Relations (III′) and (IV) of Nakamura and Schneps hold on the profinite Grothendieck–Teichmüller group, with an application to the two elliptic points of the modular curve

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Abstract

Nakamura and Schneps (2000) defined a subgroup $\Pi\Gamma$ of the profinite Grothendieck–Teichmüller group $\widehat{\mathrm{GT}}$ by two relations, (III′) and (IV), that hold for Galois elements; they proved that it acts on the profinite mapping class groups of all surfaces, and left open whether it is all of $\widehat{\mathrm{GT}}$. We prove that (IV) holds for every element of $\widehat{\mathrm{GT}}$. It follows that every element of $\widehat{\mathrm{GT}}$ is the product of a Galois element and an element of the group Λ of Hatcher, Lochak and Schneps, which acts on the whole tower of pure mapping class groups by their formulas. As an application we study a character δ with values in $\mathbf{F}_2$, defined on the larger group given by the two- and three-cycle relations alone, which compares at level twelve how an element moves the two elliptic points of the modular curve. We prove that δ vanishes on the normalized elements of $\widehat{\mathrm{GT}}$, so that every character extending it and trivial on Galois elements is trivial on $\widehat{\mathrm{GT}}$. Version 2 adds three things. First, (III′) also holds for every element of $\widehat{\mathrm{GT}}$; so $\Pi\Gamma = \widehat{\mathrm{GT}}$, and the representations of $\Pi\Gamma$ in the automorphism groups of all profinite mapping class groups, constructed by Nakamura and Schneps, are representations of $\widehat{\mathrm{GT}}$. Second, the questions that Lochak (2025) asks about his groups of compatible pairs have a positive answer: these groups are $\widehat{\mathrm{GT}}$, and by his theorem $\widehat{\mathrm{GT}}$ is then the group of inertia-preserving outer automorphisms of the procongruence mapping class group of every surface of generic type and positive genus. Third, a correction: version 1 called the strictness of the inclusion of $\widehat{\mathrm{GT}}$ in the larger group known, and we have found no published proof of it. The proofs use the theorem of Boggi and Zalesskii on finite p-subgroups of profinite completions, the action of Hatcher, Lochak and Schneps, and the theorem of Hoshi, Minamide and Mochizuki on the outer automorphisms of genus-zero configuration groups. The mathematics of version 1 was produced by GPT-6 Astra (ChatGPT Pro mode); it was checked by Claude Fable 5.2 at effort max, which wrote this text and rewrote several proofs, and by Claude Opus 5.5 at effort max as second reader. The additions of version 2 are by Claude Fable 5.2 at effort max, one of them with Claude Opus 5.5 at effort max; the new theorem and its two corollaries were checked by a second reader, Claude Opus 5.5 at effort max, in a separate run. No human has checked any of it.

1 The results

The group and the question

Let $\widehat{F}_2$ be the free profinite group on x, y. For $f \in \widehat{F}_2$ and elements a, b of a profinite group, $f(a,b)$ is the image of f under $x \mapsto a$, $y \mapsto b$. Drinfeld's group $\widehat{\mathrm{GT}}$ [10] consists of the pairs $F = (\lambda, f)$, with $\lambda \in \widehat{\mathbf{Z}}^\times$ and f in the closed commutator subgroup of $\widehat{F}_2$, such that $x \mapsto x^\lambda$, $y \mapsto f^{-1} y^\lambda f$ is an automorphism of $\widehat{F}_2$ and

(I) $f(x,y)f(y,x) = 1$;

(II) $f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$, where $xyz = 1$ and $m = (\lambda - 1)/2$;

(III) the five-cycle (pentagon) relation in the pure mapping class group $\widehat{\Gamma}_{0,5}$.

The pairs satisfying (I) and (II) form the larger group $\widehat{\mathrm{GT}}_0$ of Harbater and Schneps [13], called the gentle version in [9]. The absolute Galois group $G_{\mathbf{Q}}$ embeds in $\widehat{\mathrm{GT}}$, with λ the cyclotomic character [10, 17]. Whether $G_{\mathbf{Q}} = \widehat{\mathrm{GT}}$ is not known.

