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RESURGENCE OF THE FINITE-N CURIE–WEISS MODEL: METASTABILITY, THE SPINODAL, AND LEE–YANG ZEROS

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Abstract. We give a resurgent description of the Curie–Weiss (mean-field Ising) model at finite N below the critical temperature in a magnetic field. We argue, modulo the standard identification of lateral thimble integrals with lateral Borel sums, that the partition function is the sum of exactly two sectors, the stable and the metastable phase. Each sector's $1/N$ series is factorially divergent, with its Borel singularity at its own free-energy barrier, and the two Stokes discontinuities cancel. As a result, the median Borel sums of the two series reproduce the exact partition function, which we confirm numerically to a relative error below 10^{-28} at $N = 150$. Lateral resummation of the metastable series gives the imaginary part of the metastable free energy, with Langer's factor $\frac{1}{2}$ appearing as a half-thimble.

At distance δ below the spinodal field, the metastable singularity approaches the origin as $\frac{16}{3}\delta^{3/2}/\sqrt{\Phi_0'''(x_s)}$, where $\Phi_0(x)$ is the mean-field free-energy functional in the up-spin fraction x and x_s is the spinodal point. There we construct the two-point Chester–Friedman–Ursell uniform expansion, in which both lateral sums become $(\mathrm{Bi} \pm \mathrm{iAi})$ and $(\mathrm{Bi}' \pm \mathrm{iAi}')$ combinations with contour-independent coefficients. Tested against an exact Hubbard–Stratonovich representation of the metastable sector, the sixth-order uniform expansion achieves a relative error below 7×10^{-20} at $N = 3200$ across the tested values $\zeta_{\mathrm{lin}} \in \{0.5, 1, 2, 3, 5\}$ of the linearized Airy variable.

On the imaginary-field axis, an anti-Stokes line, the resummed two-phase transseries reproduces the exact Lee–Yang zeros at $N = 40$, with absolute errors below 2.5×10^{-18} for the zero nearest the real axis and below 3×10^{-28} for the four farthest ones. The zero at fixed filling $s = (k + \frac{1}{2})\pi/N$ has a divergent $1/N$ expansion. We determine its first two coefficients by integral inversion and in closed form, express the third in terms of the next Laplace coefficient, and compute the series through order N^{-30}. Padé poles of its Borel transform numerically approach the complex barrier action $\mathcal{A}$, where $\mathrm{Im}\,\mathcal{A} = s$ by construction. Its Borel–Padé sum reproduces the exact zeros with absolute errors decreasing from 4×10^{-9} at $N = 24$ to 2×10^{-34} at $N = 328$; from $N \approx 200$ onward, truncation at order N^{-30} is the limiting factor.

1. Introduction

Resurgence [10, 1, 23] organizes perturbative and non-perturbative contributions into a transseries whose sectors are linked by Stokes data. It is most often tested on exponential integrals and their infinite-dimensional analogs. Finite-N mean-field models sit in between: they are finite sums, so every quantity can be computed exactly to any precision, yet as $N \to \infty$ they exhibit the full phenomenology of first-order transitions, namely phase coexistence, metastability, spinodals, and Lee–Yang zeros.

This note works out that phenomenology for the Curie–Weiss model as a resurgent transseries in $1/N$ (starting from the exact representations in Section 2) and checks each statement against exact data. The underlying mathematics is that of one-dimensional Laplace integrals with three real critical points, for which exact resurgence relations [3] and

uniform expansions for coalescing saddles [8, 6] are classical. In quantum mechanics, multi-instanton transseries for potentials with degenerate minima [34, 35] and Lefschetz-thimble decompositions of path integrals [2] are the counterparts of the structure found here.

In Langer's analysis of a droplet-model integral [20, eq. (2.13)], the analytically continued free energy is $\tilde{f} \pm \mathrm{i}\Lambda$, where $\tilde{f}$, the integral up to the saddle point, is the free energy of the metastable phase, and $\mathrm{i}\Lambda$ is, to leading order for small field, purely imaginary and equal to one half of the discontinuity across the cut. Descriptions of metastability by analytic continuation were developed further in [21, 24, 30], the last for short-range models, and Penrose and Lebowitz [28] gave a rigorous treatment of metastable states in the mean-field (Kac-limit) theory of fluids. For the Curie–Weiss model itself, Newman and Schulman [25] (summarized in [31]) showed that the infinite-N free energy continued past the spinodal acquires an imaginary part proportional to the 3/2 power of the distance from the spinodal field, and related it rigorously, through Langer's prefactor and to leading order near the spinodal, to the decay rate of spin-flip dynamics. Inside the spinodal ($0 < H < H_{\mathrm{sp}}$), that continuation is real, whereas the finite-N imaginary part studied here is exponentially small in N. For a one-dimensional quartic model that approximates the Hubbard–Stratonovich exponent of the Curie–Weiss model, they also defined a finite-N continued free energy by a contour through the barrier maximum. In the window $h - h_{\mathrm{sp}} \propto N^{-2/3}$, they expressed its imaginary part through an integral with a cubic exponent [25, eq. (22)], which is the analog, at leading order and in a linearized variable, of the leading Airy term of Section 4; its large-window limit carries the factor $\frac{1}{2}$ [25, eq. (24)]. Penrose [27] continued the divergent grand-potential series of the cluster model (under a Laplace-transform hypothesis on the cluster partition functions) by a variant of Borel's method, and showed that its imaginary part on the cut is, apart from a prefactor, the Becker–Döring nucleation rate, and that its real part is approximately the series truncated at its smallest term.

Our results fall into three groups. First, the partition function equals the sum of the median Borel sums of the two phase series, with the Stokes ambiguities canceling against the barrier thimble; this yields an all-orders version of Langer's structure for the Curie–Weiss model, derived modulo a standard step and confirmed numerically, in which the imaginary part of the metastable free energy is the half-thimble (Section 3). Second, a two-point Chester–Friedman–Ursell (CFU) expansion, constructed to all orders and verified through sixth order, describes the metastable sector as the spinodal is approached, and agrees with an exact Hubbard–Stratonovich representation of that sector to 19 digits at $N = 3200$ (Section 4). Third, on the imaginary-field axis the Lee–Yang zeros at fixed filling have divergent $1/N$ expansions whose Borel transforms are, numerically, singular at a complex barrier action with $\operatorname{Im}\mathcal{A} = s$ (Section 5).

