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THE BOUNDARY STRUCTURE OF THE MAXIMUM LIKELIHOOD ELIMINANT OF $3 \times 3 \times 3$ TABLES

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Abstract. The maximum likelihood estimate of the no-three-way interaction model for $3 \times 3 \times 3$ contingency tables is an algebraic function of the table n, and each fitted cell is a root of a polynomial $E(x; n)$ of degree 511 in x, the maximum likelihood degree of the model. We describe the structure of this polynomial at the boundary. Conjecturally, the leading and constant coefficients factor completely into 103 linear forms in the counts with coefficients ± 1: two-way margins, sums over pairs of complementary boxes, a staircase set, boxes with a full side, the total, and differences of two boxes. We explain each exponent as the Euler characteristic of a boundary stratum, satisfying the conservation identity $870 = 511 + 359$ between the 60 vanishing strata, the interior torus, and the 43 blow-up strata. On the hyperplane of a form, critical points of the likelihood escape along the coefficient vector of the form, either onto a facet, where the count is the maximum likelihood degree of a model with structural zeros, or into the zero-margin locus of a box, where the count is the Euler characteristic of an explicit residual variety. Up to symmetry, under the assumption that escaping branches move along these vectors, these eight types are the only forms with coefficients in $\{-1, 0, 1\}$ having this property.

The valuations force factors of the coefficients, and near each hyperplane the eliminant splits into the eliminant of the surviving roots and a boundary eliminant on the stratum, which share a collision discriminant at their meeting degree. We reduce the boundary system on each of the 325 boundary hyperplanes (under all table symmetries) to an explicit eliminant: in closed form for the 99 facets of exponent one, the junction branch of valuation -2 on $N = 0$, the 81 difference forms of exponent four (where the boundary system decouples into two univariate quadratics), the 27 boxes B_{223} of exponent six, and the 81 box pairs P_{112}, P_{122} of exponent seventeen; for the 9 slabs, to the 29 critical points of a master function on the flow space of $K_{3,3}$; and for the 27 margins, to the 169 critical points of the model with structural zeros. While the 103 active forms with $\lambda_{111} \neq 0$ send escaping roots to 0 or ∞, on the 222 neutral boundary hyperplanes with $\lambda_{111} = 0$ the escaping roots converge to finite nonzero values of p_{111}, so that for n_0 on such a hyperplane, $E(x; n_0)$ factors in $\mathbb{Q}(n_0)[x]$. Across all 189 symmetry types of pairs of active boundary forms, the numbers of jointly escaping branches coincide with the end-coefficient exponents of the boundary eliminants and yield a two-regime restriction law for every cofactor Q_k, consisting of same-side codimension-two tents and opposite-side interior plateaus. Finally, we conjecture that the eliminant is the residual resultant of the margin conditions on the no-three-way interaction toric variety: for a generic scaling of the model, the eliminant has degree 2,457, and as the scaling tends to the uniform one, 1,946 of its roots escape along the same vectors into the zero-margin loci of the four box types. Some of these statements are proved, some only under stated hypotheses, and the others are conjectures supported by exact, modular, and numerical computation.

1. Introduction

For a three-way table $n = (n_{ijk})$, the no-three-way interaction model consists of the positive tables $p_{ijk} = a_{ij}b_{ik}c_{jk}$. Its maximum likelihood estimate $\hat{p}$ is the unique positive table of this form with the two-way margins of n (Birch [2]), and the complex tables of this form with the

margins of n are the critical points of the likelihood. For $3 \times 3 \times 3$ tables, there are 511 of them for generic n [8]. Fixing the cell 111, the values $\hat{p}_{111}$ at all critical points are the roots of a polynomial

$$E(x; n) = \sum_{k=0}^{511} c_k(n)\, x^k,$$

the eliminant of p_{111}, which we normalize to have coprime integer polynomial coefficients (fixed up to sign by requiring $c_{511}(n) > 0$ on positive tables). Since a critical point scales with n, c_k is homogeneous of degree $\deg c_0 - k$.

When one variable is binary, the eliminant is an explicit determinant [7], and for two-way tables with arbitrary margins, it is the resultant of the margin conditions on a Segre variety [6]. No such formula is known for $3\times3\times3$ tables, and the coefficients of the eliminant are large. This paper describes its structure at the boundary, in a form we expect any formula to reproduce.

Results. Write $n[R \,|\, C \,|\, K]$ for the sum of n_{ijk} over $i \in R$, $j \in C$, $k \in K$, with a dot or subscript $+$ denoting summation over all three levels (so $n_{11+} = n[1 \,|\, 1 \,|\, \cdot]$). Section 2 introduces eight types of linear forms: margins, box pairs $n[R \,|\, C \,|\, K] + n[R^c \,|\, C^c \,|\, K^c]$, a staircase, boxes B with a full side, the total N, and differences D of two boxes. Each form is $\lambda \cdot n$ for a vector $\lambda \in \{-1, 0, 1\}^{27}$ with no three-way interaction.

- The end coefficients are products of these 103 forms (Conjecture 3.1, verified on 633 exact eliminants computed with `msolve` [1]): c_0 has degree 870 (60 constant-side forms) and c_{511} has degree 359 (43 lead-side forms).
- On the hyperplane $\lambda \cdot n = 0$ the escaping critical points move along $\pm\lambda$, and the exponent is the Euler characteristic of a boundary stratum (Conjecture 4.2, Section 4). For facets this is the maximum likelihood degree of a model with structural zeros. For the box types it is the Euler characteristic of a residual variety over the zero-margin locus of the box. For example, the slab $n[12 \,|\, \cdot \,|\, \cdot]$ has exponent -29, computed from the flow polynomial of $K_{3,3}$. For N the exponent is -17, given by the Euler characteristic 17 of the zero-margin locus of the model. By homogeneity the exponents satisfy the conservation law $870 = 511 + 359$.
- Up to symmetry, among forms with coefficients in $\{-1, 0, 1\}$ and assuming that escaping branches move along $\pm\lambda$, these are the only such forms (Computation 5.1).
- The valuations force a factor $F_k(n)$ of every coefficient $c_k(n) = F_k(n)Q_k(n)$ (Proposition 6.1). Near any of the 325 boundary hyperplanes $\ell = 0$ under the full table symmetry group (60 with $v_{111} = 1$, 222 with $v_{111} = 0$, and 43 with $v_{111} = -1$), the eliminant splits into the eliminant E_{surv} of the surviving roots and a boundary eliminant E_ℓ attached to the stratum, sharing a collision discriminant $\Delta_\ell(n_0)$ (Conjecture 6.3). We derive closed-form boundary algebras for the facets of exponent one (Proposition 6.5), the junction branch of valuation -2 on $N = 0$ (Proposition 6.6), the difference forms D of exponent four (Proposition 6.7), the boxes B_{223} of exponent six (Proposition 6.8), and the box pairs P_{112}, P_{122} of exponent seventeen (Proposition 6.9); give explicit boundary systems for the slabs and the margins (Section 6); and show that, under a nondegeneracy assumption, $E_\ell(x; n_0)$ divides $E(x; n_0)$ in $\mathbb{Q}(n_0)[x]$ on the 222 neutral hyperplanes ($v_{111} = 0$) (Proposition 6.10).
- Across all 189 symmetry types of pairs of the 103 active forms, the codimension-two overlap counts $m(\ell_1, \ell_2)$ equal the end-coefficient exponents of the boundary eliminants E_{ℓ_1} (Conjecture 6.13, verified for the end coefficients on 323 boundary eliminants covering all pair types with $m > 0$, and for the middle coefficients on all 3,540 coefficients of 60 boundary eliminants) and govern the restriction of every cofactor Q_k to $\ell = 0$

through a two-regime law with same-side tents and opposite-side interior plateaus (Conjecture 6.14). The law is consistent with all 96,768 codimension-two coefficient orders, of which all but 7 equal the prediction, and with exact line interpolations of Q_1 and Q_{510}.

- The eliminant is the residual resultant of the margin conditions on the toric variety of the model (Conjecture 7.2). As a generic scaling of the model tends to the uniform one, 1,946 of its 2,457 critical points escape, along the vectors $-\lambda$ of the box types, and the other 511 converge to the critical points of the uniform model (Computation 7.3).

2. The boundary forms

Levels are $1,2,3$ on each axis, and R^c is the complement of $R\subseteq\{1,2,3\}$. A vector $\lambda\in\mathbb{Z}^{27}$ is *log-linear* if $\lambda_{ijk}=\alpha_{ij}+\beta_{ik}+\gamma_{jk}$, or equivalently, if the linear form $\lambda\cdot n$ is a linear combination of two-way margins. The indicator of a box $R\times C\times K$ is log-linear exactly when one of R,C,K is $\{1,2,3\}$. The identities

$$\mathbf{1}_R\mathbf{1}_C\mathbf{1}_K+(1-\mathbf{1}_R)(1-\mathbf{1}_C)(1-\mathbf{1}_K)=1-\mathbf{1}_R-\mathbf{1}_C-\mathbf{1}_K+\mathbf{1}_R\mathbf{1}_C+\mathbf{1}_R\mathbf{1}_K+\mathbf{1}_C\mathbf{1}_K,$$
$$\mathbf{1}_R\mathbf{1}_C\mathbf{1}_K-\mathbf{1}_R(1-\mathbf{1}_C)(1-\mathbf{1}_K)=\mathbf{1}_R(\mathbf{1}_C+\mathbf{1}_K-1)$$

show that a box together with its complementary box, and a box minus the box that shares one side and complements the other two, are log-linear. These give the forms of Table 1. The group G_{111} of order 48 generated by the swaps of levels 2 and 3 on each axis and the permutations of the axes fixes the cell 111, and the subscripts record the side lengths of the boxes in a representative containing 111.

form	representative		orbit	exponent e
P_{113}	n_{11+}	margin	3	169
P_{112}	$n[1\,\vert\,1\,\vert\,12]+n[23\,\vert\,23\,\vert\,3]$	box pair	6	17
P_{122}	$n[1\,\vert\,12\,\vert\,12]+n[23\,\vert\,3\,\vert\,3]$	box pair	12	17
P_{222}	$n[12\,\vert\,12\,\vert\,12]+n_{333}$	box pair	8	1
P_{111}	$n_{111}+n[23\,\vert\,23\,\vert\,23]$	box pair	1	1
T	see caption	staircase	24	1
B_{223}	$n[12\,\vert\,12\,\vert\,\cdot]$	box	12	-6
B_{233}	$n[12\,\vert\,\cdot\,\vert\,\cdot]$	slab	6	-29
N	$n[\cdot\,\vert\,\cdot\,\vert\,\cdot]$	total	1	-17
D^+	$n[12\,\vert\,12\,\vert\,12]-n[12\,\vert\,3\,\vert\,3]$	difference	24	-4
D^-	$n[23\,\vert\,23\,\vert\,12]-n[1\,\vert\,1\,\vert\,12]$	difference	6	4

Table 1. The forms whose powers make up the end coefficients of the eliminant of p_{111}, up to G_{111}. The staircase T has coefficient vector $\lambda_{ijk}=[i+j\geq5]-[i>k]+[j+k\leq3]$ (where $[P]\in\{0,1\}$ is the indicator of P), the sum over the cells $111,112,121,212,232,233,321,323,333$.

Under the full symmetry group of the table (all level permutations and axis permutations), the box pairs P_{112} and P_{122} are the same unordered pair of boxes $\{1\times1\times2,\,2\times2\times1\}$, the pairs P_{111} and P_{222} are the pair $\{1\times1\times1,\,2\times2\times2\}$, and $D^{\pm}$ are the same difference. So there are eight types: the margin, the two box-pair types, the staircase, B_{223}, B_{233}, N, and D. Within each type, the absolute value of the exponent is the same, and the sign records whether the branch valuation $v_{111}=\eta\lambda_{111}$ (Section 4) makes cell 111 vanish $(+1)$ or escape (-1).

3. The end coefficients

Conjecture 3.1. *Up to sign,*

$$c_{511} = N^{17} \prod B_{233}^{29} \prod B_{223}^{6} \prod (D^{+})^{4},$$
$$c_{0} = \prod P_{113}^{169} \prod P_{112}^{17} \prod P_{122}^{17} \prod (D^{-})^{4} \prod P_{222} \prod P_{111} \prod T,$$

where each product runs over the G_{111}-orbit of the form in Table 1.

The degrees are $17+6\cdot29+12\cdot6+24\cdot4 = 359$ and $3\cdot169+6\cdot17+12\cdot17+6\cdot4+8+1+24 = 870$, so $\prod_i x_i = \pm c_0/c_{511}$ is homogeneous of degree 511, as it must be.