Write $B_3 = \langle u, v \mid uvu = vuv \rangle$. Every $F \in \widehat{\mathrm{GT}}$ acts on the profinite braid groups $\widehat{B}_n$; on $\widehat{B}_3$ the action is $u \mapsto u^\lambda$, $v \mapsto f(u^2, v^2)^{-1} v^\lambda f(u^2, v^2)$. Comparing the Galois actions on $\widehat{\Gamma}_{0,5}$ and $\widehat{\Gamma}_{1,2}$, Nakamura found that Galois elements satisfy, in $\widehat{B}_3$,

$$f(u, v^4) = v^{8\rho_2}\, f(u^2, v^2)\, u^{4\rho_2}\, (uv)^{-6\rho_2}, \tag{IV}$$

where $\rho_2 = \rho_2(F) \in \widehat{\mathbf{Z}}$ is defined for every $F \in \widehat{\mathrm{GT}}$ (Definition 6) and is, for a Galois element, the Kummer cocycle of the roots of 2 [23, §1 and Definition 5.1]. Nakamura and Schneps defined $\mathrm{I}\Gamma \subset \widehat{\mathrm{GT}}$ by (IV) and a second relation (III′) in $\widehat{\Gamma}_{0,[5]}$, proved that $G_{\mathbf{Q}} \subset \mathrm{I}\Gamma$ and that $\mathrm{I}\Gamma$ acts on $\widehat{\Gamma}(\Sigma)$ for every surface Σ, compatibly with inclusions of subsurfaces [23, Theorems 1.2–1.4]; they "leave it open to settle (in-)equalities between consecutive terms of $G_{\mathbf{Q}} \subset \mathrm{I}\Gamma \subset \widehat{\mathrm{GT}}$" [23, §1]. Hatcher, Lochak and Schneps [14] had defined the group Λ of the f with $(1, f) \in \widehat{\mathrm{GT}}$ that satisfy one relation (R) in the mapping class group of a torus with two boundary circles, and had shown that Λ acts on the tower of all pure profinite mapping class groups by explicit formulas attached to pants decompositions. In February 2026 Lochak, Nakamura and Schneps [21] proved that all of $\widehat{\mathrm{GT}}$ acts on $\widehat{\Gamma}_{g,0}$ and $\widehat{\Gamma}_{g,1}$ for every $g \geq 1$, and wrote of the added condition that "it is unclear, however, whether the additional condition actually defines a proper subgroup of $\widehat{\mathrm{GT}}$" [21, §1].

Theorem 1. *Relation* (IV) *holds in* $\widehat{B}_3$ *for every* $F = (\lambda, f) \in \widehat{\mathrm{GT}}$, *with* $\rho_2 = \rho_2(F)$.

So the subgroup $\mathrm{I}\Gamma$ of Nakamura and Schneps is defined inside $\widehat{\mathrm{GT}}$ by (III′) alone; Theorem 20, added in version 2, shows that (III′) holds on $\widehat{\mathrm{GT}}$ as well. For elements with $\lambda = 1$ and $\rho_2 = 0$ relation (IV) reads $f(u, v^4) = f(u^2, v^2)$, and Schneps showed that this form implies (R) [24, p. 670]. Call $F \in \widehat{\mathrm{GT}}$ *normalized* if $\lambda_F = 1$ and $\rho_2(F) = 0$; the normalized elements form a closed subgroup Λ_0 of $\widehat{\mathrm{GT}}$.

Theorem 2. *Every* $F \in \widehat{\mathrm{GT}}$ *is a product* $F = F_* \sigma$ *with* $\sigma \in G_{\mathbf{Q}}$ *and* $F_* = (1, f_*)$ *normalized, and* $f_* \in \Lambda$ *for every normalized element. Thus* $\widehat{\mathrm{GT}} = \Lambda_0 \cdot G_{\mathbf{Q}}$ *with* $\Lambda_0 \subset \Lambda$.

Before this, the elements of Λ known explicitly were Galois: Λ contains the absolute Galois group of the field generated by all roots of unity and all roots of 2 [14, p. 3]. Theorem 2 says that, up to Galois elements, every element of $\widehat{\mathrm{GT}}$ acts on the whole pure tower by the formulas of [14].