Lee–Yang zeros [22] of the Curie–Weiss model are well studied. Kabluchko [18] identified the limiting distribution of the zeros on the imaginary-field axis for every inverse temperature, and remarked that an asymptotic result of Plancherel–Rotach type on the unit circle may be expected [18, Remark 2.6]. The density of that limiting distribution coincides with the leading term of our expansion (Remark 5.1); on the zero locus $|z| = 1$ in Kabluchko's variable $z = -\mathrm{e}^{2H}$, equation (14) below is a formal asymptotic result of that type for Z_N, from which our $1/N$ series of the zeros follows. For lattice models with convergent contour expansions, Biskup, Borgs, Chayes, Kleinwaks, and Kotecký [4, 5] derived equations for the positions of the zeros at first-order transitions, up to errors exponentially small in the system size, with the zeros concentrating on phase boundaries; the Curie–Weiss model is outside that class, and here the corrections are computed explicitly. Deger and Flindt [9] extracted the leading Lee–Yang and Fisher zeros of the Curie–Weiss model from fluctuations in finite systems. Pisani

and Smith [29] argued, for a solid-on-solid model with a wetting transition, that the Lee–Yang zeros condense as $N \to \infty$ on an anti-Stokes curve, where two asymptotic contributions to the partition function balance, and derived asymptotic expansions of the finite-size zeros, labeled by $2(n+\frac{1}{2})\pi/N$, which they compared with exact zeros. None of these Curie–Weiss studies addresses $1/N$ corrections to the zero locations, and none of the works cited above examines the Borel structure of such corrections. The relation between partition-function zeros and spinodals in long-range Ising models was studied in [14]. The closest resurgence study we know of, Hampton, Lee, and Lee [15], treats the thermal transition between bounce and sphaleron in quantum mechanics, a different system.

Code that reproduces every computation in this note is available at https://github.com/shankarsivarajan/resurgent-series.

2. Model and exact representations

The Curie–Weiss partition function with coupling $K = \beta J$ and field $H = \beta h$ is

$$Z_N(K,H) = \sum_{n=0}^{N} \binom{N}{n} \exp\Big(N\Big[\frac{Km^2}{2} + Hm\Big]\Big), \qquad m = \frac{2n}{N} - 1. \tag{1}$$

We use two exact rewritings.

Stirling form. Write $n = Nx$ and expand the binomial with Stirling's series [26, (5.11.1)]. With $P_N = \Gamma(N+1)\mathrm{e}^N N^{-N}/(2\pi)$, for fixed $x \in (0,1)$ the summand is asymptotic to $P_N N^{-1}$ times

$$\exp\Big(N\Phi_0(x) + \Phi_1(x) + \sum_{j\geq 1} N^{-j}\Phi_{j+1}(x)\Big), \tag{2}$$

where B_{j+1} are the Bernoulli numbers and

$$\Phi_0(x) = -x\log x - (1-x)\log(1-x) + \tfrac{K}{2}(2x-1)^2 + H(2x-1), \tag{3}$$

$$\Phi_1(x) = -\tfrac{1}{2}\log(x(1-x)), \tag{4}$$

$$\Phi_{j+1}(x) = -\frac{B_{j+1}}{j(j+1)}\big[x^{-j} + (1-x)^{-j}\big]. \tag{5}$$

Here Φ_0 is the mean-field free-energy functional. Its critical points solve $m = \tanh(Km+H)$, with $m = 2x-1$. For $K > 1$ and $0 < H < H_{\mathrm{sp}}$, where $H_{\mathrm{sp}} = \operatorname{atanh} m_s - Km_s$ (with $m_s = -\sqrt{1-1/K}$) is the spinodal field and $x_s = (1+m_s)/2$ is the spinodal point, there are three of them: the stable maximum x_+, the metastable maximum x_-, and the barrier (minimum) x_0 between them. Under a smooth cutoff localized near each interior critical point $x_\pm \in (0,1)$, the Euler–Maclaurin formula replaces the localized sum by the integral $\int \exp(N\Phi_0(x) + \Phi_1(x) + \sum_{j\geq 1} N^{-j}\Phi_{j+1}(x))\,\mathrm{d}x$ to all orders in $1/N$, and the all-orders Laplace method yields the two series (with $m_\pm = 2x_\pm - 1$)

$$A_\pm(N) = \sum_{r\geq 0} q_r^\pm N^{-r-1/2}, \qquad q_0^\pm = \sqrt{\frac{2\pi}{1 - K(1-m_\pm^2)}}. \tag{6}$$

The Stirling form serves only to generate these local series. Globally on $[0,1]$ it is invalid: Stirling's series fails as $x \to 0, 1$ (for instance, at $j = 3$ one has $B_4 = -1/30$ so $\Phi_4(x) = +\frac{1}{360}[x^{-3} + (1-x)^{-3}] \to +\infty$, and even at leading order, the endpoint values at $K = 1.5$, $H = 0.05$, namely $\Phi_0(0) = 0.70$ and $\Phi_0(1) = 0.80$, exceed $\Phi_0(x_0) = 0.69$, with $\Phi_0(1)$ exceeding even $\Phi_0(x_-) = 0.77$). Non-perturbative statements therefore cannot rest on this form; they rest on the following one.

