The hedgehog Skyrmion minimises the massive Skyrme energy within the Möbius family

A guided, self-contained account of a computer-assisted proof

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Abstract

For the Skyrme energy with pion mass μ the hedgehog H is expected to minimise the energy among maps of degree one; this is an open problem. We prove a restricted statement. Let $\mathcal{M}$ be the family of maps whose level surfaces are round spheres, not necessarily concentric, carrying the labels of the hedgehog composed with a Möbius transformation. A map of $\mathcal{M}$ is given by three free functions of the level (radius, position and squeeze), and the deformations may be arbitrarily large. For $\mu = 1$ and for $\mu = 0.526$ we prove $E(U) \geq E(H)$ for every $U \in \mathcal{M}$.

The proof is computer-assisted. A calibration by level sets, corrected by a "credit", reduces the claim to one pointwise inequality in three real variables; this inequality is verified by lemmas with explicit bounds and by interval arithmetic at every state of every level up to a last one, and the rest of the map is controlled at once by a far-field argument.

This paper is written to be read. Part I is the complete chain of reasoning, with every analytic step proved and every use of a computer stated as a precise mathematical claim. Part II contains the technical lemmas with their proofs and the exact statements verified by the programs. Part III is the rigorous enclosure of the hedgehog profile. The programs and their logs are available from the author. Maps outside $\mathcal{M}$ are not covered. The work was produced primarily by AI models at the direction of the author, and it has not been verified independently.

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How to read this paper

Three parts. *Part I* (Sections 1–9) is the proof. It can be read from start to finish without the other parts. Each section does one step, begins by saying what the step is for, and proves every analytic statement it makes. Where a computer is used, the text says exactly which mathematical statement the computer establishes, and Section 8 lists these statements with the programs and the logs. *Part II* (Sections 10–15) contains the lemmas with explicit bounds that are used to verify the pointwise inequality, with complete proofs, the description of the interval computations, and the technical part of the far-field argument. *Part III* (Section 16) is the computer-assisted enclosure of the hedgehog profile, on which everything else rests.

A reader who wants the idea should read Section 1.3 (one page) and then Sections 2–5. A reader who wants to check the proof needs all three parts, and the programs.

Three kinds of statements. Every theorem, proposition and lemma carries one of these labels.

PROVED — The statement is proved in this paper by a written argument. If the argument uses a CERTIFIED statement, that statement is named in the hypothesis or in the proof.

CERTIFIED — The statement is about explicit functions of finitely many real variables, and it was verified by a program that uses interval arithmetic, so that every rounding error is accounted for in the safe direction. The exact statement, the program and the log are given. No statement of this kind rests on sampling: the programs prove the inequality for *all* points of each cell of a decomposition of the domain (Section 8).

FLOAT — A floating-point computation. Such computations appear only as illustrations and as plausibility checks. No proof in this paper uses one.

Programs and logs. The programs, the data files and the logs named in this paper are available from the author. File names and paths such as `flat/siv/runs/` refer to that material.

Provenance, and who has checked this. This work was produced primarily by AI models (Claude, by Anthropic) working at the direction of the author: the arguments, the programs, and the text of this paper. No independent person has verified the proofs or the programs. The checks that were made are listed in Section 9; they are no substitute for independent verification. Section 9 also lists everything that is not proved.

Part I. The proof

1 The result

1.1 The model and the question

For maps $U : \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$ that tend to $(1,0,0,0)$ at infinity, the Skyrme energy with pion mass $\mu \geq 0$ is

$$E(U) = \int_{\mathbb{R}^3} \Big(|\mathrm{d}U|^2 + |\Lambda^2 \mathrm{d}U|^2 + 2\mu^2(1 - U_0) \Big) \mathrm{d}x, \qquad |\Lambda^2 \mathrm{d}U|^2 = \sum_{i<j} |\partial_i U \wedge \partial_j U|^2, \tag{1}$$

in the normalisation in which $E(U) \geq 12\pi^2 |\deg U|$ [1, 2]. The term with μ is the pion mass term of [3]; in these units the parameters of [3] correspond to $\mu = 0.526$ [4].

Level and label. Every point of S^3 other than $(\pm 1, 0, 0, 0)$ can be written in exactly one way as

$$U = (-\cos\theta,\ \sin\theta\, n), \qquad \theta \in (0, \pi), \quad n \in S^2.$$

We call θ the *level* and n the *label*. The vacuum $(1,0,0,0)$ is the level $\theta = \pi$, and the potential term of (1) is $V(\theta) := 2\mu^2(1 + \cos\theta)$.

The hedgehog. The candidate for the minimiser in degree one is

$$H(x) = \Big(-\cos g(|x|),\ \sin g(|x|)\, \frac{x}{|x|} \Big), \qquad g(0) = 0, \quad g' > 0, \quad g(r) \to \pi \ \ (r \to \infty) :$$

its level surfaces are the spheres about the origin, and its label is the outward normal. The profile is a solution of the Euler–Lagrange equation of E on maps of this form, which in terms of $f = \pi - g$ reads

$$(r^2 + 2\sin^2 f)\, f'' + 2r f' + \sin 2f \left(f'^2 - 1 - \frac{\sin^2 f}{r^2} \right) - \mu^2 r^2 \sin f = 0. \tag{2}$$

For $\mu = 1$ and $\mu = 0.526$ the existence of such a profile, a rigorous enclosure of it, and the value of its energy are established in Part III (Theorems 27 and 28; CERTIFIED):

$$\frac{E(H)}{12\pi^2} \in \begin{cases} [1.416223466101406227,\ 1.416223466101406358] & (\mu = 1), \\ [1.3076651860528596,\ 1.3076651860528709] & (\mu = 0.526). \end{cases} \tag{3}$$

Throughout, H denotes the hedgehog built from the solution of Theorem 27. Uniqueness of the profile is not used and not claimed.

The question. That H minimises E among all maps of degree one is a long-standing expectation. The rigorous results on existence and symmetry that we know of concern the model without pion mass: the energy has a minimum among the maps of degree one with the symmetry of the hedgehog (Kapitanskii and Ladyzhenskaya [5], as reported in [6]); it has a minimiser among all maps of degree one (Esteban [6], with the corrected proof [7], where the independent proof of Lin and Yang is described); the boundary value problem for the profile is studied in [8]; and in 2002 Manton, Schroers and Singer wrote that the hedgehog symmetry of the minimisers had not been established rigorously [9]. Two further rigorous results are close to the present subject. For the model with pion mass, Harland [10] derived a topological lower bound for the energy that depends on the pion mass and improves on the bound $12\pi^2|\deg U|$ when $\mu > 0$. For the model without pion mass, Creek, Donninger, Schlag and Snelson [11] gave a rigorous and partly computer-assisted proof of the linear stability of the hedgehog, by showing that the relevant Schrödinger operator has no negative eigenvalue. We do not know of a result that settles the question of minimality, with or without pion mass. A search of the arXiv and of the web for such a result, made in November 2026, found none; such a search cannot be exhaustive, and we make no claim of novelty beyond the statements proved here.

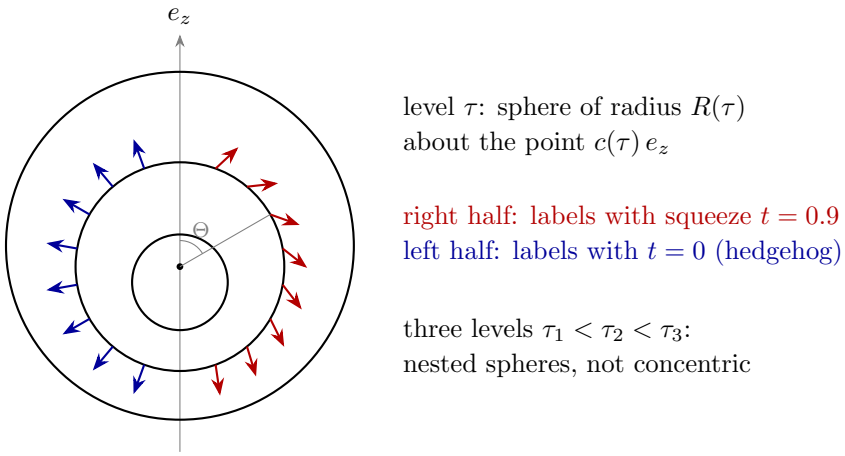

Figure 1: A map of the family $\mathcal{M}$ in a plane through the axis. The level surfaces are nested round spheres. On each of them the label at the polar angle Θ is the hedgehog's label at the polar angle α, where $\tan\frac{\alpha}{2} = e^t \tan\frac{\Theta}{2}$. The picture shows the labels on the middle sphere for $t = 0$ (left half) and for $t = 0.9$ (right half).

1.2 The family and the theorem

The theorem compares H with a family of competitors that is small compared with all maps, but is not a neighbourhood of H: the deformations may be of any size.

Definition 1 (the Möbius family $\mathcal{M}$)**.** Let R, c, t be C^1 functions on $(0, \pi)$ with $R > 0$, $|c'| < R'$, $R \to 0$ at 0 and $R - |c| \to \infty$ at π. The map U of the family given by (R, c, t) has, for every $\tau \in (0, \pi)$, the level surface $\{\theta = \tau\}$ equal to the sphere of radius $R(\tau)$ about the point $c(\tau)\, e_z$; at the point of this sphere with polar angle Θ (measured at its centre, from the direction e_z) and azimuth φ the label is

$$n = (\sin\alpha\cos\varphi,\ \sin\alpha\sin\varphi,\ \cos\alpha), \qquad \tan\tfrac{\alpha}{2} = e^{t(\tau)} \tan\tfrac{\Theta}{2}.$$

The conditions on R and c say that the balls bounded by the level surfaces are strictly nested, shrink to a point and exhaust the space, so U is a well defined C^1 map away from that point. The three functions have the following meaning (Figure 1).

- $R(\tau)$ is the *radius* of the level sphere. With $c \equiv 0$ and $t \equiv 0$ the family consists of the radial maps with the hedgehog's labels and an arbitrary increasing profile; the hedgehog itself is $R = g^{-1}$.
- $c(\tau)$ is the *position*: it moves the spheres against each other along the axis.
- $t(\tau)$ is the *squeeze*. The map $\Theta \mapsto \alpha$ is a Möbius transformation of the sphere. For $t = 0$ the label is the outward normal, as for the hedgehog. For $t > 0$ the labels turn towards the south pole, so that most of the label sphere is covered near the north pole of the level sphere: the winding is concentrated there.

The pair (sphere of radius R, squeeze t) at a level τ is called a *state*.

Theorem 1 (PROVED, from the CERTIFIED statements named below)**.** *Let $\mu = 1$ or $\mu = 0.526$. Every $U \in \mathcal{M}$ of finite energy satisfies*

$$E(U) \ \geq\ E(H).$$

The proof is completed in Section 7. It uses these computer-assisted statements: the existence and the properties of the profile (Theorems 27 and 28); the static inequality at all states of all levels up to a last level (Theorems 21 and 23); and five numerical facts about the far field (Proposition 26). Everything else is proved by hand in this paper.

1.3 The proof in one page

1. **The energy as a one-dimensional problem** (Section 2). For $U \in \mathcal{M}$ the energy is $E(U) = \int_0^\pi L \, \mathrm{d}\tau$, where the integrand L at the level τ depends on the state (R, t) and on the velocities (R', c', t'). So the question is a problem of the calculus of variations in one variable, with three unknown functions.

2. **A calibration** (Section 4). Suppose a function S of the level and the state satisfies $L \geq \frac{\mathrm{d}}{\mathrm{d}\tau} S$ along every map of the family, $S \to 0$ at the centre, and $S \to E(H)$ at infinity. Then $E(U) = \int L \geq E(H)$. A first candidate S_0 depends only on the level and on the volume enclosed by the level sphere. It is built from the hedgehog profile, and it gives

$$L - \frac{\mathrm{d}S_0}{\mathrm{d}\tau} \;\geq\; \big(\text{change of the energy of the level caused by the squeeze}\big) + \delta_Z, \qquad \delta_Z \geq 0.$$

   For spheres without squeeze the first term vanishes. This already proves the theorem for all maps of $\mathcal{M}$ with $t \equiv 0$.

3. **Why the squeeze is a real problem** (Figure 3). In the outer part of the hedgehog a squeeze *lowers* the energy carried by a single level sphere. The reason is that the energy per unit area, as a function $h_K(D)$ of the flux density D, is concave there. So S_0 alone cannot work, and the comparison of what a squeeze saves on the outer levels with what it costs elsewhere is the heart of the matter. It is not a perturbative question.

4. **The credit** (Section 5). We add to S_0 a *credit* $B = b(\tau)\, F(m)\, W(R/r)$, where m is the dipole of the flux on the level sphere (it measures the squeeze) and $b \geq 0$ is a function of the level. A squeezed state has to pay into the credit on the inner levels, where the squeeze is expensive, and may draw on it on the outer levels. After the velocities are eliminated by completing squares, the requirement $L \geq \frac{\mathrm{d}}{\mathrm{d}\tau}(S_0 + B)$ becomes one inequality

$$\Psi(r, \varrho, t) \;\geq\; 0$$

   for a function of three real variables: the level, the size of the sphere, and the squeeze (Proposition 6). We call it the *static inequality*, because no velocity is left in it.

5. **The static inequality holds at every state of every level up to a last one** (Section 6, Part II). For extreme states (large squeezes, large or small spheres, levels without credit) this follows from lemmas with explicit bounds. For the rest it is verified by interval arithmetic on about $8 \cdot 10^5$ cells. The delicate place is the neighbourhood of the hedgehog's own states, where Ψ vanishes to second order; there the program expands in Taylor series with rigorously bounded remainders.

6. **The far field** (Section 7). The calibration is stopped at a last level, at the radius $r_* = 9.53$ for $\mu = 1$ and $r_* = 7.42$ for $\mu = 0.526$, where the hedgehog is closer to the vacuum than $1.52 \cdot 10^{-5}$ resp. $3.44 \cdot 10^{-3}$. What the map does outside is not followed level by level. Its energy there is bounded below by the Yukawa energy of its boundary values, or by a comparison with an exact radial solution, and the credit accumulated up to the last level pays for the squeeze with a factor of at least 1.65 resp. 1.20 to spare.

7. **The profile** (Part III). All of the above uses the profile of the hedgehog: its existence, some sign conditions on the whole half line, and polynomial enclosures with error bounds. These are proved by a rigorous integration of the profile equation.

Figure 2 shows how the statements depend on each other.

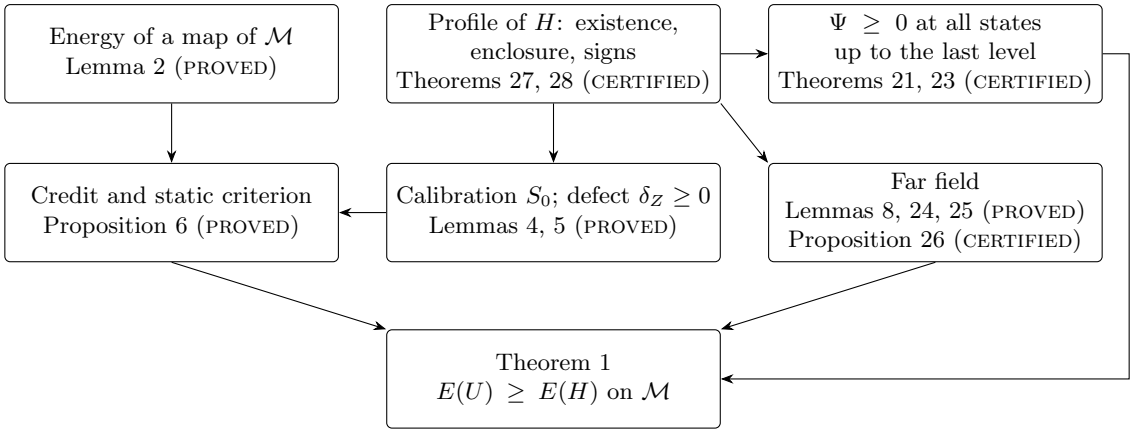

Figure 2: The structure of the proof. An arrow means "is used in".

1.4 What the theorem does not say

Nothing is proved for maps outside $\mathcal{M}$; in particular the theorem does not show that the hedgehog minimises the energy among all maps of degree one. Equality and uniqueness are not discussed. The computations were made for the two masses $\mu = 1$ and $\mu = 0.526$ only. Section 9 gives the complete list of what is open and of what has been checked.

One part of every competitor is unrestricted. The proof uses the structure of $\mathcal{M}$ only on the levels below the last level τ_*; outside the level surface of τ_* the map may be any finite-energy continuation with values in S^3 (Remark 1).

1.5 Notation

symbol	meaning	defined in
θ, n	level and label of a map, $U = (-\cos\theta, \sin\theta\, n)$	Section 1
$V(\theta)$	potential, $2\mu^2(1+\cos\theta)$	Section 1
g, $f = \pi - g$	profile of the hedgehog	(2)
τ, $s = \sin\tau$	a value of the level	Definition 1
$R(\tau)$, $c(\tau)$, $t(\tau)$	radius, position and squeeze of the level sphere	Definition 1
x, y	position cosine $\cos\Theta$ and label cosine $\cos\alpha$ on a level sphere	Section 2
j	label factor, $1/(\cosh t - x\sinh t)$	Section 2
$\bar{D}$, D	mean flux density s^2/R^2; flux density $\bar{D}\, j^2$	Section 2
ν, ω	thickness and twist of a map of $\mathcal{M}$	Lemma 2
m	dipole of the flux, $\coth t - t/\sinh^2 t$	(7)
p, q, D_*, K, κ, φ_1, Y	functions of the profile	Section 3
r_Y	the zero of Y	Section 3
$h_K(D)$	$2\sqrt{(1+2D)(D^2+2D+K)}$	(13)
$E_*(\tau)$, $v_*(\tau)$	energy of H and volume inside its level surface of τ	Section 4
r, ϱ	level radius, $\tau = g(r)$; field radius, $s/R = q(\varrho)$	Section 4
$\varrho_\Gamma(\tau)$, D_Γ	where the calibration field is frozen	Section 4
$\bar{\kappa}$, $\bar{K}$, N	the calibration field	Section 4
S_0, Z, δ_Z	calibration function, its defect density, the defect	Section 4
b, F, W, x_0, B	the credit $B = b\, F(m)\, W(R/r)$	Section 5
β, B_v, K_{eff}, Ψ_0, Ψ	the static inequality	(24)
r_i	nodes of the credit function	Theorem 9
r_*, τ_*	the last level	Section 7
$Y(R)$, $S(R)$, $\mathrm{Cal}(R)$, $B_*(R)$	far-field quantities	Section 7

Two letters have two meanings each, as in the technical version: Y is a function of the profile

in Sections 3–6 and a function of the sphere radius, $Y(R)$, in Section 7; r_i with a numerical index is a node of the credit function, while r_i with the letter i (in Section 4.4, in the captions of Figures 3 and 4, and in Part III) is the zero of $\partial_D^2 h_K(D_*)$. The context always decides.

2 Step 1: the energy of a map of the family

Purpose of this step. A map of $\mathcal{M}$ is described by three functions of one variable. We compute its energy as an integral over the level. This turns the comparison with the hedgehog into a variational problem in one variable.

Fix a level $\tau \in (0,\pi)$ and write $s = \sin\tau$. On the level sphere we use the *position cosine* $x = \cos\Theta \in [-1,1]$ and the functions

$$j = \frac{1}{\cosh t - x\sinh t}, \qquad y = \frac{x\cosh t - \sinh t}{\cosh t - x\sinh t}, \qquad \bar D = \frac{s^2}{R^2}, \qquad D = \bar D\, j^2. \tag{4}$$

Their meaning is this. The Möbius map of Definition 1 sends the polar angle Θ of the position to the polar angle α of the label; $y = \cos\alpha$ is the *label cosine*, and

$$\frac{\partial\alpha}{\partial\Theta} = \frac{\sin\alpha}{\sin\Theta} = j, \qquad \frac{\partial\alpha}{\partial t} = \sin\alpha, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = j^2. \tag{5}$$

(From $\tan\frac{\alpha}{2} = e^t\tan\frac{\Theta}{2}$: differentiating in Θ gives $\partial_\Theta\alpha = e^t\cos^2\frac{\alpha}{2}/\cos^2\frac{\Theta}{2}$, which is also $\sin\alpha/\sin\Theta$; with $u = \tan\frac{\Theta}{2}$, $x = (1-u^2)/(1+u^2)$ one finds $\sin\alpha/\sin\Theta = e^t(1+u^2)/(1+e^{2t}u^2) = 1/(\cosh t - x\sinh t)$. Differentiating in t gives $\partial_t\alpha = 2\tan\frac{\alpha}{2}\cos^2\frac{\alpha}{2} = \sin\alpha$. Finally $\mathrm{d}y = -\sin\alpha\,\mathrm{d}\alpha = j^2\,\mathrm{d}x$.) So j is the factor by which the Möbius map stretches lengths at the point x, j^2 is the factor for areas, and $D = \bar D j^2$ is the *flux density*: the area of the label sphere of radius s covered per unit area of the level sphere. The total flux does not depend on the squeeze,

$$\int_{-1}^{1} j^2\,\mathrm{d}x = \int_{-1}^{1}\mathrm{d}y = 2, \qquad \text{hence} \qquad \int_{-1}^{1}(D-\bar D)\,\mathrm{d}x = 0. \tag{6}$$

The *dipole* of the flux is its mean position in units of R,

$$m := \frac{1}{2}\int_{-1}^{1} x\, j^2\,\mathrm{d}x = \coth t - \frac{t}{\sinh^2 t}, \qquad \text{and} \qquad \frac{1}{2}\int_{-1}^{1} y\,\mathrm{d}x = -m. \tag{7}$$

(For the first integral substitute $u = \cosh t - x\sinh t$. For the second, $\frac{1}{2}\int x\, j^2\mathrm{d}x = \frac{1}{2}\int x\,\mathrm{d}y$ is the mean over the label sphere of the position cosine, i.e. the mean of the inverse Möbius map, which is the map with squeeze $-t$; and replacing t by $-t$ and x by $-x$ changes the sign of y.) The dipole increases from 0 at $t=0$ to 1 as $t\to\infty$, and $m(-t) = -m(t)$.

Lemma 2 (the energy of a map of $\mathcal{M}$; PROVED)**.** *Let $U \in \mathcal{M}$, and on the level sphere of τ put*

$$\nu = R' + c'x, \qquad \omega = s\,j\,\sqrt{1-x^2}\,\Big(t' + \frac{c'}{R}\Big).$$

Then the energy of U inside the level surface of τ_1 is $\int_0^{\tau_1} L\,\mathrm{d}\tau$ with

$$L = 2\pi R^2\int_{-1}^{1}\Big[\frac{1+2D}{\nu} + \big(2D + D^2 + V(\tau)\big)\,\nu + (1+D)\,\frac{\omega^2}{\nu}\Big]\mathrm{d}x. \tag{8}$$

The two functions on the sphere have a geometric meaning: ν is the *thickness* of the layer between neighbouring level spheres (per unit of the level), and ω measures how fast the labels turn as one crosses the layer. The first two terms of (8) say that a thin layer is expensive because of the gradient of the level, and a thick layer is expensive because of what the labels and the potential cost per unit volume.

Proof. Coordinates. The map $(\tau,\Theta,\varphi)\mapsto X=c(\tau)\,e_z+R(\tau)\,(\sin\Theta\cos\varphi,\sin\Theta\sin\varphi,\cos\Theta)$ has, in the orthonormal frame (e_r,e_Θ,e_φ) of the sphere,

$$X_\tau=\nu\,e_r-c'\sin\Theta\,e_\Theta,\qquad X_\Theta=R\,e_\Theta,\qquad X_\varphi=R\sin\Theta\,e_\varphi,$$

because $e_z=\cos\Theta\,e_r-\sin\Theta\,e_\Theta$. Its Jacobian is $\nu R^2\sin\Theta$, and $\nu\ge R'-|c'|>0$. By the conditions of Definition 1 the spheres are nested and exhaust the space, so these are coordinates on $\mathbb{R}^3$ without one point and the axis, and the volume element is $\nu R^2\sin\Theta\,\mathrm{d}\tau\,\mathrm{d}\Theta\,\mathrm{d}\varphi$.

