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BALANCED AND SUPERBALANCED HOLOGRAPHIC ENTROPY INEQUALITIES WITH ARBITRARILY LARGE CONTRACTION SCALING

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Abstract. A proof by contraction of a holographic entropy inequality may exist only after multiplication by a positive integer, the least of which is its least scaling $k_{\min}$. We show that balance (every party has the same total coefficient on both sides) and superbalance (balance after any party is exchanged with the purifier) do not bound it: there are valid balanced and superbalanced inequalities with integer coefficients without a common factor whose least scaling is finite and arbitrarily large, growing faster than any power of the number N of parties. For the balanced inequalities it is finite on the full cube of bit strings. For the superbalanced ones it is finite on the realizable domain, the bit strings on which a map already proves validity, and its finiteness on the full cube is undecided. The lower bounds hold on both domains. An explicit family for $3j+4+\binom{3j+2}{j+1}$ parties, $j \geq 1$, requires a scaling divisible by $2j+1$ whenever $2j+1$ is prime, and its balanced member for 17 parties has least scaling exactly 3. Disjoint sums of inequalities built from 3-designs on projective lines over finite fields give $\ln k_{\min} \geq (1-\varepsilon)(\ln N)^2/(50 \ln 2 \cdot \ln\ln N)$ for every $\varepsilon > 0$ and large N, and $\ln k_{\min} \geq c_0 N^{1/26}$ for some $c_0 > 0$ and large N with Linnik's theorem in the form of Xylouris. The designs are obtained by a random choice. The lower bounds come from a count of right-hand side terms on a subcube, and the finiteness follows from an explicit rational map. The results concern contraction proofs, and the constraints on holographic states remain those of the cone. The values for 17 parties rest on exact integer checks of explicit maps.

1. Introduction

In a holographic theory the entanglement entropy of a boundary region in a static state is given, to leading order in the expansion in Newton's constant G_N, by the area of a minimal bulk surface homologous to the region, divided by $4G_N$ [1–4]. Lewkowycz and Maldacena gave an argument for this formula of Ryu and Takayanagi from Euclidean gravity solutions [5]. The formula constrains the entropies of several regions at once. Bao, Nezami, Ooguri, Stoica, Sully and Walter showed that the entropy vectors of N regions obtained from it form a polyhedral cone, the holographic entropy cone HEC_N, and they describe every linear inequality that holds on the cone as a necessary condition for a state to have a smooth holographic dual [3]. Such an inequality is called valid. Subadditivity, strong subadditivity and the monogamy of mutual information are the simplest examples [6, 7], and the latter two also have proofs by bit threads [8, 9]. The cone is known completely for up to five regions [3, 10]. For six regions more than 1800 further inequalities have been found [11], and infinite families of valid inequalities are known [3, 12], two of which consist of facets of the cone [13].

The general method for proving that an inequality is valid is the proof by contraction of [3], which extends the proofs of strong subadditivity and monogamy. Physically it is a cutting and regluing of surfaces. The minimal surfaces of the regions on the left-hand side (LHS) of the inequality cut the bulk slice into pieces, and a piece is labeled by a string of bits that records, for each LHS term, on which side of the corresponding surface it lies. A proof assigns every piece to some of the regions of the right-hand side (RHS), and for each RHS term the pieces assigned to it are reassembled

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into a bulk region whose boundary is homologous to that term [3]. Akers, Hernández-Cuenca and Rath set out the proof in this geometric form [14]. The boundary of the reassembled region is a candidate surface, made of parts of the LHS surfaces, and its area is at least the entropy of the RHS term. The inequality follows if the candidate surfaces together use each part of an LHS surface at most as often as the LHS provides it. On the labels this requirement is a contraction condition: the assignment, a map from the cube of bit strings to the cube of the RHS terms, must not increase a weighted Hamming distance, and at the labels of the boundary regions its values are prescribed. The proof uses only that the bulk slice is a Riemannian manifold in which the surfaces are minimal [14]. Bao et al. carry out the proof on weighted graphs [3], and the existence of a map is a finite combinatorial question. Contraction maps and their construction have been studied further in [15–19], Bao and Naskar give an argument for the completeness of the method [17], and the method has been extended to hypergraphs [20] and to multi-entropies [21].

Whether a valid inequality has such a proof depends on how the inequality is written. Already [3] note that rescaling an inequality, or replacing an RHS term with coefficient β by β copies of the term, gives the map more freedom. In the language of this paper a map for k times the inequality assigns to each bulk piece and each RHS term an integer between 0 and k times the coefficient of the term, the number of copies of the term that receive the piece. After division by k the piece is assigned to the RHS regions in fractions with denominator k, and for an RHS term with coefficient β the product of β and its entropy is bounded by $1/k$ times the total area of $k\beta$ candidate surfaces. The least k for which a map exists, the *least scaling* $k_{\min}$ of the inequality, measures how finely a contraction proof has to subdivide the bulk pieces among the RHS terms. It is a property of the proof and leaves the inequality and its physical content unchanged.

The least scaling enters in three places. First, a proof is found by a search for a map, by a greedy extension [3], by deterministic rules [17] or by a Boolean satisfiability (SAT) solver [22], and the search space grows with k, so an inequality whose least scaling is large is out of reach of a search at $k = 1$ or at small k. For validity alone the scaling can be bypassed: if the numbers of copies are replaced by real numbers in the same ranges, the conditions on a map form a linear program whose size does not depend on k, and a solution already proves validity (Sections 2.3 and 2.4). Where an integer map itself is wanted, in the searches just named and in the results described next, whose hypothesis is an integer map, the least scaling is the smallest multiple that can be used.

Second, contraction maps are the input of several arguments beyond the static case. For time-dependent states the entropy is given by the covariant formula of Hubeny, Rangamani and Takayanagi [23, 24]. Grimaldi, Headrick and Hubeny describe it as unknown whether the inequalities hold for this formula, although significant evidence has accumulated that they do [25]. Wall proved strong subadditivity and monogamy for it with maximin surfaces, in spacetimes that obey the null curvature condition [26], and Rota and Weinberg showed that, for five or fewer regions, the maximin method in its current form proves no further inequality [27]. Several results of this kind assume a contraction map of a specified form. Czech and Dong show that in two boundary, or three bulk, dimensions all inequalities that can be proven in the static case by contraction maps also hold in holographic states with time dependence [28]. Grimaldi, Headrick and Hubeny characterize validity for the light-cone configurations they study by a majorization test on the null reductions of an inequality, the inequalities formed by the terms that contain a given party [25]. That the null reductions pass this test is proved for superbalanced contraction-provable inequalities by Grimaldi, Headrick, Hubeny and Shteyner, in a preprint [29], and for balanced inequalities that can be proven by contraction by Czech, Feng, Wu and Xie [30]. Czech and Shuai likewise state their results on the holographic renormalization group for inequalities that can be proven by contraction [31]. Section 10 discusses which of these results apply to our inequalities. In short, by our inference

those whose input is a map with one bit for each LHS term apply to the balanced ones, and for the other results and for the superbalanced inequalities the question stays undecided.

Third, the question whether the method is complete is under study. Avis and Hernández-Cuenca found that a valid five-party inequality listed by Bao et al. [3] has a map only after its RHS coefficients are split into unit copies, and asked in their Problem 3 whether this splitting makes the method necessary for validity [15]. Bao, Furuya and Naskar propose a deterministic method to find all inequalities that have contraction maps [18]. In the arXiv version of 14 July 2026 of a later article, Bao, Furuya and Naskar answer the question of completeness in the affirmative for all linear inequalities with rational coefficients, where the non-existence of a map means that none exists for any presentation obtained by merging or expanding terms and uniformly rescaling the coefficients [32]. The least scaling measures how large the rescaling that this statement allows has to be.

We study the least scaling on two classes of inequalities. An inequality is *balanced* if every party has the same total coefficient on both sides, and *superbalanced* if this remains true after any party is exchanged with the purifier [3, 33, 34]. Balance and superbalance express the cancellation of the ultraviolet divergences of the entropies in the inequality and in its purifications. He, Hubeny and Rangamani state, as their Corollary 2, that all non-redundant holographic entropy inequalities besides subadditivity are superbalanced [34].

In the first paper of this series [22] we showed that the rescaling is needed. A balanced seven-party inequality has no map at $k = 1$ and has one after doubling. Superbalanced inequalities for six to eight parties have no map at $k = 1$, on the full cube and on their realizable domains, so Problem 3, in the formulation with merged left-hand side and no scaling, has a negative answer [22]. There are also star inequalities, in which the LHS contains only single-party entropies, with $S([N]) = S(O)$ counted as that of the purifier, whose least scaling grows faster than exponentially in N [22]. These star inequalities are implied by subadditivity, however [22]. The star inequalities of [22] with large least scaling are not balanced, and a balanced star inequality has least scaling 1 or ∞ [22]. Among the balanced and superbalanced inequalities without a map at $k = 1$ the least scaling was 2 wherever it was determined [22]. This leaves the question whether balance or superbalance bounds the least scaling.

This paper answers the question in the negative. *Balance and superbalance do not bound the scaling that a contraction proof needs: there are valid balanced and superbalanced inequalities with integer coefficients of greatest common divisor* 1 *whose least scaling is arbitrarily large and finite, on the full cube for the balanced ones and on the realizable domain defined below for the superbalanced ones, and the lower bounds hold on both domains. It can grow faster than any power of the number of parties.* The statement has two parts, an explicit family and a growth bound, which we state after fixing the terms.

A map for kQ is an integer map in the sense of Definition 2.8: it assigns to every bit string a vector of integer counts, one for each RHS term, with the contraction condition and the prescribed values at the labels of the parties. The least scaling $k_{\min}(Q)$ is the least $k \geq 1$ for which kQ has such a map on the full cube of bit strings. Many bit strings are realized by no bulk region [29], for instance because of the nesting of the regions bounded by the surfaces, and by a theorem of Avis and Hernández-Cuenca [15], in the form for real maps of [22, Lemma 2.13], a map on the remaining strings already proves validity. We call the set of these strings the realizable domain, in the precise sense of Definition 2.12, and $k^{\mathcal{A}}_{\min}(Q) \leq k_{\min}(Q)$ is the least scaling on it. A lower bound on $k^{\mathcal{A}}_{\min}$ is the stronger statement, and it is the one that refers to the bit strings compatible with nesting and disjointness.

A balanced inequality is *centered* on a party that lies in all its terms [29], and $C(Q)$ denotes the superbalancing of Q, obtained by subtracting the instances of subadditivity for pairs of parties that make it superbalanced (Definition 7.1). The superbalancing of a valid inequality is valid exactly when the inequality is a superbalanced valid inequality plus a nonnegative combination of instances of subadditivity (Section 7). We prove the validity of the superbalanced inequalities below directly, by maps on their realizable domains. A *disjoint sum* of inequalities on pairwise disjoint sets of parties lists the terms of all of them, and a *padding* adds parties that lie in no term (Section 2.6). Both theorems concern *carriers*, centered inequalities attached to a set of $3j+2$ points and a family of its subsets with $j+1$ points. They are defined in (4.3) and sketched under The method below, and the carrier for $j=1$ is written out after Theorem 1.1. In this paper we give a proof of the following two statements. The first gives inequalities that can be written down, with a scaling that is forced by a prime.

Theorem 1.1 (The explicit family)**.** *For an integer $j \geq 1$ let Q_j be the carrier* (4.3) *of the set of all subsets with $j+1$ elements of a set with $3j+2$ elements, let K_j be the least positive integer for which $Q_{j,\mathrm{int}} = K_j Q_j$ has integer coefficients, and let $C_j = C(Q_{j,\mathrm{int}})$ be the superbalancing of $Q_{j,\mathrm{int}}$. Both are inequalities for $N_j = 3j+4+\binom{3j+2}{j+1}$ parties, with integer coefficients of greatest common divisor* 1*.*

(a) *$Q_{j,\mathrm{int}}$ is valid and centered on one of its parties, hence balanced, and C_j is valid and superbalanced.*

(b) *$k_{\min}(Q_{j,\mathrm{int}}) < \infty$ and $k^{\mathcal{A}}_{\min}(C_j) = k^{\mathcal{A}}_{\min}(Q_{j,\mathrm{int}}) < \infty$.*

(c) *If $kQ_{j,\mathrm{int}}$ or kC_j has an integer map on its realizable domain or on the full cube, then $(2j+1)/\gcd(2j+1, K_j)$ divides k. In particular, if $p = 2j+1$ is a prime, then*

$$p \leq k^{\mathcal{A}}_{\min}(Q_{j,\mathrm{int}}) \leq k_{\min}(Q_{j,\mathrm{int}}) < \infty\ , \qquad p \leq k^{\mathcal{A}}_{\min}(C_j) < \infty\ , \qquad p \leq k_{\min}(C_j)\ .$$

(d) *For $j=1$, where $N_1 = 17$, we have $k^{\mathcal{A}}_{\min}(Q_{1,\mathrm{int}}) = k_{\min}(Q_{1,\mathrm{int}}) = k^{\mathcal{A}}_{\min}(C_1) = 3$.*

For $j=1$ the inequality $Q_{1,\mathrm{int}}$ can be written in two lines. Its 17 parties are A, H, a party q_v for each point v of $B = \{0,1,2,3,4\}$ and a party p_T for each of the ten two-element subsets T of B. With $q_U = \{q_v \colon v \in U\}$ for $U \subseteq B$, $P_v = \{p_T \colon v \in T\}$ and F the set of all parties, and with a set of parties written as the concatenation of its parts, it reads

$$\begin{aligned} 10\sum_{b\in B} S(AHq_{B\setminus\{b\}}P_b) + 13\sum_{v\in B} S(AP_v) + 2S(Aq_B) \\ \geq\ 10S(F) + 2\sum_T \sum_{w\in T} S(F\setminus\{q_w, p_T\}) + 67S(A)\ , \end{aligned}$$

where T runs over these subsets. It has least scaling exactly 3 on both domains, its superbalancing C_1 has least scaling 3 on the realizable domain, and the coefficients of C_1 are listed in Appendix F. For $j=2$ and $j=3$ the inequalities have 66 and 343 parties and force the factors 5 and 7. In this family the forced factor $2j+1$ grows only logarithmically with the number of parties, and the second theorem trades explicitness for growth.

Theorem 1.2 (Growth)**.** (a) *For every $\varepsilon > 0$ there is an N_0 such that for every $N \geq N_0$ there are a valid balanced N-party inequality Q and a valid superbalanced N-party inequality $\widehat{Q}$, both with integer coefficients of greatest common divisor* 1*, with*

$$\ln k^{\mathcal{A}}_{\min}(Q)\ \geq\ (1-\varepsilon)\frac{(\ln N)^2}{50\ln 2\cdot \ln\ln N}\ , \qquad k_{\min}(Q) < \infty\ ,$$

and

$$\ln k^{\mathcal{A}}_{\min}(\widehat{Q}) \;\geq\; (1-\varepsilon)\frac{(\ln N)^2}{50\ln 2\cdot\ln\ln N}\;, \qquad k^{\mathcal{A}}_{\min}(\widehat{Q})<\infty\;.$$

(b) *Let $C_L \geq 1$ and $L > 1$ be constants such that for all integers $n \geq 2$ and a with $\gcd(a,n)=1$ there is a prime $p \equiv a \pmod n$ with $p \leq C_L n^L$. Then there are $c_0 > 0$ and N_0 such that for every $N \geq N_0$ there are Q and $\widehat{Q}$ as in* (a) *with the bound $c_0 N^{1/(5L+1)}$ in place of $(1-\varepsilon)\frac{(\ln N)^2}{50\ln 2\cdot\ln\ln N}$.*

In both parts $k_{\min}(Q) \geq k^{\mathcal{A}}_{\min}(Q)$ and $k_{\min}(\widehat{Q}) \geq k^{\mathcal{A}}_{\min}(\widehat{Q})$, so the lower bounds hold for the least scalings on the full cube as well, and Q is a padding of a disjoint sum of centered inequalities, $\widehat{Q}$ the padding of the disjoint sum of their superbalancings.

The two theorems rest on one mechanism. The inequalities of Theorem 1.2 are disjoint sums of smaller inequalities of the same kind as those of Theorem 1.1, built from set systems with far fewer sets (sketched under The method below), whose existence is proved by a random choice, and the forced factors of the summands multiply. The bound of part (a) already exceeds every power of N and uses only the prime number theorem, and part (b) raises it to $\exp\left(c_0 N^{1/26}\right)$, which still grows more slowly than a^N for every $a > 1$. Constants C_L and L as in Theorem 1.2(b) exist by Linnik's theorem [35, 36], and $L = 5$ is admissible by a theorem of Xylouris [37, Theorem 1], so the exponent in (b) can be taken to be $1/26$.

Whether the superbalanced inequalities of the two theorems have a finite least scaling on the full cube we have not determined. It would follow from the completeness statement of [32] in the form quoted above, which allows a uniform rescaling. Grimaldi, Headrick, Hubeny and Shteyner conclude, under the assumption that the method is complete, that an inequality with a contraction map on the realizable subset also has one on the full cube [29]. Our proofs are independent of these statements. Our results are consistent with the statement of [32]. The balanced inequalities of the two theorems define no facets of the cone, and for the superbalancings of single carriers, among them C_1, this is undetermined (Section 10).

Besides the arguments of this paper, the proofs rest on the theorem of Bao et al. that a contraction map proves validity [3], which we apply to integer maps on the full cube through [22, Lemma 2.9], and on the theorem of Avis and Hernández-Cuenca that a map on the realizable domain proves validity [15, Theorem 4], in the form for real maps of [22, Lemma 2.13]. The exponent $1/26$ comes from the theorem of Xylouris that bounds the least prime in an arithmetic progression by a constant times the fifth power of the modulus [37, Theorem 1], and two standard facts enter, the 3-transitivity of $\mathrm{PGL}(2,q)$ on the projective line and the prime number theorem, see e.g. [38] and [39, Theorem 6.9]. The validity of the superbalanced inequalities of the two theorems is proved by maps on their realizable domains (Theorem 8.16 and, for 17 parties, Appendix B). The proof of Theorem 1.1 is computer-assisted only for $j = 1$, where explicit maps are checked in exact arithmetic (Appendix B), and the proof of Theorem 1.2 uses no computation.

Use of large language models and status of the proofs. This paper was produced by large language models, instances of the Claude models of Anthropic running as autonomous agents, in October and November 2026. They found the constructions, wrote the proofs and the programs of Appendix B, ran the computations and wrote the text. The arguments were checked only by other instances of the same models, acting as referees and recomputing the certificates with programs of their own. Such checks may share the blind spots of the models that wrote the proofs. The author has read the paper, has checked that the references lead to the cited works, and follows its main line: a carrier, built from the claws of a family of blocks as described under The method below, forces a factor into every scaling at which it has a map, and a disjoint sum of carriers with pairwise

coprime forced factors forces the product of these factors, which for suitably chosen carriers grows faster than any power of the number of parties. The author has not checked the proofs in detail, and the statements attributed to the cited works were checked only by the models: each statement was matched to a verbatim passage of the cited work, and another instance of the model ruled on whether the passage supports the sentence. No human expert has checked the proofs. The first paper of this series [22], whose Lemmas 2.9 and 2.13 are inputs of the proofs, was produced and checked in the same way, and no human expert has checked its proofs either.

Code availability. The programs for the computations of Appendix B are available under the MIT license at https://github.com/ahippelainen/contraction-scaling-public.

Conventions. The set of parties is $[N] = \{1, \dots, N\}$ and O is the purifier. A term of an inequality that contains O is replaced by the entropy of the complementary set of parties, so every term is a set of parties. LHS and RHS stand for the left-hand and the right-hand side, and kQ is the inequality Q with all coefficients multiplied by k. We write $[P]$ for 1 if the statement P holds and 0 otherwise, and ln for the natural logarithm. In asymptotic statements $o(1)$ stands for a quantity that tends to 0 in the limit named there, or as $N \to \infty$ where none is named. From Section 3 on, $j \geq 1$ is an integer, $m = 3j + 2$ and B is a set of m points, the box, and the letter S without a subscript denotes both an entropy $S(I)$, with a set I of parties as argument, and a set $S \subseteq B$ of points. The letter L is the number of LHS terms, except in Theorem 1.2(b), in the paragraph after the theorems, in Section 9.3 and in Appendix E, where it is the exponent in Linnik's theorem.

The method. The lower bounds come from a counting argument on a small part of the cube. All inequalities of this paper are built from *carriers*, centered inequalities attached to a set B of $m = 3j + 2$ points, the box, and a family of blocks, subsets of B with $j + 1$ points. A block T with a distinguished point $w \in T$ defines a *claw*, the star graph that joins w to the $2j + 1$ points outside T. Among the terms of the carrier are one LHS term for each point of the box, its box term, and one RHS term for each claw, and at the bit strings that switch on two or three of the box terms a map for κ times the carrier must distribute the amounts κ and 2κ among the claws, each claw receiving at a triple at least what it receives at the pairs inside it. Counting pairs against triples gives the *claw count* of Section 3,

$$\binom{m}{3} = j\binom{m}{2} \qquad \Longrightarrow \qquad (2j+1)\sum_e \mu_e = \binom{m}{2}\kappa \ ,$$

where μ_e is the one amount that the claw e is forced to give to each of its edges. If the map has integer values, $2j + 1$ divides κ. This obstruction is arithmetic. Fractions satisfy the same count, so a map with rational values remains possible, and the next paragraph defines one.

The upper bounds need a map. We define one explicit real-valued candidate, the *water-level map* of Section 4. It treats the amount that the remaining LHS terms add as water that fills the claw terms from the lowest upward, to the level $h(x)$ determined by

$$c \sum_{e \text{ open at } x} \max\left(0, h(x) - g_e(x)\right) = \sigma(x) \ ,$$

where $g_e(x)$ is the content that the box terms give to the claw e at the bit string x, $\sigma(x)$ is the amount to be distributed, c is the coefficient of a claw term, and the sum runs over the claws that the water may enter at x, the open claws of Definition 4.8. The map has rational values, and a common denominator of these values is a scaling at which an integer map exists. Its contraction property reduces to inequalities between numbers of blocks through given points, which hold when

the blocks form a 3-design with few blocks through four or more points (Section 5). The set of all blocks gives Theorem 1.1 for $j \geq 2$, and unions of orbits of $\mathrm{PGL}(2, q)$ on the projective line give Theorem 1.2 (Section 6). For the superbalanced inequalities, a map for $kC(Q)$ restricts to a map for kQ, by the restriction argument that [25, 29, 30] apply to contraction maps at $k = 1$. This transfers the lower bounds (Section 7). On the realizable domain every map for kQ lifts to one for $kC(Q)$ by explicit formulas when the RHS coefficients of Q satisfy a capacity condition, which holds for all carriers of the two theorems except the one for 17 parties (Section 8). For 17 parties, the case $j = 1$, explicit integer maps are checked directly (Appendix B).

The rest of the paper is organized as follows. Section 2 recalls count maps, the realizable domain and the behavior of the least scaling under disjoint sums. Section 3 proves the claw count. Sections 4 and 5 define the carriers and the water-level map and prove that the map is a contraction, and Section 6 supplies the two families of designs. Sections 7 and 8 treat the superbalanced inequalities. Section 9 assembles the proofs of the two theorems. We conclude in Section 10, which discusses what the results mean for holographic states and for the arguments that take a contraction map as input, and lists what the results leave undecided. For better readability of the main line, long proofs, estimates and case lists are in the appendices: Appendix A for Section 2, Appendix C for Sections 4–6, Appendix D for Sections 7 and 8, and Appendix E for the growth bounds. Appendix B describes the computations, Appendix F writes out the 17-party inequalities, and Appendix G proves that the balanced inequalities define no facets.

What is new. As far as we could determine, the following are new. First, valid balanced inequalities with integer coefficients of greatest common divisor 1 whose least scaling is finite and exceeds any given bound, and valid superbalanced inequalities in the same normalization with this property on the realizable domain, in both classes with a least scaling that grows faster than any power of N (Theorems 1.1 and 1.2). Second, the methods behind them: the claw count, which forces a factor into every integer map of a carrier, the carriers and the water-level map, which make the carriers valid with a finite least scaling, and the lift of maps to the realizable domain of the superbalancing under a condition on the RHS coefficients (the capacity condition of Section 8.4).

In the claw count the divisibility itself is the necessary condition on the number of edges for a decomposition of a complete multigraph into stars of one size (Section 3). What is new there is that the contraction conditions force such a decomposition. Proof by contraction is due to Bao et al. [3], and the realizable domain to Avis and Hernández-Cuenca [15]. For contraction maps, the characterization of the maps of a centered inequality by grading, monotonicity and one inequality for each party is Theorem 2 of Grimaldi, Headrick, Hubeny and Shteyner [29]. The restriction of a map to a null reduction was used by Grimaldi, Headrick and Hubeny [25], by Grimaldi, Headrick, Hubeny and Shteyner [29] and by Czech, Feng, Wu and Xie [30]. The product of the maps of two summands is a construction of Grimaldi, Headrick, Hubeny and Shteyner [29]. That orbits of $\mathrm{PGL}(2, q)$ on the sets of points of the projective line, and unions of such orbits, are 3-designs is known, see e.g. [38, 40]. Integer scales of a similar kind occur in other problems: the minimum scale of hypercube embeddings [41], the fractionality of multiflow problems [42] and the number of copies in proofs of entropy inequalities by the copy lemma [43].

2. Preliminaries

In this section we restate the definitions of [22] that we use, in the same notation, and state the general facts on which the proofs of Theorems 1.1 and 1.2 rest. Every map of a centered inequality is graded, monotone and vanishes below the occurrence words of the parties outside a term (Lemma 2.14), which is the input of the lower bounds, and conversely these properties make

a function a map (Lemma 2.15), which is how we prove validity. A disjoint sum of two inequalities has as scaling spectrum the intersection of their spectra (Lemmas 2.18 and 2.19), which is how Section 9 multiplies lower bounds. The longer proofs are in Appendix A.

2.1. **The holographic entropy cone.** A *graph model* is a finite undirected graph (V, E) with nonnegative edge weights and a labeling $b\colon \partial V \to [N] \cup \{O\}$ of a set $\partial V \subseteq V$ of boundary vertices, where $[N] = \{1, \dots, N\}$ is the set of parties and O is the purifier. For $I \subseteq [N]$, an *I-cut* is a set $W \subseteq V$ with $W \cap \partial V = b^{-1}(I)$, its weight $\mathrm{cut}(W)$ is the total weight of the edges with exactly one endpoint in W, and the entropy of I is $S(I) = \min\{\mathrm{cut}(W)\colon W \text{ an } I\text{-cut}\}$. Since $\mathrm{cut}(W) = \mathrm{cut}(V \setminus W)$, the entropy of a set containing O is that of its complement in $[N] \cup \{O\}$. Bao, Nezami, Ooguri, Stoica, Sully and Walter showed that the holographic entropy cone $\mathrm{HEC}_N \subset \mathbb{R}^{2^N-1}$, defined through the Ryu–Takayanagi formula [1], is the set of entropy vectors $S = (S(I))_{\emptyset \neq I \subseteq [N]}$ of graph models, and that it is a rational polyhedral cone [3].

An entropy inequality is written

$$Q: \qquad \sum_{l=1}^{L} \alpha_l S(I_l) \;\geq\; \sum_{r=1}^{R} \beta_r S(J_r) \;, \tag{2.1}$$

with L left-hand side (LHS) terms, R right-hand side (RHS) terms, positive rational coefficients α_l, β_r and nonempty sets $I_l, J_r \subseteq [N]$, the I_l distinct, the J_r distinct and no I_l equal to a J_r. A term that contains O is replaced by the entropy of the complementary set of parties, so every term is a subset of $[N]$. The same letter denotes the coefficient vector $Q \in \mathbb{R}^{2^N-1}$, so that (2.1) reads $Q \cdot S \geq 0$, and we write $|\alpha| = \sum_l \alpha_l$, $|\beta| = \sum_r \beta_r$, and kQ for the inequality with all coefficients multiplied by k. The inequality is *valid* if $Q \cdot S \geq 0$ for every graph model, and a valid $Q \neq 0$ defines a *facet* of HEC_N if the face $\mathrm{HEC}_N \cap Q^{\perp}$ has dimension $2^N - 2$. For pairwise disjoint nonempty sets X, Y, Z of parties we write $XY = X \cup Y$ and $I(X{:}Y) = S(X) + S(Y) - S(XY)$. Subadditivity and the monogamy of mutual information (MMI) are the valid inequalities

$$I(X{:}Y) \geq 0 \;, \qquad S(XY) + S(XZ) + S(YZ) \geq S(X) + S(Y) + S(Z) + S(XYZ) \;, \tag{2.2}$$

and the second is equivalent to $I(X{:}YZ) \geq I(X{:}Y) + I(X{:}Z)$, the form in which Hayden, Headrick and Maloney proved it for the Ryu–Takayanagi formula [7]. We write $\mathrm{MMI}(X{:}Y{:}Z)$ for the coefficient vector of $S(XY) + S(XZ) + S(YZ) - S(X) - S(Y) - S(Z) - S(XYZ)$. The symmetric group $\mathfrak{S}_{N+1}$ acts on $[N] \cup \{O\}$ and, through purification, on $\mathbb{R}^{2^N-1}$, and HEC_N is invariant under it [3]. So (2.2) also holds when one of X, Y, Z contains O.

2.2. **Bell values and centered inequalities.** For distinct $i, i' \in [N] \cup \{O\}$, the *Bell-pair vector* $r_{ii'}$ is the entropy vector of the graph model with a single edge of weight 1 between a boundary vertex of i and one of i' [3, 34]: $r_{ii'}(I) = 1$ if I contains exactly one of i, i', and 0 otherwise. The *occurrence words* $x^{(i)} \in \{0,1\}^L$ and $y^{(i)} \in \{0,1\}^R$ of a party i are

$$x_l^{(i)} = [i \in I_l] \;, \qquad y_r^{(i)} = [i \in J_r] \;, \tag{2.3}$$

with the bracket $[P]$ of the Conventions, and $x^{(O)} = 0$, $y^{(O)} = 0$. On $\{0,1\}^L$ we use the coordinatewise order $x \leq x'$, the coordinatewise minimum $x \wedge x'$ and maximum $x \vee x'$, the intervals $[x, x'] = \{z \in \{0,1\}^L\colon x \leq z \leq x'\}$, the all-ones word $\mathbf{1}$, the weighted Hamming distance and the weighted size

$$d_\alpha(u, v) = \sum_{l=1}^{L} \alpha_l |u_l - v_l| \;, \qquad |u|_\alpha = d_\alpha(0, u) = \sum_{l=1}^{L} \alpha_l u_l \;, \tag{2.4}$$

and likewise d_β and $|y|_\beta$ on $\{0,1\}^R$. A term $S(I)$ equals 1 on $r_{ii'}$ exactly when I separates i from i', so the *Bell value* of Q on the pair i, i' is

$$Q \cdot r_{ii'} = d_\alpha(x^{(i)}, x^{(i')}) - d_\beta(y^{(i)}, y^{(i')}) \ . \tag{2.5}$$

A valid inequality has nonnegative Bell values, since $r_{ii'}$ is the entropy vector of a graph model. The inequality Q is *balanced* if $Q \cdot r_{iO} = |x^{(i)}|_\alpha - |y^{(i)}|_\beta = 0$ for every party i, that is, if every party has the same total coefficient on both sides [3], and *superbalanced* if $Q \cdot r_{ii'} = 0$ for all distinct $i, i' \in [N] \cup \{O\}$, the Bell condition that Bao et al. observed for their five-party inequalities and their cyclic family [3]. For a balanced Q, (2.5) gives $Q \cdot r_{ii'} = 2(|y^{(i)} \wedge y^{(i')}|_\beta - |x^{(i)} \wedge x^{(i')}|_\alpha)$, so superbalance also says that every pair of parties lies in terms of the same total coefficient on both sides.

Hubeny, Rangamani and Rota introduced the notion of superbalance: they show that a balanced inequality remains balanced under every purification exactly when its expansion in their basis of multipartite informations contains no mutual information, and they define superbalance through this expansion [33]. Our definition is equivalent to the formulation that Q and all its purifications be balanced [33, 34], because the relabeling that exchanges i' and O carries r_{iO} to $r_{ii'}$. Avis and Hernández-Cuenca prove that superbalance is equivalent to balance together with the condition on pairs [15, Lemma 4], and these two conditions are the definition of Grimaldi, Headrick, Hubeny and Shteyner [29, Definition 5].

The inequalities that carry our lower bounds have one party in every term. Following Grimaldi, Headrick, Hubeny and Shteyner [29], we call them centered.

Definition 2.6 (Centered inequalities)**.** An inequality Q as in (2.1) is *centered on* a party $A \in [N]$ if it is balanced and A lies in every term I_l and every term J_r. We call A the *central party*.

Let Q be centered on A. Then $x^{(A)} = \mathbf{1}$ and $y^{(A)} = \mathbf{1}$, so the balance of A reads $|\alpha| = |\beta|$. For a party $i \neq A$, the terms that separate A from i are those that miss i, so by (2.5) and the balance of i,

$$Q \cdot r_{Ai} = (|\alpha| - |x^{(i)}|_\alpha) - (|\beta| - |y^{(i)}|_\beta) = 0 \ . \tag{2.7}$$

With $Q \cdot r_{AO} = |\alpha| - |\beta| = 0$, every Bell value of a centered inequality that involves A or O vanishes.

