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A SHAPE OF MAXIMAL DEGREE 135 IN THE PLANE JACOBIAN PROBLEM: EXCLUSION OF THE COMPLETE CHAIN $(9,36)$, $(9,24)$, $(11/3,8)$

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Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, one shape is given by the complete chain $(9,36)$, $(9,24)$, $(11/3,8)$ with $(m,n)=(2,3)$, that is, by standard $(2,3)$-pairs of degrees $(90,135)$. We show that no such pair exists. The argument is short; it does not itself use a computer search. The divisibility theorem of Guccione–Guccione–Valqui, applied as these authors do for the chain $(8,32)$, $(8,28)$, $(11/4,7)$, shows that the vertical edge of the Newton polygon is the $3m$-th power of $x^3y^8p(y)$ with p of degree 4. The element F of their Theorem 2.6 shows that p has at most three distinct roots. A translation of y by a root of multiplicity 2, 3 or 4 then produces, respectively, a lower corner $(9,6)$ that their study of the lower side excludes, a corner on the diagonal, which their Theorem 2.6 forbids, or a standard pair with the chain $(9,36)$, $(9,12)$, which is absent from the list (this last case relies on the computer enumeration of that list). All statements are proved relative to the cited results; an ancillary script checks the arithmetic. Nothing in this note has been checked by a third party.

1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L=K[x,y]$, $L^{(l)}=K[x^{\pm 1/l},y]$ and $[P,Q]=P_xQ_y-P_yQ_x$. We use the notation of [1]: for a weight (ρ,σ), $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, $\mathrm{st}_{\rho,\sigma}$ and $\mathrm{en}_{\rho,\sigma}$ its end points; for points A,B of the plane, $A\sim B$ means that A and B are aligned with the origin [1, Definition 1.14]. *Standard (m,n)-pairs* and *regular corners* are defined in [1, Definitions 4.3, 5.5], *complete chains* in [3, Definition 2.19] and *possible last lower corners* in [2, Definition 3.17].

Standing data 1.1. $(m,n)=(2,3)$ and (P,Q) is a standard (m,n)-pair in L with $[P,Q]\in K^\times$, $\deg P=45m$, $\deg Q=45n$, which is a minimal pair and satisfies $(-1,1)<\mathrm{Succ}_P(1,0),\mathrm{Succ}_Q(1,0)<(-1,0)$, as in [1, Corollary 5.21(2),(4)], and whose complete chain given by [3, Theorem 2.20] is

$$(9,36),\ (9,24),\ (11/3,8).$$

This is the shape of degrees $(90,135)$ of [3, Section 6] with these three corners.

Theorem 1.2. *There is no pair (P,Q) as in Standing data 1.1.*

Remark 1.3. For the chain $(8,32)$, $(8,28)$, $(11/4,7)$ the four authors prove in [6] that a translation of y produces the corner $(8,4)$, impossible by [2, Proposition 3.29]; this case, the case $B_0=(8,28)$, $B_1=(8,40)$ (which we read as the chain $(8,40)$, $(8,28)$, $(11/4,7)$) and the case $B_0=(9,21)$, $B_1=(9,27)$, which leads to the corner $(9,6)$, are stated without a written proof in [2, Remark 3.31]. The present note applies the first part of that argument (Lemma 2.1 below) and adds Lemma 3.1. We found no exclusion of the shape of Standing data 1.1 in the literature we searched; its novelty is not otherwise verified. The preprint [4] of the four authors discards all cases with $\max\{\deg P,\deg Q\}\le 150$ except $(75,125)$, using a formula for the intersection number of a Jacobian pair; in the later version of that work [5] only an inequality is obtained, and the authors state that the infinite families cannot be discarded in this way. They treat in [6] the shapes of degree below 125 and obtain the bound 108.