Gradients. The level is $\theta=\tau$, so $\nabla\theta\cdot X_\tau=1$ and $\nabla\theta\cdot X_\Theta=\nabla\theta\cdot X_\varphi=0$, which gives $\nabla\theta=e_r/\nu$. The label angle $\alpha(\tau,\Theta)$ has, by (5), $\partial_\Theta\alpha=j$ and $\partial_\tau\alpha=t'\sin\alpha$. From $\partial_\Theta\alpha=\nabla\alpha\cdot X_\Theta$ and $\partial_\tau\alpha=\nabla\alpha\cdot X_\tau$ we get the components $(\nabla\alpha)_\Theta=j/R$ and $(\nabla\alpha)_r=[t'\sin\alpha+c'\sin\Theta\,j/R]/\nu=\sin\alpha\,(t'+c'/R)/\nu$, where $\sin\Theta\,j=\sin\alpha$ was used. Since $\sin\alpha=j\sqrt{1-x^2}$, this is $\nabla\alpha=(j/R)\,e_\Theta+(\omega/(s\nu))\,e_r$. The azimuth has $\nabla\varphi=e_\varphi/(R\sin\Theta)$.

The differential of U. Let $\epsilon_\theta,\epsilon_\alpha,\epsilon_\varphi$ be the derivatives of $U=(-\cos\theta,\sin\theta\,n(\alpha,\varphi))$ with respect to θ, α and φ, normalised. They are orthonormal and tangent to S^3, and $\mathrm{d}U=\mathrm{d}\theta\otimes\epsilon_\theta+s\,\mathrm{d}\alpha\otimes\epsilon_\alpha+s\sin\alpha\,\mathrm{d}\varphi\otimes\epsilon_\varphi$. With the gradients just computed, $\mathrm{d}U=a\otimes\epsilon_\theta+b\otimes\epsilon_\alpha+c\otimes\epsilon_\varphi$, where

$$a=\frac{1}{\nu}\,e_r,\qquad b=\frac{\omega}{\nu}\,e_r+\lambda\,e_\Theta,\qquad c=\lambda\,e_\varphi,\qquad \lambda=\frac{s\,j}{R},\quad \lambda^2=D.$$

The energy density. $|\mathrm{d}U|^2=|a|^2+|b|^2+|c|^2=(1+\omega^2)/\nu^2+2D$, and

$$|\Lambda^2\mathrm{d}U|^2=|a\wedge b|^2+|a\wedge c|^2+|b\wedge c|^2=\frac{D}{\nu^2}+\frac{D}{\nu^2}+D\Big(\frac{\omega^2}{\nu^2}+D\Big).$$

The sum, with the potential, is $(1+2D)/\nu^2+2D+D^2+V+(1+D)\,\omega^2/\nu^2$. Multiply by the volume element and integrate over φ. □

The hedgehog. For H the level τ sits at the radius r with $g(r)=\tau$, so $R'=1/g'(r)$, and $\nu=1/p$, $\omega=0$, $D=\bar D=\sin^2 g/r^2$. Hence the energy $E_*(\tau)$ of H inside its level surface of τ has the derivative

$$E_*'(\tau)=4\pi r^2\Big[(1+2D_*)\,p+\frac{2D_*+D_*^2+V(\tau)}{p}\Big],\qquad p=g'(r),\quad D_*=\frac{\sin^2 g(r)}{r^2}.\tag{9}$$

3 Step 2: what is used of the hedgehog profile

Purpose of this step. The calibration is built from the profile g of the hedgehog. We collect the functions of the profile that occur, and the facts about them that the proof uses. The facts are established in Part III by a rigorous numerical integration of the profile equation; here we only state them.

With $g=\pi-f$ the solution of (2) of Theorem 27, put

$$\begin{gathered}p=g',\qquad q=\frac{\sin g}{r},\qquad D_*=q^2,\qquad \varphi_1=q-p\cos g,\\ K=(1+2D_*)\,p^2-2D_*-D_*^2,\qquad \kappa=K-V(g),\qquad Y=D_*^2-D_*-\kappa.\end{gathered}\tag{10}$$

Here $V(g)=2\mu^2(1+\cos g)$ is the potential along the hedgehog, D_* is the flux density of the hedgehog's own level sphere, and $1/p$ is the thickness of its layers. The function K is defined so that

$$2D_*+D_*^2+K=(1+2D_*)\,p^2;\tag{11}$$

the reason will become clear in Section 4, where κ appears as a *price of volume*. Since $q'=(p\cos g-q)/r$, we have $q'=-\varphi_1/r$. At the centre, with $a=g'(0)$: $p,q\to a$, $K\to a^4-a^2$, $\kappa\to a^4-a^2-4\mu^2$ and $Y\to 4\mu^2$.

Facts used (all CERTIFIED; Theorems 27 and 28).

(i) *Existence and energy.* The profile exists on $(0,\infty)$ with $0 < g < \pi$, $g' > 0$ and $g \to \pi$, and the energy of H lies in the interval (3).

(ii) *Signs on the whole half line.* $\varphi_1 > 0$, so that q decreases strictly from a to 0; $p > q$; $\kappa > 0$; and $Y' < 0$ wherever $Y \geq 0$, so that $\{Y \geq 0\} = (0, r_Y]$ for a radius r_Y with

$$r_Y \in \begin{cases} [0.20731874121529259,\, 0.20731874121529282] & (\mu = 1), \\ [0.156414756367439,\, 0.156414756367667] & (\mu = 0.526). \end{cases}$$

(iii) *Enclosures.* On $[\frac{1}{16}, r_N]$, with $r_N = 10$ for $\mu = 1$ and $r_N = 16$ for $\mu = 0.526$, the functions g and g' are enclosed by piecewise polynomials of degree 12 with explicit remainder terms, and the slope $a = g'(0)$ is enclosed with an error of at most 10^{-15} (Theorem 28(6), Table 4).

(iv) *The tail.* For $r \geq r_N$ the profile decays like the solution of the linearised equation: $\bigl|p/f - m_{\mathrm{lin}}(r)\bigr| < \delta_N$ with $m_{\mathrm{lin}}(r) = \mu\bigl[1 - \frac{1}{\mu r+1} + \frac{2}{\mu r}\bigr]$ and $\delta_N < 3 \cdot 10^{-11}$, and the energy of H outside r_N is given by a formula with an explicit error bound (Theorem 27(2), Lemmas 29 and 31).

Where they are used. (ii) is what makes the calibration work: it gives the sign of the defect (Lemma 5). (iii) is the input of all interval computations of Part II. (iv) is used at the far end: in the limit $\tau \to \pi$ of Proposition 6, in Lemma 24, and in Proposition 26.

One identity is used again and again. It follows from the profile equation and the definitions alone.

Lemma 3 (PROVED)**.** $\kappa' = -\dfrac{4}{r}\,(1 + D_*)\,(p^2 - D_*)$*; in particular* $\kappa' < 0$*, by* (ii).

Proof. In terms of g, equation (2) reads $(r^2 + 2\sin^2 g)\,g'' + 2rg' + \sin 2g\,(g'^2 - 1 - \sin^2 g/r^2) + \mu^2 r^2 \sin g = 0$. Divide by r^2 and use $\sin 2g/r^2 = 2q\cos g/r$:

$$(1 + 2D_*)\,p' = -\frac{2p}{r} - \frac{2q\cos g}{r}\,(p^2 - 1 - D_*) - \mu^2 \sin g.$$

From (10), $K' = 2D_*'\,(p^2 - 1 - D_*) + 2(1 + 2D_*)\,p\,p'$, and $D_*' = 2qq' = -2q\,(q - p\cos g)/r$. Inserting, the terms with $\cos g$ cancel and

$$K' = -\frac{4}{r}\bigl[p^2 + D_*(p^2 - 1 - D_*)\bigr] - 2\mu^2 p \sin g = -\frac{4}{r}(1 + D_*)(p^2 - D_*) - 2\mu^2 p \sin g.$$

Finally $\frac{\mathrm{d}}{\mathrm{d}r}V(g) = -2\mu^2 p \sin g$. □

4 Step 3: a calibration by level sets

Purpose of this step. We construct a function S_0 of the level and of the enclosed volume such that $L - \frac{\mathrm{d}}{\mathrm{d}\tau}S_0$ is bounded below by an explicit expression without velocities. For spheres without squeeze the expression is non-negative. For squeezed spheres it need not be, and we see why.

4.1 The principle

Let $U \in \mathcal{M}$ have the states $(R(\tau), t(\tau))$, and suppose that S is a function of the level and the state with the following properties: (1) along U, $L \geq \frac{\mathrm{d}}{\mathrm{d}\tau}S$ at almost every level; (2) $S \to 0$ as $\tau \to 0$; (3) the lower limit of S as $\tau \to \pi$ is at least $E(H)$. Then, by Lemma 2,

$$E(U) = \int_0^\pi L\,\mathrm{d}\tau \;\geq\; \liminf_{\tau\to\pi} S - \lim_{\tau\to 0} S \;\geq\; E(H).$$

This is the classical method of a verification function for a variational problem in one variable. The proof of Theorem 1 is the construction of such an S and the verification of (1). The difficulty is entirely in (1), which must hold for *all* states and *all* velocities.

4.2 The price of volume and the function h_K

Let a sphere of radius R at the level τ be given, with the notation (4). By Lemma 2, dropping the term with ω^2,

$$L \;\geq\; 2\pi R^2 \int_{-1}^{1} \Big[\frac{1+2D}{\nu} + \big(2D + D^2 + V(\tau)\big)\,\nu\Big]\mathrm{d}x.$$

The thickness $\nu(x) > 0$ is at the disposal of the competitor at each point of the sphere, but its mean is tied to the growth of the enclosed volume $v = \frac{4\pi}{3}R^3$:

$$2\pi R^2 \int_{-1}^{1} \nu \,\mathrm{d}x = 4\pi R^2 R' = v'. \tag{12}$$

We therefore introduce a number $\bar\kappa$, a *price of volume*, and write $\bar K = \bar\kappa + V(\tau)$:

$$\big(2D + D^2 + V\big)\,\nu = \big(2D + D^2 + \bar K\big)\,\nu - \bar\kappa\,\nu.$$

For $A > 0$, $B \geq 0$ the infimum of $A/\nu + B\nu$ over $\nu > 0$ is $2\sqrt{AB}$. So with

$$h_K(D) := 2\sqrt{(1+2D)\,(D^2+2D+K)} \qquad (D \geq 0,\; D^2+2D+K \geq 0) \tag{13}$$

we obtain, for every number $\bar\kappa$ with $D^2 + 2D + \bar K \geq 0$ on the sphere,

$$L \;\geq\; 2\pi R^2 \int_{-1}^{1} h_{\bar K}(D)\,\mathrm{d}x - \bar\kappa\, v'. \tag{14}$$

In words: $h_K(D)$ is the least energy, per unit area and per unit of the level, of a layer with the flux density D when volume has the price $K - V$.

The hedgehog makes this an equality. For H we have $D = D_*$ and $\nu = 1/p$. The minimising thickness in (13) is $\nu = \sqrt{(1+2D)/(D^2+2D+K)}$, and by (11) this equals $1/p$ exactly when K is the function $K(r)$ of the profile. So K is *defined* by the requirement that the hedgehog's thickness is optimal, and $\kappa = K - V(g)$ is the price of volume at which the hedgehog is in equilibrium. With $v_*(\tau) = \frac{4\pi}{3}r^3$, the volume inside the hedgehog's level surface of $\tau = g(r)$, and with (9):

$$E_*'(\tau) + \kappa(r)\,v_*'(\tau) = 4\pi r^2\, h_K(D_*) = 8\pi r^2\,(1+2D_*)\,p, \qquad v_*' = \frac{4\pi r^2}{p}. \tag{15}$$

4.3 The calibration field and the function S_0

To use (14) for a competitor we must say which price $\bar\kappa$ belongs to a sphere of radius R at a level τ. We let the hedgehog decide: we look for the radius ϱ at which the hedgehog has the same flux density as our sphere, and we take the price that the hedgehog has there.

The field radius. By fact (ii) of Section 3, q decreases strictly from a to 0. So if $\bar D = s^2/R^2 < a^2$, there is exactly one radius ϱ with

$$q(\varrho) = \frac{s}{R}, \qquad \text{i.e.} \qquad D_*(\varrho) = \bar D. \tag{16}$$

We call ϱ the *field radius* of the sphere. Large spheres have large ϱ. The hedgehog's own sphere at the level $\tau = g(r)$ has $\varrho = r$.

A threshold for small spheres. At the centre $Y = 4\mu^2 = V(0)$, and Y decreases strictly to 0 on $(0, r_Y]$ (fact (ii)). So for every level $\tau \in (0,\pi)$ there is exactly one radius $\varrho_\Gamma(\tau) \in (0, r_Y)$ with $Y(\varrho_\Gamma) = V(\tau)$; put $D_\Gamma = q(\varrho_\Gamma)^2$.

The field. For the sphere of radius R at the level τ, with $v = \frac{4\pi}{3}R^3$, let

$$\text{if } \bar D \le D_\Gamma: \qquad \bar\kappa = \kappa(\varrho) \ \text{ with } \varrho \text{ from (16)}, \ \varrho \ge \varrho_\Gamma, \qquad N = 4\pi R^2\, h_{\bar K}(\bar D), \tag{17}$$

$$\text{if } \bar D > D_\Gamma: \qquad \bar\kappa = \kappa(\varrho_\Gamma(\tau)), \qquad N = 4\pi s^2\, \frac{h_{\bar K}(D_\Gamma)}{D_\Gamma}, \tag{18}$$

and in both cases $\bar K = \bar\kappa + V(\tau)$, which is positive because $\kappa > 0$. In the first case we say that the sphere lies *above* ϱ_Γ, in the second *below*; below, $\bar K = D_\Gamma^2 - D_\Gamma$ by the definition of Y. The number N is the energy per unit of the level that the calibration attributes to the sphere: above ϱ_Γ it is the area times h at the mean flux density. The second case concerns spheres that are very small for their level. There the flux density is beyond the point where the energy *per unit flux*, $h_K(D)/D$, is smallest (Lemma 5(e)), and the field is frozen at that point. The hedgehog's own sphere lies above ϱ_Γ: at $\varrho = r$, $\bar K - (\bar D^2 - \bar D) = K - D_*^2 + D_* = (1+2D_*)(p^2 - D_*) > 0$, i.e. $Y(r) < V(\tau)$.

As functions of (τ, v), $\bar\kappa$ and N are continuous and piecewise C^1. Now define

$$S_0(\tau,v) = E_*(\tau) - \int_{v_*(\tau)}^{v} \bar\kappa(\tau,v')\,\mathrm{d}v', \qquad Z = \partial_v N + \partial_\tau \bar\kappa, \qquad \delta_Z(\tau,v) = \int_{v_*(\tau)}^{v} Z(\tau,v')\,\mathrm{d}v'. \tag{19}$$

In words: S_0 is the energy that the hedgehog has spent up to the level τ, corrected by the price of the volume by which the competitor is ahead of the hedgehog or behind it. On the hedgehog's own path, $v = v_*(\tau)$, we have $S_0 = E_*$ and $\delta_Z = 0$.

Lemma 4 (calibration identity; PROVED)**.** *$\partial_v S_0 = -\bar\kappa$ and $\partial_\tau S_0 = N - \delta_Z$. Consequently, for $U \in \mathcal{M}$ with $v(\tau) = \frac{4\pi}{3}R(\tau)^3$,*

$$L - \frac{\mathrm{d}}{\mathrm{d}\tau} S_0(\tau, v(\tau)) = 2\pi R^2 \int_{-1}^{1} \Big[\frac{1+2D}{\nu} + \big(2D + D^2 + \bar K\big)\,\nu + (1+D)\,\frac{\omega^2}{\nu}\Big]\mathrm{d}x - N + \delta_Z.$$

Proof. The first formula is the definition. For the second, the derivative of the lower limit of the integral contributes $\bar\kappa(\tau, v_*)\, v_*' = \kappa(r)\, v_*'$, and by (15) $E_*' + \kappa(r)\, v_*' = 4\pi r^2 h_K(D_*) = N(\tau, v_*(\tau))$. Hence

$$\partial_\tau S_0 = N(\tau, v_*) - \int_{v_*}^{v} \partial_\tau \bar\kappa\,\mathrm{d}v' = N(\tau,v) - \int_{v_*}^{v} \big(\partial_v N + \partial_\tau \bar\kappa\big)\,\mathrm{d}v'.$$

For the last statement, $\frac{\mathrm{d}}{\mathrm{d}\tau} S_0(\tau, v(\tau)) = N - \delta_Z - \bar\kappa\, v'$; insert Lemma 2 and (12), and write $V = \bar K - \bar\kappa$. □

The whole point of the construction is that the defect has a sign.

Lemma 5 (the defect; PROVED, from facts (ii))**.** (a) $\kappa' = -\frac{4}{r}(1+D_*)(p^2 - D_*) < 0$ (Lemma 3).

(b) *At a sphere above ϱ_Γ, with the profile quantities taken at the field radius ϱ, and with $X = \cos g(r) - \cos g(\varrho)$, $\delta = 2\mu^2 X/(1+2D_*)$ and $\bar P = \sqrt{p^2 + \delta}$,*

$$Z = \frac{4(1+D_*)}{R\,\bar P\,\varphi_1}\Big\{(p^2 - D_*)\,\bar P\, X + \delta\Big[\varphi_1 + \frac{(p^2 - D_*)\cos g}{\bar P + p}\Big]\Big\},$$

which has the sign of $\varrho - r$.

(c) *At a sphere below ϱ_Γ, $Z = \kappa'(\varrho_\Gamma)\,\varrho_\Gamma'(\tau) < 0$.*

(d) *$\delta_Z \ge 0$ at every state, and for two spheres on the same side of the hedgehog's, δ_Z is larger at the one farther from it.*

(e) *For $K > 0$ the function $D \mapsto h_K(D)/D$ on $(0,\infty)$ has its only minimum at the positive root of $D^2 - D = K$. Consequently $2\pi R^2 \int_{-1}^{1} h_{\bar K}(D)\,\mathrm{d}x \ge N$ at every state below ϱ_Γ.*

Proof. (b) All profile quantities are taken at ϱ. First, $\bar K = \kappa(\varrho) + V(\tau) = K + 2\mu^2 X$, so by (11)

$$\bar P^2 = p^2 + \delta = \frac{D_*^2 + 2D_* + \bar K}{1 + 2D_*} > 0, \qquad h_{\bar K}(\bar D) = 2(1 + 2D_*)\,\bar P, \qquad \partial_K h_{\bar K}(\bar D) = \frac{1}{\bar P}.$$

Next, with $P(D) = (1+2D)(D^2 + 2D + K)$ one computes $2P - DP' = 2(1+D)(K + D - D^2)$, so that

$$h_K - D\,\partial_D h_K = \frac{2(1+D)\,(K + D - D^2)}{\sqrt{P}}, \quad \text{and at } (\bar K, \bar D): \quad \frac{2(1+D_*)(\bar P^2 - D_*)}{\bar P}. \tag{20}$$

The field radius depends on (τ, R) through $q(\varrho) = \sin\tau/R$; with $q' = -\varphi_1/\varrho$ this gives

$$\partial_R \varrho = \frac{q\,\varrho}{R\,\varphi_1}, \qquad \partial_\tau \varrho = -\frac{\cos\tau\ \varrho}{R\,\varphi_1}.$$

Now $N = 4\pi R^2 h_{\bar K}(\bar D)$ with $\bar D = s^2/R^2$ and $\partial_v = (4\pi R^2)^{-1}\partial_R$, so $\partial_v N = \frac{2}{R}(h - D\partial_D h) + \partial_K h\ \kappa'(\varrho)\,\partial_R \varrho$, and with (a)

$$\partial_v N = \frac{4(1+D_*)}{R\bar P}\Big[\bar P^2 - D_* - \frac{(p^2 - D_*)\,q}{\varphi_1}\Big], \qquad \partial_\tau \bar\kappa = \kappa'(\varrho)\,\partial_\tau \varrho = \frac{4(1+D_*)(p^2 - D_*)\cos\tau}{R\,\varphi_1}.$$

Add the two, and use $\bar P^2 = p^2 + \delta$, $\cos\tau = X + \cos g$, $\varphi_1 - q = -p\cos g$ and $\bar P - p = \delta/(\bar P + p)$:

$$(\bar P^2 - D_*)\varphi_1 - (p^2 - D_*)\,q + (p^2 - D_*)\bar P\cos\tau = \delta\,\varphi_1 + (p^2 - D_*)\Big[\bar P X + \frac{\delta\cos g}{\bar P + p}\Big].$$

This is the formula. *Sign.* The profile g increases, so X and δ have the sign of $\varrho - r$. Further $p^2 > D_*$ by (ii). The square bracket of the formula is positive: for $\cos g \ge 0$ because $\varphi_1 > 0$; for $\cos g < 0$ because $(p^2 - D_*)/(\bar P + p) < p$ and $\varphi_1 = q + p\,|\cos g|$.

(c) There N depends on τ only, and $\bar\kappa = \kappa(\varrho_\Gamma(\tau))$. Differentiating $Y(\varrho_\Gamma(\tau)) = V(\tau)$ gives $Y'\varrho_\Gamma' = V' < 0$, and $Y' < 0$ on $(0, r_Y]$; so $\varrho_\Gamma' > 0$, and $\kappa' < 0$.

(d) The spheres below ϱ_Γ are smaller than the hedgehog's. So by (b) and (c), $Z \ge 0$ for $v > v_*(\tau)$ and $Z \le 0$ for $v < v_*(\tau)$, and the integral of Z from v_* to v is non-negative and grows with the distance.

(e) By (20), $\frac{\mathrm{d}}{\mathrm{d}D}(h_K/D) = -(h_K - D\,\partial_D h_K)/D^2$ is negative before the positive root of $D^2 - D = K$ and positive after it. Below ϱ_Γ the root for $K = \bar K$ is D_Γ, so $h_{\bar K}(D) \ge h_{\bar K}(D_\Gamma)\,D/D_\Gamma$ for all $D > 0$; and $2\pi R^2 \int D\,\mathrm{d}x = 4\pi R^2 \bar D = 4\pi s^2$ by (6). □

4.4 What the calibration gives, and where it fails

Let (R, t) be the state of a map of $\mathcal{M}$ at the level τ. By Lemma 4, and by the minimisation over ν that led to (14),

$$L - \frac{\mathrm{d}}{\mathrm{d}\tau} S_0(\tau, v(\tau)) \ \ge\ G + \delta_Z, \qquad G := 2\pi R^2 \int_{-1}^{1} h_{\bar K}(D)\,\mathrm{d}x - N. \tag{21}$$

For a sphere above ϱ_Γ, $G = 2\pi R^2 \int_{-1}^{1} \big[h_{\bar K}(D) - h_{\bar K}(\bar D)\big]\mathrm{d}x$, and $\bar D$ is the mean of D by (6). So G is a *Jensen gap*: it is non-negative if $h_{\bar K}$ is convex on the range of D, and it can be negative if $h_{\bar K}$ is concave there.

Without squeeze. If $t = 0$ then $D \equiv \bar D$, and $G = 0$ above ϱ_Γ, $G \ge 0$ below (Lemma 5(e)). With $\delta_Z \ge 0$ this gives $L \ge \frac{\mathrm{d}}{\mathrm{d}\tau} S_0$: round spheres with the hedgehog's labels, of any radii and at any positions, never do better than the hedgehog. (The formal statement is Lemma 7 with Proposition 6.)

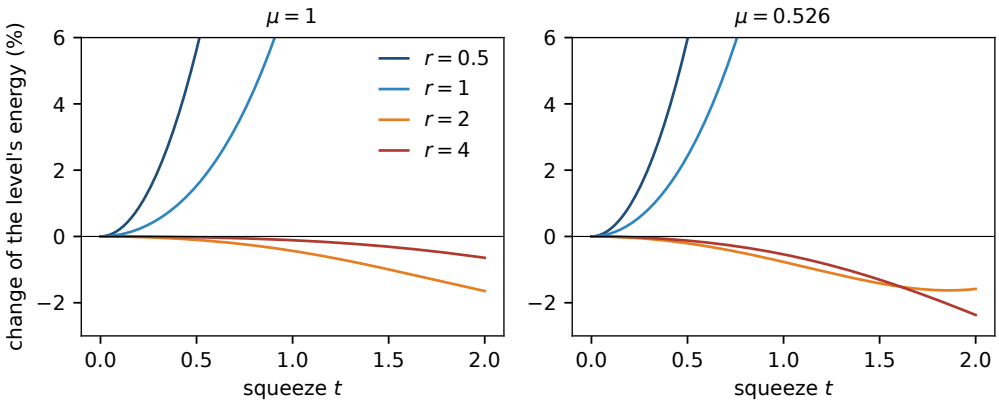

Figure 3: The relative change G/N of the energy of one level sphere of the hedgehog under a squeeze t, at four levels (the level is given by its radius r). At the inner levels a squeeze is expensive. At the outer levels, beyond $r_i \approx 1.6$ resp. 1.7, it saves energy. Floating point; an illustration only. (FLOAT; program `paper/figs/make_figs.py`.)

