THE LATTICE OF RAINBOW BASES

Abstract. Let E be the labelled union of n bases of an n-dimensional vector space over a field. We determine the integer lattice generated by the incidence vectors of its rainbow bases. It consists of the integer vectors with equal colour sums and component sums equal to the common colour sum times the component ranks, and it is saturated. In particular, the all-one vector is an integer combination of these incidence vectors. Over every field, for every sufficiently large integer k, the multiset containing k copies of each label can be partitioned into kn rainbow bases.

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1. Introduction

Rota proposed his basis conjecture in 1989: the n given bases of an n-dimensional vector space can be ordered as the rows of an $n \times n$ array whose columns are also bases. Each column then contains one vector from each prescribed row basis. Such a column is called a rainbow basis, or a transversal basis. Equal vectors in different positions are distinguished by their labels. Huang and Rota [14, Conjecture 4] published the vector-space formulation over infinite fields and recorded the corresponding conjecture for matroids.

The original approach came from invariant theory. Huang and Rota related the rearrangement of bases to a straightening coefficient in a supersymmetric bracket algebra and to the parity of Latin squares. Alon and Tarsi [2] conjectured that, for every even order, the numbers of even and odd Latin squares are different. Here the sign of a Latin square is the product of the signs of its row and column permutations. Huang and Rota proved that this nonvanishing statement implies the basis conjecture in characteristic zero. Onn [16, Theorem 1 and Corollary 1] expressed the implication through a determinant identity over arbitrary fields: for even n, the identity gives the required rearrangement whenever the integer parity difference is nonzero in the field. Drisko [7] and Glynn [10] proved the parity statement for orders $p+1$ and $p-1$, respectively, where p is an odd prime.

The matroid formulation also admits exact results for particular classes. Wild [18] proved the conjecture for strongly base orderable matroids, and Geelen and Humphries [9] proved it for paving matroids. Other work studies how many disjoint rainbow bases can be found, or how many rainbow bases suffice to cover the input. Bucić, Kwan, Pokrovskiy and Sudakov [6] obtained $(1/2 - o(1))n$ disjoint rainbow bases. Montgomery and Sauermann [15, Theorems 1.5 and 1.6] obtained $(1 - o(1))n$ disjoint rainbow bases and a cover by $(1 + o(1))n$ rainbow bases. These statements concern growing rank. A recent preprint of Arndt, Moseley, Pruhs, Swamy and Zlatin [3] treats approximation algorithms for colouring matroid intersections by common independent sets, including the partition matroid arising from Rota's problem.

Fractional decomposition has a separate history. Aharoni and Berger [1, Theorem 10.4] proved a fractional colouring theorem for the intersection of two matroids. Applied to the given matroid and the partition matroid of the colours, it expresses the all-one vector as a nonnegative rational combination of rainbow-basis incidence vectors. Clearing denominators gives a positive integer D for which D copies of every label can be partitioned into rainbow bases. Yeliussizov [19, Theorem 17] obtained this repetition statement in characteristic zero from invariants of the Levi-Civita tensor; Remark 24 of that paper identifies its equivalence with the fractional result. Repeating a decomposition gives every positive multiple of D. The present paper considers the full integer lattice and the existence of decompositions at every sufficiently large integer repetition number.

Integer lattices generated by matroid bases and constrained bases were studied by Rieder [17] and by Hochstättler and Rieder [12]. More recently, Hörsch, Imolay, Mizutani, Oki and Schwarcz [13] developed group-labelled formulations of common-base questions, including lattice

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obstructions and algebraic algorithms. For inputs consisting of full colour bases, their Theorems 3.9 and 3.14 imply that the quotient of the constraint lattice in (1) by the rainbow-basis lattice is a finite 2-group, even for arbitrary matroids; Remark 2.3 gives the deduction. Their Section 4 also uses group algebras for represented matroid intersection. Arithmetic invariants also occur in the work of Bittner, Ducey, Guo, Oh and Zweber [4], who give an elementary computation of the Smith normal form of the disjoint-transversal adjacency matrix. That matrix has all transversals of an array as both its row and column indices.

A related invariant-theoretic question concerns the degrees in which invariants occur on orbit closures. For polystable complex tensors with three factors, Bürgisser and Ikenmeyer [5, Theorem 5.3] relate the group generated by those degrees to a stabilizer period. In the present setting, a formal determinant identity over the group algebra of the rainbow-basis quotient determines the integer lattice directly. Its generators are constrained by the colours and by the connected components of the represented matroid. The formula below gives all the relations, over every field.