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2. The vertical edge

Lemma 2.1. *There are $\lambda \in K^\times$ and $p \in K[y]$ of degree 4 with $p(0) \neq 0$ such that*

$$\ell_{1,0}(P) = \lambda\, x^{9m}\big(y^8 p(y)\big)^{3m}.$$

Proof. Let $\big((P_i, Q_i), (A_i, A_i'), (\rho_i, \sigma_i), l_i\big)$ be the data of [3, Theorem 2.20], with $A_0 = (9, 36)$, $A_1 = (9, 24)$, $A_2 = (11/3, 8)$. By item (5) there (with $h = 0$, $i = 1$) and [1, Definition 5.5], $A_0 = \frac{1}{m}\operatorname{en}_{\rho_0,\sigma_0}(P_1)$ and $A_1 = \frac{1}{m}\operatorname{st}_{\rho_0,\sigma_0}(P_1)$ are the end points of $\ell_{\rho_0,\sigma_0}(P_1)$; thus $v_{\rho_0,\sigma_0}(A_0) = v_{\rho_0,\sigma_0}(A_1)$ with $A_0 \neq A_1$ on a vertical line, so $\sigma_0 = 0$, and $(\rho_0, \sigma_0) = (1, 0)$ since $(\rho_0, \sigma_0) \in](1, -1), (1, 0)]$. As (P, Q) is standard, $v_{1,-1}(\operatorname{st}_{1,0}(P)) < 0$, so the type II corner $(A_0, (1, 0))$ of (P, Q) is of type II.a) (see the classification that follows [1, Remark 5.8]); by [3, Theorem 2.20(7)], $(P_1, Q_1) = (P, Q)$ and $A_1 = \frac{1}{m}\operatorname{st}_{1,0}(P)$. Hence $\ell_{1,0}(P) = \lambda x^{9m} H(y)$ with $\lambda \in K^\times$, $H \in K[y]$ monic, $\operatorname{ord}_0 H = 24m$ and $\deg H = 36m$. By item (5) (with $h = 1$, $i = 2$), A_1 and A_2 are the end points of $\ell_{\rho_1,\sigma_1}(P_2)$, and $v_{3,-1}(A_1) = 27 - 24 = 3 = 11 - 8 = v_{3,-1}(A_2)$, so $(\rho_1, \sigma_1) = (3, -1)$. By items (4), (5) and (9) of [3, Theorem 2.20], $(A_0, (1, 0))$ and $(A_1, (3, -1))$ are regular corners of the (m, n)-pair (P_2, Q_2) in $L^{(l_2)}$, $\ell_{1,0}(P_2) = \ell_{1,0}(P)$, and $(A_2, (\rho_2, \sigma_2))$ is a regular corner of type I with $(\rho_2, \sigma_2) < (3, -1)$. In the increasing order of directions used in [1, Theorem 7.6], the corner $(A_1, (3, -1))$ therefore has an index $j > 0$ (the corner $(A_2, (\rho_2, \sigma_2))$ precedes it) and $(A_0, (1, 0))$ an index $i > j$. By [1, Theorem 7.6(3)] the element F_j of [1, Theorem 2.6] for the direction $(3, -1)$ satisfies $\operatorname{en}_{3,-1}(F_j) = \frac{p_j}{q_j} A_1$ with p_j, q_j coprime, and $v_{3,-1}(F_j) = 3 - 1 = 2$. Since $v_{3,-1}(A_1) = 3$, we get $\frac{p_j}{q_j} = \frac{2}{3}$, hence $q_j = 3$. By [1, Theorem 7.6(2),(5)], $\ell_{1,0}(P) = R^{md}$ for some $(1, 0)$-homogeneous $R \in L^{(l_2)}$ and some $d \in \mathbb{N}$ with $3 \mid d$. Writing $R = \mu x^e r(y)$ with $r \in K[y]$, we get $H = \nu\, r^{md}$ with $\nu \in K^\times$, so H is the $(3m)$-th power of a polynomial $h = y^8 p$ with $\deg h = 12$; adjusting λ we may take h, hence p, monic. □

