Freeness of the square root of the inverse different

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Abstract

Let L/K be a weakly ramified Galois extension of number fields of odd degree, with group G, and let $\mathcal{A}_{L/K}$ be the square root of its inverse different. We prove that $\mathcal{A}_{L/K} \simeq \mathbb{Z}[G]^{[K:\mathbb{Q}]}$.

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1 Introduction

Let L/K be a finite Galois extension of number fields with group G of odd order. Hilbert's formula for the different shows that there is a unique fractional $\mathcal{O}_L$-ideal $\mathcal{A}_{L/K}$ satisfying

$$\mathcal{A}_{L/K}^2 = \mathfrak{D}_{L/K}^{-1}.$$

This ideal is stable under G and is self-dual for the relative trace form. The extension is called weakly ramified if its second lower ramification group at every finite place is trivial. Erez proved that $\mathcal{A}_{L/K}$ is locally free over $\mathcal{O}_K[G]$ if and only if L/K is weakly ramified [7, Theorem 1]. Under this hypothesis, restriction of scalars gives a locally free $\mathbb{Z}[G]$-module of rank $[K:\mathbb{Q}]$. Write $[\mathcal{A}_{L/K}]$ for its class in the locally free class group $\mathrm{Cl}(\mathbb{Z}[G])$.

The study of this class follows the methods developed by Fröhlich and Taylor for rings of integers in tame extensions. Fröhlich's Hom-description expresses locally free classes by character functions with idelic values; normal-basis resolvents and Galois–Gauss sums provide the arithmetic representatives [9]. For tame Galois extensions, Taylor's proof of Fröhlich's conjecture identifies the class of the ring of integers with the class defined by symplectic Artin root numbers, which vanishes for groups of odd order [17]. Erez adapted these methods to $\mathcal{A}_{L/K}$. Besides the local criterion, he proved that, in every odd-degree weakly ramified extension, its scalar extension to a maximal $\mathbb{Z}$-order in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$ is free [7, Theorem 2]. Thus its class lies in the kernel of extension to a maximal order. He also proved its freeness over $\mathbb{Z}[G]$ for every odd-degree tame extension of number fields [7, Theorem 3]. Erez and Taylor subsequently described the associated hermitian structures in the tame case by Galois–Gauss sums twisted by the second Adams operation [8].

For weakly ramified extensions of $\mathbb{Q}$, Vinatier proved freeness when every decomposition group at a wildly ramified place is abelian [19, Théorème 1.2]. In [20, p. 395] he conjectured freeness for all odd-degree weakly ramified Galois extensions of $\mathbb{Q}$. The same paper gives an annihilation result: if $L/\mathbb{Q}$ is a weakly ramified Galois p-extension and e is its ramification index at the odd prime p, then $e[\mathcal{A}_{L/\mathbb{Q}}] = 0$ [20, Théorème 1]. For $p = 3$, he later obtained the uniform bound $3[\mathcal{A}_{L/\mathbb{Q}}] = 0$ [21, Theorem 1]. The trace form has its own obstructions: Vinatier also exhibited examples with a free underlying module whose trace lattice is not isometric to the standard lattice [20, Théorème 2].

For odd-degree weakly ramified Galois extensions over an arbitrary number field K, Pickett and Vinatier proved freeness over $\mathbb{Z}[G]$ under three conditions at each wildly ramified place v of K: a decomposition group D_v is abelian, its inertia subgroup I_v is cyclic, and $K_v/\mathbb{Q}_p$ is

unramified, where p is the residue characteristic [14, Theorem 1]. Their proof computes the norm-resolvents of explicit local generators and compares them with modified Gauss sums involving the second Adams operation [14, Theorem 2]. The construction of local generators was developed further by Johnston and by Pickett–Thomas. For every weakly ramified Galois extension F/E of p-adic fields, Johnston constructs a free generator of $\mathfrak{m}_F^n$ whenever $n \equiv 1 \pmod{|D_1|}$, with D_1 the wild inertia group; in odd degree this includes $\mathcal{A}_{F/E}$ [10, Theorem 1.1]. Using Lubin–Tate formal group exponentials, Pickett and Thomas construct explicit generators of $\mathcal{A}_{F/E}$ for abelian, totally, weakly and wildly ramified extensions of an arbitrary finite extension of $\mathbb{Q}_p$, with p odd [13]. The global class calculation further requires control of norm-resolvents and their comparison with Gauss sums.

Bley, Burns, and Hahn introduced a canonical arithmetic class $a_{L/K}$ in relative algebraic K-theory, together with a twisted unramified class $c_{L/K}$. The class $a_{L/K}$ maps to $[\mathcal{A}_{L/K}]$ and decomposes as a sum of classes induced from the completions [3, Theorems 5.2 and 7.6]. Their Proposition 9.7 makes the general freeness conjecture equivalent to the vanishing of the class-group image of $a_{F/E}$ for every odd-degree weakly ramified local extension F/E. They also proved freeness for nonabelian families. Their Theorem 10.1 covers all odd-degree weakly ramified L/K for which each wild place satisfies $K_v = \mathbb{Q}_3$ and $|D_v| \mid 27$; in particular, it covers every weakly ramified nonabelian extension of $\mathbb{Q}$ of degree 27. Their Theorem 10.5 proves freeness when each wild place satisfies $K_v = \mathbb{Q}_7$, $|D_v| = 63$, and $D_v/C_v \simeq C_3 \times C_3$, where C_v is the Sylow 7-subgroup [3].

Kuang obtained further annihilation results for the relative classes. For a weakly and wildly ramified Galois p-extension L/K, with p odd and unramified in K, he proved

$$\frac{n(L/K)}{p}\, a_{L/K} = \frac{n(L/K)}{p}\, c_{L/K} = 0 \quad \text{in } K_0(\mathbb{Z}[G], \mathbb{Q}[G]),$$

where $n(L/K)$ is the largest order of a decomposition group at a wild place [11, Theorem 1.3]. In a separate paper, he extended the relative constructions to weakly ramified extensions for which the square root exists, allowing even degree. He proved $a_{L/K} = c_{L/K}$ in $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ when every wild place has odd residue characteristic and satisfies the three hypotheses of Pickett–Vinatier [12].

The formulation of Vinatier's freeness conjecture over an arbitrary number field was already discussed by Caputo and Vinatier [4, §1.2]. It appears in the equivalent conditions of [3, Proposition 9.7] and is stated as [1, Conjecture 1.2]. We prove this general form of the conjecture.

Theorem 1.1. *Let L/K be a finite Galois extension of number fields of odd degree, let $G = \mathrm{Gal}(L/K)$, and suppose that every second lower ramification group is trivial. Then the square root of the inverse different satisfies*

$$\mathcal{A}_{L/K} \simeq \mathbb{Z}[G]^{[K:\mathbb{Q}]}.$$

The proof uses the decomposition theorem of Bley–Burns–Hahn to reduce the assertion to $\mathrm{cl}(a_{F/E}) = 0$ for local extensions. We first show that the twisted unramified term has zero image in the locally free class group. We then combine explicit generators for maximal weak elementary abelian extensions with finite-field resolvent calculations and an Artin–Hasse identity. These give integral absolute logarithms in the presence of an arbitrary cyclic unramified quotient. The image theorem for the integral group logarithm, together with a first-moment calculation, gives local class vanishing for p-groups. A conductor correction treats tame inertia, and Swan's restriction theorem completes the local argument. Odd-order cancellation then gives the asserted freeness.

Section 2 recalls the relative classes and determinant conventions. Sections 4–7 contain the integral local calculations. Their applications to pure wild inertia and to hyperelementary groups occupy Sections 8–12. Theorem 1.1 is proved in Section 13.

1.1 Notation and conventions

We fix algebraic closures $\mathbb{Q}^c \subset \mathbb{C}$ and $\mathbb{Q}_p^c$, and compatible embeddings whenever global and local characters are compared. For a local field $E \subset \mathbb{Q}_p^c$, put $\Omega_E = \mathrm{Gal}(\mathbb{Q}_p^c/E)$; for a number field K, let $\mathcal{O}_K$ denote its ring of integers. For a local field, the superscripts ur and ab denote its maximal unramified and maximal abelian extensions. We write μ_n for the group of n-th roots of unity, and $\zeta_n = \exp(2\pi i/n) \in \mathbb{Q}^c$, using its image under the fixed embeddings in local formulas. We write C_n for a cyclic group of order n and $D^{\mathrm{ab}} = D/[D,D]$ for the abelianization of a group.

For a local field $E/\mathbb{Q}_p$, let $\mathcal{O}_E$ be its valuation ring, $\mathfrak{m}_E$ its maximal ideal, $k_E = \mathcal{O}_E/\mathfrak{m}_E$ its residue field, and v_E the valuation normalized by $v_E(\pi_E) = 1$ for a uniformizer π_E. Put

$$e_E = v_E(p), \qquad d_E = v_E(\mathfrak{D}_{E/\mathbb{Q}_p}), \qquad U_E^r = 1 + \mathfrak{m}_E^r \quad (r \geq 1).$$

The normalization $v(p) = 1$ will be specified when used. We denote field trace and norm by $\mathrm{Tr}_{F/E}$ and $\mathrm{N}_{F/E}$. A principal unit means an element of U_E^1, or its counterpart in a finite coefficient extension. On principal units we use the p-adic logarithm $\log(1+x) = \sum_{n\geq 1}(-1)^{n+1}x^n/n$.

For a weakly ramified local Galois extension F/E, write

$$D = \mathrm{Gal}(F/E), \qquad I = D_0, \qquad P = D_1,$$

where D_i denotes the i-th lower ramification group. If $P \neq 1$, then p is odd in the extensions considered here, P is elementary abelian, I/P is cyclic of order prime to p, and D/I is cyclic. For a complement T of P in I, every nonidentity element of T acts freely on $P \setminus \{1\}$. Indeed, the first ramification coefficient embeds D_1/D_2 in the additive residue field, and conjugation by $t \in T$ acts through its faithful tame character. We use the ramification and trace-ideal formulas of [15, Chapters III–IV].

For a finite group D, let $\mathrm{Irr}(D)$ be its irreducible characters over the chosen algebraic closure and R_D its group of virtual characters. If χ is a character, V_χ denotes a representation affording it, $\bar{\chi}(g) = \chi(g^{-1})$ its contragredient, and $\chi^\omega(g) = \omega(\chi(g))$ its Galois conjugate. We write $\mathbf{1}$ for the trivial character and Ind and Res for character induction and restriction. For an abelian group B, write $B^\vee$ for its group of linear characters. The center of an algebra A is denoted by $\zeta(A)$. For a coefficient ring R, let

$$\varepsilon : R[D] \longrightarrow R, \quad \sum_g a_g g \longmapsto \sum_g a_g, \qquad I_{R[D]} = \ker \varepsilon.$$

Thus $I_{R[D]}$ is an augmentation ideal; the letter I alone denotes inertia unless a different local convention is explicitly introduced. For a subgroup $J \leq D$, put $e_J = |J|^{-1}\sum_{j\in J} j$.

For an invertible group-algebra element u, define $\mathrm{Det}_\chi(u) = \det(\rho_\chi(u))$, where ρ_χ affords χ. Its restriction to D factors through D^{ab}. Evaluation on an element of D^{ab} means evaluation on any lift to D; the result is independent of the lift. Determinant functions are extended multiplicatively to R_D. The group-ring resolvent and its character value are

$$r_D(a) = \sum_{g\in D} g(a)g^{-1}, \qquad (a \mid \chi) = \mathrm{Det}_\chi(r_D(a)).$$

If τ runs through chosen lifts of the embeddings $E \hookrightarrow \mathbb{Q}_p^c$, the absolute norm of a determinant function h is

$$(\mathrm{N}_{E/\mathbb{Q}_p} h)(\chi) = \prod_\tau \tau(h(\chi^{\tau^{-1}})).$$

In particular, $\mathrm{N}_{E/\mathbb{Q}_p}(a \mid \chi) = \prod_\tau \mathrm{Det}_\chi(\tau(r_D(a)))$. This is a product of scalar determinants, also when D is nonabelian. For additive group labels, the involution $[a] \mapsto [-a]$ relates resolvents with plus and inverse indices.

Local reciprocity is normalized so that a uniformizer maps to arithmetic Frobenius. We write $\mathrm{rec}_E : E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$ for this map and use the same notation for its projection to an abelian Galois quotient. Evaluation of a linear character on an element of $E^\times$ is understood through reciprocity. In this convention a p-adic unit u acts on p-power roots of unity by exponent u^{-1}. The cyclotomic character $\kappa : \Omega_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is defined by $\omega(\zeta) = \zeta^{\kappa(\omega)}$ for all such roots.

2 Relative classes and determinant functions

2.1 Relative boundaries and determinant functions

For an inclusion of orders or semisimple algebras $A \subset B$, we use the relative group $K_0(A, B)$ in its description by triples (M, ξ, N), where M, N are finite projective A-modules and $\xi : B \otimes_A M \simeq B \otimes_A N$ is an isomorphism. Here K_1 denotes the algebraic K_1-group. The localization sequence provides a boundary map from $K_1(B)$, denoted ∂_p for $A = \mathbb{Z}_p[D]$ and $B = \mathbb{Q}_p[D]$. For group rings over $\mathbb{Z}$ one has

$$K_0(\mathbb{Z}[D], \mathbb{Q}[D]) = \bigoplus_\ell K_0(\mathbb{Z}_\ell[D], \mathbb{Q}_\ell[D]),$$

where ℓ runs through the rational primes. We write $\mathrm{Cl}(\mathbb{Z}[D])$ for the locally free class group, with additive notation. If M is locally free of rank r, $[M]$ denotes its stable class relative to the free module $\mathbb{Z}[D]^r$. Let cl be the natural map from the relative group to this class group. For a local class, this notation includes its insertion into the corresponding rational-prime summand. The boundary of an element of $K_1(\mathbb{Q}[D])$ has zero image under cl.

For an odd-order group, the standard reduced-norm isomorphisms identify $K_1(\mathbb{Q}[D])$ with $\zeta(\mathbb{Q}[D])^\times$, and similarly over $\mathbb{Q}_p$; the global image is given by the Hasse–Schilling–Maass theorem [3, Section 2A1]. Write

$$\delta_{p,D} : \zeta(\mathbb{Q}_p[D])^\times \longrightarrow K_0(\mathbb{Z}_p[D], \mathbb{Q}_p[D])$$

for inverse reduced norm followed by the relative boundary. A central tuple over $\mathbb{Q}_p^c$ has the analogous boundary in $K_0(\mathbb{Z}_p[D], \mathbb{Q}_p^c[D])$; we use the same symbol for that extension, and for its coefficient extension after j^t. This extended boundary will be used in (2.6). A central tuple $f = (f(\chi))_{\chi \in \mathrm{Irr}(D)}$ is rational if

$$\omega(f(\chi)) = f(\chi^\omega) \quad (\omega \in \mathrm{Gal}(\mathbb{Q}_p^c/\mathbb{Q}_p)). \tag{2.1}$$

We extend such functions multiplicatively to the character ring. For $u \in \mathbb{Q}_p[D]^\times$, $\mathrm{Det}(u)(\chi)$ is the determinant in a representation of character χ. The kernel of $\delta_{p,D}$ is $\mathrm{Det}(\mathbb{Z}_p[D]^\times)$; the use of units instead of matrices is valid for these semilocal group rings. A rational relative class is torsion if and only if its reduced-norm representative has valuation zero in every simple central component. These are the usual determinant descriptions of relative K_0; see [3, Sections 2A1–2A3].