2.3. **Count maps.** Bao et al. proved an inequality valid by a map between Boolean cubes that respects the occurrence words and contracts weighted Hamming distances [3]. A coefficient β_r can be kept as a weight or split into β_r unit copies of $S(J_r)$ [3], and likewise on the left [44], and whether a map exists depends on this choice [15, 29]. As in [22], we merge the left-hand side and split the right-hand side, the presentation (Ln,Rs) in the classification of [29], and record a map by how many of the copies of each RHS term contain a given point.

Definition 2.8 (Count maps)**.** Let Q be as in (2.1), let $k \geq 1$ be an integer, and let $\Omega \subseteq \{0,1\}^L$ contain the word 0 and the occurrence words $x^{(i)}$ of all parties $i \in [N]$. A *real map for kQ on Ω* is a map $n = (n_r)_{r=1}^R \colon \Omega \to \prod_{r=1}^R [0, k\beta_r]$ such that

$$n_r(x^{(i)}) = k\beta_r[i \in J_r] \quad \text{for all } i \in [N] \text{ and all } r \ , \qquad n(0) = 0 \ , \tag{2.9}$$

and

$$\sum_{r=1}^{R} \big|n_r(u) - n_r(v)\big| \leq k d_\alpha(u, v) \qquad \text{for all } u, v \in \Omega \ . \tag{2.10}$$

It is an *integer map* if the coefficients $k\beta_r$ are integers and $n_r(u) \in \{0, 1, \ldots, k\beta_r\}$ for all u and r.

The conditions (2.9) are the *pins*, and the domain $\Omega = \{0,1\}^L$ is the *full cube*. We write $|n(u)| = \sum_r n_r(u)$. A real map for kQ is k times a real map for Q, so the existence of a real map is independent of k, while the existence of an integer map depends on k. By Lemma 2.9 of [22], an integer map n for kQ on Ω is the same as a map from Ω into the cube of the presentation with $k\beta_r$ unit copies of each $S(J_r)$ that sends occurrence words to occurrence words, sends 0 to 0 and contracts distances, where the copy s of J_r has the bit $[s \le n_r(u)]$ at the image of u. On the full cube such a map is a contraction map of Bao et al. for kQ, so by their theorem [3] an integer map for kQ on the full cube proves that kQ, and hence Q, is valid. This presentation is the most permissive one on the full cube: a contraction map of Bao et al. for any presentation of kQ gives an integer map for kQ [22], where the step on the right-hand side, copying the bits of repeated terms, is stated in [29]. On the full cube the contraction condition needs to be checked only on the edges of the cube, as Bao et al. note [3]. This is the case $\Omega = \{0,1\}^L$ of the irreducible-pairs lemma [22, Lemma 2.12].

Lemma 2.11 (Edges suffice)**.** *If $n\colon \{0,1\}^L \to \mathbb{R}^R$ satisfies* (2.10) *for all $u, v \in \{0,1\}^L$ at Hamming distance* 1*, it satisfies* (2.10) *for all $u, v \in \{0,1\}^L$.*

Proof. For $u, v \in \{0,1\}^L$, switch off the bits of u that are 0 in v, one at a time, and then switch on the bits of v that are 0 in u. Every coordinate in which u and v differ changes once along this path of cube edges, so the distances along the path add up to $d_\alpha(u,v)$, and the triangle inequality gives (2.10). □

2.4. **The realizable domain and least scaling.** In the proof of Bao et al., every vertex of a graph model receives the word that records which of the chosen LHS min cuts contain it [3]. If the min cuts are chosen minimal, these words satisfy two conditions, by Theorem 1 of Avis and Hernández-Cuenca on minimal min cuts [15], which they build on results of Nezami and Walter and of Bao, Cheng, Hernández-Cuenca and Su [45, 46], and a map is needed only on the words that satisfy them [15, 29].

Definition 2.12 (Realizable domain)**.** The *realizable domain* $\mathcal{A}(Q) \subseteq \{0,1\}^L$ of Q is the set of words x such that

(R1) if $x_l = x_{l'} = 1$, then $I_l \cap I_{l'} \neq \emptyset$, and
(R2) if $x_l = 1$ and $I_l \subseteq I_{l'}$, then $x_{l'} = 1$.

This is the domain of Avis and Hernández-Cuenca [15], who describe its words through pairwise intersecting upper sets of LHS terms and the antichains of their minimal elements, and it is called the realizable subset in [29]. On the role of nesting in proofs by contraction see also [19]. The domain is defined through the terms $I_l \subseteq [N]$ of (2.1), that is, after every term containing O has been replaced by its complement, and every statement about $\mathcal{A}(Q)$ refers to this presentation, with O as the purifier. The domain depends only on the LHS terms, so $\mathcal{A}(kQ) = \mathcal{A}(Q)$, and it contains 0 and every occurrence word.

By a theorem of Avis and Hernández-Cuenca, a map on $\mathcal{A}(Q)$ into the cube of the RHS terms, weighted by their coefficients, proves that Q is valid [15, Theorem 4]; see also [29]. In the form for real maps this is [22, Lemma 2.13]: a real map for Q on $\mathcal{A}(Q)$ proves that Q is valid. The lemma is stated there for integer coefficients. For rational coefficients, with D a common denominator of the coefficients, it applies to DQ and the real map Dn for DQ. All maps by which the proofs of Theorems 1.1 and 1.2 establish validity have rational values, and for such maps the lemma also follows from the proof of the theorem of Avis and Hernández-Cuenca, as noted in [22]. The

quantities that Theorems 1.1 and 1.2 bound are the least multiples of an inequality that have integer maps on the two domains.

Definition 2.13 (Scaling spectrum and least scaling)**.** The *scaling spectrum* $\operatorname{spec}(Q)$ of an inequality Q is the set of integers $k \geq 1$ for which kQ has an integer map on the full cube $\{0,1\}^L$, and $\operatorname{spec}^{\mathcal{A}}(Q)$ is the set of k for which kQ has an integer map on $\mathcal{A}(Q)$. The *least scalings* are $k_{\min}(Q) = \min \operatorname{spec}(Q)$ and $k^{\mathcal{A}}_{\min}(Q) = \min \operatorname{spec}^{\mathcal{A}}(Q)$, with $\min \emptyset = \infty$.

A map on the full cube restricts to $\mathcal{A}(Q)$, so $\operatorname{spec}(Q) \subseteq \operatorname{spec}^{\mathcal{A}}(Q)$ and $k^{\mathcal{A}}_{\min}(Q) \leq k_{\min}(Q)$. If n is a real map for $k_1 Q$ and n' one for $k_2 Q$ on the same domain, then $n + n'$ is a real map for $(k_1 + k_2)Q$, because the pins, the ranges and the contraction condition all add [22]. Hence $\operatorname{spec}(Q)$ and $\operatorname{spec}^{\mathcal{A}}(Q)$ are closed under addition, and each contains every multiple of each of its elements. If n is a real map for Q on Ω and $D \geq 1$ is an integer such that DQ has integer coefficients and all values $Dn_r(u)$ are integers, then Dn is an integer map for DQ on Ω. In particular, a rational real map for Q on the full cube proves that Q is valid and gives $k_{\min}(Q) \leq D$, and a rational real map on $\mathcal{A}(Q)$ gives $k^{\mathcal{A}}_{\min}(Q) \leq D$. An inequality with a nonempty spectrum on one of the two domains is valid, by the two statements on validity recalled above, so both spectra of an inequality that is not valid are empty and its least scalings are ∞.

An integer map for kQ, divided by k, is a real map for Q with rational values, and by the last paragraph a real map with rational values gives an integer map for a multiple of Q. So $k_{\min}(Q) < \infty$ exactly when Q has a real map with rational values on the full cube, and $k^{\mathcal{A}}_{\min}(Q) < \infty$ exactly when it has one on $\mathcal{A}(Q)$. More precisely, $k_{\min}(Q)$ is the least k for which Q has a real map on the full cube whose values are all integer multiples of $1/k$, and likewise $k^{\mathcal{A}}_{\min}(Q)$ on $\mathcal{A}(Q)$: an integer map for kQ is k times such a map, and the coefficients $k\beta_r$ are then integers because they are values of the map at the pins.

The least scalings depend on the normalization of the inequality: for an integer $t \geq 1$, $\operatorname{spec}(tQ)$ consists of the integers k with $tk \in \operatorname{spec}(Q)$, and likewise for $\operatorname{spec}^{\mathcal{A}}$. Every inequality (2.1) has exactly one positive multiple with integer coefficients of greatest common divisor 1, and Theorems 1.1 and 1.2 state least scalings for inequalities in this normalization. For the inequalities of this paper the normalization follows from one observation. If Q has a term with coefficient 1 and K is the least positive integer for which KQ has integer coefficients, then the coefficients of KQ have greatest common divisor 1: their greatest common divisor g divides the coefficient K of that term, and $(K/g)Q$ has integer coefficients, so $K/g \geq K$ and $g = 1$.

2.5. **Maps of centered inequalities.** If Q is centered on A, the word $x^{(A)} = \mathbf{1}$ is the top of the cube, and contraction toward 0 and toward $x^{(A)}$ forces every map to grow with the point by exactly the weight of the bits that are switched on. The next lemma records this for any balanced inequality on the interval $[0, x^{(A)}]$. Sections 4 and 8 apply it to centered inequalities, and Section 7 applies part (c) to a superbalanced inequality that is not centered.

Lemma 2.14 (Necessary conditions)**.** *Let Q be balanced, let $A \in [N]$, let $k \geq 1$, let $\Omega \subseteq \{0,1\}^L$ contain 0 and the occurrence words of all parties, and let n be a real map for kQ on Ω. Then for all $z, z' \in \Omega \cap [0, x^{(A)}]$:*

(a) $|n(z)| = k|z|_\alpha$;
(b) *if $z \leq z'$, then $n_r(z) \leq n_r(z')$ for every r;*
(c) $n_r(z) = 0$ *for every r with $A \notin J_r$;*
(d) $\sum_{r:\, i \in J_r} n_r(z) \geq k|z \wedge x^{(i)}|_\alpha$ *for every party i;*
(e) *if $z \leq x^{(i)} \leq x^{(A)}$ for a party i, then $n_r(z) = 0$ for every r with $i \notin J_r$.*

If Q is centered on A, then $[0, x^{(A)}]$ is the full cube and $x^{(i)} \le x^{(A)}$ for every party i, so (a), (b), (d) *and* (e) *hold on all of Ω.*

The proof is in Appendix A. We call the inequalities (d) the *party rows*. Contraction between z and 0 bounds $|n(z)|$ from above by $k|z|_\alpha$, and contraction between z and $x^{(A)}$, with the pin and the balance of A, bounds it from below by the same number, which is (a). For $z \le z'$ the contraction condition is then an equality for the sum of the counts, so no count can decrease, which is (b). Parts (c) and (e) are (b) with z' the occurrence word of A or of i, where the count of a term without that party is pinned to 0, and (d) is the contraction condition between z and $x^{(i)}$, rewritten with (a) and balance. For contraction maps the parts of the lemma appear in earlier work, which Appendix A.1 credits part by part [17, 22, 25, 29–31], and our proof follows that of Grimaldi, Headrick, Hubeny and Shteyner [29].

Conversely, for a centered inequality, a function on the full cube with the properties (a), (b) and (d) is a map. Sections 4 and 5 prove validity by constructing such a function.

Lemma 2.15 (Sufficient conditions on the full cube)**.** *Let Q be centered, let $k \ge 1$, and let $n\colon \{0,1\}^L \to \prod_{r=1}^{R} [0, k\beta_r]$ satisfy, for all $x, x' \in \{0,1\}^L$ and every party i,*

$$|n(x)| = k|x|_\alpha \ , \qquad x \le x' \ \Rightarrow \ n(x) \le n(x') \ , \qquad \sum_{r\colon\, i \in J_r} n_r(x) \ge k|x \wedge x^{(i)}|_\alpha \ . \tag{2.16}$$

The order $n(x) \le n(x')$ of count vectors is coordinatewise. Then n is a real map for kQ on the full cube.

The proof is in Appendix A. The pins follow from the third condition at the occurrence words and the ranges, the contraction condition on a cube edge from the first two conditions, and Lemma 2.11 gives it on all pairs. By Lemma 2.14 with $\Omega = \{0,1\}^L$, every real map for kQ on the full cube satisfies (2.16), so Lemmas 2.14 and 2.15 together characterize the real maps of centered inequalities on the full cube. For contraction maps this characterization is Theorem 2 of Grimaldi, Headrick, Hubeny and Shteyner [29], and Appendix A.1 states what our version adds.

2.6. **Padding and disjoint sums.** Section 9 combines inequalities on disjoint sets of parties and adds parties that lie in no term. Here we state what these operations do to validity, to balance and to the scaling spectra.

Let $\widetilde{Q}$ be obtained from Q by adding parties that lie in no term. We call $\widetilde{Q}$ a *padding* of Q, and adding one party is the trivial zero-lift of [15]. Then $\widetilde{Q}$ is valid exactly when Q is [3, 15]: a graph model for the larger set of parties becomes one for the smaller set when the boundary vertices of the new parties are labeled O, and the entropies of the sets of old parties stay the same. Q and $\widetilde{Q}$ also have the same real and integer maps [22], so $\mathrm{spec}(\widetilde{Q}) = \mathrm{spec}(Q)$ and $\mathrm{spec}^{\mathcal{A}}(\widetilde{Q}) = \mathrm{spec}^{\mathcal{A}}(Q)$, and $\widetilde{Q}$ is balanced, superbalanced or centered whenever Q is, as Avis and Hernández-Cuenca note for balance and superbalance [15]. The proof is in Appendix A: a new party has the occurrence word 0, whose pin is the pin $n(0) = 0$ of the purifier, and it is separated from every other party by the same terms as the purifier.

Definition 2.17 (Disjoint sums)**.** Let Q be an inequality on a set P of parties and Q' one on a set P' with $P \cap P' = \emptyset$, both written as in (2.1) with nonempty terms contained in P and in P', respectively. The *disjoint sum* $Q \oplus Q'$ is the inequality on the parties $P \cup P'$, with the purifier O, whose left-hand side lists the LHS terms of Q and of Q' and whose right-hand side lists the RHS terms of Q and of Q', with their coefficients.

The terms of Q and Q' are nonempty subsets of disjoint sets, so they are pairwise distinct, and $Q \oplus Q'$ is an inequality as in (2.1) with $L + L'$ LHS and $R + R'$ RHS terms, where L', R' and α' are the numbers of LHS and RHS terms and the LHS coefficients of Q'. Its coefficient vector is the sum of those of the paddings of Q and Q' to the parties $P \cup P'$, and its cube is $\{0,1\}^L \times \{0,1\}^{L'}$, with the distance $d_\alpha(x,y) + d_{\alpha'}(x',y')$ between (x,x') and (y,y').

Lemma 2.18 (Disjoint sums on the full cube)**.** *Let Q and Q' be as in Definition 2.17.*

(i) *If Q and Q' are valid, so is $Q \oplus Q'$. If both are balanced, $Q \oplus Q'$ is balanced, and if both are superbalanced, $Q \oplus Q'$ is superbalanced.*

(ii) *For every $k \geq 1$, $k(Q \oplus Q')$ has a real map on the full cube if and only if kQ and kQ' have one, and the same holds for integer maps. In particular, if Q and Q' are valid, then* $\operatorname{spec}(Q \oplus Q') = \operatorname{spec}(Q) \cap \operatorname{spec}(Q')$.

The proof is in Appendix A. The sum is valid because it is the sum of the paddings of Q and Q'. A map for $k(Q \oplus Q')$, restricted to the points $(x,0)$ and to the counts of the terms of Q, is a map for kQ, and two maps for kQ and kQ' together form a map for the sum, because the pins, the ranges and both sides of the contraction condition split over the two summands. On the realizable domain the two summands meet only at the point 0: the domain of $Q \oplus Q'$ is the union of the domains of Q and Q' glued at 0, while its full cube is the product of theirs.

Lemma 2.19 (Disjoint sums on the realizable domain)**.** *Let Q and Q' be as in Definition 2.17.*

(i) $\mathcal{A}(Q \oplus Q') = (\mathcal{A}(Q) \times \{0\}) \cup (\{0\} \times \mathcal{A}(Q'))$.

(ii) *For every $k \geq 1$, $k(Q \oplus Q')$ has a real map on $\mathcal{A}(Q \oplus Q')$ if and only if kQ has one on $\mathcal{A}(Q)$ and kQ' has one on $\mathcal{A}(Q')$, and the same holds for integer maps. In particular, if Q and Q' are valid, then* $\operatorname{spec}^{\mathcal{A}}(Q \oplus Q') = \operatorname{spec}^{\mathcal{A}}(Q) \cap \operatorname{spec}^{\mathcal{A}}(Q')$.

The proof is in Appendix A. Part (i) holds because a term of Q and a term of Q' are disjoint, so condition (R1) allows no point with a bit 1 in both, and part (ii) follows as in Lemma 2.18, with the contraction condition between $(x,0)$ and $(0,x')$ obtained through the point 0.

By induction, Lemmas 2.18 and 2.19 hold for disjoint sums $Q^{(1)} \oplus \ldots \oplus Q^{(r)}$ of inequalities on pairwise disjoint sets of parties. Let every $Q^{(t)}$ be valid. If $\operatorname{spec}(Q^{(t)}) \subseteq \ell_t\mathbb{Z}$ for pairwise distinct primes $\ell_1, \ldots, \ell_r$, then $\operatorname{spec}(Q^{(1)} \oplus \ldots \oplus Q^{(r)}) \subseteq \ell_1 \cdots \ell_r\mathbb{Z}$. If every $k_{\min}(Q^{(t)})$ is finite, their least common multiple lies in every $\operatorname{spec}(Q^{(t)})$, since a spectrum contains the multiples of its elements, and hence in the spectrum of the sum, so $k_{\min}(Q^{(1)} \oplus \ldots \oplus Q^{(r)}) \leq \operatorname{lcm}_t k_{\min}(Q^{(t)}) < \infty$. The same holds for $\operatorname{spec}^{\mathcal{A}}$ and $k^{\mathcal{A}}_{\min}$. A disjoint sum of two centered inequalities is balanced by Lemma 2.18(i), and it is centered on no party, since no party lies in terms of both summands.

3. The claw count

By Lemma 2.14, every map of a centered inequality is graded and monotone, and a count vanishes at the points below the occurrence word of a party outside its term. The lower bounds of Theorems 1.1 and 1.2 come from one arrangement of such conditions on a subcube. In this section we state this arrangement on its own, as a system of numbers attached to the pairs and triples of a finite set, and prove the count on which the lower bounds rest: if the numbers are integers, the scale of the system is divisible by $2j + 1$ (Corollary 3.8). Section 4 constructs inequalities whose integer maps for a scaling k contain such a system with scale a fixed multiple of k, and Section 9 chooses the parameters so that this forces large factors into k.

From now on, $j \geq 1$ is an integer, $m = 3j + 2$, and B is a set of m elements, called *points*. We call B the *box*.

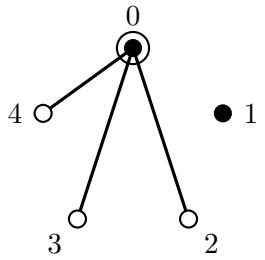

FIGURE 1. The claw $e = (0, \{0, 1\})$ for $j = 1$ on the box $B = \{0, 1, 2, 3, 4\}$. The points of the block $\{0, 1\}$ are black and the center 0 is ringed. The three lines join the center to the leaves $2, 3, 4$ and are the edges $\{0, 2\}$, $\{0, 3\}$, $\{0, 4\}$ of e, and its forks are the triples $\{0, 2, 3\}$, $\{0, 2, 4\}$, $\{0, 3, 4\}$.

Definition 3.1 (Claws)**.** A *block* is a subset of B with $j + 1$ points. A *claw* is a pair $e = (w_e, T_e)$ of a block T_e and a point $w_e \in T_e$, the *center* of e, and the $2j + 1$ points of $B \setminus T_e$ are the *leaves* of e. A set $S \subseteq B$ with two or three points is an *edge* of e if $S = \{w_e, b\}$ for a leaf b, and a *fork* of e if $S = \{w_e, b, b'\}$ for two leaves $b \neq b'$. The claw e is *available* at a set $S \subseteq B$ if $w_e \in S$ and $S \not\subseteq T_e$. For a set $\mathcal{T}$ of blocks, $\mathcal{K}(\mathcal{T})$ is the set of all claws e with $T_e \in \mathcal{T}$.

So a claw is a star graph on B that joins its center to its $2j + 1$ leaves and omits the other j points of its block (Figure 1). It is available at a pair exactly when the pair is one of its edges, and at a triple exactly when the triple contains the center and at least one leaf.

Definition 3.2 (Claw patterns)**.** Let $\mathcal{K}$ be a finite set of claws and $\kappa \geq 0$ a real number. A *claw pattern of scale κ on $\mathcal{K}$* is a family of real numbers $n_e(S) \geq 0$, one for every claw $e \in \mathcal{K}$ and every set $S \subseteq B$ with $|S| \in \{2, 3\}$, such that

(P1) $n_e(S) = 0$ unless e is available at S;
(P2) $\sum_{e \in \mathcal{K}} n_e(S) = (|S| - 1)\kappa$ for every S;
(P3) $n_e(P) \leq n_e(S)$ for every pair P contained in a triple S.

In words, every pair of points receives the total amount κ from the claws that have it as an edge, every triple receives 2κ, and a claw gives to a triple at least what it gives to each pair inside it. For $j = 1$ the blocks are pairs, a claw $(w, \{w, a\})$ has the three points other than w and a as leaves, and for the set of all 20 claws a pair $\{a, b\}$ is an edge of three claws with center a and of three with center b. The numbers $n_e(S) = \kappa/6$ for e available at S, and 0 otherwise, then form a claw pattern, since a triple has 12 available claws.

The next lemma shows that the three conditions force a claw to give the same amount to all its edges.

Lemma 3.3 (Claw count)**.** *Let $(n_e(S))$ be a claw pattern of scale κ on $\mathcal{K}$. Then for every claw $e \in \mathcal{K}$ there is a number $\mu_e \geq 0$ such that $n_e(S) = \mu_e$ for every edge S of e, and for every pair $\{a, b\} \subseteq B$*

$$\sum_{\substack{e \in \mathcal{K} \\ w_e = a,\ b \notin T_e}} \mu_e + \sum_{\substack{e \in \mathcal{K} \\ w_e = b,\ a \notin T_e}} \mu_e = \kappa \ . \tag{3.4}$$

Proof. For a claw e and a leaf b of e write $\mu_e(b) = n_e(\{w_e, b\})$. By (P1), a pair receives only from the claws of which it is an edge, so (P2) for the pair $\{a, b\}$ reads

$$\sum_{\substack{e\in\mathcal{K}\\ w_e=a,\ b\notin T_e}} \mu_e(b) + \sum_{\substack{e\in\mathcal{K}\\ w_e=b,\ a\notin T_e}} \mu_e(a) = \kappa \ . \tag{3.5}$$

Fix a triple S and add (3.5) for the three pairs inside S. A claw e enters this sum only if its center lies in S, through the pairs $\{w_e, b\}$ with $b \in S$ a leaf of e. Let b', b'' be the two points of S other than w_e. If neither is a leaf, e contributes 0. If exactly one of them is a leaf, say b', it contributes $\mu_e(b') \le n_e(S)$ by (P3). If both are leaves, that is, if S is a fork of e, it contributes

$$\mu_e(b') + \mu_e(b'') = \max\big(\mu_e(b'), \mu_e(b'')\big) + \min\big(\mu_e(b'), \mu_e(b'')\big) \le n_e(S) + \min\big(\mu_e(b'), \mu_e(b'')\big) \ ,$$

again by (P3). So every claw contributes at most $n_e(S)$, plus the minimum if S is a fork of e. The contributions add up to 3κ, the sum of the right sides of the three instances of (3.5), and $\sum_e n_e(S) = 2\kappa$ by (P2), so

$$\sum_{\substack{e\in\mathcal{K}\\ S \text{ a fork of } e}} \min\big(\mu_e(b'), \mu_e(b'')\big) \ \ge\ 3\kappa - 2\kappa = \kappa \qquad \text{for every triple } S \ . \tag{3.6}$$

Now add (3.6) over all $\binom{m}{3}$ triples. A triple is a fork of e exactly when it consists of w_e and two leaves of e, so every claw occurs once for each of the pairs $\{b', b''\}$ of its leaves, and

$$\binom{m}{3}\kappa \le \sum_{e\in\mathcal{K}} \sum_{\{b',b''\}} \min\big(\mu_e(b'), \mu_e(b'')\big) \le \sum_{e\in\mathcal{K}} \sum_{\{b',b''\}} \frac{\mu_e(b') + \mu_e(b'')}{2} = j \sum_{e\in\mathcal{K}} \sum_{b} \mu_e(b) = j\binom{m}{2}\kappa \ , \tag{3.7}$$

where $\{b', b''\}$ runs over the pairs of leaves of e and b over its leaves. The first equality holds because each of the $2j+1$ leaves lies in $2j$ pairs of leaves. The second is the sum of (3.5) over all $\binom{m}{2}$ pairs, in which every $\mu_e(b)$ occurs exactly once, for the pair $\{w_e, b\}$. Since $m = 3j + 2$,

$$\binom{m}{3} = \frac{m-2}{3}\binom{m}{2} = j\binom{m}{2} \ ,$$

so the two ends of (3.7) are equal, and every inequality in it is an equality. The second inequality was applied to each claw and each pair of its leaves separately, so $\min(\mu_e(b'), \mu_e(b'')) = \frac{1}{2}(\mu_e(b') + \mu_e(b''))$, that is, $\mu_e(b') = \mu_e(b'')$, for every claw e and any two of its leaves. So $\mu_e(b)$ has the same value μ_e for all leaves b of e, and (3.5) becomes (3.4). □

The count closes because of the identity $\binom{m}{3} = j\binom{m}{2}$, and this is where the size $m = 3j + 2$ of the box and the size $j + 1$ of the blocks enter, while the set $\mathcal{K}$ of claws is arbitrary.

Corollary 3.8. *In a claw pattern of scale κ on $\mathcal{K}$, the numbers μ_e of Lemma 3.3 satisfy*

$$(2j+1)\sum_{e\in\mathcal{K}} \mu_e = \binom{m}{2}\kappa \ . \tag{3.9}$$

If all the numbers $n_e(S)$ of the pattern are integers, then κ is an integer divisible by $2j+1$.

Proof. Add (3.4) over all pairs $\{a, b\}$. The number μ_e occurs once for every edge of e, that is, $2j+1$ times, which gives (3.9). If the $n_e(S)$ are integers, then κ is an integer by (P2) and the μ_e are integers, so $2j+1$ divides $\binom{m}{2}\kappa$, and hence $m(m-1)\kappa$. Since $2m - 3(2j+1) = 1$ and $3(2j+1) - 2(m-1) = 1$, the number $2j+1$ has no common factor with m or with $m-1$, so it divides κ. □

In graph-theoretic terms, the pair rows (3.4) say that an integer claw pattern contains a decomposition of the κ-fold complete multigraph on B into stars with $2j+1$ edges: the star of the claw e, which joins its center to its leaves, is taken μ_e times, and every pair of points is then covered κ times. Equation (3.9) counts the edges of this decomposition. That the sizes of the stars add up to the number of edges is an obvious necessary condition for a decomposition of a complete multigraph into stars, and Cameron and Horsley report a theorem of Tarsi by which some obvious numerical necessary conditions are also sufficient when all stars have one size [47]. What Lemma 3.3 adds is that the conditions at the triples make the amounts $n_e(S)$ of a claw equal on all its edges, so that an integer claw pattern has this form. Our lower bounds use only the count (3.9).

For $j=1$ equation (3.9) reads $3\sum_e \mu_e = 10\kappa$: an integer claw pattern on the five-point box has a scale divisible by 3. In Section 4 the sets S are points of a subcube of an inequality, the claws are right-hand side terms, and $n_e(S)$ is the count of the term e at the point S in a map for a multiple of the inequality.

4. Carriers and the water-level map

Section 3 showed that an integer claw pattern has a scale divisible by $2j+1$. In this section we attach to a suitable set $\mathcal{T}$ of blocks a centered inequality $Q(\mathcal{T})$, its *carrier*, in which every integer map for $\kappa Q(\mathcal{T})$ contains a claw pattern of scale κ (Proposition 4.7). This gives the lower bounds of Theorems 1.1 and 1.2. The upper bounds need the carrier to be valid with a finite least scaling, and for this we define one explicit candidate for a real map, the water-level map, and reduce the proof that it is a map to two conditions on $\mathcal{T}$ (Proposition 4.15). Section 5 verifies the two conditions, and Section 6 supplies the sets $\mathcal{T}$.

4.1. **The carrier of a block system.** The carrier is built so that its maps carry a claw pattern. It has an LHS term for each point of the box, so that a set $S \subseteq B$ is a point of its cube, and an RHS term for each claw e, whose count at S is to be the number $n_e(S)$. By Lemma 2.14(e) a count vanishes below the occurrence word of a party outside its term, and two kinds of parties turn this into availability, condition (P1): a party q_v for each point v, which lies in the LHS terms of the other points, and a party p_T for each block T, which lies in those of the points of T, while the RHS term of e leaves out exactly q_{w_e} and p_{T_e}. A party H in all these LHS terms and in every RHS term except $S(A)$ keeps the count of $S(A)$ at 0 on these points. The remaining terms are added for balance, which then fixes the coefficients: once the terms of the points and the term of all parties have coefficient 1, the balance of H, of q_v, of p_T and of A determines c, the coefficient $\frac{1}{m}$, ω and c_A of (4.3), in this order (proof of Lemma 4.5).

Definition 4.1 (Block systems)**.** A *block system of degree d* is a set $\mathcal{T}$ of blocks of B such that

(T1) every point of B lies in exactly d blocks of $\mathcal{T}$, where $d \geq 1$, and
(T2) for any two distinct points $v, b \in B$, some block of $\mathcal{T}$ contains v and misses b.

Its *claw family* is the set $\mathcal{K} = \mathcal{K}(\mathcal{T})$ of all claws e with $T_e \in \mathcal{T}$.

Every block has $j+1$ points, so a block system has $|\mathcal{T}| = md/(j+1)$ blocks and $|\mathcal{K}| = md$ claws, and every point is the center of exactly d claws.

Definition 4.2 (Carriers)**.** Let $\mathcal{T}$ be a block system of degree d. The parties are a central party A, a party H, a party q_v for every point $v \in B$ and a party p_T for every block $T \in \mathcal{T}$, so that $N = 2 + m + |\mathcal{T}|$. For $U \subseteq B$ let $q_U = \{q_v \colon v \in U\}$, for $v \in B$ let $P_v = \{p_T \colon v \in T\}$, and let F be

the set of all parties. The *carrier* of $\mathcal{T}$ is the inequality

$$\begin{aligned} Q(\mathcal{T}): \qquad & \sum_{b\in B} S\big(AHq_{B\setminus\{b\}}P_b\big) + \omega \sum_{v\in B} S(AP_v) + \frac{1}{m} S(Aq_B) \\ & \geq\ S(F) + c \sum_{e\in\mathcal{K}(\mathcal{T})} S(F\setminus\{q_{w_e}, p_{T_e}\}) + c_A S(A)\ , \end{aligned} \tag{4.3}$$

with the coefficients

$$c = \frac{m-1}{md}\ , \qquad \omega = \frac{2j+1}{j+1} - c\ , \qquad c_A = m\omega + \frac{1}{m}\ . \tag{4.4}$$

We call the three kinds of LHS terms the *box term* of b, the *star term* of v and the term Aq_B, and the RHS terms F, the *claw term* of e and the term A. The star term of v is $S(AP_v)$, whose parties other than A are the parties p_T of the d blocks T through v. Since $c<1$, all coefficients are positive. The terms are pairwise distinct, as (2.1) requires, which for the star terms uses (T2) (Appendix C.1).

For $j=1$ and the set of all 10 two-element subsets of $B=\{0,\dots,4\}$ we have $d=4$, $c=\frac{1}{5}$, $\omega=\frac{13}{10}$ and $c_A=\frac{67}{10}$, and $10Q(\mathcal{T})$ is the 17-party inequality displayed after Theorem 1.1.

The first lemma collects the properties of the carrier that the rest of the paper uses: it is centered, so that Lemmas 2.14 and 2.15 apply, its denominators are controlled, and its realizable domain is explicit.

Lemma 4.5 (Properties of the carrier)**.** *Let $\mathcal{T}$ be a block system of degree d.*

(i) *$Q(\mathcal{T})$ is centered on A.*
(ii) *The least positive integer K for which $KQ(\mathcal{T})$ has integer coefficients divides $m(j+1)d$, and the coefficients of $KQ(\mathcal{T})$ have greatest common divisor 1.*
(iii) *Write a point of the cube as $x=(S,U,\epsilon)$, where $S\subseteq B$ is the set of points whose box term has bit 1, $U\subseteq B$ the set of points whose star term has bit 1, and $\epsilon\in\{0,1\}$ the bit of Aq_B. Then $\mathcal{A}(Q(\mathcal{T})) = \{(S,U,\epsilon)\colon U\subseteq S\}$.*

The proof is in Appendix C.1. Part (i) is a count of the total coefficient of each party on the two sides, part (ii) holds because $m(j+1)d$ clears the denominators of (4.4) and the box terms have coefficient 1, and in part (iii) the only inclusion between two LHS terms is that of the star term of a point in its box term, which gives the condition $U\subseteq S$.