1.1. **Statements.** Let $\mathbb{K}$ be a field, let V be an n-dimensional $\mathbb{K}$-vector space, where $n \geq 1$, and let

$$E = E_1 \sqcup \cdots \sqcup E_n, \qquad E_i = \{(i,1), \ldots, (i,n)\}.$$

Write $[n] = \{1, \ldots, n\}$. For each $e \in E$, choose a vector $v_e \in V$, put $v_{ij} = v_{(i,j)}$, and suppose that

$$(v_{i1}, \ldots, v_{in})$$

is a basis of V for every i. The labels are distinct even when their vectors coincide. Let M be the represented matroid on E. Thus

$$r(A) = \dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}\{v_e : e \in A\} \qquad (A \subseteq E).$$

Let $\mathcal{R}$ consist of the subsets $R \subseteq E$ that contain one label from each E_i and whose vectors form a basis of V. We call these subsets *rainbow bases*.

For $z \in \mathbb{R}^E$ and $A \subseteq E$, write $z(A) = \sum_{e \in A} z_e$. Let χ_A denote the incidence vector of A, and put

$$L = \sum_{R \in \mathcal{R}} \mathbb{Z}\chi_R, \qquad S = \sum_{R \in \mathcal{R}} \mathbb{Z}_{\geq 0}\chi_R, \qquad w = (1)_{e \in E}.$$

Rota's basis conjecture is the assertion $w \in S$.

A circuit is a minimally dependent subset of labels. The connected components of M are the equivalence classes generated by pairs of labels in a common circuit; a label in no circuit is a singleton component. Write these components as $C_1, \ldots, C_c$, and put $r_a = r(C_a)$. Each E_i meets C_a in a basis of that component, so $|E_i \cap C_a| = r_a$; this is also proved in Lemma 2.2.

Theorem 1.1. *The lattice L is given by*

$$L = \left\{ z \in \mathbb{Z}^E : \begin{array}{ll} z(E_i) = d & (1 \leq i \leq n), \\ z(C_a) = d\, r_a & (1 \leq a \leq c) \end{array} \text{ for some } d \in \mathbb{Z} \right\}. \tag{1}$$

Consequently, L is saturated in $\mathbb{Z}^E$, and

$$\mathbb{Z}^E / L \cong \mathbb{Z}^{n+c-2}, \qquad \operatorname{rank} L = n^2 - n - c + 2.$$

A sublattice of $\mathbb{Z}^E$ is *saturated* if its quotient is torsion-free. The rank of a lattice is the dimension of its rational span.

Corollary 1.2. *There are integers a_R, indexed by $R \in \mathcal{R}$, such that*

$$w = \sum_{R \in \mathcal{R}} a_R \chi_R.$$

Every such expression satisfies $\sum_R a_R = n$.

Proof. The vector w satisfies $w(E_i) = n$ and $w(C_a) = n r_a$, so Theorem 1.1 applies with $d = n$. Taking the coordinate sum on any E_i gives the last assertion. □

Theorem 1.3. *For each input as above there is an integer $k_0 \geq 1$ such that*

$$kw \in S \qquad \text{for every integer } k \geq k_0.$$

Equivalently, k copies of every original label can be partitioned into kn rainbow bases for every sufficiently large integer k.

Section 2 proves Theorem 1.1 by a determinant identity over $\mathbb{K}[\mathbb{Z}^E/L]$. Section 3 represents the smallest face of the full base polytope containing w/n over the original field. Applying the lattice theorem to this representation and then an integer remainder argument proves Theorem 1.3 in Section 4. Corollary 4.2 gives a threshold depending only on n.

2. The rainbow-basis lattice

2.1. **A determinant lemma.**

Lemma 2.1. *Let A be a commutative ring with identity, and let $n \geq 2$. Suppose that $G_1, \ldots, G_n \in \mathrm{GL}_n(A)$ satisfy the formal polynomial identity*

$$\det(G_1 u_1, \ldots, G_n u_n) = \det(u_1, \ldots, u_n), \tag{2}$$

where the entries of the column vectors $u_1, \ldots, u_n$ are independent commuting indeterminates. Then there are units $\lambda_i \in A$, with $\lambda_1 = 1$, such that

$$G_i = \lambda_i G_1 \quad (1 \leq i \leq n), \qquad \det(G_1) \prod_{i=1}^{n} \lambda_i = 1. \tag{3}$$