Restriction of a determinant function from D to H evaluates it on $\mathrm{Ind}_H^D \theta$, and induction evaluates a function for H on $\mathrm{Res}_H^D \chi$. Hence

$$(\mathrm{Ind}_H^D \mathrm{Res}_H^D f)(\chi) = f(\mathrm{Ind}_H^D \mathrm{Res}_H^D \chi). \tag{2.2}$$

We will use this identity to obtain integral annihilators of local classes.

2.2 The arithmetic classes

We recall the properties of the arithmetic classes that enter the proof.

Proposition 2.1 (Bley–Burns–Hahn)**.** *For weakly ramified odd-degree extensions there are classes $a_{L/K}$ globally and $a_{F/E}$ locally with the following properties.*

(i) *$a_{L/K}$ is a torsion element of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ and $\mathrm{cl}(a_{L/K}) = [\mathcal{A}_{L/K}]$; see [3, Theorem 5.2]. The local class is rational by [3, Proposition 7.1].*

(ii) *The global classes commute with subgroup restriction and normal quotient coinflation; see [3, Theorem 6.1].*

(iii) *They have the local decomposition*

$$a_{L/K} = \sum_{p} \sum_{v \mid p} \mathrm{Ind}_{G_w}^{G} a_{L_w/K_v},$$

where v runs through the places of K above p, w is a chosen place of L above v, and G_w is its decomposition group; see [3, Theorem 7.6].

(iv) *If F/E is tame, $a_{F/E} = c_{F/E} = 0$; see [3, Theorem 8.1 and Remark 7.5]. In general*

$$c_{F/E} = \delta_{p,D}(\lambda_I), \qquad \lambda_I = 1 - e_I + \sigma^{-1} e_I, \tag{2.3}$$

where σ lifts arithmetic Frobenius in D/I.

(v) *Extension of coefficients to the integers $\mathcal{O}_p^t$ of the maximal tamely ramified extension of $\mathbb{Q}_p$ is injective on rational relative K_0:*

$$j^t : K_0(\mathbb{Z}_p[D], \mathbb{Q}_p[D]) \longrightarrow K_0(\mathcal{O}_p^t[D], \mathbb{Q}_p^c[D]). \tag{2.4}$$

This is Taylor's determinant fixed-point theorem [18, Chapter 8, Theorem 1.1], as recalled in [3, equation (7.8)].

(vi) *Any prescribed wildly and weakly ramified local extension of odd degree can be realized as the completion of a global extension with the same group, a unique wild place, and full decomposition group there. The global base can be chosen with that completion and with its roots of unity prime to the group order. This is the construction in [3, Theorem 9.1, proof, and Proposition 9.7].*

(vii) *A global class $[\mathcal{A}_{L/K}]$ vanishes if and only if $\mathcal{A}_{L/K}$ is free over $\mathbb{Z}[G]$; see [3, Lemma 9.8], using odd-order cancellation.*

For a tame local extension, (iv) gives the zero class. For a wild local extension, the torsion assertion follows from (i), (iii), and (vi). For later use set

$$x(F/E) = \mathrm{cl}(a_{F/E}) \in \mathrm{Cl}(\mathbb{Z}[D]). \tag{2.5}$$

Lemma 2.2 (Local functoriality)**.** *Let Res_H^D denote restriction to a subgroup and $\mathrm{Def}_{D/N}^D$ coinflation to a normal quotient. For $H \leq D$, $\mathrm{Res}_H^D a_{F/E} = a_{F/F^H}$. For $N \triangleleft D$, $\mathrm{Def}_{D/N}^D a_{F/E} = a_{F^N/E}$. The same identities hold for x.*

Proof. If F/E is tame, its subgroup and quotient extensions are tame, so both identities follow from Proposition 2.1(iv). Otherwise use the realization in Proposition 2.1(vi). Its unique place above the prescribed wild place has decomposition group D. Every intermediate field therefore has one place above that base place, with completion the corresponding fixed local field. Apply the global functoriality in (ii), and then the local decomposition in (iii). Every other local term is tame and vanishes by (iv). There is exactly one term at the prescribed place, so no multiplicity occurs. The relative maps commute with cl. □

2.3 Normal bases and Gauss sums

Let V_χ^I denote the subspace fixed by inertia, and let $\sigma \in D$ lift arithmetic Frobenius in D/I. For $\chi \in \mathrm{Irr}(D)$ put

$$y_1(\chi) = (-1)^{\dim V_\chi^I} \det(\sigma \mid V_\chi^I), \qquad \tau_E'(\chi) = \tau_E(\chi) y_1(\chi)^{-1}, \qquad q'(\chi) = \frac{\tau_E'(\chi)}{\tau_E'(\psi_2 \chi)}.$$

Here τ_E is the local Galois–Gauss sum with the conventions of [3, Section 4A and Remark 4.6], extended multiplicatively to R_D, and $\psi_2\chi(g) = \chi(g^2)$. Odd order makes ψ_2 a permutation of irreducible characters. Choose a $\mathbb{Z}_p$-basis $b_1, \dots, b_d$ of $\mathcal{O}_E$, with $d = [E : \mathbb{Q}_p]$, and an ordering $\tau_1, \dots, \tau_d$ of the embeddings of E in $\mathbb{Q}_p^c$. Set $\delta_E = \det(\tau_i(b_j))_{i,j}$ and $\tau_{E/\mathbb{Q}_p} = \tau_{\mathbb{Q}_p}(\mathrm{Ind}_{\Omega_E}^{\Omega_{\mathbb{Q}_p}} \mathbf{1})$, the Gauss sum of the permutation character on these embeddings. If b is an $\mathcal{O}_E[D]$-basis of $\mathcal{A}_{F/E}$, the formula at the beginning of the proof of [3, Theorem 8.1, p. 119] gives

$$j^t(a_{F/E}) = j^t \delta_{p,D} \left(\left(\frac{\delta_E}{\tau_{E/\mathbb{Q}_p}} \right)^{\chi(1)} \mathrm{N}_{E/\mathbb{Q}_p}(b \mid \chi)\, q'(\chi) \right)_\chi . \tag{2.6}$$

Breuning's separate unramified relative term is killed by j^t. The scalar $\delta_E/\tau_{E/\mathbb{Q}_p}$ is an $\mathcal{O}_p^t$-unit for every finite $E/\mathbb{Q}_p$: this is the x_1 term in the tame branch of that proof, pp. 119–120, applied to the trivial tame group. Its $\chi(1)$-power is the determinant of a scalar unit in the integral group ring. Thus this scalar can always be removed after j^t. This comparison holds for arbitrary weakly ramified F/E. We apply (2.4) after proving the rationality of each representative to which it is applied.

2.4 Duality, the principal center, and integral logarithms

For determinant functions let $f^*(\chi) = f(\bar{\chi})$. This induces an involution $*$ on the class group.

Lemma 2.3 (Arithmetic duality)**.** *For every odd-degree weak local extension,*

$$x(F/E) = -x(F/E)^*.$$

Proof. For a tame extension the class is zero, so assume F/E is wild. If M is locally free of rank r over $\mathcal{O}_K[G]$ and J is an invertible $\mathcal{O}_K$-ideal, then $[JM] = [M]$ in $\mathrm{Cl}(\mathbb{Z}[G])$. To see this directly, choose local generators j_v of J. On a χ-component the ratio of determinant representatives is

$$\prod_{v \mid p} \mathrm{N}_{K_v/\mathbb{Q}_p}(j_v)^{r\chi(1)}.$$

Write $c_p = \prod_{v \mid p} \mathrm{N}(j_v)$. These are the determinants of the scalar idèle $(c_p^r)_p$. A rational idèle is a positive rational scalar times local $\mathbb{Z}_p$-units: choose the rational scalar with its finitely many prescribed valuations. Therefore its class vanishes. This is also the base-ideal calculation in [3, Lemma 6.7]; the rank-r scalar is c_p^r.

The absolute trace dual of $\mathcal{A}_{L/K}$ is $\mathcal{A}_{L/K}\mathfrak{D}_{K/\mathbb{Q}}^{-1}$, since its relative trace dual is itself. The preceding calculation identifies their $\mathbb{Z}[G]$-classes. Contragredient duality changes a determinant representative into $f(\bar{\chi})^{-1}$, so $[\mathcal{A}_{L/K}] = -[\mathcal{A}_{L/K}]^*$. Realize F/E with one wild place as above. The local decomposition identifies its $x(F/E)$ with this global square-root class, proving the assertion. □

Lemma 2.4 (The image of the principal center)**.** *The image in* $\mathrm{Cl}(\mathbb{Z}[D])$ *of the group of rational central tuples whose character coordinates are principal p-adic units is a finite p-group.*

Proof. The principal unit group of the center of $\mathbb{Q}_p[D]$ is a finite product of pro-p groups. Its map to the finite locally free class group has open kernel. Here is an explicit verification of the last assertion. Fix a maximal order $\mathcal{M}$ containing $\mathbb{Z}_p[D]$, and choose $c \geq 1$ with $p^c\mathcal{M} \subseteq \mathbb{Z}_p[D]$. For a sufficiently deep central tuple z, set

$$b_\chi = \exp\bigl(\log(z_\chi)/\chi(1)\bigr).$$

Choose the depth larger than the convergence bound and all $v_p(\chi(1))$, so that $b \in 1 + p^c\mathcal{M}$ and is central. Writing $b = 1+h$, the inverse series $\sum_{n\geq 0}(-h)^n$ converges in $1+p^c\mathcal{M} \subseteq \mathbb{Z}_p[D]$. Thus b is a unit of $\mathbb{Z}_p[D]$, and $\mathrm{Nrd}(b)_\chi = b_\chi^{\chi(1)} = z_\chi$. Hence these sufficiently deep tuples have zero relative boundary. The finite continuous image of a pro-p group is a p-group. □

Let $\mathrm{Conj}(D)$ be the set of conjugacy classes of D. For a commutative coefficient ring R, let $R[\mathrm{Conj}(D)]$ be the free R-module with that set as basis. Define the R-linear map $\phi_D : R[D] \longrightarrow R[\mathrm{Conj}(D)]$ by sending a group element to its conjugacy class, and put $\Psi([g]) = [g^p]$. The augmentation on this class space sends each $[g]$ to 1. For $R = \mathbb{Q}_p$, character evaluation identifies this vector space with rational additive character functions. In particular, $\chi(\phi_D \log u) = \log \mathrm{Det}_\chi(u)$ whenever the logarithms are defined.

Theorem 2.5 (Chinburg–Pappas–Taylor)**.** *Let D be a finite p-group, p odd, and let $I_D = \ker(\mathbb{Z}_p[D] \xrightarrow{\varepsilon} \mathbb{Z}_p)$. For the identity Frobenius lift on $\mathbb{Z}_p$, the integral group logarithm is*

$$\mathcal{L}(u) = (p - \Psi)\phi_D(\log u), \qquad u \in 1 + I_D.$$

Then

$$\mathcal{L}(1 + I_D^2) = p\phi_D(I_D^2). \tag{2.7}$$

This is [5, Proposition 3.11]; its hypotheses include the complete torsion-free coefficient ring with a Frobenius lift, here $\mathbb{Z}_p$.

Finally, we use Swan's integral restriction theorem in the following form:

Theorem 2.6 (Swan)**.** *For a finite group D, restriction to its hyperelementary subgroups for all rational primes is jointly injective on $\widetilde{K}_0(\mathbb{Z}[D])$, and hence on $\mathrm{Cl}(\mathbb{Z}[D])$. Here $\widetilde{K}_0$ is the reduced projective class group. A group is ℓ-hyperelementary if it has a normal cyclic subgroup of order prime to ℓ with ℓ-group quotient. See [16] and [6, Theorem 2.7].*

3 The class of the twisted unramified term

Proposition 3.1. *Let D be finite with normal subgroups $P \leq I$ such that P is an abelian p-group, I/P is cyclic of order prime to p, and D/I is cyclic. Suppose a complement T of P in I acts freely on $P \setminus \{1\}$. For a lift σ of a generator of D/I, put $\lambda_I = 1 - e_I + \sigma^{-1}e_I$. The p-supported ordinary relative boundary of $[\lambda_I] \in K_1(\mathbb{Q}_p[D])$ has zero image in $\mathrm{Cl}(\mathbb{Z}[D])$. Consequently $\mathrm{cl}(c_{F/E}) = 0$ for every odd-degree weak local extension.*

Proof. The element λ_I is independent of the lift, is central, and has inverse $1 - e_I + \sigma e_I$. If $P = 1$, then $e_I \in \mathbb{Z}_p[D]$ and the relative boundary is zero. If $P \neq 1$ and $T = 1$, then $I = P$ and $\lambda_I \in \mathbb{Z}[1/p][D]^\times$. Its only possible nonzero local boundary is therefore at p, making that boundary global and proving its class vanishes.

Assume now $T \neq 1$, and put $H = N_D(T)$. Conjugacy of complements in I, by Schur–Zassenhaus, shows that every dTd^{-1} is conjugate to T by an element of P. Hence $D = PH$. If $x \in P \cap H$, then $[x, t] \in P \cap T = 1$ for all $t \in T$. Freeness of the action gives $P \cap H = C_P(T) = 1$. Thus

$$D = P \rtimes H, \qquad T \lhd H, \qquad H \cap I = T, \qquad F_0 := H/T \simeq D/I.$$

Choose $\sigma \in H$ lifting the specified generator of F_0.

Put $R = \mathbb{Z}[1/p]$ and

$$B = (1 - e_P)R[P], \qquad C = B^T.$$

The commutative R-algebra B is finite ètale. Its geometric points are the nontrivial characters of P. If $t \neq 1$ fixes such a character, that character is trivial on $(t-1)P = P$, a contradiction. Thus T acts freely on every geometric fiber. The map $\operatorname{Spec} B \to \operatorname{Spec} C$ is a finite ètale T-torsor: after a faithfully flat strictly henselian extension, B is a product of copies of the base, T permutes the factors freely, and C is the product indexed by its orbits. In particular the canonical map

$$B \otimes_C B \longrightarrow \prod_{t \in T} B, \qquad b \otimes b' \longmapsto (b\, t(b'))_t$$

is an isomorphism over R, including at its residue primes dividing $|T|$.