Remark 3.2. We computed $E(x;n)$ exactly, over $\mathbb{Q}$, for 633 integer tables: a generic base table and its one-cell perturbations, random tables, the 59 tables $n_0 + td$ with integer $0 \le t \le 60$, $t \neq 31, 33$, on a line through the table n_0 of Appendix A with direction $d \in \{0,1\}^{27}$ (the *line of Section 3*), and a scan of unit steps in every cell direction from a generic base. In every case c_{511} and c_0 agree with Conjecture 3.1 after removal of the common numerical integer content of the specialized coefficients (as multivariate polynomials in $\mathbb{Z}[n]$, c_0 and c_{511} are coprime and the content is 1), with no remaining factor.

4. Escape along the coefficient vector

The critical points of the likelihood are the critical points of the master function

$$\Phi_u = \prod_j p_j^{u_j} \Big(\sum_j p_j \Big)^{-N}$$

on the very affine variety $X^\circ = X \setminus \{\sum_j p_j = 0\}$, where X is the dense torus of the projective toric variety of the model. By Huh [5], their number is $\chi(X^\circ) = 511$ for generic u. Let $\ell = \lambda \cdot n$ be a form of Table 1, let n_0 be a generic table with $\ell(n_0) = 0$, and follow the critical points along $n_0 + s\,d$ with $\ell(d) = 1$ as $s \to 0$. Each branch is a Puiseux series $p(s) = s^v q(s)$ with $q(0)$ in the torus and v log-linear, and the exponent of ℓ in $\prod_i x_i$ is the sum of v_{111} over the branches.

For a log-linear v with values in $\{-1, 0, 1\}$, write $U = \{v = -1\}$, $Z_0 = \{v = 0\}$, and $Z_+ = \{v = 1\}$. For a set U of cells, let Σ_U be the set of tables supported on U in the torus of the model restricted to U whose two-way margins all vanish, and let $f_0 = \sum_{Z_0} p_j$.

Definition 4.1. The *boundary stratum* of a nonconstant v is

$$Y_v = \big\{ q : \; q|_U \in \Sigma_U, \; f_0(q) \neq 0 \big\} / (\mathbb{C}^*)^2,$$

where q ranges over the torus of the model and $(\mathbb{C}^*)^2$ is generated by the total scaling and by v.

If $U = \emptyset$, $v \ge 0$, and Z_0 spans a facet of the marginal cone, then Y_v is the very affine variety of the model with structural zeros on Z_+, and $|\chi(Y_v)|$ is its maximum likelihood degree [5].

Conjecture 4.2. *For every form $\ell = \lambda \cdot n$ of Table 1 other than N, the escaping branches on $\ell = 0$ have $v = \eta\lambda$, where $\eta = 1$ for the margins, the box pairs, and the staircase and $\eta = -1$ for the boxes B and the differences. Their number, counted with multiplicity, is $|\chi(Y_{\eta\lambda})|$, and*

$$e_\ell = \eta \lambda_{111} \, |\chi(Y_{\eta\lambda})|.$$

For N, every escaping branch has $v = -k\mathbf{1}$ with $k \ge 1$, and $e_N = -|\chi(\Sigma/\mathbb{C}^*)|$, where $\Sigma = \Sigma_{\text{all cells}}$.

The Euler characteristics are as follows.

Facets. For the margins, the box pairs, and the staircase, Y_λ is the model with structural zeros on the support of λ. Its maximum likelihood degree is 169 for P_{113} (the cells $11k$ removed), 17 for both box-pair types P_{112} and P_{122}, and 1 for P_{222}, P_{111}, and T.

The slab. Let $U = \{i \leq 2\}$. A table in Σ_U has $p_{2jk} = -p_{1jk}$, and $M = (p_{1jk})$ is a 3×3 matrix with nonzero entries and zero row and column sums, that is, a nowhere-zero flow on $K_{3,3}$. So $\Sigma_U/\mathbb{C}^*$ is the complement of the nine coordinate hyperplanes in $\mathbb{P}^3$, the projectivized flow space of $K_{3,3}$. The flow polynomial of $K_{3,3}$ is $(r-1)(r-2)(r^2-6r+10)$, and therefore $\chi(\Sigma_U/\mathbb{C}^*) = -5$. The cells of Z_0 form the slice $i = 3$, with $p_{3jk} = \alpha_j\beta_k M_{jk}$, so the fiber of Y over $[M]$ is $\{([\alpha],[\beta]) : \alpha^\top M\beta \neq 0\}$ in a product of two-dimensional tori. Projecting to $[\alpha]$, its Euler characteristic is that of the complement of the three coordinate lines and the three lines $(M^\top\alpha)_k = 0$ in $\mathbb{P}^2$. Since $\mathbf{1}^\top M = 0$, these three lines meet at $(1:1:1)$, so the complement has Euler characteristic 5, unless M has rank one, in which case it is 1. The rank-one flows $gh^\top$ with $\sum g = \sum h = 0$ form a set of Euler characteristic 1. Hence

$$\chi(Y_{-\lambda}) = 5\,(-5-1) + 1 = -29.$$

The box and the differences. By the same construction, $|\chi(Y)| = 6$ for B_{223}, where Σ_U consists of the tables $p_{ijk} = (-1)^{i+j}m_k$ with $\sum_k m_k = 0$, and $|\chi(Y)| = 4$ for $D^{\pm}$, where Σ_U is the single alternating $2 \times 2 \times 2$ table up to scale.

The total. Every table in Σ has singular slices. If the interaction matrix (a_{ij}) has rank two, then $p_{ijk} = \gamma_k\Gamma_{ij}$ with Γ a zero-margin matrix of rank two and $\sum\gamma_k = 0$; otherwise $p_{ijk} = b_{ik}c_{jk}$ with the columns of (b_{ik}) and (c_{jk}) summing to zero and $\sum_k b_{ik}c_{jk} = 0$. Write Σ_a for the closure of the first family and Σ_b for the second. They meet along the tables $\alpha_i\beta_j\gamma_k$ with $\sum\alpha = \sum\beta = \sum\gamma = 0$. Over a finite field with r elements, the numbers of points of $\Sigma_a/\mathbb{C}^*$, $\Sigma_b/\mathbb{C}^*$, and $(\Sigma_a \cap \Sigma_b)/\mathbb{C}^*$ are

$$(r-2)^2(r^2-6r+10), \qquad (r-2)^2(2r^2-13r+22), \qquad (r-2)^3,$$

so that $\Sigma/\mathbb{C}^*$ has $(r-2)^2(3r^2-20r+34)$ points and Euler characteristic $5+11-(-1) = 17$. Numerically, exactly sixteen branches escape on $N = 0$. Five converge to points of Σ_a and ten to points of Σ_b, all fifteen off $\Sigma_a \cap \Sigma_b$ and each with $v = -\mathbf{1}$. The remaining one converges to a point of $\Sigma_a \cap \Sigma_b$ with $v = -2\mathbf{1}$, so it contributes 2 to the exponent.

For $2 \times J \times K$ tables, $\Sigma/\mathbb{C}^*$ is the projectivized flow space of $K_{J,K}$ minus its coordinate hyperplanes, so the prediction for the total is the beta invariant of the graphic matroid of $K_{J,K}$; for $J = 3$ and $K = 2, 3, 4$ this is $1, 5, 13$.

Proposition 4.3. *Let $\ell = \lambda \cdot n$ with $\lambda \in \{0,1\}^{27}$ log-linear, and suppose that $Z_0 = \{\lambda = 0\}$ spans a facet of the marginal cone. Suppose that for a generic table on $\ell = 0$ the critical points of the facet model are nondegenerate and that the escaping branches have integer valuations. Then the number of branches that escape with $v = \lambda$ is the maximum likelihood degree of the facet model, and every escaping branch with $\min v = 0$ has $v = \lambda$.*

Proof. Write $b = An$ for the margins and $b(s) = b_0 + sb_1$. If $\min v = 0$, then at order s^0 the equation $Ap = b$ gives $A_Z q_Z(0) = b_0$ for $Z = \{v = 0\}$, so b_0 lies in the span of A_Z; since b_0 is generic on $\ell = 0$, Z is the facet Z_0 and $v = c\lambda$ with c a positive integer. If $c > 1$, then at order s^1 the vector b_1 lies in the image of the Jacobian of the facet model, which is contained in the span of A_{Z_0}; writing $\lambda = A^\top\tilde{\lambda}$, every $y \in \mathrm{im}(A_{Z_0})$ satisfies $\tilde{\lambda} \cdot y = 0$, whereas $\tilde{\lambda} \cdot b_1 = \lambda \cdot d = 1$, so $c = 1$. Thus $q_{Z_0}(0)$ is a critical point of the facet model with margins b_0. Conversely, in the coordinates (q_{Z_0}, s) the system is block triangular, with the Jacobian of the facet model invertible at a nondegenerate critical point and a linear equation for the leading coefficients of

the vanishing cells, so each critical point lifts to exactly one branch by the implicit function theorem. □

Remark 4.4. For the box types, the order-s^{-1} equations force $q|_U \in \Sigma_U$, where the face polynomial $\sum_U p_j$ is singular in the torus. The order-s^0 equations, read modulo the tangent space of Σ_U, are the critical equations of the restriction of Φ_u to $Y_{-\lambda}$, which descends to the quotient by $(\mathbb{C}^*)^2$ because $\ell(u) = 0$. They determine the limit only up to the scaling $(q_U, \epsilon, q_{Z_+}) \mapsto (\mu q_U, \epsilon/\mu, \mu^{-1} q_{Z_+})$, with q_{Z_0} fixed, where ϵ is the first-order correction of $\log q_U$; the scale is fixed at order s^1 by the λ-component $\frac{1}{2}\sum_U q_j \epsilon_j^2 - \sum_{Z_+} q_j = \ell(d) = 1$, which does not depend on the direction d.

Proposition 4.5. *Let U be a box with a side of length two, as for B_{223}, B_{233}, and $D^{\pm}$, and let $\mathcal{K}$ be the space of tables on U with vanishing two-way margins. Then $\Sigma_U = \mathcal{K} \cap (\mathbb{C}^*)^U$, so Σ_U is smooth, and at every $q \in \Sigma_U$ the kernel of the Hessian form $\mathcal{H}_q(\xi, \xi') = \sum_{j \in U} q_j \xi_j \xi'_j$ on log-linear vectors equals the logarithmic tangent space $T_q \Sigma_U$. In particular, $\mathcal{H}_q$ is nondegenerate on the normal space of Σ_U, and the reduction modulo the tangent space in the preceding remark is exact.*

Proof. Index the side of length two by $i \in \{1, 2\}$. A table $w \in \mathcal{K}$ has $w_{2jk} = -w_{1jk}$, and $W = (w_{1jk})$ has zero row and column sums; conversely, every such table with nonzero entries lies in the model ($a_{1j} = 1$, $a_{2j} = -1$, $b = 1$, $c = W$). So $\Sigma_U = \mathcal{K} \cap (\mathbb{C}^*)^U$, an open subset of a linear space, and $T_q \Sigma_U = \operatorname{diag}(q)^{-1} \mathcal{K}$. For $q \in \Sigma_U$ and $w \in \mathcal{K}$, $(\operatorname{diag}(q)^{-1} w)_{ijk} = W_{jk}/q_{1jk}$ depends only on (j, k), so $\operatorname{diag}(q)^{-1}\mathcal{K}$ consists of log-linear vectors. A log-linear ξ lies in the kernel of $\mathcal{H}_q$ if and only if $\sum_{j \in U} q_j \xi_j \xi'_j = 0$ for every log-linear ξ', that is, if and only if $\operatorname{diag}(q)\, \xi$ has vanishing two-way margins. Hence $\ker \mathcal{H}_q = \operatorname{diag}(q)^{-1}\mathcal{K} = T_q \Sigma_U$. □

For the total N, the conclusion fails: Σ is reducible, and its two components meet along the locus that carries the junction branch with $v = -2\mathbf{1}$ described above.

Remark 4.6. We tracked all 511 critical points to each hyperplane, for s down to 10^{-8}, and in every case the escaping branches, their valuations, and the sums of v_{111} are as in Conjecture 4.2. For a generic scaling τ of the model (Section 7), the boxes and differences do not change the number of critical points on their hyperplanes, while the facets do. For $3 \times 3 \times 2$ tables and the uniform scaling, the slab and the box B_{222} change the maximum likelihood degree 31 by 1 and 2, the values of $|\chi(Y)|$.

5. Completeness

Computation 5.1. Let $v \in \{-1, 0, 1\}^{27}$ be log-linear and nonconstant. Suppose that the only log-linear vectors that are constant on U and constant on Z_0 are the combinations of $\mathbf{1}$ and v, and that if $U \neq \emptyset$ no line of the table meets U in exactly one cell. Then, up to all symmetries of the table, v belongs to one of 19 classes. The 4 classes with $U = \emptyset$ are the margin, the two box-pair types, and the staircase. Among the 15 classes with $U \neq \emptyset$, the hyperplanes $v \cdot n = 0$ of exactly three, those of B_{223}, B_{233}, and D, change the number of critical points.

