hexagon:2611.00942v1

Higher weighted Lucas sums and even q-beta values

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Prepared with Codex

23 November 2026

Abstract

Let L_n be the Lucas numbers and $B_j = \sum_{m\ge1} m^{2j-1}/L_{2m}$. At least one of $B_2, \dots, B_{10}$ is linearly independent of $1, B_1$ over the rationals. The family contains infinitely many linearly independent values. More generally, at every positive real quadratic unit $q < 1$ of norm one, the rational dimension of the even q-beta values through weight $A-1$, together with 1, grows at least on the order of $\sqrt{A}$.

1 A second irrational weighted sum

Let $L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n$. With the Lucas denominators fixed, consider the sums

$$B_1 = \sum_{m\ge1} \frac{m}{L_{2m}}, \qquad B_2 = \sum_{m\ge1} \frac{m^3}{L_{2m}}, \qquad B_j = \sum_{m\ge1} \frac{m^{2j-1}}{L_{2m}}.$$

The first sum lies outside $\mathbb{Q}(\sqrt{5})$ by [5]. Can all the higher sums still be expressed as $a + bB_1$, with $a, b \in \mathbb{Q}$? This already fails among the first nine higher sums.

Theorem 1. *For some $j \in \{2, \dots, 10\}$, the numbers $1, B_1, B_j$ are linearly independent over $\mathbb{Q}$. The span of $1, B_1, B_2, \dots$ has infinite dimension. Both conclusions hold with L_{2am} in the denominator for each fixed positive integer a.*

The Lucas recurrence leads to a family with an integer parameter. Let $t \ge 3$ be an integer, let $q = (t - \sqrt{t^2-4})/2$, and write

$$V_0 = 2, \quad V_1 = t, \quad V_{m+2} = tV_{m+1} - V_m.$$

Then $V_m = q^{-m} + q^m$. These q are exactly the real quadratic units of norm one in $(0, 1)$, since $q + q^{-1}$ is their integral trace. Jouhet and Mosaki's q-beta function [1, (1.1)] gives

$$\beta_q(s) = \sum_{m\ge1} \frac{m^{s-1}q^m}{1+q^{2m}}, \qquad \beta_q(2j) = \sum_{m\ge1} \frac{m^{2j-1}}{V_m}. \tag{1}$$

All these series converge absolutely. For $t = 3$, $V_m = L_{2m}$. For the same V_m, Wang relates the sums with even numerator powers to derivatives of Eisenstein series [6, Corollary 5.2]. Here the powers in the numerator are odd.

Theorem 2. *Fix q as above. For odd $A \ge 3$ let $\rho_A(q) = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{1, \beta_q(2), \beta_q(4), \dots, \beta_q(A-1)\}$. Then $\rho_{21}(q) \ge 3$. For some $2 \le j \le 10$, the three numbers $1, \beta_q(2), \beta_q(2j)$ are linearly independent over $\mathbb{Q}$. As the weights increase,*

$$\liminf_{\substack{A\to\infty \\ A \text{ odd}}} \frac{\rho_A(q)}{\sqrt{A}} \ \ge\ \frac{\pi}{\sqrt{\pi^2+24}}. \tag{2}$$

Krattenthaler, Rivoal and Zudilin constructed small linear forms in odd q-zeta values [2]. Jouhet and Mosaki constructed the even q-beta forms used here. For $1/q \in \mathbb{Z}$, their common denominator gives infinitely many irrational even q-beta values and at least one among weights $2, 4, \ldots, 20$ [1, Theorem 1.1 and Corollaries 1.2–1.3]. At a quadratic unit, evaluating an integer polynomial usually gives a quadratic integer. Nesterenko's rational linear independence criterion requires ordinary integers.

Krisman's treatment of odd q-zeta values at quadratic units [3, §9] uses inversion of the coefficients and balances a multiplier at the two real embeddings. For the even q-beta forms every coefficient, including the constant, has the same inversion factor. We turn their cyclotomic common denominator into a reciprocal polynomial and divide by its middle power. The cleared coefficients become integer polynomials in the trace $q + q^{-1}$. This also halves the growth cost of the cyclotomic part, enough to give rank three at weight twenty.

2 Integer coefficients at every integral trace

Fix odd positive integers A, n and an integer $1 \le r < A/2$. In this section x is an indeterminate, $Q = x^2$, and $e = n(r-1)$. Half powers of Q use $Q^{1/2} = x$. We use the equation numbers of the arXiv version of [1]. Its Lemma 3.3 constructs the form

$$S_n(x^2) = \widehat{P}_{0,n}(x^2) + \sum_{\substack{2 \le j < A \\ j \text{ even}}} \widehat{P}_{j,n}(x^2)\, \beta_x(j) \qquad (0 < x < 1). \tag{3}$$

Only its coefficients, rational functions of x, will be inverted.