The application

The modular group $B = B_3/\langle (uv)^3 \rangle \cong C_2 * C_3$ is the orbifold fundamental group of the moduli of elliptic curves up to ± 1, based at the cusp. Put $s = uvu$, $r = uv$; they generate the stabilizers of the two elliptic points, the curves $y^2 = x^3 - x$ and $y^2 = x^3 + 1$ with complex multiplication by $\mathbf{Z}[i]$ and $\mathbf{Z}[\omega]$. An element $F = (1, f)$ of $\widehat{\mathrm{GT}}_0$ acts on $\widehat{B}$ and sends s and r to conjugates $\beta s \beta^{-1}$ and ArA^{-1} by unique elements β, A of the pure subgroup $\widehat{F} = \overline{\langle u^2, v^2 \rangle}$. On a subgroup H_0 of index at most three in the elements with $\lambda = 1$, the images of these two *frames* in $\mathrm{PSL}_2(\mathbf{Z}/12)$ define two characters $a_2, a_3 : H_0 \to \mathbf{F}_2$ and $\delta = a_2 + a_3$ (Section 6). Normalized elements lie in H_0.

Theorem 3. $\delta(F) = 0$ *for every normalized* $F \in \widehat{\mathrm{GT}}$.

Corollary 4. *Let $\Delta : \widehat{\mathrm{GT}} \to \mathbf{F}_2$ be a homomorphism that is trivial on $G_{\mathbf{Q}}$ and equal to δ on Λ_0. Then $\Delta = 0$.*

Proof. By Theorem 2, $\Delta(F) = \Delta(F_*) + \Delta(\sigma) = \delta(F_*) = 0$. □

Added in version 2: relation (III′) and the questions of Lochak

Three statements are added. They are stated in full and proved in Section 10, and their numbers follow those of version 1, which are unchanged.

Relation (III′) (Theorem 20). For every $F = (\lambda, f) \in \widehat{\mathrm{GT}}$,

$$f(\tau_1\tau_3, \tau_2^2) = g(x_{45}, x_{51})\, f(x_{12}, x_{23})\, f(x_{34}, x_{45}) \qquad \text{in } \widehat{\Gamma}_{0,[5]},$$

where the τ_i are the standard generators, the x_{ij} are the standard twists and $g \in \widehat{F}_2$ is the element with $f(x,y) = g(y,x)^{-1} g(x,y)$. Nakamura and Schneps prove this for Galois elements [23, Theorem 1.1]. The proof for all of $\widehat{\mathrm{GT}}$ takes one page: the pentagon says that the right side commutes with an involution of $\widehat{\Gamma}_{0,[5]}$; by the theorem of Boggi and Zalesskii it then lies in the completion of the discrete centralizer; and an element of that group is determined by its image when the fifth point is forgotten.

The group of Nakamura and Schneps (Corollary 21). $\mathrm{I}\Gamma = \widehat{\mathrm{GT}}$. So the representations of $\mathrm{I}\Gamma$ in $\mathrm{Aut}\,\widehat{\Gamma}(\Sigma)$, which Nakamura and Schneps construct for every surface Σ with a quilt decomposition and prove compatible with the inclusions of subsurfaces [23, Theorems 1.3 and 1.4], are representations of $\widehat{\mathrm{GT}}$.

The questions of Lochak (Corollary 22). Lochak calls two elements F_0, F_1 of $\widehat{\mathrm{GT}}$ a compatible pair if the outer actions they define on a piece of type $(0,5)$ and on a piece of type $(1,2)$ of a surface of type $(1,3)$ agree where the two pieces meet; this is one equation in $\widehat{B}_3$ [20, Definition 5.4]. The compatible pairs form a subgroup of $\widehat{\mathrm{GT}}$ [20, Proposition–Definition 5.6], and those with $F_0 = F_1$ are the elements that satisfy (IV) [20, pp. 58 and 60]. He proves that the group of compatible pairs is the group of inertia-preserving outer automorphisms of the procongruence mapping class group of every surface of generic type [20, Theorem 5.10], and asks whether every compatible pair has $F_0 = F_1$, whether the group of compatible pairs is $\widehat{\mathrm{GT}}$, and whether (IV) holds on all of $\widehat{\mathrm{GT}}$ [20, p. 62]. By Theorem 1 the answer to the three questions is yes, and with his theorem $\mathrm{Out}^*(\check{\Gamma}(S)) \cong \widehat{\mathrm{GT}}$ for every surface S of type (g,n) with $g \geq 1$, $3g - 3 + n > 2$ and $(g,n) \neq (2,0)$.