With squeeze. Along the hedgehog, $\partial_D^2 h_K(D_*)$ is positive for $r < r_i$ and negative for $r > r_i$, where $r_i = 1.608\ldots$ for $\mu = 1$ and $r_i = 1.715\ldots$ for $\mu = 0.526$ (Theorem 28(5)). So at the outer levels a small squeeze of the hedgehog's own sphere makes G negative: it *lowers* the energy carried by the level. Figure 3 shows the size of the effect. It is small, at most a few per cent of the energy of the level in the range shown, but it has the wrong sign, and $\delta_Z = 0$ on the hedgehog's own sphere. Hence S_0 alone does not satisfy requirement (1) of the principle. This is not a defect of the method that a better choice of the field could remove: on the outer levels a squeezed sphere really is cheaper, level for level. What saves the hedgehog is that a squeeze cannot be confined to the outer levels for free. It has to be built up through the levels, which costs the term with ω^2 in (8), or it has to be present on the inner levels, where it is expensive. The next step turns this into an inequality.

5 Step 4: the credit and the static inequality

Purpose of this step. We add to S_0 a term that depends on the squeeze, the *credit*, and we show that the requirement $L \geq \frac{\mathrm{d}}{\mathrm{d}\tau}(S_0 + B)$ follows from an inequality $\Psi \geq 0$ in which no velocity appears.

5.1 The credit

Let $b \geq 0$ be a Lipschitz function of the level. The credit of the state (R, t) at the level $\tau = g(r)$ is

$$B = b(\tau)\, F(m)\, W\Big(\frac{R}{r}\Big), \qquad F(m) = \frac{9}{8}\, m^2(2 - m^2), \qquad W(x) = \frac{x^3}{x^3 + x_0^3}, \quad x_0 = \frac{11}{20}, \tag{22}$$

where $m = m(t)$ is the dipole (7). The three factors have these roles.

- $F(m)$ measures the squeeze. It vanishes with its derivative at $t = 0$, it is even in t, it increases with $|t|$, and it is bounded by $\frac{9}{8}$. The dipole is used, and not the squeeze t itself, because the rate of change of the dipole can be written as an integral of the thickness and the twist over the sphere (formula (25) below).
- $W(R/r)$ compares the size of the sphere with the size of the hedgehog's sphere of the same level. It increases from 0 for very small spheres to 1 for very large ones.

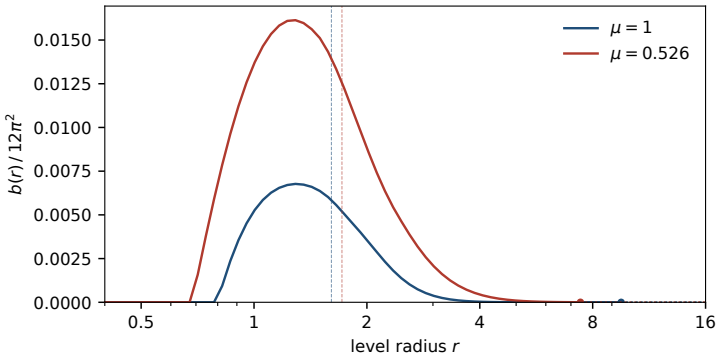

Figure 4: The credit functions b of the two data files (`flat/siv/mu1b.dat`, `mu0526b.dat`), up to the last level (dots). The dashed vertical lines are the radii r_i beyond which a squeeze of the hedgehog's own sphere saves energy. The credit is built up on the levels where a squeeze is expensive and released where it saves energy.

- $b(\tau)$ is the amount of credit at the level τ. Where b increases, a squeezed state has to pay in; where b decreases, it may draw on the credit.

The function b that is used in the proof is shown in Figure 4. It is piecewise linear in $\ln r$ between nodes r_i, it vanishes on the inner levels, it increases up to $r \approx 1.29$, and it decreases from there on. Its values are a list of numbers in the data files. They were found by a floating-point design procedure, which is no part of the proof (Section 8.4); what is proved about b is that the static inequality holds with it.

5.2 States in the coordinates (r, ϱ, t), and the inequality

A state at a level is a sphere and a squeeze. We describe it by three numbers:

$$\tau = g(r), \qquad R = \frac{\sin g(r)}{q(\varrho)}, \qquad t,$$

so r is the *level radius*, the radius at which the hedgehog has the level τ, and ϱ is the field radius (16) of the sphere. The hedgehog's own states are $\varrho = r$, $t = 0$; we call this set the *path*. (This description covers the spheres above ϱ_Γ. For smaller spheres the field is (18), and they are parametrised by R itself.) With $s = \sin\tau$ and with j, y, D, m as in (4), (7), the calibration field of Section 4 at this state is

$$\bar{D} = q(\varrho)^2, \qquad \bar{K} = \kappa(\varrho) + 2\mu^2(1 + \cos g(r)), \qquad N = 4\pi R^2\, h_{\bar{K}}(\bar{D}).$$

Let $x_R = R/r = q(r)/q(\varrho)$, and put

$$\beta = b\, W(x_R)\, F'(m), \qquad B_v = \frac{b\, F(m)\, x_R W'(x_R)}{4\pi R^3},$$
$$K_{\text{eff}} = \bar{K} - B_v - \frac{\beta}{4\pi R^3}\big[x(j^2 - 3) - m\big] - \frac{\beta^2}{64\pi^2 s^2 R^4}\,\frac{1 - y^2}{1 + D}. \tag{23}$$

K_{eff} is a function on the sphere (it depends on x). The *static inequality* is

$$\Psi(r, \varrho, t) := \Psi_0 + r\, p\, \delta_Z \;\geq 0, \qquad \Psi_0 := r\, p\,\big(\text{SI} - N\big) + b\, F\, x_R W'(x_R) - F\, W\,\frac{\mathrm{d}b}{\mathrm{d}\ln r}, \tag{24}$$

$$\text{SI} - N = 2\pi R^2 \int_{-1}^{1} \Big[h_{K_{\text{eff}}}(D) - h_{\bar{K}}(\bar{D}) - h'_{\bar{K}}(\bar{D})\,(D - \bar{D})\Big]\mathrm{d}x,$$

together with the requirement $D^2+2D+K_{\text{eff}}>0$ on the sphere. Here $p=g'(r)$, the prime on h is the derivative with respect to D, and $\delta_Z\ge 0$ is the defect (19); a formula for it in these coordinates is Lemma 11. The term with h' integrates to zero by (6); it is written because it makes the integrand small of second order in the squeeze, which matters for the computations.

How to read Ψ. It has three parts. The first, $r\,p\,(\mathrm{SI}-N)$, is the Jensen gap of Section 4.4, but computed with K_{eff} in place of $\bar K$ on the squeezed sphere; the credit enters here through K_{eff}, so holding credit is not free. The second, $b\,F\,x_R W'-F\,W\,\mathrm{d}b/\mathrm{d}\ln r$, is what the credit releases as the level advances: it is positive where b decreases. The third, $r\,p\,\delta_Z\ge 0$, is the defect of the calibration; it vanishes only if the sphere has the size of the hedgehog's. On the outer levels the first part is negative for squeezed spheres, and the second part has to make up for it. At $t=0$ all terms with the credit vanish and $\Psi=r\,p\,\delta_Z\ge 0$.

5.3 The criterion

Proposition 6 (static criterion along a map; PROVED)**.** *Let $b\ge 0$ be Lipschitz with $b=0$ near $\tau=0$, and let $U\in\mathcal{M}$ with states $(R(\tau),t(\tau))$, $v(\tau)=\frac{4\pi}{3}R(\tau)^3$.*

(a) *At every level at which b is differentiable and $D^2+2D+K_{\text{eff}}\ge 0$ on the sphere,*

$$L-\frac{\mathrm{d}}{\mathrm{d}\tau}\big[S_0(\tau,v(\tau))+B\big]\;\ge\;\frac{\Psi}{r\,p}\,,$$

where for spheres below ϱ_Γ the term $\mathrm{SI}-N$ of (24) is read as $2\pi R^2\int_{-1}^{1}h_{K_{\text{eff}}}(D)\,\mathrm{d}x-N$ with the field (18).

(b) *If the states of U satisfy $D^2+2D+K_{\text{eff}}\ge 0$ and $\Psi\ge 0$ at almost every level below τ_1, then the energy of U inside the level surface of τ_1 is at least $S_0(\tau_1,v(\tau_1))+B(\tau_1)$.*

(c) *If this holds at almost every level of $(0,\pi)$ and U has finite energy, then $E(U)\ge E(H)$.*

Proof. (a) *The rate of change of the credit.* With $x_R=R/r$ and $\mathrm{d}r/\mathrm{d}\tau=1/p$,

$$\frac{\mathrm{d}B}{\mathrm{d}\tau}=b'FW+\beta\,m'(t)\,t'+b\,F\,W'(x_R)\Big[\frac{R'}{r}-\frac{x_R}{r\,p}\Big],\qquad \beta=b\,W\,F'(m).$$

The velocities as integrals over the sphere. We have $R'=\frac12\int_{-1}^{1}\nu\,\mathrm{d}x$, and

$$m'(t)\,t'=2\pi R^2\int_{-1}^{1}\Big\{\frac{x\,(j^2-3)-m}{4\pi R^3}\,\nu+\frac{j\sqrt{1-x^2}}{4\pi R^2 s}\,\omega\Big\}\mathrm{d}x. \tag{25}$$

To see this, recall $\int j^2\mathrm{d}x=2$ and $\int x\,j^2\mathrm{d}x=2m$, and note $\int(1-x^2)\,j^2\mathrm{d}x=2m'(t)$: indeed $(1-x^2)j^2=1-y^2=-\partial_t y$, and the mean of y is $-m$ by (7). With $\nu=R'+c'x$ the first term of (25) is $\frac{c'}{2R}\big(\int x^2j^2\mathrm{d}x-2\big)=-c'm'(t)/R$ (the part with R' drops out), and with the definition of ω the second term is $(t'+c'/R)\,m'(t)$. Likewise, with B_v as in (23), $b\,F\,W'(x_R)\,R'/r=2\pi R^2\int B_v\,\nu\,\mathrm{d}x$.

Subtraction. Subtract $\mathrm{d}B/\mathrm{d}\tau$ from the identity of Lemma 4:

$$\begin{aligned}L-\frac{\mathrm{d}}{\mathrm{d}\tau}\big(S_0+B\big)=2\pi R^2\int_{-1}^{1}\Big[&\frac{1+2D}{\nu}+\Big(2D+D^2+\bar K-B_v-\beta\,\frac{x(j^2-3)-m}{4\pi R^3}\Big)\nu\\&+(1+D)\frac{\omega^2}{\nu}-\beta\,\frac{j\sqrt{1-x^2}}{4\pi R^2 s}\,\omega\Big]\mathrm{d}x-N+\delta_Z-b'FW+\frac{b\,F\,x_R W'(x_R)}{r\,p}.\end{aligned}$$

Elimination of the velocities. For fixed $\nu>0$ the two terms with ω are a quadratic polynomial in ω, which is at least $-\beta^2j^2(1-x^2)\,\nu/(64\pi^2R^4s^2(1+D))$; and $j^2(1-x^2)=1-y^2$. So the

integrand is at least $(1+2D)/\nu + (2D + D^2 + K_{\text{eff}})\,\nu$, and this is at least $h_{K_{\text{eff}}}(D)$ by the minimisation over ν of Section 4. The result is $\Psi/(r\,p)$: use $b' = \mathrm{d}b/\mathrm{d}\tau = (r\,p)^{-1}\mathrm{d}b/\mathrm{d}\ln r$, and above ϱ_Γ, $N = 2\pi R^2 \int h_{\bar K}(\bar D)\,\mathrm{d}x$ and (6).

(b) S_0 is C^1 and B is Lipschitz along the map. Integrate (a) over (τ_0, τ_1) and let $\tau_0 \to 0$. Then $E_*(\tau_0) \to 0$, $B = 0$ near 0, and $\left|\int_{v_*}^{v} \bar\kappa\right| \le \kappa(0^+)\max(v, v_*) \to 0$ because $R(\tau_0) \to 0$; so $S_0 + B \to 0$.

(c) Let $\tau_1 \to \pi$ in (b). Then $E_*(\tau_1) \to E(H)$ and $B(\tau_1) \ge 0$, so it remains to show that the positive part of $\int_{v_*(\tau)}^{v(\tau)} \bar\kappa\,\mathrm{d}v'$ tends to 0. For $v' \ge v_*(\tau)$ we have $\bar\kappa(\tau, v') = \kappa(\varrho')$ with $\varrho' \ge r$, and κ decreases; so the integral is at most $\kappa(r)\,v(\tau)$. In the far field $\kappa \le K \le (1+2D_*)p^2$ and $p \le (m_{\text{lin}} + \delta_N)f$ (fact (iv) of Section 3), hence $\kappa(r) \le C\sin^2\tau$ for τ near π, and it suffices to show $\sin^2\tau\, v(\tau) \to 0$.

Now $E(U) \ge \int V(\theta)\,\mathrm{d}x$ and $V(\tau) = 2\mu^2(1+\cos\tau) \ge \frac{4\mu^2}{\pi^2}(\pi-\tau)^2$. The region $\{\theta \le \tau\}$ has the volume $v(\tau)$, and $V \ge V(\tau)$ there, so $(\pi-\tau)^2 v(\tau) \le C_0 := \pi^2 E(U)/(4\mu^2)$. Suppose $(\pi-\tau_k)^2 v(\tau_k) \ge c > 0$ along a sequence $\tau_k \to \pi$. For each k choose an index $j > k$ with $(\pi-\tau_j)^2 \le c\,(\pi-\tau_k)^2/(2C_0)$. The shell between the levels τ_k and τ_j has the volume $v(\tau_j) - v(\tau_k) \ge c/(\pi-\tau_j)^2 - C_0/(\pi-\tau_k)^2 \ge c/(2(\pi-\tau_j)^2)$, and $V \ge \frac{4\mu^2}{\pi^2}(\pi-\tau_j)^2$ on it; so it carries at least the energy $2\mu^2 c/\pi^2$. Infinitely many disjoint such shells contradict $E(U) < \infty$. Hence $(\pi-\tau)^2 v(\tau) \to 0$, and $\sin^2\tau \le (\pi-\tau)^2$. □

In (a) the functions $\nu > 0$ and ω on the sphere were treated as arbitrary, although for a map of $\mathcal{M}$ they have only three parameters. This loses something, and it is what makes the inequality *static*: no velocity is left in it, so that it can be checked state by state.

Lemma 7 (states without squeeze; PROVED)**.** *Let $\mu = 1$ or $\mu = 0.526$ and let $b \ge 0$ be any Lipschitz function of the level. At every level $\tau \in (0,\pi)$ at which b is differentiable, and for every sphere radius, the state with $t = 0$ satisfies $K_{\text{eff}} = \bar K > 0$ and $\Psi \ge 0$.*

Proof. For $t = 0$ we have $m = 0$ and $F(m) = F'(m) = 0$, so the credit drops out of K_{eff} and of Ψ_0, whatever b is. Further $D = \bar D$, and $\Psi = r\,p\,(\mathrm{SI} - N + \delta_Z)$ with $\mathrm{SI} - N = 0$ for spheres above ϱ_Γ and $\mathrm{SI} - N \ge 0$ below (Lemma 5(e)); and $\delta_Z \ge 0$ by Lemma 5(d). □

6 Step 5: the static inequality at every state up to the last level

Purpose of this step. Proposition 6 has reduced the theorem to an inequality for an explicit function of three real variables. This section explains how the inequality is verified on the whole, unbounded set of states. The formal statements and all proofs are in Part II.

6.1 What is verified

Let $(\mu, r_*) = (1,\ r_{105} = 9.5283\ldots)$ or $(0.526,\ r_{100} = 7.4206\ldots)$, and let b be the credit function of the data file for this mass (Theorems 9 and 22 say precisely which function this is). Then:

> *At every level $0 < r \le r_*$, for every sphere radius $R > 0$ and every squeeze $t \in \mathbb{R}$:*
> *$D^2 + 2D + K_{\text{eff}} > 0$ on the sphere, and $\Psi \ge 0$.*

This is Theorem 21 for $\mu = 1$ and Theorem 23 for $\mu = 0.526$ (both CERTIFIED). With Proposition 6(b) it gives: for every $U \in \mathcal{M}$, the energy of U inside the level surface of $\tau_* = g(r_*)$ is at least $S_0 + B$ at that level.

The set of states is unbounded in three directions (large squeezes, large spheres, small spheres), it includes the limit $r \to 0$, and Ψ vanishes on the path. So no single method can cover it. The verification is a patchwork of six tools; Figure 5 shows the regions of one level, and Table 1 shows the complete cover.

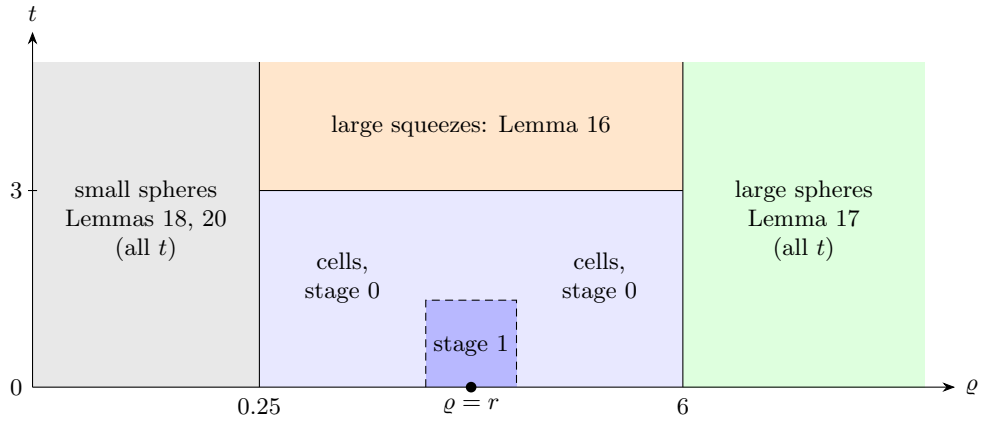

Figure 5: The states of one level with credit (schematic, for $\mu = 1$; not to scale) and the tool that proves $\Psi \geq 0$ in each region. The dot is the hedgehog's own state, where Ψ vanishes together with its first derivatives. Negative squeezes are covered by the symmetry $t \mapsto -t$.

6.2 The six tools

0. Symmetry. Ψ_0 is even in t (Lemma 10), and δ_Z does not depend on t. So only $t \geq 0$ has to be treated.

1. Levels without credit: a supporting tangent (Lemma 19). Where $b = 0$ near the level, the inequality is $\mathrm{SI} - N + \delta_Z \geq 0$ with $K_{\mathrm{eff}} = \bar{K}$. If the graph of $h_{\bar{K}}$ lies above its tangent at $\bar{D}$ on all of $[0, \infty)$, then the integrand of $\mathrm{SI} - N$ is non-negative for *every* squeeze, and nothing has to be computed in t. Lemma 19 gives two explicit conditions (i), (ii) on the pair $(\bar{K}, \bar{D})$ for this. They hold for small field radii, where h is convex enough, and they are verified by interval arithmetic on boxes in two variables. For spheres below ϱ_Γ even less is needed: there the integrand is non-negative by the choice of the field (Remark 2).

2. Large squeezes (Lemma 16). For $t \geq T$ the flux is concentrated near the north pole, and the integral SI can be bounded below by an explicit expression without any integral: the integrand is at least $\max(2\sqrt{K_{\mathrm{low}}},\, 2\sqrt{2}\, D^{3/2})$ for an explicit number K_{low}. The resulting bound for Ψ_0 is uniform in $t \geq T$, and it is evaluated in interval arithmetic on boxes in (r, ϱ). In the far field its first form says that the credit covers the cost of concentrating the whole flux in one point of the sphere.

3. Large spheres (Lemma 17). For $\varrho \geq \varrho_1$ the defect δ_Z is so large that it dominates everything else, uniformly in the squeeze: a sphere that is much too large for its level is expensive, whatever its labels do. The lemma bounds Ψ below by $r\, p\, [\delta_Z(r, \varrho_1) - 4\pi M/\sqrt{K_L}]$ minus a term with the slope of b, with explicit M and K_L.

4. Small spheres (Lemmas 18 and 20). The same idea at the other end: for $\varrho \leq \varrho_+ = 0.25$, and for all still smaller spheres, the defect dominates the credit terms. Here one uses that the credit of a small sphere is small, because of the factor $W(R/r)$, and the supporting tangent of tool 1.

5. The bulk: interval arithmetic on cells (Lemma 13). What remains is a compact set of states: ϱ in a bounded interval away from 0, and $0 \leq t \leq T$. It is divided into cells

levels	spheres	squeezes	tool
$\mu = 1$			
$0 < r \le r_{55}$ (no credit)	below ϱ_Γ	all	Remark 2
	$\varrho \le 0.8$, above ϱ_Γ	all	supporting tangent (tool 1)
	$0.8 \le \varrho \le 9.9$	$t \le 3$ / $t \ge 3$	cells (tool 5) / large squeeze (tool 2)
	$\varrho \ge 9.9$	all	large spheres (tool 3)
$r_{55} \le r \le r_{105}$ (credit)	$\varrho \le 0.25$, and all smaller	all	small spheres (tool 4)
	$0.25 \le \varrho \le 6$	$t \le T$ / $t \ge T$	cells / large squeeze; $T = 3$ or 3.75
	$\varrho \ge 6$	all	large spheres; from r_{88} on: cells and large squeeze for $6 \le \varrho \le 9.9$, large spheres beyond
$\mu = 0.526$			
$0 < r \le r_{52}$ (no credit)	below ϱ_Γ	all	Remark 2
	$\varrho \le 1.1$, above ϱ_Γ	all	supporting tangent
	$1.1 \le \varrho \le 15.9$	$t \le 3$ / $t \ge 3$	cells / large squeeze
	$\varrho \ge 15.9$	all	large spheres
$r_{52} \le r \le r_{100}$ (credit)	$\varrho \le 0.25$, and all smaller	all	small spheres
	$0.25 \le \varrho \le 10$	$t \le T$ / $t \ge T$	cells / large squeeze; $T = 3$, 3.75 or 4
	$\varrho \ge 10$	all	large spheres

Table 1: The cover of all states of all levels up to the last one. Here $r_{55} = 0.7821$, $r_{105} = 9.5283$, $r_{52} = 0.6731$, $r_{100} = 7.4206$. The levels $0 < r \le r_5$ are treated as one cell (tool 6). The precise statements are Theorems 21, 22, 23, and the logs that belong to each line are listed in Section 14.

$[r] \times [\varrho] \times [t_a, t_b]$, and on each cell the program proves $\Psi \ge 0$ for all points of the cell, in one of two ways.

- *Stage 0.* The function Ψ_0 is evaluated with the cell itself as argument, in interval arithmetic, which gives a lower bound valid on the whole cell; a lower bound of $r\,p\,\delta_Z$ on the cell is added. If the sum is non-negative, the cell is verified. Away from the path this works, because δ_Z is large there.
- *Stage 1.* Near the path a plain interval evaluation cannot work, however small the cell: on the path, Ψ and its first derivatives vanish, so the true values are small of second order, and any enclosure of a long formula on a cell of positive width contains negative numbers. Here the program uses the structure of the zero. For cells that touch $t = 0$, it uses $\Psi_0 = \partial_t \Psi_0 = 0$ at $t = 0$, so that $\Psi_0 \ge \frac{1}{2} t^2 \min \partial_t^2 \Psi_0$, and bounds the second derivative from below. For cells with $t_a > 0$ it expands Ψ_0 around the centre of the cell in a Taylor polynomial whose coefficients are evaluated exactly (with intervals of width zero up to rounding), and only the remainders are bounded on the cell.

A cell that cannot be verified is bisected, and a run counts as verified only if every start cell is exhausted by verified cells. The integral over the sphere is computed with a substitution that makes the integrand analytic in the squeeze, including $t = 0$ (Lemma 10), and with a quadrature rule with a rigorous remainder (Lemma 14).

6. The centre. As $r \to 0$ the level spheres shrink to a point, and the cells would have to become infinitely small. For the levels $r \le r_5 = 0.0642$ the inequality is divided by $\sin^2 \tau$; the result depends on the level only through $\cos \tau$, and one cell in the variable $\cos \tau$ covers all these levels at once (proof of Theorem 21).