We keep the notation $x=(S,U,\epsilon)$ and write $s=|S|$ and $u=|U|$. A point $(S,\emptyset,0)$ is a *box point*, and we write it as S. In this notation $x\leq x'$ means $S\subseteq S'$, $U\subseteq U'$ and $\epsilon\leq\epsilon'$, and the weighted size and the occurrence words (2.3) are

$$\begin{aligned} & |x|_\alpha = s + \omega u + \frac{\epsilon}{m}\ , \\ & x^{(A)} = (B,B,1)\ , \qquad x^{(H)} = (B,\emptyset,0)\ , \qquad x^{(q_v)} = (B\setminus\{v\},\emptyset,1)\ , \qquad x^{(p_T)} = (T,T,0)\ . \end{aligned} \tag{4.6}$$

A map for a multiple of $Q(\mathcal{T})$ has a count n_F for the term F, a count n_e for the claw term of each claw e, and a count n_A for the term A. By Lemma 2.14(e), a count vanishes at the points below the occurrence word of a party outside its term. The claw term of e misses q_{w_e} and p_{T_e}, and the term A misses every party but A, so by (4.6) every real map n satisfies

(V1) $n_e(x)=0$ if $U=\emptyset$ and $w_e\notin S$;
(V2) $n_e(x)=0$ if $S\subseteq T_e$, $U\subseteq T_e$ and $\epsilon=0$;
(V3) $n_A(x)=0$ if $x\leq x^{(H)}$, or $x\leq x^{(q_v)}$ for some $v\in B$, or $x\leq x^{(p_T)}$ for some $T\in\mathcal{T}$.

At a box point S these conditions say that $n_A(S) = 0$ and that $n_e(S) = 0$ unless $w_e \in S$ and $S \not\subseteq T_e$, that is, unless e is available at S in the sense of Definition 3.1. This is how the claw pattern enters.

Proposition 4.7 (The box carries a claw pattern)**.** *Let $\mathcal{T}$ be a block system, let Ω be the full cube of $Q(\mathcal{T})$ or its realizable domain, let $\kappa \geq 1$ be an integer, and let n be an integer map for $\kappa Q(\mathcal{T})$ on Ω. Then the numbers $n_e(S)$, for $e \in \mathcal{K}(\mathcal{T})$ and box points S with $|S| \in \{2, 3\}$, form a claw pattern of scale κ on $\mathcal{K}(\mathcal{T})$, and $2j+1$ divides κ. Consequently, if K is the integer of Lemma 4.5*(ii) *and $Q_{\mathcal{T},\mathrm{int}} = KQ(\mathcal{T})$, then every $k \geq 1$ for which $kQ_{\mathcal{T},\mathrm{int}}$ has an integer map on $\mathcal{A}(Q_{\mathcal{T},\mathrm{int}})$ or on the full cube is divisible by $(2j+1)/\gcd(2j+1, K)$.*

Proof. By Lemma 4.5(iii) the box points lie in Ω, and Ω contains 0 and the occurrence words. Since $Q(\mathcal{T})$ is centered, Lemma 2.14 applies on all of Ω: the map is graded, $|n(x)| = \kappa|x|_\alpha$, monotone, and satisfies (V1)–(V3). At a box point $\{b\}$ no claw is available, because a claw with center b has $b \in T_e$, and n_A vanishes, so the grading gives $n_F(\{b\}) = \kappa$. The range of n_F is $[0, \kappa]$, so by monotonicity $n_F(S) = \kappa$ at every box point $S \neq \emptyset$. Since $n_A(S) = 0$, the grading at a box point with $s \geq 1$ reads $\sum_e n_e(S) = \kappa s - \kappa$, which is (P2). Condition (P1) is the consequence of (V1) and (V2) stated above, and (P3) is the monotonicity of n_e. The values are integers, so $2j+1$ divides κ by Corollary 3.8. For the last statement, an integer map for $kQ_{\mathcal{T},\mathrm{int}}$ is an integer map for $\kappa Q(\mathcal{T})$ with $\kappa = kK$, and the two inequalities have the same realizable domain. So $2j+1$ divides kK, and after division by $g = \gcd(2j+1, K)$ the number $(2j+1)/g$ divides $k \cdot K/g$ and has no common factor with K/g, so it divides k. □

4.2. **The water-level map.** We now construct the candidate for a real map for $Q(\mathcal{T})$ on the full cube. By the claw count, a map gives the same amount to all edges of a claw, and we let the candidate treat all available claws alike on the whole box: at a box point $S \neq \emptyset$ the term F takes 1, and the remaining $s-1$ units of the weighted size are divided equally among the claws available at S. These amounts, corrected by the bit ϵ, form a *ground* above which each claw term is filled. The star bits add ωu to the weighted size, and we treat this amount as water poured onto the ground. It may enter only the claws whose block does not contain U, as (V2) demands, it fills these claws from the lowest ground upward to one common level, and the part that rises above the coefficient c of a claw term flows over into the term A. It is convenient to measure the content of a claw term in units of c, as a *height*, so that the term is full at height 1.

Definition 4.8 (The water-level map)**.** Let $\mathcal{T}$ be a block system with claw family $\mathcal{K}$. For $S \subseteq B$ let $\mathrm{Av}(S)$ be the set of claws available at S, and for $U \subseteq B$ let $\mathrm{Op}(U) = \{e \in \mathcal{K} \colon U \not\subseteq T_e\}$ be the set of claws *open* at U, and the other claws are *closed* at U. The *box level* of S is

$$\gamma(S) = \frac{|S|-1}{|\mathrm{Av}(S)|} \quad \text{if } |S| \geq 2\,, \qquad \gamma(S) = 0 \quad \text{if } |S| \leq 1\,. \tag{4.9}$$

For a point $x = (S, U, \epsilon)$, the *ground* of a claw e and the *demand* are

$$g_e(x) = \Big(1 - \frac{\epsilon}{m}\Big)\frac{\gamma(S)}{c}[e \in \mathrm{Av}(S)] + \frac{\epsilon}{m-1}[w_e \in S]\,, \qquad \sigma(x) = \begin{cases} \omega u & \text{if } S \neq \emptyset\,, \\ \max\left(0, \omega u + \frac{\epsilon}{m} - 1\right) & \text{if } S = \emptyset\,. \end{cases} \tag{4.10}$$

Here $[P]$ is 1 if P holds and 0 otherwise. For a set $\mathcal{E} \subseteq \mathcal{K}$ of claws and $h \geq 0$ let

$$\Phi_x^{\mathcal{E}}(h) = c \sum_{e \in \mathcal{E}} \max\left(0, h - g_e(x)\right) \tag{4.11}$$

be the amount that the claws of $\mathcal{E}$ hold below the level h. The *water level* $h(x)$ is the least $h \geq 0$ with $\Phi_x^{\mathrm{Op}(U)}(h) \geq \sigma(x)$, and the *height* of e is

$$h_e(x) = \max\big(g_e(x), h(x)\big) \quad \text{if } e \in \mathrm{Op}(U)\ , \qquad h_e(x) = g_e(x) \quad \text{otherwise.} \tag{4.12}$$

The *water-level map* of $\mathcal{T}$ is

$$n_F(x) = \min\big(1, |x|_\alpha\big)\ , \qquad n_e(x) = c\min\big(1, h_e(x)\big)\ , \qquad n_A(x) = c\sum_{e\in\mathcal{K}} \max\big(0, h_e(x) - 1\big)\ . \tag{4.13}$$

Condition (T2) gives $\mathrm{Av}(S) \neq \emptyset$ for $|S| \geq 2$, so that the box level is defined, and it makes $\Phi_x^{\mathrm{Op}(U)}$ unbounded whenever $\sigma(x) > 0$ (Appendix C.1). Since $\Phi_x^{\mathrm{Op}(U)}$ is continuous and nondecreasing with $\Phi_x^{\mathrm{Op}(U)}(0) = 0$, the water level exists, and

$$\Phi_x^{\mathrm{Op}(U)}\big(h(x)\big) = \sigma(x)\ , \qquad \Phi_x^{\mathrm{Op}(U)}(h) < \sigma(x) \quad \text{for } 0 \leq h < h(x)\ . \tag{4.14}$$

All values of the map are rational: c, ω and the grounds are rational, and $h(x)$ is 0 or the solution of a linear equation with rational coefficients, on the linear piece of $\Phi_x^{\mathrm{Op}(U)}$ on which it lies.

As an illustration, take $j = 1$ and the set of all two-element subsets of $B = \{0, \dots, 4\}$, where $c = \frac{1}{5}$, $\omega = \frac{13}{10}$ and there are 20 claws. At the occurrence word $(B \setminus \{v\}, \emptyset, 1)$ of q_v, the 16 claws with center other than v are available, the box level is $\frac{3}{16}$, and their ground is $\frac{4}{5} \cdot \frac{15}{16} + \frac{1}{4} = 1$, while the four claws with center v have ground 0. No star bit is set, so the heights are the grounds, and the map takes the values 1 on F, $\frac{1}{5}$ on the claw terms that contain q_v and 0 on the others, which are the pins of q_v. At the occurrence word $(T, T, 0)$ of p_T with $T = \{0, 1\}$, the six claws with center 0 or 1 and a block other than T are available, with ground $\frac{1}{6} \cdot 5 = \frac{5}{6}$, and the other claws have ground 0. The demand is $2\omega = \frac{13}{5}$, and the open claws are the 18 claws with a block other than T. Filling them to the level 1 takes $\frac{1}{5}(6 \cdot \frac{1}{6} + 12 \cdot 1) = \frac{13}{5}$, so $h = 1$: the 18 claw terms that contain p_T are full, the two claw terms of the block T are empty, nothing flows into the term A, and these are the pins of p_T. The value of ω that balance fixes in (4.4) is the one that makes this work for every block system (Step 4 of the proof of Proposition 4.15 in Appendix C.1).

Figure 2 shows a point of the same carrier at which the water rises above 1, the point $x = (\{0, 1, 2, 3\}, \{0\}, 0)$. The 16 claws with center in $S = \{0, 1, 2, 3\}$ are available, the box level is $\frac{3}{16}$, and their ground is $\frac{15}{16}$, while the four claws with center 4 have ground 0. The open claws are the 12 claws whose block misses 0, three of them with ground 0, and the demand $\sigma(x) = \omega = \frac{13}{10}$ gives $\frac{1}{5}(3h + 9(h - \frac{15}{16})) = \frac{13}{10}$, that is, $h = \frac{239}{192}$. The 12 open claw terms are full, and $n_A = \frac{1}{5} \cdot 12 \cdot \frac{47}{192} = \frac{47}{80}$.

4.3. **Reduction to two conditions.** The water-level map is graded by construction, and it vanishes where (V1)–(V3) demand. What remains is monotonicity. Along an edge of the cube three things can happen to the water. The ground can rise, more water can be poured in, and more claws can open. A higher ground under at least as much water keeps the level at least as high, so the box bits and the bit ϵ are harmless once the box levels $\gamma(S)$ grow with S. A star bit pours in the amount ω and opens new claws, into which the water spreads, and the level stays at least as high if the new claws hold at most ω below the old level. The proposition makes this precise.

Proposition 4.15 (Reduction)**.** *Let $\mathcal{T}$ be a block system with claw family $\mathcal{K}$. Suppose that the box levels satisfy*

$$\gamma(S) \leq \gamma(S \cup \{b\}) \qquad \textit{for all } S \subseteq B \textit{ and } b \in B \setminus S\ , \tag{4.16}$$

and that for every point $x = (S, U, \epsilon)$ with $U \neq \emptyset$ and every $v \in B \setminus U$

$$\Phi_x^{\mathcal{N}}\big(h(x)\big) \leq \omega\ , \qquad \textit{where } \mathcal{N} = \mathrm{Op}(U \cup \{v\}) \setminus \mathrm{Op}(U) = \{e \in \mathcal{K} \colon U \subseteq T_e,\ v \notin T_e\}\ . \tag{4.17}$$

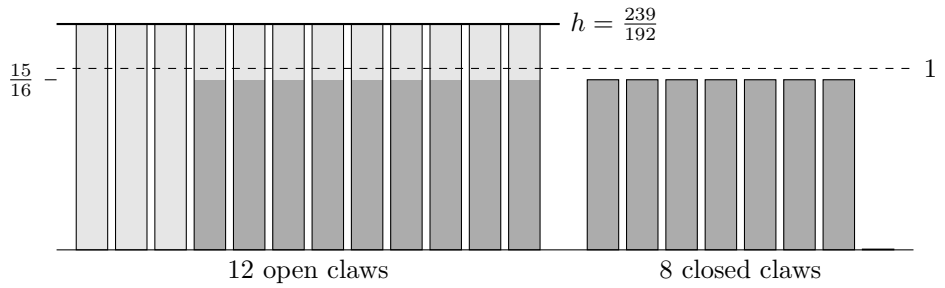

FIGURE 2. The heights of the 20 claw terms of the 17-party carrier at the point $x = (\{0,1,2,3\},\{0\},0)$. Dark: the ground, $\frac{15}{16}$ for the claws with center in $S = \{0,1,2,3\}$ and 0 for the claws with center 4. Light: the water of the star bit of 0, which enters the open claws, those whose block misses 0, and stands at the level $h = \frac{239}{192}$. A claw term has the count $\frac{1}{5}$ times its height up to 1, and the water above the dashed line, $\frac{1}{5} \cdot 12 \cdot \frac{47}{192} = \frac{47}{80}$ in all, is the count of the term A. The closed claw $(4,\{0,4\})$ has height 0.

Then the water-level map is a real map for $Q(\mathcal{T})$ on the full cube, with rational values. Consequently $Q(\mathcal{T})$ is valid and $k_{\min}(Q_{\mathcal{T},\mathrm{int}}) < \infty$.

The proof is in Appendix C.1. It verifies the conditions of Lemma 2.15. The grading holds by construction, and (V1)–(V3) follow from bounds on the grounds, where at the occurrence word of p_T the coefficient ω fills the open claws exactly to height 1. Monotonicity is the comparison just described, and the party rows follow from (V1)–(V3) and monotonicity. We call (4.16) the *box condition* and (4.17) the *star-step inequality*.

5. Validity

By Proposition 4.15, the carrier of a block system is valid with a finite least scaling once the box condition (4.16) and the star-step inequality (4.17) hold. In this section we prove both for the block systems in which every three points lie in the same number of blocks and sets of four or more points lie in few blocks (Theorem 5.7). Section 6 shows that the two families behind Theorems 1.1 and 1.2 are of this kind, and Section 9 uses the theorem for the finiteness of the least scalings in both theorems.

5.1. **Designs and the statement.** For a set $\mathcal{T}$ of blocks and $X \subseteq B$ let $\lambda(X)$ be the number of blocks of $\mathcal{T}$ that contain X. Recall that a 3-design is a collection of subsets of one size, called blocks, such that every set of three points lies in the same number of blocks, see e.g. [40]. In our setting:

Definition 5.1 (Designs)**.** Let $j \geq 2$. A nonempty set $\mathcal{T}$ of blocks of B is a 3-*design* if $\lambda(X)$ has the same value λ_3 for all $X \subseteq B$ with $|X| = 3$.

All estimates of this section run on three numbers, the numbers of blocks through one, two and three points, which the first lemma computes.

Lemma 5.2 (Counts in a design)**.** *Let $j \geq 2$ and let $\mathcal{T}$ be a 3-design. Then $\lambda(X)$ depends only on $|X|$ for $|X| \leq 3$. Writing d, λ_2, λ_3 for its values on sets of one, two and three points,*

$$\lambda_2 = \frac{j}{3j+1}d\,, \qquad \lambda_3 = \frac{j-1}{3(3j+1)}d\,, \qquad d - \lambda_2 = \frac{2j+1}{3j+1}d\,, \tag{5.3}$$

and $\mathcal{T}$ is a block system of degree d. For every $X \subseteq B$,

$$\sum_{w \in B} \lambda(X \cup \{w\}) = (j+1)\lambda(X) \ . \tag{5.4}$$

Proof. Fix two points $a \neq b$ and count the pairs (w, T) of a point $w \notin \{a, b\}$ and a block $T \supseteq \{a, b, w\}$. There are $m-2$ points w, each with λ_3 blocks, and each block that contains a and b has $j-1$ further points, so $(m-2)\lambda_3 = (j-1)\lambda(\{a,b\})$. Hence $\lambda(\{a,b\}) = 3j\lambda_3/(j-1)$ for every pair. Counting in the same way the pairs (b, T) with $b \neq a$ and $T \supseteq \{a, b\}$ gives $(m-1)\lambda_2 = j\lambda(\{a\})$, so $\lambda(\{a\}) = (3j+1)\lambda_2/j$ for every point. These two relations are (5.3). A block has at least three points, so $\lambda_3 \geq 1$ and $d \geq 1$, which is (T1), and $\lambda_2 < d$ says that for any two distinct points a, b some block through a misses b, which is (T2). In (5.4), a block that contains X is counted once for each of its $j+1$ points w. □

The further hypothesis bounds the number of blocks through four or more points:

$$|X|^2 \lambda(X) \leq d \qquad \text{for every } X \subseteq B \text{ with } |X| \geq 4 \ . \tag{5.5}$$

Since $\lambda(X) = 0$ for $|X| > j+1$, this is a condition on the sets with $4 \leq |X| \leq j+1$. In a 3-design a set of four points lies on average in $\frac{j-2}{3j-1}\lambda_3 < d/27$ blocks, and (5.5) allows $d/16$. A 3-design that satisfies (5.5) has

$$d \geq 2(j+1) \ . \tag{5.6}$$

For $j \geq 3$ a block X has $j+1 \geq 4$ points and $\lambda(X) = 1$, so $d \geq (j+1)^2$. For $j = 2$ the blocks are triples, a triple lies in a block only if it is that block, so $\lambda_3 = 1$, every triple is a block, and $d = \binom{7}{2} = 21$.

Theorem 5.7 (Validity)**.** *Let $j \geq 2$, and let $\mathcal{T}$ be a 3-design on the box B of $m = 3j+2$ points, with blocks of $j+1$ points, that satisfies* (5.5)*. Then $\mathcal{T}$ satisfies the box condition* (4.16) *and the star-step inequality* (4.17)*. Consequently the water-level map of $\mathcal{T}$ is a real map for $Q(\mathcal{T})$ on the full cube with rational values, $Q(\mathcal{T})$ is valid and centered on A, and $k_{\min}(Q_{\mathcal{T},\mathrm{int}}) < \infty$.*

The proof occupies the rest of this section. Throughout, $j \geq 2$ and $\mathcal{T}$ is a 3-design with claw family $\mathcal{K}$.

5.2. **The box condition.** The box condition (4.16) compares the box level of a set with that of the set with one more point.

Lemma 5.8 (Box levels)**.** *For $S \subseteq B$ with $s = |S| \geq 2$ we have $|\operatorname{Av}(S)| = s(d - \lambda(S))$, and if $\mathcal{T}$ satisfies* (5.5)*, the box condition* (4.16) *holds.*

Proof. A claw is available at S if its center w lies in S and its block does not contain S. Each $w \in S$ lies in d blocks, of which $\lambda(S)$ contain S. So $|\operatorname{Av}(S)| = s(d-\lambda(S))$ and $\gamma(S) = (s-1)/(s(d-\lambda(S)))$. For $s \leq 1$ the box condition holds because $\gamma(S) = 0$. For $s \geq 2$ and $b \notin S$, the inequality $\gamma(S) \leq \gamma(S \cup \{b\})$ reads $(s-1)(s+1)(d - \lambda(S \cup \{b\})) \leq s^2(d - \lambda(S))$, that is,

$$s^2\lambda(S) \leq d + (s^2-1)\lambda\big(S \cup \{b\}\big) \ . \tag{5.9}$$

For $s = 2$ this is $4\lambda_2 \leq d + 3\lambda_3$, and by (5.3) both sides equal $4jd/(3j+1)$. For $s = 3$ it follows from $9\lambda_3 = 3(j-1)d/(3j+1) \leq d$, and for $s \geq 4$ from (5.5). □

So the box levels of a pair and of a triple that contains it agree, as the claw count requires of the edges of a claw, and from there on the levels grow.

5.3. **The star-step inequality.** Fix a point $x = (S, U, \epsilon)$ with $u = |U| \geq 1$ and a point $v \in B \setminus U$, and write $h = h(x)$, $\mathcal{O} = \mathrm{Op}(U)$ for the open claws and $\mathcal{N} = \{e \in \mathcal{K} \colon U \subseteq T_e,\ v \notin T_e\}$ for the claws that the star bit of v opens. We have to show that $\Phi_x^{\mathcal{N}}(h) \leq \omega$.

The idea is a comparison. The water stands at the level h because the open claws hold the demand below it, $\Phi_x^{\mathcal{O}}(h) = \sigma(x) \leq \omega u$ by (4.14) and (4.10). So it suffices that the new claws hold at most a u-th of what the open claws hold,

$$u\Phi_x^{\mathcal{N}}(h) \leq \Phi_x^{\mathcal{O}}(h)\ , \tag{5.10}$$

and since the level is known only as the solution of an equation, we prove this for all levels $h \geq 0$ at once, wherever that is possible. What a set of claws holds below a level depends on the number of its claws and on their grounds, and a claw with a lower ground holds more. The grounds have a simple shape. Call a claw *low* at x if its center lies outside S, and write $z(\mathcal{E})$ for the number of low claws in a set $\mathcal{E}$ of claws. A low claw has ground 0 by (4.10), and if S has more points than a block, all other claws share one ground. Then (5.10) holds at every level as soon as the open claws outnumber the new claws by the factor u both among all claws and among the low claws. These are statements about numbers of blocks, and the next lemma provides them.

Lemma 5.11 (Counts for the star step)**.** *Let $\mathcal{T}$ satisfy* (5.5)*, and let x, v, $\mathcal{O}$, $\mathcal{N}$ be as at the beginning of Section 5.3. Then*

(a) $u|\mathcal{N}| \leq |\mathcal{O}|$*;*
(b) *if $s \leq j+1$, then $u|\mathcal{N}| \leq z(\mathcal{O})$;*
(c) *if $s \geq j+2$, then $uz(\mathcal{N}) \leq z(\mathcal{O})$, unless $U = \{a\}$ for a point $a \notin S$ and $m - s \in \{1, 2\}$.*

The proof, in Appendix C.2, expresses the four numbers through the counts $\lambda(X)$ and uses (5.3) for $|X| \leq 3$ and (5.5) for $|X| \geq 4$. We call the case excluded in (c) the *exceptional configuration*: one star bit a is on, the box bit of a is off, and at most one further box bit is off. There the comparison of low claws fails. Every claw with center a has a in its block and is closed, so the point a contributes no low open claw, while the claws with center a whose block misses v are new and low. In this configuration we solve the equation of the water level.

Lemma 5.12 (The exceptional configuration)**.** *Let $\mathcal{T}$ satisfy the box condition* (4.16) *and* (5.6)*. Let $x = (S, \{a\}, \epsilon)$ with $a \notin S$ and $m - s \in \{1, 2\}$, and let $v \neq a$. Then $\Phi_x^{\mathcal{N}}(h(x)) < \omega$.*

The proof is in Appendix C.3. With the two lemmas, the star-step inequality follows in a few lines.

Lemma 5.13 (Star step)**.** *If $\mathcal{T}$ satisfies* (5.5)*, the star-step inequality* (4.17) *holds.*

Proof. Let x, v, h, $\mathcal{O}$ and $\mathcal{N}$ be as above. By Lemma 5.8 the box condition holds, and (5.6) holds, so Lemma 5.12 settles the exceptional configuration, and we assume that x is not of this kind. As explained, it suffices to prove (5.10).

Let $s \leq j+1$. The grounds are nonnegative, so each new claw holds at most ch below the level h, and each low open claw holds exactly ch. By Lemma 5.11(b),

$$u\Phi_x^{\mathcal{N}}(h) \leq uc|\mathcal{N}|h \leq cz(\mathcal{O})h \leq \Phi_x^{\mathcal{O}}(h)\ .$$

Let $s \geq j+2$. No block contains S, so every claw with center in S is available at S, and by (4.10) all these claws have the same ground $g^* = (1 - \frac{\epsilon}{m})\gamma(S)/c + \frac{\epsilon}{m-1}$, while the low claws have ground 0. Hence for every set $\mathcal{E}$ of claws and every $h' \geq 0$

$$\Phi_x^{\mathcal{E}}(h') = cz(\mathcal{E})h' \quad \text{if } h' \leq g^*\ , \qquad \Phi_x^{\mathcal{E}}(h') = cz(\mathcal{E})g^* + c|\mathcal{E}|(h' - g^*) \quad \text{if } h' \geq g^*\ , \tag{5.14}$$

where the second formula adds ch' for each low claw and $c(h'-g^*)$ for each other claw. By Lemma 5.11(a) and (c), $u|\mathcal{N}| \le |\mathcal{O}|$ and $uz(\mathcal{N}) \le z(\mathcal{O})$, so both formulas give $u\Phi_x^{\mathcal{N}}(h') \le \Phi_x^{\mathcal{O}}(h')$, and $h' = h$ is (5.10). □

The smallest exceptional configuration shows what happens there. Let $j = 2$ and let $\mathcal{T}$ be the set of all three-element subsets of a box of $m = 8$ points, so that $d = 21$, $\lambda_2 = 6$, $\lambda_3 = 1$, $c = \frac{1}{24}$ and $\omega = \frac{13}{8}$, and let $x = (B \setminus \{a\}, \{a\}, 0)$. The open claws are the 105 claws whose block misses a. All of them have their center in S and the ground $g^* = \gamma(S)/c = \frac{48}{49}$, so $z(\mathcal{O}) = 0$, and the equation $\frac{1}{24} \cdot 105(h - \frac{48}{49}) = \frac{13}{8}$ gives $h = \frac{331}{245}$. The star bit of a point $v \ne a$ opens the claws of the 15 blocks that contain a and miss v, which are 15 low claws with center a and 30 claws with center in S. Below the level h they hold $\frac{1}{24}(15 \cdot \frac{331}{245} + 30 \cdot \frac{13}{35}) = \frac{513}{392}$, which is less than $\omega = \frac{637}{392}$, although $z(\mathcal{N}) = 15 > z(\mathcal{O}) = 0$.

Proof of Theorem 5.7. By Lemma 5.2, $\mathcal{T}$ is a block system of degree d. The box condition holds by Lemma 5.8 and the star-step inequality by Lemma 5.13, so Proposition 4.15 applies, and $Q(\mathcal{T})$ is centered on A by Lemma 4.5(i). □

For $j = 1$ the estimate in the proof of Lemma 5.12 is too weak, and Theorem 1.1 at $j = 1$ rests on the explicit integer maps of Appendix B.

6. Two families

So far we have shown that the carrier of a block system forces the factor $2j+1$ into the scaling of every integer map, up to the denominators of its coefficients (Proposition 4.7), and that it is valid with a finite least scaling if the block system is a 3-design that satisfies (5.5) (Theorem 5.7). This section supplies the designs. The set of all blocks gives the explicit inequalities of Theorem 1.1. Its number of blocks grows exponentially with j, while the forced factor $2j+1$ grows linearly, so for Theorem 1.2 we need designs with far fewer blocks, and we take unions of orbits of the group $\mathrm{PGL}(2,q)$ on the projective line, where the number of blocks is polynomial in j. Section 9 combines the resulting carriers.

6.1. **All blocks.** The simplest design is the set of all blocks.

Proposition 6.1 (The complete family)**.** *For $j \ge 1$ let $\mathcal{T}_j$ be the set of all subsets of B with $j+1$ points. It is a block system of degree $d = \binom{3j+1}{j}$, and for $j \ge 2$ it is a 3-design that satisfies* (5.5)*.*

Proof. A set X of $s \le j+1$ points lies in $\lambda(X) = \binom{m-s}{j+1-s}$ blocks, the number of ways to choose the remaining points. So every point lies in $d = \binom{m-1}{j}$ blocks, a point v and a point $b \ne v$ are separated by the $\binom{m-2}{j} > 0$ blocks through v that miss b, and $\mathcal{T}_j$ is a block system. For $j \ge 2$ it is a 3-design, with $\lambda_3 = \binom{m-3}{j-2}$. Since $\binom{n-1}{i-1} = \frac{i}{n}\binom{n}{i}$, the count drops by the factor $\frac{j+1-l}{m-l}$ when the $(l+1)$-st point is added, so for $|X| = s \le j+1$

$$\frac{\lambda(X)}{d} = \prod_{l=1}^{s-1} \frac{j+1-l}{m-l} \le 3^{-(s-1)} \ , \tag{6.2}$$

because $3(j+1-l) = m+1-3l \le m-l$ for $l \ge 1$. The numbers $s^2 3^{-(s-1)}$ decrease for $s \ge 4$, since $(\frac{s+1}{s})^2 \le \frac{25}{16} < 3$, and equal $\frac{16}{27}$ at $s = 4$. So $s^2\lambda(X) \le \frac{16}{27}d$ for $4 \le s \le j+1$, which is (5.5). □

We write $Q_j = Q(\mathcal{T}_j)$ for the carrier of the complete family, K_j for the least positive integer for which K_jQ_j has integer coefficients, and $Q_{j,\mathrm{int}} = K_jQ_j$. By Definition 4.2, Q_j has

$$N_j = 3j + 4 + \binom{3j+2}{j+1} \tag{6.3}$$

parties, and its coefficients are given by (4.4) with $d = \binom{3j+1}{j}$. For $j = 1$ they are the values $d = 4$, $c = \frac{1}{5}$, $\omega = \frac{13}{10}$ and $c_A = \frac{67}{10}$ computed after Definition 4.2, so that $N_1 = 17$, $K_1 = 10$, and $Q_{1,\mathrm{int}}$ is the inequality displayed after Theorem 1.1.

Corollary 6.4 (The explicit carriers)**.** *For every $j \geq 1$, the inequality $Q_{j,\mathrm{int}}$ is centered on A, and every $k \geq 1$ for which $kQ_{j,\mathrm{int}}$ has an integer map on $\mathcal{A}(Q_{j,\mathrm{int}})$ or on the full cube is divisible by $(2j+1)/\gcd(2j+1, K_j)$. For $j \geq 2$, $Q_{j,\mathrm{int}}$ is valid and $k_{\min}(Q_{j,\mathrm{int}}) < \infty$.*

Proof. The first statement is Lemma 4.5(i) and Proposition 4.7 for the block system $\mathcal{T}_j$, and the second is Theorem 5.7 with Proposition 6.1. □

6.2. **Orbits of** $\mathrm{PGL}(2,q)$ **on the projective line.** Let q be a prime power with $q \equiv 1 \pmod 3$, and put $j = (q-1)/3$, so that $m = 3j+2 = q+1$ and blocks have $j+1 = (q+2)/3$ points. As the box we take the projective line $B = \mathrm{PG}(1,q) = \mathbb{F}_q \cup \{\infty\}$ over the field $\mathbb{F}_q$ with q elements. The group $G = \mathrm{PGL}(2,q)$ consists of the maps $x \mapsto (a_1x + a_2)/(a_3x + a_4)$ of B with $a_1, \dots, a_4 \in \mathbb{F}_q$ and $a_1a_4 - a_2a_3 \neq 0$, where proportional quadruples give the same map, ∞ is mapped to a_1/a_3, and a vanishing denominator gives the value ∞. It has order $q^3 - q = (q+1)q(q-1)$ and acts 3-transitively on B, see e.g. [38]: for any two ordered triples of distinct points, some element of G maps the first to the second. The number of ordered triples of distinct points is $(q+1)q(q-1) = |G|$, so this element is unique. We use the group only through this fact.

Orbits of G on the sets of s points of the projective line, and unions of such orbits, are a known source of 3-designs, see e.g. [38, 40]. In our notation the reason is the following count. Let T_0 be a block and $X \subseteq B$ a set of $s \leq 3$ points. An element $g \in G$ has $X \subseteq gT_0$ exactly when g^{-1} maps X into T_0. Extend X to an ordered triple of distinct points. The elements of G correspond one to one to the ordered triples of distinct points that are the images of this triple under g^{-1}, and X is mapped into T_0 for $(j+1)j\cdots(j+2-s)$ choices of the images of its s points, each with $(m-s)\cdots(m-2)$ choices for the remaining $3-s$ images. Since $(m-s)\cdots(m-2) = |G|/(m(m-1)\cdots(m-s+1))$,

$$\#\{g \in G\colon X \subseteq gT_0\} = \frac{(j+1)j\cdots(j+2-s)}{m(m-1)\cdots(m-s+1)}|G| \qquad \text{for } |X| = s \leq 3\ , \tag{6.5}$$

which depends only on s. So if $T_1, \dots, T_M$ are blocks such that the $M|G|$ sets gT_i, $g \in G$, $1 \leq i \leq M$, are pairwise distinct, then these sets form a 3-design with

$$|\mathcal{T}| = M(q^3 - q)\ , \qquad d = M(j+1)q(q-1)\ , \qquad \lambda_3 = M(j+1)j(j-1)\ . \tag{6.6}$$

It has Mq^3 blocks up to lower order, against $\binom{3j+2}{j+1}$ for the complete family on the same box. What has to be arranged is (5.5), and the next lemma does this with random blocks. The lemma is an existence statement, proved by showing that a random choice of blocks fails with probability less than 1; it does not exhibit the blocks.