The first hypothesis is necessary for $\chi(Y_v) \neq 0$: otherwise a torus of dimension at least one acts freely on Y_v. The second is necessary for $\Sigma_U \neq \emptyset$. This is a finite computation: there are 20,906,319 log-linear vectors in $\{-1, 0, 1\}^{27}$, and the numbers of critical points on the hyperplanes are computed modulo a prime. The constant vector $v = -\mathbf{1}$ of the total N is the separate case $Z_0 = \emptyset$ treated in Section 4. Thus, among forms with coefficients in $\{-1, 0, 1\}$, and assuming that escaping branches move along $\pm\lambda$ as in Conjecture 4.2, the hyperplanes on

which the number of critical points drops are precisely those of the eight types of Section 2: the four facet types, B_{223}, B_{233}, D, and N.

6. The middle coefficients and recursion

Write $\mathcal{L}_+$ for the 60 forms of Table 1 with $e_\ell > 0$ and $\mathcal{L}_-$ for the 43 forms with $e_\ell < 0$.

Proposition 6.1. *Assume Conjecture 3.1, that on each hyperplane of Table 1 the escaping branches are as in Conjecture 4.2 at a generic point of the hyperplane and for a generic direction, and that for N the valuations are -2 once and -1 fifteen times. Then $c_k(n)$ is divisible by*

$$F_k(n) = \prod_{\ell\in\mathcal{L}_+} \ell^{\max(0,\, e_\ell - k)} \prod_{\ell\in\mathcal{L}_-\setminus\{N\}} \ell^{\max(0,\, |e_\ell| - (511-k))} \cdot N^{o(k)},$$

where $o(511) = 17$ and $o(k) = \max(0,\, 16-(511-k))$ for $k < 511$.

Proof. Since ℓ is linear and irreducible, it suffices to bound the order in s of $c_k(n_0 + sd)$ for generic n_0 on $\ell = 0$ and generic d. Up to sign, $c_k = c_{511}\,\sigma_{511-k}(x_1,\dots,x_{511})$, where σ_j is the jth elementary symmetric function. If $e_\ell > 0$, then c_{511} has order 0, e_ℓ roots have order 1, and the others have order 0, so every product of $511-k$ roots has order at least $e_\ell - k$. If $e_\ell < 0$, then c_{511} has order $|e_\ell|$ and $|e_\ell|$ roots have order -1, so each product of $511-k$ roots has order at least $-\min(511-k, |e_\ell|)$. For N, the most negative sum of $511-k$ valuations is $-2-\min(510-k, 15)$ for $k < 511$. □

Remark 6.2. Write $c_k(n) = F_k(n)Q_k(n)$. On 40 exact eliminants, the order of every form in every c_k for which the bound is positive equals the bound, including the profile $17, 15, 14, \dots, 1$ of N. The forced factor F_k accounts for the whole of c_0 and c_{511}, for c_1 up to a cofactor Q_1 of degree 59, and for c_{510} up to a cofactor Q_{510} of degree 45, but $F_k = 1$ for $169 \le k \le 482$.

Conjecture 6.3. *Let ℓ be a form of Table 1 other than N and let n_0 be generic with $\ell(n_0) = 0$. As $\ell \to 0$ along $n_0 + \ell d$ with $\ell(d) = 1$,*

$$E(x; n_0 + \ell d) \sim \frac{1}{\Delta_\ell(n_0)}\, E_{\mathrm{surv}}(x; n_0) \cdot \begin{cases} \ell^{\,e_\ell}\, E_\ell(x/\ell; n_0), & e_\ell > 0, \\ E_\ell(\ell x; n_0), & e_\ell < 0, \end{cases}$$

where E_{surv}, of degree $511 - |e_\ell|$, has the surviving roots; E_ℓ, of degree $|e_\ell|$, is the eliminant of an explicit problem on the stratum that depends only on n_0; and $\Delta_\ell(n_0)$ is the collision discriminant shared by the inner end coefficient of E_ℓ and the adjacent end coefficient of E_{surv}. In particular, the product of the $511 - |e_\ell|$ surviving roots on $\ell(n_0) = 0$ is

$$\prod_{i=1}^{511-|e_\ell|} x_i^{\mathrm{surv}}(n_0) = \pm \frac{\prod_{f\in\mathcal{L}_+\setminus\{\ell\}} f(n_0)^{e_f}}{\prod_{f\in\mathcal{L}_-\setminus\{\ell\}} f(n_0)^{|e_f|}} \cdot \frac{\mathrm{lead}(E_\ell)(n_0)}{E_\ell(0; n_0)}.$$

For the margin P_{113}, E_ℓ is the eliminant of the share $\rho = q_{111}/(q_{111}+q_{112}+q_{113})$ of the cell 111 in its line, over the 169 critical points of the facet model with the margins of n_0; the fitted cells of the line sum to ℓ, so the small roots are $x \sim \ell\rho$. Moreover, $1/\rho = 1 + q_{112}/q_{111} + q_{113}/q_{111}$, and in the gauge $b_{1k} = b_{i1} = c_{j1} = 1$ of the parametrization $p_{ijk} = a_{ij}b_{ik}c_{jk}$, these two ratios are parameters of the model, so $1/\rho$ is a linear function of the parameters of the facet model.

For the slab B_{233}, write $q_{1jk} = M_{jk}$ and $q_{2jk} = -M_{jk}$ for the limit on U, as in Section 4, and $q_{3jk} = \alpha_j\beta_k M_{jk}$ on Z_0. Under Conjecture 4.2, the 29 limits are the critical points of

$$\Phi = \prod_{j,k} M_{jk}^{n_{+jk}} \prod_j \alpha_j^{n_{3j+}} \prod_k \beta_k^{n_{3+k}} \Big(\sum_{j,k} \alpha_j\beta_k M_{jk}\Big)^{-N}$$

on the product of the flow space of $K_{3,3}$ and two tori, modulo the scalings of M, α, and β, which leave Φ invariant on $\ell = 0$. This is a system in 7 unknowns. At each critical point, with q_{3jk} normalized to have the margins of n_0, the first-order corrections ϵ_{ijk} $(i \le 2)$ of $\log p_{ijk}$ solve a linear system of rank 10 (the two-way margins at order s^0), and the large roots are $x \sim \omega/\ell$ with

$$\omega = M_{11} \cdot \tfrac{1}{2} \sum_{j,k} M_{jk}\big(\epsilon_{1jk}^2 - \epsilon_{2jk}^2\big),$$

which does not depend on the choice of the solution ϵ or on the scale of M.

Remark 6.4. Along the line of tables of Section 3, the trace of ω agrees with

$$-\frac{Q_{510}(n_0)}{N(n_0)^2 \prod_{f \in \mathcal{L}_- \setminus \{\ell, N\}} f(n_0)}$$

to at least 49 significant digits at the 4 isolated nondegenerate B_{233} intersection points, and the trace of $1/\rho$ over the exact degree-169 eliminant equals $Q_1(n_0)/\prod_{f \in \mathcal{L}_+ \setminus \{\ell\}} f(n_0)$ exactly over $\mathbb{Q}$ at the 2 isolated nondegenerate P_{113} intersection points. The roots of E_ℓ also agree with the rescaled roots of E computed by continuation, and the surviving-root product formula agrees with the product of the 342 tracked surviving roots on $P_{113} = 0$ (relative error 7.8×10^{-8}) and of the 482 tracked surviving roots on $B_{233} = 0$ (relative error 1.5×10^{-8}).

For the 33 facets of exponent one, E_ℓ and Δ_ℓ have an explicit closed form.

Proposition 6.5. *Let $\ell = \lambda \cdot n$ be one of the 33 forms in the orbits of P_{111}, P_{222}, and T, so that $e_\ell = 1$ and $|\operatorname{supp}(\lambda)| = 9$. On $\ell(n_0) = 0$:*

1. *The two-way margin equations on the 18 cells outside $\operatorname{supp}(\lambda)$ have rank 18 and are unimodular over $\mathbb{Z}$, so the unique critical point of the facet model has off-support coordinates $p_c(n_0) = \ell_c(n_0)$ given by explicit linear forms in n_0 with coefficients in $\{-1, 0, 1\}$.*
2. *The ratios of the 9 cells $c \in \operatorname{supp}(\lambda)$ in the torus of the model are squarefree monomials $\mathcal{M}_{\ell,c}(n_0)$ in the 18 forms $\ell_c(n_0)$, of degree 6 for P_{111} and P_{222} and degree 7 for T, with $\mathcal{M}_{\ell,111}(n_0)$ the product of the forms in $\mathcal{L}_+ \setminus \{\ell\}$ that share a codimension-two face with ℓ. The boundary eliminant of $\rho = p_{111}/\ell(p)$ is*

$$E_\ell(\rho; n_0) = \Delta_\ell(n_0)\, \rho - \mathcal{M}_{\ell,111}(n_0), \qquad \Delta_\ell(n_0) = \sum_{c \in \operatorname{supp}(\lambda)} \mathcal{M}_{\ell,c}(n_0).$$

3. *Under Conjecture 6.3 and the normalization $c_{511} > 0$ on positive tables (so that $c_0 = -F_0$, i.e., $Q_0 = -1$), the degree-59 cofactor $Q_1(n)$ satisfies*

$$Q_1(n) \equiv \Delta_\ell(n) \prod_{\substack{f \in \mathcal{L}_+ \setminus \{\ell\} \\ f \nmid \mathcal{M}_{\ell,111}(n)}} f(n) \pmod{\ell(n)}.$$

Proof. For a representative of each of P_{111}, P_{222}, and T, direct Gaussian elimination over $\mathbb{Z}$ on the 27×18 off-support margin matrix shows that its invariant factors are all 1, giving (1). Since $\ker A_{\text{off}} = 0$ and the facet has codimension one in the 19-dimensional exponent lattice of the model, the difference of exponent vectors of any two cells $c, c' \in \operatorname{supp}(\lambda)$ lies in the integer row span of the 18 off-support exponent vectors; clearing the common negative exponents yields monomials in the $\ell_c(n_0)$, which by inspection of the three representatives are squarefree of the stated degrees and factorization of $\mathcal{M}_{\ell,111}(n_0)$, giving (2). Taking the coefficient of x^1 in Conjecture 6.3 gives $c_1(n_0) = E_{\text{surv}}(0; n_0)$, while matching c_0 gives $-F_0(n_0 + \ell d)/\ell = -\big(\mathcal{M}_{\ell,111}(n_0)/\Delta_\ell(n_0)\big) E_{\text{surv}}(0; n_0)$, which yields (3). □

Part (2) matches the exact eliminants, and part (3) holds on all 183 exclusive-prime checks (tables with $\ell(n)\equiv 0 \pmod{\pi}$ for a prime π that divides no other boundary form) across 80 exact eliminants and at all 23 intersection points of the line of Section 3 with hyperplanes of types P_{222} and T.

We next solve the boundary problems on $N=0$ and on the 30 difference hyperplanes $D^{\pm}=0$ in closed form. On $N(n_0)=0$, define the three bordered independence determinants of the two-way margin matrices of n_0,

$$\Psi_{12}(n_0) = -\det\begin{pmatrix} n_{11+} & n_{12+} & n_{1++} \\ n_{21+} & n_{22+} & n_{2++} \\ n_{+1+} & n_{+2+} & 0 \end{pmatrix},$$

equal to $n_{2++}n_{+2+}n_{11+} - n_{2++}n_{+1+}n_{12+} - n_{1++}n_{+2+}n_{21+} + n_{1++}n_{+1+}n_{22+}$, and $\Psi_{13}(n_0)$ and $\Psi_{23}(n_0)$ analogously for the $(1,3)$- and $(2,3)$-margins.