What it means, and what is not proved here

In arithmetic terms the two bits compare two descriptions of one field. For a Galois element with $\lambda = 1$ both record the sign by which it moves $(3/4)^{1/4}$: this number generates the 4-division field of the curve with $j = 0$ and the field of the x-coordinates of the 3-division points of the curve with $j = 1728$, over the cyclotomic field. This description, the existence of a continuous character Δ of all of $\widehat{\mathrm{GT}}_0$ that restricts to δ and vanishes on $G_{\mathbf{Q}}$, and the existence of an element of $\widehat{\mathrm{GT}}_0$ on which it is 1, are results of the longer text behind this note (see "Provenance and checks"). *They are cited from that text and have not been refereed; they are not proved here and are not used in the proofs of Theorems 1–3.* Of the facts behind them, only the equality of the second of the two fields with $\mathbf{Q}(i, (3/4)^{1/4})$ has been verified independently. Granting them, Corollary 4 says: the two frames at the elliptic points can disagree on the hexagon group, and the pentagon relation forces them to agree. The inclusion $\widehat{\mathrm{GT}} \subset \widehat{\mathrm{GT}}_0$ is clear from the definitions [9, p. 2]. Version 1 of this note said that its strictness is known; we have found no published proof that the inclusion is strict. The research runs behind this note contain several arguments for strictness, and the nonvanishing just quoted would be one; none of them has been refereed. Granting them, what the character adds is an explicit level-twelve invariant that separates the two groups, with this meaning.

Credit

Relation (IV) is Nakamura's, and the group it defines with (III′) is Nakamura and Schneps's; that Galois elements satisfy it is their theorem [23, §1]. The group Λ, the relation (R) and the action through the complex of pants decompositions are due to Hatcher, Lochak and Schneps [14]. The implication from the normalized relation (IV) to (R) is Schneps's [24, p. 670]. What is added here is the passage from Galois elements to all of $\widehat{\mathrm{GT}}$ in Theorem 1, its consequence Theorem 2, and Theorem 3. The proofs rest on three theorems of others: the theorem of Boggi and Zalesskii that a finite p-subgroup of the profinite completion of a mapping class group of genus at most two is conjugate into the discrete group, with the expected normalizer [7]; the action of Λ [14]; and the theorem of Hoshi, Minamide and Mochizuki that $\mathrm{Out}(\widehat{\Gamma}_{0,m}) = \widehat{\mathrm{GT}} \times \mathfrak{S}_m$ for $m \geq 5$ [16, Corollary C]. For the additions of version 2: relation (III′), the group Π and its representations are due to Nakamura and Schneps [23]; compatible pairs, the groups they form and the theorem identifying them with automorphism groups are due to Lochak [20], who asked the questions answered in Corollary 22. What is added is Theorem 20 and the deduction of the two corollaries.

Context

The idea of reading relations of $\widehat{\mathrm{GT}}$ off special loci of moduli spaces, where curves have automorphisms, is that of [24], where (IV) and (III′) are derived from one special morphism for the elements that respect it; the action of $\widehat{\mathrm{GT}}$ on torsion of genus-zero mapping class groups was studied by Collas [8]. Our proof of Theorem 1 is of this kind, with the theorem of [7] supplying, for every element of $\widehat{\mathrm{GT}}$, the conjugation that the earlier works assume. The results of [21] concern the groups with at most one marked point and all of $\widehat{\mathrm{GT}}$, without the compatibility with subsurfaces; they neither use nor imply (IV), and Theorem 2 does not give their theorem. Basualdo Bonatto and Robertson [3] build a modular operad of mapping class groupoids on which the subgroup of Nakamura and Schneps acts after completion; by Corollary 21 that subgroup is $\widehat{\mathrm{GT}}$. Lochak [20] determines the automorphisms of procongruence curve complexes and asks whether (IV) holds on $\widehat{\mathrm{GT}}$; version 1 of this note did not cite his paper. Furusho's survey of 2002 records that it was not known whether (III′), (IV) and other relations satisfied by Galois elements follow from (I), (II) and (III) [12, p. 191]. Boggi's preprint [6] states that $\widehat{\mathrm{GT}}$ embeds in outer automorphism groups of profinite completions of pure mapping class groups; we come back to it in Remark 23. We know of no earlier statement of Theorems 1–3; a search of arXiv and of Hexagon on 23 November 2026 found none. For Theorem 20 a search of 24 November found none either, with one sentence of [20] to be mentioned (Remark 23).