6.3 Sizes

For $\mu = 1$ the bulk consists of 94048 start cells; 421691 cells were verified by stage 0 and 47809 by stage 1, and the large-squeeze lemma was evaluated on 40097 boxes. For $\mu = 0.526$ the numbers are 325637 cells by stage 0, 44811 by stage 1, and 56984 boxes. Stage 0 settles everything away from the path at a cost of about two milliseconds per cell. The stage-1 cells near the path are about 2 % wide in r and ϱ and 0.1 to 0.15 long in t. Their size is limited by the interval enclosures of the second derivatives, which are far wider than the true ranges; this is the price of evaluating a long formula on intervals, and it is paid only in the remainder terms. The complete verification took about four and a half hours on two processor cores when it was first run, and about twenty minutes on sixteen cores when it was repeated on a second system (Section 8.5).

7 Step 6: the far field, and the proof of Theorem 1

Purpose of this step. The calibration is stopped at a last level τ_*. Step 5 bounds the energy inside the level surface of τ_* from below. Here we bound the energy outside, whatever the map does there, and add the two bounds.

7.1 Notation, and what has to be shown

The following notation is used in this section and in Section 15. Let $\lambda_0(z) = z + 1$, $\lambda_1(z) = z + 1 + \frac{1}{1+z}$ and $m_{\mathrm{lin}}(r) = \mu\big[1 - \frac{1}{\mu r+1} + \frac{2}{\mu r}\big]$, so that $r\, m_{\mathrm{lin}}(r) = \lambda_1(\mu r)$. For a sphere of radius R at a level τ, with $s = \sin\tau$, let

$$Y(R) = 4\pi s^2 R\, \lambda_1(\mu R), \qquad S(R) = \frac{4\pi s^2 R}{1 + \mu R}.$$

We shall see that Y is the Yukawa energy of the unsqueezed boundary values and that S bounds what a squeeze can save; both increase with R. E_Ω denotes the Skyrme energy on a domain Ω.

The last level. Let $(\mu, r_*) = (1, r_{105})$ or $(0.526, r_{100})$, $\tau_* = g(r_*)$, $s = \sin\tau_*$, $f_* = \pi - \tau_*$, and let $b_* = b(r_*)$ be the value of the credit function at this node. The numbers are in Table 3; in particular $f_* \le 1.52 \cdot 10^{-5}$ for $\mu = 1$ and $f_* \le 3.44 \cdot 10^{-3}$ for $\mu = 0.526$, so beyond the last level the hedgehog is very close to the vacuum. For a sphere of radius R at the level τ_*, $v = \frac{4\pi}{3}R^3$, put

$$\begin{aligned} \mathrm{Cal}(R) = E(H) - S_0(\tau_*, v) &= \big[E(H) - E_*(\tau_*)\big] + \int_{v_*(\tau_*)}^{v} \bar\kappa(\tau_*, v')\,\mathrm{d}v', \\ B_*(R) &= \tfrac{9}{8}\, b_*\, W(R/r_*). \end{aligned} \tag{26}$$

The bracket is the energy of H outside the ball of radius r_*. Both Cal and B_* increase with R, because $\bar\kappa > 0$.

The claim. Let $U \in \mathcal{M}$ have finite energy and the state (R, t) at the level τ_*, let $m = m(t)$ be its dipole, and let Ω be the outside of its level surface of τ_*. By Step 5 and Proposition 6(b) the energy of U inside is at least $S_0(\tau_*, v) + b_* F(m)\, W(R/r_*)$, and $F(m) \ge \frac{9}{8}m^2$ because $m^2 \le 1$. So the inside contributes at least $S_0(\tau_*, v) + B_*(R)\, m^2$, and Theorem 1 follows if

$$E_\Omega(U) \;\ge\; \mathrm{Cal}(R) - B_*(R)\, m^2, \tag{27}$$

because then $E(U) \ge S_0(\tau_*, v) + \mathrm{Cal}(R) = E(H)$. In words: $\mathrm{Cal}(R)$ is what the calibration still expects from the outside; a squeezed boundary can make the outside cheaper; and the credit that the squeezed state holds at the last level has to pay for the difference.

7.2 The Yukawa bound

Lemma 8 (exterior bound; PROVED)**.** *Let $\mu > 0$, let Ω be the exterior of a ball of radius R, and let $U : \overline{\Omega} \to S^3$ have finite energy on Ω and the boundary values $U = (-\cos\tau,\ \sin\tau\, n)$ on $\partial\Omega$, where $0 < \tau < \pi$ and n are the labels of Definition 1 with squeeze t (polar angle measured at the centre of the ball). Then*

$$E_\Omega(U) \ \geq\ Y(R) - S(R)\, m^2 \ \geq\ 4\pi s^2 R\,(\mu R + 1).$$

More precisely, the minimum $Q_{\min}(t)$ of $\int_\Omega |\nabla\psi|^2 + \mu^2|\psi|^2$ over all fields $\psi : \Omega \to \mathbb{R}^3$ with the boundary values $s\,n$ satisfies $E_\Omega(U) \geq Q_{\min}(t) \geq Q_{\min}(0) - S(R)\, m^2$ and $Q_{\min}(0) = Y(R)$.

Proof. A quadratic lower bound. Let $\psi = (U_1, U_2, U_3)$. Then $|\mathrm{d}U|^2 \geq |\nabla\psi|^2$, and $2\mu^2(1 - U_0) \geq \mu^2|\psi|^2$ because $|\psi|^2 = 1 - U_0^2 = (1 - U_0)(1 + U_0) \leq 2(1 - U_0)$; the Skyrme term is non-negative. So $E_\Omega(U) \geq \int_\Omega |\nabla\psi|^2 + \mu^2|\psi|^2 \geq Q_{\min}(t)$.

The minimum by harmonics. The minimiser solves $\Delta\psi = \mu^2\psi$ in Ω and decays at infinity. Expand the components of n in spherical harmonics. The part of degree ℓ is continued into Ω by the factor $k_\ell(\mu|x|)/k_\ell(\mu R)$, where k_ℓ is the decaying solution of $z^2k'' + 2zk' = (z^2 + \ell(\ell+1))\, k$. Integrating by parts,

$$Q_{\min}(t) = -\int_{\partial\Omega} \psi \cdot \partial_r\psi \,\mathrm{d}A = s^2 R \sum_\ell \lambda_\ell(\mu R)\, c_\ell, \qquad c_\ell = \|P_\ell n\|^2_{L^2(S^2)}, \quad \lambda_\ell(z) = -\frac{z\, k_\ell'(z)}{k_\ell(z)},$$

and $\sum_\ell c_\ell = 4\pi$. For $\ell = 0, 1$ one has $k_0 = e^{-z}/z$ and $k_1 = e^{-z}(z+1)/z^2$, which gives the functions λ_0, λ_1 defined above.

λ_ℓ increases with ℓ. From the equation of k_ℓ, $z\lambda_\ell' = \lambda_\ell^2 - \lambda_\ell - z^2 - \ell(\ell+1)$. So the difference $d = \lambda_{\ell+1} - \lambda_\ell$ satisfies $zd' = d\,(\lambda_{\ell+1} + \lambda_\ell - 1) - 2(\ell+1)$, and $d' < 0$ at every zero of d. But $d \to 1$ for $z \to 0$ and $d = (\ell+1)/z + O(z^{-2}) > 0$ for $z \to \infty$. If d had a zero, it would be negative just after its largest zero and could not become positive again. Hence $d > 0$.

The part of degree 0. The components n_1, n_2 are proportional to $\cos\varphi$, $\sin\varphi$ and have no part with $\ell = 0$. The component n_3 is the label cosine y, and its mean over the sphere is $-m$ by (7). So $c_0 = 4\pi m^2$, and since $\lambda_\ell \geq \lambda_1$ for $\ell \geq 1$,

$$\sum_\ell \lambda_\ell c_\ell \ \geq\ \lambda_0 c_0 + \lambda_1(4\pi - c_0) = 4\pi\lambda_1 - 4\pi m^2(\lambda_1 - \lambda_0), \qquad \lambda_1 - \lambda_0 = \frac{1}{1+\mu R}.$$

This is $Q_{\min}(t) \geq Y(R) - S(R)\, m^2$. For $t = 0$ the labels are purely of degree 1, so $Q_{\min}(0) = Y(R)$. The last inequality of the lemma is $m^2 < 1$. □

So a squeeze can lower the exterior energy only through the mean of the labels, by at most $S(R)\, m^2$.

7.3 Near the hedgehog's own sphere: comparison with an exact solution

The Yukawa bound cannot prove (27) for spheres near the hedgehog's own. Indeed $\mathrm{Cal}(r_*)$ is the energy of H outside r_*, which is at least $Y(r_*)$ by Lemma 8 with $t = 0$; so the bound $Y - S\, m^2$ lies *below* Cal there, and the margin that is needed is of the relative order s^2. For these spheres the energy is compared with that of an exact solution. The two lemmas are stated and proved in Section 15:

- Lemma 24: for $R \geq 1/\mu$ and small $f_0 > 0$ the profile equation (2) has a solution f on $[R, \infty)$ with $f(R) = f_0$ that decays to 0, with $-f'/f$ within an explicit band around m_{lin}.

- Lemma 25: let Φ be the radial map built from this solution on the exterior Ω of the ball of radius R. Then for every U on Ω with the boundary values of squeeze t,

$$E_\Omega(U) \;\geq\; E_\Omega(\Phi) - (1+\varepsilon)\, S(R)\, m^2,$$

with an explicit small number ε. Nothing is assumed about the size of $U - \Phi$. The error is proportional to m^2 because the expansion is made around an exact solution of the field equations, so that the first-order term reduces to a boundary term.

7.4 Five numerical facts

Proposition 26 (CERTIFIED) provides, for each mass, radii $\varrho_s \leq \varrho_m < r_* < \varrho_b$ and the following facts about explicit functions of one variable. Spheres are parametrised by the field radius, $R(\varrho) = s/q(\varrho)$.

(F1) an upper bound $\bar{E}$ for the energy $E(H) - E_*(\tau_*)$ of H outside r_*;

(F2) $\mathrm{Cal}(R(\varrho_s)) \leq 0$;

(F3) $Y(R) - \mathrm{Cal}(R) \geq \max\{0,\, S(R) - B_*(R)\}$ for $\varrho_s \leq \varrho \leq \varrho_m$;

(F4) the hypotheses of Lemmas 24 and 25 hold for $f_0 = f_*$ and all $R \geq R(\varrho_m)$, with $\varepsilon \leq \bar{\varepsilon}$ on $[\varrho_m, \varrho_b]$, and $B_*(R) \geq (1+\bar{\varepsilon})\, S(R)$ for $\varrho_m \leq \varrho \leq \varrho_b$;

(F5) $\mathrm{Cal}(R) \leq 4\pi s^2 R\,(\mu R + 1)$ for all $R \geq R(\varrho_b)$.

7.5 Proof of Theorem 1

Let $U \in \mathcal{M}$ have finite energy, let (R, t) be its state at the level τ_*, m the dipole, $v = \frac{4\pi}{3}R^3$, and let Ω be the outside of the level surface of τ_*. After a translation this sphere is centred at the origin, and after a reflection $t \geq 0$. By Theorem 21 (for $\mu = 0.526$: Theorem 23) the states of U at the levels below τ_* satisfy the hypotheses of Proposition 6(b). As explained above, it remains to show (27). On $\partial\Omega$, $U = (-\cos\tau_*, \sin\tau_*\, n)$ with the labels of squeeze t. There are four cases, according to the size of the sphere.

Small spheres, $R \leq R(\varrho_s)$. Then $\mathrm{Cal}(R) \leq 0 \leq E_\Omega(U)$ by (F2), because Cal increases.

Spheres with $\varrho_s \leq \varrho \leq \varrho_m$. By Lemma 8 and (F3),

$$E_\Omega(U) \;\geq\; Y(R) - S(R)\, m^2 \;\geq\; \mathrm{Cal}(R) + \max\{0,\, S(R) - B_*(R)\} - S(R)\, m^2 \;\geq\; \mathrm{Cal}(R) - B_*(R)\, m^2;$$

the last step uses $0 \leq m^2 < 1$ (if $S > B_*$, the difference of the two sides is $(S - B_*)(1 - m^2) \geq 0$).

Spheres with $\varrho_m \leq \varrho \leq \varrho_b$. Let f be the solution of Lemma 24 with $f_0 = f_*$ and Φ the map of Lemma 25. We claim $E_\Omega(\Phi) \geq \mathrm{Cal}(R)$. Indeed, continued inside by any map of $\mathcal{M}$ with $c = 0$, $t = 0$, the map Φ is the outer part of a map of $\mathcal{M}$ whose level surfaces beyond τ_* are concentric spheres without squeeze, of radius R at τ_*. For these states $\Psi \geq 0$ with $b = 0$ (Lemma 7), so by Proposition 6(a), integrated from τ_* to τ_1, the energy of Φ between these levels is at least $S_0(\tau_1, v_\Phi(\tau_1)) - S_0(\tau_*, v)$. For $\tau_1 \to \pi$ the lower limit of $S_0(\tau_1, v_\Phi(\tau_1))$ is at least $E(H)$, as in the proof of Proposition 6(c), because Φ has finite energy. This proves the claim. By Lemma 25 and (F4),

$$E_\Omega(U) \;\geq\; \mathrm{Cal}(R) - (1+\bar{\varepsilon})\, S(R)\, m^2 \;\geq\; \mathrm{Cal}(R) - B_*(R)\, m^2.$$

Large spheres, $R \geq R(\varrho_b)$. By Lemma 8 and (F5), $E_\Omega(U) \geq 4\pi s^2 R\,(\mu R + 1) \geq \mathrm{Cal}(R)$.

This proves (27) and the theorem. □

Remark 1. (a) Outside the level surface of τ_* only the boundary values of U and the finiteness of $E_\Omega(U)$ were used. So the theorem holds for every finite-energy map that has the structure of $\mathcal{M}$ on the levels below τ_* and is an arbitrary map with values in S^3 outside.

(b) If $t(\tau) = 0$ for all $\tau \geq \tau_*$, the inequality follows from Steps 1 to 5 alone, without the far field: continue b beyond r_* by any non-negative Lipschitz function; at the levels below τ_* the hypotheses of Proposition 6 hold by Theorems 21 and 23; at the levels $\tau \geq \tau_*$ the squeeze is 0 and Lemma 7 applies; then use Proposition 6(c).

Sizes. At $R = r_*$ the squeeze can lower the outside energy by at most $S(r_*)\, m^2 = 11.4\, s^2 m^2$ for $\mu = 1$ and $19.0\, s^2 m^2$ for $\mu = 0.526$, and the credit at the last level is at least $B_*(r_*)\, m^2 = 48.9\, s^2 m^2$ resp. $32.7\, s^2 m^2$. Over the range of (F4) the ratio $B_*/((1+\bar{\varepsilon})S)$ is at least 1.65 resp. 1.20; it is smallest at the smallest sphere. For $\mu = 0.526$ the credit of the last level does not cover S at spheres much smaller than $R(\varrho_m)$; there the difference $Y - \mathrm{Cal}$ is already large, which is (F3). The bound $S\, m^2$ is crude: for small squeezes the part of degree 2 of the squeezed labels costs most of what the part of degree 0 saves.

8 The computer-assisted steps: what is certified, and how

Purpose of this section. The proof uses a computer in three places: for the profile, for the static inequality, and for five numbers of the far field. This section states what the computer establishes in each place, how the statement is guaranteed in spite of rounding errors, why the finitely many computations cover a continuum of states, and what was only *found* with the help of a computer and is no part of the proof.

8.1 Interval arithmetic

All certified statements are inequalities between explicit functions. They are verified with interval arithmetic: every number is replaced by an interval that contains it, and every operation returns an interval that contains all possible results. If the final interval lies on the right side of zero, the inequality is proved for all arguments in the input intervals. Rounding errors cannot spoil this, because each computed end point is moved outwards. Two independent implementations are used.

- *The profile* (Part III). Decimal interval arithmetic with 50 digits and directed rounding (the `decimal` module of Python), interval Taylor series of order 24. Programs: `flat/cert_lohner.py` with `flat/cert_massive.py`, `flat/cert_props.py` and the interval layer `sphere/cert_profile2.py`.
- *The static inequality and the far field* (Part II). Double precision. Each of $+, -, \times, /, \sqrt{}$ is computed in rounding to nearest and the end point is then moved outwards by at least one unit in the last place. No function of a numerical library is used: exp, sin, cos, ln are Taylor polynomials in interval arithmetic with explicit bounds for the tails. A division by an interval that contains 0, a square root of an interval that is not positive, or an overflow aborts the evaluation of the cell, which then counts as not verified. Programs: `flat/siv/si_verify.cpp` with the interval library `flat/siv/siv_num.hpp` (Section 12).

The two are connected by one data file for each mass, `mu1b.dat` and `mu0526b.dat` in the folder `flat/siv/`. Such a file contains the polynomial enclosures of the profile, with the interval ends rounded outwards to double precision numbers (program `flat/siv/export_models.py`), and the nodes and values of the credit function b.

	statement	what is checked	mode	logs
C1	Theorems 27, 28	existence of the profile, enclosures, energy, sign conditions	(`cert_lohner.py`)	`flat/logs_lohner_*.log`
C2	Theorems 9, 21–23	$\Psi \geq 0$ and $D^2 + 2D + K_{\text{eff}} > 0$ on all cells of compact blocks of states	`run`	`A*`, `B*`, `C*`, `D*`
C3	Lemma 16	its lower bound is ≥ 0 for all $t \geq T$, on boxes in (r, ϱ)	`tail`	`TAIL_*`
C4	Lemma 17	its lower bound is ≥ 0 on intervals of levels	`bigsphere`	`BIG_*`
C5	Lemmas 18, 20	their lower bound is ≥ 0 on intervals of levels	`smallsphere`	`SMALL_*`
C6	Lemma 19	conditions (i), (ii) on boxes in two variables	`supp`	`SUPP_*`
C7	Proposition 26	the numerical facts (F1)–(F5)	`farfield`	`FARFIELD_*`

Table 2: The computer-assisted statements. The logs of C2–C7 are in `flat/siv/runs/`; there are 74 of them for $\mu = 1$ and 53 for $\mu = 0.526$, each ending with `VERIFIED`. The centre cell has its own logs `CC_*` and `DC_*`. The table of Section 14 assigns the logs to the regions of Table 1.

8.2 The certified statements

Table 2 lists every statement of the proof that rests on a computation. Each line names the mathematical statement, the place in this paper where it is formulated, the mode of the program, and the logs. Every log begins with the command line that produced it and ends with the word `VERIFIED`.

8.3 Why finitely many computations cover all states

No sampling. A run of the verifier is given a product of intervals in the variables (r, ϱ, t), or in fewer variables in the modes that evaluate a lemma. It tiles this set by start cells. For each cell it either proves the inequality for all points of the cell, by the bounds of Lemma 13 or of the lemma in question, or it bisects the cell and treats the parts. It prints `VERIFIED` only if every start cell is exhausted by verified cells. At no point is the inequality tested at sample points only.

The pieces cover everything. The union of the pieces must be the set of all states of all levels up to the last one. This bookkeeping is done by a program, `flat/siv/audit_cover.py`, which does not trust file names: it reads the command line and the last line of every log, builds the list of verified pieces (levels) $\times$ (spheres) $\times$ (squeezes), and checks for every cell $[r_k, r_{k+1}]$ of the credit nodes, and for the centre cell, that the pieces cover all spheres and all squeezes $t \geq 0$. Its output is the content of the tables of Section 14. A second program, `make_cert_numbers_partial.py`, does the accounting by the rules of the proofs of Theorems 21 and 23, and the two agree.

The nodes of the credit function. The function b is piecewise linear in $\ln r$, so $\mathrm{d}b/\mathrm{d}\ln r$ jumps at the nodes. The verification is done cell by cell between the nodes, with the one-sided derivative of the cell, on the closed cell. The nodes are a null set of levels, and Proposition 6 needs the inequality at almost every level only.

8.4 What was found by computer but is not part of the proof

Several objects in the proof were found by floating-point computations. The proof does not rely on how they were found; it treats them as given data and verifies what it needs about them.

- *The slope of the profile at the centre.* A floating-point shooting method gives a number

a_0. The certificate C1 then proves that a solution with the required properties exists with a slope within ρ of a_0.

- *The credit function.* The values of b at the nodes were produced by `flat/cert_design2.py`, which integrates the leading-order form of the static inequality inwards, in floating point, with safety margins. The result is a list of numbers. What is proved about this list is C2–C6. (An independent rigorous check of the leading-order form of the inequality on the path is described in Section 9.)
- *The decomposition.* The last level, the ranges of ϱ, the thresholds T, the sizes of the cells and the rules for bisection were chosen by trial. They decide only whether the program succeeds, and how fast.
- *The radii $\varrho_s, \varrho_m, \varrho_b$ of the far field* were chosen by looking at floating-point values; (F1)–(F5) are then verified for them.

8.5 A complete re-run on a second system

The logs C2–C7 were written on Linux by binaries compiled with GCC, and by several successive versions of the verifier (the source of each version is kept, with checksums, in `flat/siv/runs/`). For this paper the whole verification was repeated on a second system: Windows, the Microsoft compiler with strict floating-point semantics, and one binary, built from the final source, for all runs. A script re-executes the command line of every log and compares the verdicts.

- All 127 runs that ended with `VERIFIED` did so again; the total run time was 20 minutes on 16 cores.
- The accounting program, applied to the new logs, again reports that all states of all levels up to r_{105} ($\mu = 1$) and r_{100} ($\mu = 0.526$) are covered.
- The outputs of the far-field runs agree with the original ones digit for digit.
- The numbers of cells differ, because the final version of the search is more economical than the early versions that wrote most of the original logs: 511935 cells verified by stage 0 and 86621 by stage 1 in the re-run, against 817711 and 92620 in the original logs (all runs of both masses, including those far from the path). The statements that are verified, i.e. the sets of states given by the command lines, are the same.
- The self-test of the interval library (`flat/siv/selftest.cpp`) reports no failure on the second system.

The profile certificate C1 was re-run on the second system as well, for both masses (about half an hour each). The results agree with the stored ones in every digit: the energy intervals, the band data, the certified properties, all boxes and all polynomial models.

A re-run on a second compiler and operating system guards against errors of the compiler, of the floating-point environment and of the hardware. It does not guard against an error in the mathematics or in the program, which are the same in both runs.

9 Checks that were made, and what is not proved

9.1 What is not proved

- *Maps outside $\mathcal{M}$.* The theorem says nothing about maps whose level surfaces are not round spheres, or whose labels are not Möbius images of the hedgehog's labels, except for the part of a map outside the last level (Remark 1). It does not prove that the hedgehog minimises the energy among all maps of degree one.

- *Equality and uniqueness* are not discussed.
- *Other masses.* The computations were made for $\mu = 1$ and $\mu = 0.526$ only. Each mass needs its own profile enclosure, credit function and runs. The case $\mu = 0$ would need a different treatment of the far field (the hedgehog then decays like a power, and Lemma 8 needs $\mu > 0$).
- *One fact about the profile equation is quoted, not proved:* for every $a > 0$ the profile equation has exactly one solution with $g(r) = ar + O(r^3)$ at the centre, and it depends continuously on a. This is a standard property of a singular initial-value problem of this type, and it is used in the proof of Proposition 30. Existence for a given slope is covered by the argument at the centre in Part III; uniqueness and continuous dependence are not reproved there.
- *Independent verification.* The arguments of this paper and the programs were produced primarily by AI models at the direction of the author, and no independent person has verified them. This is the main reservation.

9.2 Checks of the programs

None of the following is part of the proof.

1. *Self-test of the interval library* (`flat/siv/selftest.cpp`). The elementary functions against the system library at $2 \cdot 10^5$ random points, containment for interval arguments, the series used for the integration over the sphere in all ranges, and the automatic derivatives against finite differences: no failure, on both systems.
2. *Comparison with an independent floating-point program* (`flat/siv/si_ref.py`: adaptive quadrature over the sphere, and δ_Z computed in two ways). At 8 states the reference values of Ψ_0 and of δ_Z lie in the enclosures of the verifier, and the two forms of δ_Z agree to 11 digits. The reference program in turn agrees with the program in which the static inequality was first formulated, which computes the calibration field by quadrature and the geometry of the squeezed sphere from a general surface code, at 80 states with relative differences of at most $6 \cdot 10^{-11}$ (`flat/siv/cross_check_fam.py`).
3. *Derivatives.* The gradients computed by the verifier agree with central differences of point values at 5 states to 5 or 6 digits.
4. *Containment* (`flat/siv/containment.cpp`). On 141 random cells near the path the enclosures of Ψ_0, of its Taylor coefficients in t, of the gradients and of the second derivatives contain the values at three random interior points each.
5. *Negative control.* With $b \equiv 0$ the levels $r \in [1.80, 1.85]$, where the inequality is false without credit, are reported as *not* verified.
6. *Sharpness.* The value of b at the node r_{71} was raised so that the slope of b on $[r_{70}, r_{71}] = [1.656, 1.741]$ grows by 0.45, and by 0.60. In the first case the static inequality is still true, with a quarter of the margin, and the cell is verified. In the second case point evaluations show $\Psi_0 < 0$ at small t on the path for $r > 1.714$, and the run is rejected; the first failing cell is reported at $r = 1.7136$. So the verifier separates a true inequality from a slightly false one.
7. *The leading order on the path, by a different program.* The limit $t \to 0$ of the static inequality on the path is a differential inequality of Riccati type for b. It was verified for the credit function of $\mu = 1$ on $[\frac{1}{16}, 10]$ with a different program, in decimal interval arithmetic, with a margin of at least 22.8 % of the size of its terms (`flat/cert_design_leading.py`; the formulas are in the technical version, in the section on the credit function). This is the most

delicate corner of the certificate, and the stage-1 bounds of the verifier at $t \to 0$ are consistent with it.