3. The element F of the vertical edge

Lemma 3.1. *The polynomial p of Lemma 2.1 has at most three distinct roots.*

Proof. We have $(1, 0) \in \mathfrak{V}_{>0}$ and $v_{1,0}(P) = 9m > 0$. By [1, Theorem 2.6], with statement (1) there, there is a $(1, 0)$-homogeneous $F \in L$ with $v_{1,0}(F) = 1$ and $[F, \ell_{1,0}(P)] = \ell_{1,0}(P)$. Thus $F = xf(y)$ with $f \in K[y]$, and, with $H = h^{3m}$,

$$[xf, x^{9m}H] = x^{9m}\big(fH' - 9m\, f'H\big) = x^{9m}H, \qquad \text{that is} \qquad f\,\frac{H'}{H} - 9m\, f' = 1.$$

Let c be a root of H of multiplicity $k \geq 1$; write $H = (y-c)^k U$ with $U(c) \neq 0$ and $f = (y-c)^j V$ with $V(c) \neq 0$, $j \geq 0$. Then $fH'/H = (y-c)^{j-1} V\big(k + (y-c)U'/U\big)$. If $j = 0$, this has a pole at c while f' and 1 are regular there, a contradiction. If $j \geq 2$, both fH'/H and f' vanish at c, while the right hand side is 1, again a contradiction. Hence f has a simple root at every root of H, and $\deg f$ is at least the number of distinct roots of $H = (y^8 p)^{3m}$, which is $1 + s$, where s is the number of distinct roots of p (recall $p(0) \neq 0$).

By [1, Theorem 2.6(3)], $\operatorname{en}_{1,0}(F) \sim \operatorname{en}_{1,0}(P)$ or $\operatorname{en}_{1,0}(F) = (1, 1)$. Here $\operatorname{en}_{1,0}(F) = (1, \deg f)$ and $\operatorname{en}_{1,0}(P) = (9m, 36m)$, so $\deg f \in \{4, 1\}$. Since p is not constant, $s \geq 1$, so $\deg f \geq 2$; hence $\deg f = 4$ and $s \leq 3$. □

4. Translations

Lemma 4.1. *Let c be a root of p of multiplicity k, let $\varphi \in \operatorname{Aut}(L)$ be given by $\varphi(x) = x$, $\varphi(y) = y + c$, and put $(\tilde{P}, \tilde{Q}) := (\varphi(P), \varphi(Q))$. Then $[\tilde{P}, \tilde{Q}] = [P, Q]$, the pair $(\tilde{P}, \tilde{Q})$ is an (m, n)-pair in L with $\operatorname{en}_{1,0}(\tilde{P}) = \operatorname{en}_{1,0}(P)$, $\deg \tilde{P} = \deg P$, $\deg \tilde{Q} = \deg Q$, and*

$$\ell_{1,0}(\tilde{P}) = \lambda\, x^{9m}\big((y + c)^8 p(y + c)\big)^{3m}, \qquad \tfrac{1}{m}\operatorname{st}_{1,0}(\tilde{P}) = (9, 3k).$$

Moreover $\tilde{P}$ and P have the same leading forms in all directions (ρ, σ) with $\sigma > 0$, and likewise for $\tilde{Q}$ and Q; in particular $\operatorname{Succ}_{\tilde{P}}(1, 0) = \operatorname{Succ}_P(1, 0)$, $\operatorname{Succ}_{\tilde{Q}}(1, 0) = \operatorname{Succ}_Q(1, 0)$, and $\gcd(\deg \tilde{P}, \deg \tilde{Q}) = \gcd(\deg P, \deg Q)$.