Proof. Let $e_1, \ldots, e_n$ be the coordinate vectors of A^n. Fix distinct i, j, and substitute $u_i = u_j = u$ into (2). The right-hand side vanishes. Apply G_j^{-1} to all output columns on the left. Its determinant is a unit, so the resulting determinant is also zero. The other $n-2$ columns can be replaced independently by arbitrary indeterminate columns through invertible linear substitutions. Reordering columns therefore gives

$$\det(Cu, u, z_3, \ldots, z_n) = 0, \qquad C = G_j^{-1} G_i.$$

For $n = 2$, this is the only 2×2 minor of (Cu, u). For $n > 2$, fix distinct row indices a, b and substitute the coordinate vectors indexed by $[n] \setminus \{a, b\}$ for the z-columns. The determinant is, up to sign, the minor on rows a, b. Thus every 2×2 minor of (Cu, u) is zero as a polynomial in u.

Taking $u = e_b$ gives $C_{ab} = 0$ for every $a \neq b$. Hence C is diagonal. Taking $u = e_a + e_b$, with $a \neq b$, then gives $C_{aa} - C_{bb} = 0$. Thus $C = \lambda I_n$. Since C is invertible, λ is a unit: $\lambda^n = \det C$ is a unit, and

$$\lambda^{-1} = \lambda^{n-1} (\lambda^n)^{-1}.$$

Using $j = 1$ gives the first identity in (3). Evaluating (2) at $u_i = e_i$ gives the second. □

2.2. **Components in coordinates.**

Lemma 2.2. *Choose coordinates in which $v_{1j} = e_j$ for $j \in [n]$, where $e_1, \ldots, e_n$ is the coordinate basis. Form a graph on $[n]$ by joining distinct a, b whenever some input vector has nonzero a- and b-coordinates. Let $J_1, \ldots, J_c$ be its connected components. Every input vector is supported in exactly one J_a. The corresponding sets of labels are the components C_a of M, and $r_a = |J_a|$.*

Proof. Each input vector is nonzero. Its support is contained in one graph component, because any two coordinates in the support are joined. The coordinate spaces

$$V_a = \mathrm{span}_{\mathbb{K}}\{e_b : b \in J_a\}$$

give a direct sum $V = \bigoplus_a V_a$, and every input vector belongs to one summand. A circuit cannot meet two summands: the dependence would split into nonempty dependences inside the summands, contradicting minimality.

Conversely, if a label $e \notin E_1$ has expansion $v_e = \sum_{b \in T} \alpha_b e_b$, where every $\alpha_b \neq 0$, then

$$\{e\} \cup \{(1, b) : b \in T\}$$

is a circuit. Indeed, the reference vectors are independent and every coefficient in the displayed dependence is nonzero. Thus e and all the reference labels in its support belong to the same

matroid component. Every graph edge is obtained from one such support. Connectivity now shows that the labels supported in J_a form a single matroid component. The reference labels in J_a are a basis of its span, so its rank is $|J_a|$.

Since each E_i is a basis of the direct sum $\bigoplus_a V_a$, the vectors in $E_i \cap C_a$ are a basis of V_a. In particular, each such intersection has size r_a. □

2.3. **Proof of Theorem 1.1.** For $n = 1$, there is one label and one rainbow basis, so $L = \mathbb{Z}^E$ and $c = 1$, giving all assertions. Assume $n \geq 2$, and choose coordinates with $v_{1j} = e_j$. Let $A_i \in \mathrm{GL}_n(\mathbb{K})$ have columns $v_{i1}, \ldots, v_{in}$; thus $A_1 = I_n$.

Put

$$\Gamma = \mathbb{Z}^E/L, \qquad A = \mathbb{K}[\Gamma].$$

The group algebra A is the $\mathbb{K}$-vector space with basis $\{t^\gamma : \gamma \in \Gamma\}$, with multiplication $t^\gamma t^\delta = t^{\gamma+\delta}$. It is commutative, and each t^γ is a unit with inverse $t^{-\gamma}$. For $e \in E$, let $\bar{e}$ be the class of the coordinate vector at e in $\mathbb{Z}^E$, and set $t_e = t^{\bar{e}}$ and $t_{ij} = t_{(i,j)}$. Linear independence of this basis gives

$$t^\gamma = t^\delta \implies \gamma = \delta. \tag{4}$$

Define

$$D_i = \mathrm{diag}(t_{i1}, \ldots, t_{in}), \qquad G_i = A_i D_i A_i^{-1} \in \mathrm{GL}_n(A).$$

For column vectors $x_1, \ldots, x_n$ whose entries are independent commuting indeterminates, consider the formal polynomial

$$p(x_1, \ldots, x_n) = \det(A_1 x_1, \ldots, A_n x_n).$$

Its expansion is

$$p = \sum_{j_1,\ldots,j_n \in [n]} \det(v_{1j_1}, \ldots, v_{nj_n}) \prod_{i=1}^{n} x_{i,j_i}.$$