Form the skew group rings

$$A = B \# H = (1 - e_P)R[D], \qquad A' = C \# F_0.$$

In a skew group ring, multiplication is given by $(b[h])(b'[h']) = b\, h(b')[hh']$. Faithfully flat descent along the displayed torsor gives mutually inverse equivalences

$$A\text{-Mod} \longrightarrow A'\text{-Mod}, \quad M \mapsto M^T, \qquad N \longmapsto B \otimes_C N, \tag{3.1}$$

where $h \in H$ acts on the latter by $h(b \otimes n) = h(b) \otimes (hT)n$. The descent isomorphisms $B \otimes_C M^T \simeq M$ and $(B \otimes_C N)^T \simeq N$ are H-equivariant. If $\sigma^{|F_0|} = t_0 \in T$, the indicated H-action satisfies $\sigma^{|F_0|}(b \otimes n) = t_0(b) \otimes n$. It is exact, preserves finite projectives, and commutes with flat base change. Flat base change commutes with invariants, which are a finite equalizer; the assertions can equally be checked on the split torsor above.

The inverse image of the free left A'-module is the finite projective left A-module

$$Q = B \otimes_C (C \# F_0).$$

Transport right multiplication by $(\sigma T)^{-1}$ to an A-linear automorphism α of Q. The pair (Q, α) defines an element $\beta \in K_1(A)$: add a projective complement and extend α by the identity. Extend β by the identity on the e_P summand to $K_1(R[D])$.

Over $\mathbb{Q}$ put $e = (1 - e_P)e_T = e_T - e_I$. Freeness on the nontrivial P-characters makes e a full idempotent of $A_{\mathbb{Q}}$. The rational form of (3.1) is its corner equivalence, since

$$C_{\mathbb{Q}} \# F_0 \simeq eA_{\mathbb{Q}}e, \qquad c[hT] \longmapsto che.$$

It is well-defined because $te = e$. More explicitly,

$$Q_{\mathbb{Q}} \simeq A_{\mathbb{Q}}e, \qquad b \otimes c[hT] \longmapsto bche$$

is an $A_{\mathbb{Q}}$-linear isomorphism compatible with right corner multiplication. The summands $B_{\mathbb{Q}}he$ for $hT \in H/T$ have disjoint coset supports, which also verifies bijectivity of this map. It identifies α with right multiplication by $\sigma^{-1}e$. Since σ commutes with e, extending by the identity under $A_{\mathbb{Q}} = A_{\mathbb{Q}}e \oplus A_{\mathbb{Q}}(1-e)$ gives right multiplication by

$$v = 1 - e + \sigma^{-1}e. \tag{3.2}$$

Thus the rational image of β is $[v]$. This class comes from $K_1(R[D])$, so its relative boundary at every $\ell \neq p$ is zero.

At p, $e_T \in \mathbb{Z}_p[D]$, and

$$u = 1 - e_T + \sigma^{-1}e_T \in \mathbb{Z}_p[D]^\times, \qquad u^{-1} = 1 - e_T + \sigma e_T.$$

The orthogonal decomposition $e_T = e_I + (e_T - e_I)$ gives the exact group-algebra identity $u = \lambda_I v$. Hence $\partial_p[\lambda_I] = -\partial_p[v]$. The latter is the entire global boundary of $-[v]$, since its other local components vanish. Its class is zero.

For the arithmetic application the ramification facts in Section 1 verify all the hypotheses. On every simple component on which λ_I is nontrivial the character factors through the cyclic group D/I and has degree one. Thus $\mathrm{Nrd}(\lambda_I) = \lambda_I$, so the ordinary boundary just computed is exactly (2.3). □

4 Finite-field identities and integral coordinates

Fix an odd prime p, let $k = \mathbb{F}_q$, $q = p^f$, and write $P = k^+$ additively. Its group basis is denoted $[a]$, with identity $[0]$. Powers a^0 in polynomial formulas, including at zero, mean 1.

Lemma 4.1. *In $k[P][z]$ put*

$$E_q(z) = \sum_{a\in k}(1-(a-z)^{q-1})[a] = \sum_{n=0}^{q-1} c_n z^n.$$

Then $c_0 = [0]$, and

$$c_n = -\sum_{a\in k} a^{q-1-n}[a] \quad (1 \le n \le q-1).$$

One has

$$c_i c_j = \begin{cases} \binom{i+j}{i} c_{i+j}, & i+j < q, \\ 0, & i+j \ge q. \end{cases} \tag{4.1}$$

If $n = \sum_i n_i p^i$, $0 \le n_i < p$, then

$$\prod_i c_{p^i}^{n_i} = \left(\prod_i n_i!\right) c_n. \tag{4.2}$$

Moreover, with Φ acting on coefficients by Frobenius,

$$\Theta_q := \sum_{j=0}^{p-1} c_j = \sum_{a\in k}(1 - a^{q-p}(a-1)^{p-1})[a], \tag{4.3}$$

$$\prod_{i=0}^{f-1} \Phi^i(\Theta_q) = [1]. \tag{4.4}$$

Proof. For $b \in k$, $E_q(b) = [b]$. Both sides of $E_q(z)E_q(w) = E_q(z+w)$ have degree less than q in each variable and agree at every $(z,w) \in k^2$. The evaluation map on polynomials of degree less than q in each variable is injective, including for a finite-dimensional k-algebra of coefficients. This proves the polynomial identity. Coefficient comparison gives (4.1); Lucas' binomial congruence, or successive coefficient comparisons with no base-p carries, gives (4.2).

The coefficient formula uses $\binom{q-1}{n} = (-1)^n$ in k. Summing the first p coefficients gives (4.3), since $(a-1)^{p-1} = 1 + a + \cdots + a^{p-1}$ in characteristic p. Equation (4.2) gives

$$E_q(z) = \prod_{i=0}^{f-1}\left(\sum_{j=0}^{p-1} \frac{c_{p^i}^j}{j!} z^{jp^i}\right).$$

The coefficient Frobenius satisfies

$$\Phi(c_{p^i}) = c_{p^{i+1}} \quad (0 \le i < f-1), \qquad \Phi(c_{p^{f-1}}) = c_1.$$

Thus the product in (4.4) is $E_q(1) = [1]$. □

Let $E/\mathbb{Q}_p$ have residue field k, and let $\omega(a)$ be the Teichmüller lift of a, with $\omega(0) = 0$. In $\mathcal{O}_E[P]$ set

$$t_n = -\sum_{a\neq 0} \omega(a)^{-n}[a] \quad (1 \le n < q-1), \qquad t_{q-1} = q[0] - \sum_{a\in k}[a]. \tag{4.5}$$

These are an $\mathcal{O}_E$-basis of the augmentation ideal. Indeed, in the basis $[a]-[0]$, $a \neq 0$, they give the multiplicative Fourier matrix on the $q-1$ distinct Teichmüller roots, up to unit scalars. Its determinant is a unit. Under $S_u([a]) = [ua]$, $u \in k^\times$, the vector t_n has weight $\omega(u)^n$. Each augmentation weight space, including the trivial one, has rank one. Reduction modulo $\mathfrak{m}_E$ gives $t_n \mapsto c_n$. The coefficients lie in the maximal unramified subring of $\mathcal{O}_E$; reduction modulo p there followed by the natural map to k gives the same vectors. This also specifies the reduction when E is ramified.

Lemma 4.2. *For $1 \le n \le q-1$, $n = \sum_i n_i p^i$, there is a unit $a_n \in \mathcal{O}_E^\times$ such that*

$$\prod_i t_{p^i}^{n_i} = a_n t_n, \qquad \bar{a}_n = \prod_i n_i!. \tag{4.6}$$

Proof. The product has augmentation zero and weight n, so it is an integral scalar multiple of t_n. Its reduction is the nonzero multiple given by (4.2). This includes $n = q-1$, whose digits are all $p-1$. □

5 A maximal weak extension and an integral resolvent bound

5.1 The polynomial and its normal-basis generator

We first prove the valuation criterion of [10, Theorem 5.2(iii)] in the form needed below.

Lemma 5.1. *Let F/E be a totally and weakly ramified Galois p-extension, with group P of order n. If $h \equiv 1 \pmod{n}$, then every $\gamma \in F$ with $v_F(\gamma) = h$ is an $\mathcal{O}_E[P]$-basis of $\mathfrak{m}_F^h$.*

Proof. Write $h = 1 + rn$, $r \in \mathbb{Z}$. The different exponent is $2n-2$, so the trace-ideal formula gives

$$\mathrm{Tr}_{F/E}(\mathfrak{m}_F^h) = \mathfrak{m}_E^{r+1}, \qquad \mathrm{Tr}_{F/E}(\mathfrak{m}_F^{h+1}) = \mathfrak{m}_E^{r+2}.$$

Trace therefore induces a nonzero map between the one-dimensional k_E-spaces $\mathfrak{m}_F^h/\mathfrak{m}_F^{h+1}$ and $\mathfrak{m}_E^{r+1}/\mathfrak{m}_E^{r+2}$. It is an isomorphism. Consequently

$$v_F(\mathrm{Tr}_{F/E}\,\gamma) = n(r+1) = h + n - 1 < h + n,$$

so the trace of γ survives in the n-dimensional $k_E[P]$-module $V = \mathfrak{m}_F^h/\pi_E\mathfrak{m}_F^h$.

The augmentation ideal J of $k_E[P]$ is nilpotent. Indeed, for $P \neq 1$ choose a central element c of order p and argue by induction on $|P|$. If the augmentation ideal of $k_E[P/\langle c\rangle]$ has N-th power zero, then $J^N \subseteq (c-1)k_E[P]$ and $J^{Np} = 0$, since $(c-1)^p = 0$. The case $P = 1$ starts the induction. Every nonzero left ideal L of $k_E[P]$ contains $\sum_{g\in P} g$: for the largest j with $J^j L \neq 0$, the space $J^j L$ is killed by J and hence consists of P-invariants. In the regular module, invariance under all left translations forces all coefficients to be equal, so this invariant space is $k_E \sum_{g\in P} g$.

The map $k_E[P] \to V$ sending a to $a\bar{\gamma}$ is injective. A nonzero kernel would contain $\sum_g g$, whose value is the nonzero image of $\mathrm{Tr}_{F/E}\,\gamma$. Both spaces have dimension n, so the map is an isomorphism. Its integral lift $\mathcal{O}_E[P] \to \mathfrak{m}_F^h$ is surjective by Nakayama's lemma and injective because both $\mathcal{O}_E$-modules have rank n. □

Proposition 5.2. *Let $E/\mathbb{Q}_p$ be finite, p odd, with uniformizer π and residue field $k = \mathbb{F}_q$. If*

$$\beta^q - \pi\beta^{q-1} + \pi = 0, \tag{5.1}$$

then $M = E(\beta)$ is the totally and weakly ramified abelian extension with norm group $\pi^{\mathbb{Z}}\mu_{q-1}U_E^2$. It has group $P \simeq k^+$, and

$$\alpha = \beta/\pi, \qquad \mathcal{A}_{M/E} = \mathcal{O}_E[P]\alpha, \qquad \mathrm{Tr}_{M/E}(\alpha) = 1. \tag{5.2}$$

Proof. The polynomial is Eisenstein, so $[M : E] = q$ and β is a uniformizer. Put $F(X) = X^q - \pi X^{q-1} + \pi$. The polynomial

$$\beta^{-2q}F(\beta + \beta^2 Y)$$

is integral and reduces to $Y^q - Y$. For $1 \le j < q$, the term from X^q has valuation $v_M\binom{q}{j} + q + j - 2q \ge j$, since $v_M(p) \ge q$. From the second monomial the linear coefficient is $-\pi(q-1)\beta^{-q} \equiv -1 \pmod{\beta}$, using $\pi/\beta^q = (\beta^{q-1} - 1)^{-1}$. Its higher coefficients have valuations at least $j - 1$. The leading Y^q coefficient is one. Hensel's lemma gives q roots

$$\beta_a = \beta + \beta^2 Y_a, \qquad \bar{Y}_a = a \in k.$$

They all lie in M, and $v_M(\beta_a - \beta) = 2$ for $a \neq 0$. Thus M/E is Galois with $P_0 = P_1 = P$ and $P_2 = 1$. The coefficient of β^2 in $\sigma(\beta) - \beta$ adds under composition: substituting $\sigma(\beta) = \beta + a\beta^2 + O(\beta^3)$ and $\tau(\beta) = \beta + b\beta^2 + O(\beta^3)$ gives coefficient $a + b$. Its kernel is $P_2 = 1$. Since $|P| = q$, it identifies P with k^+.

Its conductor is two, so its norm group contains U_E^2. On $\mu_{q-1} \subset E$ the norm is the q-th power map, which is the identity. Also $\mathrm{N}_{M/E}(\beta) = -\pi$, and $\mathrm{N}_{M/E}(-1) = -1$ because q is odd; hence π is a norm. Since $U_E^1/U_E^2 \simeq k^+$, the displayed subgroup has index q, giving equality by local class field theory.

The different exponent is $2(q-1)$, so $\mathcal{A}_{M/E} = \mathfrak{m}_M^{1-q}$. The element β/π has that valuation, and its trace is one by the coefficient of X^{q-1}. Lemma 5.1, with $h = 1 - q$, makes it a free generator. □

For the Lubin–Tate formal $\mathcal{O}_E$-module $\mathcal{F}$ with $[\pi]_{\mathcal{F}}(X) = \pi X + X^q$, choose division points $z_1 = [\pi]_{\mathcal{F}}(z_2)$ with $z_1^{q-1} = -\pi$. Put $K_1 = E(z_1)$ and $N = E(z_2)$. Then $X^{q-1} + \pi$ is Eisenstein over E, and $X^q + \pi X - z_1$ is Eisenstein over K_1. Thus $[K_1 : E] = q - 1$, $[N : K_1] = q$, and $k_N = k_E$. With the valuation extending v_E,

$$v_E(z_1) = \frac{1}{q-1}, \qquad v_E(z_2) = \frac{1}{q(q-1)}.$$

The element $\beta = z_1/z_2$ satisfies (5.1): $\beta = \pi + z_2^{q-1}$ and $\beta^{q-1} = -\pi/(\beta - \pi)$. The tame subgroup of $\mathrm{Gal}(N/E)$ scales z_1, z_2 by the same Teichmüller root, so $E(\beta)$ is its fixed field. Thus the polynomial realizes the second-level maximal weak Lubin–Tate extension for every finite $E/\mathbb{Q}_p$.

5.2 The integral Hopf order

Put $C_0 = \mathcal{O}_{K_1}[[T]]/(\pi T + T^q)$, with comultiplication induced by $\mathcal{F}$. This is a finite free algebra of rank q. For $f(T) = \pi T + T^q$, one has

$$f'(T) = \pi(1 + (q/\pi)T^{q-1}).$$

Here q/π is integral and T is topologically nilpotent in C_0, so the second factor is a unit. The trace-dual lattice of C_0 is $f'(T)^{-1}C_0 = \pi^{-1}C_0$. To verify the trace-dual formula, Lagrange interpolation at the roots of f gives

$$\sum_{f(g)=0} \frac{g^j}{f'(g)} = \begin{cases} 0, & 0 \le j < q-1, \\ 1, & j = q-1. \end{cases}$$

Higher powers reduce integrally using the monic polynomial f. The trace-pairing matrix between $(T^i/f'(T))_{0\le i<q}$ and $(T^j)_{0\le j<q}$ therefore has entry zero for $i+j<q-1$, entry one for $i+j=q-1$, and integral remaining entries. Its determinant is a unit, proving the formula.

