m)| \le \ell \sum_{b<S,\, b|m} 1$. For the remainder of R_Z, the number of odd k with $x' < qk \le y'$, $r \mid k$ and $b \mid N - qk$ is at most $xc/(2qrb) + 1$ with $c = (b, r) \mid N$, and zero otherwise. For a fixed $t > 1$, Rankin's device in $\omega(r)$ gives

$$\sum_{\substack{\omega(r)=R_I+1\\(b,r)|N}} \frac{(b,r)}{r} \le t^{-R_I-1}(1+t)^{\omega(b)} \exp\Big(t \sum_{p\le Z} \frac{1}{p}\Big) \le t^{-R_I} \tau(b)^j L_x^{0.24t+o(1)}$$

with a fixed j such that $2^j \ge 1 + t$; and $\sum_{b<S} \ell\tau(b)^j/b \ll_j \ell^{2^j+1}$, by $\tau^j \le \tau_{2^j}$. As $t^{-R_I} \le L_x^{-C_I \log t}$, these errors are $\ll xL_x^{-P}$ once t is fixed and C_I is chosen large; the terms $+1$ total $x^{2/5+o(1)}$. For the remainder of $\mathbf{1}_H$, over the retained r with $\omega(r) \le R_I$ we have $\sum_r (b, r)/r \le 2^{\omega(b)} \prod_{2<p\le Z}(1 + 1/p)$, against the mass $L_x^{Kg'-C_B \log(C_B/(eg'))+o(1)}$ of (65); choosing C_B large makes these errors $\ll xL_x^{-P}$ too. Every $o(1)$ above is a function of x alone, and N enters only through (q, N), the puncturing, $c \mid N$, $g = (N, r_1)$ with (69), the class u_0 (covered by the maximum over reduced classes) and the height, with $N/x \le 2L^{g'}$; so the bound is uniform in even N. No smoothness of ψ is used here. □

6.7. **Type II.**

Proposition 6.13. *Assume* QRH(θ) *and Hypothesis* RE($\mathfrak{B}_{\mathrm{wide}}, \theta$). *Uniformly in even N and the blocks, the Type II sum of Proposition 2.2 satisfies $T_3(F_0) \ll xL_x^{-4}$.*

Proof. Write $T_3(F_0) = \sum_{m>U,\, n>V} \mu(m)R_Z(m)\, c_V(n)R_Z(n)\, F_0(mn)$. Delete the terms with $(mn, N) > 1$, at cost $xL^{O(1)}Z^{-1/2}$ (§6.5); then m and n are units modulo N.

Separation of variables. Let ϕ_I be smooth with $0 \le \phi_I \le 1$, equal to $\mathbf{1}_{I/x}$ outside the L_x^{-B}-neighbourhoods of the two endpoints of I/x, with $\phi_I^{(i)} \ll_i L_x^{iB}$. Replacing $\mathbf{1}_I(mn)$ by $\phi_I(mn/x)$ changes T_3 only on integers mn in two intervals of length $O(xL_x^{-B})$, where the coefficients have total size at most $L_x\tau(mn)$ and the target at most $\ell\tau(N-mn)$; by Lemma 6.11 this costs $xL_x^{O(1)-B/2}$. On the support of $\phi_I(mn/x)$, $mn \le (1-\delta)N + 2xL_x^{-B} < N$. With $\Phi(v) = \int_{\mathbb{R}} \phi_I(e^u)e^{-ivu}\,du$, integration by parts once and twice gives $|\Phi(v)| \ll \min(1, |v|^{-1}, L_x^B|v|^{-2})$, so for $1/8 \le t \le 16$

$$\phi_I(t) = \frac{1}{2\pi}\int_{|v|\le L_x^{2B}} \Phi(v)t^{iv}\,dv + O(L_x^{-B}), \qquad \int_{|v|\le L_x^{2B}} |\Phi(v)|\,dv \ll_B \log L_x.$$

The truncation costs L_x^{-B} times the full sum $\sum_{k<N} L_x\tau(k)\,\ell\tau(N-k) \ll NL^{O(1)}$ (Lemma 6.11 with $\mathcal{B} = \{1,\dots,N-1\}$). Split m and n into ranges $(M, 2M]$ and $(\mathfrak{N}, 2\mathfrak{N}]$ with $M, \mathfrak{N} \ge x^{1/5}/2$ and put $X = M\mathfrak{N}$. On the $O(L_x)$ boxes that meet the support, $x/5 < X < \min(3x, N)$ and $N/X < 5N/x \le 10L^{g'} \le L_x^{1/40}$ for large N, because $g' \le 1/(40K) < 1/40$; so $(x, M, \mathfrak{N}) \in \mathcal{R}_{\mathrm{wide}}(N)$ (Definition 2.8). The reflected target is set to zero when $mn \ge N$; this changes nothing before the separation and keeps every argument positive after it. It remains to bound, uniformly in $|v| \le L_x^{2B}$ and in the boxes,

$$\mathcal{T} = \sum_{m,n} \alpha_m\beta_n\,\Lambda^\flat(N-mn)\mathbf{1}_H(N-mn)\mathbf{1}_{mn<N},$$

$$\alpha_m = \mu(m)R_Z(m)\mathbf{1}_{(m,N)=1}m^{iv}, \qquad \beta_n = c_V(n)R_Z(n)\mathbf{1}_{(n,N)=1}n^{iv},$$

with $m \in (M, 2M]$, $n \in (\mathfrak{N}, 2\mathfrak{N}]$ and the conditions $m > U$, $n > V$ (which keep intervals). Then $\alpha \in \mathcal{C}_N^\mu(M; B)$ and, as $0 \le c_V(n) \le \log n \le 2L_x$, $\beta/2 \in \mathcal{C}_N(\mathfrak{N})$ with $C = 1$ (Definition 2.7; Remark 2.12(iii)).

Marks and Cauchy's inequality. By Lemma 6.10, at a cost $O(Ne^{-L_x^{1/10}/3})$, $\Lambda^\flat(N-mn)\mathbf{1}_H(N-mn)$ may be replaced by $\sum_A \mu(A)v_S((N-mn)/A)$ over $A \in \mathcal{A}^N$ dividing $N-mn$. The marks are squarefree products of K primes, so $\mu(A) = (-1)^K$. With the partition of unity $\sum_{Y\in 2^{\mathbb{Z}}} \eta(A/Y) = 1$ of §2.3,

$$\mathcal{T} = (-1)^K \sum_Y \sum_{h\ge 1} v_S(h)S_Y(h) + O(Ne^{-L_x^{1/10}/3}), \qquad S_Y(h) = \sum_{A\in\mathcal{A}^N} \eta(A/Y) \sum_{N-mn=Ah} \alpha_m\beta_n,$$

which is the quantity $S_Y(h)$ of Definition 2.9. Only $O_K(L_x)$ values of Y occur, and $S_Y(h) = 0$ for $h > N/Y$. By (61), $\varsigma \le 1$ and the divisor moments, $\sum_{h\le N/Y} |v_S(h)|^2 \ll (N/Y)\ell^2L^3$, so Cauchy's inequality gives

$$\Big|\sum_{h\ge 1} v_S(h)S_Y(h)\Big|^2 \ll \frac{N}{Y}\,\ell^2L^3 \sum_{h\ne 0} |S_Y(h)|^2 = \frac{N}{Y}\,\ell^2L^3\Big(\prod_i V_i^N\Big)^2 Q_Y^{\mathrm{sgn}}.$$

The range of h is $h \ll N/Y$, not X/Y, and the factor $(\prod_i V_i^N)^2$ is needed because S_Y is not normalised.

Hypothesis RE. By Remark 2.12(i) and (iii), Hypothesis $\mathrm{RE}(\mathfrak{B}_{\mathrm{wide}}, \theta)$ at $(D_*, 1)$ (precision D_*, coefficient exponent $C = 1$), the datum (D1)–(D2) and our B, together with $\mathrm{QRH}(\theta)$, gives $Q_Y^{\mathrm{sgn}} \le 8C_0\,XY\,L_x^{-D_*}$ for $x \ge x_0$. Hence each Y contributes $\ll (NX)^{1/2}\ell L^{3/2}(\prod_i V_i^N)L_x^{-D_*/2}$. Summing over the $O_K(L_x)$ values of Y and the $O(L_x)$ boxes, integrating over v with mass $O_B(\log L_x)$, and using $X < 3x$, $(N/X)^{1/2} \ll L^{g'/2}$ (charged once) and $\prod_i V_i^N = L_x^{o(1)}$, we get

$$T_3(F_0) \ll xL_x^{2+g'/2+o(1)} \cdot L^{a+3/2} \cdot L_x^{-D_*/2} + xL_x^{O(1)-B/2} \ll xL_x^{-4},$$

since $D_* > 18$, $a < 1/5$ and $g' < 1/40$, and B is chosen large. The flat weight costs exactly the extra factor $\ell L^{3/2} = L^{a+3/2}$. □

6.8. **Proof of Theorem D and Corollary D.1.** *The escape.* On a block, $0 \le 1-\mathbf{1}_H(m) \le \sum_{i=1}^K \mathbf{1}_{\omega_i(m)=0}$, so $\sum_{p\in I}\log p\,|\Lambda^\flat(N-p)|(1-\mathbf{1}_H(N-p)) \le \sum_i \sum_{p\in I}\log p\, W_S(N-p)\mathbf{1}_{\omega_i(N-p)=0}$. On the fibre $N-p=sm$ a missing band means that no prime of $\mathcal{P}_i^N$ divides m. These primes lie in $[S,z)$ by (64) and do not divide N, so Lemma 6.8 with $\mathcal{Q}=\mathcal{P}_i^N$, Lemma 6.9 and $\prod_{p\in\mathcal{P}_i^N}(1-1/(p-1)) = \exp(-V_i^N+O(e^{-cL_x^{a_1}})) \asymp L_x^{-g'}$ bound the escape on the block by

$$\ll_\psi KxL^{-g'}\mathfrak{S}^\circ(N)\log(2+\ell)+O(NL^{-P}) \ll KxL^{-g'}(\log L)^2+O(NL^{-P}). \tag{70}$$

This is an upper bound with a fixed constant. The puncturing matters here: for a band prime dividing N the two forbidden classes would coincide.