Lemma 6.7 (Existence)**.** *Let $q \geq 256$ be a prime power with $q \equiv 1 \pmod 3$, let $j = (q-1)/3$ and $m = q+1$, and let M be an integer with $5(j+1)^2(\ln m + 1) \leq M \leq m^3$. Then there are blocks $T_1, \dots, T_M$ of $\mathrm{PG}(1,q)$ such that the $M|G|$ sets gT_i are pairwise distinct and form a 3-design $\mathcal{T}$ that satisfies (5.5), with the parameters (6.6).*

The proof is in Appendix C.4. The blocks T_i are chosen independently and uniformly at random. For a set X of $s \geq 4$ points, $\lambda(X)$ is then a sum of M independent contributions, one from each orbit, and its mean is at most $3^{-(s-1)}d$ by the computation in (6.2), while (5.5) allows d/s^2, which is larger by the factor $3^{s-1}/s^2 \geq \frac{27}{16}$. An exponential tail bound for sums of independent bounded random variables, proved in the appendix as Lemma C.9, makes the probability that $\lambda(X)$ exceeds d/s^2 at most $(em)^{-s}$, with e Euler's number, when M is as large as stated, and there are fewer than m^s sets X. The sets gT_i are pairwise distinct with high probability because an element of G other than the identity fixes at most two points and therefore maps few blocks to themselves.

Corollary 6.8 (Carriers from the projective line)**.** *For every prime power $q \geq 256$ with $q \equiv 1 \pmod 3$ there is a 3-design $\mathcal{T}_q$ on $\mathrm{PG}(1,q)$ that satisfies (5.5), with $j = (q-1)/3$, such that the inequality $Q_{\mathcal{T}_q,\mathrm{int}}$*

(i) *is valid and centered on A, has integer coefficients of greatest common divisor 1, and has $k_{\min}(Q_{\mathcal{T}_q,\mathrm{int}}) < \infty$;*
(ii) *has N_q parties, where $q^3 \leq N_q \leq 2q^5 \ln q$;*
(iii) *has an integer map for $kQ_{\mathcal{T}_q,\mathrm{int}}$ on its realizable domain or on the full cube only if k is divisible by $r_q/\gcd(r_q,3)$, where $r_q = 2j+1 = (2q+1)/3$.*

The proof is in Appendix C.4. It applies Lemma 6.7 with a number M of orbits that is a power of 2 of order $q^2 \ln q$. Part (i) is then Theorem 5.7, part (ii) counts the parties of Definition 4.2, and part (iii) is Proposition 4.7, where the integer K of Lemma 4.5(ii) divides $m(j+1)d = (q+1)(j+1)^2 Mq(q-1)$, a number that shares with r_q at most the factor 3.

So each of the two families gives valid centered inequalities with a finite least scaling and a forced factor: $2j+1$ on N_j parties for the complete family, where N_j grows exponentially with j, and $(2q+1)/3$ or a third of it on at most $2q^5 \ln q$ parties for the projective line.

7. Superbalancing and the transfer of lower bounds

The carriers of Section 4 are centered and hence balanced. Theorems 1.1 and 1.2 also concern superbalanced inequalities, and in this section we attach to every inequality a superbalanced inequality on the same parties, its superbalancing (Proposition 7.3). Every map for a multiple of the superbalancing of a valid centered inequality restricts to a map for the same multiple of that inequality (Proposition 7.6), so the lower bounds of Section 4 hold for the superbalancings of the carriers. Section 8 proves a converse on the realizable domain, which gives the validity of these superbalancings and their upper bounds.

Definition 7.1 (Superbalancing)**.** Let Q be an N-party inequality with Bell values $b_{uv} = Q \cdot r_{uv}$ for distinct $u, v \in [N] \cup \{O\}$. Its *superbalancing* is

$$C(Q) = Q - \frac{1}{2} \sum_{\{u,v\}} b_{uv} I(u{:}v) \ , \tag{7.2}$$

where the sum runs over the unordered pairs of distinct elements of $[N] \cup \{O\}$ and $I(u{:}v)$ is the coefficient vector of $S(u) + S(v) - S(uv)$, with terms that contain O replaced by their complements, so that $I(u{:}O)$ is the coefficient vector of $S(u) + S([N]) - S([N] \setminus \{u\})$.

Whether the superbalancing of a valid inequality is valid turns on a decomposition into a superbalanced valid inequality and instances of subadditivity (Proposition 7.3 below), and several works state results on such decompositions. The statement of He, Hubeny and Rangamani quoted in the introduction says that every non-redundant valid inequality, that is, every facet of HEC_N, other

than an instance of subadditivity is superbalanced [34]. Grimaldi, Headrick, Hubeny and Shteyner draw from this statement the consequence that every valid inequality is a nonnegative integer combination of instances of subadditivity and a superbalanced valid inequality [29], and Czech, Feng, Wu and Xie, citing the same work, describe the valid inequalities that are not superbalanced as subadditivity and its combinations with other valid inequalities [30]. Ju and Czech give a seven-party inequality, which they prove valid by a contraction map, and a graph model on which it takes the value 0 while two of its parties have mutual information 2, and they state that it is not a superbalanced valid inequality plus a nonnegative combination of instances of subadditivity [48]. In (7.2) the instances of subadditivity are those of two single parties, with coefficients fixed by the Bell values.

Proposition 7.3. *For every inequality Q, the inequality $C(Q)$ is superbalanced. If Q has integer coefficients and all its Bell values are even, $C(Q)$ has integer coefficients, and all Bell values of Q are even whenever Q is balanced and has integer coefficients. If Q is valid, then $C(Q)$ is valid if and only if $Q = Q_0 + \sum_{(X,Y)} \nu_{XY} I(X{:}Y)$ with Q_0 superbalanced and valid and all $\nu_{XY} \geq 0$, where (X, Y) runs over the pairs of disjoint nonempty subsets of $[N] \cup \{O\}$ and terms that contain O are replaced by their complements as in Definition 7.1.*

The proof is in Appendix D.1. Each $I(u{:}v)$ has Bell value 2 on the pair $\{u, v\}$ and 0 on every other pair, so $C(Q) \cdot r_{uv} = b_{uv} - b_{uv} = 0$ for every pair, and the parity of the Bell values gives the integrality. For validity, inserting the decomposition into (7.2) leaves Q_0 plus nonnegative multiples of the differences $I(X{:}Y) - \sum_{x\in X} \sum_{y\in Y} I(x{:}y)$, each a sum of instances of MMI. Conversely, if $C(Q)$ is valid, (7.2) solved for Q is such a decomposition, since $b_{uv} \geq 0$ for a valid Q.

Theorems 1.1 and 1.2 use the first two statements only, and the validity of their superbalanced inequalities is proved by maps on the realizable domain (Theorem 8.16 and Appendix B).

Now let Q' be valid and centered on the party A, written as in (2.1) with LHS terms I_l, RHS terms J_r and coefficients α_l, β_r, and let b_{uv} be its Bell values. By (2.7), every Bell value of Q' that involves A or O vanishes. So only pairs $u, v \in [N] \setminus \{A\}$ enter (7.2), their Bell values are nonnegative because Q' is valid, and even by Proposition 7.3 if Q' has integer coefficients, and with $a_{uv} = b_{uv}/2$ the superbalancing reads

$$C(Q'): \qquad \sum_{l=1}^{L} \alpha_l S(I_l) + \sum_{\{u,v\}} a_{uv} S(uv) \;\geq\; \sum_{r=1}^{R} \beta_r S(J_r) + \sum_{u} a_u S(u) \;, \qquad a_u = \sum_{v} a_{uv} \;, \tag{7.4}$$

where u and v run over $[N] \setminus \{A\}$ and terms with coefficient 0 are omitted. The terms of Q' keep their coefficients, the added LHS terms are the pairs uv with $b_{uv} > 0$ and the added RHS terms the single parties u with $a_u > 0$, and none of the added terms contains A. So Q' is the null reduction of $C(Q')$ on A, the inequality formed by its terms that contain A [25, 29]. We write the points of the cube of $C(Q')$ as (x, z), with $x \in \{0, 1\}^L$ on the terms of Q' and $z \in \{0, 1\}^{L'}$ on the L' pair terms, and $\widehat{x}^{(u)} = (x^{(u)}, z^{(u)})$ for its occurrence words, where $z^{(u)}$ marks the pair terms that contain u. Then $\widehat{x}^{(A)} = (\mathbf{1}, 0)$, $\widehat{x}^{(O)} = 0$, and the points $(x, 0)$ form the interval between them, on which the weighted distance of $C(Q')$ is d_α. A map n for $kC(Q')$ has a count n_r for each term J_r of Q' and a count n_u for each singleton term $S(u)$ of (7.4), and its pins are

$$n_r(\widehat{x}^{(w)}) = k\beta_r[w \in J_r] \;, \qquad n_u(\widehat{x}^{(w)}) = ka_u[w = u] \;, \qquad n(0) = 0 \;, \tag{7.5}$$

for every party w, so that $n(\widehat{x}^{(A)})$ has the counts $k\beta_r$ and vanishing singleton counts.

The following restriction argument is known for unscaled maps. Grimaldi, Headrick and Hubeny restrict a contraction map in this way in the proof of their theorem on the validity of the null

reductions of contraction-provable inequalities of tripartite form [25, Theorem 2]. Grimaldi, Headrick, Hubeny and Shteyner prove that the null reduction of a superbalanced inequality with a contraction map has a contraction map, by restricting the map to the words supported on the terms that contain the reducing party [29, Theorem 1]. Czech, Feng, Wu and Xie prove the same statement with the same restriction [30]. In [29] the maps are contraction maps for $k = 1$ on the full cube, for the presentation with unit coefficients and repeated terms. Since Q' is the null reduction of $C(Q')$ on A, the following proposition is their argument applied to integer count maps of $kC(Q')$, on the full cube and on the realizable domain. The idea is that the points $(x, 0)$ lie below the occurrence word of A, where the counts of the terms without A vanish, so that on these points a map for $kC(Q')$ is a map for kQ'.

Proposition 7.6 (Transfer of lower bounds)**.** *Let Q' be valid and centered on A, let $k \geq 1$, and let either $\Omega = \{0,1\}^{L+L'}$ and $\Omega' = \{0,1\}^{L}$, or $\Omega = \mathcal{A}(C(Q'))$ and $\Omega' = \mathcal{A}(Q')$. If n is a real map for $kC(Q')$ on Ω, then $(x, 0) \in \Omega$ and $n_u(x, 0) = 0$ for every $x \in \Omega'$ and every singleton term $S(u)$, and*

$$n'(x) = \big(n_r(x, 0)\big)_{r=1}^{R} , \qquad x \in \Omega' , \tag{7.7}$$

is a real map for kQ' on Ω', which is an integer map if n is one. In particular, $\operatorname{spec}(C(Q')) \subseteq \operatorname{spec}(Q')$ and $\operatorname{spec}^{\mathcal{A}}(C(Q')) \subseteq \operatorname{spec}^{\mathcal{A}}(Q')$, so $k_{\min}(C(Q')) \geq k_{\min}(Q')$ and $k^{\mathcal{A}}_{\min}(C(Q')) \geq k^{\mathcal{A}}_{\min}(Q')$, where either side may be ∞.

Proof. The domain. On the full cube every $(x, 0)$ lies in Ω. On the realizable domain, a point $(x, 0)$ satisfies (R1) for $C(Q')$ and x satisfies it for Q', because all terms of Q' contain A. In (R2) for $(x, 0)$, a term I_l with $x_l = 1$ contains A and so lies in no pair term uv, and the condition involves only the terms of Q', where it is (R2) for x. So $(x, 0) \in \mathcal{A}(C(Q'))$ exactly when $x \in \mathcal{A}(Q')$.

The singleton counts vanish. Let $x \in \Omega'$ and $\zeta = (x, 0)$, so that $\zeta \in \Omega \cap [0, \widehat{x}^{(A)}]$. The inequality $C(Q')$ is superbalanced, hence balanced, and A lies in none of its singleton terms, so Lemma 2.14(c) for $C(Q')$ and the party A gives $n_u(\zeta) = 0$ for every singleton term $S(u)$.

The pins of Q'. The pin of kQ' at $x^{(A)} = \mathbf{1}$ is $n'_r(\mathbf{1}) = k\beta_r$ for every r, which is (7.5) at $\widehat{x}^{(A)} = (\mathbf{1}, 0)$, and the pin at 0 is $n'(0) = 0$. Let $w \neq A$ be a party and $\eta = (x^{(w)}, 0)$. It lies in Ω by the first step, because $x^{(w)} \in \mathcal{A}(Q')$, and it differs from $\widehat{x}^{(w)} = (x^{(w)}, z^{(w)})$ exactly in the pair terms that contain w, whose coefficients add up to a_w by (7.4) (if $a_w = 0$, then $\eta = \widehat{x}^{(w)}$ and there is no term $S(w)$). The singleton counts vanish at η by the second step, and at $\widehat{x}^{(w)}$ the count of $S(w)$ is ka_w and the other singleton counts vanish, by (7.5). So the contraction condition between η and $\widehat{x}^{(w)}$ reads

$$\sum_{r=1}^{R} \big|n_r(\eta) - k\beta_r[w \in J_r]\big| + ka_w \leq ka_w , \tag{7.8}$$

and $n'_r(x^{(w)}) = n_r(\eta) = k\beta_r[w \in J_r]$ for every r, which is the pin of kQ' at $x^{(w)}$.

The map. The values $n'_r(x)$ lie in $[0, k\beta_r]$, and they are integers if n is an integer map. For $x, x' \in \Omega'$ the singleton counts vanish at $(x, 0)$ and $(x', 0)$, so

$$\sum_{r=1}^{R} |n'_r(x) - n'_r(x')| = \big\|n(x, 0) - n(x', 0)\big\|_1 \leq kd_\alpha(x, x') . \tag{7.9}$$

So n' is a map for kQ' on Ω'. The statements on the least scalings follow from Definition 2.13. □

Section 8 uses the data (7.4) and, for the carriers of Section 4, bounds $k^{\mathcal{A}}_{\min}(C(Q'))$ from above, by constructing integer maps of multiples of $C(Q')$ on $\mathcal{A}(C(Q'))$ whose values at the points $(x, 0)$ have the form $(n'(x), 0)$ that Proposition 7.6 forces.

8. Maps on the realizable domain

Section 7 turned a valid centered inequality Q' into its superbalancing $C(Q')$ and showed that every map for $kC(Q')$ restricts to a map for kQ', on the full cube and on the realizable domain. The upper bounds of Theorems 1.1 and 1.2 for the superbalanced inequalities need the converse, and in this section we prove it on the realizable domain: if the RHS coefficients of Q' satisfy a capacity condition, every map for kQ' on $\mathcal{A}(Q')$ lifts to a map for $kC(Q')$ on $\mathcal{A}(C(Q'))$, so the superbalanced inequality has the same least scaling on its realizable domain as Q' (Theorem 8.16).

Grimaldi, Headrick, Hubeny and Shteyner give a superbalanced inequality that is not valid although each of its null reductions has a contraction map, so that the converse of their theorem on null reductions, recalled in Section 7, does not hold for every superbalanced inequality [29]. The lift uses the special form (7.4) of $C(Q')$, in which every term outside Q' is a pair of parties or a single party. The carriers of the two families of Section 6 satisfy the capacity condition, apart from the one for 17 parties (Lemma 8.15). The lift is given by explicit formulas, and its contraction condition reduces to local inequalities on two kinds of cube edges. Only one of these inequalities depends on the map, and the capacity condition controls it. This section defines the lift and states the one condition, and the reduction and the list of local inequalities are in Appendix D.

Throughout, Q' is centered on A, written as in (2.1), and $C(Q')$ is written as in (7.4). We work with maps for Q'. This covers every scaling, because kQ' is again centered on A, has the same realizable domain, and has the superbalancing $kC(Q')$. For parties $u \neq v$ other than A let $x^{(uv)} = x^{(u)} \wedge x^{(v)}$, the word of the LHS terms that contain both, $\alpha(uv) = |x^{(uv)}|_\alpha$, and $\beta(uv)$ the total coefficient of the RHS terms that contain both. Then $d_\alpha(x^{(u)}, x^{(v)}) = |x^{(u)}|_\alpha + |x^{(v)}|_\alpha - 2\alpha(uv)$, likewise $d_\beta(y^{(u)}, y^{(v)})$, and (2.5) with the balance of u and v gives

$$a_{uv} = \tfrac{1}{2} Q' \cdot r_{uv} = \beta(uv) - \alpha(uv) \ . \tag{8.1}$$

Since every term of Q' contains A, condition (R1) always holds and (R2) says that the positions l with $x_l = 1$ form an up-set for the inclusion of the terms I_l: with every position l the set contains every position l' with $I_l \subseteq I_{l'}$. So $\mathcal{A}(Q')$ is closed under $\wedge$ and $\vee$, and it contains 0, $\mathbf{1}$ and every occurrence word. Let g be a real map for Q' on $\mathcal{A}(Q')$. By Lemma 2.14 with $k = 1$ and $\Omega = \mathcal{A}(Q')$, g is graded, $|g(y)| = |y|_\alpha$, monotone, and satisfies

$$g_r(y) = 0 \qquad \text{if } y \leq x^{(w)} \text{ for a party } w \notin J_r \ . \tag{8.2}$$

At $y = x^{(uv)}$, (8.2) and grading give $\alpha(uv) = \sum_{r:\, u,v \in J_r} g_r(x^{(uv)}) \leq \beta(uv)$, so $a_{uv} \geq 0$. We write χ_l for the word whose only 1 is at position l. Table 1 in Appendix D lists the notation of this section.

8.1. The realizable domain of the superbalancing.

The lift is defined separately on each kind of point of $\mathcal{A}(C(Q'))$, so we first list these points. As in Section 7 a point of the cube of $C(Q')$ is (x, z), and we identify z with the set Γ of pair terms at which it is 1. The weighted distance of $C(Q')$ is then

$$\widehat{d}((x,\Gamma),(y,\Delta)) = d_\alpha(x,y) + a(\Gamma \setminus \Delta) + a(\Delta \setminus \Gamma) \ , \qquad a(\Gamma) = \sum_{uv \in \Gamma} a_{uv} \ . \tag{8.3}$$

At least two pair terms that all contain a party u form a *fan* with *center* u, whose *leaves* are the other parties of its terms. The center and the leaves of a fan are parties, whereas those of a claw (Section 3) are points of the box. Three pair terms uv, uv', vv' form a *triangle*. A point (x, Γ) is a *fan point* or a *triangle point* if Γ is a fan or a triangle.

Lemma 8.4 (Realizable points)**.** *A point (x,Γ) lies in $\mathcal{A}(C(Q'))$ if and only if $x\in\mathcal{A}(Q')$, Γ is empty, a single pair term, a fan or a triangle, and $x^{(uv)}\le x\le x^{(u)}\vee x^{(v)}$ for every $uv\in\Gamma$. For a fan with center u and leaves V the last condition reads*

$$x^-_{u,V}\le x\le x^+_{u,V}\ ,\qquad x^-_{u,V}=x^{(u)}\wedge\bigvee_{v\in V}x^{(v)}\ ,\qquad x^+_{u,V}=x^{(u)}\vee\bigwedge_{v\in V}x^{(v)}\ , \tag{8.5}$$

and for a triangle on u,v,v' it says that x is the coordinatewise median of $x^{(u)}$, $x^{(v)}$, $x^{(v')}$.

The proof is in Appendix D.2. Between two terms of Q', conditions (R1) and (R2) say that $x\in\mathcal{A}(Q')$, between two pair terms they say that the terms share a party, so that Γ is a single pair term, a fan or a triangle, and between a pair term uv and a term of Q' they say that x lies in the interval $[x^{(uv)},x^{(u)}\vee x^{(v)}]$. The distributive laws turn the intersection of these intervals into (8.5) and into the median.

At a point with a single pair term or a fan, $x^{(uv)}\le x\le x^{(u)}\vee x^{(v)}$ for the center u and each leaf v. The *shadow* of x on a party w is $\xi_w=x\wedge x^{(w)}\in\mathcal{A}(Q')$, and the distributive law gives

$$\xi_u\vee\xi_v=x\wedge(x^{(u)}\vee x^{(v)})=x\ ,\qquad \xi_u\wedge\xi_v=x\wedge x^{(uv)}=x^{(uv)}\ . \tag{8.6}$$

8.2. **The lift.** A value of the lift has a count for each RHS term J_r of Q' and one for each singleton term $S(w)$ of (7.4), and $\beta y^{(u)}$ denotes the vector of the pins $\beta_r y_r^{(u)}$ of Q' at $x^{(u)}$.

At the points without pair terms the lift equals g, the form that Proposition 7.6 forces on a map there. At a fan, all pair terms contain the center u, as do the pair terms at the occurrence word of u in $C(Q')$, and the value is placed as far from the pin of u as the contraction condition allows. The RHS terms with the center and a leaf take their caps β_r, as at the pin, those with the center and no leaf take the value of g at the shadow on the center, and those with every leaf and without the center the least value of g at the shadows on the leaves. The singleton count $\theta_{u,V}$ of the center is then defined in (8.9) as the number for which the distance of the value from the pin of u equals the distance of the point from the occurrence word of u, so that the contraction condition between these two points holds with equality. That this number lies in the range $[0,a_u]$ of the count is part of Proposition D.14 in Appendix D.4. At a single pair term uv the same number can have either sign, and its positive part is the count of $S(u)$ and its negative part that of $S(v)$. At a triangle point the value is the coordinatewise median of the pins (7.5) at $\widehat{x}^{(u)}$, $\widehat{x}^{(v)}$, $\widehat{x}^{(v')}$.

Definition 8.7 (The lift)**.** Let g be a real map for Q' on $\mathcal{A}(Q')$. For a party $u\ne A$, a nonempty set V of parties v with $a_{uv}>0$ and $x\in\mathcal{A}(Q')$ in the interval (8.5), let

$$o_r^{u,V}(x)=\begin{cases} g_r(\xi_u) & \text{if } u\in J_r \text{ and } J_r\cap V=\emptyset\ ,\\ \beta_r & \text{if } u\in J_r \text{ and } J_r\cap V\ne\emptyset\ ,\\ \min_{v\in V} g_r(\xi_v) & \text{if } u\notin J_r \text{ and } V\subseteq J_r\ ,\\ 0 & \text{if } u\notin J_r \text{ and } V\not\subseteq J_r\ , \end{cases} \tag{8.8}$$

$$\theta_{u,V}(x)=a_u(V)-d_\alpha(x,x^{(u)})+\big\|o^{u,V}(x)-\beta y^{(u)}\big\|_1\ ,\qquad a_u(V)=\sum_{v\in V}a_{uv}\ , \tag{8.9}$$

and $G_{uv}(y)=\sum_{r:\ u,v\in J_r}g_r(y)$. The *lift* $\widehat{g}$ of g takes at $(x,\Gamma)\in\mathcal{A}(C(Q'))$ the following values, with all singleton counts not mentioned equal to 0, where $t^+=\max(t,0)$ and $t^-=\max(-t,0)$ for a real number t:

(i) if $\Gamma=\emptyset$, the counts $g_r(x)$;

(ii) if $\Gamma = \{uv\}$, the counts $o_r^{u,\{v\}}(x)$, the count $(G_{uv}(\xi_u) - G_{uv}(\xi_v))^+$ of $S(u)$ and the count $(G_{uv}(\xi_v) - G_{uv}(\xi_u))^+$ of $S(v)$;
(iii) if Γ is a fan with center u and leaves V, the counts $o_r^{u,V}(x)$ and the count $\theta_{u,V}(x)$ of $S(u)$;
(iv) if Γ is the triangle on u, v, v', the count β_r for every J_r that contains at least two of u, v, v', and 0 for the other J_r.

Exchanging u and v permutes the four cases of (8.8) for $V = \{v\}$, so $o^{u,\{v\}} = o^{v,\{u\}}$ and the values in (ii) are symmetric in u and v. If g is rational, so is $\widehat{g}$.

For example, $S(AB)+S(AC)+S(AD) \geq 2S(A)+S(ABCD)$ is centered on A with $a_{BC} = a_{BD} = a_{CD} = 1$ by (8.1), and $\mathcal{A}(Q') = \{0,1\}^3$ because its LHS terms are pairwise incomparable. The counts $g(0) = (0,0)$ and $g(x) = (|x|_\alpha - 1, 1)$ for $x \neq 0$ of $S(A)$ and $S(ABCD)$ satisfy the conditions of Lemma 2.15, so they form a real map: they are graded and monotone, and the only RHS term that contains B, C or D is $ABCD$, whose count 1 at $x \neq 0$ is at least $|x \wedge x^{(w)}|_\alpha \leq 1$ for $w = B, C, D$. At the fan point $(0, \{BC, BD\})$, $o^{B,\{C,D\}}(0) = (0,1) = \beta y^{(B)}$ and $\theta_{B,\{C,D\}}(0) = 2 - 1 + 0 = 1$, so the lift is $(0,1)$ with singleton counts $(1,0,0)$ of B, C, D. At the triangle point $(0, \{BC, BD, CD\})$ it is $(0,1)$ with singleton counts 0, so the values at these two points, which are at distance $a_{CD} = 1$, differ by 1. At $(x^{(B)}, \{BC, BD\})$, the occurrence word of B in $C(Q')$, the rule gives $(0,1)$ with $(2,0,0)$, the pin of B.

8.3. **When the lift is a map.** The lift has the pins and the ranges of a map for $C(Q')$ for every g, and what has to be decided is the contraction condition. On the full cube this condition on edges gives it on all pairs (Lemma 2.11). On $\mathcal{A}(C(Q'))$ two kinds of cube edges suffice: the edges between two points whose pair terms all contain one party u, and the three edges that join a triangle point to the points with one pair term fewer (Lemma D.6 in Appendix D.2). Such an edge switches one pair term at fixed x or changes one position of x at fixed Γ, and Proposition D.14 in Appendix D.4 computes the change of the lift along each of them from the grading and the monotonicity of g and from (8.2). Along every one of these edges the lift changes by at most the length of the edge, whatever the map g is, with one exception. The exception occurs at a fan with center u and leaves V, when a position l with $x_l = 0$ is switched on that lies outside $x^{(u)}$ and in every leaf word $x^{(v)}$, $v \in V$. We call such a position *shared*, and we write $x + \chi_l$ for the word x with the position l switched on. At a shared position every shadow ξ_v on a leaf grows by χ_l, the counts $\min_v g_r(\xi_v)$ of the terms that contain all leaves and miss the center grow, and the lift contracts along the edge exactly when their total growth is at most α_l.

Proposition 8.10 (When the lift is a map)**.** *The lift $\widehat{g}$ of a real map g for Q' on $\mathcal{A}(Q')$ is a real map for $C(Q')$ on $\mathcal{A}(C(Q'))$ if and only if, at every fan point (x, Γ) with center u and leaves V and every shared position l with $(x + \chi_l, \Gamma) \in \mathcal{A}(C(Q'))$,*

$$\sum_{r\colon u \notin J_r,\ V \subseteq J_r} \Big(\min_{v \in V} g_r(\xi_v + \chi_l) - \min_{v \in V} g_r(\xi_v) \Big) \leq \alpha_l \ . \tag{8.11}$$

We call (8.11) the *shared-position inequality*. The proof of the proposition is at the end of Appendix D.4, after the two statements just named.

8.4. **Capacity and the lifting theorem.** Each summand in (8.11) is at most the cap β_r, and the following condition bounds the total of these caps by the coefficient of the shared position.

Definition 8.12 (Capacity condition)**.** An inequality Q' centered on A satisfies the *capacity condition* if for every party $u \neq A$ and every LHS term I_l with $u \notin I_l$

$$\sum_{r:\, u\notin J_r,\ J_r\neq\{A\}} \beta_r \leq \alpha_l \ . \tag{8.13}$$

The condition is unchanged when Q' is multiplied by k.

Lemma 8.14 (Capacity)**.** *If Q' satisfies the capacity condition, every real map for Q' on $\mathcal{A}(Q')$ satisfies the shared-position inequality* (8.11) *at every fan point and every shared position.*

Proof. A shared position l lies outside $x^{(u)}$, so $u \notin I_l$. Since g is monotone and $\xi_v \leq \xi_v + \chi_l$ for every leaf v, each summand of (8.11) lies between 0 and $\min_v g_r(\xi_v + \chi_l) \leq \beta_r$. Each J_r in the sum contains the leaves, so $J_r \neq \{A\}$, and the sum is at most the left side of (8.13). □

The carriers that Section 9 uses are the carriers $Q(\mathcal{T})$ of block systems $\mathcal{T}$ of degree d (Definition 4.2), which are centered on A by Lemma 4.5. A block system has $|\mathcal{T}| = md/(j+1)$ blocks, so the condition $d \geq (j+1)(m-1)$ of the next lemma says $|\mathcal{T}| \geq m(m-1)$.

Lemma 8.15 (Capacity of carriers)**.** *If $d \geq (j+1)(m-1)$, the carrier $Q(\mathcal{T})$ satisfies the capacity condition. This holds for the system of all blocks of B if $j \geq 2$, and for the block systems of Lemma 6.7.*

The proof is in Appendix D.5. For each party $u \neq A$ it compares the total coefficient of the RHS terms other than A that miss u with the least coefficient of an LHS term that misses u. Every RHS term other than A contains H. With c as in (4.4), the two numbers are $dc < 1$ and 1 for q_w, and $(j+1)c \leq 1/m$ and $1/m$ for p_T. The bound on d holds for the system of all blocks by a binomial estimate and for the block systems of Lemma 6.7 by (6.6).

For $j = 1$ and all blocks, $d = 4 < 8 = (j+1)(m-1)$, so the capacity condition fails at the parties p_T, where $(j+1)c = \frac{2}{5} > \frac{1}{5} = \frac{1}{m}$, and the validity of $C(Q_{1,\mathrm{int}})$ and the bound $k^{\mathcal{A}}_{\min}(C(Q_{1,\mathrm{int}})) \leq 3$ that Theorem 1.1 needs come from the computation of Appendix B, which checks the lift of an integer map for $3Q_{1,\mathrm{int}}$ on all pairs of points of $\mathcal{A}(C(Q_{1,\mathrm{int}}))$.

Theorem 8.16 (Lifting)**.** *Let Q' be centered on A, satisfy the capacity condition, and have a real map on $\mathcal{A}(Q')$. Then*

(a) *for every $k \geq 1$, the lift of a real map for kQ' on $\mathcal{A}(Q')$ is a real map for $kC(Q')$ on $\mathcal{A}(C(Q'))$, and the lift of an integer map is an integer map;*
(b) *Q' and $C(Q')$ are valid,* $\mathrm{spec}^{\mathcal{A}}(C(Q')) = \mathrm{spec}^{\mathcal{A}}(Q')$ *and* $k^{\mathcal{A}}_{\min}(C(Q')) = k^{\mathcal{A}}_{\min}(Q')$*;*
(c) *if Q' has a real map with rational values on $\mathcal{A}(Q')$, this least scaling is finite.*

Proof. (a) The inequality kQ' is centered, satisfies the capacity condition and has the superbalancing $kC(Q')$, so Lemma 8.14 and Proposition 8.10 apply to it. If g is an integer map for kQ', then kQ' has integer coefficients. The $k\beta_r$ are integers by Definition 2.8. For a position l let $x^{[l]} \in \{0,1\}^L$ be the word with 1 exactly at the positions l' with $I_{l'} \supseteq I_l$. Both $x^{[l]}$ and $x^{[l]} - \chi_l$ lie in $\mathcal{A}(Q')$, because their positions form up-sets: if $I_{l'} \supseteq I_l$ with $l' \neq l$ and $I_{l''} \supseteq I_{l'}$, then $I_{l''} \supseteq I_l$, and $l'' \neq l$, because $l'' = l$ would give $I_{l'} = I_l$, while the LHS terms are distinct. So the grading of g gives $k\alpha_l = |g(x^{[l]})| - |g(x^{[l]} - \chi_l)|$, an integer. Hence the Bell values of kQ' are even by Proposition 7.3, and the coefficients ka_{uv}, ka_w of $kC(Q')$ are integers. The values of the lift are values of g, caps, minima, sums and differences of these, $ka_u(V)$ and $kd_\alpha(x, x^{(u)})$, all integers.

(b) A real map on the realizable domain proves validity [22, Lemma 2.13], for Q' and, through its lift, for $C(Q')$. By (a), $\mathrm{spec}^{\mathcal{A}}(Q') \subseteq \mathrm{spec}^{\mathcal{A}}(C(Q'))$, and Proposition 7.6 gives the opposite inclusion.

(c) If D is a common denominator of the coefficients of Q' and the values of a rational real map g, then Dg is an integer map for DQ', and $k^{\mathcal{A}}_{\min}(C(Q')) = k^{\mathcal{A}}_{\min}(Q') \le D$. □

Section 9 applies Theorem 8.16 to the carriers of Theorem 1.1 with $j \ge 2$ and to the centered summands of the disjoint sums of Theorem 1.2, which satisfy the capacity condition by Lemma 8.15, and so carries their lower bounds and finite least scalings on the realizable domain over to the superbalancings.