Hubbard–Stratonovich form. Linearizing the quadratic term [16, 32] gives

$$Z_N = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{N\Psi(y)}\,\mathrm{d}y, \qquad \Psi(y) = -\frac{Ky^2}{2} + \log\bigl(2\cosh(Ky+H)\bigr), \tag{7}$$

whose integrand $(2\cosh(Ky+H))^N \mathrm{e}^{-NKy^2/2}$ is *entire*. The critical points of Ψ are again $y = m$ with $m = \tanh(Km+H)$; we write $y_0 = m_0$ and $y_\pm = m_\pm$, and $\Psi(m_i) = \Phi_0(x_i)$. Representation (7) is exact, so it provides independent high-precision values of each sector (Section 4). Modulo the identification of lateral thimble integrals with lateral Borel sums (Section 3), it also shows that Z_N has exactly the two sectors below and no "discreteness" (Poisson) sectors.

3. Two-phase transseries and median reconstruction

Terminology. For a series $A(N) = \sum_r q_r N^{-r-1/2}$ with factorially growing coefficients, the Borel transform $\sum_r q_r \xi^{r-1/2}/\Gamma(r+\frac{1}{2})$ is continued from the origin, and the Laplace integral back along a ray gives the Borel sum. When singularities lie on the integration ray, the integrals just above and below it, the lateral sums $\mathcal{S}_\uparrow$ and $\mathcal{S}_\downarrow$, differ by exponentially small terms, and their average is the median sum $\mathcal{S}_{\mathrm{med}}$. For real series, the difference (the Stokes jump) is $\mathcal{S}_\uparrow - \mathcal{S}_\downarrow = 2\mathrm{i}\,\mathrm{Im}\,\mathcal{S}_\uparrow$; we call $|\,\mathrm{Im}\,\mathcal{S}_\uparrow| = |\,\mathrm{Im}\,\mathcal{S}_\downarrow|$ the half-jump, and write $\mathcal{S}_\tau$ with $\tau = +1$ for $\mathcal{S}_\uparrow$ and $\tau = -1$ for $\mathcal{S}_\downarrow$. A transseries is a sum of such series, each multiplied by its own exponential $\mathrm{e}^{N\Phi}$; its sectors are linked by these jumps. A thimble is a steepest-descent contour through a critical point of the exponent. A Stokes direction is one in which a Borel singularity lies on the integration ray; on an anti-Stokes line, the exponentials of two sectors have equal modulus.

Result 3.1 (derived; confirmed numerically)**.** For $K > 1$ and $0 < H < H_{\mathrm{sp}}$,

$$\frac{Z_N}{P_N} = \mathrm{e}^{N\Phi_0(x_+)}\,\mathcal{S}_{\mathrm{med}}[A_+](N) + \mathrm{e}^{N\Phi_0(x_-)}\,\mathcal{S}_{\mathrm{med}}[A_-](N), \tag{8}$$

where $\mathcal{S}_{\mathrm{med}}$ is the median (average of the two lateral) Borel sum. The Borel transform of $A_\pm$ has its leading singularity at the barrier height $\Phi_0(x_\pm) - \Phi_0(x_0)$. Let $T = (\sqrt{NK/2\pi}/P_N)\,|\int_{\bar{y}_*}^{y_*} \mathrm{e}^{N\Psi(y)}\,\mathrm{d}y| > 0$ be the barrier-thimble integral in the Hubbard–Stratonovich form (7), with $y_* = (\mathrm{i}\pi/2 - H)/K$. Then $\mathcal{S}_\tau[A_\pm] - \mathcal{S}_{\mathrm{med}}[A_\pm] = \tau\,\frac{\mathrm{i}}{2}\,T\,\mathrm{e}^{-N\Phi_0(x_\pm)}$ for both phases, so that $Z_N/P_N = \mathrm{e}^{N\Phi_0(x_+)}\mathcal{S}_\tau[A_+] + \mathrm{e}^{N\Phi_0(x_-)}\mathcal{S}_{-\tau}[A_-]$ for either τ: the weighted jumps cancel between opposite lateral sums of the two phases.

Proof sketch. In (7) the real line is homologous to $\mathcal{J}_+ + \mathcal{J}_-$:

- $\mathcal{J}_-$ runs from $-\infty$ along the real axis to the barrier y_0, then along the barrier thimble to the zero y_* of the integrand (or to $\bar{y}_*$);
- $\mathcal{J}_+$ runs back along the same barrier leg from y_* (or $\bar{y}_*$) to y_0 and then along the real axis to $+\infty$.

Each lateral thimble integral is the corresponding lateral Borel sum. Since $\mathcal{J}_-$ and $\mathcal{J}_+$ traverse the barrier leg in opposite directions, the leg ending at y_* corresponds to the lateral sum $\mathcal{S}_\tau$ for one phase and $\mathcal{S}_{-\tau}$ for the other; averaging over the choice of leg (y_* or $\bar{y}_*$) gives (8). That the two phases' jumps have the same sign, and hence cancel between opposite lateral sums, also follows by continuity from $H = 0$, where $A_+ = A_-$; we confirmed it numerically. The HS and Stirling expansions agree sector by sector as formal series: the coefficients $q_r^\pm$ are analytic in H on both sides, uniqueness of the asymptotic expansion identifies the dominant sector, and continuing in H through $H < 0$, where the other maximum dominates, identifies the subdominant one. The identification extends to the Borel sums: the Stirling series of P_N

is Borel summable along the real axis to P_N itself (Binet's formula [26, §5.9] writes $\log\Gamma$ as a Laplace integral whose Borel transform is singular only at $\pm 2\pi i k$), and lateral Borel summation in a fixed direction is multiplicative, so the lateral Borel sums of the Stirling sectors times P_N are the HS thimble integrals. The step from lateral thimble integrals to lateral Borel sums is the standard one for Laplace integrals along steepest-descent paths and is not proved in detail here. Numerically, the steepest-descent path from y_0 ends at y_*. The results below support the full statement.