The nonzero coefficients are indexed exactly by the rainbow bases. For each such basis R, its incidence vector belongs to L, so $\prod_{e\in R} t_e = 1$. It follows coefficient by coefficient that

$$p(D_1 x_1, \ldots, D_n x_n) = p(x_1, \ldots, x_n).$$

With $u_i = A_i x_i$, this is (2). Lemma 2.1 gives

$$G_i = \lambda_i D_1, \qquad \lambda_1 = 1, \qquad \left(\prod_{i=2}^{n} \lambda_i\right)\left(\prod_{b=1}^{n} t_{1b}\right) = 1. \tag{5}$$

For a label $e = (i, j)$, we have $G_i v_e = t_e v_e$. Comparing a nonzero coordinate b of v_e in (5) gives

$$t_e = \lambda_i t_{1b}. \tag{6}$$

The coordinate is a nonzero element of $\mathbb{K}$, hence a unit in A, so this comparison is valid over A. Taking two nonzero coordinates of the same vector shows that t_{1b} is constant on every component J_a from Lemma 2.2. Write this monomial as t^{β_a}. Equation (6), applied to any vector in colour i, also shows that λ_i is a group monomial. Write

$$\lambda_i = t^{\alpha_i}, \qquad \alpha_1 = 0.$$

Equations (4)–(6) imply

$$\bar{e} = \alpha_i + \beta_a \quad (e \in E_i \cap C_a), \qquad \sum_{i=2}^{n} \alpha_i + \sum_{a=1}^{c} r_a \beta_a = 0. \tag{7}$$

To verify that these are all the relations, let H be the abelian group with generators

$$\xi_2, \ldots, \xi_n,\ \eta_1, \ldots, \eta_c$$

and the single relation

$$\sum_{i=2}^{n} \xi_i + \sum_{a=1}^{c} r_a \eta_a = 0, \tag{8}$$

and put $\xi_1 = 0$. Define a homomorphism $\mathbb{Z}^E \to H$ by sending the coordinate vector at $e \in E_i \cap C_a$ to $\xi_i + \eta_a$. Every rainbow basis meets each colour once and component C_a exactly r_a times, so its incidence vector maps to zero by (8). We therefore obtain a homomorphism $\phi : \Gamma \to H$.

Conversely, (7) gives a homomorphism $\psi : H \to \Gamma$ with $\psi(\xi_i) = \alpha_i$ and $\psi(\eta_a) = \beta_a$. For $e \in E_i \cap C_a$,

$$\psi\phi(\bar{e}) = \alpha_i + \beta_a = \bar{e},$$

so $\psi\phi$ is the identity. Choose a reference label $e_a \in E_1 \cap C_a$. Then $\beta_a = \bar{e}_a$, and hence $\phi\psi(\eta_a) = \phi(\bar{e}_a) = \eta_a$. For $i \geq 2$, choose $e \in E_i \cap C_a$, which exists by Lemma 2.2. Since $\alpha_i = \bar{e} - \beta_a$,

$$\phi\psi(\xi_i) = \phi(\bar{e}) - \phi(\beta_a) = (\xi_i + \eta_a) - \eta_a = \xi_i.$$

Thus $\phi\psi$ is also the identity, and $\Gamma \cong H$.

For $z \in \mathbb{Z}^E$, its image in H is

$$\sum_{i=2}^{n} z(E_i)\xi_i + \sum_{a=1}^{c} z(C_a)\eta_a.$$

It vanishes precisely when there is an integer d such that

$$z(E_i) = d \ (2 \leq i \leq n), \qquad z(C_a) = dr_a \ (1 \leq a \leq c).$$

Since $\sum_a r_a = n$ and both partitions have total coordinate sum $z(E)$, these equations also imply $z(E_1) = d$. This proves (1). Finally, the coefficient of ξ_2 in (8) is 1, so H is free of rank $n + c - 2$. The saturation and rank assertions follow.

Remark 2.3. Let M be any rank-n matroid on a disjoint union of n bases, let L be its rainbow-basis lattice, and let Λ be the lattice on the right of (1). Retain E_i, C_a, r_a for its colours, components and component ranks. Then Λ/L is a finite 2-group, as follows from Theorems 3.9 and 3.14 of [13].