8. *Accounting* (`flat/siv/audit_cover.py`), see Section 8.3.
9. *The re-run on a second system*, see Section 8.5.

9.3 Checks of the mathematics by floating-point experiments

Again none of this is part of the proof. All of it is FLOAT.

1. *The energy formula* (`flat/siv/check_lagrangian.py`). The energy density of Lemma 2 against the definition (1) at 2000 random points of random maps of $\mathcal{M}$, with the Jacobians of the parametrisation by finite differences: relative difference at most $4 \cdot 10^{-9}$. The identity (25) for random velocities: $6 \cdot 10^{-12}$. Lemma 5(a),(b) against finite differences at about 600 states per mass, with the sign rule.
2. *The static inequality at random states* (`flat/siv/validate_float2.py`, for each mass). Ψ at 1500 random states on all levels up to r_*, half of them within 15 % of the path, and comparisons of the bounds of Lemmas 16, 17, 20 and of the conclusion of Lemma 19 with direct quadratures at 380 states: no violation.
3. *The theorem itself* (`flat/siv/test_family_energy.py`, $\mu = 1$). The energy of 236 random maps of $\mathcal{M}$ with deformations of order one, computed from Lemma 2: the smallest value of $E(U) - E(H)$, divided by the square of the size of the deformation, is $+1.4 \cdot 10^{-2}$. Local minimisations of $E(U) - E(H)$ with a squeeze prescribed at one level and all other parameters free end at positive values; the smallest, for a squeeze of 0.3 at $r = 6.5$, is $8.5 \cdot 10^{-7} \cdot 12\pi^2$. So the inequality is true in these experiments, and it is tight in the far field.
4. *The far field* (`flat/siv/farfield_float.py`, `flat/siv/test_cmp.py`). The numbers of Proposition 26 in floating point, with an independent profile, agree with the certified bounds. For five choices of (μ, R, f_0) the radial solution of Lemma 24, computed by shooting, lies in the band of the lemma. On about 390 random axially symmetric trial maps for each choice, with squeezes up to 3.5, the exact expansion (28) and the bounds of Lemmas 8 and 25 hold; the closest cases of (28) have relative margins between $3 \cdot 10^{-11}$ and $4 \cdot 10^{-5}$, and the difference of its two sides is of second order in the deformation, which tests the boundary term.
5. *The profile.* The checks of the profile certificate are listed at the end of Part III.

Part II. Technical lemmas and certified statements

About this part. Part II contains what Step 5 and Step 6 of Part I rely on: the precise statements that were verified by interval arithmetic (Theorems 9, 21, 22, 23 and Proposition 26), and the lemmas, with complete proofs, that reduce the infinite set of states to these statements. The text of this part is taken from the technical version of this work with small editorial changes; it is denser than Part I. Each lemma is PROVED by the argument printed after it. Each theorem is CERTIFIED: its proof consists of the lemmas together with computations whose logs are named.

10 The certified block of states

What this section does. It states the central computer-assisted result for $\mu = 1$: the static inequality on the compact block of states around the path that no lemma covers. The nodes r_i

and the values b_i of the credit function are fixed here once and for all, as the numbers stored in a data file.

Theorem 9 (certified block; CERTIFIED)**.** *Let $\mu = 1$ and let b be the function that is linear in $\ln r$ between the nodes r_i, $i = 0, \dots, 115$, with the values b_i stored in* `flat/siv/mu1b.dat` *(r_i are the double-precision numbers stored there, equal to $\frac{1}{20}e^{i/20}$ up to a relative error below $3 \cdot 10^{-14}$; $b_i = 0$ for $i \le 55$, $\max b_i = 6.77 \cdot 10^{-3} \cdot 12\pi^2$ at $r_{65} = 1.2895\dots$). Then for every*

$$r \in [r_{46}, r_{105}] = [0.49870\dots,\ 9.52831\dots], \qquad \varrho \in [0.25,\ 6], \qquad t \in \mathbb{R},$$

$D^2 + 2D + K_{\text{eff}} > 0$ on the whole sphere and $\Psi(r, \varrho, t) \ge 0$; at the nodes this holds with either one-sided derivative of b.

The proof is the computation of Section 13, by the method of Sections 11, 12; negative t by the symmetry of Lemma 10.

11 Lemmas

What this section does. The first six lemmas are the tools of the interval verification on cells. Lemma 10 introduces a variable w on the sphere in which the integrand is analytic in the squeeze, including $t = 0$, and shows that Ψ_0 is even in t and vanishes to second order at $t = 0$. Lemma 11 gives the defect δ_Z in the coordinates (r, ϱ) as an integral with an integrand of known sign, so that lower Riemann sums are lower bounds. Lemma 12 writes the integrand of $\mathrm{SI} - N$ without cancellation. Lemma 13 is the statement that a cell is verified, in the two stages described in Section 6. Lemmas 14 and 15 are Taylor's formula with remainder, for the quadrature over the sphere and for the functions of the squeeze. The last three lemmas, 16, 17 and 18, are the explicit bounds for large squeezes, large spheres and the smallest spheres.

Lemma 10 (the sphere variable w; PROVED)**.** *For $t \neq 0$ put $x = (\cosh t - e^{-tw})/\sinh t$, $w \in [-1, 1]$. Then $j = e^{tw}$, $y = (e^{tw} - \cosh t)/\sinh t$, $1 - y^2 = (1 - x^2)j^2$, $\mathrm{d}x = t\, e^{-tw}\mathrm{d}w/\sinh t$, and $\int_{-1}^{1}(j^2 - 1)\,\mathrm{d}x = 0$. With $e_1(z) = (1 - e^{-z})/z$,*

$$x = \tanh\tfrac{t}{2} + \frac{t}{\sinh t}\, w\, e_1(tw), \qquad y = \frac{t}{\sinh t}\, w\, e_1(-tw) - \tanh\tfrac{t}{2},$$
$$j^2 - 1 = tw\, e_1(-tw)\,(e^{tw} + 1),$$

so the integrand of $\mathrm{SI} - N$ times $\mathrm{d}x/\mathrm{d}w$ is analytic in (t, w) including $t = 0$. Ψ_0 is even in t, and $\Psi_0 = \partial_t \Psi_0 = 0$ at $t = 0$ for all (r, ϱ).

Proof. Substitution; $\int_{-1}^{1} j^2 \mathrm{d}x = \big[1/(\sinh t\,(\cosh t - x \sinh t))\big]_{-1}^{1} = 2$. The map $(t, w) \mapsto (-t, -w)$ leaves tw, j, D and $\mathrm{d}x$ unchanged and changes the signs of x, y, m and $F'(m)$, hence leaves K_{eff} and F unchanged; so Ψ_0 is even. At $t = 0$: $m = F = \beta = 0$, $j = 1$, $K_{\text{eff}} = \bar{K}$, $D = \bar{D}$. □

Lemma 11 (the defect; PROVED)**.** *For $r, \varrho > r_Y$,*

$$\delta_Z = 16\pi s^2 \int_r^{\varrho} \frac{1 + D_*}{\bar{P}\,\varrho'\,q^3} \Big\{ (p^2 - D_*)\bar{P}X + \delta\Big[\varphi_1 + \frac{(p^2 - D_*)\cos g}{\bar{P} + p}\Big]\Big\} \mathrm{d}\varrho',$$

$X = \cos g(r) - \cos g(\varrho')$, $\delta = 2\mu^2 X/(1 + 2D_*)$, $\bar{P} = \sqrt{p^2 + \delta}$ (profile quantities at ϱ'). The integrand has the sign of $\varrho' - r$; in particular $\delta_Z \ge 0$, and it is bounded from below by any sum of terms (lower bound of the integrand on a cell) × (length) over disjoint cells between r and ϱ.

Proof. $\delta_Z = \int Z\, 4\pi R'^2 \mathrm{d}R'$ from the radius r of the path to R (Section 4); put $R' = s/q(\varrho')$, $\mathrm{d}R' = s\,\varphi_1\,\mathrm{d}\varrho'/(\varrho' q^2)$, and insert the closed form of Z (Lemma 5), whose hypotheses are $\varphi_1 > 0$, $p > q$, $\kappa \geq 0$ and $\bar{K} \geq D_*^2 - D_*$; the last one holds because $\bar{K} \geq \kappa(\varrho') > D_*^2 - D_*$ for $\varrho' > r_Y$. $\bar{P}^2 = (\bar{K} + 2D_* + D_*^2)/(1 + 2D_*) > 0$. Sign: g is increasing, so X has the sign of $\varrho' - r$, and so has δ; $p^2 > D_*$; and the square bracket is positive: for $\cos g \geq 0$ trivially, for $\cos g < 0$ because $(p^2 - D_*)/(\bar{P} + p) < p$ and $\varphi_1 = q + p|\cos g|$. □

Lemma 12 (Jensen remainder without cancellation; PROVED)**.** *Let $P(D) = (1 + 2D)(D^2 + 2D + K)$, $h = 2\sqrt{P}$. Then*

$$h(D) - h(\bar{D}) - h'(\bar{D})(D - \bar{D}) = \frac{(D - \bar{D})^2}{(\sqrt{P} + \sqrt{\bar{P}})\sqrt{\bar{P}}} \Big[2Q_2 \sqrt{\bar{P}} - \frac{P'(\bar{D})\, Q_1}{\sqrt{P} + \sqrt{\bar{P}}} \Big],$$

$\bar{P} = P(\bar{D})$, $Q_1 = 2(D^2 + D\bar{D} + \bar{D}^2) + 5(D + \bar{D}) + 2 + 2K$, $Q_2 = 2D + 4\bar{D} + 5$.

Proof. $P(D) - P(\bar{D}) = (D - \bar{D})Q_1$ and $Q_1 - P'(\bar{D}) = (D - \bar{D})Q_2$ ($P = 2D^3 + 5D^2 + (2 + 2K)D + K$); then $h(D) - h(\bar{D}) = 2(D - \bar{D})Q_1/(\sqrt{P} + \sqrt{\bar{P}})$ and $h'(\bar{D}) = P'(\bar{D})/\sqrt{\bar{P}}$. □

Here $D - \bar{D} = \bar{D}(j^2 - 1)$ carries the explicit factor t of Lemma 10. The program uses this form for $|tw| \leq \frac{3}{4}$ and the plain difference otherwise (for large $D/\bar{D}$ the bracket loses accuracy).

Lemma 13 (cells; PROVED)**.** *Let $B = [r] \times [\varrho]$ be a box inside one cell of the credit nodes, with centre c and half-widths h_r, h_ϱ, let $[t_a, t_b] \subset [0, \infty)$ have centre t_c and half-length h_t, and let $d \geq 0$ be a lower bound of $r\, p\, \delta_Z$ on B.*

(a) Stage 0. *If $\underline{\Psi}$ is a lower bound of Ψ_0 on $B \times [t_a, t_b]$ and $\underline{\Psi} + d \geq 0$, then $\Psi \geq 0$ on the cell.*

(b) Stage 1, $t_a > 0$. *Let $a_0, \ldots, a_4$ be the Taylor coefficients of $t \mapsto \Psi_0(c, t)$ at t_c and A_5 a bound of $|\partial_t^5 \Psi_0(c, s)|/5!$ for $s \in [t_a, t_b]$; let $\gamma^0, \ldots, \gamma^{n-1} \in \mathbb{R}^2$ be the Taylor coefficients of $t \mapsto \nabla\Psi_0(c, t)$ at t_c (∇: gradient in (r, ϱ)) and $G = (G_r, G_\varrho)$ a componentwise bound of $|\partial_t^n \nabla\Psi_0(c, s)|/n!$ on $[t_a, t_b]$; let $H_{rr}, H_{r\varrho}, H_{\varrho\varrho}$ bound the absolute values of the second derivatives of Ψ_0 in (r, ϱ) on $B \times [t_a, t_b]$. For the four sign pairs σ let $P_\sigma(u) = \sum_{k<5} a_k u^k + \sum_{k<n} (\sigma_r h_r \gamma_r^k + \sigma_\varrho h_\varrho \gamma_\varrho^k) u^k$. Then on the cell*

$$\Psi_0 \;\geq\; \min_{\sigma} \min_{|u| \leq h_t} P_\sigma(u) - A_5 h_t^5 - (G_r h_r + G_\varrho h_\varrho) h_t^n - \tfrac{1}{2}\big(H_{rr} h_r^2 + 2H_{r\varrho} h_r h_\varrho + H_{\varrho\varrho} h_\varrho^2\big).$$

(c) Stage 1, $t_a = 0$. *Let the data of (b) be given for $\partial_t^2 \Psi_0$ in place of Ψ_0 and let $\underline{\gamma}$ be the resulting lower bound of $\partial_t^2 \Psi_0$ on $B \times [0, t_b]$. Then $\Psi_0(x, t) \geq \frac{1}{2} t^2 \underline{\gamma}$ on the cell; so $\Psi \geq 0$ there if $\underline{\gamma} \geq 0$ or $d + \frac{1}{2} t_b^2\, \underline{\gamma} \geq 0$.*

Proof. (b) For $x = c + \Delta \in B$ and fixed t, Taylor's formula in (r, ϱ) along the segment from c to x gives $\Psi_0(x, t) = \Psi_0(c, t) + \nabla\Psi_0(c, t) \cdot \Delta + \frac{1}{2}\Delta^\top \mathrm{Hess}\, \Psi_0(\xi, t) \Delta$; Ψ_0 is twice differentiable on B because b is a smooth function of r on the closed cell of the nodes. For fixed Δ, Taylor's formula in t for the two functions $\Psi_0(c, \cdot)$ and $\nabla\Psi_0(c, \cdot) \cdot \Delta$, with Lagrange remainders, gives the polynomial with Δ in place of $(\sigma_r h_r, \sigma_\varrho h_\varrho)$ and the two remainder bounds. The polynomial part is affine in Δ for each t, so its minimum over the box is attained at a corner.

(c) By Lemma 10, $\Psi_0(x, t) = \int_0^t (t - s)\, \partial_t^2 \Psi_0(x, s)\, \mathrm{d}s \geq \frac{1}{2} t^2 \underline{\gamma}$. □

In (b) the program uses $n = 2$ for $t_a < 1.2$ and $n = 1$ above; in (c) the gradient of $\partial_t^2 \Psi_0$ at c is expanded to the same order as its value (degree 2, remainder from the fifth coefficients). Minima of polynomials over $|u| \leq h_t$ are bounded on 8 pieces by the value at the midpoint minus the bound of the derivative on the piece times the half-length.

Lemma 14 (quadrature; PROVED)**.** *If f is six times continuously differentiable on $[w_c-h, w_c+h]$ with Taylor coefficients f_k at w_c, and the interval F_6 contains $f^{(6)}(w)/6!$ for all w of the cell, then $\int_{w_c-h}^{w_c+h} f\,\mathrm{d}w \in 2\big(f_0 h + \frac{1}{3}f_2 h^3 + \frac{1}{5}f_4 h^5\big) + \frac{2}{7}h^7 F_6$. The same holds for the derivatives of the integral with respect to parameters, with the derivatives of f_k and of F_6.*

Lemma 15 (functions of the squeeze on an interval; PROVED)**.** *Let ϕ be analytic on $[t_c-h, t_c+h]$ with Taylor coefficients $c_k(t) = \phi^{(k)}(t)/k!$. Then for $|u| \le h$ and $k < J$*

$$c_k(t_c+u) \in \sum_{j<J-k} \binom{k+j}{k} c_{k+j}(t_c)\, u^j + \binom{J}{k} C_J\, u^{J-k},$$

where C_J is any interval containing c_J on the whole interval.

Both lemmas are Taylor's formula with Lagrange remainder. Lemma 15 is applied to $t/\sinh t$, $\tanh\frac{t}{2}$, m, F, $F'(m)$ with $J = k_{\max}+9$ on pieces of length ≤ 0.1 (and the hull over the pieces is taken): the direct evaluation of $m = (\sinh 2t - 2t)/(\cosh 2t - 1)$ on a t-interval loses all accuracy.

Lemma 16 (large squeeze; PROVED)**.** *Let $T \ge 1$, $F_T = F(m(T))$, $F'_T = F'(m(T))$, and at a state (r, ϱ) with $b \ge 0$ let*

$$B_\infty = \frac{9}{8}\,\frac{b\,x_R W'}{4\pi R^3}, \quad c_1 = \frac{b\,W\,F'_T}{4\pi R^3}, \quad c_2 = \frac{(b\,W\,F'_T)^2}{64\pi^2 s^2 R^4}, \quad K_{\mathrm{low}} = \bar K - B_\infty - 3c_1 - c_2, \quad a_1 = \frac{c_1}{\bar D}.$$

If $K_{\mathrm{low}} > 0$ and $a_1 \le 2$, then for all $t \ge T$: $D^2 + 2D + K_{\mathrm{eff}} > 0$ on the sphere and

$$\Psi_0 \;\ge\; r\,p\,2\pi R^2\big(\Lambda - 2h_{\bar K}(\bar D)\big) + \min\big(F_T\,C, \tfrac{9}{8}C\big), \qquad C = b\,x_R W' - W\,\frac{\mathrm{d}b}{\mathrm{d}\ln r},$$

with $\Lambda = 4\sqrt{K_{\mathrm{low}}}$; and if moreover $\bar D e^{-2T} \le D_0 \le \bar D e^{2T}$ for $D_0 = (K_{\mathrm{low}}/2)^{1/3}$, also with

$$\Lambda = 4\sqrt{K_{\mathrm{low}}} + \frac{2\sqrt{2}\,\bar D^{3/2} e^{2T} - \sqrt{\bar D}\,\big(2\sqrt{2}\,D_0 + 4\sqrt{K_{\mathrm{low}}/D_0}\big)}{2\sinh T}.$$

Proof. $m' = 2(t\cosh t - \sinh t)/\sinh^3 t > 0$, $m < 1$, and $m(1) = 0.5889 > 1/\sqrt{3}$. F increases on $[0,1]$ and $F'(m) = \frac{9}{2}m(1-m^2)$ decreases on $[1/\sqrt{3}, 1]$. So for $t \ge T$: $F \in [F_T, \frac{9}{8})$ and $0 \le F'(m) \le F'_T$, hence $B_v \le B_\infty$ ($x_R W' = 3W(1-W) \ge 0$) and the coefficients of the two other terms of K_{eff} are at most c_1, c_2. On the sphere $|x| \le 1$, so $x(j^2-3) - m \le \max(j^2-3, 3) \le j^2+3$, and $0 \le (1-y^2)/(1+D) \le 1$. With $j^2 = D/\bar D$ this gives $K_{\mathrm{eff}} \ge K_{\mathrm{low}} - a_1 D$, so $D^2 + 2D + K_{\mathrm{eff}} \ge D^2 + (2-a_1)D + K_{\mathrm{low}} > 0$ and

$$h_{K_{\mathrm{eff}}}(D) \;\ge\; \tilde h(D) := 2\sqrt{(1+2D)(D^2 + (2-a_1)D + K_{\mathrm{low}})} \;\ge\; \max\big(2\sqrt{K_{\mathrm{low}}},\, 2\sqrt{2}\,D^{3/2}\big).$$

By Lemma 10, $D = \bar D e^{2tw}$ and $\mathrm{d}x = c_t\, D^{-3/2}\mathrm{d}D$ on $[\bar D e^{-2t}, \bar D e^{2t}]$, $c_t = \sqrt{\bar D}/(2\sinh t)$, of total mass 2. Hence $\int \tilde h\,\mathrm{d}x \ge 4\sqrt{K_{\mathrm{low}}}$, and, since $\tilde h - 2\sqrt{K_{\mathrm{low}}} \ge 0$ everywhere and $\ge 2\sqrt{2}D^{3/2} - 2\sqrt{K_{\mathrm{low}}} \ge 0$ for $D \ge D_0$ (and $[D_0, \bar D e^{2t}]$ lies in the range of D for every $t \ge T$),

$$\int \tilde h\,\mathrm{d}x \;\ge\; 4\sqrt{K_{\mathrm{low}}} + c_t \int_{D_0}^{\bar D e^{2t}} \big(2\sqrt{2} - 2\sqrt{K_{\mathrm{low}}} D^{-3/2}\big)\mathrm{d}D \;\ge\; 4\sqrt{K_{\mathrm{low}}} + \frac{\alpha e^{2t} - \gamma}{2\sinh t},$$

$\alpha = 2\sqrt{2}\,\bar D^{3/2}$, $\gamma = \sqrt{\bar D}\,(2\sqrt{2}D_0 + 4\sqrt{K_{\mathrm{low}}/D_0})$. The last expression increases in t as soon as $\tanh t \ge \frac{1}{2}$: its derivative is $[\alpha e^{2t}(2\sinh t - \cosh t) + \gamma\cosh t]/(2\sinh^2 t)$. Finally $\mathrm{SI} - N = 2\pi R^2(\int h_{K_{\mathrm{eff}}}(D)\,\mathrm{d}x - 2h_{\bar K}(\bar D))$ and $\Psi_0 = r\,p\,(\mathrm{SI} - N) + F\,C$. □

The bound of the lemma involves no integral; it is evaluated in interval arithmetic on boxes in (r, ϱ) (with bisection) and added to the lower bound of $r\,p\,\delta_Z$. In the far field its first form is the statement that the credit covers the cost of concentrating the whole flux in a point of the sphere.

Lemma 17 (large spheres; PROVED)**.** *Fix a level r and $\varrho_1 > r$, and put $s = \sin g(r)$, $V = 2\mu^2(1+\cos g(r))$, $q_1 = q(\varrho_1)$, $\kappa_1 = \kappa(\varrho_1)$, $R_1 = s/q_1$,*

$$K_L = V - \Big(\frac{27}{32} + 3\sqrt{3}\Big)\frac{b}{4\pi R_1^3} - \frac{3b^2}{64\pi^2 s^2 R_1^4},$$

$$M = s^2\big[2 + 2(V+\kappa_1) + 5q_1^2 + 2q_1^4\big] + \Big(\frac{27}{32} + 3\sqrt{3}\Big)\frac{b}{4\pi R_1} + \frac{3b^2}{64\pi^2 s^2 R_1^2}.$$

If $b \geq 0$, $K_L > 0$ and $\sqrt{3}\,b/(4\pi R_1 s^2) \leq 2$, then for all $\varrho \geq \varrho_1$ and all t: $D^2 + 2D + K_{\text{eff}} > 0$ on the sphere and

$$\Psi(r,\varrho,t) \;\geq\; r\,p\,\Big[\delta_Z(r,\varrho_1) - \frac{4\pi M}{\sqrt{K_L}}\Big] - \frac{9}{8}\max\Big(\frac{\mathrm{d}b}{\mathrm{d}\ln r}, 0\Big).$$

Proof. q and κ are positive and decreasing (Theorem 28), so for $\varrho \geq \varrho_1$: $\bar{D} \leq q_1^2$, $V < \bar{K} \leq V + \kappa_1$, $R \geq R_1$; $0 < W < 1$, $0 \leq x_R W' = 3W(1-W) \leq \frac{3}{4}$; and for all t: $0 \leq F \leq \frac{9}{8}$, $0 \leq F'(m) \leq \sqrt{3}$. As in the proof of Lemma 16 (with these bounds in place of F_T'), $K_{\text{eff}} \geq K^* - a^* D$ with $K^* = \bar{K} - \frac{9}{8}\frac{b\,x_R W'}{4\pi R^3} - \frac{3\sqrt{3}\,bW}{4\pi R^3} - \frac{3(bW)^2}{64\pi^2 s^2 R^4} \geq K_L$ and $a^* = \sqrt{3}\,bW/(4\pi R^3 \bar{D}) = \sqrt{3}\,bW/(4\pi R s^2) \leq 2$, hence $\int h_{K_{\text{eff}}}(D)\,\mathrm{d}x \geq 4\sqrt{K^*}$. With $\bar{P} = (1+2\bar{D})(\bar{D}^2 + 2\bar{D} + \bar{K})$, $\sqrt{\bar{P}} - \sqrt{K^*} \leq (\bar{P} - K^*)/(2\sqrt{K_L})$ and $R^2(\bar{P} - K^*) = s^2[2 + 2\bar{K} + 5\bar{D} + 2\bar{D}^2] + R^2(\bar{K} - K^*) \leq M$ (because $R^2\bar{D} = s^2$). So $\mathrm{SI} - N \geq -4\pi M/\sqrt{K_L}$. Further $F\,C \geq -\frac{9}{8}\max(\mathrm{d}b/\mathrm{d}\ln r, 0)$, and $\delta_Z(r,\varrho) \geq \delta_Z(r,\varrho_1)$ by Lemma 11. □

Remark 2 (smaller spheres, levels without credit). For $\varrho \leq r_Y$ the calibration field is continued differently (Section 4): for every level there is a radius $\varrho_\Gamma < r_Y$, the root of $Y(\varrho) = 2\mu^2(1+\cos\tau)$, and for spheres smaller than the one of ϱ_Γ the field is $\bar{K} = \kappa(\varrho_\Gamma) + 2\mu^2(1+\cos\tau) = D_\Gamma^2 - D_\Gamma$ and $N = 4\pi s^2\, h_{\bar{K}}(D_\Gamma)/D_\Gamma$ (the infimum over surfaces is attained with more area than the sphere). D_Γ is the unique minimum point of $D \mapsto h_{\bar{K}}(D)/D$, so there $\mathrm{SI} - N = 2\pi R^2 \int [h_{K_{\text{eff}}}(D) - h_{\bar{K}}(D_\Gamma)D/D_\Gamma]\,\mathrm{d}x$ has a non-negative integrand whenever $K_{\text{eff}} = \bar{K}$. Hence at the levels where b vanishes near the level ($r < r_{55} = 0.782$) the inequality holds at all these states, for every squeeze, with no computation (also $\delta_Z \geq 0$ there, by the same remark). What is *not* covered below $\varrho = 0.2074$ are the spheres between ϱ_Γ and 0.2074 at all levels, and the spheres below ϱ_Γ at the levels with $b > 0$.