9. Proofs of Theorems 1.1 and 1.2

All parts of Theorems 1.1 and 1.2 are now in place: the forced factors and the validity of the carriers (Corollaries 6.4 and 6.8), the transfer of lower bounds to the superbalancing (Proposition 7.6), the lift of maps to its realizable domain (Theorem 8.16), and the behavior of the scaling spectra under disjoint sums (Lemmas 2.18 and 2.19). This section assembles the proofs. Theorem 1.1 concerns a single carrier of the complete family. Theorem 1.2 uses disjoint sums of carriers from the projective line with pairwise coprime divisors of their forced factors, so that these divisors multiply while the numbers of parties add.

9.1. **Proof of Theorem 1.1.** We write $C_j = C(Q_{j,\mathrm{int}})$ for the superbalancing of $Q_{j,\mathrm{int}}$. The only arithmetic needed is that a prime $2j+1$ survives the passage from Q_j to $Q_{j,\mathrm{int}} = K_j Q_j$.

Lemma 9.1. *If $p = 2j+1$ is a prime, then p does not divide K_j.*

Proof. By Lemma 4.5(ii), K_j divides $m(j+1)\binom{3j+1}{j}$ with $m = 3j+2$. Since $2m - 3p = 1$ and $2(j+1) - p = 1$, the prime p divides neither m nor $j+1$. In $\binom{3j+1}{j} = \frac{(3j+1)!}{j!(2j+1)!}$, the only multiple of p among $1, \dots, 3j+1$ is p itself, because $2p = 4j+2 > 3j+1$. So p divides $(3j+1)!$ and $(2j+1)!$ exactly once and does not divide $j!$, and it does not divide the binomial coefficient. □

Proof of Theorem 1.1. Validity. For $j \ge 2$, $Q_{j,\mathrm{int}}$ is valid with $k_{\min}(Q_{j,\mathrm{int}}) < \infty$ by Corollary 6.4. For $j = 1$, where $N_1 = 17$ and $K_1 = 10$, Appendix B gives an integer map for $3Q_{1,\mathrm{int}}$ on the full cube, so $Q_{1,\mathrm{int}}$ is valid and $k_{\min}(Q_{1,\mathrm{int}}) \le 3$. By Corollary 6.4, $Q_{j,\mathrm{int}}$ is centered on A, hence balanced, on the N_j parties of (6.3), and its coefficients have greatest common divisor 1 by Lemma 4.5(ii). By Proposition 7.3, C_j is superbalanced and has integer coefficients, and by (7.4) the terms of $Q_{j,\mathrm{int}}$ occur in C_j with their coefficients, so the coefficients of C_j have greatest common divisor 1 as well. The inequality C_j is valid for $j \ge 2$ by Theorem 8.16(b), whose hypotheses are verified in the third step below, and for $j = 1$ by the integer map for $3C_1$ on $\mathcal{A}(C_1)$ of Appendix B, which is a real map for $3C_1$ on its realizable domain and so proves that $3C_1$, and hence C_1, is valid by [22, Lemma 2.13].

The lower bounds. By Corollary 6.4, every k for which $kQ_{j,\mathrm{int}}$ has an integer map on $\mathcal{A}(Q_{j,\mathrm{int}})$ is divisible by $(2j+1)/\gcd(2j+1, K_j)$. A map on the full cube restricts to the realizable domain, so the same holds for the full cube. By Proposition 7.6, an integer map for kC_j on $\mathcal{A}(C_j)$ or on the full cube gives an integer map for $kQ_{j,\mathrm{int}}$ on $\mathcal{A}(Q_{j,\mathrm{int}})$ or on the full cube, so the same divisibility holds for C_j on both domains. If $p = 2j+1$ is prime, then $\gcd(p, K_j) = 1$ by Lemma 9.1, and p divides every such k.

Finiteness for $j \ge 2$. The water-level map of $\mathcal{T}_j$, multiplied by K_j and restricted to $\mathcal{A}(Q_{j,\mathrm{int}})$, is a real map for $Q_{j,\mathrm{int}}$ with rational values (Theorem 5.7), and $Q_{j,\mathrm{int}}$ satisfies the capacity condition by Lemma 8.15. So Theorem 8.16 gives the validity of C_j and $k^{\mathcal{A}}_{\min}(C_j) = k^{\mathcal{A}}_{\min}(Q_{j,\mathrm{int}}) \le k_{\min}(Q_{j,\mathrm{int}}) < \infty$.

The case $j = 1$. We have $k^{\mathcal{A}}_{\min}(Q_{1,\mathrm{int}}) \le k_{\min}(Q_{1,\mathrm{int}}) \le 3$ from the map above, and by Appendix B, $3C_1$ has an integer map on $\mathcal{A}(C_1)$, so $k^{\mathcal{A}}_{\min}(C_1) \le 3$. Since $3 = 2j + 1$ is prime, the lower bounds give $k^{\mathcal{A}}_{\min}(Q_{1,\mathrm{int}}) = k_{\min}(Q_{1,\mathrm{int}}) = k^{\mathcal{A}}_{\min}(C_1) = 3$. □

9.2. **Disjoint sums of carriers.** Both parts of Theorem 1.2 use the same construction, which the next proposition isolates. We call an inequality a *design carrier* if it is $Q_{\mathcal{T},\mathrm{int}}$ for a 3-design $\mathcal{T}$ of Corollary 6.8.

Proposition 9.2 (Sums with coprime forced factors)**.** *Let $Q^{(1)}, \dots, Q^{(r)}$ be design carriers on pairwise disjoint sets of parties, and let $f_1, \dots, f_r$ be pairwise coprime positive integers such that f_t divides every k for which $kQ^{(t)}$ has an integer map on $\mathcal{A}(Q^{(t)})$. Let N be at least the total number of parties, and let Q and $\widehat{Q}$ be the paddings to N parties of*

$$Q^{(1)} \oplus \ldots \oplus Q^{(r)} \qquad \text{and} \qquad C(Q^{(1)}) \oplus \ldots \oplus C(Q^{(r)}) \ .$$

Then Q is a valid balanced N-party inequality and $\widehat{Q}$ a valid superbalanced N-party inequality, both with integer coefficients of greatest common divisor 1, and

(9.3)
$$f_1 \cdots f_r \le k^{\mathcal{A}}_{\min}(Q) \le k_{\min}(Q) < \infty \ , \qquad f_1 \cdots f_r \le k^{\mathcal{A}}_{\min}(\widehat{Q}) < \infty \ , \qquad k_{\min}(\widehat{Q}) \ge k^{\mathcal{A}}_{\min}(\widehat{Q}) \ .$$

Proof. Each $Q^{(t)}$ is valid and centered with $k_{\min}(Q^{(t)}) < \infty$, and has integer coefficients of greatest common divisor 1, by Corollary 6.8(i). By Lemma 2.18, extended to r summands as described after Lemma 2.19, the sum $Q^{(1)} \oplus \ldots \oplus Q^{(r)}$ is valid and balanced, and its least scaling on the full cube is at most the least common multiple of the $k_{\min}(Q^{(t)})$. The coefficients of each summand occur among those of the sum, so the sum has integer coefficients of greatest common divisor 1. By Lemma 2.19, $\mathrm{spec}^{\mathcal{A}}$ of the sum is the intersection of the $\mathrm{spec}^{\mathcal{A}}(Q^{(t)})$, so each of its elements is divisible by every f_t, and hence by $f_1 \cdots f_r$, since the f_t are pairwise coprime. With $k^{\mathcal{A}}_{\min} \le k_{\min}$ this is the first chain of (9.3) for the sum.

The inequality $Q^{(t)}$ has a real map with rational values on $\mathcal{A}(Q^{(t)})$, the restriction of a multiple of the water-level map, and it satisfies the capacity condition by Lemma 8.15. So $C(Q^{(t)})$ is valid and $k^{\mathcal{A}}_{\min}(C(Q^{(t)})) < \infty$ by Theorem 8.16. It is superbalanced with integer coefficients by Proposition 7.3, and its coefficients have greatest common divisor 1 because the terms of $Q^{(t)}$ occur in it with their coefficients, by (7.4). The disjoint sum of the $C(Q^{(t)})$ is therefore valid and superbalanced by Lemma 2.18, with integer coefficients of greatest common divisor 1. By Proposition 7.6, $\mathrm{spec}^{\mathcal{A}}(C(Q^{(t)})) \subseteq \mathrm{spec}^{\mathcal{A}}(Q^{(t)})$, so every element of $\mathrm{spec}^{\mathcal{A}}(C(Q^{(t)}))$ is divisible by f_t. By Lemma 2.19, $\mathrm{spec}^{\mathcal{A}}$ of the sum of the $C(Q^{(t)})$ is the intersection of the $\mathrm{spec}^{\mathcal{A}}(C(Q^{(t)}))$. It contains the least common multiple of the $k^{\mathcal{A}}_{\min}(C(Q^{(t)}))$, and all its elements are divisible by $f_1 \cdots f_r$. The last inequality of (9.3) holds for every valid inequality.

Padding preserves validity, balance, superbalance, the spectra and the coefficients (Section 2.6), so all statements hold for Q and $\widehat{Q}$. □

9.3. **The two growth bounds.** Proposition 9.2 reduces Theorem 1.2 to a supply of design carriers on few parties with large pairwise coprime divisors of their forced factors. By Corollary 6.8, for a prime power $q \ge 256$ with $q \equiv 1 \pmod 3$ the design carrier over the field with q elements has at most $2q^5 \ln q$ parties and forces the factor $r_q / \gcd(r_q, 3)$ with $r_q = (2q + 1)/3$. The two parts of the theorem differ in the choice of the fields. We describe the two choices here, and the complete proofs, with the estimates from the prime number theorem, are in Appendix E.

For part (a) we take the fields with $q_n = 2^{n-1}$ elements, where $n \ge 11$ is a prime. Their forced factors are the numbers $(2^n + 1)/3$, which are pairwise coprime (Lemma E.1), because a common

divisor of 2^n+1 and $2^{n'}+1$ for distinct primes n, n' divides 3. The disjoint sum over the primes n with $11 \le n \le n_0$ has at most $n_0 2^{5n_0}$ parties, and the logarithm of the product of its forced factors is about $\ln 2$ times the sum of these primes, which by the prime number theorem is at least $(\frac{\ln 2}{2} - o(1))n_0^2/\ln n_0$ as $n_0 \to \infty$. Since $\ln N \le (5\ln 2 + o(1))n_0$ for the number N of parties, this gives the bound of part (a).

The fields $\mathbb{F}_{2^n-1}$ are sparse, and their forced factors are small compared with the number of parties. For part (b) we therefore choose the field for the prime that it is to force. For a prime $\ell > 512$ let $q(\ell)$ be the least prime q with $q \equiv 1 \pmod 3$ and $2q+1 \equiv 0 \pmod \ell$. These two conditions define one residue class modulo 3ℓ, so the hypothesis of part (b), which holds by Linnik's theorem, gives $q(\ell) \le C_L(3\ell)^L$, and the design carrier over the field with $q(\ell)$ elements forces a multiple of ℓ. The disjoint sum of these carriers over the primes ℓ with $512 < \ell \le Y$ forces the product of all these primes, whose logarithm is $(1+o(1))Y$ as $Y \to \infty$, on at most a constant times Y^{5L+1} parties. This gives the bound of part (b).

In both parts the inequalities are sums of carriers of one kind, and the exponents reflect two costs: a carrier on the projective line over $\mathbb{F}_q$ has about $q^5 \ln q$ parties for a forced factor of size about q, and the field that forces a given prime ℓ can be as large as ℓ^L.

10. Discussion and conclusions

We have constructed valid balanced and superbalanced holographic entropy inequalities whose contraction proofs need a large scaling. The explicit family of Theorem 1.1 starts with a 17-party inequality of least scaling exactly 3, and every odd prime occurs in it as a forced factor $2j+1$. The disjoint sums of Theorem 1.2 have least scalings that grow faster than any power of the number N of parties. For the balanced inequalities the least scaling was shown to be finite on the full cube. For the superbalanced ones it was shown to be finite on the realizable domain, and the lower bounds were proved on both domains. All lower bounds were obtained from one count, the claw count of Section 3, and all upper bounds from the water-level map and its lift, with the values at 17 parties checked on explicit integer maps (Appendix B).

Consequences for contraction proofs. A contraction proof distributes the bulk pieces cut out by the LHS surfaces among the RHS terms. Our inequalities are valid because a distribution in fractions exists, and a distribution of whole pieces among the copies of the RHS terms of kQ exists only when the forced factors divide k. Three consequences follow from the theorems. First, a search for a map at $k=1$, or at any fixed k, cannot decide validity for balanced or for superbalanced inequalities, since for every k there are valid inequalities in both classes with least scaling above k. Second, the completeness statement of [32] allows a uniform rescaling of the inequality before a map is sought, and this rescaling cannot be bounded by a power of N on either class: the balanced inequalities of Theorem 1.2 have a finite least scaling on the full cube that grows faster than any power of N, and their superbalanced companions have the same lower bounds. Third, the lower bounds persist on the realizable domain, that is, for maps defined only on the bit strings compatible with nesting and disjointness. For $k=1$, the first and the third consequence and the need for a rescaling were shown in [22]. The present paper extends the first and the third consequence to every fixed k and adds the second, for balanced and for superbalanced inequalities.

The growth faster than any power of N comes from disjoint sums, since the factor forced in a single carrier is smaller than its number of parties (Section 6). This factor is a lower bound, and the least scaling of a single carrier is known only for 17 parties. A disjoint sum is valid because its summands are (Lemma 2.18), and each summand has a map at its own scaling. What grows in

Theorem 1.2 is the scaling at which one map for the whole inequality exists, which has to lie in the spectrum of every summand.

Holographic states. The inequalities of this paper are valid. The facets form a minimal set of inequalities that defines the cone [3], so each of our inequalities follows from the facet inequalities. The results therefore concern certificates of validity, and the constraints on states remain those that define the cone.

Which of our inequalities define facets themselves is settled in part. The carriers are centered. Grimaldi, Headrick, Hubeny and Shteyner note that all centered valid inequalities are redundant [29]. Bao and Ferko prove in a preprint that a nonzero valid centered inequality has a pair of parties that lies in terms of different total coefficients on the two sides [49, Theorem 2.4], so that it is not superbalanced. For the carriers of Section 4 this is explicit: by (2.5), (4.6) and (4.4), the Bell value on the pair H, q_v is $1 + \frac{1}{m} - dc = \frac{2}{m}$, since the two parties are separated by the box term of v and the term Aq_B on the left and by the d claw terms with center v on the right, where d is the degree of the block system, so that $dc = \frac{m-1}{m}$. For the balanced inequalities of our theorems the conclusion has a direct proof, given in Appendix G. Its idea is that if an inequality that defines a facet is the sum of two valid inequalities, both are multiples of it, while each of our balanced inequalities is the sum of a valid inequality and a positive multiple of the instance $I(H{:}q_v) \geq 0$ of subadditivity. The same argument covers the superbalanced disjoint sums with two or more summands. It leaves open the superbalancings of single carriers, among them the inequalities C_j of Theorem 1.1 (the fourth question below).

A proof by contraction uses only that the bulk slice is a Riemannian manifold with boundary [14], so the inequalities and our statements about their proofs hold for static states of every background in which each entropy is the minimal area of a surface homologous to its region. They are the same for a confining theory and for a conformal one. For gravity duals of confining gauge theories, Klebanov, Kutasov and Murugan use a generalization of the formula to ten-dimensional geometries with a varying dilaton ϕ, the area in the string-frame metric weighted by $e^{-2\phi}$ [50], and this functional is the area in the metric rescaled by the factor $e^{-\phi/2}$, so, minimized over the surfaces homologous to the region, it is of the kind to which the proof applies. For renormalization group flows that are holographic at every scale, Bao and Ferko find that inequalities beyond strong subadditivity constrain entanglement along the flow [49], and our inequalities enter such statements only through their validity.

Two limitations of the setting apply to our results as to every statement about the cone. The formula captures the leading order of the expansion in G_N. Faulkner, Lewkowycz and Maldacena describe its one-loop correction [4], Engelhardt and Wall propose the quantum extremal surface formula for all orders [51], and once general quantum corrections are included the inequalities are no longer satisfied [14]. For time-dependent states the validity of the inequalities is undecided, although significant evidence has accumulated that they hold [25], and Grimaldi and Headrick review what is known beyond static states in classical gravity [52]. In both directions there are results that take a contraction map as their input, and the next subsection relates our maps to them.

Arguments that take a contraction map as input. Several results beyond the static classical setting are proved for inequalities that have a contraction map (Section 1). What our theorems imply for them depends on the form of the map that each result takes as its input, and two forms occur. The results below take a map of the inequality as their input, in their statements or, for Akers, Hernández-Cuenca and Rath, in the argument for their theorem. What they give when only a multiple kQ has one is, in every case, our inference from their hypotheses.

The first form is a map from the full cube with one bit for each LHS term that contracts the weighted Hamming distances, for an inequality with positive coefficients. It is the input of the theorem of Czech and Dong for time-dependent states in three bulk dimensions [28]. Akers, Hernández-Cuenca and Rath use it in the argument by which they prove that the entropies given by the quantum extremal surface formula obey all inequalities of the cone if the bulk entropies do [14, Theorem 2]: for each inequality the map also proves the bulk inequality from which the inequality for these entropies follows. Bousso and Kaya assume it in their theorem that extends every inequality with such a map and positive integer coefficients to the areas of static generalized entanglement wedges of bulk regions in arbitrary spacetimes, under an independence condition on the regions [53, Theorem 12].

An integer map for kQ on the full cube is a map of this form for kQ written with $k\beta_r$ unit copies of each RHS term (Section 2.3), and the conclusion of each of these theorems for kQ is its conclusion for Q. Our inference is that the three theorems, in the scope that each work states, cover the balanced inequalities of Theorems 1.1 and 1.2, whose multiples have integer maps on the full cube. The inference assumes that an RHS term may be listed several times in the hypotheses of these theorems, which their statements leave unrestricted, and we have not examined the proofs for this case.

Where a theorem of this kind holds for every facet, it also holds for our balanced inequalities, which follow from the facet inequalities, whatever their least scaling. What our lower bounds mean for these theorems depends also on the fifth question below, whether a facet can have a least scaling above 1.

The second form is a map from the cube with one bit for each copy of an LHS term, in the presentation with unit coefficients and repeated terms. It is the form of the contraction-provable inequalities of Grimaldi, Headrick, Hubeny and Shteyner [29, Definition 3] and of the maps of Czech, Feng, Wu and Xie [30]. For maps of this form it is proved that the null reductions pass the majorization test, by Grimaldi, Headrick, Hubeny and Shteyner [29] for superbalanced inequalities and by Czech, Feng, Wu and Xie [30] for balanced ones, and that the null reductions of superbalanced inequalities are valid. Grimaldi, Headrick and Hubeny assume unit coefficients in the same way in the proof of their theorem on null reductions [25, Theorem 2]. Our maps for kQ have one bit for each distinct LHS term, with the weight $k\alpha_l$, so on the full cube they are defined on a part of the domain of a map of the second form, the points at which all copies of a term agree. Whether multiples of our balanced inequalities have maps of the second form is the sixth question below. Whether the majorization theorem of Czech, Feng, Wu and Xie applies to them rests on this question, and the theorems on null reductions assume superbalance. For the results of Czech and Shuai [31] we have not examined the form of map they take as input.

For the superbalanced inequalities we have maps on the realizable domain only. On this domain the two forms coincide, as Grimaldi, Headrick, Hubeny and Shteyner note [29], while the theorems above take maps on the full cube, so whether they apply to the superbalanced inequalities rests on the first question below.

Open questions. Multiple lines of generalization lie open. First, whether the superbalanced inequalities C_j and $\widehat{Q}$ of the two theorems have integer maps on the full cube at some scaling. For contraction maps, Grimaldi, Headrick, Hubeny and Shteyner pose the question whether a map on the realizable subset can be extended directly to one on the full cube, and they announce further work on it [29]. Bao and Naskar observe that for all valid inequalities known to them contraction maps exist whether or not the bit strings outside the realizable domain are included, and they leave for future work a proof that the conditions from these bit strings are redundant [17].

Second, the true growth of the least scaling. Our lower bound $\exp\left(c_0 N^{1/26}\right)$ of Theorem 1.2(b) is the bound of one construction, and the true growth may be faster. This bound grows faster than any power of N but more slowly than a^N for every $a > 1$, so the question of [22], whether the least scaling of balanced or superbalanced inequalities can grow exponentially with N, stays undecided, while for star inequalities the largest finite least scaling is at least $N^{(1/4-o(1))N}$ [22]. We expect to return to this question in a later paper of this series.

Third, whether a balanced or superbalanced inequality that is not a padding of a disjoint sum can have a least scaling larger than every power of N. Fourth, whether the superbalancing of a single carrier, such as the inequalities C_j of Theorem 1.1, can define a facet of HEC_N. Fifth, whether some facet of HEC_N has least scaling above 1, on the full cube or on the realizable domain, a question left undecided in [22].

Sixth, whether multiples of our balanced inequalities have maps on the full cube with one bit for each copy of an LHS term, the form in which the theorems on majorization and on null reductions are stated. Such a map for kQ restricts, at the points where all copies of a term agree, to a map of our kind for kQ (Section 2.3), so our lower bounds hold in this form, while the maps that our theorems give on the full cube have merged LHS terms. Grimaldi, Headrick, Hubeny and Shteyner discuss multiplication in this form. In their discussion of primitive inequalities, those that cannot be written as a positive integer combination of other valid inequalities, they consider a candidate inequality written with unit coefficients and repeated terms that has no contraction map with one bit for each copy of an LHS term and of an RHS term, and remark that, empirically, multiplying both sides by a positive integer does not produce such a map [29].

Seventh, whether the proofs of the theorems that take a map with one bit for each LHS term extend to inequalities with repeated RHS terms, as our inference for the balanced inequalities assumes.

Appendix A. Proofs for Section 2

This appendix proves the necessary and sufficient conditions for maps of centered inequalities (Lemmas 2.14 and 2.15), the statements on padding made in Section 2.6, and the two lemmas on disjoint sums (Lemmas 2.18 and 2.19). The notation is that of Section 2.

A.1. **Maps of centered inequalities.** For contraction maps, that is, integer maps at $k = 1$ on the full cube, the parts of Lemma 2.14 appear in earlier work. Bao and Naskar show that a contraction map sends every point on a shortest path between two points whose distance it preserves to a shortest path between their images, which fixes the coordinates in which the two images agree [17, Lemma 1 and Rule 1]. Balance makes a map preserve the distances from 0 to $x^{(A)}$ and to $x^{(i)}$, and this gives (a), (b), (c) and (e). The first paper of this series uses the same argument between the pins of two parties with Bell value 0 [22]. Part (c) is a step of the proofs of [25, Theorem 2] and [29, Theorem 1]. Czech, Feng, Wu and Xie [30] derive (a), (c), (d) and (e) from two facts on the collinearity of a point with 0 and an occurrence word, which they attribute to Czech and Shuai [31]. Grimaldi, Headrick, Hubeny and Shteyner derive (a), (b) and (d) for centered inequalities in the necessity half of the proof of their Theorem 2 [29], and our proof follows theirs.

Proof of Lemma 2.14. For a party i let $Y_i = \{r\colon i \in J_r\}$, so that the pin of i reads $n_r(x^{(i)}) = k\beta_r[r \in Y_i]$, $|n(x^{(i)})| = k|y^{(i)}|_\beta$, and balance reads $|y^{(i)}|_\beta = |x^{(i)}|_\alpha$. Let $z, z' \in \Omega \cap [0, x^{(A)}]$.

(a) Contraction between z and 0, where $n(0) = 0$, gives $|n(z)| \le k d_\alpha(0, z) = k|z|_\alpha$. Since $z \le x^{(A)}$, we have $d_\alpha(z, x^{(A)}) = |x^{(A)}|_\alpha - |z|_\alpha$, and contraction between z and $x^{(A)}$, the pin of A

and balance give

$$|n(z)| \geq |n(x^{(A)})| - \sum_r |n_r(x^{(A)}) - n_r(z)| \geq k|x^{(A)}|_\alpha - k(|x^{(A)}|_\alpha - |z|_\alpha) = k|z|_\alpha \ .$$

(b) If $z \leq z'$, then $d_\alpha(z, z') = |z'|_\alpha - |z|_\alpha$, so by (a) and contraction

$$\sum_r (n_r(z') - n_r(z)) = k|z'|_\alpha - k|z|_\alpha = kd_\alpha(z, z') \geq \sum_r |n_r(z') - n_r(z)| \ .$$

So the sum over r of the nonnegative numbers $|n_r(z') - n_r(z)| - (n_r(z') - n_r(z))$ is at most 0, every one of them vanishes, and $n_r(z') \geq n_r(z)$ for every r.

(c) The word $x^{(A)}$ lies in $\Omega \cap [0, x^{(A)}]$, so (b) with $z' = x^{(A)}$ gives $0 \leq n_r(z) \leq n_r(x^{(A)}) = 0$ for $r \notin Y_A$.

(d) Since $0 \leq n_r(z) \leq k\beta_r$ and the pin of i is a vertex of the product $\prod_r [0, k\beta_r]$, the left side of the contraction condition between z and $x^{(i)}$ is, as in [22, Lemma 6.5],

$$\sum_{r \notin Y_i} n_r(z) + \sum_{r \in Y_i} (k\beta_r - n_r(z)) = |n(z)| + k|y^{(i)}|_\beta - 2\sum_{r \in Y_i} n_r(z) \ ,$$

and its right side is $kd_\alpha(z, x^{(i)}) = k|z|_\alpha + k|x^{(i)}|_\alpha - 2k|z \wedge x^{(i)}|_\alpha$. By (a) and balance, the contraction condition becomes $\sum_{r \in Y_i} n_r(z) \geq k|z \wedge x^{(i)}|_\alpha$.

(e) If $z \leq x^{(i)} \leq x^{(A)}$, then $x^{(i)} \in \Omega \cap [0, x^{(A)}]$, and (b) gives $0 \leq n_r(z) \leq n_r(x^{(i)}) = 0$ for $r \notin Y_i$. If Q is centered on A, then $x^{(A)} = \mathbf{1}$ by Definition 2.6. □

For maps with values in $\{0, 1\}$ of the presentation in which every term has coefficient 1, the characterization of Section 2.5 is Theorem 2 of Grimaldi, Headrick, Hubeny and Shteyner [29], where maps with the three properties (2.16) are called inclusion dominance maps. Lemmas 2.14 and 2.15 are their statement in the language of count maps, extended to scaled and real maps, to the presentation with rational coefficients and without repeated terms, and, for the necessary conditions, to balanced inequalities on the interval $[0, x^{(A)}]$ and to every domain that contains 0 and the occurrence words. For sufficiency they argue through the meet $u \wedge u'$ of two points, and our proof of Lemma 2.15 checks the contraction condition on the edges of the cube.

Proof of Lemma 2.15. The ranges hold by assumption. *Pins.* At $x = 0$ the first condition gives $|n(0)| = 0$, so $n(0) = 0$. At $x = x^{(i)}$ the third condition and balance give $\sum_{r \in Y_i} n_r(x^{(i)}) \geq k|x^{(i)}|_\alpha = \sum_{r \in Y_i} k\beta_r$, and since $n_r \leq k\beta_r$, every $r \in Y_i$ has $n_r(x^{(i)}) = k\beta_r$. The first condition gives $|n(x^{(i)})| = k|x^{(i)}|_\alpha = \sum_{r \in Y_i} k\beta_r$, so the values $n_r(x^{(i)}) \geq 0$ with $r \notin Y_i$ sum to 0 and vanish. *Edges.* If x' is obtained from x by switching on the bit l, then $n(x') \geq n(x)$, and $\sum_r |n_r(x') - n_r(x)| = |n(x')| - |n(x)| = k\alpha_l = kd_\alpha(x, x')$. *All pairs.* Lemma 2.11 gives (2.10) for all pairs. □

A.2. **Padding and disjoint sums.** *Padding.* Let $\widetilde{Q}$ be a padding of Q. Its validity is equivalent to that of Q by the argument given in Section 2.6. The terms of Q and $\widetilde{Q}$ are the same sets, so the two inequalities have the same cube, the same realizable domain and the same occurrence words of the old parties, and a new party has the occurrence words 0, whose pin is the pin $n(0) = 0$ of the purifier. Hence Q and $\widetilde{Q}$ have the same real and integer maps, $\mathrm{spec}(\widetilde{Q}) = \mathrm{spec}(Q)$ and $\mathrm{spec}^{\mathcal{A}}(\widetilde{Q}) = \mathrm{spec}^{\mathcal{A}}(Q)$. A new party i_0 is separated from every old party i by the same terms as the purifier, so $\widetilde{Q} \cdot r_{i_0 i} = Q \cdot r_{Oi}$, and no term separates i_0 from O or from another new party, so these Bell values vanish. Hence $\widetilde{Q}$ is balanced, superbalanced or centered whenever Q is.

Disjoint sums. Let Q and Q' be as in Definition 2.17. Since the terms of Q' are subsets of P', the occurrence words of $i \in P$ in $Q \oplus Q'$ are $(x^{(i)}, 0)$ and $(y^{(i)}, 0)$, and those of $i' \in P'$ are $(0, x'^{(i')})$

and $(0, y'^{(i')})$. A real map n on a subset of this cube has components $n = (n^Q, n^{Q'})$ on the RHS terms of Q and of Q'.

Proof of Lemma 2.18. (i) The paddings of Q and Q' to $P \cup P'$ are valid, and so is their sum $Q \oplus Q'$. For $i_1, i_2 \in P \cup \{O\}$, the terms that separate i_1 from i_2 are terms of Q, so $(Q \oplus Q') \cdot r_{i_1 i_2} = Q \cdot r_{i_1 i_2}$, and likewise for $i_1, i_2 \in P' \cup \{O\}$. For $i \in P$ and $i' \in P'$, the terms that separate i from i' are the terms of Q that contain i and the terms of Q' that contain i', so $(Q \oplus Q') \cdot r_{ii'} = Q \cdot r_{iO} + Q' \cdot r_{i'O}$. All these values vanish if Q and Q' are superbalanced, and those with $i_2 = O$ vanish if both are balanced.

(ii) Let n be a real map for $k(Q \oplus Q')$ on the full cube, and put $g(x) = n^Q(x, 0)$ for $x \in \{0,1\}^L$. The ranges of g are those of n^Q. The pin of $i \in P$ in $Q \oplus Q'$ gives $g_r(x^{(i)}) = n^Q_r(x^{(i)}, 0) = k\beta_r[i \in J_r]$, and $g(0) = n^Q(0,0) = 0$. Dropping the components of Q' lowers the left side of the contraction condition, so

$$\sum_{r=1}^{R} |g_r(x) - g_r(y)| \le \sum |n(x,0) - n(y,0)| \le k d_\alpha(x,y) \ ,$$

where the middle sum runs over all $R + R'$ components. So g is a real map for kQ, and an integer map if n is one. The same argument gives a map for kQ'. Conversely, let g and g' be real maps for kQ and kQ' on the full cubes, and put $n(x, x') = (g(x), g'(x'))$. The pin of $i \in P$ holds, $n(x^{(i)}, 0) = (g(x^{(i)}), 0)$, because the RHS terms of Q' are subsets of P', likewise for $i' \in P'$, and $n(0,0) = 0$. The contraction condition holds because both sides split into a sum over the coordinates of Q and one over those of Q'. The map n is an integer map if g and g' are. □

Proof of Lemma 2.19. (i) Let $(x, x') \in \mathcal{A}(Q \oplus Q')$. A term I_l of Q and a term $I'_{l'}$ of Q' are disjoint, so by (R1) at most one of the bits x_l and $x'_{l'}$ is 1. Hence $x = 0$ or $x' = 0$. If $x' = 0$, the conditions (R1) and (R2) for two positions of Q are those of Q, so $x \in \mathcal{A}(Q)$, and likewise if $x = 0$. Conversely, let $x \in \mathcal{A}(Q)$. In $(x, 0)$ every pair of positions with value 1 is a pair of positions of Q, so (R1) holds. If $x_l = 1$ and $I_l \subseteq K$ for an LHS term K of $Q \oplus Q'$, then K is a term of Q, because I_l is a nonempty subset of P, and (R2) for Q gives the bit of K the value 1. So $(x, 0) \in \mathcal{A}(Q \oplus Q')$, and likewise $(0, x')$ for $x' \in \mathcal{A}(Q')$.