Over K_1 the evaluation algebra is a product indexed by $\mathcal{F}[\pi]$. The functional represented by a function d under the trace pairing corresponds to $\sum_g d(g)[g]$. The dual of the formal-group comultiplication is multiplication in the group algebra. Hence the Cartier dual is the commutative Hopf order

$$\mathcal{H} = \left\{ \pi^{-1} \sum_{g\in\mathcal{F}[\pi]} b(g)[g] : b \in \mathcal{O}_{K_1}[[T]]/(\pi T + T^q) \right\} \subseteq \pi^{-1}\mathcal{O}_{K_1}[P]. \tag{5.3}$$

This is the order in [2, Proposition 6.5, $m = r = 1$]. The element $[0]$ is the image of $\pi + T^{q-1}$ in (5.3). Replacing the highest monomial by this element and subtracting the augmentation of each remaining vector gives the basis $[0]$ together with $\pi^{-1}\sum_g g^i([g]-[0])$, $0 \le i \le q-2$. For $1 \le i \le q-2$ this vector is $z_1^{-(q-1-i)}t_{q-1-i}$; for $i=0$ it is $-\pi^{-1}t_{q-1}$. Using $z_1^{q-1} = -\pi$, the basis reads

$$\mathcal{H} = \mathcal{O}_{K_1}[0] \oplus \bigoplus_{n=1}^{q-1} \mathcal{O}_{K_1} d_n, \qquad d_n = z_1^{-n} t_n. \tag{5.4}$$

In particular its augmentation ideal $J_{\mathcal{H}}$ is the sum of the d_n lines. For an integral coefficient extension S of $\mathcal{O}_{K_1}$, write $\mathcal{H}_S = S \otimes_{\mathcal{O}_{K_1}} \mathcal{H}$ and $J_{\mathcal{H}_S} = S \otimes_{\mathcal{O}_{K_1}} J_{\mathcal{H}}$. For a finite extension N/K_1, we abbreviate $\mathcal{H}_{\mathcal{O}_N}$ by $\mathcal{H}_N$, with the same convention for its augmentation ideal. For $n = q-1$ the equality follows from $d_{q-1} = -\pi^{-1}(q[0] - \sum_a [a])$; the term $q[0]$ gives the augmentation-zero vector in the trivial weight.

Lemma 5.3. *For every finite integral coefficient extension $\mathcal{O}_{K_1} \subseteq S$ and $v \in J_{\mathcal{H}} \otimes S$,*

$$v^p \in (p/\pi)(J_{\mathcal{H}} \otimes S). \tag{5.5}$$

Proof. Equation (4.6) implies that $J_{\mathcal{H}}$ is generated as an ideal by $d_1, d_p, \dots, d_{p^{f-1}}$. Weight considerations give

$$t_{p^i}^p = b_i t_{p^{i+1}} \ (i < f-1), \qquad t_{p^{f-1}}^p = b_{f-1} t_1.$$

Each $b_i \in p\mathcal{O}_E$. Indeed the p-th power of an augmentation-zero element of $\mathcal{O}_E[P]/p$ is zero, because P has exponent p, and each target vector has a unit coefficient. Write $b_i = pu_i$. Dividing by the powers of z_1, and using $z_1^{q-1} = -\pi$ at the last index, gives

$$d_{p^i}^p = p u_i d_{p^{i+1}} \ (i < f-1), \qquad d_{p^{f-1}}^p = -(p/\pi) u_{f-1} d_1.$$

The quotient by $(p/\pi)\mathcal{H}_S$ has characteristic p (and is zero if p/π is a unit). Its commutative Frobenius kills all these augmentation generators, and hence every element of the augmentation ideal. The direct sum $\mathcal{H}_S = S[0] \oplus J_{\mathcal{H}_S}$ identifies $J_{\mathcal{H}_S} \cap (p/\pi)\mathcal{H}_S$ with $(p/\pi)J_{\mathcal{H}_S}$, proving the assertion. □

Proposition 5.4 (Uniform resolvent power bound)**.** *For the generator in* (5.2)*, either indexing convention for its group resolvent R satisfies*

$$R^p - 1 \in (p/\pi) I_{\mathcal{O}_E[P]}. \tag{5.6}$$

In particular R^p is an augmentation-one unit of $\mathcal{O}_E[P]$.

Proof. Put $f(X) = \pi X + X^q$ and use the labels $g_a = \omega(a) z_1$ on $\mathcal{F}[\pi]$. Consider the group-valued series

$$U(X) = \pi^{-1} \sum_{a \in k} \frac{f(X)}{g_a +_{\mathcal{F}} X}[a].$$

This notation denotes a formal power series. To verify its integral meaning, write

$$h(T, X) = \frac{f(T +_{\mathcal{F}} X)}{T +_{\mathcal{F}} X} = \pi + (T +_{\mathcal{F}} X)^{q-1} = \sum_{j \geq 0} h_j(T) X^j.$$

All $h_j(T)$ have integral coefficients. At a torsion point g_a, the numerator equals $f(X)$. Formula (5.3) therefore places every coefficient of $U(X)$ in $\mathcal{H}$. Also $U(0) = [0]$.

There is an exact augmentation identity. The q series $g_a +_{\mathcal{F}} X$ are the roots of $Y^q + \pi Y - f(X)$. Their $(q-1)$-st Newton sum is $-(q-1)\pi$. Thus $\varepsilon(U(X)) = 1$, and

$$U(X) - 1 \in X J_{\mathcal{H}}[[X]].$$

Evaluation at z_2 converges in the finite free $\mathcal{O}_N$-module $\mathcal{H}_N$. Let $\tau_a \in \mathrm{Gal}(N/K_1)$ be defined by $\tau_a(z_2) = g_a +_{\mathcal{F}} z_2$. Since $N = MK_1$, restriction identifies this group with P, and

$$\tau_a(\alpha) = \frac{z_1}{\pi(g_a +_{\mathcal{F}} z_2)}, \qquad U(z_2) = \sum_{a \in k} \tau_a(\alpha)[a].$$

Thus $U(z_2)$ is the resolvent with these additive labels. The involution $[a] \mapsto [-a]$ gives inverse indices and preserves the lattices in the calculation. Hence $R = 1 + z_2 v$ with $v \in J_{\mathcal{H}_N}$. Lemma 5.3 and the binomial expansion give

$$R^p - 1 \in p z_2 J_{\mathcal{H}_N} + (p/\pi) z_2^p J_{\mathcal{H}_N} \subseteq (p/\pi) \mathfrak{m}_N \mathcal{H}_N. \tag{5.7}$$

All coefficients of R lie in M. Coefficient action of $\tau_b \in P$ gives $\tau_b(R) = [-b]R$ with plus indices; hence $R^p \in E[P]$. By (5.3), every coefficient of $\mathfrak{m}_N \mathcal{H}_N$ has E-normalized valuation strictly greater than -1. A coefficient lying also in E therefore has nonnegative integral valuation. Consequently

$$(\mathfrak{m}_N \mathcal{H}_N) \cap E[P] \subseteq \mathcal{O}_E[P].$$

Divide (5.7) by p/π and intersect with $E[P]$. Its augmentation is zero, proving (5.6). An augmentation-one element of a local coefficient ring's p-group ring is a unit; alternatively its reduction lies in $1 + I_{k[P]}$, a nilpotent augmentation ideal. □

6 An unramified residue formula

Assume in this section that $E/\mathbb{Q}_p$ is unramified, with residue field $k = \mathbb{F}_q$, $q = p^f$, and take $\pi = p$ in the preceding construction. We determine the reduction of the p-th power of the resolvent in $k[P]$.

Theorem 6.1. *Label P by $\overline{(\sigma_a(\beta) - \beta)/\beta^2} = a \in k$, and put*

$$R_+ = \sum_{a \in k} \sigma_a(\beta/p)[a].$$

Then

$$R_+^p \in \mathcal{O}_E[P], \qquad R_+^p \bmod p = \Theta_q. \tag{6.1}$$

6.1 Gauss sums and carries

Put $\lambda = \zeta_p - 1$ and set

$$\psi(a) = \zeta_p^{\mathrm{Tr}_{k/\mathbb{F}_p}(a)}, \qquad \mathcal{G} = \sum_{a\neq 0} \omega(a)^{-1}\psi(a), \qquad J = \mathcal{G}^{p-1}.$$

Lemma 6.2. *One has $J \in p\mathcal{O}_E^\times$, $J/p \equiv -1 \pmod p$, and*

$$t_{p^i}^p = J t_{p^{i+1}} \quad (i < f-1), \qquad t_{p^{f-1}}^p = J t_1. \tag{6.2}$$

For $1 \le n \le q-1$, let $s(n)$ be its base-p digit sum, and let $\mathrm{rot}(n)$ be the cyclic digit rotation, equivalently pn modulo $q-1$ represented in $\{1,\dots,q-1\}$. Then

$$t_n^p = p^{s(n)}\kappa_n t_{\mathrm{rot}(n)}, \qquad \kappa_n \in \mathcal{O}_E^\times, \quad \bar\kappa_n = (-1)^{s(n)}. \tag{6.3}$$

Proof. The constant coefficient of the λ-expansion of $\mathcal{G}$ is zero. The coefficient of λ reduces to

$$\sum_{a\neq 0} a^{-1}\,\mathrm{Tr}_{k/\mathbb{F}_p}(a) = \sum_{i=0}^{f-1}\sum_{a\neq 0} a^{p^i-1} = -1.$$

Only $i=0$ contributes. Hence $\mathcal{G} \equiv -\lambda$ modulo λ^2. For $b \in \mathbb{F}_p^\times$, the automorphism $\zeta_p \mapsto \zeta_p^b$ multiplies $\mathcal{G}$ by $\omega(b)$. Since E is unramified, it is linearly disjoint from $\mathbb{Q}_p(\zeta_p)$, so $J \in E$. The cyclotomic equation gives $\lambda^{p-1}/p \equiv -1 \pmod\lambda$. Thus $J \in p\mathcal{O}_E^\times$ and $J/p \equiv -1 \pmod\lambda$. Since $(\lambda)\cap\mathcal{O}_E = p\mathcal{O}_E$, the latter congruence is $J/p \equiv -1 \pmod p$.

For all i, changing variables by Frobenius and using invariance of the finite-field trace gives $\sum_{a\neq0}\omega(a)^{-p^i}\psi(a) = \mathcal{G}$. At the additive character $\psi_b(a) = \psi(ba)$, $b \neq 0$,

$$\psi_b(t_{p^i}) = -\omega(b)^{p^i}\mathcal{G}.$$

All these vectors vanish at the trivial character. Character evaluation separates the characteristic-zero group algebra and therefore proves (6.2), including the wrap $\omega(b)^q = \omega(b)$.

Apply (6.2) to (4.6). This gives

$$t_n^p = J^{s(n)}\frac{a_{\mathrm{rot}(n)}}{a_n^p}t_{\mathrm{rot}(n)}.$$

Rotation preserves the product of digit factorials, which lies in $\mathbb{F}_p^\times$. The asserted leading unit follows. The case $n = q-1$ is included; it also follows directly from $t_{q-1}^p = q^{p-1}t_{q-1}$. □

6.2 A weighted integral algebra

Use $N = E(z_2)$ as in Section 5, and normalize $v(p) = 1$. Then $v(\beta) = 1/q$. Put

$$v_0 = [0], \qquad v_n = \beta^{-n}t_n \ (1 \le n \le q-1), \qquad B = \bigoplus_{n=0}^{q-1}\mathcal{O}_N v_n.$$

Lemma 6.3. *The lattice B is a commutative ring, $pB \subseteq \mathfrak{m}_N\mathcal{O}_N[P]$, and every element of B has integral p-th power. The rule*

$$B/\mathfrak{m}_N B \longrightarrow k[P], \qquad b \longmapsto b^p \bmod \mathfrak{m}_N\mathcal{O}_N[P] \tag{6.4}$$

is a well-defined Frobenius-semilinear ring map. Moreover, for $1 \le n \le q-1$,

$$v_n^p \bmod \mathfrak{m}_N\mathcal{O}_N[P] = \begin{cases} c_j, & n = jq/p,\ 1 \le j < p, \\ 0, & \text{otherwise} \end{cases} \quad \text{in } k[P]. \tag{6.5}$$

Proof. For $m+n \le q-1$, weight and augmentation show that $t_m t_n$ is an integral multiple of t_{m+n}. For $m+n > q-1$, write $l = m+n-(q-1)$. The same argument gives $t_m t_n = b t_l$, and (4.1) shows $b \in p\mathcal{O}_E$. Then $b\beta^{-(q-1)}$ is integral, since its valuation is at least $1/q$. These two cases prove that B is a ring. Also $v(p\beta^{-n}) = 1 - n/q > 0$ for all its basis vectors, so $pB \subseteq \mathfrak{m}_N \mathcal{O}_N[P]$.

Equation (6.3) gives

$$v_n^p = p^{s(n)} \beta^{-pn} \kappa_n t_{\mathrm{rot}(n)}.$$

Its coefficients are integral because

$$s(n) - \frac{pn}{q} = \sum_{i=0}^{f-1} n_i (1 - p^{i+1-f}) \ge 0.$$

Equality is possible exactly when every nonzero digit of n is the top digit, namely $n = jq/p$. For a general element of B, the pure terms in its p-th power are integral by this estimate. Every mixed multinomial coefficient is divisible by p, so every mixed term lies in $pB \subseteq \mathfrak{m}_N \mathcal{O}_N[P]$. If the element belongs to $\mathfrak{m}_N B$, its pure terms have strictly positive valuation as well. These facts prove the well-defined additive and semilinear assertions in (6.4); multiplicativity follows from commutativity.

For $n = jq/p$, its digit sum is j and its rotation is j. Thus $v_n^p = (p/\beta^q)^j \kappa_n t_j$. Equation (5.1) gives $p/\beta^q \equiv -1 \pmod{\mathfrak{m}_N}$, while $\kappa_n \equiv (-1)^j$. This proves (6.5). □

6.3 The formal-group calculation

Write

$$\mathcal{F}(X,Y)^{q-1} = \sum_{m,j \ge 0} h_{m,j} X^m Y^j.$$

The Teichmüller endomorphisms are $X \mapsto \omega(u)X$: they commute with $pX + X^q$, and uniqueness of the associated Lubin–Tate endomorphism gives this assertion. Hence $\mathcal{F}(\omega(u)X, \omega(u)Y) = \omega(u)\mathcal{F}(X,Y)$ and

$$h_{m,j} = 0 \quad \text{unless } m + j \equiv 0 \pmod{q-1}. \tag{6.6}$$

The degree-$q-1$ homogeneous part is $(X+Y)^{q-1}$.