Lemma 18 (smallest spheres; PROVED)**.** *Fix a level r with $b \geq 0$ and $\varrho_+ \in [r_Y, r)$, and put $s = \sin g(r)$, $V = 2\mu^2(1+\cos g(r))$, $\omega = 2^{2/3}/(3x_0 r)$,*

$$K_S = \kappa(\varrho_+) + V - \Big(\frac{27}{8} + 3\sqrt{3}\Big)\frac{b}{4\pi r^3 x_0^3} - \frac{2^{4/3}}{3}\,\frac{b^2}{64\pi^2 s^2 r^4 x_0^4}, \qquad \nu^2 = \frac{1}{K_S} + \frac{1}{\sqrt{K_S}}.$$

If $K_S > 0$ and $\sqrt{3}\,b\,\omega/(4\pi s^2) \leq 2$, then at every state of this level whose sphere is smaller than the sphere of ϱ_Γ (Remark 2), for every t: $D^2 + 2D + K_{\text{eff}} > 0$ on the sphere and

$$\Psi \;\geq\; r\,p\,\Big[\delta_Z(r,\varrho_+) - \nu\Big(\big(\tfrac{27}{8} + 4\sqrt{3}\big)\,b\,\omega + \frac{3\,b^2\omega^2}{16\pi s^2}\Big)\Big] - \frac{9}{8}\,W\Big(\frac{q(r)}{q(\varrho_+)}\Big)\max\Big(\frac{\mathrm{d}b}{\mathrm{d}\ln r}, 0\Big).$$

Proof. For all $x > 0$: $W(x)/x^3 \leq x_0^{-3}$, $W(x)/x \leq 2^{2/3}/(3x_0)$, $W(x)/x^2 \leq 2^{2/3}/(3x_0^2)$; so $W/R^3 \leq 1/(r^3 x_0^3)$, $W/R \leq \omega$, $W^2/R^4 \leq 2^{4/3}/(9r^4 x_0^4)$. Also $x_R W' = 3W(1-W) \leq 3W$, $0 \leq F \leq \frac{9}{8}$, $0 \leq F'(m) \leq \sqrt{3}$. As in the proof of Lemma 16, $K_{\text{eff}} \geq \bar{K} - \Delta - a\,D$ with $\Delta = B_v + 3c_1 + c_2$, $c_1 = \sqrt{3}\,bW/(4\pi R^3)$, $c_2 = 3b^2W^2/(64\pi^2 s^2 R^4)$, $a = c_1 R^2/s^2 \leq \sqrt{3}\,b\,\omega/(4\pi s^2) \leq 2$; and $\bar{K} - \Delta \geq K_S$, because $\bar{K} = \kappa(\varrho_\Gamma) + V$, $\varrho_\Gamma \leq r_Y \leq \varrho_+$ and κ decreases. Hence $D^2 + 2D + K_{\text{eff}} \geq D^2 + (2-a)D + K_S > 0$. Since D_Γ minimises $h_{\bar{K}}(D)/D$, the integrand of $\mathrm{SI} - N$ in Remark 2 is at least

$$h_{K_{\text{eff}}}(D) - h_{\bar{K}}(D) \;\geq\; -\frac{(\Delta + aD)\sqrt{1+2D}}{\sqrt{D^2 + (2-a)D + K_S}} \;\geq\; -\nu\,(\Delta + aD),$$

because $(1+2D)/(D^2+K_S) \le 1/K_S + 1/\sqrt{K_S}$. Integrating, with $\int D\,\mathrm{d}x = 2s^2/R^2$: $\mathrm{SI} - N \ge -4\pi R^2\nu\,(B_v + 4c_1 + c_2)$, and $4\pi R^2 B_v \le \frac{27}{8}\,b\,\omega$, $4\pi R^2 c_1 \le \sqrt{3}\,b\,\omega$, $4\pi R^2 c_2 \le 3b^2\omega^2/(16\pi s^2)$. The defect grows as the sphere shrinks ($Z \le 0$ below the path, also in the region of Lemma 5), so $\delta_Z \ge \delta_Z(r, \varrho_+)$; and $FC \ge -\frac{9}{8} W \max(\mathrm{d}b/\mathrm{d}\ln r, 0)$ with $W \le W(q(r)/q(\varrho_+))$. □

Remark 3 (spheres between ϱ_Γ and r_Y). There the formulas of Section 5 hold unchanged (Lemma 5). The program admits them with the switch `SIV_SMALL=1`; a box on which $Y(\varrho_{\max}) > V$ for all its levels lies below the curve of Remark 2 (Y decreases on $(0, r_Y]$ by Theorem 28(4)) and is left to Lemma 18.

12 Implementation

What this section does. It describes the program that proves Theorem 9: the interval arithmetic, the automatic derivatives, the quadrature, the use of the profile enclosures, and the search over cells.

Interval arithmetic. Double precision, rounding to nearest. Each of $+, -, \times, /, \sqrt{\ }$ is correctly rounded (IEEE 754), so the computed endpoint $\tilde{z}$ differs from the exact one by at most half a unit in the last place; it is replaced by $\tilde{z} \mp (|\tilde{z}|2^{-52} + 10^{-300})$, which moves it outwards by at least one unit (for $|\tilde{z}| \ge 2^{-970}$ the product $|\tilde{z}|2^{-52}$ is exact and not smaller than the unit in the last place; below, 10^{-300} exceeds every possible rounding error). No library function is used: exp, sin, cos, ln are Taylor polynomials in interval arithmetic with explicit tail bounds (argument halving for exp). Division by an interval containing 0, the square root of an interval that is not positive, and overflow abort the evaluation of the cell, which then counts as not verified.

Derivatives. Three coefficient types: intervals; intervals with gradient in (r, ϱ); intervals with gradient and second derivatives (forward differentiation, all components intervals). On top of any of them, truncated Taylor polynomials ('jets') in t, and on top of those truncated Taylor polynomials of length 7 in w (for Lemma 14); products, quotients, square roots and exponentials by the usual recurrences. An evaluation at a point uses thin intervals, an evaluation on a cell the cell itself as argument; in the latter case each coefficient is an enclosure of the corresponding Taylor coefficient at every point of the cell.

The function e_1. For $y \ge 0$: $e_1^{(k)}(y) = (-1)^k I_k(y)$, $I_k = \int_0^1 s^k e^{-ys}\mathrm{d}s = e^{-y} S_k(y)$, $S_k = \sum_{j\ge 0} y^j k!/(k+1+j)!$, $S_{k-1} = (yS_k + 1)/k$; and $\frac{\mathrm{d}^k}{\mathrm{d}y^k} e_1(-y) = N_k(y) = \sum_{j\ge 0} y^j/(j!\,(k+1+j))$. All series have positive terms; they are summed up to 18–96 terms depending on $y \le 12.5$, and the neglected tails (with their first three t-derivatives) are below 10^{-27}, which is added.

Quadrature in w. Lemma 14 on 4 to 128 equal start cells (depending on t); a cell whose remainder term exceeds 10^{-3} of its contribution is halved, up to 5 times.

Profile. On each of the 292 steps of $[\frac{1}{16}, 10]$, g and g' are the polynomials of degree 12 plus remainder term of Theorem 28(6) (interval coefficients, rounded outwards to doubles by `export_models.py`); g'' from the profile equation, g''' by differentiating it along the solution. The proven signs $\kappa > 0$, $\varphi_1 > 0$, $p > q$ are used to cut the enclosures at 0. δ_Z: lower Riemann sum of Lemma 11 with 48 cells between the box sides.

Search. Start cells: (cell of the credit nodes in r) × (64 geometric cells in $\varrho \in [0.25, 6]$; 4 resp. 8 in the two outer ranges) × (cells in t: $[0, 0.15]$, then ratio 1.4, length ≤ 0.15, $t = 1.2$ a boundary). For each cell: stage 0 if the box does not meet the path; if it fails by less than 3 times the δ_Z-term, bisection (in t, or in the relatively wider of r, ϱ); otherwise stage 1, provided the box is at most 3 % wide in r and ϱ; if stage 1 fails, bisection in the direction with the largest loss. A run is 'verified' only if every start cell is exhausted by verified cells.

13 The runs for $\mu = 1$

What this section does. It lists the runs that constitute the proof of Theorem 9, with the numbers of cells. The times are those of the original runs.

levels	ϱ	squeeze	cells verified by stage 0 / stage 1	time (2 cores)
$[r_{46}, r_{105}]$	$[0.25, 6]$	$[0, 3]$	310016 / 35517	121 min
$[r_{62}, r_{82}]$	$[0.25, 6]$	$[3, 3.75]$	68883 / 2836	24 min
$[r_{46}, r_{52}]$	$[0.2074, 0.25]$	$[0, 3]$	8452 / 611	6 min
$[r_{82}, r_{105}]$	$[6, 9.9]$	$[0, 3]$	34340 / 8845	17 min
$[r_{32}, r_{105}]$	$[0.25, 6]$	$t \geq 3$ (3.75 on $[r_{62}, r_{82}]$)	Lemma 16: 21740 boxes	seconds
$[r_{32}, r_{105}]$	$[0.2074, 0.25]$	$t \geq 3$	Lemma 16: 292 boxes	seconds
$[r_{32}, r_{105}]$	$[6, 9.9]$	$t \geq 3$	Lemma 16: 17872 boxes	seconds
$[r_{32}, r_{105}]$	$[\frac{1}{16}, 0.2074]$	$t \geq 3$	Lemma 16: 193 boxes	seconds
$[r_{32}, r_{105}]$	≥ 9.9	all t	Lemma 17	seconds
$[r_{32}, r_{105}]$	below ϱ_Γ	all t	Lemma 18	seconds

Totals of the interval verification on bounded squeezes: 94048 start cells in 47 invocations of `si_verify` (two cells of the credit nodes each for $0.25 \leq \varrho \leq 6$), 421691 cells verified by stage 0 and 47809 by stage 1, with 1017417 and 222507 evaluations, deepest bisection 9, 168 min of wall time on two cores. Large squeezes: 9 invocations of the mode `tail`, 40097 boxes, 21 s. Only invocations that end with `VERIFIED` are counted: `make_cert_numbers_partial.py` reads the logs, keeps the credit cells for which both the runs on bounded squeezes and Lemma 16 are verified, and writes the block of Theorem 9 and this table.

Where the work is. Stage 0 settles everything away from the path: there the δ_Z-term is large and a plain enclosure of Ψ_0 costs about 2 ms. Near the path stage 1 is needed; its cells are about 2 % wide in r, 1–2 % in ϱ and 0.1–0.15 long in t, and cost 0.05–0.4 s each. Their size is limited by the enclosures of the second derivatives, which are far too wide: for $\partial_t^2 \Psi_0$ near $r = 1.5$ the true second derivatives are 8–16, the enclosures on a box of half-width 0.01 are 3000–6000. This is the price of evaluating a long formula with intervals that are several per cent wide; it is paid only in the remainder.

14 All states of all levels up to the last one

What this section does. It extends the block to *all* states: all sphere sizes, all squeezes, and all levels from the centre to the last level, for both masses. Two more lemmas are needed, the supporting tangent (Lemma 19) and a second lemma on small spheres (Lemma 20). The proofs of the theorems say which tool covers which states, and the tables say which logs contain the corresponding computations.

Lemma 19 (supporting tangent; PROVED)**.** *Let $K > 0$, $P(D) = (1 + 2D)(D^2 + 2D + K)$ and $G_K(D) = 2PP'' - P'^2 = 12D^4 + 40D^3 + 24(1 + K)D^2 + 24KD - 4(K^2 - 3K + 1)$. If $\bar{D} > 0$ satisfies*

$$\text{(i)}\ \ G_K(\bar{D}) \geq 0, \qquad \text{(ii)}\ \ (1 + \bar{D})(K + \bar{D} - \bar{D}^2) \leq \sqrt{K\, P(\bar{D})},$$

then $h_K(D) \geq h_K(\bar{D}) + h_K'(\bar{D})(D - \bar{D})$ for all $D \geq 0$.

Proof. $h_K'' = G_K/(2P^{3/2})$ and G_K increases on $[0, \infty)$, so h_K is concave on some $[0, D_i]$ and convex on $[D_i, \infty)$, $D_i \leq \bar{D}$. On $[D_i, \infty)$ the tangent at $\bar{D}$ lies below h_K. On $[0, D_i]$ the difference of h_K and the tangent is concave, non-negative at D_i, and at $D = 0$ it equals $2\sqrt{K} - (h_K(\bar{D}) - \bar{D}h_K'(\bar{D}))$, where $h_K - Dh_K' = (2P - DP')/\sqrt{P} = 2(1+D)(K+D-D^2)/\sqrt{P}$; this is (ii). □

At a level without credit $K_{\text{eff}} = \bar{K}$ and $\Psi = r\,p\,(\mathrm{SI} - N + \delta_Z)$; so if (i), (ii) hold at $(\bar{K}, \bar{D})$, then $\mathrm{SI} - N \geq 0$ for every squeeze and $\Psi \geq 0$ with no further computation. With credit the same lemma gives:

Lemma 20 (small spheres above ϱ_Γ; PROVED)**.** *In the situation of Lemma 18, the bound stated there also holds, for every t, at every state of the level with $\varrho_\Gamma \leq \varrho \leq \varrho_+$ at which $(\bar{K}, \bar{D})$ satisfies* (i) *and* (ii)*.*

Proof. Now $\bar{K} = \kappa(\varrho) + V \geq \kappa(\varrho_+) + V$, so $K_{\text{eff}} \geq \bar{K} - \Delta - aD$ and $\bar{K} - \Delta \geq K_S$ as before, and $h_{K_{\text{eff}}}(D) \geq h_{\bar{K}}(D) - \nu(\Delta + aD) \geq h_{\bar{K}}(\bar{D}) + h'_{\bar{K}}(\bar{D})(D - \bar{D}) - \nu(\Delta + aD)$ by Lemma 19. The rest is unchanged: $\delta_Z(r, \varrho) \geq \delta_Z(r, \varrho_+)$ by Lemma 11, and $R \leq s/q(\varrho_+)$. □

Theorem 21 ($\mu = 1$: all Möbius states of the levels up to r_{105}; CERTIFIED)**.** *Let $\mu = 1$ and b as in Theorem 9. At every level $0 < r \leq r_{105}$ ($r_{105} \geq 9.5283$), for every sphere radius $R > 0$ and every squeeze t: $D^2 + 2D + K_{\text{eff}} > 0$ on the sphere and $\Psi \geq 0$ (for spheres smaller than the sphere of ϱ_Γ with $\bar{K}$, N, δ_Z as in Remark 2; at the nodes with either one-sided derivative of b).*

Proof. Spheres are parametrised by ϱ down to $\varrho_\Gamma(r)$; the path $\varrho = r$ lies above ϱ_Γ, because there $\bar{K} - (\bar{D}^2 - \bar{D}) = (1 + 2D_*)(p^2 - D_*) > 0$ by (P2) of Theorem 28.

Levels without credit, $0 < r \leq r_{55}$. Spheres below ϱ_Γ: Remark 2. Spheres with $\varrho \leq 0.8$ above ϱ_Γ: (i) and (ii) hold at $(\bar{K}, \bar{D})$ for all these levels and all $0 < \varrho \leq 0.8$. This is verified on boxes in (V, ϱ), $V = 2\mu^2(1 + \cos\tau) \in [V(g(r_{55})), 4\mu^2]$; for $\varrho \leq \frac{1}{16}$, where there is no polynomial model, with $q(\frac{1}{16}) \leq q \leq a$ and $\kappa(\frac{1}{16}) \leq \kappa \leq a^4 - a^2 - 4\mu^2$ (q and κ decrease, $a = g'(0) = 2.574173668139034358 \pm 10^{-17}$, all from Theorems 27 and 28). Spheres with $0.8 \leq \varrho \leq 9.9$: interval runs for $t \leq 3$ and Lemma 16 for $t \geq 3$. Spheres with $\varrho \geq 9.9$: Lemma 17.

The limit $r \to 0$. For $b = 0$, $\Psi/(r\,p\,s^2) = 2\pi\,q(\varrho)^{-2} \int [h_{\bar{K}}(D) - h_{\bar{K}}(\bar{D}) - h'_{\bar{K}}(\bar{D})(D - \bar{D})]\mathrm{d}x + \delta_Z/s^2$. The first term depends on the level only through $c = \cos\tau$ (in $\bar{K} = \kappa(\varrho) + 2\mu^2(1 + c)$). For all levels $r \leq r_5$ and $\varrho \geq r_5$, $\delta_Z/s^2 \geq 16\pi \int_{r_5}^{\varrho} (\ldots)\,\mathrm{d}\varrho'$ with the integrand of Lemma 11, which is non-negative between r and r_5 and also depends on the level only through c. The program (switch `SIV_CENTRE`) verifies the resulting inequality, and the bounds of Lemmas 16 and 17 in the same scaled form, on the cell $c \in [\cos g(r_5), 1]$, which contains $\cos g(r)$ for all $0 < r \leq r_5$.

Levels with credit, $r_{55} \leq r \leq r_{105}$. All spheres with $\varrho \leq 0.25$ and all smaller ones: Lemmas 18 and 20 with $\varrho_+ = 0.25$; the conditions (i), (ii) are verified for all levels $r \leq r_{105}$ and all $\varrho \leq 0.25$. $0.25 \leq \varrho \leq 6$: Theorem 9. $\varrho \geq 6$: Lemma 17 with $\varrho_1 = 6$ for the levels up to r_{88}; beyond, runs for $6 \leq \varrho \leq 9.9$ and Lemma 17 with $\varrho_1 = 9.9$. Negative t by symmetry. □

levels	spheres	logs in `flat/siv/runs/` (all end with `VERIFIED`)
$(0, r_{55}]$	$\varrho \leq 0.8$	`SUPP_C_r55_rho0.8`
$(0, r_5]$	$\varrho \geq 0.8$	`CC_run_t0_3`, `CC_tail_T3`, `CC_big_rho9.9`
$[r_5, r_{34}]$	$\varrho \geq 0.8$	`C_n5_34_t0_3`, `TAIL_C_n5_34_T3`, `BIG_C_n5_34_rho9.9`
$[r_{34}, r_{46}]$	$\varrho \geq 0.8$	`C_n34_46_t0_3`, `TAIL_A_n32_46_T3`, `TAIL_AH_n32_105_T3`, `BIG_A_n32_105_rho9.9`
$[r_{46}, r_{55}]$	$\varrho \geq 0.8$	runs `A`; `CH_n46_55_t0_3`, `TAIL_AH_n32_105_T3`, `BIG_A_n32_105_rho9.9`
$[r_{55}, r_{105}]$	$\varrho \leq 0.25$	`SMALL_A_n55_105_rho0.25`, `SUPP_C_r105_rho0.25`
$[r_{55}, r_{88}]$	$\varrho \geq 6$	`BIG_A_n32_82_rho6`, `BIG_A_n82_88_rho6`
$[r_{88}, r_{105}]$	$\varrho \geq 6$	runs `AH` with `TAIL_AH_n32_105_T3`; `BIG_A_n32_105_rho9.9`
$[r_{55}, r_{105}]$	$0.25 \leq \varrho \leq 6$	runs `A` with `TAIL_A_*` (Section 13)

The runs `C`, `CH`, `CC` are far from the path and were settled by stage 0 alone (about two minutes in all); `flat/siv/run_centre.sh` and `run_centre2.sh` reproduce them. `make_cert_numbers_partial.py` computes r_{105} from the logs by the rules of this proof.

Theorem 22 ($\mu = 0.526$, levels without credit; CERTIFIED)**.** *Let $\mu = 0.526$ and let b be the credit function of* `flat/siv/mu0526b.dat` *(linear in $\ln r$ between the nodes r_i, $i = 0, \dots, 128$, which are the same numbers as in Theorem 9 for $i \le 115$; $b_i = 0$ for $i \le 52$). At every level $0 < r \le r_{52} = 0.6731\dots$, for every sphere radius and every squeeze, $\Psi \ge 0$ (and $K_{\text{eff}} = \bar{K} > 0$).*

Proof. As for the levels without credit in Theorem 21, with the profile of Theorem 27 for this mass ($a = 2.2620678312003673796 \pm 10^{-15}$, models on $[\frac{1}{16}, 16]$): conditions (i), (ii) for all levels $r \le r_{52}$ and all $0 < \varrho \le 1.1$ (`SUPP_D_r52_rho1.1`); runs, Lemma 16 and Lemma 17 for $\varrho \ge 1.1$ on the levels $[r_5, r_{52}]$ (`D_n5_52_t0_3`, `TAIL_D_n5_52_T3`, `BIG_D_n5_52_rho15.9`) and on the centre cell (`DC_run_t0_3`, `DC_tail_T3`, `DC_big_rho15.9`); script `run_inner.sh`, about three minutes. □

Status of these logs: all seven end with `VERIFIED`.

Theorem 23 ($\mu = 0.526$, levels with credit; CERTIFIED)**.** *Let $\mu = 0.526$ and b as in Theorem 22. At every level $0 < r \le r_{100}$ ($r_{100} \ge 7.4206$), for every sphere radius $R > 0$ and every squeeze t: $D^2 + 2D + K_{\text{eff}} > 0$ on the sphere and $\Psi \ge 0$, with the conventions of Theorem 21.*

Proof. Levels $r \le r_{52}$: Theorem 22. Levels $r_{52} \le r \le r_{100}$, as for the levels with credit in Theorem 21. All spheres with $\varrho \le 0.25$ and all smaller ones: Lemmas 18 and 20 with $\varrho_+ = 0.25$ (`SMALL_B_n52_114_rho0.25`); the conditions (i), (ii) are verified for all levels $r \le r_{114}$ and all $\varrho \le 0.25$ (`SUPP_D_r114_rho0.25`). Spheres with $0.25 \le \varrho \le 10$: interval runs `B` for $0 \le t \le T$ and Lemma 16 for $t \ge T$ (`TAIL_B_*`), with $T = 3$ on $[r_{52}, r_{70}]$ and $[r_{88}, r_{100}]$, $T = 3.75$ on $[r_{70}, r_{76}]$ and $[r_{82}, r_{88}]$, $T = 4$ on $[r_{76}, r_{82}]$. Spheres with $\varrho \ge 10$: Lemma 17 with $\varrho_1 = 10$ (`BIG_B_n52_96_rho10`, `BIG_B_n96_100_rho10`). Negative t by symmetry. □

levels ($\mu = 0.526$)	ϱ	squeeze	cells verified by stage 0 / stage 1	time (2 cores)
$[r_{52}, r_{100}]$	$[0.25, 10]$	$[0, 3]$	256068 / 39944	77 min
$[r_{70}, r_{76}]$	$[0.25, 10]$	$[3, 3.75]$	17892 / 1042	5 min
$[r_{82}, r_{88}]$	$[0.25, 10]$	$[3, 3.75]$	24742 / 1403	6 min
$[r_{76}, r_{82}]$	$[0.25, 10]$	$[3, 4]$	26935 / 2422	14 min
$[r_{52}, r_{100}]$	$[0.25, 10]$	$t \ge T$, $T = 3$, 3.75, 4	Lemma 16: 56984 boxes	seconds

All runs for the levels up to r_{100} are finished: 33 invocations, 325637 cells verified by stage 0 and 44811 by stage 1, 101 min of wall time on two cores; they are the lines of `runs/queue9.sh`. A second program, `audit_cover.py`, reads the command line and the last line of every log (not the file names) and checks for each credit cell, and for the centre cell, that the logs cover all spheres and all squeezes; for both masses it reproduces the accounting of Theorems 21, 22 and 23.