The flat bound. On a block, the hit part $\sum_{p\in I}\log p\,\Lambda^\flat(N-p)\mathbf{1}_H(N-p)$ differs from $\sum_{x<n\le 2x}\Lambda(n)F_0(n)$ by the prime powers, $O(N^{1/2+o(1)})$, and Proposition 2.2 writes the latter as $T_1(F_0)-T_2(F_0)+T_3(F_0)+N^{o(1)}$. Propositions 6.12 and 6.13 give $O(xL_x^{-4})$. Summing this and (70) over the blocks with $\sum_I x \le N$, the $O(\log L)$ errors $O(NL^{-P})$ with $P>g'+3$, the end intervals (63), the primes $p\mid N$ (cost $N^{o(1)}$) and the exceptional sets of §6.5, we obtain

$$\Big|\sum_{p<N}\log p\;\Lambda^\flat(N-p)\Big| \ll_{\theta,K,(a_i),g',a,\psi} KNL^{-g'}(\log L)^2,$$

which is (54).

The reduction. In (59), $T^\flat$ has $\Lambda(n)$ on the prime side; the $O(N^{1/2}L)$ proper prime powers $n<N$ contribute $O(N^{1/2+o(1)})$ by (61), so $T^\flat = \sum_{p<N}\log p\,\Lambda^\flat(N-p)+O(N^{1/2+o(1)}) = o(N)$ by (54). With Proposition 6.4 this proves (55). Under (I'_a) the error term is $L\cdot C_H NL^{-1-\varepsilon} = C_H NL^{-\varepsilon}$, which proves Corollary D.1. All $o(N)$ terms are uniform over even N with the parameters fixed in the stated order. The thresholds are ineffective, if only because the constants of Hypothesis RE need not be effective. For Theorem D(ii), the exponents chosen after a satisfy (D2) and $a<a_1$, which is all the proof uses. □

6.9. **What (I'_a) asks for.** Hypothesis (I'_a) bounds the counts of the single primes $N-dk$ in progressions, averaged absolutely over $d<Q_a$, against an explicit model, with all endpoints. It contains no product of two prime indicators. In Theorem D the parity-sensitive input is the flat bound (54), which comes from Hypothesis RE, not from (I'_a). Table 3 sets this beside three classical frames.

Remark 6.14 (Selberg's example). For general sequences, information on progressions alone does not detect primes at this level either. Let $a_n = 1+\lambda(n)$, with λ Liouville's function. Then $a_n\ge 0$ and $a_p=0$ for every prime p. Since λ is completely multiplicative, $A_d(t) := \sum_{n\le t,\, d\mid n} a_n = \lfloor t/d\rfloor + \lambda(d)\sum_{m\le t/d}\lambda(m)$. From $\lambda(n)=\sum_{u^2v=n}\mu(v)$, the bound $\sum_{m\le y}\mu(m)\ll_B y(\log y)^{-B}$, and a split at $u=y^{1/4}$, one gets $\max_{y\le Y}|\sum_{m\le y}\lambda(m)| \ll_B Y^{1/2}+Y(\log Y)^{-B}$. Since $x/d>x/Q$ for $d<Q$, for fixed $a>0$ and $Q=xe^{-(\log x)^a/2}$ this gives

$$\sum_{d<Q}\max_{t\le x}\Big|A_d(t)-\frac{t}{d}\Big| \ll_B x\log x\,(\log(x/Q))^{-B}+Q \ll_P x(\log x)^{-P}$$

for every P, taking $B\ge (P+1)/a$. This is Selberg's example [39], [14, p. 1045], [11, §1], at the level of (I'_a); only the level differs from the classical statement. The example has the density $1/d$ rather than the model (51). A presieved form of the example removes this mismatch: for each even N, with $W_a=\exp(L^{a/8})$, $V_a=\prod_{p\le W_a}(1-1/p)$ and $P^-(1)=\infty$, the row $a_n^* = V_a^{-1}\mathbf{1}_{P^-(n)>W_a}(1+\lambda(N-n))$, $1\le n<N$, is nonnegative and vanishes whenever $N-n$ is prime, and with a_n^* in place of $\Lambda(n)$ in $\Psi(y;d,N)$ it satisfies (I'_a) against the model (51), indeed with the bound $\ll_{a,P} NL^{-P}$ for every fixed P, while $\sum_{n<N}a_n^*\Lambda(N-n)=O(N^{1/2}L^{a/8})$. In outline: for $(d,N)=1$ the $n\le \min(y,N-1)$ with $n\equiv N \pmod d$ are the $n=N-kd$ with k in an interval $J_y\subset[1,N/d)$

TABLE 3. Distribution level and parity-sensitive input. $R(\nu)$ is the level-of-distribution hypothesis of [11, p. 1663] at level x^ν with every logarithmic saving.

Result	Level, and where parity-sensitive information enters
Bombieri's asymptotic sieve [5], as stated in [11, p. 1664]	In the dimension-one setting, $R(\nu)$ for all $\nu < 1$, together with mild growth conditions on A and regularity conditions on g, gives $\sum_{n\le x} a_n\Lambda_k(n) \sim kHA(x)(\log x)^{k-1}$ for $k \ge 2$; the functions Λ_k, $k \ge 2$, weight the two parities equally. By Selberg's example (Remark 6.14) no prime asymptotic follows.
Ford [11, Theorem 1]	For each fixed $\nu \in (0,1)$ there is a sequence satisfying $R(\nu)$ (with $A(x) = x$, $g(d) = 1/d$) for which these asymptotic formulas fail for every $k \ge 1$.
Friedlander–Iwaniec [14, Theorem 1]	Type I information at a level $x^{2/3} < D < x$ and their bilinear condition for the Möbius function, under density, regularity and squarefree-support hypotheses, give the prime sum with relative error $O(\log\delta/\log\Delta)$; parity enters through the bilinear condition.
Theorem D	Type I information at the level $Q_a = Ne^{-L^a/2} = N^{1-o(1)}$ with one saving $L^{-1-\varepsilon}$ against the model (51), i.e. (I'_a); parity enters through (54), from Hypothesis RE.

containing $\min(y, N-1)/d + O(1)$ integers, and $\lambda(N-n) = \lambda(d)\lambda(k)$. Modulo each prime $p \le W_a$, the condition $P^-(N-kd) > W_a$ excludes one class of k if $p \nmid d$ and none if $p \mid d$. So the fundamental lemma of sieve theory [24, Theorem 19.1], with weights $\rho^\pm$ for the primes $p \le W_a$, $p \nmid d$, of level $e^{L^{a/4}} = W_a^{L^{a/8}}$, gives $V_a^{-1}\sum_{k\in J_y} \mathbf{1}_{P^-(N-kd)>W_a} = (1+O(Le^{-L^{a/8}}))\min(y,N-1)/\varphi(d) + O(Le^{L^{a/4}})$, the primes $p \mid d$ with $p \le W_a$ supplying $d/\varphi(d)$ up to a factor $1 + O(L/W_a)$. In the part with $\lambda(k)$, replacing $\mathbf{1}_{P^-(N-kd)>W_a}$ by $(1*\rho^+)(N-kd)$ costs at most $\sum_{k\in J_y}(1*(\rho^+-\rho^-))(N-kd)$, bounded in the same way. For a modulus q of ρ^+ put $g = (q, N)$: as q is squarefree and coprime to d, $q \mid N - kd$ means $k = gk'$ with k' in one reduced class modulo q/g, and $\lambda(k) = \lambda(g)\lambda(k')$. As above, $\lambda(m) = \sum_{u^2v=m}\mu(v)$, Lemma 2.14 at the heights t/u^2 with $u \le t^{1/10}$, and the trivial bound for larger u give, for every $P' > 0$,

$$\sum_{q\le t^{1/10}} \max_{(c,q)=1} \max_{t'\le t} \Big| \sum_{\substack{m\le t'\\ m\equiv c \pmod q}} \lambda(m)\Big| \ll_{P'} t(\log t)^{-P'}.$$

Applied at $t = N/(dg)$ to the moduli $q/g \le e^{L^{a/4}} \le t^{1/10}$, with $\log t \ge L^a/4$ and $\sum_{g\mid N}\mu^2(g)/g \ll \log L$, this gives $\ll_{P'} (N/d)L^{1+a/8-aP'}$ for what remains, the factor $V_a^{-1} \ll L^{a/8}$ included. Summing over $d < Q_a$ with P' large, where $Q_a e^{L^{a/4}} = Ne^{-L^a/2+L^{a/4}}$, and passing from $\min(y, N-1)/\varphi(d)$ to $M_N(y;d)$ as at the end of the *Known limitations* below proves the bound, uniformly in y; its constants, as proved here, are ineffective (Siegel's theorem, through Lemma 2.14). In $\sum_{n<N} a_n^*\Lambda(N-n)$ only the $N - n = p^{2j}$ remain, since $\lambda(p^j) = (-1)^j$, and $a_n^* \le 2V_a^{-1}$. So (I'_a), read as information about a nonnegative row in place of Λ, does not imply the asymptotic for $G(N)$ read in the same way, and in Theorem D the parity-sensitive input is (54).

Related reductions. Huang and Li, continuing work of Pan (see [22]), fix $A > 0$ and $0 < \theta_{\mathrm{HL}} < 1$, and assume the Elliott–Halberstam conjecture for the primes at level $N^{\theta_{\mathrm{HL}}}(\log N)^{2A+8}$ together with its variant twisted by the Möbius function, for $\Lambda(n)\mu(N-n)$, at level $N^{1-\theta_{\mathrm{HL}}}$. They obtain $G(N) \ge \mathfrak{S}(N)\big(1 - \prod_{p\nmid N,\, p>2}(1 - \frac{1}{p(p-1)})\big)N + O(N(\log N)^{-A})$ for every even N, and they note that under these hypotheses $G(N) \sim \mathfrak{S}(N)N$ is equivalent to $\sum_{n<N}\Lambda(n)\mu(N-n) = o(N)$ [22, Theorem 1]; they remark that the proof needs their hypotheses only for the residue class $n \equiv N \pmod q$ (the remark after [22, Corollary 1]). For a fixed nonzero even h, Murty and Vatwani prove

the analogous lower bound for $\sum_{n\le x}\Lambda(n)\Lambda(n+h)$ under the analogous hypotheses for the shift h [31, Theorem 1.1]. In Theorem D the split is (59). Its Type I half needs the primes in the class N to moduli up to $Q_a = N^{1-o(1)}$, with the single saving $L^{-1-\varepsilon}$, against the exact model (51). Its other half is the smoothed reflected sum (54), built from Möbius sums of the quotients $(N-p)/b$ with $b < S = e^{L^a}$; Theorem D derives it from Hypothesis RE instead of assuming it. We found no earlier statement of this form.