For the plus labels consider

$$U_+(X) = p^{-1} \sum_{a \in k} (p + \mathcal{F}(X, -\omega(a) z_1)^{q-1})[a].$$

The same Newton-sum argument as before gives $U_+(0) = [0]$ and $\varepsilon(U_+(X)) = 1$ exactly. Its value at z_2 is R_+, since $[p]_{\mathcal{F}}(\mathcal{F}(z_2, -\omega(a)z_1)) = z_1$ and hence $p + \mathcal{F}(z_2, -\omega(a)z_1)^{q-1} = z_1/\mathcal{F}(z_2, -\omega(a)z_1)$. We verify that this translation gives the stated labels. Every nonlinear monomial $X^i Y^j$ of $\mathcal{F}(X,Y)$ has $i, j \ge 1$. Teichmüller equivariance gives $i + j \equiv 1 \pmod{q-1}$, and hence $i + j \ge q$. Since $v(z_1) = q v(z_2)$ and $v(\beta) = (q-1)v(z_2)$, its value at $(X,Y) = (z_2, -\omega(a)z_1)$, divided by z_2, has valuation at least

$$(q-2)v(z_2) + v(z_1) = 2v(\beta).$$

Consequently

$$\frac{\mathcal{F}(z_2, -\omega(a)z_1)}{z_2} = 1 - \omega(a)\beta + O(\beta^2), \qquad \frac{z_1}{\mathcal{F}(z_2, -\omega(a)z_1)} = \beta + \omega(a)\beta^2 + O(\beta^3).$$

Reduction of $\omega(a)$ gives the positive first ramification coefficient a in Theorem 6.1.

For $1 \le n < q-1$, projecting onto the weight-n augmentation line shows that the coefficient of t_n is

$$z_1^{-n} X^n \sum_{k,l \ge 0} (-1)^n h_{n+k(q-1),\, q-1-n+l(q-1)} (-p)^l X^{k(q-1)}. \tag{6.7}$$

Indeed (6.6) restricts the contributing powers to $j = q-1-n+l(q-1)$ and $m = n+k(q-1)$. The sum over $a \neq 0$ of $\omega(a)^j[a]$ is $-t_n$, so its contribution is

$$-p^{-1}(-1)^j z_1^j h_{m,j} X^m = z_1^{-n} X^n (-1)^n h_{m,j} (-p)^l X^{k(q-1)}.$$

Here $(-1)^j = (-1)^n$ and $z_1^{q-1} = -p$. Every coefficient in the inner series is integral. After removing the common leading factor and setting $X = z_2$, each remaining scalar has valuation $v(p^l z_2^{k(q-1)}) = l + k/q$. This proves convergence of the double series, and every summand other than $k = l = 0$ has positive valuation; the leading residue is $(-1)^n \binom{q-1}{n} = 1$.

For the trivial weight, compute the coefficient at the identity. The coefficient at $[0]$ of $U_+(X)$ is exactly $1 + X^{q-1}/p$. The vectors t_n for $n < q-1$ have zero $[0]$-coefficient, and t_{q-1} has coefficient $q-1$. Using augmentation one, the entire positive-degree trivial weight is therefore

$$\frac{X^{q-1}}{p(q-1)}\, t_{q-1}.$$

At z_2 this is $-\beta^{-(q-1)} t_{q-1}/(q-1)$, with leading coefficient one. Combining all weights proves

$$R_+ = 1 + \sum_{n=1}^{q-1} A_n v_n, \qquad A_n \in \mathcal{O}_N, \quad A_n \equiv 1 \pmod{\mathfrak{m}_N}. \tag{6.8}$$

In particular $R_+ \in B$ and $R_+ - (1 + \sum_{n=1}^{q-1} v_n) \in \mathfrak{m}_N B$.

Proof of Theorem 6.1. Apply (6.4) to (6.8). Equation (6.5) gives

$$R_+^p \in \mathcal{O}_N[P], \qquad \overline{R_+^p} = [0] + \sum_{j=1}^{p-1} c_j = \Theta_q \quad \text{in } k[P].$$

The resolvent covariance $\sigma_b(R_+) = [-b]R_+$ implies $R_+^p \in E[P]$. Intersecting gives integrality in $\mathcal{O}_E[P]$. Since E is unramified, $\mathfrak{m}_N \cap \mathcal{O}_E = p\mathcal{O}_E$, so this is exactly (6.1). □

Remark 6.4. The calculation uses successively the lattices B and $\mathcal{O}_N[P]$: the resolvents are compared modulo $\mathfrak{m}_N B$, and their p-th powers modulo $\mathfrak{m}_N \mathcal{O}_N[P]$. Descent to E then changes the latter modulus to $p\mathcal{O}_E[P]$.

7 Semilinear descent and the absolute logarithm

Let $E/\mathbb{Q}_p$ be arbitrary, let K/E be an unramified extension of odd degree m, choose $\pi \in E$ a uniformizer, and put $q = |k_K|$. If E is absolutely unramified, choose $\pi = p$. Construct $M = K(\beta)$ with $\beta^q - \pi\beta^{q-1} + \pi = 0$. This polynomial is Eisenstein over both E and K. Thus $E(\beta)/E$ is totally ramified and disjoint from K/E. Every element of $\mathrm{Gal}(K/E)$ extends uniquely to M fixing β, and these extensions form a subgroup. Consequently

$$\Gamma = \mathrm{Gal}(M/E) = P \rtimes C, \qquad P = k_K^+, \quad C = \langle \sigma \rangle \simeq C_m, \tag{7.1}$$

where σ acts on P by residue Frobenius and fixes $\alpha = \beta/\pi$. The extension is weakly ramified and has inertia exactly P.

Let δ be a trace-one integral normal-basis generator of K/E. Such a generator is obtained by lifting a normal basis of k_K/k_E: its trace is nonzero, since a zero trace would be a nontrivial linear relation among the conjugates. The lift has unit trace, and division by that trace gives δ. The element $b = \alpha\delta$ is an $\mathcal{O}_E[\Gamma]$-basis of $\mathcal{A}_{M/E}$: its conjugates are $g(\alpha)\sigma^j(\delta)$, the product of the two successive bases. With inverse indices one has the exact factorization

$$r_\Gamma(b) = U_C r_P, \quad U_C = \sum_{j=0}^{m-1} \sigma^j(\delta)\sigma^{-j} \in \mathcal{O}_K[C]^\times, \quad r_P = \sum_{g\in P} g(\alpha)g^{-1}. \tag{7.2}$$

The order of the factors matters: the term on the right indexed by (g, j) is $g(\alpha)\sigma^j(\delta)\sigma^{-j}g^{-1}$, as required.

The unit assertion for U_C follows from the trace pairing. Let δ^* generate the normal basis dual to δ. The extension K/E is unramified, so $\delta^* \in \mathcal{O}_K$. Then

$$U_C^{-1} = \sum_{j=0}^{m-1} \sigma^j(\delta^*)\sigma^j. \tag{7.3}$$

Indeed the coefficient of σ^k in the product is $\mathrm{Tr}_{K/E}(\delta\,\sigma^k(\delta^*))$, which is 1 for $k = 0$ and 0 otherwise.

By the Hopf bound, $r_P^p \in \mathcal{O}_K[P]$ has augmentation one. In a finite splitting field, every value of a linear character of the p-group P on a group element has residue one. Hence every character value of r_P^p has residue one. A p-th root of such a value is a unit with residue one, so every character value of r_P is principal. Its logarithm $\ell = \log r_P$ is therefore defined in the split group algebra. Coefficient action by $g \in P$ changes r_P to $r_P g$; the logarithm kills the commuting finite-order element g, giving $\ell \in K[P]$. Since $\sigma(\alpha) = \alpha$, coefficient action by σ changes r_P to $\sigma^{-1}r_P\sigma$ on its abstract group labels. Thus

$$z = \phi_\Gamma(\ell) \in E[\mathrm{Conj}(\Gamma)], \qquad y_0 = \mathrm{Tr}_{E/\mathbb{Q}_p}(z). \tag{7.4}$$

If $\ell = \sum_{g\in P} a_g g$, the coefficient of a Γ-conjugacy class $\mathcal{C}$ in z is $\sum_{g\in\mathcal{C}\cap P} a_g$. Coefficient action by σ permutes these summands, so this sum belongs to E. A lower valuation bound for the a_g is preserved by these sums.

Theorem 7.1. *For every canonical model* (7.1)*,*

$$y_0 \in \phi_\Gamma(I_{\mathbb{Z}_p[P]}). \tag{7.5}$$

Proof. The support is contained in the classes lying in P, and the augmentation is zero. It remains to prove integral coefficients. Suppose first $e = e_E \geq 2$. By the Hopf bound, $r_P^p = 1 + w$ with $w \in \mathfrak{m}_K^{e-1} I_{\mathcal{O}_K[P]}$. The coefficient action of σ on w^n is again label conjugation, so $\phi_\Gamma(w^n)$ has coefficients in $\mathfrak{m}_E^{n(e-1)}$. The coefficients of the n-th term of $\phi_\Gamma(p^{-1}\log(1+w))$ therefore lie in $\mathfrak{m}_E^{n(e-1)-e(1+v_p(n))}$. The trace-ideal formula places their absolute traces in

$$p^{\lfloor (n(e-1)-e(1+v_p(n))+d_E)/e \rfloor}\mathbb{Z}_p.$$

They are integral if

$$n(e-1) + d_E \geq e(1 + v_p(n)).$$

As $d_E \geq e - 1$, the difference is at least

$$n(e-1) - 1 - ev_p(n) \geq n - 1 - 2v_p(n) \geq 0.$$

Indeed, for $a = v_p(n)$,

$$n(e-1) - 1 - ea = (n - 1 - 2a) + (e-2)(n-a) \geq 0,$$

because $n \geq p^a \geq 1 + 2a$ and $n \geq a$. The valuations tend to infinity, so the series and its termwise trace converge in the integral conjugacy-class lattice.

Suppose now $E/\mathbb{Q}_p$ is unramified of degree f_0, and put $[K : \mathbb{Q}_p] = f = f_0 m$. We also write Φ for the arithmetic Frobenius of $K/\mathbb{Q}_p$, acting coefficientwise on $\mathcal{O}_K[P]$ and lifting the finite-field Frobenius. Put $r_+ = \sum_{g \in P} g(\alpha) g$. The involution $g \mapsto g^{-1}$ sends r_+ to r_P and commutes with coefficient trace and passage to conjugacy classes. It suffices to prove integrality of $\mathrm{Tr}_{E/\mathbb{Q}_p} \phi_\Gamma(\log r_+)$. Theorem 6.1 gives $r_+^p \bmod p = \Theta_q = \sum_{j<p} c_1^j / j!$.

The Artin–Hasse series

$$\mathrm{AH}(X) = \exp\left(\sum_{n \geq 0} \frac{X^{p^n}}{p^n}\right) = \prod_{\substack{d \geq 1 \\ p \nmid d}} (1 - X^d)^{-\mu(d)/d}$$

has p-integral coefficients; here μ is the Möbius function. Taking formal logarithms of the product proves the identity: the sum of the Möbius function over the prime-to-p divisors of an integer is zero unless that integer is a power of p. Each product exponent is in $\mathbb{Z}_p$, making its binomial expansion integral. The element t_1 is topologically nilpotent in $\mathcal{O}_K[P]$, since its reduction belongs to the nilpotent augmentation ideal. Therefore $\mathrm{AH}(t_1)$ converges in the original order, has augmentation one, and is a unit. Since $t_1^p \in p\mathcal{O}_K[P]$,

$$\mathrm{AH}(t_1) \bmod p = \sum_{j<p} c_1^j / j! = \Theta_q.$$

Hence $r_+^p \, \mathrm{AH}(t_1)^{-1} \in 1 + p\mathcal{O}_K[P]$ and

$$\log(r_+^p) \equiv \log \mathrm{AH}(t_1) \pmod{p\mathcal{O}_K[P]}. \tag{7.6}$$

To evaluate the formal logarithmic identity for AH at t_1, observe that $t_1^n \in p^{\lfloor n/p \rfloor} \mathcal{O}_K[P]$. In the expansion of the logarithm of an integral series in $1 + X\mathbb{Z}_p[[X]]$, a term of total degree n has denominator $j \leq n$. Its evaluated valuation is therefore at least $\lfloor n/p \rfloor - \lfloor \log_p n \rfloor$, which is nonnegative and tends to infinity. For nonnegativity, if $a = \lfloor \log_p n \rfloor \geq 1$, then $\lfloor n/p \rfloor \geq p^{a-1} \geq a$. There are only finitely many terms of each total degree, so the logarithmic expansion and its rearrangement converge in the integral group ring. Thus

$$\log \mathrm{AH}(t_1) = \sum_{n \geq 0} \frac{t_1^{p^n}}{p^n} \in \mathcal{O}_K[P].$$

Together with (7.6), this proves $p \log r_+ \in \mathcal{O}_K[P]$.

Iterating (6.2) gives

$$t_1^{p^n} = J^{(p^n-1)/(p-1)} t_{p^n \bmod (q-1)}.$$

For $n \geq 2$, after division by p^n the scalar has valuation $(p^n - 1)/(p-1) - n \geq p - 1$. The convergent Artin–Hasse logarithm and $J/p \equiv -1$ thus give

$$p \log r_+ \equiv t_1 - \Phi(t_1) \pmod{p\mathcal{O}_K[P]}. \tag{7.7}$$

For $f = 1$, the coefficient Frobenius fixes t_1.

The class coefficients of $\log r_+$ belong to E by the same orbit-sum calculation used in (7.4). The embeddings of $E/\mathbb{Q}_p$ extend to Φ^i on K, $0 \le i < f_0$. Applying the class quotient and summing (7.7) gives, in the integral class lattice,

$$\begin{aligned} p \operatorname{Tr}_{E/\mathbb{Q}_p} \phi_\Gamma(\log r_+) &= \sum_{i=0}^{f_0-1} \Phi^i(\phi_\Gamma(p \log r_+)) \\ &\equiv \phi_\Gamma\big(t_1 - \Phi^{f_0}(t_1)\big) \pmod p. \end{aligned}$$

Since σ acts on P by $a \mapsto a^{p^{f_0}}$, $\Phi^{f_0}(t_1) = \sigma^{-1} t_1 \sigma$. The last class vector is therefore zero. The left side belongs to $\mathbb{Z}_p[\mathrm{Conj}(\Gamma)]$, and $p\mathcal{O}_K \cap \mathbb{Q}_p = p\mathbb{Z}_p$. Dividing by p proves the required integrality for plus indices. The group involution then gives $y_0 \in \mathbb{Z}_p[\mathrm{Conj}(\Gamma)]$.