15 The far field: technical part

What this section does. It proves the two lemmas on radial solutions that Step 6 uses near the hedgehog's own sphere, and it states and proves the five numerical facts (F1)–(F5). The notation λ_0, λ_1, m_{lin}, $Y(R)$, $S(R)$ is that of Section 7.

Lemma 24 (radial solutions outside a ball; PROVED)**.** *Let $\mu > 0$, $R \ge 1/\mu$, $0 < f_0 < \frac{\pi}{2}$, and let $\alpha, \beta, \gamma, c, \delta$ be the numbers of the band lemma (Lemma 29) for $r_1 = R$:*

$$\alpha = \frac{2(2+\mu^2R^2)}{R^4} + \frac{4}{3}\,\frac{1 + f_0^2/R^2}{R^2} + \frac{2}{R^4} + \frac{\mu^2}{6}, \qquad \beta = \frac{4}{R^3}, \qquad \gamma = \frac{\frac{4}{3}f_0^2 + 2}{R^2},$$
$$c = \alpha + \frac{\beta}{2} + 2\Big(\gamma + \frac{\beta}{2}\Big)m_{\text{lin}}(R)^2,$$

$\delta = f_0^2 c/\mu$; *assume* $4\delta < \mu$ *and* $4(\gamma + \frac{\beta}{2})f_0^2 < 1$. *Then* (2) *has a solution* f *on* $[R,\infty)$ *with* $f(R) = f_0$, $f > 0$, $f' < 0$, *and*

$$\Big| -\frac{f'(r)}{f(r)} - m_{\mathrm{lin}}(r)\Big| < \delta \qquad \text{for all } r \geq R;$$

in particular $f(r) \leq f_0\, e^{-(\mu-\delta)(r-R)}$.

Proof. Lemma 29 says, under the stated assumptions at a point r_1 with $f(r_1) = f_1 > 0$, $f'(r_1) < 0$ and with $u = -f'/f - m_{\mathrm{lin}}$: (i) if $u(r_1) \geq \delta_1$ then f has a zero $r_2 > r_1$ with $f' < 0$ on $[r_1, r_2]$; (ii) if $u(r_1) \leq -\delta_1$ then $f'(r_2) > 0$ for some $r_2 > r_1$ with $f > 0$ on $[r_1, r_2]$. The numbers α, β, γ, c decrease when r_1 increases or f_1 decreases, so the assumptions hold, with $\delta_1 \leq \delta$, at every $r_1 \geq R$ with $0 < f_1 \leq f_0$. For $\sigma \in [-\delta, \delta]$ let f_σ solve (2) with $f_\sigma(R) = f_0$, $f'_\sigma(R) = -f_0(m_{\mathrm{lin}}(R) + \sigma)$. The set Σ_+ of those σ for which f_σ has a zero r_2 with $f'_\sigma < 0$ on $[R, r_2]$ and the set Σ_- of those for which $f'_\sigma(r_2) > 0$ at some r_2 with $f_\sigma > 0$ on $[R, r_2]$ are open in $[-\delta, \delta]$, disjoint, and contain δ and $-\delta$ respectively; so some σ belongs to neither. For this σ, $f = f_\sigma$ satisfies $f > 0$ and $f' < 0$ as long as it exists: at a first point with $f = 0$ one would have $f' < 0$ ($f = f' = 0$ is excluded by uniqueness), i.e. $\sigma \in \Sigma_+$; at a first point with $f' = 0 < f$, (2) gives $f'' > 0$, i.e. $\sigma \in \Sigma_-$. At every $r_1 \geq R$ in the interval of existence $|u(r_1)| < \delta_1 \leq \delta$, because otherwise (i) or (ii), applied at r_1, would put σ into Σ_+ or Σ_-. So f'/f stays bounded, the solution exists on $[R,\infty)$, and $-f'/f \geq m_{\mathrm{lin}} - \delta > \mu - \delta$. □

Lemma 25 (comparison with a radial solution; PROVED)**.** *In the situation of Lemma 24 let* $\tau = \pi - f_0$, $s = \sin\tau$, $\Omega = \{|x| > R\}$, $\Phi(x) = (\cos f(|x|),\ \sin f(|x|)\, x/|x|)$, $M = m_{\mathrm{lin}}(R) + \delta$,

$$\eta = f_0^2 \max\Big\{\frac{2M}{R} + \frac{\sqrt{2}}{R^2},\ \ \frac{1}{2} + \frac{M^2 + 2/R^2}{\mu^2} + \frac{2f_0^2}{\mu^2}\Big(\frac{2M^2}{R^2} + \frac{1}{R^4}\Big)\Big\},$$
$$\varepsilon = 20\,(1+\mu R)\Big[\lambda_1(\mu R)\Big(\frac{2s^2}{R^2} + \eta\Big) + R\,\delta\,\Big(1 + \frac{2s^2}{R^2}\Big)\Big],$$

and assume $\eta < 1$. *Then* Φ *has the boundary values* $(-\cos\tau, \sin\tau\, x/|x|)$, *and for every* $U : \overline{\Omega} \to S^3$ *of finite energy with the boundary values* $(-\cos\tau, \sin\tau\, n)$, n *the labels with squeeze* t *and dipole* m,

$$E_\Omega(U) \ \geq\ E_\Omega(\Phi) - (1+\varepsilon)\, S(R)\, m^2.$$

Proof. Step 1: an exact expansion. Φ is smooth on $\overline{\Omega}$, its derivatives decay exponentially, and it is a critical point of E_Ω among S^3-valued maps with respect to variations with compact support in Ω: (2) is the Euler–Lagrange equation of the energy restricted to maps of the form $(\cos f(|x|), \sin f(|x|)x/|x|)$, and these are the fixed points of an isometric action of $O(3)$ that leaves E invariant (symmetric criticality). Let $\zeta = U - \Phi$ (with values in $\mathbb{R}^4$), $A_{ij} = \partial_i\Phi \wedge \partial_j\Phi$, $T_{ij} = \partial_i\Phi \wedge \partial_j\zeta + \partial_i\zeta \wedge \partial_j\Phi$, $C_{ij} = \partial_i\zeta \wedge \partial_j\zeta$. Then $|\mathrm{d}U|^2 = |\mathrm{d}\Phi|^2 + 2\,\mathrm{d}\Phi : \mathrm{d}\zeta + |\mathrm{d}\zeta|^2$, $2\mu^2(1 - U_0) = 2\mu^2(1 - \Phi_0) - 2\mu^2\zeta_0$, and

$$\sum_{i<j} |A_{ij} + T_{ij} + C_{ij}|^2 \ \geq\ \sum_{i<j} |A_{ij}|^2 + 2\sum_{i<j} A_{ij} \cdot T_{ij} - \sqrt{2}\,|\Lambda^2\mathrm{d}\Phi|\,|\mathrm{d}\zeta|^2,$$

because $2A \cdot C \geq -2|A||C|$ and $\sum_{i<j} |C_{ij}|^2 \leq \frac{1}{2}|\mathrm{d}\zeta|^4$. So $E_\Omega(U) \geq E_\Omega(\Phi) + \ell(\zeta) + \int_\Omega (1 - \sqrt{2}|\Lambda^2\mathrm{d}\Phi|)|\mathrm{d}\zeta|^2$ with the linear functional $\ell(\zeta) = \int_\Omega 2\,\mathrm{d}\Phi : \mathrm{d}\zeta + 2\sum_{i<j} A_{ij} \cdot T_{ij} - 2\mu^2\zeta_0$. Integration by parts (all terms are integrable: ζ is bounded, $\mathrm{d}\zeta \in L^2$, $\zeta_0 \in L^1$ and $|\zeta|^2 \in L^1$ because $1 - U_0$ and $1 - \Phi_0$ are integrable, and the boundary terms at infinity vanish) gives $\ell(\zeta) = \int_\Omega \mathcal{L} \cdot \zeta + \int_{\partial\Omega} \mathcal{N} \cdot \zeta\, \mathrm{d}A$, where $\mathcal{L}$ is the unconstrained gradient of the energy at Φ. By criticality $\mathcal{L}$ is normal to the sphere, $\mathcal{L} = \lambda\Phi$ with $\lambda = \mathcal{L} \cdot \Phi = 2|\mathrm{d}\Phi|^2 + 4|\Lambda^2\mathrm{d}\Phi|^2 - 2\mu^2\Phi_0$ (use

$\Phi\cdot\partial_i\Phi=0$; the three parts of the energy are homogeneous of degree 2, 4 and, in Φ_0, 1). Since $|U|=|\Phi|=1$, $\Phi\cdot\zeta=-\frac12|\zeta|^2$. Hence

$$E_\Omega(U)\;\ge\;E_\Omega(\Phi)+\int_{\partial\Omega}\mathcal{N}\cdot\zeta\,\mathrm{d}A \\ +\int_\Omega\Big[\big(1-\sqrt{2}|\Lambda^2\mathrm{d}\Phi|\big)|\mathrm{d}\zeta|^2+\big(\mu^2\Phi_0-|\mathrm{d}\Phi|^2-2|\Lambda^2\mathrm{d}\Phi|^2\big)|\zeta|^2\Big]\mathrm{d}x. \quad (28)$$

Step 2: the coefficients. By Lemma 24, $0<f\le f_0$ and $|f'|\le Mf$ on $[R,\infty)$. So $|\mathrm{d}\Phi|^2=f'^2+2\sin^2 f/r^2\le f_0^2(M^2+2/R^2)$, $|\Lambda^2\mathrm{d}\Phi|^2=2f'^2\sin^2 f/r^2+\sin^4 f/r^4\le f_0^4(2M^2/R^2+1/R^4)$, $\Phi_0=\cos f\ge 1-\frac12 f_0^2$, and the integrand of (28) is at least $(1-\eta)(|\mathrm{d}\zeta|^2+\mu^2|\zeta|^2)$. Dropping the component ζ_0 and using the Dirichlet principle for the other three, which lie in $H^1(\Omega)$ and have the boundary values $s\,(n-x/|x|)$,

$$\int_\Omega(|\mathrm{d}\zeta|^2+\mu^2|\zeta|^2)\;\ge\;\mathcal{Q}:=s^2R\sum_\ell\lambda_\ell(\mu R)\,\|P_\ell(n-x/|x|)\|^2_{L^2(S^2)}.$$

Let $\chi=\chi(t)=1-\frac{1}{4\pi}\int_{S^2}n\cdot\frac{x}{|x|}$. As $x/|x|$ is purely $\ell=1$, $\mathcal{Q}=s^2R\,[\sum_\ell\lambda_\ell c_\ell-4\pi\lambda_1+8\pi\lambda_1\chi]$, and by the proof of Lemma 8

$$\mathcal{Q}\;\ge\;8\pi R\,\lambda_1(\mu R)\,s^2\chi-S(R)\,m^2. \quad (29)$$

Step 3: the boundary term. With the outward normal $-x/|x|$ of Ω, $\int_\Omega 2\,\mathrm{d}\Phi:\mathrm{d}\zeta$ contributes $-2\,\partial_r\Phi\cdot\zeta$ to $\mathcal{N}\cdot\zeta$. For the Skyrme term, $\sum_{i<j}A_{ij}\cdot T_{ij}=\sum_j\Sigma_j\cdot\partial_j\zeta$ with $\Sigma_j=\sum_{i\ne j}[\,|\partial_i\Phi|^2\partial_j\Phi-(\partial_i\Phi\cdot\partial_j\Phi)\,\partial_i\Phi\,]$, and in the radial direction $\Sigma_r=(2\sin^2 f/r^2)\,\partial_r\Phi$, because the tangential derivatives of Φ are orthogonal to $\partial_r\Phi$ and have squared length $\sin^2 f/r^2$ each. So $\mathcal{N}=-2(1+2s^2/R^2)\,\partial_r\Phi$ on $\partial\Omega$. There $\zeta=(0,\,s\,(n-x/|x|))$ and $\partial_r\Phi=f'(R)\,(-\sin f_0,\cos f_0\,x/|x|)$, so $\mathcal{N}\cdot\zeta=-2(1+2s^2/R^2)\,|f'(R)|\cos f_0\;s\,(1-n\cdot x/|x|)\le 0$, and with $|f'(R)|\le Mf_0$, $f_0\cos f_0\le\sin f_0=s$, $RM=\lambda_1(\mu R)+R\delta$:

$$\int_{\partial\Omega}\mathcal{N}\cdot\zeta\,\mathrm{d}A\;\ge\;-8\pi R\,s^2\chi\Big(1+\frac{2s^2}{R^2}\Big)\big(\lambda_1(\mu R)+R\,\delta\big).$$

Step 4. From (28), (29) and Step 3,

$$E_\Omega(U)-E_\Omega(\Phi)\;\ge\;-(1-\eta)\,S(R)\,m^2-8\pi R\,s^2\chi\Big[\Big(1+\frac{2s^2}{R^2}\Big)(\lambda_1+R\delta)-(1-\eta)\lambda_1\Big],$$

and the bracket is at most $\lambda_1(2s^2/R^2+\eta)+R\delta(1+2s^2/R^2)$. Since $8\pi Rs^2=2(1+\mu R)\,S(R)$, the lemma follows from

$$\chi(t)\;\le\;10\,m(t)^2\qquad(t\ge0). \quad (30)$$

With the position cosine x, the label cosine $y=(x\cosh t-\sinh t)/u$ and $u=\cosh t-x\sinh t$ one has $n\cdot x/|x|=xy+(1-x^2)/u$, and an elementary integration gives $\chi=1-(t+\sinh t)/(\sinh t\,(\cosh t+1))<1$. Next, $\chi\le\frac12 t^2$, because $1-n\cdot x/|x|=1-\cos(\alpha-\Theta)\le\frac12(\alpha-\Theta)^2$ and $0\le\alpha-\Theta\le t$ ($\partial_t\alpha=\sin\alpha$). Finally $x-y=\sinh t\,(1-x^2)/u\ge\sinh t\,e^{-t}(1-x^2)$, and the mean of $x-y$ over the sphere is m, so $m\ge\frac13(1-e^{-2t})$. For $t\le\frac12$ this gives $m\ge\frac13 t$ and $\chi\le\frac92 m^2$; for $t\ge\frac12$, $m\ge m(\frac12)=0.3226\ldots$ and $\chi<1<10\,m^2$. □

In Lemma 25 nothing is assumed about the size of $U-\Phi$; the error terms are proportional to m^2 because the expansion is made around an exact solution.

The last level. Let $(\mu,r_*)=(1,r_{105})$ or $(0.526,r_{100})$, $\tau_*=g(r_*)$, $s=\sin\tau_*$, $f_*=\pi-\tau_*$, and let $b_*=b(r_*)$ be the value of the credit function of Theorem 9 resp. Theorem 22 at this node. For a sphere of radius R at the level τ_*, $v=\frac{4\pi}{3}R^3$, put

$$\mathrm{Cal}(R)=E(H)-S_0(\tau_*,v)=\big[E(H)-E_*(\tau_*)\big]+\int_{v_*(\tau_*)}^{v}\bar\kappa(\tau_*,v')\,\mathrm{d}v',\qquad B_*(R)=\tfrac98\,b_*W(R/r_*).$$

The bracket is the energy of H outside the ball of radius r_*; Cal and B_* increase with R ($\bar\kappa > 0$). Spheres are parametrised as in Section 5: $R(\varrho) = s/q(\varrho)$, which increases with ϱ; for $\varrho > r_Y$ the sphere lies above ϱ_Γ, $\bar\kappa = \kappa(\varrho)$ and $\mathrm{d}v = 4\pi s^3\varphi_1/(\varrho\, q^4)\,\mathrm{d}\varrho$ (because $q' = -\varphi_1/\varrho$). By Lemma 8, $\mathrm{Cal}(r_*) \geq Y(r_*)$: the Yukawa bound alone cannot settle the spheres near the hedgehog's own, and there Lemma 25 is used.

Proposition 26 (far-field numbers; CERTIFIED)**.** *For each of the two masses, with the numbers $\varrho_s \leq \varrho_m < r_* < \varrho_b \leq r_N$ of Table 3 (r_N is the end of the profile models of Theorem 28(6)):*

(F1) $E(H) - E_*(\tau_*) \leq \bar E$;

(F2) $\mathrm{Cal}(R(\varrho_s)) \leq 0$;

(F3) $Y(R) - \mathrm{Cal}(R) \geq \max\{0,\, S(R) - B_*(R)\}$ *for* $R = R(\varrho)$, $\varrho_s \leq \varrho \leq \varrho_m$;

(F4) *with* $R_m \leq R(\varrho_m)$ *and* $R_b \geq R(\varrho_b)$ *as in the table: the hypotheses of Lemmas 24 and 25 hold for* $f_0 = f_*$ *and every* $R \geq R_m$; *the number* ε *of Lemma 25 is at most* $\bar\varepsilon$ *for* $R_m \leq R \leq R_b$; *and* $B_*(R) \geq (1+\bar\varepsilon)\,S(R)$ *for* $R = R(\varrho)$, $\varrho_m \leq \varrho \leq \varrho_b$;

(F5) $\mathrm{Cal}(R) \leq 4\pi s^2 R\,(\mu R + 1)$ *for all* $R \geq R(\varrho_b)$.

Table 3: The numbers of Proposition 26, from the logs `FARFIELD_A_n105.log` and `FARFIELD_B_n100.log`, rounded in the safe direction. $R_s \leq R(\varrho_s)$, $R_m \leq R(\varrho_m)$, $R_b \geq R(\varrho_b)$; $(x)_+ = \max\{0, x\}$; δ_N bounds the half-width of the band of Theorem 27(2) at r_N.

	$\mu = 1$	$\mu = 0.526$
last level r_*	$r_{105} = 9.52831\ldots$	$r_{100} = 7.42065\ldots$
$s = \sin\tau_* \geq$	$1.5100241 \cdot 10^{-5}$	0.0034393636
$f_* = \pi - \tau_* \leq$	$1.5100242 \cdot 10^{-5}$	0.0034393705
b_*/s^2	50.7493	33.8843
$\varrho_s,\ \varrho_m,\ \varrho_b$	8.8, 8.8, 9.9	5.42, 6.82, 8.22
$R_s,\ R_m,\ R_b$	3.89265, 3.89265, 14.9702	1.28535, 4.48933, 14.1450
r_N, δ_N	10, $2.1 \cdot 10^{-11}$	16, $2.8 \cdot 10^{-11}$
(F1) $\bar E$	$2.90304 \cdot 10^{-7}$ ($\leq 1273.17\, s^2$)	0.00573424 ($\leq 484.752\, s^2$)
(F1) bound of $\lvert e_N\rvert$ in the tail lemma	$5.8 \cdot 10^{-20}$	$6.5 \cdot 10^{-19}$
(F2) lower sum over $[\varrho_s, r_*]$	$3.05557 \cdot 10^{-7}$ ($\geq 1340.06\, s^2$)	0.00726002 ($\geq 613.736\, s^2$)
(F3) smallest $[Y - \mathrm{Cal} - (S - B_*)_+]/Y$	(trivial: $\varrho_s = \varrho_m$)	≥ 0.162
(F4) $\delta \leq,\ \eta \leq$	$2.32 \cdot 10^{-10},\ 5.36 \cdot 10^{-10}$	$7.60 \cdot 10^{-6},\ 3.86 \cdot 10^{-5}$
(F4) $M \leq,\ \lambda_1(\mu R_b) \leq$	1.310, 16.04	0.8151, 8.559
(F4) $\bar\varepsilon$	$4.01 \cdot 10^{-6}$	0.0756
(F4) smallest $B_*/((1+\bar\varepsilon)S)$	≥ 1.65	≥ 1.20
(F5) $\mathrm{Cal}(R(\varrho_b)) \leq$	$6.13951 \cdot 10^{-7}$ ($\leq 2692.56\, s^2$)	0.0151447 ($\leq 1280.28\, s^2$)
(F5) $4\pi s^2 R(\mu R+1)$ at $R(\varrho_b)$, $\geq$	$6.85038 \cdot 10^{-7}$ ($\geq 3004.32\, s^2$)	0.0177468 ($\geq 1500.25\, s^2$)
(F5) $\Theta \leq$	0.23009	0.077513
(F5) $r_N f(r_N)\Theta \leq$	$2.0563 \cdot 10^{-5}$	$1.9327 \cdot 10^{-5}$
(F5) $2\mu s(1 - f(r_N)^2/6) \geq$	$3.0200 \cdot 10^{-5}$	0.0036182

Proof. Interval arithmetic with the profile models of Theorem 28(6), mode `farfield` of the program; all sums have 400 cells per interval. (F1): the energy density of H is $4\pi r^2[(1+2D_*)p^2 + 2D_* + D_*^2 + 2\mu^2(1+\cos g)]$; an upper Riemann sum on $[r_*, r_N]$, and for $r > r_N$ Lemma 31: the energy outside r_N is $4\pi[r_N^2 f(r_N)p(r_N) + e_N]$ with the bound of $\lvert e_N\rvert$ in the table. (F2): a lower Riemann sum of $4\pi s^3 \int_{\varrho_s}^{r_*} \kappa\varphi_1/(\varrho q^4)\,\mathrm{d}\varrho$ is at least $\bar E$, and $\varrho_s > r_Y$. (F3): on each cell of

$[\varrho_s, \varrho_m]$, Cal is at most $\bar{E}$ minus the lower sum from the upper end of the cell to r_*, and Y, S, B_* are monotone in R; for $\mu = 1$, where $\varrho_s = \varrho_m$, (F3) follows from (F2) and (F4). (F4): δ, M, η decrease and $\lambda_1(\mu R)$, $1 + \mu R$ increase with R, so the hypotheses are checked at R_m and $\bar{\varepsilon} = 20(1+\mu R_b)[\lambda_1(\mu R_b)(2s^2/R_m^2+\eta)+R_b\,\delta\,(1+2s^2/R_m^2)]$ with δ, η at R_m; the last inequality is verified on cells. (F5): an upper Riemann sum gives $\mathrm{Cal}(R(\varrho_b)) \le \bar{E} + 4\pi s^3 \int_{r_*}^{\varrho_b} \kappa\varphi_1/(\varrho q^4)\,\mathrm{d}\varrho \le 4\pi s^2 R(\mu R+1)$ at $R = R(\varrho_b)$. For larger spheres compare the derivatives with respect to the volume: that of Cal is $\kappa(\varrho)$, that of $4\pi s^2 R(\mu R+1)$ is $s^2(2\mu R+1)/R^2 = 2\mu s\, q(\varrho) + q(\varrho)^2$. On $[\varrho_b, r_N]$ the inequality $\kappa \le 2\mu s q + q^2$ is verified on cells. For $\varrho \ge r_N$, with $f_N = f(r_N)$ and $m^+ = m_{\mathrm{lin}}(r_N) + \delta_N \ge p/f$ (Theorem 27(2)), $\kappa - D_* \le (1+2D_*)p^2 - 2\mu^2(1-\cos f) \le f^2\Theta$ with $\Theta = (1 + 2f_N^2/r_N^2)(m^+)^2 - \mu^2(1 - \frac{1}{12}f_N^2)$, while $2\mu s\, q \ge 2\mu s\, f(1 - \frac{1}{6}f_N^2)/\varrho$; as $\varrho f(\varrho)$ decreases, it suffices that $r_N f_N \Theta \le 2\mu s(1 - \frac{1}{6}f_N^2)$. □

Part III. The hedgehog profile

16 A rigorous enclosure of the profile and the energy of the massive hedgehog

What this part does. It proves that the profile equation has a solution with the boundary values of the hedgehog for $\mu = 1$ and $\mu = 0.526$, encloses this solution and its energy, and proves the sign conditions that the calibration needs (facts (i)–(iv) of Section 3). The method is a rigorous numerical integration of the equation for a whole interval of slopes at the centre, together with an analytic lemma that decides, from the values at a finite radius r_N, whether a solution overshoots or turns back. The text is taken from the technical version of this work with small editorial changes.