Known limitations. Here b_0 and $b_\pm$ denote residues, and for $(q,b_0)=1$ we write, as in [13, (1.1)],

$$\sum_{\substack{p\le x\\ p\equiv b_0 \pmod q}} \log p = \frac{x}{\varphi(q)}\big(1+\Delta(x;q,b_0)\big).$$

Friedlander and Granville showed that for every fixed $B>0$ there are arbitrarily large b_0 and x with

$$\sum_{\substack{q<x(\log x)^{-B}\\ (q,b_0)=1}} \Big|\sum_{\substack{n\le x\\ n\equiv b_0 \pmod q}} \Lambda(n) - \frac{x}{\varphi(q)}\Big| \gg_B x$$

[12]; see also [13, p. 32]. Friedlander, Granville, Hildebrand and Maier showed that for $\varepsilon, A>0$ and $x > x_0(\varepsilon,A)$ there is an integer b_0 such that $\sum_{q<Q,\,(q,b_0)=1}\frac{x}{\varphi(q)}|\Delta(x;q,b_0)| > x(\log x)^{-A}$ for all $Q \ge x\exp(-(1-\varepsilon)A(\log_2 x)^2/\log_3 x)$ [13, Corollary 2.2], where $\log_k$ is the k-fold iterated logarithm; the integers produced by their proof grow with x [13, p. 32]. Hypothesis (I'_a) is not contradicted by these results, which reach only the levels $x(\log x)^{-B}$ and $x\exp(-(1-\varepsilon)A(\log_2 x)^2/\log_3 x)$, for residues constructed together with x. Hypothesis (I'_a) concerns the residue class N itself at the lower level $N\exp(-(\log N)^a/2)$, where every progression has more than $\exp((\log N)^a/2)$ terms, against the exact local model (51). The level $Q = x\exp(-(\log x)^a/2)$ lies in the range $x\exp(-c\sqrt{\log x}) < Q \le x(\log x)^{-1-\varepsilon}$ of [13, Theorem A3]; there $\log(x/Q) = (\log x)^a/2$, the exponent $\delta_3(x,x/Q)$ of that theorem is $a+o(1)$, and the lower bound it gives for $x\,|\sum_{q\sim Q_\pm,\,(q,b_\pm)=1}\Delta(x;q,b_\pm)/\varphi(q)|$ is $x\exp(-(1+\varepsilon)(a/2+o(1))(\log x)^a)/\log_2 x$, smaller than $x(\log x)^{-B}$ for every fixed B. We know of no result that contradicts (I'_a). Hypothesis (I'_a) would follow from the Hypothesis of [13, p. 33, (2.17)–(2.18)]: for each fixed $\eta>0$, $|\Delta(x;q,b_0)| \ll_\eta (x/q)^{-(1-\eta)\delta}$ with $\delta = \min\big(\frac12, \log(\log(x/q)/\log\log x)/\log\log x\big)$, for all $q \ge q_0(\eta)$, $(b_0,q)=1$ and $x \ge q(\log q)^{1+\eta}$, read with the implied constant depending only on η. Indeed, for $d < Ne^{-L^a/2}$ and $de^{L^a/4} < y \le N$ one has $\delta \ge a/2$ at $(x,q)=(y,d)$ once N is large, so with $\eta=\frac12$ these y contribute $O(NL\exp(-aL^a/16))$ to $\mathcal{E}'_N(Ne^{-L^a/2})$, while the remaining y, the prime powers, the finitely many $d < q_0(\frac12)$ and the passage from $y/\varphi(d)$ to $M_N(y;d)$ contribute $O_B(NL^{-B})$ for every fixed B, the last by $\big|\#\{k\le v:(k,N)=1\} - v\varphi(N)/N\big| \le 2v^{1-\sigma/2}\mathcal{H}_N + v^{1/2}$ (Rankin's trick, with σ and $\mathcal{H}_N$ as in the proof of Lemma 6.2).

Appendix A. Source contracts

This appendix lists, for each preprint on which a proof in this paper rests, the numbered statements used, with their quantifiers, the version read, and the place where the paper uses them. Hypotheses are restated, not extended; where a proof goes beyond a statement's printed hypotheses, the entry names the place where the extension is proved. No fixed-shift theorem of these papers is used at the shift $-N$. Within an entry the notation (x, $L=\log x$, bands, lists, constants) is the source's, except that a scale which a source calls N is written H. Pages are those of the PDF files unless a printed page is named.

A.1. **Versions.** We read [34] in its version of 30 September 2026 (199 pp.), [33] and [35] in their versions of 24 September 2026 (74 and 68 pp.), and [42], [21] and [28] in version 1 on arXiv (arXiv:2610.40439v1, 2611.12756v1 and 2610.20242v1; 44, 31 and 52 pp.). Version 2 of [42] (12 November 2026) keeps the numbering of its theorems, lemmas and equations; the page numbers below are those of version 1. Version 2 of [28] (19 November 2026, 75 pp.) is revised: most equations from (2.18) on are renumbered, and some results that keep their numbers, such as Propositions 4.2 and 5.3, are restated. Proposition 2.1 and (3.1),

which we use as stated, are the same in both versions. All numbers, pages and quotations of [28] in this paper are those of version 1. The three OpenAI preprints carry no version numbers; the files read have the SHA-256 digests

```
[QRH]  8fe93046f8cf5ef1ba5969c89addc02d76311adc4ee907509ff9cd96f7ec99e7
[PD]   4b54d35506a1928022ac21e6d270d30f6533423ddc5990975270a1e39a87bce7
[WDG]  351692885d112277e969901e75fe56eeee0d263d5d5847d0b980fe2c8c7744c0
```

A.2. **The inputs of Theorem A.** [34, Theorem 1.1, p. 4]**.** "Every finite-order Hecke L-function over $F=\mathbb{Q}(\sqrt{-3})$ has no zero in $\Re s>7/8$. The same holds for every Dirichlet L-function, including $\zeta(s)$. A pole at $s=1$ for a principal character is allowed." Only the second and third sentences are used, as QRH(7/8) of Definition 2.1. *Used in* §3.4, and through it in the major terms of §§3.5, 3.6 and 3.8. The result announced in [1] (§2.1) would also give QRH(1/2), hence QRH(7/8); no step of this paper uses that announcement.

From [33]. Here $W=\exp(L^{0.24})$, $V(W)=\prod_{p\le W}(1-1/p)$, and an integer is rough if it has no prime factor $\le W$.

Lemma 2.1, p. 6. The prime number theorem holds with an error smaller than every fixed negative power of the logarithm; $\sum_{p\le y}1/p=\log\log y+c_M+o(1)$ and $\prod_{p\le y}(1-1/p)\sim e^{-\gamma_E}/\log y$; and for fixed $A,C>0$, uniformly for $r\le(\log y)^C$ and $(a,r)=1$, $\#\{p\le y:p\equiv a\pmod r\}=\mathrm{Li}(y)/\varphi(r)+O_{A,C}(y/(\log y)^A)$. The constants need not be effective. *Used in* §3.2, only for the classical estimates it states (Mertens' theorem gives the band masses there).

Lemmas 2.2–2.5, pp. 6–8. Divisor moments and Dirichlet mean squares; a long prime polynomial; logarithmic phases on progressions; rough integers in long intervals. They enter Theorem A only through the proofs in [33] of the statements below, and each concerns moduli at most a fixed power of L.

Lemma 2.6, p. 9 = [35, Lemma 2.9]**, pp. 8–9 (block sieve).** Let finitely many objects carry nonnegative weights, with a bad condition at each designated prime $p\le z$, and suppose that the weight on which the conditions at all primes dividing a squarefree d hold is $Xg(d)+r(d)$, where $X\ge 0$, g is multiplicative, $0\le g(p)\le 1-\eta_0$ and $\sum_{w<p\le w^2}g(p)\le C_0$ $(w>1)$. Then for every sufficiently large even h the weight avoiding all designated conditions is $X\prod_{p\le z}(1-g(p))(1+O(e^{-h}))+O\big(\sum_{d\le z^{4h+2},\ d\text{ squarefree}}|r(d)|\big)$, over designated primes only, with constants depending only on η_0,C_0 and uniform in h. *Used in* §3.3, with $g(p)=1/p$ for $y<p\le z$, $p\nmid N$.

Theorem 3.1, p. 10 (not applied). The one-sided marked Type II estimate at $mn-1$, for the bands $\mathcal{P}_i=\{p:\exp(L^{a_i})\le p\le\exp(2L^{a_i})\}$, $0.1<a_1<\cdots<a_K<0.2$ (3.2), with K large in terms of the fixed parameters and not of the a_i. §3.5 transports its proof (the next entry) to $N-mn$, on the bands $\{p\in\mathcal{P}_i:p\nmid N\}$, and checks that [33, §§3–4] use the bands only through properties that these subsets inherit: disjointness, the window $[\exp(L^{a_i}),\exp(2L^{a_i})]$, two-sided bounds on the masses V_i, and identities valid for any finite set of primes.

Lemmas 3.2–3.4, Proposition 4.1 and (3.12)–(4.64), pp. 11–33 (the part of the proof of Theorem 3.1 in §§3–4). Here $J=\lfloor L^{0.01}\rfloor$, $R=\lceil L^{0.5}/2\rceil$ (3.12), $U=d_0YH_m$ and $V=H_n$ (3.13). Lemma 3.2: for any fixed lists, the set of $r\in\mathbb{R}/\mathbb{Z}$ failing goodness has measure at most $\exp(-cL^{0.1})$, for some $c>0$ and all large x. Lemma 3.3 (pp. 15–17): there is $c_\delta>0$ such that for squarefree $S\le\exp(C_KL^{0.98})$, $g_0\in\mathrm{SL}_2(\mathbb{Z}/S\mathbb{Z})$ and intervals $I_1\subset[1,16]$, $I_2\subset[1,2]$, $I_3\subset[0,1]$, the number of primitive roots (u,v) with $u/U\in I_1$, $v/V\in I_2$, $r=c/u\in I_3$ and $g\equiv g_0\pmod S$, where $g=\left(\begin{smallmatrix}u&c\\v&d\end{smallmatrix}\right)$, $ud-vc=1$, $0\le c<u$ (3.22), is $UV|I_1||I_2||I_3|/(\zeta(2)|\mathrm{SL}_2(\mathbb{Z}/S\mathbb{Z})|)+O(UVx^{-c_\delta})$, uniformly in the intervals and in g_0; its proof gives (3.24), $|K_m(h,k)|\ll m^{3/4+o(1)}(h,k,m)^{1/4}$. Lemma 3.4 (pp. 17–18): in the normalized path expansion one may replace the root average by $\zeta(2)^{-1}\,d(u/U)\,d(v/V)\,dr$ on $[1,16]\times[1,2]\times[0,1]$ and by independent uniform lines in $\mathbb{P}^1(\mathbb{F}_p)$ at the band primes, retaining all box and good-state conditions, with error $O_A(L^{-2AR})$ for every fixed A; likewise, with polynomial logarithmic coefficient bounds, for one edge and for failure of either endpoint's goodness. Proposition 4.1 (p. 19): for every fixed $E_0>0$ one can choose A_0 and then K, in terms of E_0, q and the fixed parameters of Theorem 3.1 and not of the a_i, so that $(UV)^{-1}\sum_{P\in\Omega\text{ primitive}}\langle u_P,(AA^*)^Ru_P\rangle\le L^{-2RE_0}$; consequently $|Q_Y-Q_Y^{\mathrm{maj}}|\ll_A XYL^{-A}$ for every fixed A. *Used in* §3.5: the root count for $N-mn$ at every fixed δ rests on (3.24) and the counting scheme (3.25)–(3.29); the determinant comparison transports (3.12)–(4.64) to lattices of index N; and §3.8 does the same for odd N.