Finally, integral augmentation-zero sums of classes supported in P are precisely the image of the integral augmentation ideal of $\mathbb{Z}_p[P]$: choose one representative from each class in P. This proves (7.5). □

8 Determinants for pure wild inertia

8.1 Gauss-sum identities

We recall the Gauss-sum identities in the normalization of Section 2. For a representation V let $a_E(V)$ be its Artin conductor exponent. If η is an unramified linear character, then

$$\tau_E(\chi\eta) = \eta(\sigma)^{-a_E(\chi) - d_E\chi(1)} \tau_E(\chi). \tag{8.1}$$

Gauss sums are inductive in degree zero. The standard additive character $\psi_{\mathbb{Q}_p}$ is trivial on $\mathbb{Z}_p$ and sends p^{-n} to ζ_{p^n}; put $\psi_E = \psi_{\mathbb{Q}_p} \circ \operatorname{Tr}_{E/\mathbb{Q}_p}$. Choose $c_{E,2}\mathcal{O}_E = \pi_E^2 \mathfrak{D}_{E/\mathbb{Q}_p}$. For a representation V with character χ, define

$$\tau_E^\star(\chi) = \tau_E(\chi) y_2(\chi)^{-1} \operatorname{Det}_\chi(\mathrm{rec}_E(c_{E,2})), \qquad y_2(\chi) = (-1)^{\dim V^P} \det(\sigma^2 \mid V^P). \tag{8.2}$$

For a weak linear character θ of conductor two, there exists c_θ generating $\pi_E^2 \mathfrak{D}_{E/\mathbb{Q}_p}$ such that $\theta(1+y)^{-1} = \psi_E(c_\theta^{-1} y)$ for $y \in \pi_E \mathcal{O}_E$, and

$$\tau_E^\star(\theta - \theta^2) = \theta\left(\mathrm{rec}_E\left(\frac{c_\theta}{4c_{E,2}}\right)\right) \psi_E(c_\theta^{-1})^{-1}. \tag{8.3}$$

For weak linear characters of conductor zero or two, this modification is unchanged under unramified twists. It equals -1 on an unramified linear character. These formulas are [14, Definition 3.6 and Propositions 3.8–3.9]. When θ is a linear character of a possibly nonabelian inducing subgroup H, we apply them to the abelian quotient $H/\ker\theta$. Equivalently, for an extension M/B with group H, the formulas concern the abelian extension $M^{\ker\theta}/B$. Reciprocity, the conductor, and the Gauss sum of θ all factor through this quotient. Inductivity in degree zero and the Gauss-sum conventions are reviewed in [14, Section 3.2] and [3, Section 4A and Remark 4.6]. Formula (8.1) is the ordinary unramified Gauss-twist formula in those conventions; its exponent includes the absolute different of the additive character.

Lemma 8.1 (Covariance in the monomial models)**.** *Suppose every irreducible character of an odd-order local Galois group D is induced from a linear character of a subgroup. Put $q(\chi) = \tau_E(\chi)/\tau_E(\psi_2\chi)$. Then*

$$\omega(q(\chi)) = q(\chi^\omega) \operatorname{Det}_{\chi^\omega}(\mathrm{rec}_E(\kappa(\omega))). \tag{8.4}$$

Multiplying τ_E by a rational determinant function does not change this covariance for the resulting twisted ratio.

Proof. For a ramified linear character θ of a local field B, write the standard finite Gauss sum as

$$\tau_B(\theta) = \sum_u \theta(\mathrm{rec}_B(u/c))\psi_B(u/c),$$

where c generates its conductor times the absolute different and the sum uses the usual representatives of the unit quotient. Applying ω and replacing u by $\kappa(\omega)u$ gives

$$\omega(\tau_B(\theta)) = \theta^\omega(\mathrm{rec}_B(\kappa(\omega)))^{-1}\tau_B(\theta^\omega).$$

The same formula holds for unramified θ, since its value on this unit is one. Dividing the formulas for θ and θ^2 gives the positive factor $\theta^\omega(\mathrm{rec}_B(\kappa(\omega)))$.

If $\chi = \mathrm{Ind}_H^D \theta$ and $B = F^H$, then

$$\psi_2 \,\mathrm{Ind}_H^D \theta = \mathrm{Ind}_H^D(\theta^2). \tag{8.5}$$

Indeed an element lies in a conjugate of H if and only if its square does: an inverse of 2 modulo the exponent of D recovers the element from its square. The induction character formulas then agree term by term. Inductivity of Gauss sums for $\theta - \theta^2$, a degree-zero virtual character, gives $\tau_E(\chi - \psi_2\chi) = \tau_B(\theta - \theta^2)$.

The determinant of an induced linear character is its transfer character times the sign of the permutation action on D/H. The latter is trivial for odd D. Local reciprocity identifies this transfer with inclusion $E^\times \hookrightarrow B^\times$. Thus

$$\theta^\omega(\mathrm{rec}_B(\kappa(\omega))) = \mathrm{Det}_{\chi^\omega}(\mathrm{rec}_E(\kappa(\omega))),$$

which proves (8.4). Rational modification factors and their Adams transforms are $\Omega_{\mathbb{Q}_p}$-equivariant, proving the final assertion. □

Lemma 8.2 (The twist exponent in the models used here)**.** *Formula* (8.1) *follows from the linear Gauss-sum formula and degree-zero induction for both the pure wild models* (7.1) *and the tame models* (10.2)*.*

Proof. For a ramified linear character θ of B, the finite sum in Lemma 8.1 has $v_B(c) = a_B(\theta) + d_B$. An unramified linear character ν_B is trivial on every summation unit u, so twisting multiplies the sum by $\nu_B(\pi_B)^{-a_B(\theta)-d_B}$. For an unramified θ, the formula $\tau_B(\theta) = \theta(\mathrm{rec}_B(\pi_B))^{-d_B}$ gives the same assertion with conductor zero.

Write Γ for the group of either model, and let ν be an unramified linear character of Γ. If $\chi = \mathrm{Ind}_H^\Gamma \theta$ and $B = M^H$, put $\nu_B = \mathrm{Res}_H^\Gamma \nu$. Degree-zero induction applied to $\theta\nu_B - \theta$ cancels the induction constant. Here $\nu_B(\pi_B)$ abbreviates evaluation on $\mathrm{rec}_B(\pi_B)$, and similarly over E. Since $\nu_B(\pi_B) = \nu(\pi_E)^{[k_B:k_E]}$, its exponent is $[k_B : k_E](a_B(\theta) + d_B)$. For a nontrivial wild irreducible in the pure wild model, B/E is unramified of degree $r = \chi(1)$, $a_B(\theta) = 2$, $d_B = d_E$, and $a_E(\chi) = 2r$. The exponent is thus $r(2 + d_E) = a_E(\chi) + d_E\chi(1)$.

For a wild irreducible in the tame model, put $Q = T \times C_n$, $A = \ker(Q \to \mathrm{Aut}(P))$, $H = P \times A$, and $b = |Q/A| = \chi(1)$. The field B/E has ramification index $t = |T|$ and residue degree b/t. It is tame, so $d_B = td_E + t - 1$, while $a_B(\theta) = 2$. The lower ramification formula gives $a_E(\chi) = b + b/t$. Consequently

$$\frac{b}{t}(2 + d_B) = b + \frac{b}{t} + d_E b = a_E(\chi) + d_E\chi(1).$$

All other irreducibles in the two models are linear and were covered by the first paragraph. This proves (8.1) in the two families of models. □

8.2 A weak modification

Return to $\Gamma = P \rtimes C_m$ in (7.1), and put

$$q^\star(\chi) = \frac{\tau_E^\star(\chi)}{\tau_E^\star(\psi_2\chi)}.$$

Lemma 8.3. *The function $q^\star$ is invariant under unramified linear character twists and takes p-power root-of-unity values. On every P-trivial irreducible it equals one. It has the covariance* (8.4).

Proof. Weakness and $I = P$ give $a_E(V) = 2(\dim V - \dim V^P)$. For an unramified twist, the exponent contributed by (8.1), by y_2^{-1}, and by the determinant of $c_{E,2}$ is respectively

$$-a_E(V) - d_E \dim V, \qquad -2\dim V^P, \qquad (d_E + 2)\dim V.$$

Their sum is zero. Thus $\tau_E^\star$, and hence $q^\star$, is twist invariant.

Every irreducible is induced from a linear character of $H = P \rtimes C_\lambda$, where C_λ is the stabilizer in C of a linear character λ of P. An invariant λ extends to H by giving its cyclic complement value one; all extensions are unramified twists. This is the elementary Clifford description for an abelian normal subgroup with cyclic complement. If $\lambda \neq 1$, put $B = M^H$; it is unramified over E. Choose $c_{B,2} = c_{E,2}$, which generates the requisite ideal since the absolute different extends along an unramified extension. Degree-zero induction, (8.5), and the determinant transfer formula identify

$$q_E^\star(\chi) = \tau_B^\star(\theta - \theta^2).$$

There are no inertia invariants in these nontrivial wild characters. By the unramified-twist part of [14, Proposition 3.8], remove the unramified part of θ. The remaining character has order p and conductor two. Formula (8.3) then expresses the ratio as a product of a p-power root from the character and a p-power root from the additive character. If $\lambda = 1$, χ is unramified and linear; twist invariance gives $q^\star(\chi) = q^\star(1) = 1$. The displayed Clifford description also verifies the monomial hypothesis of Lemma 8.1. The modification factors in (8.2) are rational, proving its covariance claim. □

8.3 Unramified determinant factors

Choose representatives τ for the absolute embeddings of E in $\mathbb{Q}_p^c$, with coefficient lifts, and define

$$F_0(\chi) = \prod_\tau \mathrm{Det}_\chi(\tau(r_P)). \tag{8.6}$$

Put $\bar{P} = P/(\sigma - 1)P$, so $\Gamma^{\mathrm{ab}} = \bar{P} \times C$. Let $h_P : \Omega_E \longrightarrow \bar{P}$ be the projection of the restriction homomorphism to Γ^{ab}, and write $h_{P,E} : E^\times \to \bar{P}$ for its reciprocity version.

For coefficient action by $g \in P$, r_P changes to $r_P g$; for coefficient action by σ it changes by label conjugation. Determinants discard the latter. Hence the determinant cocycle of r_P is precisely h_P. If $\omega\tau = \tau'\gamma_{\omega,\tau}$ with $\gamma_{\omega,\tau} \in \Omega_E$, the cocycle factors in (8.6) multiply to $\prod_\tau h_P(\gamma_{\omega,\tau})$. This is the corestriction of h_P. Permuting the coset representatives therefore gives

$$\omega(F_0(\chi)) = F_0(\chi^\omega)\,\mathrm{Det}_{\chi^\omega}\big((\mathrm{cor}\, h_P)(\omega)\big). \tag{8.7}$$

Here cor is corestriction of the finite abelian-valued Galois character. It is the transfer corresponding under reciprocity to the inclusion $\mathbb{Q}_p^\times \hookrightarrow E^\times$. On inertia of $\mathbb{Q}_p$ it therefore satisfies

$$(\mathrm{cor}\, h_P)(\omega) = h_{P,E}(\kappa(\omega)^{-1}). \tag{8.8}$$

Since a unit has image in inertia, Lemma 8.3 implies that the cocycle of $F_0 q^\star$ is

$$\eta(\omega) = (\operatorname{cor} h_P)(\omega) h_{P,E}(\kappa(\omega)) \in \bar{P}. \tag{8.9}$$

It is an unramified homomorphism by (8.8). Pick $d \in P$ above its value at arithmetic Frobenius. Sending Frobenius to d defines an unramified homomorphism $h_U : \Omega_{\mathbb{Q}_p} \to P$. Let $S/\mathbb{Q}_p$ be unramified of degree $|\langle d \rangle|$, write Fr_S for its arithmetic Frobenius, and choose a trace-one integral normal-basis generator δ_0. The resolvent

$$U = \sum_{j=0}^{|\langle d \rangle|-1} \mathrm{Fr}_S^j(\delta_0) d^{-j}$$

satisfies

$$U \in \mathcal{O}_S[P]^\times, \qquad \varepsilon(U) = 1, \qquad \omega(U) = U h_U(\omega) \tag{8.10}$$

by the trace-dual inverse calculation (7.3) and the identity $\mathrm{Fr}_S(U) = Ud$. Its determinant in Γ has cocycle (8.9).

There is a different unramified determinant in (7.2). The absolute norm of $\mathrm{Det}(U_C)$ is the determinant of

$$V = \prod_\tau \tau(U_C).$$

This product is formed inside the commutative C-group algebra. Both V and its inverse are integral. Its coefficient cocycle is the corestriction of the unramified C-valued character for K/E. Under reciprocity this is the restriction of an unramified valuation character to $\mathbb{Q}_p^\times$, hence is unramified. Thus inertia fixes V coefficientwise, and

$$V \in \mathcal{O}_{\mathbb{Q}_p^{\mathrm{ur}}}[C]^\times. \tag{8.11}$$

Proposition 8.4 (A rational representative)**.** *For every canonical pure wild model with arbitrary odd m, the function*

$$f^\star = F_0 q^\star / \operatorname{Det}(U) \tag{8.12}$$

is rational in the sense of (2.1)*, has principal coordinates, satisfies $f^\star(\mathbf{1}) = 1$, and has the properties*

$$a_{M/E} - c_{M/E} = \delta_{p,\Gamma}(f^\star), \tag{8.13}$$

$$f^\star(\chi\nu) = f^\star(\chi) \qquad \text{for every unramified linear character } \nu, \tag{8.14}$$

$$\operatorname{logdet}(f^\star) \in \phi_\Gamma(I_{\mathbb{Z}_p[P]}). \tag{8.15}$$

Here $\operatorname{logdet} f$ *is the unique rational class vector satisfying* $\chi(\operatorname{logdet} f) = \log(f(\chi))$ *for every* $\chi \in \mathrm{Irr}(\Gamma)$.

Proof. Equations (8.7)–(8.10) cancel the cocycle and prove rationality of $f^\star$. Every linear-character value of r_P is principal by the Hopf bound. Restriction to P makes each determinant in F_0 a product of these values. Lemma 8.3 supplies principal values for $q^\star$. The augmentation-one unit $U \in \mathcal{O}_S[P]^\times$ has principal values on P, hence principal determinant values in Γ. The trivial coordinate of all three factors is one. Both resolvent factors are supported in P, and $q^\star$ is twist invariant; this proves (8.14).

To identify the arithmetic class, the two modifications y_1, y_2 give the exact determinant identity

$$\frac{q'}{q^\star} = \operatorname{Det}(\mathrm{rec}_E(c_{E,2}))\operatorname{Det}(\lambda_P), \qquad \lambda_P = 1 - e_P + \sigma^{-1} e_P. \tag{8.16}$$

On P-trivial characters the quotient of the y-factors is $\chi(\sigma)^{-1}$; on all others it is one. The ratio of the determinant modification and its Adams transform is $\mathrm{Det}_\chi(\mathrm{rec}_E(c_{E,2}))^{-1}$, giving the stated sign when it is moved to $q'/q^\star$. Apply (2.6) and factor the normal basis as in (7.2). The scalar there, V in (8.11), U in (8.10), and any lift to Γ of the reciprocity image in (8.16) are integral group-ring unit determinants after tame coefficient extension. The remaining factor is precisely $f^\star\lambda_P$. Consequently

$$j^t(a_{M/E} - c_{M/E}) = j^t\delta_{p,\Gamma}(f^\star).$$

The left side is rational by Proposition 2.1; the right side is rational by the explicit cocycle calculation above. Taylor injectivity (2.4) proves (8.13).