Numerical confirmation. We test these results at $K = 1.5$, $H = 0.05$ using 60 coefficients per series (the sign of the jumps was also checked at $H = 0.05$ and 0.001 with $N = 40$, 40 coefficients, and rays at $\arg\xi = \pm 0.1$):

- *Actions.* Large-order estimates of the actions agree with the barrier heights 0.161134539514 (stable) and 0.0753642625393 (metastable) to better than 2×10^{-11} and 10^{-13}, respectively.
- *Reconstruction.* The median reconstruction (8), evaluated with 60-term Borel–Padé sums, agrees with the exact Z_N to relative errors below 3×10^{-12}, 9×10^{-17}, 3×10^{-19}, 2×10^{-25}, and 10^{-28} at $N = 20, 40, 60, 100, 150$. The canceled ambiguity is as large as 1.65% of Z_N (at $N = 20$).
- *Stokes jumps.* The weighted half-jumps $\mathrm{e}^{N\Phi_0(x_\pm)}|\operatorname{Im}\mathcal{S}_\uparrow[A_\pm]|$ of the two phases agree to at least eight significant digits. With the Gaussian approximation $T_0\,\mathrm{e}^{N\Phi_0(x_0)}$ to T, where $T_0 = \mathrm{e}^{\Phi_1(x_0)}\sqrt{2\pi/(N\Phi_0''(x_0))}$, the ratio $|\operatorname{Im}\mathcal{S}_\uparrow[A_\pm]|/\big(T_0\,\mathrm{e}^{-N(\Phi_0(x_\pm)-\Phi_0(x_0))}\big)$ tends to $\frac{1}{2}$ (0.454, 0.473, 0.481, 0.488, 0.492 for $N = 20$–150).

The imaginary part of the metastable free energy. Define the metastable free energy by

$$f_- = -\frac{1}{N}\log\Big(\mathrm{e}^{N\Phi_0(x_-)}\mathcal{S}_\tau[A_-]\Big). \tag{9}$$

Lateral resummation then gives

$$\operatorname{Im} f_- = -\tau\,\frac{1}{N}\arctan\frac{|\operatorname{Im}\mathcal{S}_\uparrow[A_-]|}{\operatorname{Re}\mathcal{S}_{\mathrm{med}}[A_-]}. \tag{10}$$

To leading order, $\operatorname{Im} f_- \simeq -\tau\frac{1}{2N}T_0\,\mathrm{e}^{-N(\Phi_0(x_-)-\Phi_0(x_0))}/\mathcal{S}_{\mathrm{med}}[A_-]$; the ratio of $\operatorname{Im} f_-$ to this leading estimate is 0.905, 0.947, 0.962, 0.976 at $N = 20, 40, 60, 100$, consistent with $1+O(1/N)$. The factor $\frac{1}{2}$ is the median prescription: each lateral sum carries half of the barrier thimble, as in Langer's analysis [20].

4. Approaching the spinodal: uniform Airy expansion to all orders

Fold scaling. As $H \to H_{\mathrm{sp}}$, the metastable maximum and the barrier merge. With $\delta = H_{\mathrm{sp}} - H$, the metastable Borel singularity approaches the origin as

$$\Phi_0(x_-) - \Phi_0(x_0) = \frac{16}{3}\frac{\delta^{3/2}}{\sqrt{\Phi_0'''(x_s)}}\,(1 + O(\delta)). \tag{11}$$

Writing $\Delta\Phi_-$ for this action, numerically $\Delta\Phi_-/\delta^{3/2}$ decreases from 1.2052 at $\delta \approx 0.1575$ ($H = 0.05$) to 1.1708 at $\delta = 0.0045$, against the limit $16/(3\sqrt{\Phi_0'''(x_s)}) = 4/\big(3K(1-1/K)^{1/4}\big) = 1.1698$ (at $K = 1.5$). Once $N\Delta\Phi_- = O(1)$, the Gaussian Stokes normalization fails, and a uniform treatment is needed. (For long-range models in finite dimension, nucleation near the spinodal

was analyzed field-theoretically in [33]. At leading order, the $N^{-2/3}$ window and the cubic-exponent integral appear in [25, eq. (22)] for a quartic model.)

Two-point CFU expansion. Define the exact Airy variable and the midpoint value by

$$\rho^{3/2} = \tfrac{3}{4}(\Phi_0(x_-) - \Phi_0(x_0)), \qquad \zeta = N^{2/3}\rho, \qquad \bar{\Phi} = \tfrac{1}{2}(\Phi_0(x_-) + \Phi_0(x_0)). \tag{12}$$

To leading order in δ, ζ equals the linearized variable $\zeta_{\rm lin} = 2\delta N^{2/3}/(3\gamma)^{1/3}$ with $\gamma = \Phi_0'''(x_s)/6$, which we use to choose test fields. Introduce the cubic change of variables $\Phi_0(x) = \bar{\Phi} + \rho u - u^3/3$, which sends $x_- \mapsto u = +\sqrt{\rho}$ and $x_0 \mapsto u = -\sqrt{\rho}$. Let $G(u) = g(x(u))\,|\mathrm{d}x/\mathrm{d}u|$ (here $\mathrm{d}x/\mathrm{d}u < 0$), with $g = \exp(\Phi_1 + \sum_j N^{-j}\Phi_{j+1})$, and expand $G = \sum_j N^{-j}G^{(j)}$ in the Stirling corrections. Applying Bleistein's integration-by-parts recursion [6], $G_\ell^{(j)} = c_{j,\ell} + d_{j,\ell}u + (u^2 - \rho)R_{j,\ell}$ and $G_{\ell+1}^{(j)} = R_{j,\ell}'$ with $G_0^{(j)} = G^{(j)}$, and using the contour integrals for Ai and Bi [26, (9.5.4)–(9.5.5)], we obtain as $N \to \infty$ at fixed ζ for either lateral contour

$$\mathrm{e}^{N\Phi_0(x_-)}\mathcal{S}_\tau[A_-] \simeq \pi\,\mathrm{e}^{N\bar{\Phi}}N^{-1/3}\Big[\mathcal{C}(N)\,(\mathrm{Bi} + \mathrm{i}\tau\mathrm{Ai})(\zeta) + N^{-1/3}\mathcal{D}(N)\,(\mathrm{Bi}' + \mathrm{i}\tau\mathrm{Ai}')(\zeta)\Big], \tag{13}$$

with $\mathcal{C} = \sum_{j,\ell} c_{j,\ell}N^{-j-\ell}$ and $\mathcal{D} = \sum_{j,\ell} d_{j,\ell}N^{-j-\ell}$, truncated at total order $t = j + \ell$; $\mathcal{C}$ and $\mathcal{D}$ are the same for both contours. The coefficients require only Taylor data at the two critical points, obtained from local square-root maps and series reversion, so no disk containing both points is needed. As in [6], we assume that the cubic change of variables is one-to-one along the contours; we have not proved this globally, and its numerical support is the agreement in Table 1 over the tested range $\zeta_{\rm lin} \le 5$ ($\zeta < 5.11$).