Proof. This is essentially [1, Proposition 3.9] with $l = 1$, $k = 0$; we recall the argument. φ has Jacobian 1, so $[\tilde{P}, \tilde{Q}] = [P, Q]$. For a weight (ρ, σ) with $\sigma \geq 0$, φ maps a monomial $x^i y^j$ to $x^i y^j$ plus terms of strictly smaller (ρ, σ)-degree when $\sigma > 0$, and of the same $(1, 0)$-degree when $\sigma = 0$; hence $\ell_{\rho,\sigma}(\tilde{P}) = \ell_{\rho,\sigma}(P)$ for $\sigma > 0$ and $\ell_{1,0}(\tilde{P}) = \varphi(\ell_{1,0}(P))$. This gives the displayed formula, whose lowest power of y is y^{3mk}, so $\frac{1}{m}\operatorname{st}_{1,0}(\tilde{P}) = (9, 3k)$; the highest one is unchanged. The total degree is $v_{1,1}$, and $(1, 1)$ has $\sigma > 0$, so degrees are unchanged; the conditions defining an (m, n)-pair [1, Definition 4.3] only involve $[\cdot, \cdot]$, $v_{1,1}$, $v_{1,0}$ and $\operatorname{en}_{1,0}$, which are preserved. The sets of directions of P and $\tilde{P}$ (and of Q and $\tilde{Q}$) agree on $](1, 0), (-1, 0)[$, where $\sigma > 0$; since $\operatorname{Succ}_P(1, 0)$ and $\operatorname{Succ}_Q(1, 0)$ lie in this interval, they are also the successors of $(1, 0)$ for $\tilde{P}$ and $\tilde{Q}$. □

Lemma 4.2. *No root of p has multiplicity 2, 3 or 4.*

Proof. Let c, k and $(\tilde{P}, \tilde{Q})$ be as in Lemma 4.1.

$k = 2$: then $\frac{1}{m}\operatorname{st}_{1,0}(\tilde{P}) = (9, 6)$ lies below the main diagonal, so $m(9, 6)$ is a corner of $H(\tilde{P})$ below the main diagonal. Since $(\tilde{P}, \tilde{Q})$ is an (m, n)-pair in L (the setting of [2, Section 2]), running along $H(\tilde{P})$ counterclockwise from its point $(t, 0)$ as in [2, Remark 3.20] up to this corner (the edges met before it have directions in $](0, -1), (1, -1)[$ by [2, Proposition 2.1], so their corners lie below the main diagonal) gives an admissible chain with end point $(9, 6)$, so $(9, 6)$ is a possible last lower corner [2, Definition 3.17]. But $(9, 6) = 3\,(3, 2)$ is not a possible last lower corner by [2, Proposition 3.29] (with $\wp = 3$, $n' = 3$). This is the argument of [6] for the chain $(8, 32)$, $(8, 28)$, $(11/4, 7)$.

$k = 3$: then $\operatorname{st}_{1,0}(\tilde{P}) = m(9, 9) \sim (1, 1)$, while $(1, 0) \in \mathfrak{V}_{>0}$, $v_{1,0}(\tilde{P}) = 9m > 0$ and $[\tilde{P}, \tilde{Q}] \in K^\times$; this contradicts [1, Theorem 2.6(4)].