For $n = 1$, both lattices are $\mathbb{Z}^E$. Suppose $n \geq 2$, and let P be the partition matroid whose independent sets contain at most one label from each E_i. Write r_M, r_P for the rank functions of M, P. The components of P are the sets E_i, and its common bases with M are the rainbow bases. A common basis exists: before choosing from colour i, the chosen independent set has rank $i - 1 < n$; since E_i is a basis, some element of E_i can be added while preserving independence.

For every nonempty proper subset $X \subset E$,

$$r_M(X) + r_P(E \setminus X) > n.$$

Indeed, if X contains an entire colour class, then $r_M(X) = n$ and $r_P(E \setminus X) \geq 1$. Otherwise, $E \setminus X$ meets every colour, so $r_P(E \setminus X) = n$, while $r_M(X) \geq 1$ because M has no loops.

The cited theorems give the following label-splitting criterion: for two rank-n matroids having a common basis and satisfying this strict rank inequality, labels in an abelian group with no element of order two whose sum is zero on every common basis split as the sum of two labellings, each constant on the components of one matroid.

Put $\Gamma = \mathbb{Z}^E/L$ and

$$T_2 = \{g \in \Gamma : 2^s g = 0 \text{ for some integer } s \geq 0\}.$$

Since Γ is finitely generated, T_2 is finite. The group Γ/T_2 has no element of order two: if $2g \in T_2$, then $2^{s+1}g = 0$ for some s, so $g \in T_2$. Label each e by the image $\rho(e)$ of its coordinate vector in this quotient. All common-basis label sums are zero. The criterion therefore gives $\alpha_i, \beta_a \in \Gamma/T_2$ such that

$$\rho(e) = \alpha_i + \beta_a \quad (e \in E_i \cap C_a), \qquad \sum_i \alpha_i + \sum_a r_a\beta_a = 0.$$

The second identity is the label sum on any common basis. Thus, for $z \in \Lambda$ with colour sums d,

$$\sum_e z_e\rho(e) = d\left(\sum_i \alpha_i + \sum_a r_a\beta_a\right) = 0.$$

Hence $\Lambda/L \subseteq T_2$. Conversely, Λ is the kernel of the integer linear map whose coordinates are

$$z(E_i) - z(E_1) \ (2 \leq i \leq n), \qquad z(C_a) - r_a z(E_1) \ (1 \leq a \leq c).$$

It is therefore saturated: if a nonzero integer multiple of z lies in Λ, all these integer coordinates of z vanish. Since $L \subseteq \Lambda$, every torsion class of Γ lies in Λ/L. Consequently $\Lambda/L = T_2$, proving the assertion.

3. Base polytopes and faces

For a finite set of real vectors, conv denotes its convex hull and cone its nonnegative real span. We write aff P for the affine span of P, and relint P for its interior relative to aff P. A nonempty face of a polytope is the set on which a real linear functional attains its maximum. The first lemma is a special case of Edmonds' matroid-intersection theorem [8].

3.1. **Integral colour constraints.** For a finite matroid N on F, write r_N for its rank function. We use its submodularity:

$$r_N(A) + r_N(B) \geq r_N(A \cap B) + r_N(A \cup B).$$

For a represented matroid, put $U_A = \operatorname{span}\{v_e : e \in A\}$ and $U_B = \operatorname{span}\{v_e : e \in B\}$. The dimension formula gives

$$r_N(A) + r_N(B) = \dim(U_A + U_B) + \dim(U_A \cap U_B) \geq r_N(A \cup B) + r_N(A \cap B).$$

Lemma 3.1. *Let $F = P_1 \sqcup \cdots \sqcup P_s$ be a partition and let b_i be nonnegative integers with $\sum_i b_i = r_N(F)$. If nonempty, the polytope*

$$Q_b = \{y \in \mathbb{R}^F_{\geq 0} : y(A) \leq r_N(A) \ (A \subseteq F),\ y(P_i) = b_i \ (1 \leq i \leq s)\} \tag{9}$$

is the convex hull of the incidence vectors of the bases B satisfying $|B \cap P_i| = b_i$ for every i.

Proof. The singleton rank inequalities give $0 \leq y_e \leq 1$, so Q_b is bounded. We prove that every vertex y is integral.