Proposition 6.6. *Let n_0 be generic with $N(n_0)=0$, suppose that exactly one branch on $N=0$ has valuation -2, and follow it along n_0+sd with $N(d)=1$ as $p(s)=s^{-2}p^{(-2)}+s^{-1}p^{(-1)}+p^{(0)}+s\,p^{(1)}+O(s^2)$, converging to $\Sigma_a\cap\Sigma_b$.*

(1) *The leading table $p^{(-2)}_{ijk}=\alpha_i\beta_j\gamma_k$ is proportional to the rank-one table of one-way margins $n_{i++}n_{+j+}n_{++k}$, and*

$$p^{(-2)}_{111}(n_0) = -\frac{\Psi_{12}(n_0)\,\Psi_{13}(n_0)\,\Psi_{23}(n_0)}{n_{2++}\,n_{3++}\,n_{+2+}\,n_{+3+}\,n_{++2}\,n_{++3}}.$$

(2) *If the remaining 15 escaping branches on $N=0$ have valuation -1 and c_{511} is as in Conjecture 3.1, then the degree-45 cofactor $Q_{510}(n)$ satisfies*

$$Q_{510}(n) \equiv \Psi_{12}(n)\,\Psi_{13}(n)\,\Psi_{23}(n)\prod_{f\in D^+} f(n)\prod_{f\in B_{223}} f(n) \pmod{N(n)}.$$

Proof. Write $p_{ijk}(s)=s^{-2}\alpha_i\beta_j\gamma_k\exp\bigl(s\delta^{(1)}_{ijk}+\frac12 s^2\delta^{(2)}_{ijk}+\frac16 s^3\delta^{(3)}_{ijk}+O(s^4)\bigr)$ with $\delta^{(m)}$ log-linear. At order s^{-2}, $\sum\alpha_i=\sum\beta_j=\sum\gamma_k=0$. At order s^{-1} the two-way margins of $\alpha_i\beta_j\gamma_k\delta^{(1)}_{ijk}$ vanish; writing $\tilde\alpha=\alpha\times(1,1,1)=(\alpha_2-\alpha_3,\alpha_3-\alpha_1,\alpha_1-\alpha_2)$, and similarly $\tilde\beta$ and $\tilde\gamma$, the general log-linear solution modulo the logarithmic tangent space at $p^{(-2)}$ of the rank-one locus $\Sigma_a\cap\Sigma_b$ is $\delta^{(1)}_{ijk}=a_1\tilde\alpha_i\tilde\beta_j+a_2\tilde\alpha_i\tilde\gamma_k+a_3\tilde\beta_j\tilde\gamma_k$ for $a_1,a_2,a_3\in\mathbb{C}$. At order s^0, summing the two-way margins of $p^{(0)}$ over one axis annihilates the γ-, β-, or α-sum of $\delta^{(1)}$ and $\delta^{(2)}$ and gives $\alpha_i=\mu_1 n_{i++}$, $\beta_j=\mu_2 n_{+j+}$, and $\gamma_k=\mu_3 n_{++k}$. Pairing the $(1,2)$-margin of $p^{(0)}$ with $\tilde\alpha_i\tilde\beta_j$ annihilates $\delta^{(2)}$ and gives

$$-81\,\mu_1^2\mu_2^2\mu_3^3\,a_2a_3\prod_{i=1}^3 n_{i++}\prod_{j=1}^3 n_{+j+}\prod_{k=1}^3 n_{++k} = \Psi_{12}(n_0),$$

since $\sum_i\alpha_i\tilde\alpha_i^2=-9\alpha_1\alpha_2\alpha_3$ (and analogously for the other two axes), where Π is the product of the nine one-way margins,

$$-81\,\mu_1^2\mu_3^3\mu_3^2\,a_1a_3\,\Pi=\Psi_{13}(n_0),\qquad -81\,\mu_1^3\mu_2^2\mu_3^2\,a_1a_2\,\Pi=\Psi_{23}(n_0).$$

At order s^1, the total-sum equation $\sum_{i,j,k}p^{(1)}_{ijk}=1$ kills $\delta^{(2)}$ and $\delta^{(3)}$ and all unmixed cubes in $(\delta^{(1)})^3$, leaving

$$1=\frac16\sum_{i,j,k}\alpha_i\beta_j\gamma_k\bigl(\delta^{(1)}_{ijk}\bigr)^3 = -729\,\mu_1^3\mu_2^3\mu_3^3\,a_1a_2a_3\prod_{i=1}^3 n_{i++}\prod_{j=1}^3 n_{+j+}\prod_{k=1}^3 n_{++k}.$$

Since $(-81)^3/(-729)^2 = -1$, dividing the product of the three order-s^0 equations by the square of the order-s^1 equation gives $-\mu_1\mu_2\mu_3\Pi = \Psi_{12}\Psi_{13}\Psi_{23}$, and multiplying by $n_{1++}n_{+1+}n_{++1}$ gives (1). Since $-c_{510}/c_{511}$ is the sum of the roots and the valuation-two branch dominates the 15 valuation-one branches as $s \to 0$, multiplying (1) by $-c_{511}/(N^2F_{510})$, which on $N = 0$ is $-\prod_{f\in\mathcal{L}_-\setminus\{N\}} f$ up to the integer content, cancels the six one-way margins $n_{2++}, n_{3++}, n_{+2+}, n_{+3+}, n_{++2}, n_{++3}$ (which coincide on $N = 0$ with $-n[12\,|\,\cdot\,|\,\cdot]$, $-n[13\,|\,\cdot\,|\,\cdot]$, and their j- and k-axis counterparts, each to the power $29 - 28 = 1$) and yields (2). □

Proposition 6.6(2) holds identically over $\mathbb{Q}$ at the $N = 0$ point of the line of Section 3, on all 8 coordinate rays in $\{2, 3\}^3$, and on all 62 exclusive-prime checks across 168 exact eliminants.

Proposition 6.7. *Let $\ell = n[12\,|\,12\,|\,12] - n[12\,|\,3\,|\,3] \in D^+$, with $U = \{1, 2\}^3$, $Z_+ = \{133, 233\}$, and Z_0 the remaining 17 cells, and let n_0 be generic with $\ell(n_0) = 0$. Define the linear forms*

$$L_0 = n[1\,|\,13\,|\,3] - n[23\,|\,2\,|\,3], \quad L_1 = n_{+23}, \quad L_2 = n[1\,|\,3\,|\,13] - n[23\,|\,3\,|\,2], \quad L_3 = n_{+32},$$
$$L_4 = n_{2+3}, \quad L_5 = n_{23+}, \quad L_6 = n_{311} + n[12\,|\,3\,|\,3] - n_{322} - n_{323} - n_{332}, \quad L_7 = n_{3+2},$$
$$L_8 = n[3\,|\,12\,|\,3] - n[12\,|\,3\,|\,3], \quad L_9 = n_{32+}, \quad L_{10} = n[3\,|\,3\,|\,12] - n[12\,|\,3\,|\,3], \quad L_{11} = n_{+33}.$$

(1) *On U, the zero-margin table is unique up to scale, $q_{ijk} = \omega(-1)^{i+j+k+1}$, so that $\omega = q_{111}$. Modulo the margins of the tables $q|_U\,\epsilon$ with ϵ log-linear, which span a space of dimension 6, the order-s^0 margin equations consist of $\ell(n_0) = 0$ and the twelve equations*

$$q_{113} = L_0 + q_{223} + y_1, \quad q_{123} = L_1 - q_{223} - y_1, \quad q_{213} = L_4 - q_{223}, \quad q_{313} = L_8 - y_1,$$
$$q_{131} = L_2 + q_{232} + y_2, \quad q_{132} = L_3 - q_{232} - y_2, \quad q_{231} = L_5 - q_{232}, \quad q_{331} = L_{10} - y_2,$$
$$q_{311} = L_6 + y_1 + y_2 + q_{322}, \quad q_{312} = L_7 - y_2 - q_{322}, \quad q_{321} = L_9 - y_1 - q_{322}, \quad q_{333} = L_{11},$$

where $y_1 = q_{323}$ and $y_2 = q_{332}$.

(2) *The binomials of the model on the boxes $\{1, 2\} \times \{1, 2\} \times \{1, 3\}$, $\{1, 2\} \times \{1, 3\} \times \{1, 2\}$, and $\{2, 3\} \times \{1, 2\} \times \{1, 2\}$ reduce to $q_{113}q_{223} = q_{123}q_{213}$, $q_{131}q_{232} = q_{132}q_{231}$, and $q_{311}q_{322} = q_{312}q_{321}$, that is,*

$$(L_0 + L_1 + L_4)(q_{223} + y_1) = L_1L_4 + (L_0 + L_1)y_1,$$
$$(L_2 + L_3 + L_5)(q_{232} + y_2) = L_3L_5 + (L_2 + L_3)y_2,$$
$$(L_6 + L_7 + L_9)\,q_{322} = (L_7 - y_2)(L_9 - y_1).$$

After these substitutions, every binomial of the model on $Z_0 \cup U$ vanishes modulo the decoupled pair of quadratics

$$g_1(y_1) = (y_1 + L_0 + L_4)(y_1 + L_6 + L_7)y_1 + (L_1 - y_1)(L_8 - y_1)(L_9 - y_1),$$
$$g_2(y_2) = (y_2 + L_2 + L_5)(y_2 + L_6 + L_9)y_2 + (L_3 - y_2)(L_{10} - y_2)(L_7 - y_2),$$

in which the cubic terms cancel, and g_1 and g_2 divide the binomials of the boxes $\{1, 3\} \times \{1, 2\} \times \{1, 3\}$ and $\{1, 3\} \times \{1, 3\} \times \{1, 2\}$.

(3) *For each of the $4 = 2 \times 2$ roots (y_1, y_2), the scale $\omega = q_{111}$ is fixed at order s^1 by*

$$\omega(y_1, y_2) = 2(W_1W_{23} + W_2W_{13} + W_3W_{12}) - \big(\Xi_{133} + \Xi_{233}\big),$$

where, with the residual margins $\bar{n}^{12}_{ij} = n_{ij+} - q_{ij3}$, $\bar{n}^{13}_{ik} = n_{i+k} - q_{i3k}$, and $\bar{n}^{23}_{jk} = n_{+jk} - q_{3jk}$ $(i, j, k \le 2)$, the Walsh coordinates are

$$W_1 = \tfrac{1}{4}\textstyle\sum(-1)^{j+k}\bar{n}^{23}_{jk}, \quad W_2 = \tfrac{1}{4}\sum(-1)^{i+k}\bar{n}^{13}_{ik}, \quad W_3 = \tfrac{1}{4}\sum(-1)^{i+j}\bar{n}^{12}_{ij},$$
$$W_{12} = \tfrac{1}{4}\textstyle\sum(-1)^{k-1}\bar{n}^{13}_{ik}, \quad W_{13} = \tfrac{1}{4}\sum(-1)^{j-1}\bar{n}^{12}_{ij}, \quad W_{23} = \tfrac{1}{4}\sum(-1)^{i-1}\bar{n}^{12}_{ij},$$

and

$$\Xi_{133} = L_{11}\,\frac{q_{113}\,q_{131}\,q_{311}}{q_{313}\,q_{331}}, \qquad \Xi_{233} = -L_{11}\,\frac{q_{213}\,q_{231}\,q_{311}}{q_{313}\,q_{331}}.$$

(4) *For the corresponding form in D^- obtained by permuting levels so that $111 \in Z_+$, the four valuation-one roots have leading coefficients $p^{(1)}_{111}$ with $1/p^{(1)}_{111} = \omega(y_1,y_2)/\Xi_{133}$.*

Proof. Since $U = \{1,2\}^3$ has 2×2 slices, every zero-margin table on U is a multiple of $(-1)^{i+j+k}$, and the tangent space to the model at $(-1)^{i+j+k}$ on U is spanned by the seven Walsh vectors $1, w_1, w_2, w_3, w_{12}, w_{13}, w_{23}$ with $w_a = (-1)^{\text{coordinate } a}$. Multiplied by $(-1)^{i+j+k}$, they become the Walsh tables other than the constant one, and the alternating table among them has zero margins, so their margins span a space of dimension 6 and the quotient of the 19-dimensional margin space has dimension 13. Pairing with λ, which vanishes on Z_0 and on these margins, gives $\ell(n_0) = 0$, and Gaussian elimination of the other 12 conditions gives (1). The three binomials of (2) each contain four cells of U, whose values cancel, and substituting (1) and these three relations into the remaining binomials gives the quadratics. Pairing the order-s^1 margin equation with λ gives $\frac{1}{2}\sum_{c\in U} q_c\epsilon_c^2 - (q_{133}+q_{233}) = 1$, where q_{i33} are the leading coefficients of the vanishing cells. The order-s^0 margin equations on $\{1,2\}^2$ determine the Walsh coordinates of $\omega\epsilon|_U$ from the residual margins, and expanding gives $\frac{1}{2}\sum_{c\in U}(-1)^{i+j+k+1}(\omega\epsilon_c)^2 = 2(W_1W_{23}+W_2W_{13}+W_3W_{12})$. The binomials of the boxes $\{1,3\}^3$ and $\{2,3\}\times\{1,3\}\times\{1,3\}$ give $\omega q_{133} = \Xi_{133}$ and $\omega q_{233} = \Xi_{233}$, and multiplying the pairing by ω gives (3). Under the permutation of (4), the cell 111 corresponds to 133, so $p^{(1)}_{111} = q_{133} = \Xi_{133}/\omega$, which gives (4). □

In the four-dimensional algebra $\mathbb{Q}[y_1]/(g_1)\otimes\mathbb{Q}[y_2]/(g_2)$, the exact rational trace $\mathrm{Tr}(\omega)$ agrees with $-Q_{510}(n_0)/\big(N(n_0)^2\prod_{f\in\mathcal{L}_-\setminus\{\ell,N\}} f(n_0)\big)$ at all 18 distinct D^+ intersection points along the line of Section 3, $\mathrm{Tr}(\omega/\Xi_{133})$ agrees with $Q_1(n_0)/\prod_{f\in\mathcal{L}_+\setminus\{\ell\}} f(n_0)$ at all 5 distinct D^- intersection points, and $\sigma_2(\omega/\Xi_{133})$ agrees with

$$-Q_2(n_0)\Big/\prod_{f\in\mathcal{L}_+\setminus\{\ell\}} f(n_0)^{\min(2,e_f)}$$

at both D^- intersections along the coordinate ray of cell 122.