$k = 4$: then $p = (y - c)^4$ and $\frac{1}{m}\operatorname{st}_{1,0}(\tilde{P}) = (9, 12)$, which lies above the main diagonal, so $(\tilde{P}, \tilde{Q})$ is a standard (m, n)-pair [1, Definition 4.3]; by Lemma 4.1 it has the degrees $(90, 135)$, it is a minimal pair, and it satisfies the conditions of [1, Corollary 5.21(4)]; so it satisfies Standing data 1.1 except possibly for the chain. By [3, Theorem 2.20] it has a complete chain with $A_0 = (9, 36)$, and the corner $(A_0, (\rho_0, \sigma_0))$ is a regular corner of type II with $\frac{1}{m}\operatorname{en}_{\rho_0,\sigma_0}(\tilde{P}) = (9, 36)$; since for $(\rho, \sigma) < (1, 0)$ in $](1, -1), (1, 0)]$ the point $\operatorname{en}_{\rho,\sigma}(\tilde{P})$ is $\operatorname{st}_{1,0}(\tilde{P})$ or lies below it [1, Proposition 3.7], and $\ell_{1,0}(\tilde{P})$ is not a monomial, we get $(\rho_0, \sigma_0) = (1, 0)$. In the notation (5.4) of [1, Section 5], $\ell_{1,0}(\tilde{P}) = \lambda x^{9m}\mathfrak{p}(y)$ with $\mathfrak{p}(y) = y^{12m}(y + c)^{24m}$, which has two distinct factors, and $v_{1,-1}(\operatorname{st}_{1,0}(\tilde{P})) = m(9 - 12) < 0$; so this corner is of type II.a) in the classification that follows [1, Remark 5.8], and by [3, Theorem 2.20(7)] $A_1 = A_0' = \frac{1}{m}\operatorname{st}_{1,0}(\tilde{P}) = (9, 12)$. But [3, Section 6] lists the complete chains of the possible counterexamples with $\max\{\deg P, \deg Q\} \leq 150$ (the cases with m and n exchanged being obtained by swapping m and n); those with first corner $(9, 36)$ are $(9, 36), (17/9, 4)$, for $(m, n) = (3, 2)$ and $(2, 3)$, and $(9, 36), (9, 24), (11/3, 8)$, for $(m, n) = (2, 3)$. None has second corner $(9, 12)$. This step relies on the completeness of the list of [3, Section 6], which is the output of the algorithm of that paper. As a cross-check, an independent reimplementation of the algorithms of [3, Sections 1–2] (ancillary file `anc/ggv4_chains_check.py`) finds that the valid edge $((9, 36), (9, 12))$ starts no complete chain, and recovers the published chains through $(9, 24)$, $(8, 28)$, $(12, 33)$ and $(12, 30)$. □

Proof of Theorem 1.2. By Lemma 2.1, p has degree 4; by Lemma 3.1, it has at most three distinct roots. The multiplicities of its roots therefore form a partition of 4 into at most three parts, that is (4), $(3, 1)$, $(2, 2)$ or $(2, 1, 1)$. Each of them has a part equal to 2, 3 or 4, which Lemma 4.2 excludes. □

Remark 4.3. The same method applied to the chain $(12, 36)$, $(12, 33)$, $(11/3, 8)$ of degree 144 gives $\ell_{1,0}(P) = \lambda\big(x^4 y^{11}(y - c)\big)^{3m}$ and, after translation, the last lower corner $(12, 3)$, which is a possible last lower corner; it does not settle that chain.

5. Scope

- **Proved here, relative to the cited results:** Theorem 1.2, from [1, Theorems 2.6, 7.6, Definitions 1.14, 4.3, 5.5, the classification following Remark 5.8, Propositions 3.7, 3.9], [2, Proposition 2.1, Remark 3.20, Definition 3.17, Proposition 3.29] and [3, Theorem 2.20 and the list of Section 6]. The case $k = 4$ of Lemma 4.2 relies on the completeness of that list, which is the output of a computer enumeration in [3]. The standing data are those of the standard pairs attached to a minimal counterexample in [1, Corollary 5.21]; the theorem is stated under them.
- **Verified by exact computation** (ancillary file `anc/verify_936.py`): the value $q_j = 3$, the classification of the points $(9, 3k)$, the partitions, a transcription of the rows of [3, Section 6] used in Lemma 4.2 (this checks the transcription, not the completeness of the list), and, as a control on examples, the conclusion of Lemma 3.1. The ancillary file `anc/ggv4_chains_check.py` gives the cross-check described at the end of the proof of Lemma 4.2; it is not used in the proof.
- **Not claimed:** anything about the other shapes of the list, about the conjecture itself, or a new bound on the degree of a counterexample.
- Nothing in this note has been checked by a third party; it has not been validated by professionals.

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