Call $A \subseteq F$ tight if $y(A) = r_N(A)$. Both $\varnothing$ and F are tight. If A, B are tight, then

$$\begin{aligned} r_N(A) + r_N(B) &= y(A) + y(B) \\ &= y(A \cap B) + y(A \cup B) \\ &\leq r_N(A \cap B) + r_N(A \cup B) \\ &\leq r_N(A) + r_N(B). \end{aligned}$$

Thus the tight sets are closed under union and intersection. Choose an inclusion-maximal chain of tight sets

$$\varnothing = T_0 \subsetneq T_1 \subsetneq \cdots \subsetneq T_m = F, \qquad D_j = T_j \setminus T_{j-1}.$$

Every tight set A is a union of the blocks D_j. Indeed, the tight set $T_{j-1} \cup (A \cap T_j)$ lies between consecutive members of the chain. Maximality implies that it is one endpoint, so $A \cap D_j$ is empty or all of D_j.

Form a bipartite multigraph with one vertex for each part P_i, one for each block D_j, and an edge for each e with $y_e > 0$, joining its part to its block. Give that edge weight y_e. The incident weight sums are the integers

$$b_i \quad \text{at } P_i, \qquad r_N(T_j) - r_N(T_{j-1}) \quad \text{at } D_j.$$

If the graph contains a cycle, it has even length; two parallel edges are allowed as a cycle of length two. Perturb its edge weights alternately by $+\varepsilon$ and $-\varepsilon$. This preserves every part sum and block sum, hence every tight rank equation. It leaves zero coordinates unchanged. All other rank inequalities have positive slack, and there are finitely many of them. For sufficiently small $|\varepsilon| > 0$, both signs of the perturbation preserve all inequalities and keep positive coordinates positive. This contradicts that y is a vertex.

The graph is therefore a forest. At a leaf, the unique incident edge has integer weight. Delete it and subtract its weight from the integer sum at the other endpoint. Continuing through each tree proves that every edge weight is an integer. Hence y is integral.

Since $0 \leq y_e \leq 1$, we have $y = \chi_B$ for some B. Furthermore,

$$|B| = y(F) = r_N(F), \qquad |B| = y(B) \leq r_N(B) \leq |B|.$$

Thus B is a basis, with the prescribed counts. The converse inclusion follows directly from the rank inequalities. □

Writing $P(N) = \operatorname{conv}\{\chi_B : B \text{ is a basis of } N\}$, the one-part instance of the lemma gives

$$P(N) = \{y \in \mathbb{R}^F_{\geq 0} : y(F) = r_N(F),\ y(A) \leq r_N(A)\ (A \subseteq F)\}. \tag{10}$$

For a rank-n matroid on $E = E_1 \sqcup \cdots \sqcup E_n$, all quotas equal to 1 give

$$P(N) \cap \{y : y(E_i) = 1 \text{ for all } i\} = \operatorname{conv}\{\chi_R : R \text{ is a rainbow basis of } N\}. \tag{11}$$

3.2. **Representing a face.**

Lemma 3.2. *Every nonempty face of the base polytope of a matroid represented over $\mathbb{K}$ is the base polytope of a matroid represented over $\mathbb{K}$ on the same labelled ground set. Its bases are exactly the original bases whose incidence vectors lie in the face.*

Proof. If the ground set is empty, the base polytope is $\{0\}$ and the assertion holds with the same matroid. Assume that the ground set is nonempty. Let M be the original represented matroid. Let its vectors span a space of dimension r, and choose a weight vector ω exposing the face by maximization. List its distinct values as $\gamma_1 > \cdots > \gamma_t$. Put

$$T_0 = \varnothing, \qquad T_j = \{e : \omega_e \geq \gamma_j\}, \qquad T_t = E, \qquad d_j = r_M(T_j).$$

For any original basis B,

$$\omega(B) = \gamma_t r + \sum_{j=1}^{t-1} (\gamma_j - \gamma_{j+1}) |B \cap T_j| \leq \gamma_t r + \sum_{j=1}^{t-1} (\gamma_j - \gamma_{j+1}) d_j.$$

All bounds can be attained simultaneously: choose a basis from T_1, extend it to a basis from T_2, and continue. Since the coefficients in the sum are positive, a basis maximizes the weight precisely when

$$|B \cap T_j| = d_j \qquad (1 \leq j \leq t). \tag{12}$$

Let U_j be the span of the original vectors labelled by T_j, and set $U_0 = 0$. In the r-dimensional space

$$W = \bigoplus_{j=1}^{t} U_j / U_{j-1},$$

replace v_e, for $e \in T_j \setminus T_{j-1}$, by its class in the j-th summand. Each layer spans U_j/U_{j-1}, so the resulting represented matroid N has rank r.