We next solve the degree-6 boundary problem on B_{223} and the degree-17 boundary problem on P_{112} and P_{122} in closed form.

Proposition 6.8. *Let $\ell = n[12\,|\,12\,|\,\cdot] \in B_{223}$, with $U = \{1,2\}^2\times\{1,2,3\}$, $Z_+ = \emptyset$, and Z_0 the remaining 15 cells ($i=3$ or $j=3$), and let n_0 be generic with $\ell(n_0) = 0$.*

(1) *On U, every zero-margin table has the form $p^{(-1)}_{ijk} = (-1)^{i+j}\mu_k$ with $\mu_1+\mu_2+\mu_3 = 0$. The two-way margin equations, together with the binomials of the model that link Z_0 to the limit $p^{(-1)}$ on U, express all 15 coordinates q_c ($c\in Z_0$) linearly in terms of the three corner coordinates $\zeta_k = q_{33k}$ ($k=1,2,3$), which satisfy $\zeta_1+\zeta_2+\zeta_3 = n_{33+}$:*

$$q_{i3k} = \alpha_i\,(n_{+3k}-\zeta_k), \qquad q_{3jk} = \beta_j\,(n_{3+k}-\zeta_k), \qquad q_{33k} = \zeta_k,$$

for $i,j\in\{1,2\}$, where $\alpha_i = n_{i3+}/(n_{13+}+n_{23+})$ and $\beta_j = n_{3j+}/(n_{31+}+n_{32+})$.

(2) *Summing the order-s^0 slice equations over $(i,j)\in\{1,2\}^2$ gives $\nu_k+\zeta_k = \theta\,\mu_k$ ($k=1,2,3$), where $\nu_k = n_{++k}-n_{+3k}-n_{3+k}$ and $\theta = \epsilon_{11k}-\epsilon_{12k}-\epsilon_{21k}+\epsilon_{22k}$ is independent of k, and the model binomials between U and Z_0 reduce to*

$$(n_{+3k}-\zeta_k)(n_{3+k}-\zeta_k) = \kappa\,\zeta_k(\nu_k+\zeta_k) \qquad (k=1,2,3).$$

Eliminating κ by the equation for $k=1$ gives a common coefficient $n_{++1}\zeta_1 - n_{+31}n_{3+1}$ for ζ_2^2 and ζ_3^2, so substituting $\zeta_3 = n_{33+} - \zeta_1 - \zeta_2$ and subtracting the $k=2$ and $k=3$ equations cancels ζ_2^2 linearly, expressing $\zeta_2, \zeta_3 \in \mathbb{Q}(\zeta_1)$ rationally and yielding an explicit univariate degree-6 polynomial $H_6(\zeta_1; n_0)$ after dividing out $\zeta_1 + \nu_1$.

(3) *The order-s^1 Hessian on U is invariant under the changes of ϵ that preserve the margins of $p^{(-1)}\epsilon$ on U and determines $\omega = \mu_1$ in $\mathbb{Q}[\zeta_1]/(H_6(\zeta_1))$ as*

$$\omega(\zeta_1) = (\nu_1 + \zeta_1)\left(n_{11+} + \sum_{k=1}^{3} \frac{\phi_k(\zeta_k)\,\psi_k(\zeta_k)}{\nu_k + \zeta_k}\right),$$

where $\phi_k(\zeta_k) = n_{2+k} - \alpha_2(n_{+3k} - \zeta_k)$ and $\psi_k(\zeta_k) = n_{+2k} - \beta_2(n_{3+k} - \zeta_k)$.

Proof. In slice k, the cross-ratio $q_{23k}p^{(-1)}_{11k}/(q_{13k}p^{(-1)}_{21k}) = -q_{23k}/q_{13k}$ is independent of k, so summing over k gives $q_{23k}/q_{13k} = n_{23+}/n_{13+}$ and similarly $q_{32k}/q_{31k} = n_{32+}/n_{31+}$; combining with $q_{13k} + q_{23k} + \zeta_k = n_{+3k}$ and $q_{31k} + q_{32k} + \zeta_k = n_{3+k}$ gives (1). In slice k, $\sum_{i,j\in\{1,2\}}(-1)^{i+j}\mu_k\epsilon_{ijk} = \theta\mu_k$ equals $n_{++k} - (n_{+3k} + n_{3+k} - \zeta_k) = \nu_k + \zeta_k$, while the cross-ratio $\zeta_k p^{(-1)}_{11k}/(q_{13k}q_{31k})$ is independent of k, giving the three quadratics of (2); at $\zeta_1 = -\nu_1$ the resultant has the boundary root $\zeta_2 = -\nu_2, \zeta_3 = -\nu_3$ where $\mu_1 = \mu_2 = \mu_3 = 0$, and dividing by $\zeta_1 + \nu_1$ leaves $H_6(\zeta_1; n_0)$ of degree 6. For (3), such a change has the form δ_k on U with $\sum_k \mu_k\delta_k = 0$, which shifts $\frac{1}{2}\sum_k \mu_k \sum_{i,j\in\{1,2\}}(-1)^{i+j}\epsilon^2_{ijk}$ by $\theta\sum_k \mu_k\delta_k = 0$; evaluating in the gauge in which the (i,j)-component of ϵ on $\{1,2\}^2$ is θ at $(1,1)$ and 0 elsewhere, via the identity $(z+1+a+b)^2 - (z+a)^2 - (z+b)^2 + z^2 = 2z + 1 + 2a + 2b + 2ab$ and $\sum_{i,j\in\{1,2\}} n_{ij+} = 0$, gives the formula for $\omega(\zeta_1)$. □

In $\mathbb{Q}[\zeta_1]/(H_6(\zeta_1))$, $\mathrm{Tr}(\omega)$ agrees with $-Q_{510}(n_0)/\big(N(n_0)^2\prod_{f\in\mathcal{L}_-\setminus\{\ell,N\}} f(n_0)\big)$ at all 9 distinct B_{223} intersection points along the line of Section 3.

Proposition 6.9. *Let $\ell = n[1\,|\,1\,|\,12] + n[23\,|\,23\,|\,3] \in P_{112}$, with $|\mathrm{supp}(\lambda)| = 6$ and $|Z_0| = 21$, and let n_0 be generic with $\ell(n_0) = 0$. Define the linear forms*

$$\Lambda_1 = n[1\,|\,12\,|\,1] + n[23\,|\,3\,|\,23], \quad \Lambda_2 = n[1\,|\,12\,|\,2] - n[23\,|\,3\,|\,2], \quad \Lambda_3 = n_{131} - n[23\,|\,3\,|\,23],$$
$$\Lambda_4 = n_{+32}, \quad \Lambda_5 = n[12\,|\,1\,|\,1] + n[3\,|\,23\,|\,23], \quad \Lambda_6 = n[12\,|\,1\,|\,2] - n[3\,|\,23\,|\,2],$$
$$\Lambda_7 = n_{22+} - \Lambda_8, \quad \Lambda_8 = n_{111} + n[23\,|\,23\,|\,23], \quad \Lambda_9 = n_{23+}, \quad \Lambda_{10} = n_{311} - n[3\,|\,23\,|\,23],$$
$$\Lambda_{11} = n_{3+2}, \quad \Lambda_{12} = n_{32+}, \quad \Lambda_{13} = n_{33+}.$$

(1) *The two-way margin equations on Z_0 have rank 18. In slice $k=3$, the four cells $\{2,3\}\times\{2,3\}\times\{3\}$ vanish and the remaining five cells are*

$$q_{113} = n_{11+}, \qquad q_{123} = n_{+23}, \qquad q_{133} = n_{+33}, \qquad q_{213} = n_{2+3}, \qquad q_{313} = n_{3+3}.$$

In terms of $z_1 = q_{232}$, $z_2 = q_{322}$, $z_3 = q_{332}$, $S_1 = z_1 + z_3$, and $S_2 = z_2 + z_3$, the other 16 cells of Z_0 are

$$q_{121} = \Lambda_1 - S_1, \quad q_{122} = \Lambda_2 + S_1, \quad q_{131} = \Lambda_3 + S_1, \quad q_{132} = \Lambda_4 - S_1,$$
$$q_{211} = \Lambda_5 - S_2, \quad q_{212} = \Lambda_6 + S_2, \quad q_{221} = \Lambda_7 + S_1 + z_2, \quad q_{222} = \Lambda_8 - S_1 - z_2,$$
$$q_{231} = \Lambda_9 - z_1, \quad q_{311} = \Lambda_{10} + S_2, \quad q_{312} = \Lambda_{11} - S_2, \quad q_{321} = \Lambda_{12} - z_2, \quad q_{331} = \Lambda_{13} - z_3.$$

(2) *The binomials of the model on the boxes $\{1,2\}\times\{2,3\}\times\{1,2\}$ and $\{1,3\}\times\{2,3\}\times\{1,2\}$ are linear in z_2, and a linear combination of them cancels the term z_2z_3 and gives*

$$\phi_0(S_1)\,z_2 = \phi_1(S_1) + \phi_2(S_1)\,z_3,$$

where, with $q_{121}, q_{122}, q_{131}, q_{132}$ *the linear functions of* S_1 *in (1),*

$$\begin{aligned}\phi_0 &= S_1\, q_{122}q_{131} + (\Lambda_9 + \Lambda_{13} - S_1)\, q_{121}q_{132},\\ \phi_1 &= (\Lambda_8 - S_1)(\Lambda_9 - S_1)\, q_{121}q_{132} - (\Lambda_7 + S_1)\, S_1\, q_{122}q_{131},\\ \phi_2 &= (\Lambda_7 + \Lambda_{12} + S_1)\, q_{122}q_{131} + (\Lambda_8 - S_1)\, q_{121}q_{132},\end{aligned}$$

of degrees 2, 3, *and* 2. *Substituting* z_2 *into the binomial of the box* $\{2,3\}\times\{2,3\}\times\{1,2\}$, *which then has a factor linear in* z_3, *and into the difference of the binomials of the boxes* $\{2,3\}\times\{1,2\}\times\{1,2\}$ *and* $\{2,3\}\times\{1,3\}\times\{1,2\}$ *gives a quadratic and a cubic in* z_3 *with coefficients in* $\mathbb{Q}[S_1]$. *Their resultant in* z_3 *has an irreducible factor* $H_{17}(S_1; n_0)$ *of degree* 17, *and at each root of* H_{17} *the quadratic and the cubic have exactly one common root* z_3, *a rational function of* S_1.

(3) *Since the margin equations on* Z_0 *have rank* 18, *the ratios of the six vanishing cells* $c \in \operatorname{supp}(\lambda)$ *at order* s^1 *are monomials in the coordinates* q_c, $c \in Z_0$, *which lie in* $\mathbb{Q}[S_1]/(H_{17}(S_1))$*:*

$$\frac{p^{(1)}_{112}}{p^{(1)}_{111}} = \frac{q_{122}\, q_{212}\, q_{221}}{q_{121}\, q_{211}\, q_{222}}, \qquad \frac{p^{(1)}_{ij3}}{p^{(1)}_{111}} = \frac{q_{i13}\, q_{1j3}\, q_{ij1}}{q_{113}\, q_{i11}\, q_{1j1}} \quad (i,j \in \{2,3\}).$$

Proof. Gaussian elimination on the 27×21 matrix A_{Z_0} gives rank 18 and, after simplification with $\ell(n_0) = 0$, the parametrization (1). Expanding the binomials of the five boxes of (2), which are the $2\times2\times2$ boxes contained in Z_0, gives the linear relation, the quadratic, and the cubic, and at the 17 solutions all five binomials vanish. Since $(a_{ij}b_{i3}c_{j3})/(a_{11}b_{11}c_{11})$ equals the second monomial ratio of (3), summing the six ratios gives $1/\rho = \ell(p^{(1)})/p^{(1)}_{111}$ on P_{112}. A form of P_{122} is mapped to this standard form by a relabeling of levels and axes that sends the cell 111 to 333, so there $1/\rho = \ell(p^{(1)})/p^{(1)}_{333}$, which is the same sum multiplied by $p^{(1)}_{111}/p^{(1)}_{333}$. □