It remains to check the integrality of the ordinary logarithm of U. Put $I = I_{\mathcal{O}_S[P]}$. For $x \in I$ one has $x^p \in pI$, since P has exponent p. Hence x^n/n is integral, using $\lfloor n/p \rfloor \geq v_p(n)$, and these terms tend to zero. Thus $\log(1+I) \subseteq I$ for this elementary abelian group. The covariance of U is multiplication by finite-order commuting group elements, so its logarithm is Galois invariant. Therefore

$$\log U \in I_{\mathbb{Z}_p[P]}.$$

Lemma 8.3 gives $\log q^\star = 0$, and Theorem 7.1 computes $\operatorname{logdet} F_0 = y_0$. It follows that $\operatorname{logdet} f^\star = y_0 - \phi_\Gamma(\log U)$ is in the claimed lattice. □

Proposition 3.1 also gives

$$\mathrm{cl}(c_{M/E}) = 0. \tag{8.17}$$

9 Local p-groups and pure wild inertia

9.1 First moments and the integral logarithm

Proposition 9.1. *If m in the canonical model (7.1) is a power of p, then $x(M/E) = 0$.*

Proof. Put $\Lambda = \mathbb{Z}_p[\Gamma]$, $I = \ker\varepsilon$, and $y = \operatorname{logdet} f^\star$. Proposition 8.4 gives $y \in \phi_\Gamma(I_{\mathbb{Z}_p[P]})$; its augmentation is zero and $\Psi(y) = 0$, since every element of P has p-th power one.

Conjugacy differences belong to I^2. Thus

$$\phi_\Gamma(I)/\phi_\Gamma(I^2) \simeq I/I^2 \simeq \mathbb{Z}_p \otimes_{\mathbb{Z}} \Gamma^{\mathrm{ab}}. \tag{9.1}$$

The last isomorphism sends $g-1$ to g additively; the relation $gh - 1 \equiv (g-1) + (h-1) \pmod{I^2}$ gives the standard augmentation presentation. Since $\Gamma^{\mathrm{ab}} = \bar{P} \times C$, the moment of y lies in the exponent-p summand $\bar{P}$.

For every character λ of $\bar{P}$, extend it trivially on C to Γ. It has order dividing p. Rationality of $f^\star$ places its coordinate in $\mathbb{Q}_p(\zeta_p)$, and that coordinate is principal. For odd p, the logarithm of a principal unit of $\mathbb{Q}_p(\zeta_p)$ lies in $(\zeta_p-1)^2\mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$. Indeed U^1/U^2 is generated by the image of ζ_p, so $U^1 = \mu_p U^2$, and logarithm on U^2 takes values in the second ideal. Write an integral augmentation-zero lift of y supported on P as $\sum_g y_g g$, and write $\lambda(g) = \zeta_p^{a_g}$. Modulo $(\zeta_p-1)^2$,

$$\lambda(y) = (\zeta_p - 1)\sum_g y_g a_g.$$

The coefficient sum belongs to $\mathbb{Z}_p$, and $(\zeta_p-1)\mathcal{O}_{\mathbb{Q}_p(\zeta_p)} \cap \mathbb{Z}_p = p\mathbb{Z}_p$. Every such first moment is therefore zero modulo p. The characters separate $\bar{P}$, so its moment in (9.1) is zero. Hence

$$y \in \phi_\Gamma(I^2). \tag{9.2}$$

Theorem 2.5 now supplies $v \in 1 + I^2$ with $\mathcal{L}(v) = py$, because $(p - \Psi)y = py$. On the augmentation-zero rational conjugacy-class space, Ψ is nilpotent: if the exponent of Γ divides

p^s, then Ψ^s sends every class to [1] and hence kills every augmentation-zero vector. Thus $p - \Psi$ is invertible there. It follows that

$$\phi_\Gamma(\log v) = y, \qquad z := f^\star / \operatorname{Det}(v) \quad \text{satisfies } \log z = 0.$$

The coordinates of $\operatorname{Det}(v)$ are principal: on a stable lattice over a splitting valuation ring, every composition factor modulo p of a p-group representation is trivial, and an augmentation-one unit has determinant one on this reduction. Hence z is a rational tuple of principal units with zero logarithm. Each coordinate is a p-power root of unity.

We next view z as a global central element. Every character field of Γ is contained in $\mathbb{Q}(\zeta_{p^s})$. Its completion lies in $\mathbb{Q}_p(\zeta_{p^s})$, whose p-power roots of unity have order at most p^s. Thus every root coordinate of z is already a root in this common cyclotomic field. The prime p is unique in $\mathbb{Q}(\zeta_{p^s})$, and the global and local cyclotomic Galois groups identify. Their actions on characters therefore have the same orbits and stabilizers. The local equivariance of z is exactly global equivariance, so

$$z \in \zeta(\mathbb{Q}[\Gamma])^\times.$$

Choose its global K_1 preimage by reduced norm. For $\ell \neq p$, the order $\mathbb{Z}_\ell[\Gamma]$ is maximal, and root-of-unity reduced-norm coordinates are units. Its local boundary is zero there. Its p-boundary is therefore a global boundary and has zero class. The factor $\operatorname{Det}(v)$ already has zero p-boundary. Equations (8.13) and (8.17) give $x(M/E) = 0$. □

9.2 A universal unramified cover

Lemma 9.2. *Let F/E be an odd-degree weak local extension with nontrivial inertia equal to its wild inertia P. It is a Galois subextension of a canonical model M/E of the form* (7.1)*. If its group is a p-group, the canonical group can also be taken to be a p-group.*

Proof. Put $K = F^P$; then K/E is unramified and cyclic. The extension F/K is totally weak elementary abelian, with reciprocity character $h : K^\times \to P$ killing U_K^2 and the prime-to-p roots of unity. Choose a uniformizer π_E of E, taking p if E is unramified. Enlarge K by an unramified extension K'/K of degree p. The extension FK'/K' is still totally and weakly ramified and has reciprocity map $h' = h \circ \mathrm{N}_{K'/K}$. In particular,

$$h'(\pi_E) = h(\pi_E)^p = 1.$$

It kills $U_{K'}^2$ and prime-to-p roots as well. Its norm subgroup therefore contains $\pi_E^{\mathbb{Z}} \mu_{|k_{K'}|-1} U_{K'}^2$. Proposition 5.2 and local class field theory show that FK' lies in the maximal weak field $M = K'(\beta)$ for this subgroup. Since $\pi_E \in E$ and K'/E is unramified, this is the canonical semilinear model. The original F/E is Galois, so its group is a quotient of the canonical group. If $[K : E]$ is a power of p, then so is $[K' : E]$, proving the last claim. Only unramified base changes have been made. □

Corollary 9.3. *For every weak local extension with odd p-group Galois group, $x(F/E) = 0$.*

Proof. If its wild inertia is trivial, the extension is tame and the assertion follows from Proposition 2.1(iv). Otherwise its tame inertia quotient is trivial. Apply Lemma 9.2, Proposition 9.1, and normal deflation from Lemma 2.2. □

Theorem 9.4. *For every odd-degree weak local extension with $I = P$, one has $x(F/E) = 0$.*

Proof. If $P = 1$, the extension is tame, so $x(F/E) = 0$. Assume $P \neq 1$; then p is odd. The universal cover reduces to the canonical model with $m = p^a n$, $(n, p) = 1$. Let $H = P \rtimes C_{p^a}$; it is normal of index n in Γ. Restriction of $x(M/E)$ to H is zero by Corollary 9.3. The tensor identity for induction and restriction gives

$$\operatorname{Ind}_H^\Gamma \operatorname{Res}_H^\Gamma \chi = \sum_{\nu \in (\Gamma/H)^\vee} \chi\nu.$$

All these characters ν are unramified. By (8.14) and (2.2),

$$\operatorname{Ind}_H^\Gamma \operatorname{Res}_H^\Gamma f^\star = (f^\star)^n.$$

As $x(M/E) = \operatorname{cl}\delta(f^\star)$ by (8.13) and (8.17), it follows that $nx(M/E) = 0$. On the other hand $f^\star$ has principal coordinates, so Lemma 2.4 makes its class p-primary. Coprimality gives $x(M/E) = 0$, and deflation proves the general assertion. □

Corollary 9.5. *For every odd-degree abelian weak local extension, $x(F/E) = 0$, for every finite base field $E/\mathbb{Q}_p$.*

Proof. The tame case is Proposition 2.1(iv). Otherwise a nontrivial tame inertia complement would act both trivially, by abelianness, and faithfully without fixed points on the nontrivial wild inertia. Thus it is trivial, so $I = P$ and Theorem 9.4 applies. □

10 Hyperelementary local groups and an inertia basis

Lemma 10.1. *Let D be an odd-order hyperelementary group occurring in a weak local extension of residue characteristic p with nontrivial wild inertia P. Then one of the following holds:*

(i) *$I = P$;*

(ii) *D is abelian;*

(iii)

$$D = P \rtimes Q, \qquad P = C_p, \qquad Q \text{ abelian of odd order prime to } p, \qquad I = P \rtimes T, \tag{10.1}$$

where $T \leq Q$ is nontrivial cyclic, acts faithfully on P, and Q/T is cyclic.

Proof. First any normal p'-subgroup N of D is central, cyclic, and unramified. Indeed $[N, P] \subseteq N \cap P = 1$. An element of $N \cap I$ would induce a tame action centralizing P; freeness of that action forces its image in I/P to be trivial, whence $N \cap I = 1$. Thus N embeds in the cyclic group D/I. Finally $[D, N] \subseteq N \cap I = 1$, proving centrality.

Write $D = C_n \rtimes Q_\ell$, with $(n, \ell) = 1$ and Q_ℓ an ℓ-group; the splitting follows from Schur–Zassenhaus. If $\ell = p$, the preceding paragraph makes C_n central and unramified. Thus $D = C_n \times Q_p$ and its inertia is a p-group, giving (i).

Suppose $\ell \neq p$. The image of the normal p-group P in D/C_n is trivial, so $P \subset C_n$. As it is nontrivial elementary abelian, $P = C_p$. Write $C_n = C_{p^a} \times C_s$, with $(s, p) = 1$. The factor C_s is central and unramified. If $a > 1$, every nontrivial ℓ-power automorphism of C_{p^a} is nontrivial modulo p: the kernel of $\operatorname{Aut}(C_{p^a}) \to \operatorname{Aut}(C_p)$ is a p-group. Such an automorphism has commutator subgroup all of C_{p^a}. But $[D, D] \subseteq I$, because D/I is cyclic, whereas the p-Sylow of I is just $P = C_p$. This contradiction shows that Q_ℓ acts trivially on C_{p^a}. Thus P is central; the tame complement in I is trivial, and Q_ℓ embeds in D/I, so Q_ℓ is cyclic. In this case D is abelian.

It remains that $a = 1$. All other prime Sylows lie in the central unramified factor C_s; thus I has only p- and ℓ-factors and lies in $P \rtimes Q_\ell$. Put $T = Q_\ell \cap I$. It is a complement of P in I, cyclic, and faithful on C_p, so $|T| \mid p-1$. Conjugation in D on tame inertia is a power of residue Frobenius. The residue cardinality of E is a power of p, which is 1 modulo $|T|$. Consequently T is central in Q_ℓ. Also Q_ℓ/T is cyclic, as it embeds in D/I; hence Q_ℓ is abelian. Take $Q = C_s \times Q_\ell$. If $T = 1$ we are in (i); otherwise all assertions in (iii) follow. □

Assume now the remaining case (10.1). An unramified field extension gives the following split presentation.

Lemma 10.2. *There is an unramified extension M/F, Galois over E, such that*

$$\Gamma = \mathrm{Gal}(M/E) = P \rtimes (T \times C_n) = (P \rtimes T) \rtimes C_n, \tag{10.2}$$

where n is odd and prime to p, the inertia remains $I = P \rtimes T$, and a generator σ of C_n centralizes T. The projection $\Gamma \to D$ is surjective.

Proof. Choose $\sigma_0 \in Q$ projecting to the specified arithmetic Frobenius in Q/T, and let n be its order. Let E'/E be the unramified extension of degree n, and put $M = FE'$. Since $|Q/T|$ divides n, its maximal common unramified subfield with F is F^I. Thus the Galois group is the fiber product of D and C_n over D/I. The element $(\sigma_0, \mathrm{Frob}_{E'/E})$ has order n and splits the unramified quotient. It centralizes the embedded T, since Q is abelian. The inertia is unchanged and its intersection with this cyclic complement is trivial, proving (10.2). Weak ramification is preserved by this unramified base change. □

Lemma 10.3 (A fixed inertia generator)**.** *Put $K = M^I = E'$ in* (10.2)*. There exists $\alpha \in M^{C_n}$ such that*

$$\mathcal{A}_{M/K} = \mathcal{O}_K[I]\alpha.$$

If δ is a trace-one integral normal-basis generator of K/E, then $\alpha\delta$ is an $\mathcal{O}_E[\Gamma]$-basis of $\mathcal{A}_{M/E}$ and

$$r_\Gamma(\alpha\delta) = U_C r_I(\alpha), \qquad U_C = \sum_{j=0}^{n-1} \sigma^j(\delta)\sigma^{-j}. \tag{10.3}$$

Proof. Put $t = |T|$ and $h = 1 - p(t+1)/2$. The different exponent is $p(t+1) - 2$, so $\mathcal{A}_{M/E} = \mathcal{A}_{M/K} = \mathfrak{m}_M^h$ and $h \equiv 1 \pmod p$. Choose $a_0, b_0 \in \mathbb{Z}$ with $a_0 p + b_0 t = 1$, and put $T_0 = T \times C_n$ and $S_0 = P \rtimes C_n$.

The extension M^{S_0}/E is cyclic, totally tamely ramified, and of degree t. Since $t \mid p-1$, the field E contains μ_t. Choose a generator s of T whose tame character has value ζ_t. For a uniformizer u of M^{S_0}, set

$$\pi_s = \frac{1}{t} \sum_{j=0}^{t-1} \zeta_t^{-j} s^j(u).$$

Then $\pi_s/u \equiv 1$ modulo the maximal ideal, and $s(\pi_s) = \zeta_t \pi_s$. Thus π_s is a uniformizer and π_s^t is a uniformizer of E. Choose also a uniformizer π_t of M^{T_0}. Both π_s and π_t are fixed by C_n.