16.1 Statement

With $g = \pi - f$ and $p = g'$ the profile equation is

$$(r^2 + 2\sin^2 g)\, g'' + 2rg' + \sin 2g \left(g'^2 - 1 - \frac{\sin^2 g}{r^2} \right) + \mu^2 r^2 \sin g = 0, \tag{31}$$

and the energy of the hedgehog is $E = 4\pi \int_0^\infty \big[r^2p^2 + 2\sin^2 g\, p^2 + 2\sin^2 g + \sin^4 g/r^2 + 2\mu^2 r^2(1+\cos g)\big]\mathrm{d}r$. For $a > 0$ there is exactly one solution with $g(r) = ar + O(r^3)$ at the centre, and it depends continuously on a (standard for this singular initial-value problem; used, not reproved here).

Theorem 27 (CERTIFIED)**.** *Let (μ, a_0, ρ, r_N) be one of the rows of Table 4. There is $a_* \in [a_0 - \rho, a_0 + \rho]$ such that the solution g of (31) with $g \sim a_* r$ exists on $(0, \infty)$ and satisfies $0 < g < \pi$, $g' > 0$, $g \to \pi$. Moreover, with $f = \pi - g$:*

1. *at each of the step ends between $r = \frac{1}{16}$ and r_N the point $(g(r), g'(r))$ lies in the box listed in the data file (largest widths in Table 4);*
2. *for $r \ge r_N$: $\big| -f'/f - m_{\mathrm{lin}}(r) \big| < \delta(r)$ with $m_{\mathrm{lin}} = \mu\big[1 - \frac{1}{\mu r+1} + \frac{2}{\mu r}\big]$ and $\delta(r) \le \delta(r_N)\, f(r)^2/f(r_N)^2$, $\delta(r_N)$ as in the table; in particular $f(r) \le f(r_N)\, e^{-\mu(r-r_N)}$;*
3. *the energy lies in the interval of the table.*

	$\mu = 1$	$\mu = 0.526$
a_0	2.574173668139034358	2.2620678312003673796
ρ	10^{-17}	10^{-15}
r_N	10	16
$f(r_N) \in$	$8.9371485^{451}_{199} \cdot 10^{-6}$	$1.558294^{426}_{322} \cdot 10^{-5}$
$u = p/f - m_{\mathrm{lin}}$ at r_N for $a_0 - \rho$	$-2.77^{306}_{296} \cdot 10^{-9}$	$-2.852^{375}_{230} \cdot 10^{-8}$
u at r_N for $a_0 + \rho$	$2.779^{676}_{587} \cdot 10^{-9}$	$4.056^{827}_{682} \cdot 10^{-8}$
band half-width $\delta(r_N)$	$2.05 \cdot 10^{-11}$	$2.77 \cdot 10^{-11}$
largest width of the g-boxes, p-boxes	$2.5 \cdot 10^{-14}$, $2.3 \cdot 10^{-14}$	$1.0 \cdot 10^{-12}$, $4.8 \cdot 10^{-13}$
$E/12\pi^2 \in$	$1.41622346610140^{358}_{227}$	$1.3076651860528^{709}_{596}$
r_Y (zero of Y) $\in$	$2.07318741215292^{813}_{590} \cdot 10^{-1}$	$1.56414756367^{667}_{439} \cdot 10^{-1}$
r_i (zero of $\partial_D^2 h_K$ at D_*) $\in$	$1.6081056237276^{980}_{757}$	$1.715831570678^{754}_{469}$
steps (centre + mean-value integrator); time	480 + 292; 21 min	416 + 388; 23 min

Table 4: The certified runs (50 decimal digits, Taylor order 24). Super- and subscripts are the upper and lower ends of an interval. For comparison, floating point (shooting with `scipy`, energy integrated as a third unknown, tail from Lemma 31): $E/12\pi^2 = 1.41622346613$ for $\mu = 1$ and 1.30766518607 for $\mu = 0.526$.

Theorem 28 (CERTIFIED)**.** *Let f be the solution of Theorem 27 for one of the two masses, $p = -f'$, $q = \sin f/r$, $D_* = q^2$,*

$$K = (1+2D_*)p^2 - 2D_* - D_*^2, \qquad \kappa = K - 2\mu^2(1-\cos f),$$
$$Y = D_*^2 - D_* - \kappa, \qquad h_K(D) = 2\sqrt{(1+2D)(D^2+2D+K)}.$$

Then on $(0,\infty)$:

(P1) *$q + p\cos f > 0$; equivalently $q' < 0$, so q decreases strictly from a_* to 0;*

(P2) *$p > q$;*

(3) *$\kappa' < 0$ and $\kappa > 0$;*

(4) *$Y' < 0$ wherever $Y \geq 0$; hence $\{Y \geq 0\} = (0, r_Y]$, with r_Y in the interval of Table 4;*

(5) *$\partial_D^2 h_K(D_*) > 0$ for $r < r_i$ and < 0 for $r > r_i$, with r_i in the interval of Table 4;*

(6) *on each step $[r_k, r_k + h_k]$ of the integrator between $\frac{1}{16}$ and r_N, $g(r_k + t) \in \sum_{j \leq 12} G_j t^j + R\, t^{13}$ for $0 \leq t \leq h_k$, and likewise g', with the intervals G_j, R of the data file.*

(P1), (P2), $\kappa > 0$ and (4) are the hypotheses of Lemma 5; (5) separates the levels at which a squeeze of the hedgehog's own sphere costs energy ($r < r_i$) from those at which it saves energy ($r > r_i$), see Figure 3; (6) is the input of the interval computations of Part II (the models are read by `flat/profile_oracle.py` and exported to the verifier by `flat/siv/export_models.py`).

16.2 Method

(a) Centre. Put $q = \sin g/r$, $e = g' - q$, $\gamma = 1 - \cos g$ and $w = 2q/(1+2q^2)$, so that $p = q + e$. A direct computation shows that (31) is equivalent to

$$q' = \frac{e - p\gamma}{r}, \qquad (r^3 e)' = r^2 G - \frac{\mu^2}{2} r^4 w, \qquad g' = p, \qquad G = \gamma\,(p - w) + w\,\big(\gamma\,(2qe + e^2) - e^2\big).$$

For a solution with $g \sim ar$ at the centre we have $q \to a$, $e \to 0$, and

$$r^3 e = \int_0^r s^2 \Big(G - \frac{\mu^2}{2}\, s^2 w\Big)\mathrm{d}s, \qquad q = a + \int_0^r \frac{e - p\gamma}{s}\,\mathrm{d}s, \qquad g = \int_0^r p\,\mathrm{d}s.$$

Let $w_0 = 2a/(1+2a^2)$, $G_0 = a^3(2a^2-1)/(2(1+2a^2))$, $e_2 = (G_0 - \frac{\mu^2}{2} w_0)/5$ and $q_2 = (a^3/2 - e_2)/2$. Since $|G| \le C(s^2+e^2)$, the first identity gives $e = O(r^2)$, and then $e/r^2 \to e_2$ and $(q-a)/r^2 \to -q_2$. Now suppose that $0 \le e \le \bar{e}\, s^2$ and $a - \bar{q}\, s^2 \le q \le a$ for $0 < s \le r$, where $r \le r_1$, $\bar{e} = 1.001\, e_2$ and $\bar{q} = 1.001\, q_2$. Then p, g/s, γ/s^2 and w lie in explicit intervals, so there are computable constants with $G_l \le (G - \frac{\mu^2}{2} s^2 w)/s^2 \le G_h$ and $Q_l \le (e - p\gamma)/s^2 \le Q_h$, and the integral identities give $G_l s^2/5 \le e \le G_h s^2/5$ and $Q_l s^2/2 \le q - a \le Q_h s^2/2$. The program (`centre` in `flat/cert_massive.py`) checks in interval arithmetic, with the whole interval of slopes as a, that

$$G_h/5 < \bar{e}, \qquad G_l > 0, \qquad Q_h < 0, \qquad -Q_l/2 < \bar{q}$$

for $r_1 = 2^{-19}$ ($\mu = 1$) resp. $r_1 = 2^{-17}$ ($\mu = 0.526$). So the supposed bounds improve themselves strictly, and by continuity they hold on all of $(0, r_1]$. They give $q > 0$ and $e > 0$ there (hence $p > q$), and a box for $(q, r^3 e, g)$ at r_1 of width $O(r_1^4)$.

The same inequalities give the existence of a solution with the slope a: the set of continuous triples (q, e, g) on $[0, r_1]$ that satisfy the bounds just derived (and the corresponding bounds for g) is convex and closed, the right-hand sides of the three integral identities map it into itself, and this map is compact; by Schauder's theorem it has a fixed point. For a fixed point $(rq - \sin g)' = q + rq' - p\cos g = 0$, so $q = \sin g/r$, and g solves (31) with $g \sim ar$.

From r_1 to $r_c = \frac{1}{16}$ the plain interval Taylor method is used: order 24 in r for $(q, r^3 e, g)$ and the energy inside the ball of radius r; steps $h \le r/32$; an a priori enclosure B of the solution over the step from the Picard operator $(Y_0 + [0,h]\, F([r_0, r_0+h], B) \subset B)$; Taylor coefficients by power-series arithmetic with interval coefficients; the Lagrange remainder evaluated on B. This is done for the slope interval and for its two end points separately. On every step $q > 0$, $r^3 e \ge 0$ (hence $p > 0$) and $0 < g < \pi$ are checked on the a priori box.

(b) Mean-value integrator on $[r_c, r_N]$. The plain interval method is useless on this range: measured for $\mu = 1$, it widens an enclosure by $5 \cdot 10^9$ between $r = 0.05$ and $r = 3.4$ and by $e^{2.1}$ per unit radius beyond. Instead the image of a box is kept as $\hat{y}_k + A_k[S_k]$ with a point $\hat{y}_k$, a point matrix A_k and an interval vector $[S_k]$. For one step φ of length h: $\varphi(y) \in \varphi(\hat{y}_k) + [J](y - \hat{y}_k)$, where $[J]$ encloses the Jacobian of φ on the hull of all tracked sets; $[J]$ is obtained from the variational equation by the Taylor method (dual numbers on power series with interval coefficients, a priori boxes from the Picard operator, Lagrange remainders on these boxes). Then $\hat{y}_{k+1}$ is the midpoint of the enclosure of $\varphi(\hat{y}_k)$, A_{k+1} the midpoint of $[J]A_k$, and $[S_{k+1}] = (A_{k+1}^{-1}[J]A_k)[S_k] + A_{k+1}^{-1}(\varphi(\hat{y}_k) - \hat{y}_{k+1})$. Three sets are carried: the box of the slope interval and the boxes of its end points. The energy integral over $(0, r]$ is advanced with the Taylor expansion of the density on the hull. On the a priori box of every step, $p > 0$ and $0 < g < \pi$ are checked. Widths now grow as the true dynamics dictates (Table 4).

(c) Deciding the side of an end point at r_N.

Lemma 29 (band; PROVED)**.** *Let $r_1 \ge 1/\mu$ and let f solve* (31) *near r_1 with $0 < f(r_1) =: F$, $f'(r_1) < 0$. Put $m = -f'/f$, $u = m - m_{\mathrm{lin}}$,*

$$\alpha = \frac{2(2+\mu^2 r_1^2)}{r_1^4} + \frac{4}{3}\frac{1 + F^2/r_1^2}{r_1^2} + \frac{2}{r_1^4} + \frac{\mu^2}{6}, \qquad \beta = \frac{4}{r_1^3}, \qquad \gamma = \frac{\frac{4}{3}F^2 + 2}{r_1^2},$$
$$c = \alpha + \tfrac{\beta}{2} + 2\big(\gamma + \tfrac{\beta}{2}\big) m_{\mathrm{lin}}(r_1)^2,$$

$\delta = F^2 c/\mu$, and assume $4\delta < \mu$, $4(\gamma + \frac{\beta}{2})F^2 < 1$. (i) *If $u(r_1) \ge \delta$, then f has a zero $r_2 > r_1$ with $f' < 0$ on $[r_1, r_2]$.* (ii) *If $u(r_1) \le -\delta$, then there is $r_2 > r_1$ with $f'(r_2) > 0$ and $f > 0$ on $[r_1, r_2]$.*

Proof. From (31), $m' = m^2 - N/(r^2 + 2\sin^2 f)$ with $N = 2rm + \frac{\sin 2f}{f}\big(1 + \frac{\sin^2 f}{r^2} - m^2 f^2\big) + \mu^2 r^2 \frac{\sin f}{f}$. For the linearised equation $N_0 = 2rm + 2 + \mu^2 r^2$, the denominator is r^2, and m_{lin} is a solution

(it is the logarithmic derivative of $e^{-x}(x+1)/x^2$, $x=\mu r$). Hence

$$u' = \lambda u + u^2 + \varepsilon, \qquad \lambda = 2m_{\rm lin} - \tfrac{2}{r} = 2\mu\big(1+\tfrac{1}{x(x+1)}\big) \ge 2\mu, \qquad \varepsilon = \frac{2\sin^2 f\, N_0 - (N-N_0)\, r^2}{r^2\,(r^2+2\sin^2 f)}.$$

With $|\frac{\sin 2f}{f} - 2| \le \frac{4}{3}f^2$, $|\frac{\sin f}{f} - 1| \le \frac{1}{6}f^2$, $\sin^2 f \le f^2$ one finds, for $m \ge 0$, $|\varepsilon| \le f^2(\alpha + \beta m + \gamma m^2) \le f^2\big(\alpha + \frac{\beta}{2} + (\gamma + \frac{\beta}{2})m^2\big)$, where α, β, γ are evaluated at the current r and f; they decrease in r and increase in f, as does $m_{\rm lin}$ in r.

(i) As long as $u \ge \delta$: $m > 0$, so f decreases and the constants stay below their values at r_1; with $m^2 \le 2m_{\rm lin}^2 + 2u^2$, $|\varepsilon| \le F^2 c + 2(\gamma + \frac{\beta}{2})F^2u^2 \le \frac{1}{2}\lambda u + \frac{1}{2}u^2$. So $u' \ge \frac{1}{2}\lambda u + \frac{1}{2}u^2 > 0$: u stays above δ and blows up at a finite r_2. Then $\int^{r_2} m = \infty$, i.e. $f(r_2) = 0$, with $f' = -mf < 0$ before and $f'(r_2) \ne 0$ by uniqueness.

(ii) While $0 \le m \le m_{\rm lin} - \delta$: f decreases, $|\varepsilon| \le F^2 c$, and $u' \le -|u|(\lambda - |u|) + F^2 c$. The concave function $|u|(\lambda - |u|)$ on $[\delta, m_{\rm lin}]$ is at least $\min\{\delta(2\mu - \delta),\, m_{\rm lin}(m_{\rm lin} - \frac{2}{r})\} \ge \min\{\frac{7}{4}F^2c,\, \frac{\mu^2}{2}\} > F^2c$ (as $m_{\rm lin} - \frac{2}{r} = \frac{\mu x}{x+1} \ge \frac{\mu}{2}$ and $F^2 c = \mu\delta < \mu^2/4$). So u decreases at a rate bounded away from zero until $m = 0$; there $m' = -N/(r^2 + 2\sin^2 f) < 0$, so $m < 0$, i.e. $f' > 0$, immediately after. □

(d) The bracket.

Proposition 30 (PROVED)**.** *Let $A = [a^-, a^+]$ and $r_N \ge 1/\mu$. Suppose that for every $a \in A$ the solution satisfies $0 < g < \pi$, $g' > 0$ on $(0, r_N]$ and $f(r_N) < \pi/2$, that the solution with slope a^- satisfies the hypothesis of Lemma 29*(ii) *at r_N and the solution with slope a^+ that of* (i) *(or the other way round). Then some $a_* \in A$ gives a solution with $f > 0$, $f' < 0$ on $(0, \infty)$ and $f \to 0$, and $|u(r)| < \delta(r)$ for all $r \ge r_N$.*

Proof. Let A^+ be the set of $a \in A$ for which there is $r_2 > r_N$ with $f(r_2) < 0$ and $f' < 0$ on $[r_N, r_2]$, and A^- the set for which there is $r_2 > r_N$ with $f'(r_2) > 0$ and $f > 0$ on $[r_N, r_2]$. Both are open in A (continuous dependence on compact intervals, strict inequalities) and they are disjoint. By Lemma 29 the end points lie in different sets, so some a_* lies in neither. For a_* let $R = \sup\{r \ge r_N :\ f > 0,\ f' < 0 \text{ on } [r_N, r)\}$. If $R < \infty$ then either $f(R) = 0$, and $f'(R) < 0$ by uniqueness, so $a_* \in A^+$; or $f(R) > 0 = f'(R)$, and then by (31) $f''(R) = [\sin 2f\,(1 + \sin^2 f/R^2) + \mu^2 R^2 \sin f]/(R^2 + 2\sin^2 f) > 0$ because $0 < f(R) < \pi/2$, so $a_* \in A^-$. Hence $R = \infty$. A positive decreasing solution has a limit $L < \pi/2$; if $L > 0$ then by (31) $f'' \ge c > 0$ for large r wherever $|f'| \le 1$, which is incompatible with $f' < 0$ and f bounded. The bound on u is the contrapositive of Lemma 29 applied at r. □

(e) Energy beyond r_N.

Lemma 31 (PROVED)**.** *Let f be as in Proposition 30, $F = f(r_N)$, $M = m_{\rm lin}(r_N) + \delta(r_N)$. Then the energy in $r > r_N$ equals $4\pi\,[\,r_N^2\, f(r_N)\,|f'(r_N)| + \varrho\,]$ with*

$$|\varrho| \le F^4\Big[\big(c_3 + 2M^2 + \tfrac{2}{3} + \tfrac{1}{r_N^2}\big)\frac{1}{4\mu} + \frac{\mu^2}{4}\Big(\frac{r_N^2}{4\mu} + \frac{r_N}{8\mu^2} + \frac{1}{32\mu^3}\Big)\Big],$$

$c_3 = 2c_2 + 2M^2 + \frac{4}{3} + \frac{2}{r_N^2}$, $c_2 = \frac{2M}{r_N} + \frac{2(M^2F^2 + 1 + F^2/r_N^2)}{r_N^2} + \mu^2$.

Proof. By Proposition 30, $m_{\rm lin} - \delta < m < m_{\rm lin} + \delta \le M$ for $r \ge r_N$; since $m_{\rm lin} - \mu \ge 1/r$ and $\delta(r)\, r \le \delta(r_N)\, r_N < 1$ as long as $f \le Fe^{-\mu(r - r_N)}$ (the program checks $\delta(r_N) r_N < 1$), a continuity argument gives $m > \mu$, so $f \le Fe^{-\mu(r-r_N)}$ and $|f'| \le Mf$ throughout. Integrating by parts, $\int_{r_N}^{\infty} r^2 f'^2 = r_N^2 f|f'|\big|_{r_N} - \int f\,(r^2 f'' + 2rf')$, and (31) gives $r^2 f'' + 2rf' = (2 + \mu^2 r^2)f + \varrho_3$ with $|\varrho_3| \le f^3[2c_2 + 2M^2 + \frac{4}{3} + \frac{2}{r^2} + \frac{\mu^2 r^2}{6}]$ ($|f''| \le c_2 f$, again from the equation). So the quadratic part $\int r^2 f'^2 + 2f^2 + \mu^2 r^2 f^2$ of the energy equals the boundary term up to $\int f\varrho_3$, and the remaining terms of the density are bounded by $f^4[2M^2 + \frac{2}{3} + \frac{1}{r^2} + \frac{\mu^2 r^2}{12}]$. Integrate $e^{-4\mu(r-r_N)}$ and $e^{-4\mu(r-r_N)} r^2$. □

(f) Properties on whole intervals (Theorem 28). On a step the solution is a Taylor polynomial of degree 24 with interval coefficients (valid for every point of the hull) plus a Lagrange remainder whose coefficient is evaluated on the a priori box. For a function $\Phi(r, g, p)$ the Taylor coefficients of $r \mapsto \Phi(r, g(r), p(r))$ follow from those of g, p by series arithmetic, and the same is true of the remainder coefficient; so $\Phi(r_k + t) \in \sum_{j \le 24} \Phi_j t^j + \Phi^{\mathrm{rem}} t^{25}$ on the step, and Horner's scheme with an interval t gives ranges on subintervals. This is applied to $e = p - q$, to $\varphi_1 = q + p\cos f$ $(= -rq')$, to Y and $\psi = rY' = -2q(2D_* - 1)\varphi_1 + 4(1 + D_*)(p^2 - D_*)$, and to $H = 2P''P - P'^2$, which has the sign of $\partial_D^2 h_K$ at D_* ($P(D) = (1+2D)(D^2+2D+K)$, $h_K = 2\sqrt{P}$), together with $H' = 24P\,D_*' + [2(12D_* + 10)(1 + 2D_*) - 4P']K'$. On every step the program requires $e > 0$, $\varphi_1 > 0$, and "$\psi < 0$ or $Y < 0$" (bisecting the step if necessary), and it classifies the sign of H; a step on which a function changes sign is accepted only if the function is positive at the left end, negative at the right end and its derivative is negative throughout, and the zero is then located by bisection with point evaluations. Near the centre the same is done in the variables $(q, r^3 e, g)$, where e and $p^2 - D_* = e(2q + e)$ involve no cancellation, and on the cone $(0, r_1]$ with the bounds of the cone argument: there $Y'/r \in [-223.16, -223.15]$, $Y \in [3.999999, 4.000001]$ for $\mu = 1$ (at the centre $Y = 4\mu^2$). Beyond r_N: $m > m_{\mathrm{lin}} - \delta > 1/r \ge \sin f/(fr)$ gives (P2); $\cos f > 0$ gives (P1); $H \le 2(12D_* + 10)(1 + 2D_*)^2 p^2 - 4 < 0$ because $P' \ge 2$; and $Y = D_*(D_* - 1) - \kappa < 0$. Finally (3): $\kappa' = -\frac{4}{r}(1 + D_*)(p^2 - D_*)$ (Lemma 5(a); the program re-checks this identity, and $q' = -\varphi_1/r$ and the formula for H', against the first Taylor coefficients at every step: largest relative mismatch $7 \cdot 10^{-28}$), so $\kappa' < 0$ by (P2), and $\kappa \to 0$.

16.3 Checks, and what is missing

Checks (none is part of the proof). (1) An independent enclosure of the solution with slope $a_0 + \rho$ of the first run for $\mu = 1$ ($\rho = 10^{-11}$), by the plain interval method with a finer start ($r_1 = 2^{-20}$, steps $r/64$, 60 digits), has width $5 \cdot 10^{-22}$ at $r = 1$ and lies inside the box of that end point carried by the mean-value integrator (width $2.5 \cdot 10^{-16}$), at its centre to within 10^{-3} of the half-width; the end point is $7.7 \cdot 10^{-12}$ away from the centre of the main box, so the Jacobians are right to at least seven digits there. (2) A floating-point integration with the energy as a third unknown (`scipy`, DOP853, tolerance $3 \cdot 10^{-14}$) gives energies on $(0, r_N]$ inside the certified intervals of the first runs ($\mu = 1$, $r_N = 7$: 1.4162230377644; $\mu = 0.526$, $r_N = 13$: 1.3076650864864). (3) Negative control: a run for $\mu = 1$ whose slope interval ($\rho = 10^{-15}$ around an extrapolated slope) did not contain a_* ended with both end points on the same side of the band and was reported as not verified. (4) The interval layer has its own self-tests (`cert_profile2.selftest`). (5) The polynomial models of consecutive steps intersect at all step ends, and at 200 random radii they agree with a floating-point integration to the accuracy of the latter (`flat/profile_oracle.py`). (6) The certified r_i agree with the floating-point values 1.608 ($\mu = 1$) and 1.716 ($\mu = 0.526$). (7) The program `flat/siv/export_models.py`, applied to the stored results of the two runs, reproduces the enclosures in the data files of the verifier bit for bit. The profile certificate C1 was re-run on the second system as well, for both masses (about half an hour each). The results agree with the stored ones in every digit: the energy intervals, the band data, the certified properties, all boxes and all polynomial models.

Missing. (i) A second pair of eyes on the lemmas and on the program. (ii) The uniqueness and the continuous dependence on a of the regular solution at the centre are quoted, not proved (see the beginning of this section and Section 9). (iii) Uniqueness of the monotone solution; this is not needed for Theorem 1, which concerns the hedgehog built from the solution of Theorem 27. (iv) Polynomial models on $(0, \frac{1}{16})$; there only the sign conditions are certified. (v) Other masses; $\mu = 0$ needs a different tail lemma, because the profile then decays like a power.

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Note on the references. The references are given for orientation. The proofs of this paper do not use any result from them beyond the definition of the model and its normalisation. The descriptions of [10] and [11] in Section 1 follow the abstracts of these papers.