Lemma 3. *For $j = 0, 2, 4, \ldots, A-1$, the rational function $x^e \widehat{P}_{j,n}(x^2)$ is invariant under $x \mapsto x^{-1}$.*

Proof. Suppress n in the coefficients. The poles in [1, (3.8)] are x^{1-2k}, $0 \le k \le n$. Its scalar exponent of x is $-A(n^2-1) - ((A-2r)n + A - 2)/2$, an integer. The partial fraction coefficients $d_{s,k}(Q)$ and the finite sums (3.15)–(3.18) in that paper therefore belong to $\mathbb{Q}(x)$. Their denominators have no zeros at the real $x = q, q^{-1}$ considered here.

For $1 \le s \le A$ put $P_s(Q) = \sum_{k=0}^{n} (-1)^k Q^{1/2-k} d_{s,k}(Q)$. Inverting the base interchanges pole k with pole $n-k$. The residue identity [1, (3.12)] is

$$d_{s,n-k}(1/Q) = -Q^{n(r-2)+1} d_{s,k}(Q).$$

Reindexing the sum for $P_s(1/Q)$ gives $P_s(1/Q) = Q^e P_s(Q)$, since the sign from $n-k$ cancels the displayed minus sign for odd n. Each $\widehat{P}_j$ with positive even j is a rational linear combination of these P_s by [1, (3.16)].

The constant coefficient contains the finite-tail rational function $P_0(Q) = P_{0,n}(1, Q)$ of [1, (3.17)]. Its complete expression is

$$\widehat{P}_0(Q) = P_0(Q) + Q^{-e} P_0(1/Q) - \tfrac{1}{2} P_1(Q).$$

The first two summands exchange after inversion and multiplication by Q^{-e}. The last one has the same transformation law. Thus $\widehat{P}_j(1/Q) = Q^e \widehat{P}_j(Q)$ for every index, which gives the lemma. □

Let Φ_k be the k-th cyclotomic polynomial. In the notation of [1, (4.2)–(4.4)] put

$$d_m(X) = \prod_{k \le m} \Phi_k(X), \qquad \Delta_n(X) = \prod_{\substack{k < 2n \\ k \text{ odd}}} \Phi_k(X), \qquad \varphi_n(X) = \prod_{k=1}^{n} \Phi_{2k}(X)^{\lfloor n/k \rfloor}.$$

All products over k start at 1. Write

$$F_n(X) = \varphi_n(X)^{2r} d_{2n}(X)^{A-1} \Delta_n(X), \qquad G_n(X) = (X-1)F_n(X).$$

Jouhet and Mosaki's denominator lemma [1, Lemma 4.5] provides an integer k_n such that

$$D_n(x) = (A-1)!\, x^{k_n} F_n(x^{-1}), \qquad D_n(x)\widehat{P}_{j,n}(x^2) \in \mathbb{Z}[x^{-1}] \tag{4}$$

for every index including zero. This is (4.15) of their arXiv version, or (4.17) of the authors' corrected text.

For $A = 3,\ r = n = 1$, the polynomial $G_1(X)$ is $(X^2-1)^4$. Multiplication by $2!\, x^{-4}$ changes it into $2(x - x^{-1})^4 = 2((x + x^{-1})^2 - 4)^2$, an integer polynomial in the trace. Centering the degree in this way works for the whole family.

Lemma 4. *The polynomial G_n is reciprocal of even degree $2h_n$. Set $C_n(x) = (A-1)!\, x^{-h_n} G_n(x)$. For every coefficient in (3),*

$$C_n(x)\, x^e \widehat{P}_{j,n}(x^2) \ \in \ \mathbb{Z}[x + x^{-1}].$$

Proof. Among the cyclotomic factors only $\Phi_1 = X - 1$ changes sign under reversal. Its multiplicity in G_n is $A + 1$, which is even. The other linear factor $\Phi_2 = X + 1$ has even multiplicity $A - 1 + 2rn$. Every other factor is reciprocal of even degree. Hence $G_n(x) = x^{2h_n} G_n(x^{-1})$, so C_n is invariant.