(ii) If n is a real map for $k(Q \oplus Q')$ on $\mathcal{A}(Q \oplus Q')$, then by (i) the function $g(x) = n^Q(x, 0)$ is defined on $\mathcal{A}(Q)$, and it is a real map for kQ there by the argument of Lemma 2.18(ii). Likewise for kQ'. Conversely, let g and g' be real maps for kQ on $\mathcal{A}(Q)$ and for kQ' on $\mathcal{A}(Q')$. By (i), every point of $\mathcal{A}(Q \oplus Q')$ is of the form $(x, 0)$ or $(0, x')$, and we put $n(x, 0) = (g(x), 0)$ and $n(0, x') = (0, g'(x'))$, which agree at $(0, 0)$ because $g(0) = 0$ and $g'(0) = 0$. The pins and ranges hold as in Lemma 2.18, and the contraction condition between two points of the same form is that of g or of g'. For $x \in \mathcal{A}(Q)$ and $x' \in \mathcal{A}(Q')$, using $g(0) = 0$ and $g'(0) = 0$,

$$\sum |n(x,0) - n(0,x')| = \sum_r |g_r(x) - g_r(0)| + \sum_{r'} |g'_{r'}(x') - g'_{r'}(0)| \le k d_\alpha(x,0) + k d_{\alpha'}(x',0) \ ,$$

which is k times the distance between $(x, 0)$ and $(0, x')$. The map n is an integer map if g and g' are. □

Appendix B. The computations

This appendix describes three computations. The second and the third are used by the proof of Theorem 1.1 at $j = 1$, and the first checks the data printed in Appendix F. They concern the 17-party inequality $Q_{1,\mathrm{int}}$ displayed after Theorem 1.1 and its superbalancing. Every check is done in exact integer arithmetic: rational numbers are multiplied by a common denominator before they

are compared, and no floating-point number enters a check. The two integer maps below were found by solving integer programs. The proof uses only the maps themselves, and each map is checked against Definition 2.8 directly, from the list of terms of the inequality. The repository named in the introduction contains the two maps and the programs that perform the checks of this appendix.

The data of $Q_{1,\mathrm{int}}$ and $C(Q_{1,\mathrm{int}})$ (Appendix F). The program builds $Q_{1,\mathrm{int}}$ from the definition of Section 4 with $j = 1$, checks that every term contains A, and computes all Bell values (2.5) on the 153 pairs of the 17 parties and the purifier. They vanish on every pair that contains A or O, so $Q_{1,\mathrm{int}}$ is centered on A, and they are even. The program then forms $C(Q_{1,\mathrm{int}})$ by the construction of Section 7, and checks that its coefficients are integers and that all its Bell values vanish. The program reproduces the numbers $L = 131$ and $R = 38$ of LHS and RHS terms of $C(Q_{1,\mathrm{int}})$, and the coefficients of its pair and singleton terms are the weights a_{uv} and a_u of Table 2. The same statements are proved for every j in Sections 4 and 7, so this computation checks only the data printed in Appendix F.

$k_{\min}(Q_{1,\mathrm{int}}) \le 3$ *(Theorem 1.1 at $j = 1$).* The certificate is an integer map n for $3Q_{1,\mathrm{int}}$ on the full cube $\{0,1\}^{11}$, a table of the values $n_r(u)$ at the 2,048 points u for the 22 RHS terms. The program checks the pins (2.9) at the occurrence words of the 17 parties and at 0, the ranges $n_r(u) \in \{0, 1, \dots, 3\beta_r\}$, and the contraction condition (2.10) on the 11,264 edges of the cube, which suffices on the full cube by Lemma 2.11. It also checks the condition directly on all 2,096,128 pairs of points. With the obstruction of Sections 3 and 4, by which 3 divides every k for which $kQ_{1,\mathrm{int}}$ has an integer map on $\mathcal{A}(Q_{1,\mathrm{int}})$, this gives $k^{\mathcal{A}}_{\min}(Q_{1,\mathrm{int}}) = k_{\min}(Q_{1,\mathrm{int}}) = 3$.

$k^{\mathcal{A}}_{\min}(C(Q_{1,\mathrm{int}})) \le 3$ *(Theorem 1.1 at $j = 1$).* The certificate is an integer map g for $3Q_{1,\mathrm{int}}$ on $\mathcal{A}(Q_{1,\mathrm{int}})$, a table of values at its 486 points. The program checks the pins, the ranges and the contraction condition on all 117,855 pairs of points of $\mathcal{A}(Q_{1,\mathrm{int}})$. It then rebuilds $\mathcal{A}(C(Q_{1,\mathrm{int}}))$ from Definition 2.12, which has 679,944 points, computes the lift of g by the rule of Section 8, and checks that the lift is an integer map for $3C(Q_{1,\mathrm{int}})$ on $\mathcal{A}(C(Q_{1,\mathrm{int}}))$: the pins at the occurrence words of the 17 parties and at 0, the ranges, and the contraction condition on all 231,161,581,596 pairs of points. Since every pair is checked, the statement rests on this computation alone, independently of Lemma D.6 and of the capacity condition, which fails for $Q_{1,\mathrm{int}}$ (Section 8.4). With the transfer of Section 7 and the obstruction of Sections 3 and 4, it gives $k^{\mathcal{A}}_{\min}(C(Q_{1,\mathrm{int}})) = 3$. As a real map for $3C(Q_{1,\mathrm{int}})$ on its realizable domain, the lift also proves that $C(Q_{1,\mathrm{int}})$ is valid (Section 2.4).

Appendix C. Proofs for the carriers, the validity theorem and the existence lemma

This appendix proves the properties of the carrier and the reduction of Section 4 (Lemma 4.5 and Proposition 4.15), and the three lemmas that Sections 5 and 6 state without proof: the counts for the star step (Lemma 5.11), the exceptional configuration (Lemma 5.12), and the existence of designs from $\mathrm{PGL}(2, q)$ that satisfy (5.5) (Lemma 6.7), with the carriers that it gives (Corollary 6.8).

C.1. **Proofs for Section 4.** Here $\mathcal{T}$ is a block system of degree d with claw family $\mathcal{K}$, and $Q(\mathcal{T})$ is its carrier (Definitions 4.1 and 4.2).

The terms of $Q(\mathcal{T})$ are pairwise distinct. A box term contains H, which the other LHS terms do not, and the box terms of $b \neq b'$ differ in q_b. The star terms of two points differ by (T2), and a star term contains a party p_T, which Aq_B does not. On the right, F contains all parties, a claw term contains H and misses exactly one q_w and one p_T, and the term A is a single party. A box term misses, besides q_b, the $|\mathcal{T}| - d = (2j + 1)d/(j + 1)$ parties p_T with $b \notin T$, which are at least two,

since by (T2) the blocks that miss b cover the $m-1=3j+1>j+1$ points other than b, so it is no claw term, and the other LHS terms miss H and have at least two parties.

Proof of Lemma 4.5. (i) Every term contains A. The total coefficients of the terms that contain a given party are, with $mdc=m-1$:

$$\begin{aligned} H: &\quad m &&= 1+mdc\ ,\\ q_v: &\quad (m-1)+\tfrac{1}{m} &&= 1+(m-1)dc\ ,\\ p_T: &\quad (j+1)+(j+1)\omega &&= 1+\big(md-(j+1)\big)c\ ,\\ A: &\quad m+m\omega+\tfrac{1}{m} &&= 1+mdc+c_A\ . \end{aligned}$$

On the left, H lies in the m box terms, q_v in the box terms of the $m-1$ points other than v and in Aq_B, and p_T in the box terms and the star terms of the $j+1$ points of T. On the right, H lies in F and in all md claw terms, q_v in F and in the claw terms of the $(m-1)d$ claws with center other than v, and p_T in F and in the claw terms of all claws except the $j+1$ claws with block T. The first equation holds as stated, the second because $(m-1)dc=(m-1)^2/m=m-2+\frac{1}{m}$, the third because its right side is $m-(j+1)c$ and $(j+1)(1+\omega)=3j+2-(j+1)c$ by (4.4), and the fourth by the definition of c_A.

(ii) The numbers $mdc=m-1$, $m(j+1)d\omega=md(2j+1)-(j+1)(m-1)$ and $m(j+1)dc_A$ are integers, so $m(j+1)dQ(\mathcal{T})$ has integer coefficients. The integers D for which $DQ(\mathcal{T})$ has integer coefficients form a subgroup of $\mathbb{Z}$, which consists of the multiples of its least positive element K. The box terms have coefficient 1 in $Q(\mathcal{T})$, so the coefficients of $KQ(\mathcal{T})$ have greatest common divisor 1 by the observation on the normalization in Section 2.4.

(iii) All terms contain A, so (R1) holds at every point, and $\mathcal{A}(Q(\mathcal{T}))$ is the set of points that satisfy (R2) for every inclusion between two LHS terms. The star term of v is contained in the box term of v, which gives the condition $U\subseteq S$. There is no other inclusion. A box term lies in no other LHS term, because it contains H and the box terms of $b\neq b'$ differ in $q_{b'}$. The term Aq_B lies in no box term, since the box term of b misses q_b, and in no star term, since star terms contain no q_v. A star term contains some p_T, so it does not lie in Aq_B, and the star term of v lies in the box term or the star term of a point $b\neq v$ only if every block that contains v contains b, which (T2) excludes. □

The water level is defined. If $|S|\geq 2$, take $v\neq b$ in S. By (T2) some block T contains v and misses b, and the claw (v,T) is available at S, so $\mathrm{Av}(S)\neq\emptyset$. The function $\Phi_x^{\mathrm{Op}(U)}$ is continuous and nondecreasing with $\Phi_x^{\mathrm{Op}(U)}(0)=0$, since the grounds are nonnegative. If $\sigma(x)>0$, then $U\neq\emptyset$, some block misses a point of U by (T2), so $\mathrm{Op}(U)\neq\emptyset$ and $\Phi_x^{\mathrm{Op}(U)}$ is unbounded. In every case the water level exists, and (4.14) holds by its definition and the continuity of $\Phi_x^{\mathrm{Op}(U)}$.

Proof of Proposition 4.15. Let n be the water-level map. We verify, in this order, bounds on the grounds, the grading, monotonicity, (V1)–(V3), and the ranges, and then apply Lemma 2.15.

Step 1: the grounds. No block contains B or a set $B\setminus\{b\}$, since these have more than $j+1$ points. So $\mathrm{Av}(B)=\mathcal{K}$ has md claws, $\mathrm{Av}(B\setminus\{b\})$ consists of the $(m-1)d$ claws with center other than b, and

$$\gamma(B)=\frac{m-1}{md}=c\ ,\qquad \gamma(B\setminus\{b\})=\frac{m-2}{(m-1)d}=\frac{m(m-2)}{(m-1)^2}c\ .$$

Every $S \neq B$ is contained in some $B \setminus \{b\}$, so adding points one at a time and using (4.16) gives $\gamma(S) \leq \gamma(B \setminus \{b\})$ for $S \neq B$ and $\gamma(S) \leq c$ for all S. With (4.10), for every claw e,

$$\begin{aligned} &g_e(x) \leq \Big(1 - \frac{\epsilon}{m}\Big)\frac{m(m-2)}{(m-1)^2} + \frac{\epsilon}{m-1} \leq 1 \quad \text{if } S \neq B \ , \\ &g_e(B,U,0) = 1 \ , \qquad g_e(B,U,1) = 1 - \frac{1}{m} + \frac{1}{m-1} > 1 \ . \end{aligned} \tag{C.1}$$

For $S \neq B$ the bound is $m(m-2)/(m-1)^2 < 1$ at $\epsilon = 0$ and $\frac{m-2}{m-1} + \frac{1}{m-1} = 1$ at $\epsilon = 1$.

Step 2: the grading. Since $\min(1,h) + \max(0,h-1) = h$, the counts of the claw terms and of the term A add up to $c\sum_e h_e(x)$, and by (4.12) and (4.14)

$$c\sum_{e\in\mathcal{K}} h_e(x) = c\sum_{e\in\mathcal{K}} g_e(x) + \Phi_x^{\mathrm{Op}(U)}(h(x)) = c\sum_{e\in\mathcal{K}} g_e(x) + \sigma(x) \ .$$

By (4.9) the box levels of the claws available at S add up to $s-1$ if $s \geq 1$ and to 0 if $s = 0$, and sd claws have their center in S. With $cd/(m-1) = 1/m$,

$$c\sum_{e\in\mathcal{K}} g_e(x) = \Big(1 - \frac{\epsilon}{m}\Big)(s-1) + \frac{\epsilon s}{m} = s - 1 + \frac{\epsilon}{m} \quad \text{if } s \geq 1 \ , \qquad c\sum_{e\in\mathcal{K}} g_e(x) = 0 \quad \text{if } s = 0 \ .$$

If $s \geq 1$, then $|x|_\alpha \geq 1$, so $n_F(x) = 1$ and $|n(x)| = 1 + (s - 1 + \frac{\epsilon}{m}) + \omega u = |x|_\alpha$. If $s = 0$, then $n_F(x) = \min(1, |x|_\alpha)$ and $\sigma(x) = \max(0, |x|_\alpha - 1)$, which add up to $|x|_\alpha$.

Step 3: monotonicity. The count n_F is nondecreasing in x, and n_e and n_A are nondecreasing functions of the heights. So it suffices to show $h_e(x) \leq h_e(x')$ for every claw e whenever x' is obtained from x by switching on one bit. Suppose first that the bit is a box bit or the bit ϵ. Then U and the set $\mathrm{Op}(U)$ are unchanged, and we claim that $g_e(x) \leq g_e(x')$ for every e and $\sigma(x) \leq \sigma(x')$. For a box bit, $S' = S \cup \{b\}$: a claw available at S is available at S', so the first term of the ground does not decrease by (4.16), and neither does $[w_e \in S]$. The demand is unchanged if $S \neq \emptyset$, and for $S = \emptyset$ it goes from $\max(0, \omega u + \frac{\epsilon}{m} - 1)$ to ωu. For the bit ϵ, the ground changes by $\frac{1}{m-1}[w_e \in S] - \frac{\gamma(S)}{mc}[e \in \mathrm{Av}(S)]$, which is nonnegative because an available claw has its center in S and $\gamma(S)/(mc) \leq \frac{1}{m} < \frac{1}{m-1}$ by Step 1, and the demand does not decrease. With the claim, $\Phi_{x'}^{\mathrm{Op}(U)}(h) \leq \Phi_x^{\mathrm{Op}(U)}(h)$ for every h, so by (4.14)

$$\Phi_{x'}^{\mathrm{Op}(U)}(h) \leq \Phi_x^{\mathrm{Op}(U)}(h) < \sigma(x) \leq \sigma(x') \qquad \text{for } 0 \leq h < h(x) \ .$$

Hence $h(x') \geq h(x)$, and all heights (4.12) are at least as large at x' as at x.

Suppose now that the bit is a star bit, $U' = U \cup \{v\}$. The grounds depend only on S and ϵ and are unchanged, and $\mathrm{Op}(U')$ is the disjoint union of $\mathrm{Op}(U)$ and the set $\mathcal{N}$ of (4.17). If $\sigma(x) = 0$, then $h(x) = 0$, every height at x equals the ground, and the heights at x' are at least the grounds. If $\sigma(x) > 0$, then $U \neq \emptyset$ and $\sigma(x') = \sigma(x) + \omega$: for $S \neq \emptyset$ by (4.10), and for $S = \emptyset$ because the maximum in (4.10) is then attained by its second entry at x and at x'. For $0 \leq h < h(x)$, by (4.14) and (4.17),

$$\Phi_{x'}^{\mathrm{Op}(U')}(h) = \Phi_x^{\mathrm{Op}(U)}(h) + \Phi_x^{\mathcal{N}}(h) < \sigma(x) + \Phi_x^{\mathcal{N}}\big(h(x)\big) \leq \sigma(x) + \omega = \sigma(x') \ ,$$

so $h(x') \geq h(x)$. The heights of the claws in $\mathrm{Op}(U)$ do not decrease, those of the claws in $\mathcal{N}$ go from $g_e(x)$ to $\max(g_e(x), h(x'))$, and the others are unchanged.

Step 4: the conditions (V1)–(V3). In (V1), $U = \emptyset$, so no claw is open and $h_e(x) = g_e(x)$, and $w_e \notin S$ makes both terms of the ground vanish. In (V2), e is neither available at S nor open at U and $\epsilon = 0$, so $h_e(x) = g_e(x) = 0$. For (V3) we show that all heights are at most 1. If $x \leq x^{(H)}$ or

$x \le x^{(q_v)}$, then $U = \emptyset$, the heights are the grounds, and either $\epsilon = 0$ or $S \neq B$, so they are at most 1 by (C.1). If $x \le x^{(p_T)}$, then $n_A(x) \le n_A(x^{(p_T)})$ by Step 3, so it suffices to treat $x = x^{(p_T)} = (T, T, 0)$. There the grounds are at most 1 by (C.1), the demand is $(j+1)\omega$, and the open claws are the $md - (j+1)$ claws with a block other than T. They include the claws available at T, whose box levels add up to j, and the other open claws have ground 0. So

$$\Phi_x^{\mathrm{Op}(T)}(1) = c(md-(j+1)) - j = (m-1) - (j+1)c - j = (2j+1) - (j+1)c = (j+1)\omega = \sigma(x) \ ,$$

and $h(x) \le 1$ by the definition of the water level. So no height exceeds 1, and $n_A(x^{(p_T)}) = 0$.

Step 5: ranges and conclusion. By (4.13), $0 \le n_F \le 1$, $0 \le n_e \le c$ and $n_A \ge 0$. At the top $x^{(A)} = (B, B, 1)$ all grounds exceed 1 by (C.1), so $n_F = 1$ and $n_e = c$ for all md claws, and the grading gives $n_A(x^{(A)}) = m + m\omega + \frac{1}{m} - 1 - (m-1) = c_A$. By Step 3, $n_A \le c_A$ everywhere. For the party rows of (2.16), let i be a party and x a point, and put $z = x \wedge x^{(i)}$. The counts of the terms that miss i vanish at z, by (V1)–(V3) and (4.6), and for $i = A$ every term contains A. So by Steps 2 and 3

$$\sum_{r\colon i \in J_r} n_r(x) \ \ge \ \sum_{r\colon i \in J_r} n_r(z) = |n(z)| = |z|_\alpha \ ,$$

where r runs over the RHS terms J_r of $Q(\mathcal{T})$. Lemma 2.15 with $k = 1$ now shows that n is a real map for $Q(\mathcal{T})$ on the full cube, and its values are rational. Let D be a common denominator of these values that is a multiple of K. By the remarks after Definition 2.13, $Q(\mathcal{T})$ is valid and Dn is an integer map for $DQ(\mathcal{T}) = (D/K)Q_{\mathcal{T},\mathrm{int}}$, so $k_{\min}(Q_{\mathcal{T},\mathrm{int}}) \le D/K$. □

C.2. **Proof of Lemma 5.11.** Here $j \ge 2$, $\mathcal{T}$ is a 3-design that satisfies (5.5), $x = (S, U, \epsilon)$ is a point with $u = |U| \ge 1$, $v \in B \setminus U$, $\mathcal{O}$ is the set of claws e with $U \not\subseteq T_e$, and $\mathcal{N}$ the set of claws with $U \subseteq T_e$ and $v \notin T_e$. A point w is the center of d claws, one for each block through w. Of these, $\lambda(U \cup \{w\})$ have a block that contains U, and $\lambda(U \cup \{w, v\})$ have a block that contains U and v. So the sets $\mathcal{O}_w$ and $\mathcal{N}_w$ of claws with center w in $\mathcal{O}$ and in $\mathcal{N}$ have the sizes

$$|\mathcal{O}_w| = d - \lambda\big(U \cup \{w\}\big) \ , \qquad |\mathcal{N}_w| = \lambda\big(U \cup \{w\}\big) - \lambda\big(U \cup \{w, v\}\big) \ , \tag{C.2}$$

and by (5.4)

$$|\mathcal{O}| = \sum_{w \in B} |\mathcal{O}_w| = md - (j+1)\lambda(U) \ , \qquad |\mathcal{N}| = \sum_{w \in B} |\mathcal{N}_w| = (j+1)\big(\lambda(U) - \lambda(U \cup \{v\})\big) \le (j+1)\lambda(U) \ . \tag{C.3}$$

The low claws are those with center outside S, so $z(\mathcal{O}) = \sum_{w \notin S} |\mathcal{O}_w|$ and $z(\mathcal{N}) = \sum_{w \notin S} |\mathcal{N}_w|$. We use (5.3) for $\lambda(X)$ with $|X| \le 3$ and (5.5) in the form $\lambda(X) \le d/|X|^2$ for $|X| \ge 4$. The following inequality carries parts (a) and (b):

$$(u+1)(j+1)\lambda(U) \le (2j+1)d \qquad \text{if } u \ge 2 \ . \tag{C.4}$$

For $u = 2$ the left side is $3(j+1)\lambda_2 = \frac{3j(j+1)}{3j+1}d$, and $3j^2 + 3j \le (3j+1)(2j+1)$. For $u = 3$ it is $4(j+1)\lambda_3 = \frac{4(j+1)(j-1)}{3(3j+1)}d$, and $4(j^2-1) \le 3(3j+1)(2j+1)$. For $u \ge 4$ it is at most $\frac{u+1}{u^2}(j+1)d \le \frac{5}{16}(j+1)d < (2j+1)d$.

Proof of Lemma 5.11. Part (a). By (C.3), $u|\mathcal{N}| \le |\mathcal{O}|$ follows from $(u+1)(j+1)\lambda(U) \le md$. For $u \ge 2$ this is (C.4), since $2j+1 \le m$, and for $u = 1$ it reads $2(j+1)d \le (3j+2)d$.

Part (b). Let $s \le j+1$, so that at least $m - (j+1) = 2j+1$ points lie outside S. If $u \ge 2$, then by (C.2), (5.4), (C.4) and (C.3)

$$z(\mathcal{O}) = (m-s)d - \sum_{w \notin S} \lambda\big(U \cup \{w\}\big) \ \ge \ (2j+1)d - (j+1)\lambda(U) \ \ge \ u(j+1)\lambda(U) \ \ge \ u|\mathcal{N}| \ .$$

If $u = 1$ and $U = \{a\}$, then $|\mathcal{O}_w| = d - \lambda_2$ for every $w \neq a$, so $z(\mathcal{O}) \geq 2j(d - \lambda_2)$, while $|\mathcal{N}| = (j+1)(d-\lambda_2)$ by (C.3), and $j + 1 \leq 2j$.

Part (c). Let $s \geq j + 2$. By (C.2),

$$z(\mathcal{O}) - uz(\mathcal{N}) = \sum_{w \notin S} D_w \,, \qquad D_w = |\mathcal{O}_w| - u|\mathcal{N}_w| = d - (u+1)\lambda\big(U \cup \{w\}\big) + u\lambda\big(U \cup \{w, v\}\big) \,. \tag{C.5}$$

We show that $D_w \geq 0$ for every w, with one exception at $u = 1$. If $u \geq 4$, the set $U \cup \{w\}$ has at least u points, so $D_w \geq d - \frac{u+1}{u^2} d > 0$. If $u = 3$ and $w \notin U$, then $U \cup \{w\}$ has four points and $D_w \geq d - \frac{4}{16} d > 0$. If $u = 3$ and $w \in U$, then $D_w \geq d - 4\lambda_3 > 0$, because $4(j-1) < 3(3j+1)$. If $u = 2$ and $w \notin U$, then $D_w \geq d - 3\lambda_3 > 0$. If $u = 2$ and $w \in U$, then $U \cup \{w\} = U$ is a pair and $U \cup \{w, v\}$ a triple, and by (5.3)

$$D_w = d - 3\lambda_2 + 2\lambda_3 = \frac{3(3j+1) - 9j + 2(j-1)}{3(3j+1)} d = \frac{2j+1}{3(3j+1)} d > 0 \,.$$

Let $u = 1$ and $U = \{a\}$. For $w \notin \{a, v\}$,

$$D_w = d - 2\lambda_2 + \lambda_3 = \frac{3(3j+1) - 6j + (j-1)}{3(3j+1)} d = \frac{2(2j+1)}{3(3j+1)} d = \frac{2}{3}(d - \lambda_2) \,.$$

For $w = v$ we have $|\mathcal{N}_v| = 0$ and $D_v = d - \lambda_2$. For $w = a$ we have $|\mathcal{O}_a| = 0$, $|\mathcal{N}_a| = d - \lambda_2$ and $D_a = -(d - \lambda_2)$. So if $a \in S$, every term of the sum in (C.5) is nonnegative. If $a \notin S$ and $m - s \geq 3$, the sum contains D_a and at least two further terms, each at least $\frac{2}{3}(d - \lambda_2)$, and it is at least $\frac{1}{3}(d - \lambda_2) > 0$. The remaining case, $a \notin S$ and $m - s \in \{1, 2\}$, is the one excluded in (c). □

C.3. **Proof of Lemma 5.12.** Here $j \geq 2$, $\mathcal{T}$ is a 3-design with $d \geq 2(j+1)$, as in (5.6), that satisfies the box condition, $x = (S, \{a\}, \epsilon)$ with $a \notin S$ and $m - s \in \{1, 2\}$, and $v \neq a$. We write $h = h(x)$, and $\mathcal{O}$ and $\mathcal{N}$ are the sets of claws e with $a \notin T_e$ and with $a \in T_e$, $v \notin T_e$, respectively. Since $s \geq m - 2 = 3j \geq j + 2$, the formulas (5.14) hold, with the ground g^* of the claws with center in S, and $g^* \leq 1$ by (C.1), because $S \neq B$.

Proof of Lemma 5.12. The counts. By (C.3) and (5.3), $|\mathcal{O}| = md - (j+1)d = (2j+1)d$ and $|\mathcal{N}| = (j+1)(d - \lambda_2)$. The points outside S are a and at most one further point w. By (C.2), $|\mathcal{O}_a| = 0$ and $|\mathcal{O}_w| = d - \lambda_2$, and $|\mathcal{N}_a| = d - \lambda_2$ and $|\mathcal{N}_w| \leq \lambda_2 - \lambda_3$. So

$$z(\mathcal{O}) \leq d \,, \qquad z(\mathcal{N}) \leq d - \lambda_3 \,, \qquad \frac{|\mathcal{N}|}{|\mathcal{O}|} = \frac{(j+1)(d-\lambda_2)}{(2j+1)d} = \frac{j+1}{3j+1} \,. \tag{C.6}$$

The coefficient ω. Since $c = \frac{m-1}{md} < \frac{1}{d} \leq \frac{1}{2(j+1)}$, we have by (4.4)

$$\omega = \frac{2j+1}{j+1} - c > \frac{4j+1}{2(j+1)} \geq 1 \,. \tag{C.7}$$

The water level lies above g^*. The demand is $\sigma(x) = \omega$, because $S \neq \emptyset$ and $u = 1$. By (5.14) and (C.6), $\Phi_x^{\mathcal{O}}(g^*) = cz(\mathcal{O})g^* \leq cd = \frac{m-1}{m} < 1 < \omega$. Since $\Phi_x^{\mathcal{O}}$ is nondecreasing and $\Phi_x^{\mathcal{O}}(h) = \omega$ by (4.14), this gives $h > g^*$. Then the second formula of (5.14) for $\mathcal{O}$ reads $\omega = cz(\mathcal{O})g^* + c|\mathcal{O}|(h - g^*)$, so

$$c(h - g^*) = \frac{\omega - cz(\mathcal{O})g^*}{|\mathcal{O}|} \leq \frac{\omega}{|\mathcal{O}|} \,.$$

The new claws. By the second formula of (5.14) for $\mathcal{N}$, with $g^* \leq 1$ and (C.6),

$$\Phi_x^{\mathcal{N}}(h) = cz(\mathcal{N})g^* + |\mathcal{N}|c(h - g^*) \leq c(d - \lambda_3) + \frac{|\mathcal{N}|}{|\mathcal{O}|}\omega = \frac{4(2j+1)}{3(3j+2)} + \frac{j+1}{3j+1}\omega \,,$$

where $c(d-\lambda_3) = \frac{3j+1}{3j+2} \cdot \frac{3(3j+1)-(j-1)}{3(3j+1)} = \frac{4(2j+1)}{3(3j+2)}$ by (5.3). The right side is less than ω exactly when

$$\frac{4(2j+1)}{3(3j+2)} < \frac{2j}{3j+1}\omega \ , \qquad \text{that is,} \qquad \omega > \frac{2(2j+1)(3j+1)}{3j(3j+2)} \ .$$

By (C.7) it suffices that $3j(3j+2)(4j+1) \geq 4(j+1)(2j+1)(3j+1)$. The two sides are $36j^3+33j^2+6j$ and $24j^3+44j^2+24j+4$, and their difference is

$$12j^3 - 11j^2 - 18j - 4 = j\big(j(12j-11)-18\big) - 4 \ \geq \ 2(2\cdot 13 - 18) - 4 = 12 \qquad \text{for } j \geq 2 \ .$$

So $\Phi_x^{\mathcal{N}}(h) < \omega$. For $j = 1$ the difference is -21, and the estimate is too weak. □

C.4. **Proofs of Lemma 6.7 and Corollary 6.8.** We use the notation of Section 6.2: $q \geq 256$ is a prime power with $q \equiv 1 \pmod 3$, $j = (q-1)/3$, $m = q+1$, the box is $B = \mathrm{PG}(1,q)$, $G = \mathrm{PGL}(2,q)$ has order $|G| = (q+1)q(q-1)$, and for any two ordered triples of distinct points exactly one element of G maps the first to the second. The integer M satisfies $5(j+1)^2(\ln m + 1) \leq M \leq m^3$. Let $T_1, \dots, T_M$ be independent random blocks, each uniformly distributed over the $\binom{m}{j+1}$ blocks of B, and put $d = M(j+1)q(q-1)$. For $X \subseteq B$ let

$$\lambda'(X) = \sum_{i=1}^{M} \lambda'_i(X) \ , \qquad \lambda'_i(X) = \#\{g \in G \colon X \subseteq gT_i\} \ , \tag{C.8}$$

the number of pairs (g,i) with $X \subseteq gT_i$. We show that with positive probability the $M|G|$ sets gT_i are pairwise distinct and $|X|^2\lambda'(X) \leq d$ for all X with $|X| \geq 4$. For such a choice, $\mathcal{T} = \{gT_i\}$ has $M|G|$ blocks and $\lambda = \lambda'$, so (5.5) holds, and by (6.5) $\mathcal{T}$ is a 3-design of degree $M(j+1)|G|/m = d$.

The tail estimate is the following exponential bound for sums of independent random variables with values in $[0,1]$. It is a bound of Chernoff type: see e.g. [54] for sums of independent observations of one random variable, and [55, Theorem 5.1] for Hoeffding's extension to summands with values in $[0,1]$. We include the short proof.

Lemma C.9. *Let $Y_1, \dots, Y_M$ be independent random variables with values in $[0,1]$, and let $0 < \overline{\mu} < a$ with $\sum_i \mathbb{E}Y_i \leq \overline{\mu}$. Then*

$$\Pr\Big[\sum_{i=1}^{M} Y_i \geq a\Big] \leq \exp\big(-a\varphi(a/\overline{\mu})\big) \ , \qquad \varphi(y) = \ln y - 1 + \frac{1}{y} \ . \tag{C.10}$$

Proof. Let $t > 0$. The function $y \mapsto e^{ty}$ is convex, so $e^{ty} \leq 1 + y(e^t - 1)$ for $y \in [0,1]$, and $\mathbb{E}e^{tY_i} \leq 1 + (e^t-1)\mathbb{E}Y_i \leq \exp((e^t-1)\mathbb{E}Y_i)$. By Markov's inequality and independence,

$$\Pr\Big[\sum_i Y_i \geq a\Big] \leq e^{-ta} \prod_i \mathbb{E}e^{tY_i} \leq \exp\big(-ta + (e^t-1)\overline{\mu}\big) \ ,$$

and $t = \ln(a/\overline{\mu})$ gives the exponent $-a\ln(a/\overline{\mu}) + a - \overline{\mu} = -a\varphi(a/\overline{\mu})$. □

Proof of Lemma 6.7. Sets of four or more points. Fix $X \subseteq B$ with $|X| = s$ and $4 \leq s \leq j+1$. For a fixed $g \in G$, the set $g^{-1}X$ is a fixed set of s points, and it lies in the uniformly random block T_i with probability $\binom{m-s}{j+1-s}/\binom{m}{j+1} = \prod_{l=0}^{s-1} \frac{j+1-l}{m-l}$. Summing over g and i,

$$\mathbb{E}\lambda'(X) = M|G| \prod_{l=0}^{s-1} \frac{j+1-l}{m-l} = d \prod_{l=1}^{s-1} \frac{j+1-l}{m-l} \leq 3^{-(s-1)}d \ ,$$

because $M|G|(j+1)/m = d$ and $3(j+1-l) = m+1-3l \le m-l$ for $l \ge 1$. Each $\lambda_i'(X)$ is at most $\lambda_{\max} = (j+1)j(j-1)$: an element g with $X \subseteq gT_i$ maps three fixed points of X, by g^{-1}, to an ordered triple of distinct points of T_i, and it is determined by this triple. The variables $Y_i = \lambda_i'(X)/\lambda_{\max}$ are independent, since the T_i are, and have values in $[0,1]$. We apply Lemma C.9 with $a = d/(s^2\lambda_{\max})$ and $\overline{\mu} = 3^{-(s-1)}d/\lambda_{\max}$, so that $a/\overline{\mu} = 3^{s-1}/s^2 \ge \frac{27}{16}$. With $j = (q-1)/3$ and $j-1 = (q-4)/3$,

$$a = \frac{M(j+1)q(q-1)}{s^2(j+1)j(j-1)} = \frac{9Mq}{s^2(q-4)} \;\ge\; \frac{9M}{s^2} \;.$$

We claim that $45\varphi(3^{s-1}/s^2) \ge s$ for every $s \ge 4$. For $s=4$, $\varphi(\frac{27}{16}) = 3\ln 3 - 4\ln 2 - \frac{11}{27} > 0.523 - 0.408$, and $45 \cdot 0.115 > 4$. For $s \ge 5$ we use $\varphi(y) \ge \ln y - 1$: the function $45\big((s-1)\ln 3 - 2\ln s - 1\big) - s$ is at least $45 \cdot 0.175 - 5 > 0$ at $s=5$, and its derivative $45(\ln 3 - \frac{2}{s}) - 1$ is positive for $s \ge 5$. With the claim, the exponent in (C.10) is at least $\frac{9M}{s^2} \cdot \frac{s}{45} = \frac{M}{5s} \ge \frac{(j+1)^2}{s}(\ln m + 1) \ge s(\ln m + 1)$, so

$$\Pr\big[s^2\lambda'(X) > d\big] \le \Pr\Big[\sum_i Y_i \ge a\Big] \le (em)^{-s} \;.$$

There are at most $m^s/s!$ sets X with s points, and sets with more than $j+1$ points lie in no block. So the probability that $|X|^2\lambda'(X) > d$ for some X with $|X| \ge 4$ is at most $\sum_{s\ge 4} e^{-s}/s! < 10^{-3}$.