Scope

We do not decide whether $G_{\mathbf{Q}} = \widehat{\mathrm{GT}}$. We do not compare the actions now available for every element of $\widehat{\mathrm{GT}}$ on mapping class groups of positive genus: that of [23], that of [21] on the groups with at most one marked point, that of [20] on procongruence completions, and the one obtained from [14] by Theorem 2. Theorem 5.10 of [20] and Theorems 1.3 and 1.4 of [23] are used as published; we have not checked their proofs.

2 Proof overview

The difficulty. Relation (IV) compares f on a half twist and a full twist, $f(u, v^4)$ against $f(u^2, v^2)$. For Galois elements it expresses that two maps between moduli spaces are defined over $\mathbf{Q}$. An element of $\widehat{\mathrm{GT}}$ is only an automorphism of profinite groups, and such maps are not part of the structure it is known to respect.

The idea. Use a locus where curves have automorphisms. In the moduli of ten points on the sphere, the configurations invariant under the Klein four-group $\{\pm t^{\pm 1}\}$ form a thrice-punctured sphere. This locus maps to the moduli of four points in two unordered pairs in two ways, by remembering two different quartets, and the two maps differ by an automorphism S of the target that exchanges a half-twist cusp with a full-twist cusp. An element F of $\widehat{\mathrm{GT}}$ acts on $\widehat{\Gamma}_{0,[10]}$. By the theorem of Boggi and Zalesskii the image of the four-group is conjugate to a discrete four-group, hence, after a correction that both forgetting maps kill, equal to it; F then acts on its centralizer, compatibly with both maps. So the standard action on the four-point group commutes with S up to an inner automorphism (Proposition 8). Centralizers in a free product turn this into one equation, and abelianization on a subgroup of index two gives (IV) with a coefficient c; a second character computation shows $c = \rho_2$ (Section 4).

From (IV) to the tower. Multiplying by a Galois element makes $\lambda = 1$ and $\rho_2 = 0$; Schneps's computation gives (R), so the action of [14] is available (Section 5). We extend it to mapping classes that permute marked points and check that it commutes with forgetting a point (Section 7).

The character. For Theorem 3 the same method is used in genus one, twice. The translations by points of order three form a group C_3^2 in the mapping class group of a torus with nine marked points; its normalizer contains a copy of $\mathrm{SL}_2(\mathbf{Z})$, on which a normalized F acts as on $\widehat{\Gamma}_{1,1}$. Dividing by the elliptic involution and forgetting three branch points leads to a sphere with five marked points, and the double cover branched at the images of the points of order three is again a torus with two marked points. A classical identity makes this torus the curve $y^2 = x^3 + 1$, whatever the curve one starts from: the modular orbifold is identified with the quotient of the punctured hexagonal curve by its automorphism group μ_6. So F acts compatibly on the modular group and on a subgroup G_3 of $\widehat{\Gamma}_{1,2}$ isomorphic to it (Proposition 18). Under this isomorphism the frame of F at the point of order three becomes a six-fold symmetry of a lattice, the frame at the point of order two a translation, and the two bits are read off the same matrix modulo 4 and modulo 3 (Section 9). The hypothesis $\rho_2 = 0$ is used to have the action of [14]; the pentagon is used through [14] and [16].