Exact reference. In (7) the metastable median is $\int_{-\infty}^{y_0} + \mathrm{Re}\int_{y_0}^{y_*}$ and the half-jump is $|\,\mathrm{Im}\int_{y_0}^{y_*}\,|$, each multiplied by $\sqrt{NK/2\pi}/P_N$. Both are ordinary quadratures of an entire function and are independent of the path from y_0 to y_*. They can be evaluated to high precision at any N, and they agree with the independent values $Z_N/P_N - \mathrm{e}^{N\Phi_0(x_+)}\mathcal{S}_{\rm med}[A_+]$ and the stable half-jump to better than 10^{-14} at $N = 50, 100$ (60-term stable series, $\zeta_{\rm lin} = 0.5$). That subtraction becomes unusable for large N, since it requires the stable sum to relative accuracy $\mathrm{e}^{-N(\Phi_0(x_+)-\Phi_0(x_-))}$.

TABLE 1. Relative error of the uniform expansion (13) truncated at total order 0 and 6, against the exact HS thimble integrals. The worse of the real and imaginary parts is shown. $K = 1.5$, with $H = H_{\rm sp} - \zeta_{\rm lin}(3\gamma)^{1/3}/(2N^{2/3})$. Entries marked n/a have $H < 0$, outside the regime.

$\zeta_{\rm lin}$	$N = 50$: order $0 \to 6$	$N = 400$: order $0 \to 6$	$N = 3200$: order $0 \to 6$
0.5	$9.8 \times 10^{-3} \to 4.1 \times 10^{-7}$	$1.0 \times 10^{-3} \to 8.5 \times 10^{-14}$	$1.2 \times 10^{-4} \to 2.9 \times 10^{-20}$
1	$1.2 \times 10^{-2} \to 1.5 \times 10^{-6}$	$1.1 \times 10^{-3} \to 1.2 \times 10^{-13}$	$1.2 \times 10^{-4} \to 3.2 \times 10^{-20}$
2	$1.7 \times 10^{-2} \to 2.9 \times 10^{-5}$	$1.2 \times 10^{-3} \to 2.4 \times 10^{-13}$	$1.3 \times 10^{-4} \to 4.0 \times 10^{-20}$
3	n/a	$1.3 \times 10^{-3} \to 4.8 \times 10^{-13}$	$1.3 \times 10^{-4} \to 4.8 \times 10^{-20}$
5	n/a	$1.6 \times 10^{-3} \to 2.0 \times 10^{-12}$	$1.4 \times 10^{-4} \to 6.9 \times 10^{-20}$

Results. Table 1 and Figure 1(a) show numerically that truncation at total order t is accurate to $O(N^{-t-1})$ at fixed ζ; the sixth-order residual falls by a factor between 139 and 215 from $N = 1600$ to $N = 3200$, and these factors decrease toward the asymptotic value $2^7 = 128$ (from about 143 to 285 between $N = 800$ and 1600). With ζ defined exactly through the barrier height, the leading term is accurate to $O(1/N)$. The linearized variable $\zeta_{\rm lin}$ is a poor substitute: with it, the ratio $|\,\mathrm{Im}\,|/\,\mathrm{Re}$ approaches $\mathrm{Ai}(\zeta_{\rm lin})/\mathrm{Bi}(\zeta_{\rm lin})$ only slowly (at $\zeta_{\rm lin} = 3$, the