Distinctness. If $gT_i = g'T_{i'}$ for two different pairs (g,i) and (g',i'), then either $i = i'$ and the element $g^{-1}g' \ne 1$ maps T_i to itself, or $i \ne i'$ and $T_{i'}$ is one of the at most $|G|$ sets hT_i with $h \in G$. An element $h \ne 1$ of G fixes at most two points, since an element that fixes three points is the identity. A set that h maps to itself is a union of cycles of the permutation h of B, which has at most two cycles of length 1 and hence at most $2 + \frac{m-2}{2}$ cycles, so at most $2^{m/2+1}$ sets are mapped to themselves by h. Since each T_i is uniform and $T_{i'}$ is independent of T_i, summing over the M indices i and the $\binom{M}{2}$ pairs $i < i'$, the probability that the sets gT_i are not pairwise distinct is at most

$$\frac{M|G|2^{m/2+1} + \binom{M}{2}|G|}{\binom{m}{j+1}} \;. \tag{C.11}$$

For the denominator, $\binom{3j+2}{j+1} = \binom{3j}{j}\frac{(3j+2)(3j+1)}{(j+1)(2j+1)} \ge \binom{3j}{j}$. In the expansion $1 = \sum_{l=0}^{3j}\binom{3j}{l}(\frac{1}{3})^l(\frac{2}{3})^{3j-l}$ the ratio of the term $l+1$ to the term l is $\frac{3j-l}{2(l+1)}$, which is at least 1 exactly for $l \le j-1$. So the term $l=j$ is the largest of the $3j+1$ terms and is at least $\frac{1}{3j+1}$, which gives $\binom{3j}{j} \ge \frac{1}{3j+1}(\frac{27}{4})^j$. With $M \le m^3 \le 2^{m/2+1}$, $|G| < m^3$, $3j+1 < m$ and $2^{m/2+1} = 4 \cdot 8^{j/2}$, (C.11) is at most

$$2(3j+1)M|G| \cdot 4 \cdot 8^{j/2}\Big(\frac{4}{27}\Big)^j \le 8m^7\Big(\frac{4\sqrt{8}}{27}\Big)^j < 8m^7(0.42)^{(m-2)/3} \;.$$

The logarithm of the right side is $\ln 8 + 7\ln m + \frac{m-2}{3}\ln 0.42$. At $m = 257$ it is less than $2.1 + 38.9 - 73.7 < -30$, and it decreases for $m \ge 257$, because its derivative $\frac{7}{m} + \frac{1}{3}\ln 0.42$ is negative for $m \ge 25$. So (C.11) is less than 10^{-3}.

Both events together have probability less than 1, so some choice of $T_1, \dots, T_M$ avoids both. □

Proof of Corollary 6.8. Let M be the power of 2 with $M_0 \le M < 2M_0$, where $M_0 = 5(j+1)^2(\ln m + 1)$. Since $j+1 = (m+1)/3$ and $m \ge 257$, we have $M < \frac{10}{9}(m+1)^2(\ln m + 1) \le m^3$, so Lemma 6.7 gives a 3-design $\mathcal{T}_q$ with (5.5) and the parameters (6.6).

(i) is Theorem 5.7, where $j \ge 85$, with Lemma 4.5(ii) for the coefficients.

(ii) By Definition 4.2, $N_q = 2 + m + |\mathcal{T}_q| = q + 3 + M(q^3 - q)$, which is at least q^3. For $q \geq 256$ we have $(q+2)^2 \leq 1.02q^2$ and $\ln(q+1)+1 \leq 1.2 \ln q$, so $M < \frac{10}{9}(q+2)^2(\ln(q+1)+1) \leq 1.36q^2 \ln q$ and $N_q \leq q + 3 + 1.36q^5 \ln q \leq 2q^5 \ln q$.

(iii) By Proposition 4.7, k is divisible by $r_q/\gcd(r_q, K)$, where K divides $m(j+1)d = (q+1)(j+1)^2 Mq(q-1)$ by Lemma 4.5(ii) and (6.6). The identities

$$2(q+1) - 3r_q = 1\ , \qquad 2(j+1) - r_q = 1\ , \qquad 3r_q - 2q = 1\ , \qquad 3r_q - 2(q-1) = 3$$

show that r_q has no common factor with $q+1$, $j+1$ or q, and that $\gcd(r_q, q-1)$ divides 3. The number $r_q = 2j+1$ is odd and M is a power of 2. So $\gcd(r_q, K)$ divides 3, and $r_q/\gcd(r_q, K)$ is a multiple of $r_q/\gcd(r_q, 3)$. □

Appendix D. Proofs for Sections 7 and 8

This appendix proves the statements of Sections 7 and 8 whose proofs the main text omits. Appendix D.1 proves Proposition 7.3 on the superbalancing. The rest of the appendix proves Proposition 8.10, by which the lift of a real map g for Q' is a real map for $C(Q')$ on $\mathcal{A}(C(Q'))$ exactly when the shared-position inequality (8.11) holds, and Lemma 8.15, by which the carriers of the two families of Section 6, apart from the one for 17 parties, satisfy the capacity condition. Theorem 8.16 combines Proposition 8.10 with Lemma 8.14, and Section 9 applies it to the carriers through Lemma 8.15. The proof of Proposition 8.10 checks the contraction condition edge by edge. Appendix D.2 proves the description of the points of $\mathcal{A}(C(Q'))$ (Lemma 8.4) and shows that the condition on two kinds of cube edges implies it on all pairs of points (Lemma D.6). Appendix D.4 computes the change of the lift along each of these edges, together with its ranges and pins (Proposition D.14). Along every one of these edges except those that switch on a shared position at a fan, the lift contracts for every map g. Most of these computations use only the grading and the monotonicity of g and (8.2). Items (iii), (iv) and (v) of the proposition compare values of g at the shadows on several parties at once, and Appendix D.3 prepares them by writing each count of g as the measure of a set of threshold sets, the cuts, so that the comparisons reduce to inclusions between sets of cuts and the identity of Lemma D.10(c). The example after Definition 8.7 gives the values of the lift at a fan point, a triangle point and an occurrence word for a four-party inequality, and Table 1 lists the objects of this appendix with the place of their definition.

D.1. **Proof of Proposition 7.3.**

Proof of Proposition 7.3. For disjoint nonempty $X, Y \subseteq [N] \cup \{O\}$ write $I(X{:}Y)$ for the coefficient vector of $S(X) + S(Y) - S(XY)$. A term $S(Z)$ takes the value 1 on r_{uv} exactly when Z separates u from v, that is, contains exactly one of them, and this is unchanged when Z is replaced by its complement. If $u \in X$ and $v \in Y$, then $S(X)$ and $S(Y)$ separate u from v and $S(XY)$ does not. If $u \in X$ and $v \notin XY$, then $S(X)$ and $S(XY)$ separate them and enter with opposite signs. If $u, v \in X$ or $u, v \notin XY$, no term separates them. The remaining cases follow by exchanging X with Y or u with v, and together they give

$$I(X{:}Y) \cdot r_{uv} = 2[u \in X,\ v \in Y \text{ or } u \in Y,\ v \in X]\ . \tag{D.1}$$

For $X = \{u'\}$ and $Y = \{v'\}$ this says that $I(u'{:}v')$ has Bell value 2 on the pair $\{u', v'\}$ and 0 on every other pair, so $C(Q) \cdot r_{uv} = b_{uv} - \frac{1}{2} b_{uv} \cdot 2 = 0$ for all pairs, and $C(Q)$ is superbalanced. Now let $Q = Q_0 + \sum_{(X,Y)} \nu_{XY} I(X{:}Y)$ with Q_0 superbalanced and valid and all $\nu_{XY} \geq 0$. Since the Bell

object	meaning	defined in
$C(Q')$, a_{uv}, a_u	superbalancing of Q' and the coefficients of its pair terms $S(uv)$ and singleton terms $S(u)$	(7.4), (8.1)
$x^{(uv)}$, $\alpha(uv)$, $\beta(uv)$	word of the LHS terms that contain u and v, and the total LHS and RHS coefficients of the terms that contain both	Section 8
(x,Γ), $\widehat{d}$	point of the cube of $C(Q')$ with its set Γ of pair terms, and the distance between such points	(8.3)
fan, center, leaves, triangle	at least two pair terms that all contain the center u, the other parties of these terms, and three pair terms uv, uv', vv'	Section 8.1
$x^-_{u,V}$, $x^+_{u,V}$	bounds on x at a fan with center u and leaves V	(8.5)
ξ_w	shadow $x \wedge x^{(w)}$ of x on the party w	(8.6)
χ_l, shared position	word with its only 1 at position l, and a position that lies in every leaf word and outside $x^{(u)}$	Sections 8, 8.3
$o^{u,V}$, $\theta_{u,V}$, $a_u(V)$	counts of the lift on the RHS terms of Q' at a single pair term or a fan, the count of $S(u)$ at a fan, and $\sum_{v\in V} a_{uv}$	(8.8), (8.9)
G_{uv}	total value of g on the RHS terms that contain u and v	Definition 8.7
$\widehat{g}$	lift of g, a function on $\mathcal{A}(C(Q'))$ with values in the count vectors of $C(Q')$	Definition 8.7
$\mathcal{F}_u$	points of $\mathcal{A}(C(Q'))$ whose pair terms all contain u	Appendix D.2
cut Z, J_Z, μ, $E(y,y')$	threshold set of a count of g, its term, the measure on cuts, and the number defined in Lemma D.10(c)	Appendix D.3
$\lvert y\rvert$	weighted size $\lvert y\rvert_\alpha$ of a word y	Appendix D.3

TABLE 1. The objects used in the proof of Proposition 8.10, with the place of their definition. Throughout, Q' is centered on A and g is a real map for Q' on $\mathcal{A}(Q')$.

values of Q_0 vanish, (D.1) shows that b_{uv} is twice the sum of the ν_{XY} over the pairs (X,Y) with one of u,v in X and the other in Y. Inserting this into (7.2) gives

$$C(Q) = Q_0 + \sum_{(X,Y)} \nu_{XY}\Big(I(X{:}Y) - \sum_{x\in X}\sum_{y\in Y} I(x{:}y)\Big)\ . \tag{D.2}$$

For disjoint nonempty X_1, X_2, Y, expanding the mutual informations shows, as in the proof of [15, Proposition 8],

$$I(X_1X_2{:}Y) - I(X_1{:}Y) - I(X_2{:}Y) = \mathrm{MMI}(X_1{:}X_2{:}Y)\ , \tag{D.3}$$

and MMI holds for disjoint nonempty subsets of $[N]\cup\{O\}$ by the invariance of HEC_N under $\mathfrak{S}_{N+1}$ and the coarse-graining of [3] (see also [15]). For $X=\{x_1,\dots,x_h\}$, applying (D.3) with $X_1=\{x_i\}$ and $X_2=\{x_{i+1},\dots,x_h\}$ for $i=1,\dots,h-1$ and adding gives

$$I(X{:}Y) - \sum_{i=1}^{h} I(x_i{:}Y) = \sum_{i=1}^{h-1} \mathrm{MMI}(x_i{:}x_{i+1}\cdots x_h{:}Y)\ , \tag{D.4}$$

and the same identity applied to each $I(x_i{:}Y)$, in its second argument, writes the bracket in (D.2) as a sum of instances of MMI. So $C(Q)$ is valid. Conversely, if Q and $C(Q)$ are valid, then (7.2) writes Q as the superbalanced valid inequality $C(Q)$ plus $\frac{1}{2}\sum_{\{u,v\}} b_{uv} I(u{:}v)$, where $b_{uv} = Q \cdot r_{uv} \geq 0$ because r_{uv} is the entropy vector of a graph model, and this is a decomposition of the stated form. For the integrality, a term $S(I)$ with $I \subseteq [N]$ separates u from v exactly when $[u \in I] + [v \in I] - 2[u, v \in I] = 1$, and separates u from O exactly when $u \in I$. Summing over the terms with their coefficients gives

$$b_{uv} \equiv b_{uO} + b_{vO} \pmod 2 \qquad (u, v \in [N]) \,, \tag{D.5}$$

so for a balanced Q with integer coefficients, where $b_{uO} = 0$ for every party u, all Bell values are even, and then $C(Q)$ has integer coefficients by (7.2). □

D.2. **The realizable domain of the superbalancing.** Here Q' is centered on A and $C(Q')$ is written as in (7.4), with the notation of Section 8.1.

Proof of Lemma 8.4. The pair terms are sets of two parties other than A, and the terms of Q' contain A. For two terms of Q', (R1) always holds and (R2) is the condition $x \in \mathcal{A}(Q')$. For two pair terms in Γ, (R1) says that they share a party, and (R2) is void. For $uv \in \Gamma$ and a term I_l of Q', (R1) says that $x_l = 1$ implies $I_l \cap \{u, v\} \neq \emptyset$, that is, $x \leq x^{(u)} \vee x^{(v)}$. In (R2), $I_l \not\subseteq \{u, v\}$ because $A \in I_l$, and $\{u, v\} \subseteq I_l$ must imply $x_l = 1$, that is, $x \geq x^{(uv)}$.

Let every two terms of Γ share a party and $|\Gamma| \geq 2$, with $uv, uv' \in \Gamma$ and $v \neq v'$. If all terms contain u, Γ is a fan. A term missing u meets uv and uv', so it is vv', and then every term meets uv, uv' and vv', so it contains two of u, v, v', and Γ is the triangle. For a fan, $\bigwedge_{v \in V} [x^{(uv)}, x^{(u)} \vee x^{(v)}]$ is (8.5) by the distributive laws. For a triangle, $x^{(uv)} \vee x^{(uv')} \vee x^{(vv')}$ has a 1 exactly where at least two of the three words do, and $(x^{(u)} \vee x^{(v)}) \wedge (x^{(u)} \vee x^{(v')}) \wedge (x^{(v)} \vee x^{(v')})$ has a 0 exactly where at least two have a 0, so both bounds are the median, which lies in $\mathcal{A}(Q')$ as a combination of occurrence words by $\wedge$ and $\vee$. □

On the full cube the contraction condition on edges gives it on all pairs, and on $\mathcal{A}(C(Q'))$ the next lemma names the edges that suffice. For a party $u \neq A$, let $\mathcal{F}_u$ be the set of points (x, Γ) of $\mathcal{A}(C(Q'))$ at which every term of Γ contains u.

Lemma D.6 (Edges that suffice on the realizable domain of $C(Q')$)**.** *Let n be a function on $\mathcal{A}(C(Q'))$ with values in the count vectors of $C(Q')$ such that $\|n(P) - n(P')\|_1 \leq \widehat{d}(P, P')$, with $\widehat{d}$ as in* (8.3)*, for every two points that differ in one coordinate and lie in a common set $\mathcal{F}_u$, and for every triangle point P and each of the three points P' obtained by removing one of its pair terms. Then $\|n(P) - n(P')\|_1 \leq \widehat{d}(P, P')$ for all $P, P' \in \mathcal{A}(C(Q'))$.*

The proof joins any two points of $\mathcal{A}(C(Q'))$ by a path of such steps along which every coordinate changes at most once, so that the lengths of the steps add up to the distance of the two points.

Proof of Lemma D.6. If the condition holds along a path $P = P_0, \ldots, P_s = P'$ in $\mathcal{A}(C(Q'))$ of length $\sum_t \widehat{d}(P_{t-1}, P_t) = \widehat{d}(P, P')$, then $\|n(P) - n(P')\|_1 \leq \sum_t \|n(P_{t-1}) - n(P_t)\|_1 \leq \widehat{d}(P, P')$. A path of single-coordinate steps has this length if every coordinate changes at most once and only where P and P' differ. Let $P = (x, \Gamma)$ and $P' = (y, \Delta)$.

(a) $P, P' \in \mathcal{F}_u$. Let $\Theta = \Gamma \cap \Delta$. Remove the terms of $\Gamma \setminus \Theta$ one at a time. By Lemma 8.4 every intermediate point lies in $\mathcal{F}_u$. Then change x into y at Θ. Here a word is identified with the set of its positions with bit 1. The words between $x \wedge y$ and $x \vee y$ satisfy the interval conditions of Θ, and we keep them in $\mathcal{A}(Q')$ as follows. Switch off the positions of x outside y, each time one whose term is minimal under inclusion among the remaining ones. If switching off l broke the up-set property,

a position $l'' \in x$ with $I_{l''} \subsetneq I_l$ would still be on. By minimality $l'' \in x \wedge y$, and since $x \wedge y$ is an up-set, $l \in x \wedge y$, which is false. Then switch on the positions of y outside x, each time one whose term is maximal among the remaining ones. If switching on l broke the up-set property, some l''' with $I_{l'''} \supsetneq I_l$ would be off. Since y is an up-set, l''' is a remaining position, against maximality. Finally add the terms of $\Delta \setminus \Theta$.

(b) Neither point is a triangle and no set $\mathcal{F}_u$ contains both. Then $\Gamma, \Delta \neq \emptyset$. If $\Gamma \cap \Delta = \emptyset$, the path through $(x, \emptyset)$ and $(y, \emptyset)$, which lie in every $\mathcal{F}_u$, consists of three pairs covered by (a) and has length $a(\Gamma) + d_\alpha(x, y) + a(\Delta) = \widehat{d}(P, P')$. Let $e \in \Gamma \cap \Delta$. If $\Gamma = \{e\}$, then Δ is $\{e\}$ or a fan with center in e, so both points lie in $\mathcal{F}_w$ for a party w of e, which this case excludes, and the same holds if $\Delta = \{e\}$. So Γ and Δ are fans, with centers u and u' in e. Equal centers would put both points in one $\mathcal{F}_u$, so $u \neq u'$, $e = uu'$ and $\Gamma \cap \Delta = \{uu'\}$. The point $(x, \{uu'\})$ is realizable, lies in $\mathcal{F}_u \cap \mathcal{F}_{u'}$, and the pairs $P, (x, \{uu'\})$ and $(x, \{uu'\}), P'$, covered by (a), have lengths adding up to $\widehat{d}(P, P')$.

(c) P is a triangle. If $\Delta \supseteq \Gamma$, then $\Delta = \Gamma$ and $y = x$ by Lemma 8.4, so $P' = P$. Otherwise take $e \in \Gamma \setminus \Delta$ and $P_1 = (x, \Gamma \setminus \{e\})$, a fan. The pair P, P_1 satisfies the condition by hypothesis, and $\widehat{d}(P, P') = a_e + \widehat{d}(P_1, P')$. If P' is a triangle too, the same step at P', with a term $e' \in \Delta \setminus (\Gamma \setminus \{e\})$, gives a fan P_1' with $\widehat{d}(P_1, P') = \widehat{d}(P_1, P_1') + a_{e'}$. The remaining pair is covered by (a) or (b). □

D.3. **The cut language.** This subsection prepares the proof of Proposition D.14: two identities for the shadows, used in items (i) and (iii), and the cuts of g.

The setting is that of Section 8: Q' is centered on A, g is a real map for Q' on $\mathcal{A}(Q')$, graded, monotone and with the property (8.2), and $\widehat{g}$ is its lift (Definition 8.7). At a point with a single pair term or a fan with center u and leaves V, the word x lies in $\mathcal{A}(Q')$ and in the interval (8.5), $\xi_w = x \wedge x^{(w)}$ is the shadow on a party w, and (8.6) holds for every leaf v. We write $|y|$ for $|y|_\alpha$. The weighted size is additive over positions, so $|y| + |y'| = |y \wedge y'| + |y \vee y'|$ for all words, and by (8.6)

$$|\xi_u| + |\xi_v| - |x| = \alpha(uv) \qquad \text{for every leaf } v \ . \tag{D.7}$$

For parties $u \neq v$ we also use, from grading and (8.2) at $\xi_u \leq x^{(u)}$,

$$\sum_{r\colon u \in J_r,\ v \notin J_r} g_r(\xi_u) = |\xi_u| - G_{uv}(\xi_u) \ , \tag{D.8}$$

and the same with u and v exchanged.

Items (iii), (iv) and (v) of Proposition D.14 below compare values of g at several shadows at once, and we do this by writing g as a measure on threshold sets, as in the proof of [22, Lemma 2.13], where the thresholds are taken on the vertices of a graph model. For an RHS term J_r and $0 \leq \tau < \beta_r$, the set $Z_{r,\tau} = \{y \in \mathcal{A}(Q')\colon g_r(y) > \tau\}$ is the *cut* of J_r at τ, and for a cut $Z = Z_{r,\tau}$ we write $J_Z = J_r$ for its term. A cut is labeled by its pair (r, τ), and cuts of different pairs are counted separately even when they are equal as sets. Let μ be the measure on the pairs (r, τ) that is Lebesgue measure in τ for each r, and write $\mu\{Z\colon \dots\}$ for the measure of the set of pairs (r, τ) whose cut $Z_{r,\tau}$ satisfies the condition after the colon. We also write this measure without the prefix $Z\colon$, as in $\mu\{Z \ni y\}$. Since $0 \leq g_r(y) \leq \beta_r$, the numbers $\tau \in [0, \beta_r)$ with $g_r(y) > \tau$ form the interval $[0, g_r(y))$, so

$$g_r(y) = \mu\{Z\colon J_Z = J_r,\ Z \ni y\} \ , \qquad |y| = |g(y)| = \mu\{Z\colon Z \ni y\} \ . \tag{D.9}$$

Lemma D.10 (Cuts)**.** (a) *If $y \in Z$ and $y \leq y' \in \mathcal{A}(Q')$, then $y' \in Z$.*

(b) *$x^{(w)} \in Z$ if and only if $w \in J_Z$. Hence, if $y \in Z$ and $y \leq x^{(w)}$, then $w \in J_Z$.*

(c) *For $y, y' \in \mathcal{A}(Q')$, $\mu\{Z\colon Z \ni y,\ Z \ni y',\ Z \not\ni y \wedge y'\} = \mu\{Z\colon Z \ni y \vee y',\ Z \not\ni y,\ Z \not\ni y'\}$. We call this number $E(y, y')$.*

(d) *For parties $u \neq v$ other than A, $a_{uv} = \mu\{Z\colon u, v \in J_Z,\ Z \not\ni x^{(uv)}\}$.*

Proof. (a) g_r is monotone. (b) By the pins of g, $g_r(x^{(w)}) = \beta_r > \tau$ if $w \in J_r$ and $g_r(x^{(w)}) = 0 \leq \tau$ otherwise. If $y \in Z$ and $y \leq x^{(w)}$, then $x^{(w)} \in Z$ by (a). (c) By (a), a cut that contains $y \wedge y'$ contains y and y', and a cut that contains y or y' contains $y \vee y'$. Let E and E' be the two sides of the claim. Then

$$\begin{aligned}\mu\{Z \ni y\} + \mu\{Z \ni y'\} &= \mu\{Z \ni y \text{ and } Z \ni y'\} + \mu\{Z \ni y \text{ or } Z \ni y'\} \\ &= \left(\mu\{Z \ni y \wedge y'\} + E\right) + \left(\mu\{Z \ni y \vee y'\} - E'\right) .\end{aligned}$$

By (D.9) the left side is $|y| + |y'| = |y \wedge y'| + |y \vee y'| = \mu\{Z \ni y \wedge y'\} + \mu\{Z \ni y \vee y'\}$, so $E = E'$. (d) For a term J_r that contains u and v, the cuts of J_r that do not contain $x^{(uv)}$ have measure $\beta_r - g_r(x^{(uv)})$. Summing over these terms gives $\beta(uv) - \sum_{r\colon u,v\in J_r} g_r(x^{(uv)}) = \beta(uv) - \alpha(uv)$, because by (8.2) the other counts vanish at $x^{(uv)} \leq x^{(u)}, x^{(v)}$ and $|g(x^{(uv)})| = \alpha(uv)$. This is a_{uv} by (8.1). □

Lemma D.11 (Fans in the cut language)**.** *Let u be a party, V a nonempty set of leaves, x as above and $o = o^{u,V}(x)$. Then*

$$\sum_{r=1}^{R} \left|o_r - \beta_r y_r^{(u)}\right| = \mu(\Psi_1) + \mu(\Psi_2) \tag{D.12}$$

and

$$d_\alpha(x, x^{(u)}) = \mu\{Z\colon u \in J_Z,\ Z \not\ni \xi_u\} + \mu\{Z\colon Z \ni x,\ Z \not\ni \xi_u\} , \tag{D.13}$$

where $\Psi_1 = \{Z\colon u \in J_Z,\ J_Z \cap V = \emptyset,\ Z \not\ni \xi_u\}$ and $\Psi_2 = \{Z\colon u \notin J_Z,\ V \subseteq J_Z,\ Z \ni \xi_v$ for every $v \in V\}$.

Proof. We go through the four cases of (8.8). If $u \in J_r$ and $J_r \cap V = \emptyset$, the pin is β_r and $\beta_r - g_r(\xi_u)$ is the measure of the cuts of J_r that do not contain ξ_u. If $u \in J_r$ meets V, the count equals the pin. If $u \notin J_r \supseteq V$, the pin is 0, and $\min_v g_r(\xi_v)$ is the measure of the τ with $\tau < g_r(\xi_v)$ for every v, that is, of the cuts of J_r that contain every ξ_v. In the last case both numbers are 0. Adding up gives (D.12). For (D.13), $\xi_u = x \wedge x^{(u)}$ gives $d_\alpha(x, x^{(u)}) = (|x^{(u)}| - |\xi_u|) + (|x| - |\xi_u|)$. Every cut that contains ξ_u contains $x^{(u)}$ and x, so by (D.9) the two brackets are the measures of the cuts that contain $x^{(u)}$, respectively x, and not ξ_u, and a cut contains $x^{(u)}$ exactly when $u \in J_Z$. □

D.4. **The local inequalities.** A cube edge inside a set $\mathcal{F}_u$ either switches one pair term at fixed x or changes one position of x at fixed Γ. Switching a pair term joins $\Gamma = \emptyset$ to a single pair term, or a fan to the set with one leaf fewer. A position changes at $\Gamma = \emptyset$ along an edge of $\mathcal{A}(Q')$, where $\widehat{g} = g$ contracts, at a single pair term inside $[x^{(uv)}, x^{(u)} \vee x^{(v)}]$, or at a fan inside (8.5). The free positions of a fan, those of $x^+_{u,V}$ outside $x^-_{u,V}$, are the positions of $x^{(u)}$ in no leaf word and the shared positions, those outside $x^{(u)}$ in every leaf word. The proposition covers these edges, the triangle edges, the ranges and the pins.

Proposition D.14 (Local inequalities)**.** *Let g be a real map for Q' on $\mathcal{A}(Q')$ and $\widehat{g}$ its lift. All points below lie in $\mathcal{A}(C(Q'))$, v, v' are leaves, and $x + \chi_l$ switches on a position l with $x_l = 0$.*

(i) Switching on a single pair term *at a point $(x, \emptyset)$, where the lift equals g:* $\|\widehat{g}(x, \{uv\}) - \widehat{g}(x, \emptyset)\|_1 \leq a_{uv}$.

(ii) Moving x at a single pair term*:* $\|\widehat{g}(x + \chi_l, \{uv\}) - \widehat{g}(x, \{uv\})\|_1 = \alpha_l$.

(iii) The singleton count of a fan. *Let $\theta_{u,V}$ be given by* (8.9) *for every nonempty V. Then $\theta_{u,\{v\}}(x) = G_{uv}(\xi_u) - G_{uv}(\xi_v)$, so the counts of $S(u)$ and $S(v)$ in* (ii) *are the positive and the negative part of $\theta_{u,\{v\}}(x)$. If $|V| \geq 2$ and $v \in V$, then $\theta_{u,V\setminus\{v\}}(x) \leq \theta_{u,V}(x)$ and $\theta_{u,V}(x) \geq 0$.*

(iv) Removing a leaf*: for a fan Γ with center u, $\|\widehat{g}(x,\Gamma) - \widehat{g}(x,\Gamma\setminus\{uv\})\|_1 = a_{uv}$.*

(v) Ranges*: $0 \leq \widehat{g}_r \leq \beta_r$ and $0 \leq \widehat{g}_w \leq a_w$ for every singleton term $S(w)$.*

(vi) Moving x at a fan on a position of $x^{(u)}$ in no leaf word*: $\|\widehat{g}(x+\chi_l,\Gamma) - \widehat{g}(x,\Gamma)\|_1 = \alpha_l$.*

(vii) Moving x at a fan on a shared position*, the one condition that holds only for some maps g: for the fan Γ with center u and leaves V, $\|\widehat{g}(x+\chi_l,\Gamma) - \widehat{g}(x,\Gamma)\|_1 \leq \alpha_l$ if and only if the shared-position inequality* (8.11) *holds.*

(viii) Triangles*: for the triangle point P on u, v, v', $\|\widehat{g}(P) - \widehat{g}(x,\{uv,uv'\})\|_1 = a_{vv'}$.*

(ix) Pins*: $\widehat{g}$ has the pins* (7.5) *with $k = 1$ at every party and at the purifier.*

Items (i), (ii), (vi), (vii) and (ix) are direct computations with the grading, the monotonicity and (8.2). Items (iii), (iv) and (v) compare values of g at several shadows through the cuts of Appendix D.3, and item (viii) uses the nonnegativity of the singleton count proved in item (iii). We prove the items in their order.

(i) *Switching on a single pair term.* Write $G = G_{uv}$. By monotonicity $g_r(\xi_u), g_r(\xi_v) \leq g_r(x) \leq \beta_r$, so the counts of the terms J_r at $(x,\{uv\})$ and $(x,\emptyset)$ differ by

$$\sum_{r\colon u\in J_r,\ v\notin J_r} (g_r(x) - g_r(\xi_u)) + \sum_{r\colon u\notin J_r,\ v\in J_r} (g_r(x) - g_r(\xi_v)) + \sum_{r\colon u,v\in J_r} (\beta_r - g_r(x)) + \sum_{r\colon u,v\notin J_r} g_r(x)$$
$$= |x| - 2G(x) - (|\xi_u| - G(\xi_u)) - (|\xi_v| - G(\xi_v)) + \beta(uv) = a_{uv} - 2G(x) + G(\xi_u) + G(\xi_v)\ ,$$

by (D.8), (D.7) and (8.1). The two singleton counts at $(x,\{uv\})$ add up to $|G(\xi_u) - G(\xi_v)|$ and vanish at $(x,\emptyset)$. Since $G(\xi_u) + G(\xi_v) + |G(\xi_u) - G(\xi_v)| = 2\max(G(\xi_u), G(\xi_v))$, the total is

$$a_{uv} - 2\big(G(x) - \max(G(\xi_u), G(\xi_v))\big) \leq a_{uv}\ ,$$

because G is a sum of monotone counts and $\xi_u, \xi_v \leq x$.

(ii) *Moving x at a single pair term.* The position l lies in $x^{(u)} \vee x^{(v)}$ and not in $x^{(uv)}$, since $x^{(uv)} \leq x < x+\chi_l \leq x^{(u)} \vee x^{(v)}$, so it lies in exactly one of $x^{(u)}$ and $x^{(v)}$, say in $x^{(u)}$. Then the shadow on u changes from ξ_u to $\xi_u + \chi_l$, which lies in $\mathcal{A}(Q')$ (it is $(x+\chi_l) \wedge x^{(u)}$), and the shadow on v stays. In (8.8) only the terms with $u \in J_r$, $v \notin J_r$ change, by $\delta_1 = \sum_{r\colon u\in J_r,\ v\notin J_r} (g_r(\xi_u+\chi_l) - g_r(\xi_u)) \geq 0$, and $\theta = G(\xi_u) - G(\xi_v)$ grows by $\delta_2 = G(\xi_u + \chi_l) - G(\xi_u) \geq 0$. The singleton counts are θ^+ and θ^-, and since θ^+ is nondecreasing, θ^- nonincreasing and $\theta^+ - \theta^- = \theta$, they change by δ_2 in total. By (8.2) the counts of terms without u vanish at ξ_u and $\xi_u + \chi_l$, so grading gives $\delta_1 + \delta_2 = |\xi_u + \chi_l| - |\xi_u| = \alpha_l$, the distance of the two points.