From [35].

Definitions 3.1–3.3, pp. 10–12. Small and big prime groups in $\exp([L^a,L^b])$ and $\exp([L^c,L^d])$, $0<a<b<c<d<0.47$, with $c_0\log L\le|S|,|B|\le C_0\log L$ and $v_-\le V_g\le v_+$; $M=J+\ell$, $|C_g|\le m$, C_g empty on small groups; laws $\mu_g(p)=1/(V_gp)$ and vertex weight $\prod_g V_g^{-M}$; shifts $n'=n+kD$ with $0<|k|\le L^{C_2}$; coefficient mass $\le L^{C_1}$. Only the displacement restriction is changed, in §3.6.

Lemma 3.4 and Theorem 3.5, p. 12. Absolute row and column sums are at most $L^{C_{\rm abs}}$ with $C_{\rm abs}$ independent of J. For every fixed $E>0$ there is $E_{\rm id}$, depending only on E and the data of [35, Theorem 3.1], such that if $\sup_\Theta\|T(\Theta)\|\le L^{-E_{\rm id}}$, then for every sufficiently large fixed J, with $R=\lceil L^{0.52}\rceil$, every fixed $\gamma>0$ and $I=[X,2X)$, $x^\gamma\le X\le x^{1/\gamma}$: $|I\cap\mathbb{Z}|^{-1}\sum_{n\in I}\langle u_n,(AA^*)^Ru_n\rangle\le L^{-2RE}$ for large x. Theorem 3.5 is printed for $|k|\le L^{C_2}$; §3.6 proves its extension to displacements of size N, re-checking each use of the displacement.

Lemmas 3.6–3.10, pp. 13–23. Compatible paths have root average equal to the independent residue expectation times $1+O(\exp(-An_e\log L))$, incompatible ones 0 (3.6); the exact memory identity, whose compatibility uses $p\nmid k$ (3.7); memory row and column bounds L^{C_a}, C_a independent of J, with truncation at $\lceil L^2\rceil$ costing $\exp(-\Omega(L^2))$ (3.8); the dirty norm $\le 2L^{C_a}(B/\mathcal{D})^{1/2}$, using that k is invertible modulo p (3.9); and the clean norm $\le 2L^{-E_{\rm id}+C_s}$, C_s independent of m,C_1,J, with the order of choices (3.37) (3.10). *Used in* §3.6 (large-displacement root moment), where $p\nmid k$ comes from $p\nmid N$. Corollary 3.11 (p. 26) is not used; §3.6 replaces its blocks of size $\max|kD|$ by progression blocks.

Theorem 4.1, p. 27. For fixed $E_{\rm id}>0$ and $C_4\ge0$ there are fixed m, J_0, C_1, depending only on these, ℓ and the group bounds, such that for every fixed $J\ge J_0$ there is a family consisting of the raw pattern ($C_g=\emptyset$ for all g, $K_0=1$) and signed comparison patterns with $C_g=\emptyset$ ($g\in S$), $|C_g|\le m$ ($g\in B$) and $\sum_\nu\sup|K_\nu|\le L^{C_1}$, for which the symmetrized ideal operator satisfies $\sup_{\Theta\in\mathbb{R},\ |\zeta|\le L^{C_4}}\|T(\Theta)\|_{2\to2}\le L^{-E_{\rm id}}$ for large x; the defining parameters do not depend on J. *Used in* §3.6, for the original interval banks; the banks with the primes dividing N deleted are treated there as a perturbation.

From [42]. Its Theorem 1.1 (fixed h) is not used.

Lemmas 2.1–2.3, pp. 8–9. Lemma 2.1: for $n\asymp x$ and $1\le A\le\exp(\sqrt{L})$, $M_{j,S}(n/A)/V=V^{j-1}L^{j-1}(1+O_j(L^{-1/2}))/(j-1)!$. Lemma 2.2: for fixed a,r,c and $0<\rho<C$, uniformly for $x^\rho\le T\le x^C$, with $\theta=L^{-2/5}$: $\sum_{n\le cT}d_a(n)^rn_S^{r\theta}\ll TL^{a^r-1}$ and $\sum_{n\le cT}d_a(n)^r\mathbf{1}_{n_S>E}\ll TL^{a^r-1}e^{-rL^{1/10}}$. Lemma 2.3: for fixed M,r and $H=x^{1+o(1)}$, $\sum_{n\asymp H}\sum_{\mathbf{p}\in\mathcal{L}_M(n)}w_M(|\alpha_{\mathcal{P}}(n)|^r+|\beta_{\mathcal{P}}(n)|^r)\ll HL^{C_r}$, $w_M=\prod_gV_g^{-M}$, with C_r and the constant independent of M, of the active family and of the lists (the threshold may depend on M), for active sets that are unions of the banks of (2.1), p. 6. *Used in* §3.2 (moment bounds and deletions) and §3.6. For banks with the primes dividing N deleted, Lemma 2.3 is used through its proof (p. 9), which needs only three facts that such subsets inherit; Lemma 3.18 in §3.6 proves this.

Lemma 5.1, p. 20 (not applied). Smooth twisted prime sums $P^{-1}\sum_p\chi(p)w(p/P)p^{it}$ for $P\ge\exp(L^a)$, $q\le L^B$, $|t|\le x^C$, with main term for principal χ and error $O(P^{-1/16}L^D)$. §3.4 re-proves this for moduli up to a power of N under QRH(7/8).

Lemma 5.2, p. 21. Fix $C>0$, integers $j,K\ge1$ and $D\ge0$; let $R=x^{1+o(1)}$, $q\le L^C$ be S-smooth, and for every S-smooth A let b_A be q-periodic with $|b_A|\le1$ and ψ_A supported in one fixed compact subinterval of $(0,\infty)$ with $\sup_A\|(u\partial_u)^m\psi_A\|_\infty\ll_m L^{C_m}$. With $S_A(\tau)=\sum_{(v,S)=1}b_A(v)\psi_A(Av/R)(Av)^{i\tau}\{d_j(v)-M_{j,S}(v)/V\}$, there are $c,C'>0$ such that $R^{-1}\sum_A d_K(A)|S_A(\tau)|\ll L^{C'}\exp(-cL^{1/10})$ uniformly for $|\tau|\le L^D$, the sum over all S-smooth A. *Used in* §3.6 (comparison major arcs, the principal exception), with $q=1$ and $j=K=k$.

Lemma 6.2, pp. 25–26. For $O(J)$ disjoint sets of primes with $V_g\asymp1$, independent products D_1,D_2 of at most m draws of law $1/(V_gp)$ per bank satisfy $\mathbb{P}(D_i=n)\le L^C/n$, and for $|b_i|\le1$ supported on $[H_i,2H_i)$, $(a,r)=1$ and $|\theta-a/r|\le r^{-2}$: $|\mathbb{E}b_1(D_1)b_2(D_2)e(\theta D_1D_2)|^2\ll L^C\big(\frac1r+\frac1{H_1}+\frac{\log2r}{H_2}+\frac{r\log2r}{H_1H_2}\big)$, C depending only on m and the implied bounds. *Used in* §3.6 (comparison minor arcs).

Lemma 7.1, p. 36; Theorem 7.2, p. 37, with (7.1)–(7.14). Each comparison kernel splits as $K_\nu=\sum_{j\in I_\nu}c_{\nu j}a_{\nu j}(d_1)b_{\nu j}(d_2)u_{\nu j}(\mathbf{p})v_{\nu j}(\mathbf{q})$ with factors of modulus ≤1, endpoint factors using only unshared big labels, and $\sum_{\nu\ne0}(|I_\nu|+\sum_j|c_{\nu j}|)\le L^C$, C independent of M. Theorem 7.2 (fixed h) is used only as a template: its clipping proof and the compression (7.14), $A=P_MA^{\rm rep}P_M$. *Used in* §3.6 (ideal comparison, shared-label comparison and large-displacement pairing).

From [21]. Its Theorem 1.1 (fixed h) is not used at $h=-N$; its Lemma 4.1 and Proposition 4.2 are replaced by §§3.5–3.6.

Lemmas 2.1–2.2, p. 4. For $n \asymp x$ with $n_S \le E$, $D_k(n) = D_k^0(n)(1+O(L^{-1/2}))$; for $n \le x^2$, $0 \le D_k(n) \ll L^{k-1}\tau_k(n)$ and $D_k(n) \ll D_k^0(n)$. For fixed integers $a, r \ge 1$ and constants $c > 0$, $0 < \delta \le C$, uniformly for $x^\delta \le H \le x^C$, $\sum_{n\le cH} \tau_a(n)^r n_S^{rL^{-2/5}} \ll HL^{a^r-1}$ and hence $\sum_{n\le cH} \tau_a(n)^r \mathbf{1}_{n_S>E} \ll HL^{a^r-1}\exp(-rL^{1/10})$. *Used in* §3.2.
(3.2), p. 6, and Lemmas 3.1–3.3 and Proposition 3.4, pp. 6–10. For fixed j, $(b,q) = 1$ and smooth compactly supported ψ, $\sum_{m\equiv b\,(q)} \tau_j(m)\psi(m/Y_0) = \varphi(q)^{-1}\operatorname{Res}_{s=1}\big[\zeta(s)^j \prod_{p\mid q}(1-p^{-s})^j Y_0^s \widehat{\psi}(s)\big] + O(Y_0^{1/2}q^{j/2})$. Lemmas 3.1–3.3 (fixed h) give the small-modulus remainder up to x^{c_0}, the sieve coefficients ($P^-(r) > y$, $r \le x^{c_0/2}$) and the positive model main term. Proposition 3.4: for a sufficiently large even s, $\gamma = c_0/(8s+8)$ and $z = x^\gamma$, the normalized sifted sum (3.10) has $\limsup_{x\to\infty} \ll_{k,h,\Phi} e^{-s}/\gamma$; "The implied constant is independent of s, K and the fixed choice of auxiliary prime sets. We take the limit with all these parameters fixed" ($\Phi \ge 0$ by the convention of p. 2). *Used in* §3.3, where the reduced classes, the multiplicities of gcd's and the local densities are redone for $N-u$, and the independence of the constant is proved for the reflected sum.
(5.18)–(5.19), pp. 26–27. The exact identities $\mathbf{1}_{\sum_g I_g \ge r_0} = \sum_{r=r_0}^{n_b}(-1)^{r-r_0}\binom{r-1}{r_0-1}\sum_{|T|=r}\prod_{g\in T} I_g$ and $\widetilde{a}^{*j}(v) = j^r B(v)$, $B(v) = \widetilde{a}^{*j}(v_{\mathcal{P}'})$, the second for squarefree selected single-prime records, r the number of selected banks; total coefficient mass $\le 3^{n_b}$. *Used in* §3.7.
Proposition 6.2 with (6.4) and Remark 6.3, pp. 28–29, and (2.7), p. 5. For each fixed collection of sets defining $\mathcal{W}$, the normalized prime sum of (6.4) has $\limsup_x \ll_{k,h,\Phi} e^{-s}/\gamma$, and "The implied constant depends neither on K nor on the exponents defining the prime sets"; the constants of (2.7), $0 < c_{K,k} \le m_{\mathcal{W}} \le C_{K,k}$ and $\langle(\mathcal{W}/m_{\mathcal{W}})^2\rangle_* \le C_{K,k}$, "are independent of the fixed exponents a_i; the threshold for x may depend on them". *Used in* §3.7 (removal of the damping; limits in x, then in the number of mark systems, then in s).