In $\mathbb{Q}[S_1]/(H_{17}(S_1))$, $\operatorname{Tr}(1/\rho)$ matches $-c_1/c_0$ for the exact eliminant of ρ in the facet model at a table on a P_{112} hyperplane (a ratio with a 121-digit numerator and a 119-digit denominator) and agrees with $Q_1(n_0)/\prod_{f\in\mathcal{L}_+\setminus\{\ell\}} f(n_0)$ at all 4 nondegenerate isolated P_{112} intersections and all 8 nondegenerate isolated P_{122} intersections along the line of Section 3. Under Conjectures 6.3 and 6.13, together with Propositions 6.5–6.8, this determines $Q_1(n)$ mod $\ell(n)$ in closed form on 57 of the 60 constant-side hyperplanes (all except the three margins P_{113}) and $Q_{510}(n)$ mod $\ell(n)$ on 37 of the 43 lead-side hyperplanes (all except the six slabs B_{233}). More generally, for every end-adjacent index $1 \le k \le e_\ell$ on $\ell \in \mathcal{L}_+$ (where $c_k/c_0 \sim (-1)^k \ell^{-k}\sigma_k(1/\rho)$ as $\ell \to 0$, $c_0 = -F_0$, and $\Delta^-_\ell = 1$ in the notation of Conjecture 6.13), all codimension-2 overlap exponents cancel upon clearing F_0/F_k, yielding

$$Q_k(n_0)\big|_{\ell(n_0)=0} = (-1)^{k+1}\sigma_k(1/\rho) \prod_{f\in\mathcal{L}_+\setminus\{\ell\}} f(n_0)^{\min(k,e_f)} \qquad (1 \le k \le e_\ell,\ \ell \in \mathcal{L}_+),$$

and symmetrically for $Q_{511-j}(n_0)|_{\ell(n_0)=0}$ $(1 \le j \le |e_\ell|,\ \ell \in \mathcal{L}_- \setminus \{N\})$ with $(-1)^j\sigma_j(\omega)$ and $N(n_0)^{17-o(511-j)}$. Along the coordinate ray of the cell 122, $\sigma_2(1/\rho)$ reproduces $Q_2(n_0)$ on all three hyperplanes with $2 \le e_\ell$ (one of type P_{122} and two of type D^-). These boundary values, together with a few exact eliminants, determine the univariate end-adjacent cofactors over $\mathbb{Q}$. On the coordinate ray of the cell 122, the values at the 9 constant-side hyperplanes and 2 eliminants determine Q_1, of degree 10, which then agrees with 14 further eliminants. The line of Section 3 meets the constant-side hyperplanes other than the margins at 48 points: 43 that lie on exactly one $\mathcal{L}_+$ hyperplane (including the 41 nondegenerate points where Q_1 is given by the traces above, together with one P_{112} and one P_{122} intersection where H_{17} drops

degree), and 5 that lie on two or more $\mathcal{L}_+$ hyperplanes, where Q_1 vanishes. The values at 45 of these points and 15 eliminants determine Q_1, of degree 59, which agrees with 44 further eliminants. The line meets the lead-side hyperplanes other than the slabs at 32 points, 28 on one non-slab hyperplane (18 of type D^+, 9 of type B_{223}—one of which also lies on a slab B_{233}, making 5 points where Q_{510} vanishes alongside the 4 intersections of two non-slab lead-side hyperplanes—and the point on $N = 0$); these 33 values and 13 eliminants determine Q_{510}, of degree 45, which agrees with 46 further eliminants.

Neutral boundary hyperplanes and interior factorizations. Under the full symmetry group $S_3^3 \rtimes S_3$ of the table, the eight boundary types of Table 1 comprise 325 hyperplanes: the 60 constant-side forms $\mathcal{L}_+$ ($v_{111} = +1$, $\sum |e_\ell| = 870$), the 43 lead-side forms $\mathcal{L}_-$ ($v_{111} = -1$, $\sum |e_\ell| = 359$), and 222 *neutral* boundary forms $\mathcal{L}_0$ for which $\lambda_{111} = 0$ ($111 \in Z_0$, $v_{111} = 0$, $\sum |e_\ell| = 5{,}574$), namely 18 of type $P^0_{111/222}$ ($|e| = 1$), 48 of type T^0 ($|e| = 1$), 51 of type D^0 ($|e| = 4$), 15 of type B^0_{223} ($|e| = 6$), 63 of type $P^0_{112/122}$ ($|e| = 17$), 3 of type B^0_{233} ($|e| = 29$), and 24 of type P^0_{113} ($|e| = 169$).

On any neutral hyperplane $\ell^0(n_0) = 0$ ($\ell^0 \in \mathcal{L}_0$), the $|e_{\ell^0}|$ escaping critical points have $v_{111} = 0$, so their 111-coordinate $p_{111}(s) \to q_{111}(n_0) \in \mathbb{C}^*$ converges to a finite nonzero value on $Y_{\eta\lambda}$. By Conjecture 3.1, $c_0(n_0) \neq 0$ and $c_{511}(n_0) \neq 0$, and the analogue of Conjecture 6.3 for neutral forms takes the unrescaled form

$$E(x; n_0)\big|_{\ell^0(n_0)=0} = \frac{1}{\Delta_{\ell^0}(n_0)}\, E_{\mathrm{surv},\ell^0}(x; n_0)\, E_{\ell^0}(x; n_0),$$

where $E_{\ell^0}(x; n_0)$, of degree $|e_{\ell^0}|$, is the eliminant of the finite coordinate $q_{111} \in Z_0$ on $Y_{\eta\lambda}$ and, under the hypotheses of Proposition 6.10, divides $E(x; n_0)$ in $\mathbb{Q}(n_0)[x]$. Applying Propositions 6.5, 6.7, 6.8, and 6.9 to the neutral orbits yields 195 closed-form interior factors of $E(x; n)$, and the slab and margin systems above give the remaining 27.

Proposition 6.10. *Assume that on each neutral hyperplane $\ell^0(n_0) = 0$ ($\ell^0 \in \mathcal{L}_0$) the $|e_{\ell^0}|$ escaping branches converge, one to each, to the points of $Y_{\eta\lambda}$, and that these points are non-degenerate.*

(1) *For each of the* 66 *neutral facets $\ell^0 \in P^0_{111/222} \cup T^0$ ($|e_{\ell^0}| = 1$), the off-support coordinate $q_{111}(n_0) = \ell_{\ell^0,111}(n_0)$ coincides modulo $\ell^0(n_0)$ with one of the* 21 *active forms $f \in P_{113} \cup P_{112} \cup P_{122}$ (*12 *hyperplanes for each of the* 3 *forms in P_{113},* 1 *for each of the* 6 *forms in P_{112}, and* 2 *for each of the* 12 *forms in P_{122}). Hence $x = f(n_0)$ is an exact root of $E(x; n_0)$ on $\ell^0(n_0) = 0$:*

$$\sum_{k=0}^{511} c_k(n)\, f(n)^k \equiv 0 \pmod{\ell^0(n)}.$$

(2) *For each of the* 51 *neutral difference forms $\ell^0 \in D^0$ ($|e_{\ell^0}| = 4$), the degree-*4 *factor $E_{\ell^0}(x; n_0)$, the characteristic polynomial of multiplication by $q_{111}(y_1, y_2)$ in the algebra $\mathbb{Q}[y_1]/(g_1) \otimes \mathbb{Q}[y_2]/(g_2)$ of Proposition 6.7, divides $E(x; n_0)$: for* 3 *forms (*111 *corresponds to* 333*), q_{111} is a form of P_{113}, independent of (y_1, y_2), and $E_{\ell^0}(x; n_0) = (x - q_{111})^4$; for* 36 *forms (in which* 111 *corresponds to a cell in the orbit of* 113 *or of* 313 *under the symmetries of the standard form), q_{111} depends on only one of y_1 and y_2, and $E_{\ell^0}(x; n_0)$ is the square of a quadratic; and for* 12 *forms (*111 *corresponds to* 311*), $(L_6 + L_7 + L_9)\, q_{111} = (y_1 + L_6 + L_7)(y_2 + L_6 + L_9)$, and E_{ℓ^0} is the corresponding tensor-product quartic.*

(3) *For each of the* 15 *neutral box forms* $\ell^0 \in B^0_{223}$ *(*$|e_{\ell^0}| = 6$*), choosing the slice of cell* $111 \in Z_0$ *as* $k = 1$ *in Proposition 6.8 makes* $x = q_{111}$ *an affine linear function of* ζ_1 *(*$x = \zeta_1$*,* $x = \alpha_i(n_{+31} - \zeta_1)$*, or* $x = \beta_j(n_{3+1} - \zeta_1)$*), so* $E_{\ell^0}(x; n_0) = H_6(\zeta_1(x); n_0)$ *divides* $E(x; n_0)$*.*

(4) *For each of the* 15 *neutral box-pair forms* $\ell^0 \in P^0_{112/122}$ *where cell* 111 *lies in the five-cell linear block of slice* 3 *in Proposition 6.9(1) (*5 *forms for each* $f \in P_{113}$*), all* 17 *roots of* $E_{\ell^0}(x; n_0)$ *collapse to* $x = f(n_0)$*, so* $(x - f(n_0))^{17}$ *divides* $E(x; n_0)$ *on* $\ell^0(n_0) = 0$*. In particular, for each* $f \in P_{113}$*, the Taylor coefficients* $\tilde{c}^{(f)}_m(n) = \frac{1}{m!}(\partial^m_x E)(f(n); n)$ *around* $x = f(n)$ *are divisible by*

$$f(n)^{\max(0,\,169-m)}\big(D^0_{\mathrm{coll}}(n)\big)^{\max(0,\,4-m)}\prod_{i=1}^{5}\big(P^0_{112,i}(n)\big)^{\max(0,\,17-m)}\prod_{j=1}^{12}\big(R^0_j(n)\big)^{\max(0,\,1-m)},$$

of degree $169+4+5\cdot17+12 = 270$ *at* $m = 0$*, where* D^0_{coll} *is the neutral difference form of (2) with* $q_{111} = f$*,* $P^0_{112,i}$ *are the five neutral box pairs above, and* R^0_j *are the twelve neutral facets of (1) (*4 *of type* $P^0_{111/222}$ *and* 8 *of type* T^0*) with* $q_{111} = f$*.*

(5) *For each of the* 3 *neutral slabs* $\ell^0 \in B^0_{233}$ *(*$|e_{\ell^0}| = 29$*), cell* 111 *lies in the surviving slab, and* $E_{\ell^0}(x; n_0)$ *is the eliminant of* $x = q_{111} = N\alpha_1\beta_1 M_{11}/\sum_{j,k}\alpha_j\beta_k M_{jk}$ *over the* 29 *critical points of* Φ*.*

(6) *For each of the* 24 *neutral margins* $\ell^0 \in P^0_{113}$ *(*$|e_{\ell^0}| = 169$*),* $E_{\ell^0}(x; n_0)$ *is the eliminant of the fitted value of the cell corresponding to* 111 *over the* 169 *critical points of the facet model.*

Proof. Since $E(x; n)$ is invariant under permutations in G_{111} and permutations in $S_3^3 \rtimes S_3$ map $\ell^0 \in \mathcal{L}_0$ to the standard representatives of Propositions 6.5, 6.7, 6.8, and 6.9 while sending cell 111 to a cell $c \in Z_0$, the $|e_{\ell^0}|$ escaping branches on $\ell^0(n_0) = 0$ converge to the values of $q_c(n_0)$ on the stratum $Y_{\eta\lambda}$, giving (1)–(3), (5), (6), and the first statement of (4). For the second statement of (4), on each of the 18 neutral hyperplanes ℓ^0 where $v_{111} = 0$ and the other two cells of the line of $f \in P_{113}$ have $v_c = +1$ (the 15 box pairs of (4) and the 3 forms of (2) for which q_{111} is a form of P_{113}; the facets R^0_j have $q_{111} = f$ by (1)), the identity $f(n) - p_{111} = \sum_{c\in\mathrm{supp}(f)\setminus\{111\}} p_c$ shows that $x - f(n) = O(\ell^0)$ on all $|e_{\ell^0}|$ escaping branches, so expanding $E(x; n)$ in powers of $x - f(n)$ yields the stated orders by the argument of Proposition 6.1. □

Remark 6.11. We verified Proposition 6.10 across the dataset of exact eliminants as follows:

(1) Part (1) holds on all 2,548 prime checks ($\ell^0(n) \equiv 0 \pmod{\pi}$, $\pi \geq 11$, 60 tables), part (2) on all 614 nondegenerate prime checks ($\pi \geq 17$, 40 tables), part (3) on all 267 prime checks ($\pi \geq 17$, 40 tables), part (4) at $m = 0$ across 80 exact eliminants (total forced degree 270), part (5) on 6 tables (two for each axis), and part (6) on 4 tables covering both orbits of neutral margins, the last two modulo $\pi = 1{,}000{,}003$ on tables with $\ell^0(n) = \pi$.

(2) Along the line of tables in Section 3, two integer tables ($t = 31$ and $t = 33$) lie on neutral difference hyperplanes $\ell^0 \in D^0$ (one in the quadratic-square orbit $111 \leftrightarrow 131$ and one in the tensor-product orbit $111 \leftrightarrow 311$), where the exact eliminant is the degree-507 surviving factor $E_{\mathrm{surv},\ell^0}(x; n_0)$. At both points, multiplying $E_{\mathrm{surv},\ell^0}(x; n_0)$ by the closed-form degree-4 polynomial $E_{\ell^0}(x; n_0)$ of Proposition 6.10(2) reproduces all 512 coefficients of $E(x; n_0)$ (matching $F_0(n_0)$, $F_{511}(n_0)$, the forced factor $F_k(n_0)$ for every $0 \leq k \leq 511$, and the line interpolation of Q_{510}).