A basis of N supplies a basis of each quotient. Its original vectors are independent: project any dependence to the highest occupied quotient, obtain zero coefficients in that layer, and descend. It also satisfies (12). Conversely, for an original basis satisfying these counts, $B \cap T_j$ is a basis of U_j, so the vectors from its j-th layer give a basis of U_j/U_{j-1}. Thus the bases of N are exactly the maximizing original bases. Taking convex hulls proves the assertion. □

Proposition 3.3. *For the input of Section 1 there is a representation N over $\mathbb{K}$ on E such that each original colour class is still a basis, every N-basis is an M-basis, and*

$$w/n \in \operatorname{relint} \operatorname{conv}\{\chi_R : R \in \mathcal{R}_N\},$$

where $\mathcal{R}_N$ is the family of rainbow bases of N. Moreover, for every positive integer k,

$$kw \in \sum_{R \in \mathcal{R}} \mathbb{Z}_{\geq 0} \chi_R \quad \Longleftrightarrow \quad kw \in \sum_{R \in \mathcal{R}_N} \mathbb{Z}_{\geq 0} \chi_R.$$

Proof. Put $x = w/n = (1/n) \sum_i \chi_{E_i} \in P(M)$. In the finite inequality description (10), impose as equalities all rank inequalities tight at x, and let F be the resulting face. The sum of these inequalities exposes F. Since all coordinates of x are positive and all other inequalities are strict there, a small ball about x in the common affine solution space lies in F. Hence that space is $\operatorname{aff} F$, and $x \in \operatorname{relint} F$. Any linear functional maximized at x is constant on this relative neighbourhood and therefore on F; thus F is the smallest face containing x.

Choose ω exposing F, with maximum β. Then

$$\beta = \omega \cdot x = \frac{1}{n} \sum_i \omega \cdot \chi_{E_i}, \qquad \omega \cdot \chi_{E_i} \le \beta.$$

Every summand equals β; hence $\chi_{E_i} \in F$ for every i. Lemma 3.2 gives a representation N with $P(N) = F$, preserving these bases.

Let $\mathcal{H} = \{y : y(E_i) = 1 \text{ for all } i\}$. It contains x. Intersecting a small relative ball about x in F with $\mathcal{H}$ gives a ball in $\operatorname{aff} F \cap \mathcal{H}$. Consequently,

$$\operatorname{aff}(F \cap \mathcal{H}) = \operatorname{aff} F \cap \mathcal{H}, \qquad x \in \operatorname{relint}(F \cap \mathcal{H}).$$

Equation (11) identifies this intersection with the required convex hull.

Suppose $kw = \sum_{R \in \mathcal{R}} m_R \chi_R$, with $m_R \ge 0$. A colour sum gives $\sum_R m_R = kn$. After division by kn, this is a convex expression for x using vertices of $P(M)$. Applying ω shows that every positive summand lies in F, and hence belongs to $\mathcal{R}_N$. The reverse implication holds because every N-basis is an original basis. □

4. All sufficiently large repetition numbers

4.1. **An elementary arithmetic lemma.**

Lemma 4.1. *For vectors $a_1, \dots, a_m \in \mathbb{Z}^s$, where $s \ge 0$ and $m \ge 1$ are integers, put*

$$C = \operatorname{cone}\{a_1, \dots, a_m\}, \qquad L_0 = \sum_j \mathbb{Z} a_j.$$

If $u \in L_0 \cap \operatorname{relint} C$, then $ku \in \sum_j \mathbb{Z}_{\ge 0} a_j$ for every sufficiently large integer k.

Proof. Set $a = \sum_j a_j$. Since $u \in \operatorname{relint} C$ and $a \in \operatorname{span}_{\mathbb{R}} C$, some positive rational ε satisfies $u - \varepsilon a \in C$. Choose an expression

$$u - \varepsilon a = \sum_{j \in J} c_j a_j, \qquad c_j > 0,$$

with $|J|$ minimal. Its active columns are linearly independent. Indeed, if $\sum_{j \in J} h_j a_j = 0$ is a nonzero real relation, change its sign so that some $h_j > 0$, and put

$$t = \min_{j \in J,\ h_j > 0} \frac{c_j}{h_j}.$$

The coefficients $c_j - t h_j$ are nonnegative and at least one is zero, contrary to minimality. Since $u - \varepsilon a$ is rational, an invertible square row minor of the active integer columns now shows that all c_j are rational. For $J = \varnothing$ the expression is zero. Giving the omitted columns coefficient zero and then adding ε to every coefficient yields a strictly positive rational expression for u.