(iii) *The singleton count of a fan.* The identity: for $V = \{v\}$, by (8.9), (8.8), (D.8) for ξ_u and for ξ_v, $d_\alpha(x, x^{(u)}) = |x| + |x^{(u)}| - 2|\xi_u|$ and the balance $|x^{(u)}| = \sum_{r\colon u\in J_r} \beta_r$,

$$\begin{aligned}\theta_{u,\{v\}}(x) &= a_{uv} - d_\alpha(x, x^{(u)}) + \sum_{r\colon u\in J_r,\ v\notin J_r} (\beta_r - g_r(\xi_u)) + \sum_{r\colon u\notin J_r,\ v\in J_r} g_r(\xi_v)\\ &= a_{uv} - |x| - |x^{(u)}| + 2|\xi_u| + (|x^{(u)}| - \beta(uv)) - (|\xi_u| - G_{uv}(\xi_u)) + |\xi_v| - G_{uv}(\xi_v)\\ &= a_{uv} - \beta(uv) + \alpha(uv) + G_{uv}(\xi_u) - G_{uv}(\xi_v) = G_{uv}(\xi_u) - G_{uv}(\xi_v)\ ,\end{aligned}$$

by (D.7) and (8.1).

Monotonicity in the leaves: let $|V| \geq 2$, $v \in V$ and $\Lambda = V \setminus \{v\}$. Comparing (8.8) for V and Λ case by case, $o^{u,V}$ and $o^{u,\Lambda}$ differ only on the terms with $u \in J_r$ and $J_r \cap V = \{v\}$, where they are

β_r and $g_r(\xi_u)$, and on the terms with $u \notin J_r \supseteq \Lambda$, where they are $\min_{w\in V} g_r(\xi_w)$ and $\min_{w\in\Lambda} g_r(\xi_w)$ if $v \in J_r$, and 0 and $\min_{w\in\Lambda} g_r(\xi_w)$ if $v \notin J_r$. In the cut language the differences are the measures of

$$\begin{aligned}\mathcal{U} &= \{Z\colon u \in J_Z,\ J_Z \cap V = \{v\},\ Z \not\ni \xi_u\}\ ,\\ \mathcal{B} &= \{Z\colon u \notin J_Z,\ \Lambda \subseteq J_Z,\ Z \ni \xi_w \text{ for every } w \in \Lambda,\ Z \not\ni \xi_v\}\ ,\end{aligned}$$

where for $v \notin J_Z$ the condition $Z \not\ni \xi_v$ holds by Lemma D.10(b). On both kinds of terms the count moves away from the pin $\beta_r y_r^{(u)}$ when v is removed, so with $D_v = \mu(\mathcal{U}) + \mu(\mathcal{B})$,

$$\left\|o^{u,V} - o^{u,\Lambda}\right\|_1 = D_v\ , \qquad \theta_{u,\Lambda}(x) = \theta_{u,V}(x) - a_{uv} + D_v\ . \tag{D.15}$$

We show $D_v \le a_{uv}$, which gives $\theta_{u,\Lambda} \le \theta_{u,V}$. Let $\mathcal{C} = \{Z\colon Z \ni \xi_u,\ Z \ni \xi_v,\ Z \not\ni x^{(uv)}\}$, of measure $E(\xi_u, \xi_v)$, since $\xi_u \wedge \xi_v = x^{(uv)}$. By Lemma D.10, every cut in $\mathcal{U}$ or in $\mathcal{C}$ has $u, v \in J_Z$ and does not contain $x^{(uv)}$ (for $\mathcal{U}$ because $x^{(uv)} \le \xi_u$), and $\mathcal{U} \cap \mathcal{C} = \emptyset$, so $\mu(\mathcal{U}) + \mu(\mathcal{C}) \le a_{uv}$ by Lemma D.10(d). A cut in $\mathcal{B}$ contains $\xi_w \le x$ for some $w \in \Lambda$, which exists because $|V| \ge 2$, so it contains x. It does not contain ξ_u, because $u \notin J_Z$, and it misses ξ_v. So $\mathcal{B} \subseteq \{Z\colon Z \ni \xi_u \vee \xi_v,\ Z \not\ni \xi_u,\ Z \not\ni \xi_v\}$, whose measure is $E(\xi_u, \xi_v) = \mu(\mathcal{C})$ by Lemma D.10(c). Hence $D_v \le \mu(\mathcal{U}) + \mu(\mathcal{C}) \le a_{uv}$.

Nonnegativity: removing leaves one at a time, monotonicity gives $\theta_{u,V}(x) \ge \theta_{u,\{v,v'\}}(x)$ for any two leaves $v, v' \in V$, and x lies in the interval (8.5) of $\{v, v'\}$. So let $V = \{v, v'\}$. We write $\theta_{u,V}(x)$ as a first bracket of measures minus the measures of two disjoint sets $\mathcal{X}$ and $\mathcal{Y}$ of cuts, and show that the cuts of $\mathcal{X}$ and $\mathcal{Y}$ lie in three sets whose measures add up to at most the first bracket. By (D.12) and (D.13), after splitting the cuts that contain x and not ξ_u according to whether $u \in J_Z$ (if $u \notin J_Z$, the cut cannot contain ξ_u),

$$\theta_{u,V}(x) = \Big[a_{uv} + a_{uv'} - \mu\{u \in J_Z,\ Z \not\ni \xi_u\} + \mu(\Psi_1)\Big] + \Big[\mu(\Psi_2) - \mu\{u \notin J_Z,\ Z \ni x\}\Big] - \mu(\mathcal{X})\ ,$$

with $\mathcal{X} = \{Z\colon u \in J_Z,\ Z \ni x,\ Z \not\ni \xi_u\}$. In the first bracket, Lemma D.10(d) and the split according to whether $Z \ni \xi_u$ (a cut without ξ_u has no $x^{(uv)} \le \xi_u$) give $a_{uv} = \mu\{u, v \in J_Z,\ Z \not\ni \xi_u\} + \mu\{u, v \in J_Z,\ Z \ni \xi_u,\ Z \not\ni x^{(uv)}\}$, likewise for v', and by inclusion and exclusion over $J_Z \cap \{v, v'\}$ among the cuts with $u \in J_Z$ and $Z \not\ni \xi_u$,

$$\begin{aligned}&\mu\{u, v \in J_Z,\ Z \not\ni \xi_u\} + \mu\{u, v' \in J_Z,\ Z \not\ni \xi_u\} - \mu\{u \in J_Z,\ Z \not\ni \xi_u\} + \mu(\Psi_1)\\ &= \mu\{u, v, v' \in J_Z,\ Z \not\ni \xi_u\}\ .\end{aligned}$$

So the first bracket is $\mu\{u, v, v' \in J_Z,\ Z \not\ni \xi_u\} + \mu\{u, v \in J_Z,\ Z \ni \xi_u,\ Z \not\ni x^{(uv)}\} + \mu\{u, v' \in J_Z,\ Z \ni \xi_u,\ Z \not\ni x^{(uv')}\}$. In the second bracket, a cut that contains ξ_v and $\xi_{v'}$ has $v, v' \in J_Z$ and contains x, so $\Psi_2 = \{Z\colon u \notin J_Z,\ Z \ni \xi_v,\ Z \ni \xi_{v'}\}$ is a subset of $\{Z\colon u \notin J_Z,\ Z \ni x\}$, and the second bracket is $-\mu(\mathcal{Y})$ with $\mathcal{Y} = \{Z\colon u \notin J_Z,\ Z \ni x,\ Z \not\ni \xi_v \text{ or } Z \not\ni \xi_{v'}\}$. Now let $\mathcal{W}_v = \{Z\colon Z \ni x,\ Z \not\ni \xi_u,\ Z \not\ni \xi_v\}$, of measure $E(\xi_u, \xi_v) = \mu\{Z \ni \xi_u,\ Z \ni \xi_v,\ Z \not\ni x^{(uv)}\} \le \mu\{u, v \in J_Z,\ Z \ni \xi_u,\ Z \not\ni x^{(uv)}\}$ by Lemma D.10(c) and (b), and likewise $\mathcal{W}_{v'}$. A cut in $\mathcal{X}$ with $v, v' \in J_Z$ lies in $\{u, v, v' \in J_Z,\ Z \not\ni \xi_u\}$, and one with $v \notin J_Z$ does not contain ξ_v and lies in $\mathcal{W}_v$, likewise for v'. A cut in $\mathcal{Y}$ does not contain ξ_u, because $u \notin J_Z$, so it lies in $\mathcal{W}_v \cup \mathcal{W}_{v'}$. The sets $\mathcal{X}$ and $\mathcal{Y}$ are disjoint, so

$$\mu(\mathcal{X}) + \mu(\mathcal{Y}) \le \mu\{u, v, v' \in J_Z,\ Z \not\ni \xi_u\} + \mu(\mathcal{W}_v) + \mu(\mathcal{W}_{v'})\ ,$$

which is at most the first bracket. Hence $\theta_{u,\{v,v'\}}(x) \ge 0$.

(iv) *Removing a leaf.* Let Γ have center u and leaves V, and $\Lambda = V \setminus \{v\}$. If $|\Lambda| \ge 2$, both points are fans, and by (D.15) and $D_v \le a_{uv}$ the counts differ by $D_v + |\theta_{u,V} - \theta_{u,\Lambda}| = D_v + (a_{uv} - D_v) = a_{uv}$.

If $\Lambda = \{v'\}$, the point $(x, \{uv'\})$ has the counts $o^{u,\{v'\}}$ and, by the identity of item (iii), the singleton counts $\theta^+_{u,\Lambda}$ of $S(u)$ and $\theta^-_{u,\Lambda}$ of $S(v')$. The fan has $\theta_{u,V}$ and 0 there. Since $\theta_{u,V} \geq \theta_{u,\Lambda}$ and $\theta_{u,V} \geq 0$, we have $|\theta_{u,V} - \theta^+_{u,\Lambda}| + \theta^-_{u,\Lambda} = \theta_{u,V} - \theta_{u,\Lambda}$, both when $\theta_{u,\Lambda} \geq 0$ and when $\theta_{u,\Lambda} < 0$. The total is again $D_v + a_{uv} - D_v = a_{uv}$.

(v) *Ranges.* The counts of the terms J_r are values of g_r, minima of such values, β_r or 0, so they lie in $[0, \beta_r]$. At a fan, $\theta_{u,V} \geq 0$ by item (iii). In (D.12), Ψ_1 lies in the first set of (D.13), and a cut in Ψ_2 contains $\xi_v \leq x$ and misses ξ_u, because $u \notin J_Z$, so Ψ_2 lies in the second set. Hence $\sum_r |o_r - \beta_r y_r^{(u)}| \leq d_\alpha(x, x^{(u)})$ and $\theta_{u,V} \leq a_u(V) \leq a_u$. The same bound for $V = \{v\}$ and the identity of item (iii) give $G_{uv}(\xi_u) - G_{uv}(\xi_v) \leq a_{uv}$, and exchanging u and v gives $G_{uv}(\xi_v) - G_{uv}(\xi_u) \leq a_{uv}$, so the singleton counts of a single pair term lie in $[0, a_{uv}]$. All other singleton counts are 0.

(vi) *Moving x at a fan on a position of the center.* The position l lies in $x^{(u)}$ and in no leaf word, so ξ_u becomes $\xi_u + \chi_l$, which lies in $\mathcal{A}(Q')$ (it is $(x + \chi_l) \wedge x^{(u)}$), and every ξ_v stays. Only the counts with $u \in J_r$ and $J_r \cap V = \emptyset$ change, by $\delta_0 = \sum_{r\colon u \in J_r,\ J_r \cap V = \emptyset} (g_r(\xi_u + \chi_l) - g_r(\xi_u))$, which by monotonicity and grading satisfies $0 \leq \delta_0 \leq |\xi_u + \chi_l| - |\xi_u| = \alpha_l$. These counts move toward their pins β_r, and $d_\alpha(x, x^{(u)})$ drops by α_l, so $\theta_{u,V}$ grows by $\alpha_l - \delta_0 \geq 0$. The total change is $\delta_0 + (\alpha_l - \delta_0) = \alpha_l$.

(vii) *Moving x at a fan on a shared position.* The position l lies in every leaf word and not in $x^{(u)}$, so ξ_u stays and every ξ_v becomes $\xi_v + \chi_l$. Only the counts with $u \notin J_r \supseteq V$ change, each by $\delta_r = \min_v g_r(\xi_v + \chi_l) - \min_v g_r(\xi_v) \geq 0$, away from the pin 0. With $\delta = \sum_r \delta_r$ the left side of (8.11), $\sum_r |o_r - \beta_r y_r^{(u)}|$ grows by δ, and $d_\alpha(x, x^{(u)})$ grows by α_l, so $\theta_{u,V}$ changes by $\delta - \alpha_l$. The total change is $\delta + |\delta - \alpha_l|$, which equals α_l if $\delta \leq \alpha_l$ and exceeds α_l if $\delta > \alpha_l$.

(viii) *Triangles.* Let x be the median of $x^{(u)}, x^{(v)}, x^{(v')}$. Let R_1 be the set of r with $u \in J_r$ and $v, v' \notin J_r$, R_2 the set with $u \notin J_r$ and $v, v' \in J_r$, R_3 the set with $u, v, v' \in J_r$, and L_1, L_2, L_3 the sets of positions l defined in the same way by I_l. At the fan point $(x, \{uv, uv'\})$ and at the triangle point the counts agree on every term except those of R_1, where they are $g_r(\xi_u)$ and 0, and those of R_2, where they are $\min(g_r(\xi_v), g_r(\xi_{v'}))$ and β_r. The singleton count of u is $\theta_{u,\{v,v'\}}(x) \geq 0$ at the fan and 0 at the triangle, by item (iii). Inserting (8.9), the values of g cancel and the distance is

$$\sum_{r \in R_1} g_r(\xi_u) + \sum_{r \in R_2} \Big(\beta_r - \min_{w=v,v'} g_r(\xi_w)\Big) + \theta_{u,\{v,v'\}}(x) = \beta(R_1) + \beta(R_2) + a_{uv} + a_{uv'} - d_\alpha(x, x^{(u)}) \ ,$$

where $\beta(R_i) = \sum_{r \in R_i} \beta_r$ and $\alpha(L_i) = \sum_{l \in L_i} \alpha_l$. The median x differs from $x^{(u)}$ exactly at the positions of L_1 and L_2, so $d_\alpha(x, x^{(u)}) = \alpha(L_1) + \alpha(L_2)$. By (8.1) and the balance $\sum_{r\colon u \in J_r} \beta_r = |x^{(u)}|$ of u,

$$a_{uv} + a_{uv'} = \beta(uv) + \beta(uv') - \alpha(uv) - \alpha(uv') = \big(|x^{(u)}| - \beta(R_1) + \beta(R_3)\big) - \big(|x^{(u)}| - \alpha(L_1) + \alpha(L_3)\big) \ ,$$

because a term containing u is counted in $\beta(uv) + \beta(uv')$ once for each of v, v' it contains, and likewise on the left. Inserting, the distance is $\beta(R_2) + \beta(R_3) - \alpha(L_2) - \alpha(L_3) = \beta(vv') - \alpha(vv') = a_{vv'}$.

(ix) *Pins.* The party A lies in no pair term, so its occurrence word in $C(Q')$ is $(\mathbf{1}, \emptyset)$, where $\widehat{g}(\mathbf{1}, \emptyset) = g(\mathbf{1})$ is the pin, with singleton counts 0, and the purifier has $(0, \emptyset)$ and $g(0) = 0$. A party w in no pair term has $(x^{(w)}, \emptyset)$ and no singleton term, and $\widehat{g}(x^{(w)}, \emptyset) = g(x^{(w)})$ is its pin. Let V be the nonempty set of parties v with $a_{wv} > 0$, so that $a_w = a_w(V)$ and the occurrence word of w in $C(Q')$ is $(x^{(w)}, \Gamma)$ with $\Gamma = \{wv\colon v \in V\}$, a single pair term or a fan with center w. Here $\xi_w = x^{(w)}$ and $\xi_v = x^{(wv)} \leq x^{(w)}$. In (8.8) with center w, the counts of the terms with $w \in J_r$ are β_r in both cases of (8.8), by the pin $g_r(x^{(w)}) = \beta_r$ in the first, and those of the terms with $w \notin J_r$ are 0 or values $g_r(x^{(wv)}) = 0$ by (8.2), so $o^{w,V}(x^{(w)}) = \beta y^{(w)}$. If V has at least two elements, $\theta_{w,V}(x^{(w)}) = a_w(V) - 0 + 0 = a_w$. If $V = \{v\}$, then $G_{wv}(x^{(w)}) = \beta(wv)$ by the

pins of g and $G_{wv}(x^{(wv)}) = |g(x^{(wv)})| = \alpha(wv)$ by (8.2) and grading, so the count of $S(w)$ is $\beta(wv) - \alpha(wv) = a_{wv} = a_w$ and that of $S(v)$ is 0. These are the pins (7.5).

Proof of Proposition 8.10. Items (v) and (ix) give the ranges and the pins. If (8.11) holds, the contraction of g and items (i), (ii), (iv), (vi), (vii) and (viii) give the contraction condition on every edge that Lemma D.6 asks for, hence on all pairs. Conversely, contraction on the edge of item (vii) is (8.11). □

D.5. **Proof of Lemma 8.15.**

Proof of Lemma 8.15. By (4.4), $dc = (m-1)/m$, and if $d \geq (j+1)(m-1)$, then $(j+1)c \leq 1/m$ and $\omega = \frac{2j+1}{j+1} - c \geq \frac{2j+1-1/m}{j+1} > 1$. We check (8.13) party by party, using Definition 4.2. Every RHS term other than the term A contains H, so for $u = H$ the left side of (8.13) is 0. The RHS terms other than A that miss q_w are the claw terms of the d claws with center w, of total coefficient $dc < 1$, and the LHS terms that miss q_w are the box term of w and the star terms, with coefficients 1 and $\omega > 1$. The RHS terms other than A that miss p_T are the claw terms of the $j+1$ claws with block T, of total coefficient $(j+1)c \leq 1/m$, and $1/m$ is the least LHS coefficient.

For the system of all blocks, $d = \binom{3j+1}{j}$. For $j \geq 2$ we have $2 \leq j \leq (3j+1)/2$, so $\binom{3j+1}{j} \geq \binom{3j+1}{2} = \frac{3j(3j+1)}{2} \geq (j+1)(3j+1) = (j+1)(m-1)$. The block system of Lemma 6.7 lives on the box $B = \mathrm{PG}(1,q)$ with $m = q+1$ and consists of $M \geq 1$ times $|\mathrm{PGL}(2,q)|$ pairwise distinct blocks gT_i, and its degree is $d = M(j+1)q(q-1) \geq (j+1)q = (j+1)(m-1)$. □

Appendix E. Proofs of the growth bounds

This appendix proves the two parts of Theorem 1.2 from Proposition 9.2 and Corollary 6.8, by the two choices of fields described in Section 9.3.

E.1. **The elementary bound.** For part (a) of Theorem 1.2 we use the fields with $q = 2^{n-1}$ elements, n a prime, where the forced factors are the numbers $(2^n+1)/3$.

Lemma E.1. *For a prime $n \geq 5$ the number $(2^n+1)/3$ is an integer that 3 does not divide, and the numbers $(2^n+1)/3$ and $(2^{n'}+1)/3$ are coprime for distinct primes $n, n' \geq 5$.*

Proof. The powers of 2 modulo 9 are $2, 4, 8, 7, 5, 1$ and repeat with period 6, so $2^n \equiv -1 \pmod 9$ exactly when $n \equiv 3 \pmod 6$, and $2^n \equiv -1 \pmod 3$ for every odd n. A prime $n \geq 5$ is odd and is not divisible by 3, so 3 divides 2^n+1 and 9 does not. Let D be a common divisor of 2^n+1 and $2^{n'}+1$. Then D is odd, $2^{2n} \equiv 1$ and $2^{2n'} \equiv 1 \pmod D$, so the order of 2 modulo D divides $\gcd(2n, 2n') = 2$, and D divides $2^2 - 1 = 3$. A common divisor of $(2^n+1)/3$ and $(2^{n'}+1)/3$ is such a D, and 3 does not divide $(2^n+1)/3$. □

The growth estimates rest on the prime number theorem in the form $\vartheta(x) = \sum_{p \leq x} \ln p = (1+o(1))x$ as $x \to \infty$, where p runs over the primes, see e.g. [39, Theorem 6.9]. We use three consequences. First, for every $\delta > 0$ and all sufficiently large x there is a prime in the interval $(x, (1+\delta)x]$, because $\vartheta((1+\delta)x) - \vartheta(x) = (\delta + o(1))x$. Second, the number $\pi(x)$ of primes up to x satisfies $\pi(x) \leq \sqrt{x} + \vartheta(x)/\ln\sqrt{x} \leq 3x/\ln x$ for all sufficiently large x, since each prime between $\sqrt{x}$ and x contributes at least $\ln\sqrt{x}$ to $\vartheta(x)$. Third, summation by parts gives

$$\sum_{p \leq x} p \;\geq\; \frac{1}{\ln x} \sum_{p \leq x} p \ln p = \frac{1}{\ln x}\Big(x\vartheta(x) - \int_2^x \vartheta(y)dy\Big) = \Big(\frac{1}{2} + o(1)\Big)\frac{x^2}{\ln x}\,. \tag{E.2}$$

Proof of Theorem 1.2(a). For a prime $n \geq 11$ let $q_n = 2^{n-1}$. Then $q_n \geq 256$, and $q_n \equiv 1 \pmod 3$ because $n-1$ is even. Let $Q^{(n)} = Q_{\mathcal{T}_{q_n},\mathrm{int}}$ be the design carrier of Corollary 6.8 for $q = q_n$, with the parties of different n chosen disjoint. Its forced factor $r_{q_n}/\gcd(r_{q_n}, 3)$ of Corollary 6.8(iii) equals $r_{q_n} = (2q_n + 1)/3 = (2^n + 1)/3$ by Lemma E.1, and its number of parties lies between $2^{3(n-1)}$ and $2q_n^5 \ln q_n = 2^{5n-4}(n-1)\ln 2 \leq n2^{5n-4}$. For a prime $n_0 \geq 11$ we apply Proposition 9.2 to the inequalities $Q^{(n)}$ with n prime, $11 \leq n \leq n_0$, and the factors r_{q_n}, which are pairwise coprime by Lemma E.1. Successive primes differ by at least 2, so the total number of parties is

$$N[n_0] \leq \sum_{i \geq 0} n_0 2^{5(n_0-2i)-4} \leq n_0 2^{5n_0} \ , \qquad N[n_0] \geq 2^{3(n_0-1)} \ . \tag{E.3}$$

The primes below 11 add up to 17, and there are at most n_0 primes up to n_0, so by (E.2), with $o(1)$ denoting quantities that tend to 0 as $n_0 \to \infty$,

$$\ln \prod_n r_{q_n} \ \geq \ \sum_n (n\ln 2 - \ln 3) \ \geq \ \ln 2 \Big(\sum_{p \leq n_0} p - 17 \Big) - n_0 \ln 3 \ \geq \ \Big(\frac{\ln 2}{2} - o(1) \Big) \frac{n_0^2}{\ln n_0} \ . \tag{E.4}$$

Now let $N \geq N[11]$, let n_0 be the largest prime with $N[n_0] \leq N$, and let n_1 be the next prime. Proposition 9.2 with this N gives a valid balanced inequality Q and a valid superbalanced inequality $\widehat{Q}$ on N parties, both with integer coefficients of greatest common divisor 1, that satisfy (9.3) with the product of the factors f_t equal to $\prod_n r_{q_n}$. As $N \to \infty$ we have $n_0 \to \infty$ and $n_1/n_0 \to 1$. By (E.3), $N < N[n_1] \leq n_1 2^{5n_1}$ gives $\ln N \leq (5\ln 2 + o(1))n_0$, and $2^{3(n_0-1)} \leq N$ gives $n_0 \leq \ln N$ for large N, so that $\ln n_0 \leq \ln\ln N$. With these two bounds, (E.4) becomes

$$\ln \prod_n r_{q_n} \ \geq \ \Big(\frac{\ln 2}{2} - o(1) \Big) \frac{(\ln N)^2}{(5\ln 2)^2 \ln\ln N} = (1 - o(1)) \frac{(\ln N)^2}{50 \ln 2 \cdot \ln\ln N} \ ,$$

which with (9.3) is part (a). □

E.2. **The bound with Linnik's theorem.** Part (b) rests on Linnik's theorem on the least prime in an arithmetic progression [35, 36]: there is an absolute constant L_0 such that for all integers $n \geq 2$ and a with $1 \leq a \leq n-1$ and $\gcd(a, n) = 1$ the least prime $p \equiv a \pmod n$ satisfies $p < n^{L_0}$. By a theorem of Xylouris [37, Theorem 1], for all coprime positive integers a and n the least prime $p \equiv a \pmod n$ is less than $C_X n^5$, with an effectively computable constant C_X. We use the theorem in the following form: there are constants $C_L \geq 1$ and $L > 1$ such that for all integers $n \geq 2$ and a with $\gcd(a, n) = 1$ there is a prime $p \equiv a \pmod n$ with $p \leq C_L n^L$. In this subsection L denotes this exponent, as in Theorem 1.2(b). Applied to the residue of a modulo n in $\{1, \ldots, n-1\}$, Linnik's theorem gives this form with $C_L = 1$ and $L = \max(L_0, 2)$, and the theorem of Xylouris gives it with $C_L = \max(C_X, 1)$ and $L = 5$.

Proof of Theorem 1.2(b). Let $\ell > 512$ be a prime. The conditions $q \equiv 1 \pmod 3$ and $2q + 1 \equiv 0 \pmod \ell$ define one residue class modulo 3ℓ, and its elements have no common factor with 3ℓ, because $2q \equiv -1 \pmod \ell$. Let $q(\ell)$ be the least prime in this class. By the hypothesis of part (b), applied with $n = 3\ell$, $q(\ell) \leq C_L(3\ell)^L$, and since ℓ divides $2q(\ell) + 1$, we have $q(\ell) \geq (\ell - 1)/2 \geq 256$. So Corollary 6.8 applies to $q = q(\ell)$. Its forced factor $r_q/\gcd(r_q, 3)$ is divisible by ℓ, because $\ell \neq 3$ divides $2q + 1 = 3r_q$.

For $Y \geq 521$ let $\mathcal{Q}(Y)$ be the set of the distinct primes $q(\ell)$ with ℓ prime, $512 < \ell \leq Y$, and for $q \in \mathcal{Q}(Y)$ let f_q be the product of the primes $\ell \in (512, Y]$ with $q(\ell) = q$. Each of them divides $r_q/\gcd(r_q, 3)$, so f_q does, and the f_q are pairwise coprime, being products over disjoint sets of

primes. We apply Proposition 9.2 to the design carriers $Q_{\mathcal{T}_q,\mathrm{int}}$, $q \in \mathcal{Q}(Y)$, on disjoint sets of parties, with the factors f_q. Their product is the product of all primes in $(512, Y]$, so

$$\ln \prod_{q\in\mathcal{Q}(Y)} f_q = \vartheta(Y) - \vartheta(512) \;\geq\; \frac{Y}{2} \qquad \text{for large } Y \ . \tag{E.5}$$

Every $q \in \mathcal{Q}(Y)$ is at most $X = C_L(3Y)^L$, and there are at most $\pi(Y) \leq 3Y/\ln Y$ of them. By Corollary 6.8(ii), and since $\ln X \leq 2L \ln Y$ for large Y, the total number of parties is

$$N[Y] \;\leq\; \frac{3Y}{\ln Y} \cdot 2X^5 \ln X \;\leq\; C_* Y^{5L+1} \ , \qquad C_* = 12L \cdot 3^{5L} C_L^5 \ , \tag{E.6}$$

for all $Y \geq Y_0$, with a suitable Y_0. The numbers $N[Y]$ do not decrease with Y and are unbounded. Let $N \geq N[Y_0]$, and let Y be the largest integer with $N[Y] \leq N$. Proposition 9.2 with this N gives Q and $\widehat{Q}$ on N parties that satisfy (9.3), and $N < N[Y+1] \leq C_*(Y+1)^{5L+1}$ gives $Y + 1 > (N/C_*)^{1/(5L+1)}$. With (E.5), the logarithm of the product of the forced factors is at least $\frac{1}{2}((N/C_*)^{1/(5L+1)} - 1) \geq c_0 N^{1/(5L+1)}$ for large N, with $c_0 = \frac{1}{3} C_*^{-1/(5L+1)}$. This is part (b). □

Appendix F. The 17-party inequalities

This appendix writes out the superbalancing $C(Q_{1,\mathrm{int}})$ of the 17-party inequality $Q_{1,\mathrm{int}}$ displayed after Theorem 1.1, in the notation used there, with the ten parties p_T written $p_{01}, p_{02}, \ldots, p_{34}$. The inequality $Q_{1,\mathrm{int}}$ is 10 times the carrier of Definition 4.2 for the set of all two-element subsets of $B = \{0, 1, 2, 3, 4\}$, and 10 is the least integer that makes its coefficients integers. It has $L = 11$ LHS terms and $R = 1 + 20 + 1 = 22$ RHS terms. Every term contains A, and every party has the same total coefficient on both sides, 117 for A, 50 for H, 42 for each q_v and 46 for each p_T, so $Q_{1,\mathrm{int}}$ is centered on A.

By (2.7), the Bell values of $Q_{1,\mathrm{int}}$ that involve A or O vanish, and the others are listed in Table 2. Each is the total coefficient of the terms that separate the two parties, LHS minus RHS, by (2.5). For example, H and q_0 are separated by the box term with $b = 0$ and by $S(Aq_B)$ on the left, with coefficients 10 and 2, and by the four terms $S(F \setminus \{q_0, p_T\})$ with $0 \in T$ on the right, with coefficient 2 each, so $Q_{1,\mathrm{int}} \cdot r_{Hq_0} = 12 - 8 = 4$. The other rows follow by the same count, and Appendix B describes the exact computation of all Bell values. All 120 pairs of parties other than A have a positive Bell value, so by (7.4)

$$C(Q_{1,\mathrm{int}}) = Q_{1,\mathrm{int}} + \sum_{\{u,v\}} a_{uv}\big(S(uv) - S(u) - S(v)\big) \ , \tag{F.1}$$

where $\{u, v\}$ runs over these 120 pairs and $a_{uv} = \frac{1}{2} Q_{1,\mathrm{int}} \cdot r_{uv}$ is given in Table 2. The superbalancing adds the 120 pair terms $a_{uv}S(uv)$ on the left and the 16 terms $a_u S(u)$ on the right, with $a_u = \sum_v a_{uv}$ equal to 270 for $u = H$, 238 for $u = q_v$ and 380 for $u = p_T$, so $C(Q_{1,\mathrm{int}})$ has $L = 131$ LHS terms and $R = 38$ RHS terms. It is superbalanced by Proposition 7.3 and valid by the integer map for $3C(Q_{1,\mathrm{int}})$ on its realizable domain that Appendix B describes.

Appendix G. The balanced inequalities define no facets

This appendix proves the statement of Section 10 that no balanced inequality of Theorems 1.1 and 1.2 defines a facet of the cone, and that the same holds for a superbalanced inequality of Theorem 1.2 that is a padding of a disjoint sum with two or more summands.

pair $\{u, v\}$	condition	pairs	$Q_{1,\text{int}} \cdot r_{uv}$	a_{uv}
H, q_v		5	4	2
H, p_T		10	52	26
q_v, q_w		10	4	2
q_v, p_T	$v \in T$	20	60	30
q_v, p_T	$v \notin T$	30	36	18
p_T, $p_{T'}$	$\lvert T \cap T' \rvert = 1$	30	38	19
p_T, $p_{T'}$	$T \cap T' = \emptyset$	15	84	42

TABLE 2. The Bell values of $Q_{1,\text{int}}$ on the 120 pairs of parties other than A, by class of pair, and the coefficients a_{uv} of the pair terms of its superbalancing (F.1). The column "pairs" counts the pairs in each class. The parties are those of $Q_{1,\text{int}}$.

If a valid inequality Q that defines a facet is the sum $X + Y$ of two valid inequalities, then $X \cdot S = Y \cdot S = 0$ for every S in the face $\text{HEC}_N \cap Q^\perp$, because the two numbers are nonnegative and add up to $Q \cdot S = 0$. By the definition in Section 2.1 this face spans the hyperplane $Q^\perp$, so X and Y are multiples of Q. Now let Q be a balanced inequality of Theorems 1.1 and 1.2. It is a multiple $Q^{(1)}$ of a carrier or a padding of a disjoint sum of such multiples $Q^{(1)}, \dots, Q^{(r)}$. By (7.2), $Q^{(1)} = C(Q^{(1)}) + \sum_{\{u,v\}} a_{uv} I(u{:}v)$ with $a_{uv} = \frac{1}{2} Q^{(1)} \cdot r_{uv} \geq 0$, where $C(Q^{(1)})$ is valid by Theorem 8.16 or, for 17 parties, by Appendix B, and $a_{Hq_v} > 0$ by the Bell value $\frac{2}{m}$ of a carrier on the pair H, q_v computed in Section 10. So $Q = X + a_{Hq_v} I(H{:}q_v)$, where X is the sum of the paddings of $C(Q^{(1)})$, of the other terms $a_{uv} I(u{:}v)$ and of $Q^{(2)}, \dots, Q^{(r)}$, all of which are valid. If Q defined a facet, the instance $I(H{:}q_v)$ of subadditivity, which has three terms, would be a multiple of Q, which has more than three. So Q defines no facet. A superbalanced inequality of Theorem 1.2 that is a padding of a disjoint sum with two or more summands, which is the case for all large N in the constructions of Section 9, defines no facet either, by the same argument: it is the sum of two valid inequalities with terms on disjoint sets of parties, and neither of them is a multiple of the sum.

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