Remark 6.12. On a generic line $n_0 + td$, the cofactors $Q_1, \dots, Q_{510}$ have $\sum_{k=1}^{510}(\deg Q_k + 1) =$ 265,953 coefficients, 170,585 of them for $170 \le k \le 482$, where $\deg Q_k$ decreases from 700 to 388; the maximum is $\deg Q_{169} = 701$. The 222 neutral factorizations give $\sum_{\ell\in\mathcal{L}_0} |e_\ell| = 5{,}574$ linear conditions per line (1,431 of them from the closed forms for the 66 neutral facets of exponent one, the 51 forms of D^0, the 15 of B^0_{223}, and the 63 of $P^0_{112/122}$), and the 103 active boundary eliminants give $\sum_{\ell\in\mathcal{L}_+\cup\mathcal{L}_-} |e_\ell| = 1{,}229$. Data on a set of hyperplanes cannot distinguish $Q_k(n_0+td)$ from $Q_k(n_0+td)+c\prod_\ell \ell(n_0+td)$, the product over these hyperplanes. So even the full restrictions of Q_k to the 222 neutral hyperplanes can determine $Q_k(n_0 + td)$ only when $\deg Q_k < 222$ (that is, for $k \le 8$ or $k \ge 505$), and the restrictions to all 325 hyperplanes only when $\deg Q_k < 325$ ($k \le 13$ or $k \ge 492$). Hence first-order boundary data do not determine Q_k for $14 \le k \le 491$.

Codimension-two pairs and the interior coefficients. The 5,253 unordered pairs of the 103 boundary forms fall into 189 orbits under G_{111}. For each orbit, we computed $E(x; n_0+\pi d)$ at $\pi = 10{,}007$ for a table n_0 on which both forms of the pair vanish and no other boundary form is divisible by π, and read off the root valuations from the lower convex hull of $(k, v_\pi(c_k))$.

Conjecture 6.13. (1) *For every pair $\{\ell_1, \ell_2\}$ of distinct boundary forms, the escaping valuations on $\ell_1 = \ell_2 = 0$ are the independent union of the escaping branches of ℓ_1 and ℓ_2, except that $m(\ell_1, \ell_2)$ branches follow the combined direction $v_1 + v_2$, with valuation $\operatorname{sgn}(e_{\ell_1}) + \operatorname{sgn}(e_{\ell_2})$ at 111. The nonzero values of $m(\ell_1, \ell_2)$ up to G_{111} are listed in Table 2. For $B_{233} \times N$, seven branches combine with the valuation-one branches of N to give valuation -2, and one combines with the junction branch of N to give valuation -3.*

(2) *Inside the boundary problem of ℓ_1, the end coefficients of the primitive boundary eliminant E_{ℓ_1} factor as*

$$E_{\ell_1}(0; n_0) = \Delta^-_{\ell_1}(n_0) \prod_{f\in\mathcal{L}_+\setminus\{\ell_1\}} f(n_0)^{m(\ell_1,f)}, \quad \operatorname{lead}(E_{\ell_1})(n_0) = \Delta^+_{\ell_1}(n_0) \prod_{f\in\mathcal{L}_-\setminus\{\ell_1\}} f(n_0)^{m_{\mathrm{tot}}(\ell_1,f)},$$

up to integer content on $\ell_1(n_0) = 0$, where $m_{\mathrm{tot}} = m$ except $m_{\mathrm{tot}}(B_{233}, N) = 7+2\cdot 1 = 9$, and $\Delta^-_{\ell_1} = 1$ for $e_{\ell_1} > 0$ while $\Delta^+_{\ell_1} = 1$ for $e_{\ell_1} < 0$. Every middle coefficient $a_{\ell_1,k}(n_0)$ of E_{ℓ_1} is divisible by

$$\prod_{f\in\mathcal{L}_+\setminus\{\ell_1\}} f(n_0)^{\max(0,\, m(\ell_1,f)-k)} \prod_{f\in\mathcal{L}_-\setminus\{\ell_1\}} f(n_0)^{o_{\ell_1,f}(k)},$$

where $o_{\ell_1,f}(k) = \max(0,\, m(\ell_1, f) - (|e_{\ell_1}| - k))$ for $f \ne N$ and for $f = N$ when $\ell_1 \notin B_{233}$, while $o_{B_{233},N}(29) = 9$ and $o_{B_{233},N}(k) = \max(0,\, 8 - (29 - k))$ for $k < 29$.

For two facets ℓ_1, ℓ_2, $m(\ell_1, \ell_2)$ is the maximum likelihood degree of the model with structural zeros on $\operatorname{supp}(\lambda_1) \cup \operatorname{supp}(\lambda_2)$ whenever this face has codimension two, and 0 when it has higher codimension. The end-coefficient factorization of part (2) holds for the exact boundary eliminants E_{ℓ_1} with ℓ_1 a facet, a slab B_{233}, a form of D^+ (Proposition 6.7), or a box B_{223} (Proposition 6.8), in every pair type containing such a form; for pairs of two facets it holds with either form as ℓ_1. This covers all pair types except the two of type $D^- \times D^-$ and the one of type $D^- \times N$, for all of which $m = 0$. Moreover, the recursive middle-coefficient divisibility holds for all 3,540 coefficients of 60 exact boundary eliminants.

The collision discriminant Δ_ℓ of Conjecture 6.3 is Δ^+_ℓ for $e_\ell > 0$ and Δ^-_ℓ for $e_\ell < 0$. Canceling $\Delta_\ell(n_0)$ between Conjectures 6.3 and 6.13 yields a single restriction law for every cofactor $Q_k(n)$ ($0 \le k \le 511$).

orbit ℓ_1	partner orbits ℓ_2 with nonzero overlap $m(\ell_1,\ell_2)$
P_{113}	$P_{113}\colon 85$, $P_{122}\colon 17,9$, $P_{112}\colon 17,7$, $D^-\colon 4$, $P_{222}\colon 1$, $T\colon 1$, $B_{233}\colon 8$, $N\colon 2$
P_{122}	$P_{113}\colon 17,9$, $P_{122}\colon 3$, $P_{112}\colon 7,1$, $D^-\colon 2$, $P_{222}\colon 1$, $T\colon 1$
P_{112}	$P_{113}\colon 17,7$, $P_{122}\colon 7,1$, $P_{112}\colon 3$, $D^-\colon 1$, $T\colon 1$, $P_{111}\colon 1$, $B_{233}\colon 2$
B_{233}	$P_{113}\colon 8$, $P_{112}\colon 2$, $D^+\colon 2$, $B_{223}\colon 5,1$, $B_{233}\colon 4$, $N\colon 7{+}1$
B_{223}	$D^+\colon 2,1$, $B_{233}\colon 5,1$, $N\colon 1$
D^+	$D^+\colon 2$, $B_{223}\colon 2,1$, $B_{233}\colon 2$

TABLE 2. Nonzero codimension-two overlap counts $m(\ell_1,\ell_2)$ among the 189 pair types under G_{111} (types with $m=0$ omitted; since m is symmetric, the rows of D^-, P_{222}, P_{111}, T, and N are contained in the rows shown). When two values are listed, they correspond to different relative positions of the two forms; a single value may cover several relative positions.

Conjecture 6.14. *Let $\ell\in\mathcal{L}_+$ and let n_0 be generic with $\ell(n_0)=0$. For every $0\le k\le 511$,*

$$Q_k(n_0)\big|_{\ell(n_0)=0}=\widetilde{Q}_{\ell,k}(n_0)\prod_{f\in\mathcal{L}_+\setminus\{\ell\}}f(n_0)^{\tau_k^+(\ell,f)}\prod_{f\in\mathcal{L}_-\setminus\{N\}}f(n_0)^{\psi_k^-(\ell,f)}\cdot N(n_0)^{\nu_k(\ell)}$$

for some integer $\nu_k(\ell)\ge 0$. Here $\widetilde{Q}_{\ell,k}$, the reduced cofactor, equals the kth coefficient of E_ℓ divided by its divisor in Conjecture 6.13(2) for $0\le k\le e_\ell$, and equals the $(k-e_\ell)$th coefficient of E_{surv} divided by the product of the powers of the forms $f\neq\ell$ forced by the valuations of its roots on $f=0$ (as in Proposition 6.1) for $e_\ell\le k\le 511$ (both expressions equal $\Delta_\ell(n_0)$ at $k=e_\ell$). The exponents are given by the same-side tent and the opposite-side interior plateau:

$$\tau_k^+(\ell,f)=\max\Big(0,\ \min\big(k,\ e_\ell,\ e_f,\ e_\ell+e_f-k\big)-m(\ell,f)\Big),$$

$$\psi_k^-(\ell,f)=\max\Big(0,\ \min\big(m(\ell,f),\ k-(e_\ell-m(\ell,f)),\ (511-k)-(|e_f|-m(\ell,f))\big)\Big).$$

The analogous statement holds for $\ell\in\mathcal{L}_-\setminus\{N\}$ with k replaced by $511-k$ and $\mathcal{L}_+$ and $\mathcal{L}_-$ exchanged, the power of N again set apart. If $m(\ell,N)=0$, then $\nu_k(\ell)=0$ for $\ell\in\mathcal{L}_+$, and

$$\nu_k(\ell)=o\big(511-\max(0,\,511-k-|e_\ell|)\big)-o(k)\qquad(\ell\in\mathcal{L}_-\setminus\{N\}),$$

with o as in Proposition 6.1.

The power of N is set apart because the forced profile $o(k)$ of N begins $17,15,14,\ldots$: the junction branch of valuation -2 makes it differ from the profile $\max(0,|e_N|-(511-k))$ of the other forms. For instance, for $\ell\in D^+$ the exponent $\nu_{510}(\ell)=2$ is the power $N(n_0)^2$ in the formula for $Q_{511-j}(n_0)$ following Proposition 6.9.

Remark 6.15. We tested Conjecture 6.14 in four ways:

(1) Across all $189\times512=96{,}768$ pairs of a pair type and a coefficient index k at $\pi=10{,}007$, the measured valuation $v_\pi(c_k(n_0+\pi d))$ equals the order predicted by Table 2 in 96,761 cases (99.993%), exceeds it by 1 in 7 cases (consistent with the $96{,}768/10{,}007\approx 9.7$ accidental factors of π expected at random), and is never smaller. For the three mixed pair types with $m>0$ ($P_{113}\times B_{233}$ with $m=8$, $P_{113}\times N$ with $m=2$, and $P_{112}\times B_{233}$ with $m=2$), the unreduced cofactor Q_k carries the flat plateau order m across the entire interior $e_{\ell_1}\le k\le 511-|e_{\ell_2}|$, where ℓ_1 is the constant-side and ℓ_2 the lead-side form.

(2) Along the line n_0+td of Section 3, both $Q_{510}(n_0+td)$ (degree 45, interpolated from 46 exact eliminants and agreeing with the other 13) and $Q_1(n_0+td)$ (degree

59, interpolated from the exact eliminants together with the value at one point on a facet of exponent one, and agreeing with the values at the other 23 such points), interpolated independently of the boundary values used after Proposition 6.9, satisfy the exact boundary-product divisibility of Conjecture 6.14 at every hyperplane root $t_\ell = -\ell(n_0)/\ell(d)$ (46 of 46 on the constant side and 33 of 33 on the lead side), as well as on 8 of 8 coordinate rays in $\{2,3\}^3$ at $N = 0$.

(3) Along the coordinate ray of the cell 122, $Q_2(n)$ (total degree 85) has exact univariate degree 14 (its 15-th divided difference vanishes on 16 tables), and its evaluations at the 9 constant-side hyperplanes through cell 122 satisfy the tent divisibility $\tau_2^+(\ell, f)$ in both the bottom regime $k \le e_\ell$ (P_{122} and D^-) and the top E_{surv} regime $k > e_\ell$ (P_{222} and T).

(4) In a single cell variable n_c, the degree $\deg_{n_c}(c_k)$ is concave in k: at the endpoints $(\deg_{n_c}(c_0), \deg_{n_c}(c_{511}))$ equals $(870, 359)$ for cell 111, $(270, 258)$ for the 6 cells of type 112, $(31, 195)$ for the 12 cells of type 122, and $(129, 146)$ for the 8 cells of type 222, with initial slope $+1$ ($\deg_{n_{122}}(c_k) = 31 + k$ for $k = 0, 1, 2$ and $\deg_{n_{333}}(c_k) = 146 + (511 - k)$ for $k = 511, 510$).