Put $B = M^P$. The extensions B/M^{S_0} and M^T/M^{T_0} are unramified. Consequently π_s and π_t are uniformizers of B and M^T, respectively, and $v_M(\pi_s) = p$, $v_M(\pi_t) = t$. The element

$$w = \pi_t^{hb_0} \pi_s^{ha_0}$$

has valuation $h(tb_0 + pa_0) = h$. The extension M/B is totally weakly ramified with group $P = C_p$. Lemma 5.1 therefore gives

$$\mathfrak{m}_M^h = \mathcal{O}_B[P]w.$$

Choose $u_0, \dots, u_{t-1} \in \mathcal{O}_E^\times$ and put

$$\gamma = \pi_s^{ha_0} \sum_{i=0}^{t-1} u_i \pi_s^i, \qquad \alpha = \pi_t^{hb_0} \gamma. \tag{10.4}$$

The polynomial $X^t - \pi_s^t$ is Eisenstein over K, so $B = K(\pi_s)$ and $\mathcal{O}_B = \bigoplus_{i=0}^{t-1} \mathcal{O}_K \pi_s^i$. Relative to the basis $(\pi_s^{ha_0+i})_{0 \le i < t}$ of $\pi_s^{ha_0}\mathcal{O}_B$, the conjugates $(s^j\gamma)_{0 \le j < t}$ have matrix

$$(u_i \zeta_t^{j(ha_0+i)})_{i,j}.$$

Removing the unit row factors u_i and column factors $\zeta_t^{jha_0}$ leaves the Fourier matrix $(\zeta_t^{ij})_{i,j}$. Its Vandermonde determinant $\prod_{i<j}(\zeta_t^j - \zeta_t^i)$ is a unit, since t is prime to p. Hence $\mathcal{O}_K[T]\gamma = \pi_s^{ha_0}\mathcal{O}_B$. Substitution into the wild basis identity gives

$$\begin{aligned} \mathfrak{m}_M^h &= \bigoplus_{g \in P} g(\pi_t^{hb_0}) \pi_s^{ha_0} \mathcal{O}_B \\ &= \bigoplus_{g \in P,\ \tau \in T} \mathcal{O}_K \, g(\pi_t^{hb_0}) \tau(\gamma) = \mathcal{O}_K[I]\alpha. \end{aligned}$$

The last equality uses that T fixes π_t and P fixes B. This is Johnston's inertia generator [10, Proposition 7.1]; every factor in (10.4) is fixed by C_n.

Finally, the elements $i(\alpha)\sigma^j(\delta)$, $i \in I$ and $0 \le j < n$, form an $\mathcal{O}_E$-basis of $\mathfrak{m}_M^h$. Since σ fixes α and I fixes K, they are the Γ-conjugates of $\alpha\delta$. With inverse indices,

$$r_\Gamma(\alpha\delta) = \sum_{i,j} i(\alpha)\sigma^j(\delta)\sigma^{-j}i^{-1} = \left(\sum_{j=0}^{n-1} \sigma^j(\delta)\sigma^{-j} \right) r_I(\alpha),$$

proving (10.3). □

11 A conductor correction

Continue with (10.2) and its fixed generator. Since σ commutes with T, put

$$v_T = 1 - e_T + \sigma e_T \in \mathbb{Z}_p[\Gamma]^\times, \qquad \lambda_I = 1 - e_I + \sigma^{-1} e_I. \tag{11.1}$$

The inverse of v_T is $1 - e_T + \sigma^{-1}e_T$. Choose $c_{E,1}$ generating $\pi_E \mathfrak{D}_{E/\mathbb{Q}_p}$ and define

$$\tau_E^\sharp(\chi) = \tau_E(\chi) \det_\chi(\mathrm{rec}_E(c_{E,1})) \det_\chi(v_T) \det_\chi(\lambda_I)^2, \tag{11.2}$$

$$q^\sharp(\chi) = \tau_E^\sharp(\chi)/\tau_E^\sharp(\psi_2\chi). \tag{11.3}$$

Lemma 11.1 (The conductor identity)**.** *For every representation V of Γ,*

$$a_E(V) = \dim V + \dim V^T - 2\dim V^I. \tag{11.4}$$

Consequently $\tau_E^\sharp$ and $q^\sharp$ are invariant under unramified linear character twists.

Proof. The lower ramification formula gives

$$a_E(V) = \dim V - \dim V^I + \frac{\dim V - \dim V^P}{|T|}.$$

On the sum of nontrivial P-weight spaces, T acts freely on the weights, so its invariant dimension is $(\dim V - \dim V^P)/|T|$. On V^P its invariant space is V^I. Thus $\dim V^T = \dim V^I + (\dim V - \dim V^P)/|T|$, proving (11.4).

For an unramified linear character ν, the four factors in (11.2) contribute powers of $\nu(\sigma)$ with exponents

$$-a_E(V) - d_E \dim V, \quad (d_E + 1)\dim V, \quad \dim V^T, \quad -2\dim V^I.$$

They sum to zero by the conductor identity. This proves invariance of $\tau_E^\sharp$; applying it also to $\psi_2\chi$ and ν^2 proves the assertion for $q^\sharp$. □

Lemma 11.2 (The correction boundary)**.** *The quotient of* (11.3) *by the usual BBH ratio satisfies*

$$\delta_{p,\Gamma}(q^\sharp/q') = -c_{M/E}. \tag{11.5}$$

Proof. Write $(1 - \psi_{2,*})u = u/(u \circ \psi_2)$ for multiplicative character functions. One has

$$y_1(\chi) = (-1)^{\dim V_\chi^I} \operatorname{Det}_\chi(\lambda_I)^{-1}.$$

The dimensions of I-invariants are preserved by ψ_2, so the signs cancel. Hence

$$\frac{q^\sharp}{q'} = (1 - \psi_{2,*}) \left(\operatorname{Det}(\operatorname{rec}_E(c_{E,1}))\operatorname{Det}(v_T)\operatorname{Det}(\lambda_I)\right). \tag{11.6}$$

In this identity each of the three Adams transforms is its square. For a genuine group element this is the determinant identity for squared eigenvalues. For v_T, restrict a representation to the abelian subgroup $\langle T, \sigma\rangle$. On a character weight, v_T acts by the value of σ if the weight is T-trivial and by one otherwise. Squaring the character preserves T-triviality because $|T|$ is odd, and squares the former value. Thus

$$\psi_{2,*}\operatorname{Det}(v_T) = \operatorname{Det}(v_T)^2.$$

For λ_I, an irreducible either has no I-invariants, giving value one, or factors through the cyclic unramified quotient and is linear; its Adams transform likewise squares the value. Therefore (11.6) is the product of two determinants of integral units and $\operatorname{Det}(\lambda_I)^{-1}$. The boundary of the two integral unit determinants is zero, which proves (11.5). □

Proposition 11.3 (A representative for the tame model)**.** *There is a rational central tuple f such that*

$$a_{M/E} - c_{M/E} = \delta_{p,\Gamma}(f). \tag{11.7}$$

All its coordinates are units, and

$$f(\chi\nu) = f(\chi) \quad \textit{for every unramified linear character } \nu. \tag{11.8}$$

Proof. Put

$$F_I(\chi) = \mathrm{N}_{E/\mathbb{Q}_p} \operatorname{Det}_\chi(r_I(\alpha)).$$

Since α is C_n-fixed, the coefficient action of σ on r_I is label conjugation, while an element $g \in I$ multiplies its determinant by $\operatorname{Det}(g)$. The nontrivial faithful action of T on $P = C_p$ gives $[I, I] = P$. Since $T \times C_n$ is abelian,

$$\Gamma^{\mathrm{ab}} = T \times C_n.$$

Thus the reduced inertia determinant cocycle is the homomorphism $h_T : \Omega_E \to T$ obtained by projecting this abelianization. Write $h_{T,E} : E^\times \to T$ for its reciprocity version. The cocycle of F_I is $\operatorname{cor} h_T$.

Every irreducible of Γ is monomial. Indeed, if $Q = T \times C_n$ and $A = \ker(Q \to \operatorname{Aut}(P))$, the P-trivial irreducibles are the linear characters of Q; every other irreducible is induced from a linear character of $P \times A$. The stabilizer of each nontrivial P-character is exactly A. Lemma 8.1 applies. The modification factors in (11.2) are rational group-algebra determinants, so its covariance holds for $q^\sharp$ as well. On inertia of $\mathbb{Q}_p$, reciprocity gives

$$(\operatorname{cor} h_T)(\omega) = h_{T,E}(\kappa(\omega)^{-1}).$$

Since $\operatorname{rec}_E(\kappa(\omega))$ lies in inertia, the Gauss factor has the inverse determinant cocycle there. Therefore $F_I q^\sharp$ has an unramified T-valued determinant cocycle.

Construct the trace-one normal integral resolvent U_T of the unramified T-Galois algebra by (8.10), with $d \in T$ equal to that Frobenius value, and define

$$f = F_I q^\sharp / \operatorname{Det}(U_T). \tag{11.9}$$

The entire cocycle cancels, proving $\Omega_{\mathbb{Q}_p}$-equivariance of f. Both resolvent factors are supported on I, and Lemma 11.1 makes $q^\sharp$ twist invariant. This proves (11.8).

The factorization (10.3) and the reasoning for (8.11) show that the absolute norm of its U_C factor is a determinant of an integral unit over a finite unramified coefficient field. The same is true of U_T. Apply (2.6); remove these two factors, the base scalar, and Breuning's separate term after j^t. Lemma 11.2 gives

$$j^t(a_{M/E} - c_{M/E}) = j^t \delta_{p,\Gamma}(f).$$

Both sides are rational, so Taylor injectivity proves (11.7). The class $a_{M/E}$ is torsion by the realization argument after Proposition 2.1. The element λ_I has finite order, so $c_{M/E}$ is torsion as well. Consequently $\delta(f)$ is torsion; the reduced-norm valuation description of relative K_0 makes each $f(\chi)$ a local unit. □

12 The class for tame hyperelementary groups

Theorem 12.1. *In the remaining tame hyperelementary case* (10.1)*, one has* $x(F/E) = 0$.

Proof. By Lemma 10.2 and deflation it suffices to prove the assertion for the split cover M/E. Write $Q = T \times C_n$, $A = \ker(Q \to \operatorname{Aut}(P))$, and

$$H = P \times A, \qquad b = |Q/A|.$$

The group A is central, and $A \cap T = 1$ because T acts faithfully on P. Hence it embeds in the cyclic unramified quotient Γ/I, so it is cyclic and unramified. The group H is abelian. The integer b is odd and prime to p, indeed $b \mid p-1$. Every wild irreducible has the form

$$\chi_{\lambda,\eta} = \operatorname{Ind}_H^\Gamma(\lambda \otimes \eta), \quad \lambda \in P^\vee \setminus \{1\}, \quad \eta \in A^\vee, \qquad \chi_{\lambda,\eta}(1) = b. \tag{12.1}$$

We first normalize the P-trivial coordinates of the representative in Proposition 11.3. Deflation of $a_{M/E}$ to $\Gamma/P = Q$ is zero because the quotient extension is tame. Deflation of $c_{M/E}$ is zero as a relative class too: e_I becomes e_T, and the resulting $1 - e_T + \sigma^{-1} e_T$ is a $\mathbb{Z}_p[Q]$-unit. Thus the restriction of f to characters of Q is $\operatorname{Det}(u_Q)$ for a unit $u_Q \in \mathbb{Z}_p[Q]^\times$. Its values depend only on the restriction of a character to T by (11.8). Map u_Q through the quotient $Q = T \times C_n \to T$ to obtain $u_T \in \mathbb{Z}_p[T]^\times$. Its determinant, viewed in Γ, agrees with

f on all P-trivial irreducibles. Divide f by this determinant. This changes neither its relative class nor its twist invariance, since an unramified linear character is trivial on T. We may therefore assume

$$f(\chi) = 1 \quad \text{for every } P\text{-trivial irreducible.} \tag{12.2}$$

By Corollary 9.5 and local functoriality, $\operatorname{Res}^{\Gamma}_{H} x(M/E) = 0$. For the character (12.1), restriction to H is the sum of its b conjugate inducing characters. Inducing each back gives the same χ. For a P-trivial character, the analogous induction is a sum of b characters of Q, on which $f = 1$. Thus on every irreducible,

$$\operatorname{Ind}^{\Gamma}_{H} \operatorname{Res}^{\Gamma}_{H} f = f^{b}.$$

Since $\operatorname{cl}(c_{M/E}) = 0$, equation (11.7) identifies $x(M/E) = \operatorname{cl}\delta(f)$. It follows that

$$b\, x(M/E) = 0. \tag{12.3}$$

Now choose ω_p in local inertia acting by inversion on μ_p and fixing every root of unity of order prime to p. Such an element exists from the action of inertia on $\mathbb{Q}_p(\zeta_p)$; all prime-to-p roots lie in the maximal unramified extension. For (12.1),

$$\chi^{\omega_p} = \operatorname{Ind}^{\Gamma}_{H}(\lambda^{-1} \otimes \eta), \qquad \bar{\chi} = \operatorname{Ind}^{\Gamma}_{H}(\lambda^{-1} \otimes \eta^{-1}).$$

Because A embeds in the cyclic unramified quotient, η^{-2} extends to an unramified linear character ν of Γ. Thus $\bar{\chi} = \chi^{\omega_p}\nu$. Rationality and twist invariance give

$$f(\bar{\chi}) = f(\chi^{\omega_p}) = \omega_p(f(\chi)). \tag{12.4}$$

Inertia acts trivially on residue fields. As $f(\chi)$ is a unit, its quotient by $f(\bar{\chi})$ has residue one. The P-trivial coordinates are already one by (12.2). Hence f/f^{*} is a principal central tuple.

By Lemmas 2.3 and 2.4, its class is

$$x(M/E) - x(M/E)^{*} = 2x(M/E),$$

and is p-primary. It is killed by b from (12.3); since $(b, p) = 1$ it is zero. Finally b is odd, so $2x = 0$ and $bx = 0$ together give $x = 0$. Deflation proves the theorem for the original extension. □

13 Integral detection and global freeness

Theorem 13.1. *For every odd-degree weakly ramified Galois extension F/E of local fields,*

$$\operatorname{cl}(a_{F/E}) = 0.$$

Proof. Let $D = \operatorname{Gal}(F/E)$. For every hyperelementary subgroup H of D, the extension F/F^{H} is weakly ramified: its lower ramification groups are intersections with those of D. If its wild inertia is trivial it is covered by the tame theorem. Otherwise Lemma 10.1 reduces it to Theorem 9.4, Corollary 9.5, or Theorem 12.1. Every such local class vanishes. By Lemma 2.2 this says that every hyperelementary restriction of $x(F/E)$ is zero. Swan's integral all-primes theorem, Theorem 2.6, implies $x(F/E) = 0$. □

Proof of Theorem 1.1. The local decomposition of Proposition 2.1(iii) and Theorem 13.1 give

$$[\mathcal{A}_{L/K}] = \operatorname{cl}(a_{L/K}) = \sum_{p} \sum_{v \mid p} \operatorname{Ind}^{G}_{G_w} \operatorname{cl}(a_{L_w/K_v}) = 0.$$

Odd-order cancellation, equivalently [3, Lemma 9.8], makes the projective $\mathbb{Z}[G]$-module $\mathcal{A}_{L/K}$ free. Its $\mathbb{Z}$-rank is $[L : \mathbb{Q}] = |G|[K : \mathbb{Q}]$; hence its free $\mathbb{Z}[G]$-rank is $[K : \mathbb{Q}]$. □

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