Clear denominators and use $u \in L_0$ to obtain

$$Du = \sum_j b_j a_j, \quad b_j \in \mathbb{Z}_{>0}, \quad D \in \mathbb{Z}_{>0}, \qquad u = \sum_j z_j a_j, \quad z_j \in \mathbb{Z}. \tag{13}$$

Put $Z = \max_j \max(0, -z_j)$ and $Q = (D-1)Z$. For $k \ge \max(1, DQ)$, write $k = qD + r$, where $0 \le r < D$. Then $q \ge Q$, and

$$ku = \sum_j (q b_j + r z_j) a_j.$$

Each coefficient is at least $q - (D-1)Z \ge 0$. □

Proof of Theorem 1.3. Apply Proposition 3.3 to obtain N. Every colour is still a basis of an n-dimensional space over $\mathbb{K}$. Applying Corollary 1.2 to this new representation gives

$$w \in \sum_{R \in \mathcal{R}_N} \mathbb{Z} \chi_R.$$

Put

$$Q_N = \operatorname{conv}\{\chi_R : R \in \mathcal{R}_N\}, \qquad C_N = \operatorname{cone}\{\chi_R : R \in \mathcal{R}_N\}, \qquad \ell(y) = y(E_1).$$

Every generator has $\ell = 1$, so

$$\operatorname{aff} Q_N = \{y \in \operatorname{span}_{\mathbb{R}} C_N : \ell(y) = 1\}.$$

Indeed, a linear combination of the generators lies in this hyperplane exactly when its coefficients sum to one. The map $(\lambda, p) \mapsto \lambda p$ is a homeomorphism from $(0,\infty) \times \operatorname{aff} Q_N$ to $\{y \in \operatorname{span}_{\mathbb{R}} C_N : \ell(y) > 0\}$, with inverse $y \mapsto (\ell(y), y/\ell(y))$. Since $x = w/n \in \operatorname{relint} Q_N$, a neighbourhood of (n, x) maps into C_N. Thus $w \in \operatorname{relint} C_N$.

Lemma 4.1 now gives $kw \in \sum_{R\in\mathcal{R}_N} \mathbb{Z}_{\geq 0}\chi_R$ for every sufficiently large integer k. The bases of N are original bases, so $kw \in S$. Taking a colour sum shows that the decomposition uses exactly kn rainbow bases. □

Corollary 4.2. *For each positive integer n, there is an integer $K(n)$ that works as k_0 in Theorem 1.3 for every field and every input of rank n.*

Proof. On $[n] \times [n]$ there are n^n transversals, so only finitely many possible families $\mathcal{R}$. The semigroup and the feasibility of kw depend only on this family. Take the maximum of the thresholds from Theorem 1.3 over the families realizable over some field. □

For a fixed input, the two expressions in (13) give the sufficient threshold $\max\{1, D(D-1)Z\}$, where $Z = \max_j \max(0, -z_j)$.

4.2. **Examples.**

Example 4.3. Let $E_1 = \{a, b\}$, $E_2 = \{c, d\}$, with

$$v_a = v_c = e_1, \qquad v_b = e_2, \qquad v_d = e_1 + e_2.$$

Over every field the rainbow bases are $\{a, d\}$, $\{b, c\}$, and $\{b, d\}$. The rank-one set $\{a, c\}$ is tight at $w/2$. In any nonnegative expression

$$kw = \alpha\chi_{\{a,d\}} + \beta\chi_{\{b,c\}} + \gamma\chi_{\{b,d\}},$$

the a- and c-coordinates give $\alpha = \beta = k$, and the b-coordinate then gives $\gamma = 0$. Every such expression is therefore supported on $\{a, d\}$ and $\{b, c\}$.

The face representation lies in $\mathbb{K}e_1 \oplus (V/\mathbb{K}e_1)$. The labels b, d become parallel in its second summand, leaving exactly the rainbow bases $\{a, d\}$ and $\{b, c\}$. On this support, $w = \chi_{\{a,d\}} + \chi_{\{b,c\}}$.

Example 4.4. Consider the common-base configuration in Harvey, Király and Lau [11, Appendix A]. In the graphic matroid of the complete graph K_4, whose bases are its spanning trees, colour the edges by

$$\{12, 34\}, \qquad \{13, 24\}, \qquad \{14, 23\}.$$

The rainbow bases are precisely the four vertex stars T_v; the other four transversals are triangles. The all-one vector h on the six edges satisfies

$$h = \frac{1}{2}\sum_{v=1}^{4} \chi_{T_v},$$

but is not an integer combination of these vectors. Indeed, $h = \sum_v c_v\chi_{T_v}$ would give $c_u + c_v = 1$ on every edge uv, forcing all $c_v = 1/2$. Here the three colour classes have size two, while the graphic matroid has rank three.

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