Classical inputs of Theorem A.
The local formulas (L1)–(L3) [40, Proposition 19, Exercise 45(i)–(ii), display (46)]**.** For fixed $C, \varepsilon > 0$, uniformly in $q \ge 1$, in non-principal χ of period q (primitive or not), in $t_0 \in \mathbb{R}$ and in $s = \sigma + it$ with $\varepsilon \le \sigma \le C$: (L1) $L(\cdot,\chi)$ has $O_\varepsilon(\log(q(2+|t_0|)))$ zeros, with multiplicity, in $\{\varepsilon \le \sigma \le 1,\ |t-t_0| \le 1\}$; (L2) $-L'/L(s,\chi) = -\sum_{|s-\rho|\le\varepsilon/2}(s-\rho)^{-1} + O_{C,\varepsilon}(\log(q(2+|t|)))$, over the zeros ρ of $L(s,\chi)$; (L3) $-\zeta'/\zeta(s) = -\sum_{|s-\rho|\le\varepsilon/2}(s-\rho)^{-1} + (s-1)^{-1} + O_{C,\varepsilon}(\log(2+|t|))$; and $L(s,\chi) = \zeta(s)\prod_{p\mid q}(1-p^{-s})$ for principal χ modulo q. Versions of (L1)–(L3) for ζ and for primitive non-principal χ, with the zeros $|\gamma - t| \le 1$ in place of $|s-\rho| \le \varepsilon/2$ (and, in the formulas, s at distance at least $1/2$ from the trivial zeros), are [24, Lemma 8.2 and Lemma 11.4]. *Used in* §3.4, with $\varepsilon = 1/16$ and $C = 2$, where under QRH(7/8) the zero sums are empty for $\sigma \ge 15/16$; (L1) is not needed.
[20, New Theorem 5, printed p. 377]**.** Let $k \ge 1$, $Q_1, \dots, Q_k \in \mathbb{Z}[X]$ primitive, $Q = Q_1 \cdots Q_k$, $R_1, \dots, R_r$ the irreducible factors of Q, $g = \deg Q$, $\alpha, \delta \in (0,1]$, $A, B \ge 1$, $0 < \varepsilon < \alpha/(50g(g+\delta^{-1}))$ and $F \in M_k(A,B,\varepsilon)$. Then, uniformly in $x \ge C_0\|Q\|^\delta$ and $x^\alpha < y \le x$,

$$\sum_{x<n\le x+y} F(|Q_1(n)|,\dots,|Q_k(n)|) \ll y \prod_{g<p\le x}\Big(1-\frac{\rho(p)}{p}\Big)\sum_{n_1\cdots n_r\le x}\frac{\widetilde{F}(n_1,\dots,n_r)\,\breve{\rho}_R(n_1,\dots,n_r)}{[n_1\kappa(n_1),\dots,n_r\kappa(n_r)]},$$

with C_0 and the implied constant depending at most on g, α, δ, A, B. Here $\breve{\rho}_R(a_1,\dots,a_r)$ counts the n modulo $[a_1\kappa(a_1),\dots,a_r\kappa(a_r)]$ such that, for every i, $a_i \,\|\, R_i(n)$, and $p \nmid R_i(n)$ whenever $p \mid \prod_{j\ne i} a_j$ and $p \nmid a_i$ (erratum, (0·1)–(0·2), p. 375); $\kappa(n)$ is the squarefree kernel, and the rest of the notation is that of the original paper. The corrected density $\breve{\rho}_R$, not that of the original Theorem 5, is used. *Used in* §3.3 (positive envelopes, for two linear forms of resultant $\pm N$, one of negative slope) and §3.8.

A.3. **The inputs of Theorems B–D from** [28]**.** The version read is arXiv:2610.20242v1, submitted 14 October 2026; its first page is dated 15 October 2026. In [28], $L = \log x$, $Z = \exp(L^{0.24})$, $U = V = x^{1/5}$ and $R_Z(n) = \mathbf{1}_{P^-(n)>Z}$ (2.1), and we write $\mathfrak{N}$ for their N. They use C_1 for two exponents: the one in the proof of Lemma 4.1, written C_1^{LWW} in Definition 2.5, and the one in §5.3. Theorems B–D use [28] only as follows.
Used as stated: Proposition 2.1, p. 4. Restated as Proposition 2.2, with its proof recalled. *Used in* §§4.3–4.4 (Theorem B, with $F = G_N\mathbf{1}_{(x,y]}$, $y \in (x, 2x]$, so that $\|F\|_\infty \le 1$), §5.2 (the same for Theorem C), and §§6.6–6.8 (Theorem D, on each block, with the target function of §6.6 and the Type I forms of Remark 2.3).

Used as stated: (3.1), p. 10. Restated as Lemma 2.14; [28] cite it to [32, Proposition 3]. *Used in* §6.6, through Lemma 2.15.

Written out, with [28] **as template: the proof of Proposition 3.1, pp. 10–14.** The inclusion–exclusion for $R_Z(k)$ truncated at $R=\lceil C\log L\rceil$ prime factors, with $C\log(C/e)>A+2$ (pp. 11–12), the bound (3.3), the factorisation (3.4), the split of da at $x^{1/20}$, Bombieri–Vinogradov for μ at the moduli $2qr$ grouped by the product qr with the weight $\tau(qr)$ (pp. 12–13), and Rankin's bounds (3.5) (p. 13). Proposition 3.1 itself, which concerns $\mu(n+2)\mathcal{W}(n+2)$, is not used. *Used in* §4.3 (with a prime cofactor, and Lemma 2.13 for primes), hence §5.2, and §6.6 (with Lemmas 2.15 and 2.16).

Written out, with [28] **as template: §4.1 and Lemma 4.1, pp. 14–16.** The smoothing of $\mathbf{1}_{x<mn\le 2x}$ with transitions of width L^{-B}, the Mellin separation over $|v|\le L^{2B}$, the dyadic split, the expansion of the band weight with removal of the terms with $(a,h)>1$ (4.3), and Cauchy's inequality in h, (4.4)–(4.5). Lemma 4.1: "In (4.5), the total absolute contribution of terms for which a,b share a prime is $O_{A,K}(XYL^{-A})$, for every fixed $A>0$"; its proof uses (4.6) for odd $\ell\asymp Y$, and an exponent $C_1^{\mathrm{LWW}}>0$ that "depend[s] only on the original coefficient-size exponent, not on K". *Used in* §4.4, hence §5.2, and §6.7, at $N-mn$, where Lemma 4.1 is re-proved with ℓ a product of band primes coprime to N in place of an odd ℓ; C_1^{LWW} enters c_{poly} in Definition 2.5.

Models for Hypothesis RE: (4.5), (4.8), (4.9), Propositions 4.2 and 5.3. The square Q_Y (4.5), the determinant equation $bmn-ars=2(a-b)$ (4.8), and the arcs $\mathfrak{M}_1$, the cutoff ψ_1, the kernel H_{-2} and Q_Y^{maj} (4.9), with the pairs (a,b) sharing a prime included (pp. 15–16). Proposition 4.2 (p. 17) gives $|Q_Y-Q_Y^{\mathrm{maj}}|\ll_A XYL^{-A}$ for $M,\mathfrak{N}\ge x^{\delta}$, $M\mathfrak{N}\asymp x$ and coefficients supported on sifted integers in subintervals of their dyadic intervals with $|\alpha_m|,|\beta_n|\le L^C$, after choosing A_0 and then K, independently of the a_i. Proposition 5.3 (p. 42) gives $|Q_Y^{\mathrm{maj}}|\ll_D XYL^{-D}$ for every fixed D, for $\alpha_m=\mu(m)R_Z(m)$ and $|\beta_n|\le L^C$ supported on Z-sifted integers, $M,\mathfrak{N}\ge x^{\delta}$, also with Mellin factors, interval restrictions or fixed smooth cutoffs. *Used in* Definition 2.9 and Remark 2.11 only. Neither proposition is used in the proofs of Theorems B–D, nor are the operator argument of §§4.2–4.7 and Appendix A, Proposition 2.2 (p. 5), [30, Lemma 3.9(ii)] or [33], through which [28] prove them; Hypothesis RE takes their place.

The order of choices and the constants. "choosing $A_0>G+3$ and then $K>200(G+3)$" (before (4.51), p. 31); "Take $G=E_0+5$" (proof of Proposition 4.5, p. 32); "Choose $E_0>A+4C+2$ first, and then choose A_0,K" (p. 39); in §5.3 (p. 45), $D_*>2(A_*+C_1)$ and $E_0>D_*+4C+2$, with C_1 there bounding the dyadic and Mellin factors. The exponents C_3 and C_4 of (4.74) and (4.76) (p. 38) are fixed and depend on the coefficient bounds; [28] print no values for C_1^{LWW}, C_3, C_4. *Used in* Definition 2.4, the hand-offs to Hypothesis RE in §4.4, §5.2 and §6.7 (Remark 2.12(i)), and c_{poly} in Definition 2.5.