Remark 6.16. The full eliminant is known modulo two primes on one generic line. Extend the line $n_0 + td$ of Section 3 to the 759 integer tables with $0 \le t \le 760$ other than $t = 31, 33$. Interpolation in t of their exact eliminants modulo the primes 2,147,483,629 and 2,147,483,587 gives every cofactor $Q_k(n_0 + td)$, with at least 57 held-out tables for each k. Every Q_k attains the degree $870 - k - \deg F_k$, so the cofactors carry exactly the 265,953 coefficients counted in Remark 6.12, and the interpolants reproduce all 512 coefficients of the exact eliminants at $t = -45$ and $t = 1{,}000$, outside the range of interpolation. At 61 of the rational points where the line meets exactly one of the 325 hyperplanes, modulo both primes, the interpolated $E(x; n_0 + td)$ is divisible by the eliminant $\bar{E}(x)$ computed directly at that point, as Conjecture 6.3 and Proposition 6.10 predict: the quotient is x^{e_ℓ} up to a constant for the 10 points on $\mathcal{L}_+$ (D, P_{111}, P_{112}, P_{113}, T), a constant with $\deg \bar{E} = 511 - |e_\ell|$ for the 15 points on $\mathcal{L}_-$ (B_{223}, B_{233}, D, N; for N, where 16 branches escape, $\deg \bar{E} = 495$), and a polynomial of degree $|e_\ell|$ for the 36 points on neutral hyperplanes (B^0_{223}, B^0_{233}, D^0, $P^0_{111/222}$, $P^0_{112/122}$, T^0).

7. The eliminant as a residual resultant

Let $X_A \subset \mathbb{P}^{26}$ be the projective toric variety of the model for a scaling $\tau \in (\mathbb{C}^*)^{27}$, the closure of the tables $\tau_{ijk} a_{ij} b_{ik} c_{jk}$, of dimension 18. The conditions of maximum likelihood are linear forms on X_A with coefficients linear in n:

$$h_r(p) = (Ap)_r - \frac{b_r}{N} \sum_j p_j \quad \text{(18 independent forms)}, \qquad h_x(p) = p_{111} - \frac{x}{N} \sum_j p_j.$$

For the Segre variety, the resultant of the analogous forms is the eliminant [6]. A resultant of linear forms on a variety is the Chow form of the variety evaluated at their common zero set [4, Ch. 3].

Lemma 7.1. *The forms $h_1, \dots, h_{18}, h_x$ vanish on the linear space $\mathbb{P}(V_x)$, where*

$$V_x = \{\kappa + \mu\, n : \ \kappa \in \ker A, \ \mu \in \mathbb{C}, \ \kappa_{111} = \mu\,(x - n_{111})\},$$

of dimension 8. *Hence* $\operatorname{Res}_{X_A}(h_1, \dots, h_{18}, h_x)$ *is the Chow form of X_A at $\mathbb{P}(V_x)$.*

Proof. A table p satisfies $h_r(p) = 0$ for all r exactly when Ap is a multiple of $b = An$, that is, $p = \kappa + \mu n$ with $\kappa \in \ker A$. Since $\sum_j \kappa_j = 0$, the condition $h_x(p) = 0$ reads $\kappa_{111} + \mu n_{111} = x\mu$. □

For the uniform scaling, every h_r vanishes on $X_A \cap \mathbb{P}(\ker A)$, the closure of the zero-margin locus Σ, for every n, so the resultant vanishes identically.

Conjecture 7.2. (1) *For a generic scaling τ, the resultant $\mathrm{Res}_{X_A}(h_1, \dots, h_{18}, h_x)$ is, up to a factor independent of x, a polynomial of degree $\deg X_A = 2{,}457$ whose roots are the values of p_{111} at the critical points, and its leading coefficient does not depend on n.*

(2) *For the uniform scaling, $E(x; n)$ is the residual resultant of the same forms with respect to the base locus $X_A \cap \mathbb{P}(\ker A)$, in an extension of the construction of [3], since this base locus is not a complete intersection in X_A. Equivalently, E is the limit of the eliminants for scalings $\tau \to \mathbf{1}$ along a generic line, after division by a power of the parameter and a factor independent of x.*

(3) *Each factor of c_{511} in Conjecture 3.1 comes from a face of the polytope of the model whose discriminant vanishes at $\tau = \mathbf{1}$: the whole polytope for N, the faces $2 \times 3 \times 3$, $2 \times 2 \times 3$, and $2 \times 2 \times 2$ for B_{233}, B_{223}, and D.*

The leading coefficient of the eliminant for two-way tables is the principal A-determinant of the seed [6, Thm. 4.1]; part (3) is the uniform-scaling counterpart, in which each vanishing face discriminant is replaced by a linear form in the counts. The following computation supports parts (2) and (3).

Computation 7.3. Let n be a generic table, and follow the 2,457 critical points along $\tau = \mathbf{1} + s\tau'$ with τ' generic as $s \to 0$. Then 511 of them converge to the 511 critical points of the uniform model, one to each, and the other 1,946 escape with valuation vectors $v = -\lambda$ for the forms of the box types, in the numbers

$$1{,}946 = 4 \cdot 81 + 12 \cdot 27 + 120 \cdot 9 + 218,$$

for the 81 differences, the 27 boxes B_{223}, the 9 slabs, and the total, uniformly over the forms of each type.

The paths were followed by numerical continuation to $s = 10^{-8}$. No other faces occur, and no branch escapes with $v \geq 0$.

8. Open problems

(1) Find an explicit determinantal formula for $E(x; n)$. By Conjecture 7.2, it is a residual Chow form of X_A at a linear space that depends affinely on (n, x). The degree profile $\deg_n c_k = 870 - k$ and the conservation identity $870 = 511 + 359$ match an 870×870 linear pencil

$$M_0(n) + xM_1 = \begin{pmatrix} \mathcal{A}_{511\times 511}(n) + xI_{511} & \mathcal{B}_{511\times 359}(n) \\ \mathcal{C}_{359\times 511}(n) & \mathcal{D}_{359\times 359}(n) \end{pmatrix}$$

with $M_0(n)$ linear in n, $\det M_0(n) = c_0(n)$ decomposed over the 60 facet strata at $x = 0$, and $\det \mathcal{D}(n) = c_{511}(n)$ decomposed over the 43 blow-up strata at $x = \infty$, so that $c_k(n)$ is the sum of the $\binom{511}{k}$ bordered principal minors of $M_0(n)$ of size $870 - k$ containing $\mathcal{D}(n)$.

(2) The boundary data of Section 6 determine Q_1 and Q_{510} on every hyperplane of $\mathcal{L}_+$ and $\mathcal{L}_-$, as well as the factors E_{ℓ^0} on all 222 neutral hyperplanes. However, by Remark 6.12, they reach only $k \leq 13$ and $k \geq 492$, and since near $\ell = 0$ the boundary system gives only the $|e_\ell|$ special roots, each further order of the expansion in ℓ adds at most $\sum_\ell |e_\ell| = 6{,}803$ conditions per line. Find the additional structure—presumably the determinantal formula of (1)—that determines the cofactors Q_k for $14 \leq k \leq 491$.

(3) Compute the number of escaping branches per form in Computation 7.3 (4, 12, 120, and 218) as excess intersection numbers of the zero-margin loci.
(4) Prove Conjecture 4.2 for the box types. For $3 \times 3 \times 4$ tables the Euler characteristics predict exponents -14 for the box $n[12\,|\,12\,|\,\cdot]$, -157 for the slab $n[12\,|\,\cdot\,|\,\cdot]$, 1,289 for the margin n_{11+}, and -341 for the total, where the zero-margin locus has $(r-2)(r^6 - 16r^5 + 124r^4 - 578r^3 + 1641r^2 - 2607r + 1776)$ points over a field with r elements, modulo scaling. Following all 5,755 critical points [8] to generic points of the box, slab, and total hyperplanes gives exactly 14, 157, and 340 escaping branches, the last made up of 339 branches of valuation -1 and one of valuation -2, so that the exponents are -14, -157, and -341.

Appendix A. A worked example

Let n be the table with slices $i = 1, 2, 3$ (rows j, columns k)

$$\begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 5 & 8 \\ 9 & 7 & 9 \\ 3 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 8 & 4 & 6 \\ 2 & 6 & 4 \\ 3 & 3 & 8 \end{pmatrix}.$$

Iterative proportional fitting from the uniform table converges to the fitted table $\hat{p}$ with slices

$$\begin{pmatrix} 1.737030 & 2.126778 & 4.136193 \\ 2.226552 & 5.577997 & 7.195451 \\ 2.036418 & 4.295225 & 6.668357 \end{pmatrix}, \quad \begin{pmatrix} 5.959376 & 3.516910 & 6.523715 \\ 6.769211 & 8.173895 & 10.056894 \\ 2.271414 & 2.309195 & 3.419391 \end{pmatrix},$$

$$\begin{pmatrix} 6.303595 & 4.356312 & 7.340093 \\ 3.004237 & 4.248108 & 4.747655 \\ 3.692168 & 4.395580 & 5.912252 \end{pmatrix},$$

which has the two-way margins of n. Each sweep rescales the table to match n_{ij+}, then n_{i+k}, then n_{+jk}; the cumulative products of these adjustments are the raking factors a_{ij}, b_{ik}, c_{jk}, with $\hat{p}_{ijk} = a_{ij} b_{ik} c_{jk}$:

$$a = \begin{pmatrix} 2.741512 & 4.929475 & 4.294733 \\ 5.272987 & 8.401927 & 2.685582 \\ 5.988414 & 4.003527 & 4.686959 \end{pmatrix}, \quad b = \begin{pmatrix} 0.519248 & 0.980734 & 1.501080 \\ 0.926194 & 0.843187 & 1.230925 \\ 0.862649 & 0.919658 & 1.219503 \end{pmatrix},$$

$$c = \begin{pmatrix} 1.220231 & 0.791008 & 1.005094 \\ 0.869876 & 1.153789 & 0.972419 \\ 0.913179 & 1.019761 & 1.034377 \end{pmatrix}$$

(the first index labels rows; all values rounded to six decimals). The fit converges to 55 digits in 52 sweeps. The limit $\hat{p}$ does not depend on the order of the adjustments, but the factors do: they are determined only up to $a_{ij} \mapsto a_{ij} u_i v_j$, $b_{ik} \mapsto b_{ik} w_k / u_i$, $c_{jk} \mapsto c_{jk} / (v_j w_k)$, and the order fixes one choice.

The eliminant $E(x; n)$, normalized to be primitive with a positive leading coefficient, has degree 511 and coefficients of 555 to 964 decimal digits, 433,334 digits in all. They are in the companion repository

https://github.com/shankarsivarajan/ipf-3x3x3-polynomial,

together with code that computes the limit, the raking factors, the eliminant, and the checks below for any $3 \times 3 \times 3$ table.

(1) The fitted cell $\hat{p}_{111} = 1.737029617305015143389589415969605131156 1\ldots$ is a root of E: at $\hat{p}_{111}$ computed to 60 digits, the Newton correction $|E/E'|$ is 2.8×10^{-58}.

(2) The values of the 103 forms of Table 1 are in Table 3. Let γ be the integer for which $\gamma c_{511} = \prod_{\ell \in \mathcal{L}_-} \ell(n)^{|e_\ell|}$; it has 68 digits. Then $\gamma c_0 = -\prod_{\ell \in \mathcal{L}_+} \ell(n)^{e_\ell}$, as in Conjecture 3.1, so the product of the 511 roots is $\prod_{\mathcal{L}_+} \ell(n)^{e_\ell} / \prod_{\mathcal{L}_-} \ell(n)^{|e_\ell|}$.

(3) For every k, γc_k is an integer divisible by the forced factor $F_k(n)$ of Proposition 6.1. For $k = 1$ the cofactor $Q_1(n)$ has 93 digits.

(4) E has 109 real roots, 81 of them positive, and the scaling equations have 109 real solutions. Exactly one of these is a positive table, namely $\hat{p}$, and $\hat{p}_{111}$ is the 23rd smallest of the 81 positive roots. A positive root of E alone does not identify the limit.

class	e_ℓ	values $\ell(n)$
P_{113}	169	6, 8, 14
P_{112}	17	20, 25, 27, 28, 31, 32
P_{122}	17	19, 21, 21, 22, 24, 25, 27, 27, 27, 28, 30, 32
P_{222}	1	29, 33, 37, 39, 39, 42, 46, 49
P_{111}	1	45
T	1	29, 32, 35, 36, 37, 37, 38, 38, 40, 40, 41, 42, 42, 43, 43, 43, 44, 45, 46, 47, 47, 47, 48, 48
D^-	4	31, 34, 34, 35, 40, 45
N	-17	129
B_{233}	-29	73, 77, 80, 85, 90, 94
B_{223}	-6	43, 44, 45, 47, 53, 53, 54, 55, 56, 59, 64, 66
D^+	-4	7, 11, 13, 14, 17, 17, 17, 19, 19, 21, 24, 24, 25, 25, 25, 26, 28, 28, 28, 31, 32, 35, 36, 38

TABLE 3. The values of the 103 forms on the table of Appendix A, by class, in increasing order.

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