Using the reversed expression for G_n gives the exact quotient

$$\frac{C_n(x)}{D_n(x)} = x^{h_n - k_n}(x^{-1} - 1).$$

Equation (4) puts each cleared coefficient in $\mathbb{Z}[x, x^{-1}]$. Lemma 3 makes it invariant. To express an invariant integer Laurent polynomial in $\mathbb{Z}[x + x^{-1}]$, pair its terms of degrees $k, -k$ and use the recurrence $U_0 = 2,\ U_1 = z,\ U_{k+2} = zU_{k+1} - U_k$ to express $x^k + x^{-k} = U_k(x + x^{-1})$. □

For the rest of the proof specialize $x = q$. Multiplying (3) by $C_n(q)q^e$ now gives a linear form $\mathcal{L}_n$ with ordinary integer coefficients. This evaluates each coefficient polynomial at the integer t. It requires no action of conjugation on the values of the infinite series.

3 The cost of the common denominator

Write $\ell = \log(1/q) > 0$. The relevant scale for the linear forms is $n^2\ell$, with A, r, q fixed. We claim that

$$\frac{\log C_n(q)}{n^2\ell} \longrightarrow H = \frac{6A-2}{\pi^2} + \frac{2r}{3}. \tag{5}$$

Indeed let $b = 1/q > 1$. Möbius inversion in $X^k - 1$ gives

$$|\log \Phi_k(b) - \deg(\Phi_k)\log b| \le \sum_{d \ge 1} |\log(1 - b^{-d})| < \infty.$$

The bound is uniform in k. There are $O_{A,r}(n\log(2n))$ cyclotomic factors of G_n, counted with multiplicity. Its logarithm at b therefore equals $\deg(G_n)\ell + O_{A,r,q}(n\log(2n))$.

For completeness the three leading degrees can be read directly from the products, in agreement with [1, Lemma 5.4 and (5.9)]. Write $\varphi(k)$ for Euler's totient. Summing $\varphi(k) = \sum_{d|k} \mu(d)k/d$ through y gives $3y^2/\pi^2 + O(y\log(2y))$. Restricting to odd k gives $2y^2/\pi^2 + O(y\log(2y))$, since the

sum of the odd integers through v is $v^2/4 + O(v+1)$ and $\sum_{d \text{ odd}} \mu(d)/d^2 = 8/\pi^2$. Taking $y = 2n$ gives the leading degrees of d_{2n} and Δ_n. For the remaining product,

$$\deg \varphi_n = \sum_{m \le n} \sum_{k \mid m} \varphi(2k).$$

If $m = 2^h u$ with u odd, the inner sum is $(1 + 2 + \cdots + 2^h)u = 2m - u$. The sum of the odd parts u through $m \le n$ is $n^2/3 + O(n)$, by summing the odd integers through $n/2^h$ over h. Thus $\deg \varphi_n = (2/3)n^2 + O(n)$, and

$$\frac{\log G_n(b)}{n^2 \ell} \longrightarrow (A-1)\frac{12}{\pi^2} + \frac{8}{\pi^2} + \frac{4r}{3}.$$

Since $C_n(q) = (A-1)!\, b^{-h_n} G_n(b) > 0$, exactly half of this limit remains in (5).

Jouhet and Mosaki's Lemmas 5.1 and 5.2 apply at $Q = q^2$ for arbitrary $0 < q < 1$. They give

$$\lim_{n \to \infty} \frac{\log |S_n(q^2)|}{n^2 \ell} = -r(A - 2r),$$
$$\limsup_{\substack{n \to \infty \\ n \text{ odd}}} \frac{\log |\widehat{P}_{j,n}(q^2)|}{n^2 \ell} \le (A + 4r^2)/4.$$

The first is an exact rate. In their series the terms through index rn vanish, and for large n the ratios in the subsequent tail have absolute value below $1/3$. Hence the remainder is nonzero and comparable to its first surviving term. For the second bound the Cauchy estimate at pole k gives a power of Q with exponent $Ak(k-n)/2 - r^2 n^2/2 + O_{A,r}(n)$. Its quadratic part is at least $-(A/8 + r^2/2)n^2$ uniformly in k. In the constant coefficient, (3.12) returns the residues at $1/Q$ to the same estimate, and the finite-tail factors cost only $\exp(O_{A,r,q}(n))$. The factor two in both displayed rates comes from $\log(1/Q) = 2\ell$.

Set

$$a = r(A - 2r) - H, \qquad b_0 = (A + 4r^2)/4 + H.$$

Since $e = O_r(n)$, the factor q^e costs zero on this scale. For odd n, $\log |\mathcal{L}_n|/(n^2\ell) \to -a$, and the logarithm of the maximum absolute integer coefficient has limsup at most b_0 after division by $n^2\ell$.