Displays (4.54)–(4.62), pp. 33–35. The rank decomposition of the moment in the proof of Proposition 4.5. *Used in* Appendix B only, as reference points for the evidence recorded there.

Appendix B. Polylogarithmic bands: the evidence for RE($\mathfrak{B}_{\mathrm{poly}},7/8$)

Theorem C assumes Hypothesis RE($\mathfrak{B}_{\mathrm{poly}},7/8$) of §2.3. This appendix records the evidence for it, and where each entry of the band condition $c_1>c_{\mathrm{poly}}$ of Definition 2.5 comes from. It is evidence, not a proof (Remark 2.12(v)). Estimates of the form (RE1) and (RE2) are modelled on [28, Propositions 4.2 and 5.3], at the shift $+2$ and with bands of primes of size $\exp(L^{a_i})$ (Remark 2.11). Below we record each place where the size of the band primes enters these arguments, with the condition it imposes at $\mathfrak{B}_{\mathrm{poly}}$. Steps marked † occur only in the reflected setting; for them we state the bound and the condition it imposes, without proof. At two places the printed argument does not carry over: the rank split in the moment bound (§B.1) and the second good-state test (§B.2). For these we give statements with complete proofs. The other places are bookkeeping and are listed in Table 4 (§B.3). That no other use of the band scale exists is a census, not a proof. The deduction of Theorem C from RE($\mathfrak{B}_{\mathrm{poly}},7/8$) (§5.2) does not use this appendix.

Conventions. Throughout this appendix $L=\log x$ (this is L_x of §2.3), a datum for $\mathfrak{B}_{\mathrm{poly}}$ at (A,C) is fixed (Definitions 2.4 and 2.5), $S_i:=c_1+\cdots+c_i$ for $0\le i\le K$ (so $S_0=0$, and S_K is as in Remark 2.6(iii)), $J=\lfloor L^{0.01}\rfloor$ and $R=\lceil L^{1/2}/2\rceil$ as in [28, (4.10)]. Equation numbers of the form (4.n) and (5.n) refer to version 1 of [28]. Only the dyadic Y with $\eta(a/Y)\ne 0$ for some $a\in\mathcal{A}^N$ matter, since for the others both squares of Definition 2.9 vanish. As $\eta(a/Y)\ne 0$ forces

$a/4 < Y < a$, and $L^{S_K} < a \le L^{2S_K}$, these Y satisfy

$$S_K \log L - \log 4 < \log Y \le 2S_K \log L, \qquad \text{that is,} \qquad L^{S_K}/4 < Y < a \le L^{2S_K}. \tag{71}$$

The upper bounds control size bookkeeping, and the lower bound is used in the Y-dependent estimates, including those that feed the edge bound (4.51).

B.1. **The rank split.**

Lemma B.1 (Rank split)**.** *Let G, A_0, $C^\sharp$, $b_\sharp = 5 + 2G + 2C^\sharp$ and $h_\sharp = C^\sharp + E_0 + 4$ be as in Definitions 2.4 and 2.5, let c_1 be real and $C_{\mathrm{hi}}, C_{\mathrm{lo}} \ge 1$, and put $\Upsilon := 4C_{\mathrm{hi}}(13K)^2$ and $\Theta := \max\{\log(2\Upsilon), 2\log C_{\mathrm{lo}}\}$. Let $L > 1$, $\ell := \log L$, let $R \ge 1$ be an integer with $2R \ge L^{1/2}$, and let $1 \le Q \le 13KL^{5/2}$ and $0 \le \Delta \le 4L^{-c_1}$. Let $\mathcal{S}$ be a finite set with maps $r\colon \mathcal{S} \to \mathbb{Z}_{\ge 0}$ and $w, I\colon \mathcal{S} \to [0,\infty)$ such that*

(S1) $\sum_{r(s)=k} w(s) \le Q^{2k}$ *for every integer $k \ge 0$;*
(S2) $I(s) \le L^{C^\sharp(2R+1)}(C_{\mathrm{hi}}\Delta)^{r(s)}$ *for every s;*
(S3) $I(s) \le 2C_{\mathrm{lo}}^{2R+1} L^{-G(2R-2r(s))+2C^\sharp r(s)}$ *for every s with $r(s) < R$,*

and put $\Sigma := \sum_s w(s)I(s)$. Let $0 < \gamma < 1/2$, put $\varpi_{\mathrm{hi}} := \gamma(c_1 - 5) - h_\sharp$ and $\varpi_{\mathrm{lo}} := 1 - \gamma b_\sharp$, assume that both are positive, and put $\varpi := \min\{\varpi_{\mathrm{hi}}, \varpi_{\mathrm{lo}}\}$ and $r_0 := \lceil 2\gamma R \rceil$. If $L \ge \max\{\exp(6\Theta/\varpi), (6C^\sharp/\varpi_{\mathrm{hi}})^2\}$, then

(i) $\Sigma_{\mathrm{hi}} := \sum_{r(s)\ge r_0} w(s)I(s) \le L^{-2R(E_0+4+\varpi_{\mathrm{hi}}/2)}$;
(ii) $\Sigma_{\mathrm{lo}} := \sum_{r(s)< r_0} w(s)I(s) \le L^{-2R(E_0+4+\varpi_{\mathrm{lo}}/2)}$;
(iii) $\Sigma \le 2L^{-2R(E_0+4+\varpi/2)} \le L^{-2R(E_0+4)}$.

The proof uses of the datum only $b_\sharp = 5 + 2G + 2C^\sharp$, $h_\sharp = C^\sharp + E_0 + 4$, $C^\sharp \ge 6$, $E_0 > 0$ and $K \ge 1$; in particular $b_\sharp \ge 27$, $h_\sharp > 10$ and $\log(2\Upsilon) \ge \log 1352 > 7$.

Proof. Step 0. Since $\varpi_{\mathrm{lo}} < 1$, we have $\varpi < 1$, so $\ell \ge 6\Theta/\varpi > 42$ and $L > 2$. Next, $L^{1/2} = e^{\ell/2} \ge e^{3\Theta/\varpi} \ge e^{21/\varpi} \ge 12/\varpi$, so $2R \ge 12/\varpi$ and $2R\ell \ge 12/\varpi$. From $\varpi_{\mathrm{hi}} > 0$ and $\gamma < 1/2$ we get $c_1 - 5 > h_\sharp/\gamma > 2h_\sharp > 20$. Finally $2\gamma R \le r_0 < 2\gamma R + 1$, so $r_0 \ge 1$.

Step 1 (high rank). For $r(s) \ge r_0$, (S2) gives $w(s)I(s) \le w(s)L^{C^\sharp(2R+1)}(C_{\mathrm{hi}}\Delta)^{r(s)}$. Grouping by rank and using (S1), $\Sigma_{\mathrm{hi}} \le L^{C^\sharp(2R+1)} \sum_{k\ge r_0} y^k$ with $y := C_{\mathrm{hi}}Q^2\Delta$. The ranges of Q and Δ give $y \le y_0 := \Upsilon L^{5-c_1}$, and $(c_1 - 5)\ell > 20\ell > \Theta \ge \log(2\Upsilon)$ gives $y_0 < 1/2$. Hence $\sum_{k\ge r_0} y^k \le y_0^{r_0}/(1 - y_0) \le 2y_0^{r_0} \le 2y_0^{2\gamma R}$, and

$$\begin{aligned}\log \Sigma_{\mathrm{hi}} &\le \log 2 + C^\sharp(2R+1)\ell + 2\gamma R \log \Upsilon - 2\gamma R(c_1 - 5)\ell \\ &= -2R(E_0 + 4 + \varpi_{\mathrm{hi}})\ell + \big[\log 2 + C^\sharp \ell + 2\gamma R \log \Upsilon\big],\end{aligned}$$

because $C^\sharp - \gamma(c_1 - 5) = C^\sharp - h_\sharp - \varpi_{\mathrm{hi}} = -(E_0 + 4) - \varpi_{\mathrm{hi}}$. Each bracketed term is at most $2R\ell\varpi_{\mathrm{hi}}/6$: the third because $\gamma < 1$ and $0 < \log\Upsilon < \Theta \le \varpi\ell/6 \le \varpi_{\mathrm{hi}}\ell/6$; the second because $2R \ge L^{1/2} \ge 6C^\sharp/\varpi_{\mathrm{hi}}$; the first because $\log 2 < 2 \le 2R\ell\varpi_{\mathrm{hi}}/6$, as $2R\ell \ge 12/\varpi$. This gives (i).

Step 2 (low rank). If $r(s) < r_0$, then $r(s) \le r_0 - 1 < 2\gamma R < R$, so (S3) applies, and with (S1)

$$\Sigma_{\mathrm{lo}} \le \sum_{k=0}^{r_0-1} Q^{2k}\, 2C_{\mathrm{lo}}^{2R+1} L^{-G(2R-2k)+2C^\sharp k} = 2C_{\mathrm{lo}}^{2R+1} L^{-2GR} \sum_{k=0}^{r_0-1} W^k, \qquad W := Q^2 L^{2G+2C^\sharp}.$$

Since $Q \ge 1$ and $L > 2$, $W \ge L^{10} \ge 2$, so $\sum_{k<r_0} W^k < W^{r_0}/(W-1) \le 2W^{r_0-1}$. Also $W \le (13K)^2 L^{b_\sharp}$ and $0 \le r_0 - 1 < 2\gamma R$, so $W^{r_0-1} \le ((13K)^2 L^{b_\sharp})^{2\gamma R}$. As $G - \gamma b_\sharp = G - 1 + \varpi_{\mathrm{lo}} = E_0 + 4 + \varpi_{\mathrm{lo}}$,

$$\log \Sigma_{\mathrm{lo}} \le -2R(E_0 + 4 + \varpi_{\mathrm{lo}})\ell + \big[\log 4 + (2R+1)\log C_{\mathrm{lo}} + 4\gamma R \log(13K)\big].$$

Each bracketed term is at most $2R\ell\varpi_{\rm lo}/6$: $(2R+1)\log C_{\rm lo} \le 4R\log C_{\rm lo} \le 2R\Theta$ and $4\gamma R\log(13K) < 2R\log((13K)^2) \le 2R\Theta$, with $\Theta \le \varpi\ell/6 \le \varpi_{\rm lo}\ell/6$; and $\log 4 < 2 \le 2R\ell\varpi_{\rm lo}/6$. This gives (ii).