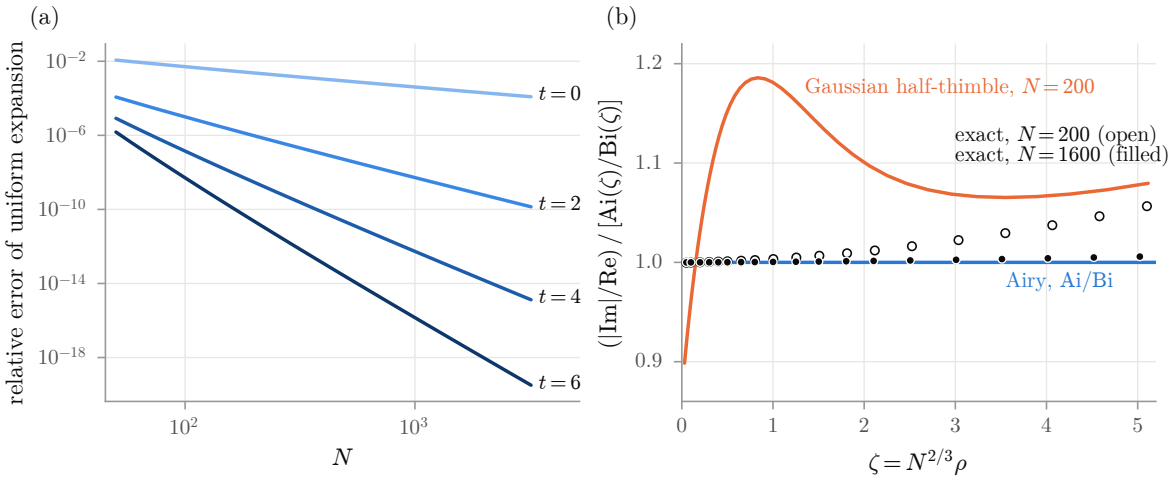

FIGURE 1. The metastable sector near the spinodal ($K = 1.5$). (a) Relative error of the uniform expansion (13) truncated at total order t, against the exact Hubbard–Stratonovich thimble integrals, at fixed $\zeta_{\mathrm{lin}} = 1$; the worse of the real and imaginary parts is shown. (b) Ratio $|\mathrm{Im}|/\mathrm{Re}$ of the metastable sector divided by its uniform limit $\mathrm{Ai}(\zeta)/\mathrm{Bi}(\zeta)$, as a function of the exact Airy variable ζ. Exact values at $N = 200$ (open) and $N = 1600$ (filled) approach the Airy limit as N grows; the Gaussian half-thimble approximation (half the normalized Gaussian barrier thimble $T_0 \mathrm{e}^{-N(\Phi_0(x_-)-\Phi_0(x_0))}$ divided by the leading metastable term $q_0^- N^{-1/2}$), shown at $N = 200$, overshoots it by up to about 19% near $\zeta \approx 0.85$ and undershoots it as $\zeta \to 0$.

ratio reaches only 0.938 of that limit at $N = 400$ and 0.983 at $N = 3200$). The signed real and imaginary parts agree as well: the real part is the positive median, and the imaginary part, positive for $\tau = +1$, is the half-jump of $\mathcal{S}_\uparrow$, consistent with Result 3.1. Physically, (13) continues the imaginary part of the metastable free energy toward the spinodal (tested for $0.5 \le \zeta_{\mathrm{lin}} \le 5$): near the fold $\mathrm{Im}\, f_-$ is governed by $\mathrm{Ai}(\zeta)/\mathrm{Bi}(\zeta)$ rather than by half a Gaussian thimble. Figure 1 shows both the order-by-order convergence of (13) and the approach of the exact ratio to Ai/Bi, which the Gaussian half-thimble misses by up to about 19% at $N = 200$.

5. Lee–Yang zeros

The imaginary-field axis is an anti-Stokes line. For $H = \mathrm{i}\eta$, the two phases are complex conjugates: $x_- = 1 - \bar{x}_+$, so $\Phi_0(x_-) = \overline{\Phi_0(x_+)}$ and $q_r^- = \overline{q_r^+}$. For $\eta > 0$, the barrier action $\mathcal{A} = \Phi_0(x_+) - \Phi_0(x_0)$ is complex, the real Borel ray passes on opposite sides of $\mathcal{A}$ and $\bar{\mathcal{A}}$ for the two phases, consistent with $\mathcal{S}[A_-] = \overline{\mathcal{S}[A_+]}$, and the singularities leave the real ray at once, since the imaginary part of the barrier action, $\mathrm{Im}\,\Phi_0(x_+(\mathrm{i}\eta)) = \int_0^\eta \mathrm{Re}\, m_+(\mathrm{i}\eta')\,\mathrm{d}\eta'$, is positive for $\eta > 0$. By Remark 5.1, $\mathrm{Re}\, m_+/\pi$ is the limiting density of zeros, whose support is all of $\mathbb{R}$ for $K \ge 1$ [18, Remark 2.8]. Continuing the opposite-side combination of Result 3.1 from $H = 0$ then gives (derived; confirmed numerically below)

$$\frac{Z_N}{P_N} = 2\,\mathrm{Re}\left[\mathrm{e}^{N\Phi_0(x_+)}\,\mathcal{S}[A_+]\right], \tag{14}$$

with $x_+(\mathrm{i}\eta)$ continued from $H = 0$. Hence the plain Borel sum is unambiguous.

At $N = 40$, $K = 1.5$, with a 40-term series, the right side of (14) matches the exact $Z_N(\mathrm{i}\eta)$ to relative errors below 10^{-18} at $\eta = 0.1$ and 10^{-24} at $\eta = 0.3$ and 0.5. Its zeros reproduce the seven exact Lee–Yang zeros with $\eta < 0.6$ ($\eta_k = 0.0474759392\ldots$, $0.1400296650\ldots$, $0.2294970478\ldots$, …, $0.5634587992\ldots$) with absolute errors below 2.5×10^{-18}, 1.6×10^{-22}, and 2×10^{-25} for the three zeros nearest the real axis, and below 3×10^{-28} for the other four. The accuracy is lowest for the zero nearest the real axis, where the barrier singularity approaches the positive Borel axis. At $\eta = 0$, the barrier singularity lies on the positive real axis; there the lateral sums carry imaginary parts of relative size 5×10^{-3}, and the median sum gives Z_N to relative error 5×10^{-15}.

The $1/N$ expansion of the zeros at fixed filling. Itzykson, Pearson, and Zuber [17] interpreted lines of zeros as boundaries where two asymptotic behaviors of the partition function have comparable magnitude, which they called Stokes lines (anti-Stokes lines in the terminology used here), with the discontinuity of the imaginary part of the free energy proportional to the density of zeros. Pisani and Smith [29] developed this picture for a wetting model. Kanazawa and Tanizaki [19] linked anti-Stokes lines and Lee–Yang zeros in thimble decompositions of zero-dimensional fermionic models. Write $\varepsilon = 1/N$ and $\log\sum_r q_r^+\varepsilon^r = \sum_j l_j\varepsilon^j$. The zeros of (14) satisfy $N\,\mathrm{Im}\,\Phi_0(x_+) + \arg\mathcal{S}[A_+] = (k+\frac12)\pi$, that is, at filling $s = (k+\frac12)\pi/N$,

$$F(\eta,\varepsilon) \equiv \mathrm{Im}\,\Phi_0\big(x_+(\mathrm{i}\eta)\big) + \sum_{j\ge0}\varepsilon^{j+1}\,\mathrm{Im}\,l_j(\eta) = s. \tag{15}$$