4 From integer forms to independent sums

For clarity, Nesterenko's criterion [4], in the form of [1, Proposition 1.4], says that if integer linear forms on a fixed real vector have logarithmic magnitude $-\alpha k^2 + o(k^2)$ and coefficients bounded by $\exp(\gamma k^2 + o(k^2))$, with $\alpha, \gamma > 0$, the coordinates span a rational space of dimension at least $1 + \alpha/\gamma$.

In our case the coordinate 1 is present. One can see the criterion directly by writing the coordinates in a rational basis $1, \theta_1, \ldots, \theta_{d-1}$ and clearing a fixed denominator. Dimension one is impossible since a nonzero integer cannot tend to zero. For $d \ge 2$, simultaneous Dirichlet approximation gives unbounded integers M and integers p_i with $\max_i |M\theta_i - p_i| \le M^{-1/(d-1)}$. For each such M, choose the first k beyond eventual nonvanishing for which the form has magnitude at most $1/(2M)$. The exact decay at k and $k-1$ gives $k^2 \sim \log M/\alpha$ and $M|\mathcal{L}_k| = M^{o(1)}$, using $\mathcal{L}_k$ here for the fixed-denominator forms. Replacing each $M\theta_i$ by p_i turns $M\mathcal{L}_k$ into an integer at distance at most $M^{-\delta + o(1)}$, where $\delta = 1/(d-1) - \gamma/\alpha$. If $\delta > 0$, the integer has magnitude at most $1/2 + o(1)$ and must vanish. Then $M|\mathcal{L}_k| \le M^{-\delta + o(1)}$, while the first-crossing estimate gives $M|\mathcal{L}_k| \ge M^{-\delta/2}$ eventually. This contradiction gives $d \ge 1 + \alpha/\gamma$.

Apply the criterion with $n = 2k + 1$. The two exponents on scale k^2 are $4a\ell$ and $4b_0\ell$, so whenever $a > 0$,

$$\rho_A(q) \;\geq\; 1 + \frac{r(A - 2r) - H}{(A + 4r^2)/4 + H}. \tag{6}$$

The right side simplifies to twice the function $f(r; A)$ in Jouhet and Mosaki's integer-base bound [1, Theorem 1.1]. For $A = 21,\ r = 3$, the right side is

$$1 + \frac{45 - H}{57/4 + H}, \qquad H = 2 + \frac{124}{\pi^2}.$$

It exceeds 2 precisely when $\pi^2 > 992/107$, which follows already from $\pi > 31/10$. Its value is approximately 2.05630. The dimension is an integer, hence at least three. The weight-two theorem [5, Theorem 1] applies to $q^{-1} > 1$, a quadratic algebraic integer whose other conjugate q is less than one. Thus $\beta_q(2) \notin \mathbb{Q}(q)$ for every trace $t \geq 3$. Some higher weight through twenty must lie outside the span of $1, \beta_q(2)$.

For the asymptotic, take $r = \lfloor u\sqrt{A} \rfloor$ for fixed $u > 0$ and then let the odd integer A grow. In (6) the numerator is $uA^{3/2} + O_u(A)$, and the denominator is $(1/4 + 6/\pi^2 + u^2)A + O_u(\sqrt{A})$. The leading ratio divided by $\sqrt{A}$ is largest at $u = \sqrt{1/4 + 6/\pi^2}$, giving (2). This completes the proof of Theorem 2.

Finally take $\phi = (1 + \sqrt{5})/2$ and $q = \phi^{-2a}$ for a fixed positive integer a. Then $q + q^{-1} = L_{2a}$ and $V_m = L_{2am}$. The finite and infinite assertions of Theorem 2 now give Theorem 1 at each fixed stride.

Verification

An independent AI review of the 23 November draft checked the coefficient descent at every odd index, the analytic estimates, and the finite and asymptotic rank bounds. The reviewer also checked the exposition through the first theorem with the abstract covered.

The program `checks/check_higher_weight_forms.py` in the accompanying source tree uses Python and SymPy to reconstruct the partial fractions at both roots of $X^2 - tX + 1$. At $t = 3, A = 3, r = n = 1$, it gives $5S_1(q^2) = -21 + 22B_1$. At $A = 21, r = 3$ it checks all eleven coefficients after clearing for $t = 3,\ n = 1, 3, 5$ and $t = 4,\ n = 3$. The proof of the limits uses the analytic estimates cited above.

References

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