Step 3. $\Sigma = \Sigma_{\rm hi} + \Sigma_{\rm lo} \le 2L^{-2R(E_0+4+\varpi/2)}$, and $2L^{-R\varpi} \le 1$ because $R\varpi\ell \ge 6 > \log 2$. □

Corollary B.2. *Let $c_1 > 5 + h_\sharp b_\sharp$ and $\gamma_* := (h_\sharp+1)/(c_1-5+b_\sharp)$. Then $0 < \gamma_* < 1/b_\sharp < 1/2$, and $\varpi_{\rm hi} = \varpi_{\rm lo} = \varpi_* := (c_1-5-h_\sharp b_\sharp)/(c_1-5+b_\sharp) \in (0,1)$ at $\gamma = \gamma_*$. Hence, under the other hypotheses of Lemma B.1, $\Sigma \le 2L^{-2R(E_0+4+\varpi_*/2)} \le L^{-2R(E_0+4)}$ for every $L \ge L_1 := \max\{\exp(6\Theta/\varpi_*), (6C^\sharp/\varpi_*)^2\}$ and every integer R with $2R \ge L^{1/2}$.*

Proof. $\gamma_*(c_1-5)-h_\sharp = [(h_\sharp+1)(c_1-5)-h_\sharp(c_1-5+b_\sharp)]/(c_1-5+b_\sharp) = (c_1-5-h_\sharp b_\sharp)/(c_1-5+b_\sharp)$, and $1-\gamma_* b_\sharp = [c_1-5+b_\sharp-(h_\sharp+1)b_\sharp]/(c_1-5+b_\sharp)$ is the same number. It is positive by hypothesis, which is the inequality $\gamma_* < 1/b_\sharp$, and it is less than 1 because $-h_\sharp b_\sharp < b_\sharp$. Apply Lemma B.1. □

Proposition B.3. *If $c_1 \le 5 + h_\sharp b_\sharp$, then for every real $L \ge 16b_\sharp^8$ and every integer R with $2R \ge L^{1/2}$ there are Q, Δ in the ranges of Lemma B.1 and data $(\mathcal{S}, r, w, I)$ satisfying* (S1)–(S3) *with $\Sigma > 2L^{-2R(E_0+4)}$.*

Proof. Take $Q := \lfloor 13KL^{5/2}\rfloor$, $\Delta := 4L^{-c_1}$, $k_\flat := \lfloor 2R/b_\sharp\rfloor$, and one element s with $r(s) = k_\flat$, $w(s) = Q^{2k_\flat}$ and $I(s) := \min\{L^{C^\sharp(2R+1)}(C_{\rm hi}\Delta)^{k_\flat},\ 2C_{\rm lo}^{2R+1}L^{-G(2R-2k_\flat)+2C^\sharp k_\flat}\}$. Since $b_\sharp > 2$, $k_\flat < R$, so (S3) applies and holds; (S1) and (S2) hold by construction. As $KL^{5/2} \ge 1$, $Q \ge 12KL^{5/2}$ and $Q^2 \ge 144K^2L^5$. Now $\Sigma = \min\{T_1, T_2\}$ with $T_1 := L^{C^\sharp(2R+1)}(4C_{\rm hi}Q^2L^{-c_1})^{k_\flat}$ and $T_2 := 2C_{\rm lo}^{2R+1}L^{-2GR}(Q^2L^{2G+2C^\sharp})^{k_\flat}$.

$T_1 \ge L^{C^\sharp(2R+1)}L^{(5-c_1)k_\flat}$. If $c_1 \le 5$ the last factor is at least 1; otherwise $0 < c_1 - 5 \le h_\sharp b_\sharp$ and $k_\flat \le 2R/b_\sharp$ give $(c_1-5)k_\flat \le 2h_\sharp R$. Either way $T_1 \ge L^{C^\sharp(2R+1)-2h_\sharp R} = L^{C^\sharp}L^{-2R(E_0+4)} > 2L^{-2R(E_0+4)}$, since $C^\sharp \ge 6$ and $L > 2$.

$T_2 \ge 2\cdot 144^{k_\flat}L^{-2GR+b_\sharp k_\flat}$, and $k_\flat > 2R/b_\sharp - 1$ gives $b_\sharp k_\flat > 2R - b_\sharp$, so $T_2 > 2L^{-2R(E_0+4)}\cdot 144^{k_\flat}L^{-b_\sharp}$. Put $u := L^{1/4} \ge 2b_\sharp^2$. Then $k_\flat > u^2/b_\sharp - 1 \ge u^2/(2b_\sharp)$, so $k_\flat \log 144 > 2.48\,u^2/b_\sharp$; and $u^2 \ge 2b_\sharp^2 u \ge 2e\,b_\sharp^2\log u$, so $2.48\,u^2/b_\sharp \ge 13b_\sharp\log u > b_\sharp\log L$. Hence $144^{k_\flat} > L^{b_\sharp}$ and $T_2 > 2L^{-2R(E_0+4)}$. □

Proposition B.3 is a limit of the bounds (S1)–(S3), obtained by saturating (S1) at one rank and taking I to be the smaller of the two caps. It is not a lower bound for the sums to which the lemma is applied, and it says nothing about the operator method itself.

Remark B.4 (Where the lemma enters). In the proof of [28, Proposition 4.5] the distinct-prime matrix element is expanded by (4.55) into a sum over collections $\mathcal{C}_0$ of disjoint index sets $C \subseteq \mathcal{B}$ with $|C| \ge 2$, with weight $w(\mathcal{C}_0) = \prod_C(|C|-1)!$, rank $r(\mathcal{C}_0) = \sum_C(|C|-1)$ and $I(\mathcal{C}_0) = |\mathcal{I}(\mathcal{C}_0)|$; this sum must be $\ll L^{-2R(E_0+4)}$. (4.56) is (S1) with $Q = |\mathcal{B}|$, and by (4.54) and (4.10), $|\mathcal{B}| \le K(J+1) + 2KR + K(2R+1)\lceil L^2\rceil \le 13KL^{5/2}$ for $L \ge 1$. The display after (4.58) is (S2), and the display before (4.62) is (S3), using $\|b\|^2 \le 2$. Here $C^\sharp = 2K + 3A_0 + 4$ is taken for the exponent C_* of (4.41), together with the fixed enlargement made before the first of these displays; this is admissible since $\lceil L^2\rceil^K(1+L^{3A_0}) \le L^{2K+3A_0+1}$ for large L; the enlargement absorbs the bounds for the two occurrences of b and for the G_j, which are $O_{K,\rho}(1)$ by (4.41) and $\|b\|^2 \le 2$, and so costs at most a factor L^3 per factor $L^{C^\sharp}$ for large L. And $C_{\rm hi}$, $C_{\rm lo}$ denote the constants written $C_{K,\rho}$ there, which are fixed before the band exponents. By (4.35), $b'_p \ge 1 - (2R+1)/(p+1)$, so $\nu_i(p) \le 4/p$ when $p+1 \ge 2(2R+1)$ and $V_i \ge 1/2$; at $\mathfrak{B}_{\rm poly}$, $\Delta \le 4L^{-c_1}$ for large L.

[28] split at the r_0 of (4.59), with $r_0/(2R) \to 0$, using $C_{K,\rho}|\mathcal{B}|^2\Delta \le e^{-L^{0.1}/2}$, which rests on $p_{\min} \ge e^{L^{0.1}}$. At $\mathfrak{B}_{\rm poly}$, $C_{\rm hi}|\mathcal{B}|^2\Delta \le \Upsilon L^{5-c_1}$ is only a fixed power of L: the high-rank sum is small only for $r_0 \gtrsim 2Rh_\sharp/(c_1-5)$, the low-rank sum only for $r_0 \lesssim 2R/b_\sharp$, and the two ranges meet exactly when $c_1 > 5 + h_\sharp b_\sharp$; Proposition B.3 shows that (S1)–(S3) give nothing at or below that threshold. Corollary B.2 with $r_0 = \lceil 2\gamma_* R\rceil$ replaces (4.59)–(4.62). It uses (S3) for $r < r_0$, where $r_0 < 2R/b_\sharp + 1$, and not only for $r = o(R)$; the derivation of the display before (4.62) uses that at

most $2r$ of the operators E_j are modified and that the others satisfy (4.51), and we use it in this range (†). The bounds $\Delta \le 4L^{-c_1}$, $\|b\|^2 \le 2$, (4.41) and (4.51) at $\mathfrak{B}_{\mathrm{poly}}$ are rows of Table 4. The order of choices is $E_0, G, A_0, K, C^\sharp, C_{\mathrm{hi}}, C_{\mathrm{lo}}$; then $c_1 > 5 + h_\sharp b_\sharp$; then γ_* and L_1, which depend only on these; then $L \to \infty$.

B.2. **The second good-state test.** In [28, §4.2] the second good-state test fixes a nonempty $I \subseteq \{1, \dots, K\}$ and $i_* = \max I$; D runs over the products omitting one entry from each list, Z over the products $\prod_{i \in I \setminus \{i_*\}} q_i$ with $q_i \in \mathcal{P}_i$, and each $T_f = \prod_{i \notin I} p_i$, $p_i \in \mathcal{P}_i$, has weight $\prod_{i \notin I} (p_i V_i)^{-1}$. Except for choices of T_f of total weight at most $\exp(-L^{a_{i_*}}/4)$, every pair with $DZ \ne D'Z'$ must satisfy (4.15), $\|(DZ - D'Z')T_f r_P\| \ge 100 \exp(-L^{a_{i_*}})$; the measure estimate after (4.15) uses $\#\{DZ\} = \exp(o(L^{a_{i_*}}))$. At $\mathfrak{B}_{\mathrm{poly}}$ both the number of products DZ and the band primes are powers of L, and we use the same products and weights (bands punctured at N) with the thresholds

(72)
$$\|(DZ - D'Z')T_f r_P\| \ge 100 L^{-c_{i_*}}, \qquad \text{exceptional weight at most } L^{-\tau}, \qquad \tau := 2K + 2G + \varepsilon_\sharp/2.$$

Lemma B.5. *Let $A'' := A + 2 + 4C + 3A_0$, so that $\varkappa = 2.02K + 2G + A'' + \varepsilon_\sharp$, and let the bands be $\mathcal{P}_i = \{p : L^{c_i} < p \le L^{2c_i}\}$, punctured at N or not, with exponents satisfying $c_i \ge 4S_{i-1} + \varkappa$ for $1 \le i \le K$. For each nonempty I and each fixed choice of the lists, the set of $r \in \mathbb{T}$ at which the test* (72) *fails for I, with r in place of r_P, has measure at most $200 \cdot 4^K L^{-A'' - \varepsilon_\sharp/2}$ for $L \ge 1$.*