Solving order by order gives $\eta_k = \sum_n a_n(s)N^{-n}$. Write $F_0(\eta) = \mathrm{Im}\,\Phi_0(x_+(\mathrm{i}\eta))$ and $F_1(\eta) = \mathrm{Im}\,l_0(\eta)$. The envelope identity $\mathrm{d}\Phi_0/\mathrm{d}H = m$ gives $\partial_\eta\,\mathrm{Im}\,\Phi_0 = \mathrm{Re}\,m_+(\mathrm{i}\eta)$. Write $m(\eta) = m_+(\mathrm{i}\eta)$ and $\kappa(\eta) = 1 - K(1-m^2)$. Then, with m and κ evaluated at $\eta = a_0$,

$$\int_0^{a_0}\mathrm{Re}\,m(\eta')\,\mathrm{d}\eta' = s, \tag{16}$$

$$a_1 = \frac{\arg\kappa}{2\,\mathrm{Re}\,m}, \tag{17}$$

$$a_2 = -\frac{\frac12F_0''a_1^2 + F_1'a_1 + \mathrm{Im}(q_1^+/q_0^+)}{\mathrm{Re}\,m}, \tag{18}$$

$$F_0'' = -\,\mathrm{Im}\,\frac{1-m^2}{\kappa}, \qquad F_1 = -\tfrac12\arg\kappa, \qquad F_1' = -\,\mathrm{Re}\,\frac{Km(1-m^2)}{\kappa^2}. \tag{19}$$

Here q_1^+ is the next coefficient of the Laplace expansion of A_+. Correspondingly, the zero density is $\nu(\eta) = \pi^{-1}\big[N\,\mathrm{Re}\,m(\eta) - \frac12\partial_\eta\arg\kappa + O(1/N)\big]$.

Remark 5.1. The leading term agrees with Kabluchko's limit law [18, Corollary 2.7, Remark 2.8], whose density of zeros at $H = \mathrm{i}\eta$ is $(\pi\beta)^{-1}\,\mathrm{Im}\,\theta_\beta(\eta)$, where we write $\theta_\beta(\eta)$ for his solution in the closed upper half-plane of $\theta - \beta\tan\theta = \eta$; his β and h are our K and H. To see the agreement, put $w = Km + \mathrm{i}\eta$ in $m = \tanh(Km + \mathrm{i}\eta)$ and write $w = \mathrm{i}\theta$. Then $\theta = \eta + K\tan\theta$ and $\theta = \eta - \mathrm{i}Km$. The branch with $\mathrm{Im}\,\theta > 0$ is $m = m_-$, for which $\mathrm{Im}\,\theta/K = -\,\mathrm{Re}\,m_- = \mathrm{Re}\,m_+$. Hence his density equals $\mathrm{Re}\,m_+(\mathrm{i}\eta)/\pi$ per unit η and per spin.

We computed higher orders as follows. For each $\mathcal{G} \in \{\Phi_0(x_+), l_0, l_1, \ldots\}$, viewed as an analytic function of η, the combination $\mathrm{Im}\,\mathcal{G}(\eta) = (\mathcal{G}(\eta) - \overline{\mathcal{G}(\bar\eta)})/2\mathrm{i}$ is itself analytic. We sampled it at 80 points on a circle of radius 0.04 around a_0, extracted Taylor coefficients by the Cauchy integral, and inverted the series formally. At $K = 1.5$, $s = \pi/16$, this gives $a_0 = 0.2249856434$, $a_1 = 0.1610133589$, $a_2 = 0.7499786$, $a_3 = 2.0878070$, $a_4 = -29.43$, $a_5 = -586.1,\ldots$ (through order N^{-30}).

Resurgence of the zero series. We apply Padé approximants in ξ to the Borel transform $\sum_{n\geq 1} a_n \xi^{n-1}/(n-1)!$ of the zero series; the Borel sum is a_0 plus the Laplace integral back along the positive real axis. The series diverges factorially. Numerically, its Borel singularities lie at the complex barrier actions $\mathcal{A} = 0.05750 + 0.19635\,\mathrm{i}$ and $\bar{\mathcal{A}}$, evaluated at $\eta = a_0$: the nearest $[M/M]$ Padé poles approach $\mathcal{A}$, from $0.0617 + 0.2028\,\mathrm{i}$ at $M = 8$ to $0.0587 + 0.1988\,\mathrm{i}$ at $M = 14$, and

$$\operatorname{Im}\mathcal{A} = s. \tag{20}$$

This follows directly from the definition of a_0, by which $\operatorname{Im}\Phi_0(x_+) = s$ at $\eta = a_0$, and from the reality of $\Phi_0(x_0)$: at $H = \mathrm{i}\eta$ the barrier root continued from $H = 0$ stays purely imaginary, $m_0 = \mathrm{i}\mu$ with $\mu = \tan(K\mu + \eta)$ on the branch $|K\mu + \eta| < \pi/2$, so x_0 and $1 - x_0$ are complex conjugates. The singularity controls the large-order growth of the series, factorial at rate $|\mathcal{A}|^{-1}$ up to oscillation, so optimal truncation leaves an error of order $\mathrm{e}^{-N|\mathcal{A}|}$ up to a prefactor. It should not produce corrections to the Borel sum along the real axis, which is not a Stokes direction for $\eta > 0$; numerically, the Borel–Padé sum agrees with the exact zeros far below $\mathrm{e}^{-N\operatorname{Re}\mathcal{A}}$ (Table 2). Note that the diagonal [15/14] Padé approximant used in Table 2 and Figure 2 carries two spurious poles on the positive real axis (at $\xi \approx 0.534$, with residue $\approx 1.5 \times 10^{-3}$, and $\xi \approx 0.953$), so the real-axis integral requires a regularization (numerical quadrature stepping past the poles, principal value, or a real-pole-free approximant such as [14/14]). Table 2 and Figure 2 use quadrature stepping past the poles; for $N \geq 56$, this choice changes the error by less than 10^{-13}, whereas at $N = 24$ and 40 the error depends on it at the 10^{-9}–10^{-11} level.

Zero k at $N = 8(2k+1)$ lies at exactly $s = \pi/16$, so a single series can be checked against many exact zeros (Table 2 and Figure 2).