Proof. There are at most $(J+1)^K \le 2^K L^{0.01K}$ products D, and $\#\mathcal{P}_i \le L^{2c_i}$, so the number of products DZ is at most $N_I := 2^K L^{0.01K + 2S_{i_*-1}}$. For a nonzero integer m, $\{r \in \mathbb{T} : \|mr\| < 100 L^{-c_{i_*}}\}$ has measure at most $200 L^{-c_{i_*}}$, and $m = (DZ - D'Z')T_f$ is nonzero for every pair with $DZ \ne D'Z'$. Let $F(r)$ be the total weight of the T_f for which some pair fails at r. The weights sum to 1, so $\int_{\mathbb{T}} F \le 200 N_I^2 L^{-c_{i_*}}$, and by Markov's inequality $\{F > L^{-\tau}\}$ has measure at most $200 N_I^2 L^{\tau - c_{i_*}} \le 200 \cdot 4^K L^{0.02K + 4S_{i_*-1} + \tau - c_{i_*}}$. The exponent is at most $-A'' - \varepsilon_\sharp/2$ when $c_{i_*} \ge 4S_{i_*-1} + \varkappa$. □

The test enters three estimates. In the bad-angle bound (4.70), applied at precision $A+2$ as in [28, §4.7], the measure is multiplied by $d_0 Y M \mathfrak{N} L^{4C+3A_0}$; this is why A'' appears. In the dirty raw edges (4.47)–(4.48), the exceptional term $\lceil L^2 \rceil^K e^{-cL^{0.1}}$ becomes $\lceil L^2 \rceil^K O(L^{-\tau})$, whose square root is $\ll_K L^{K - \tau/2} = L^{-G - \varepsilon_\sharp/4}$, while the other term $L^{1 - 0.005K}$ is below L^{-G-2} since $K > 200(G+3)$. In the separation step after (4.47), the admissible interval has diameter at most $32/p_* \le 32 L^{-c_{i_*}}$ for $p_* \in \mathcal{P}_{i_*}$, which is less than the separation $100 L^{-c_{i_*}}$, so it contains at most one product DZ, as in [28]. Like the printed test, (72) depends only on the lists and on r_P and is invariant under list permutations, so the operator identities that use these properties are unchanged. For $\mathfrak{B}_{\mathrm{poly}}$ the hierarchy holds because $c_i = 5^{i-1} c_1 = 4S_{i-1} + c_1$ and $c_1 > c_{\mathrm{poly}} \ge \varkappa$. It is sufficient for the union bound above; we do not know whether it is necessary for the test. Adjacent bands, and ratios $c_{i+1}/c_i \le 4$, do not satisfy it, and the argument above says nothing about them.

B.3. **The census.** Table 4 lists the places where the size of the band primes enters the arguments behind (RE1) and (RE2), with the condition each imposes at $\mathfrak{B}_{\mathrm{poly}}$ at precision A and coefficient exponent C. Rows 1–20 follow [28, §§4.2–4.7] at the bands of $\mathfrak{B}_{\mathrm{poly}}$. Rows 21–24 concern the reflected major arcs under $\mathrm{QRH}(\theta)$, modelled on [28, §5]. There one band $\mathcal{P}_e$ is extracted from the marks as in (5.15), and the set $\mathcal{E}$ of characters and unit intervals on which the extracted Dirichlet polynomial exceeds $L^{-A_{\mathrm{s}}}$ is counted as in [28, Lemma 5.2]; the exponent must satisfy $2A_{\mathrm{s}} \ge A + 20A_0 + 2C + 14$, and the off-diagonal terms of the following mean-value step save $\mathfrak{N}^{-(1-\theta)/2}$ (†). "None" means that the band scale enters only through sizes that are smaller at $\mathfrak{B}_{\mathrm{poly}}$ than at the bands of [28]. Steps marked † occur only in the reflected setting. The list is a census, not a proof: a use not phrased through the band scale can be missed.

Remark B.6 (The entries of c_{poly}). Each entry of c_{poly} comes from one row: $5 + h_\sharp b_\sharp$ from row 11 (Corollary B.2); $\varkappa$ from rows 7 and 14 (Lemma B.5); $4K + 6A_0 + 2G + 2$ from row 8; $G + 3A_0 + 2$ from

TABLE 4. Uses of the band scale in the arguments behind $\mathrm{RE}(\mathfrak{B}_{\mathrm{poly}}, 7/8)$. Equation and section numbers are those of version 1 of [28]; †: only in the reflected setting. $p_{\min} > L^{c_1}$ is the least band prime, Y is as in (71), and C_1, C_3, C_4 are C_1^{LWW}, C_3^{LWW}, C_4^{LWW} of Definition 2.5.

	Step	At $\mathfrak{B}_{\mathrm{poly}}$	Condition
1	(4.10)–(4.13): marks, Y	$\log(d_0 Y) \le 2(J+1)S_K \log L = o(L)$	none
2	(4.35)–(4.36); $\|b\|^2$	$\Delta \le 4L^{-c_1}$; $s_i = 1 + O_K(RL^{-c_1})$, so $\|b\|^2 \le 2$	$c_1 > 0.51$
3	puncturing (G1)†	$\sum_{p\mid N,\, p\in\mathcal{P}_i} 1/p \le \omega(N)L^{-c_1} \le 2L^{1-c_1}$	$c_1 > 2$
4	§4.3: root averaging	path mass $\ll L^{0.51+o(1)}$; $\log S \ll c_K L^{0.76}\log L$	none
5	(4.46): clean edges	$Y > L^{2A_0}$; $L^{0.2-A_0}$ becomes $(2S_K \log L + O(1))L^{-A_0}$; $Y^{-0.1}\log Y \le L^{-G-1}$	$S_K \ge 10G + 11$
6	collisions, before (4.46)	$(1 + L^{3A_0})\Delta \ll L^{3A_0 - c_1} \le L^{-G-1}$	$c_1 \ge G + 3A_0 + 2$
7	(4.47)–(4.48): dirty raw	separation by the second test (§B.2)	$c_i \ge 4S_{i-1} + \varkappa$
8	(4.50): dirty comparison	$L^{C_\dagger} \ll L^{2K+3A_0\dagger}$; $p_{\min}^{-1} + Y^{-0.1} \le 2.2L^{-\min(c_1, 0.1S_K)}$	$\min(c_1, 0.1S_K) \ge 4K + 6A_0 + 2G + 2$
9	(4.51): edge bound	rows 5–8; $L^{1-0.005K} < L^{-G-2}$	none beyond $K > 200(G+3)$
10	(4.53): memory	$\log p \le 2c_K \log L$ in the path mass	none
11	(4.54)–(4.62): rank sum	the rank split (§B.1)	$c_1 > 5 + h_\sharp b_\sharp$
12	§4.6: pairings	$\lvert t\rvert \ll Y = o(Z)$; $(d_0 Y)^{1/(2R)} = 1 + o(1)$	none
13	(4.16), (4.70): test (i)	$2(J+1)^K Y^{-1/2} \le 2^{K+2} L^{0.01K - S_K/2}$, at precision $A+2$	$S_K \ge 0.02K + 2(A + 4C + 3A_0 + 2)$
14	(4.70): test (ii)	Lemma B.5	as row 7
15	§4.7: sum over d_0	$O(S_K L^{0.01}\log L)$ intervals, not $O_K(L)$	none
16	(4.73)–(4.76): pads	$1 - w \ll L^{0.02-c_1}$, times XYL^{C_3} and $XYL^{C_4+3A_0}$	$c_1 \ge A + \max(C_3, C_4 + 3A_0) + 1$
17	Lemma 4.1, (4.7); $(a,b) > 1$	$\ll Y^2 L^{-c_1}$ pairs share a prime; each $\ll (X/Y)L^{C_1}$, resp. $XL^{C_4+3A_0}/Y$	$c_1 \ge A + \max(C_1, C_4 + 3A_0) + 1$
18	(4.6) at $\ell \asymp Y$	needs $Y \ll X^{1/4}L$; here $Y \le L^{2S_K}$	none
19	disjoint bands; lists	$J+2$ primes per band; unique factorisation	$c_{i+1} > 2c_i$
20	band primes in $(2, Z)$	$2 < L^{c_1} < p \le L^{2c_K} < \exp(L^{0.24})$	none
21	units mod $k \le L^{A_0}$†	band primes exceed $2L^{A_0}$	$c_1 > A_0 + 1$
22	disjoint major arcs†	$Y > 4L^{3A_0}$	$S_K > 3A_0 + 1$
23	(5.15): extraction†	$O(c_e \log L)$ dyadic blocks of $\mathcal{P}_e$	none
24	Lemma 5.2: large values†	$\#\mathcal{E} \le x^{(1+2A_{\mathrm{s}})/c_e + o(1)}$ against $\mathfrak{N}^{-(1-\theta)/2}$, $\mathfrak{N} \ge x^{1/5}/2$	$c_e > 10(1 + 2A_{\mathrm{s}})/(1-\theta)$

row 6, which also gives row 21; Γ^{LWW} from rows 16–17; and $80(A + 20A_0 + 2C + 15) = 80(1 + 2A_{\mathrm{s}})$ from row 24 at $\theta = 7/8$, with the least admissible A_{s} and $e = 1$. The other conditions are implied. The conditions on S_K in rows 5, 8, 13 and 22 are $S_K \ge 10G + 11$, $S_K \ge 20(2K + 3A_0 + G + 1)$, $S_K \ge 0.02K + 2(A + 4C + 3A_0 + 2)$ and $S_K > 3A_0 + 1$. Since $G < K/200$, $A_0 \le G + 5$ and $A + 4C < G - 7$ (Definition 2.4), and $K > 2000$, each right-hand side is less than $41K < 48K^2 < 6h_\sharp b_\sharp < 6c_1 \le S_K$; here $h_\sharp b_\sharp > 2(C^\sharp)^2 > 8K^2$ and $S_K = c_1(5^K - 1)/4 \ge 6c_1$. Rows 2, 3 and 19 hold because $c_1 > 8K^2$ and $c_{i+1} = 5c_i$. The entry $5 + h_\sharp b_\sharp$ need not be the largest, since $\varkappa$ grows with $\varepsilon_\sharp$ and C_1^{LWW}, C_3^{LWW}, C_4^{LWW} have no printed values, so the full maximum is used. Lower bounds for $2S_K = c_1(5^K - 1)/2$, the exponent in the size of the marks, use only $c_1 > 5 + h_\sharp b_\sharp > 8K^2$ and the lower bound for K. Row 24 is the only row whose condition involves θ. With $\theta = 1/2$ in that condition, which QRH(1/2) would allow (it would follow from the result announced in [1]; §2.1), the entry would be $20(1 + 2A_{\mathrm{s}})$ in place of $80(1 + 2A_{\mathrm{s}})$; the lower bounds for $2S_K$ do not use this entry and would not change.

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