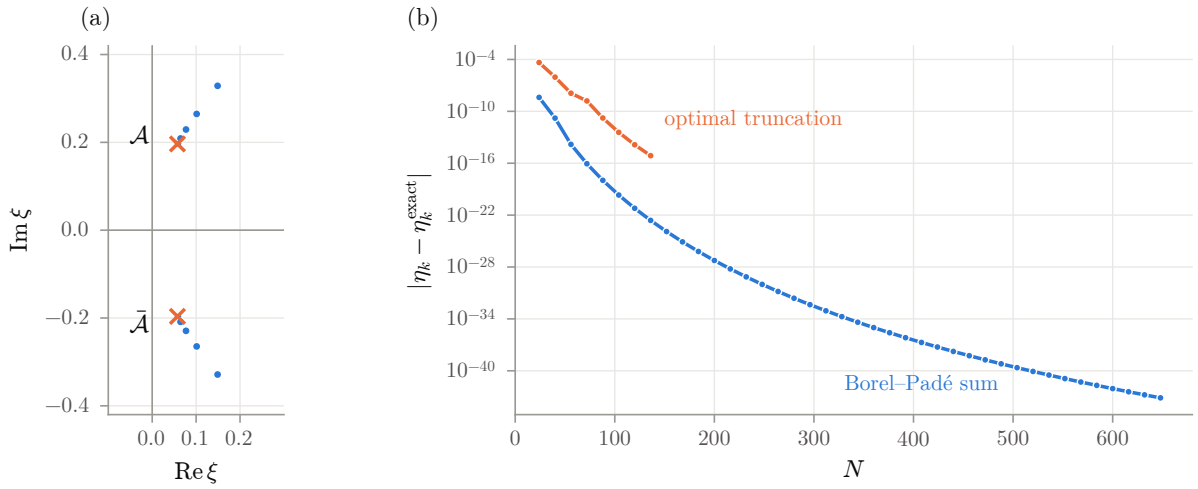

FIGURE 2. The $1/N$ series of the Lee–Yang zero at filling $s = \pi/16$ ($K = 1.5$, through order N^{-30}). (a) Borel plane: poles of the [15/14] Padé approximant of the Borel transform (dots), the nearest lying at distance 0.0026 from the complex barrier actions $\mathcal{A}$ and $\bar{\mathcal{A}}$ (crosses; $\operatorname{Im}\mathcal{A} = s$), with further poles beyond them. (b) Absolute error against the exact zeros at $N = 8(2k+1)$, $k = 1, \ldots, 40$: optimal truncation (for $N|\mathcal{A}| < 30$, where the smallest term is among the computed orders) and the Borel–Padé sum.

TABLE 2. The $1/N$ series of the Lee–Yang zero at filling $s = \pi/16$ ($K = 1.5$; through order N^{-30}) against exact zeros (rounded to 18 digits). Absolute errors are shown. Optimal truncation is evaluated for $N|\mathcal{A}| < 30$, summing up to, but not including, the smallest term (order in parentheses). Because the coefficients oscillate (two conjugate singularities), the smallest term can fall on an anomalously small coefficient, as at order 23 for $N = 88$ and 136 (and even at order 28 for $N = 200$, though the average turning point $N|\mathcal{A}|$ with $|\mathcal{A}| = 0.205$ lies beyond the computed orders for $N = 200$ and 328). For $N \gtrsim 200$, the Borel–Padé error decays roughly like N^{-31} (local exponents 30 to 32 between $N = 200$ and 648), reflecting the truncation at order N^{-30} rather than the resummation.

N	k	exact η_k	optimal truncation (order)	Borel–Padé
24	1	0.233029526069293597	4.4×10^{-5} (6)	4.1×10^{-9}
40	2	0.229497047871485927	9.0×10^{-7} (11)	1.6×10^{-11}
56	3	0.228108088420070792	1.2×10^{-8} (13)	1.5×10^{-14}
88	5	0.226914660426714395	1.7×10^{-11} (23)	1.1×10^{-18}
136	8	0.226210845282279990	7.5×10^{-16} (23)	2.4×10^{-23}
200	12	0.225809700417462746	n/a	5.7×10^{-28}
328	20	0.225483565311543875	n/a	1.8×10^{-34}

6. DISCUSSION

In the finite-N Curie–Weiss model, the cancellation of Stokes jumps between the two phases, together with the reality of Z_N, selects the median prescription and hence the factor $\frac{1}{2}$ in the imaginary part of the metastable free energy. The collision of the metastable maximum with the barrier at the spinodal produces an Airy uniformization, constructed to all orders in $1/N$ and verified through sixth order for $\zeta_{\text{lin}} \in \{0.5, 1, 2, 3, 5\}$. On the imaginary-field axis, an anti-Stokes line, the Lee–Yang zeros at fixed filling have resurgent $1/N$ expansions whose Borel singularity is a complex barrier action with imaginary part equal to the filling. The Hubbard–Stratonovich representation makes each sector separately computable at any N, and, modulo the identification of lateral thimble integrals with lateral Borel sums (Section 3), it shows that the discreteness of the spin sum generates no further sectors.

The same methods should apply to the zeros in the complex temperature plane (studied by finite-size scaling for infinite-range models in [12]), to the critical point $K \to 1$, where three critical points merge and the uniform expansion involves the Pearcey integral, and to escape times of the Glauber dynamics, whose exponential rates are set by the same barrier heights because the magnetization chain is reversible with respect to the Gibbs measure. Relaxation times growing exponentially with N were found for this model in [13]. Newman and Schulman [25] proved an upper bound on the metastable decay rate inside the spinodal and derived the exact large-N decay rate per spin beyond it, and sharp metastable time scales for reversible mean-field dynamics, including the random-field Curie–Weiss model, were derived in [7]. Decay rates and imaginary parts of continued free energies need not correspond in general: Gaveau and Schulman [11] gave a counterexample with two barriers, whereas the Curie–Weiss magnetization has a single barrier.

Computations. The code cited in the introduction regenerates every number above.

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