Relation (III′) *and the questions of Lochak (version 2).* Relation (III′) compares the value of f on a group of mapping classes that commute with an involution ε of the sphere with five marked points, the left side, with a product of values of f on twists, the right side. Applying ε to the right side and using the pentagon gives the same element back; this is how Nakamura and Schneps deduce the pentagon from (III′), read backwards. So the right side commutes with ε, and by the theorem of Boggi and Zalesskii it lies in the completion of the discrete centralizer, as the left side does. On the pure part of that centralizer, forgetting the fifth point, which ε fixes, is injective, and it sends both sides to $f(u^2, v^2)$ (Section 10). The action of F on $\widehat{\Gamma}_{0,[5]}$ is not used. Lochak's questions are then a matter of rewriting (IV): his compatibility condition for the pair (F, F) is relation (IV), because $(uv^2)^2 = (uv)^3$ is central in B_3.

3 Conventions

Groups

Completions are profinite completions; $\overline{X}$ is the closure of X. We multiply in $\widehat{\mathrm{GT}}_0$ so that $F \mapsto \Phi_F$ is a homomorphism, where $\Phi_F(x) = x^\lambda$, $\Phi_F(y) = f^{-1}y^\lambda f$. For $\sigma \in G_{\mathbf{Q}}$, λ is the cyclotomic character. $\Gamma_{g,[n]}$ is the mapping class group of an oriented surface of genus g with n marked points, which may be permuted; $\Gamma_{g,n}$ is its pure subgroup. A subscript such as $[9] + [3]$ means that two sets of marked points are preserved separately.

Braids

B_n has generators $\sigma_1, \dots, \sigma_{n-1}$; in B_3 we write u, v, and in B_4 we write a, b, c. Put $y_i = \sigma_{i-1} \cdots \sigma_1^2 \cdots \sigma_{i-1}$. We use the following known facts.

(B1) For $F \in \widehat{\mathrm{GT}_0}$ the formulas $u \mapsto u^\lambda$, $v \mapsto f(u^2, v^2)^{-1} v^\lambda f(u^2, v^2)$ define an automorphism Φ_F of $\widehat{B}_3$, with $\Phi_F((uv)^3) = (uv)^{3\lambda}$ [22, 13].

(B2) For $F \in \widehat{\mathrm{GT}}$ and $n \geq 3$ the formulas $\sigma_1 \mapsto \sigma_1^\lambda$, $\sigma_i \mapsto f(y_i, \sigma_i^2)^{-1} \sigma_i^\lambda f(y_i, \sigma_i^2)$ define an automorphism A_F of $\widehat{B}_n$ [10, 22], [21, Theorem 2.1]. It induces the identity on permutations and descends to the quotient $\widehat{\Gamma}_{0,[n]}$ and to the quotient by the centre, which is the stabilizer of one point in $\widehat{\Gamma}_{0,[n+1]}$.

(B3) *Forgetting.* Let $M_S \subset \Gamma_{0,[n]}$ be the stabilizer of a set S of marked points and $p_S : M_S \to \Gamma_{0,[S]}$ the map forgetting the others. For $F \in \widehat{\mathrm{GT}}$ there is a pure $g \in \widehat{\Gamma}_{0,S}$ with $p_S \circ A_F = \mathrm{Ad}(g) \circ A_F \circ p_S$ on $\widehat{M}_S$.

For (B3), see [13, §3.1]; the proof is short. If S is the set of the first k strands, p_S is induced by deleting the last $n-k$ strands of braids that preserve the first k; this group is generated by the σ_i with $i \neq k$ and by σ_k^2. For $i < k$ the formula for $A_F(\sigma_i)$ is the same word in $\widehat{B}_k$; for $i > k$, and for σ_k^2, the generator dies and its image under A_F, a conjugate of a power of it by a pure braid, dies too. So p_S commutes with A_F. In general conjugate S to the first k strands by a braid b, and note that $A_F(b)b^{-1}$ is pure.

(F) *Completion of a free normal subgroup.* Let Π be a normal subgroup of a group A that is free of finite rank $r \geq 2$. Then the profinite topology of A induces the full profinite topology on Π, and $1 \to \widehat{\Pi} \to \widehat{A} \to \widehat{A/\Pi} \to 1$ is exact; see also [1].

For (F), let N be a characteristic subgroup of finite index of Π, $D = \Pi/N$, ℓ a prime larger than $|D|(r-1)$, and $N_1 = [N,N]N^\ell$. On the cover with group D of a rose with r petals, a nontrivial element of D has trace 0 on the cellular chains, hence trace 1 on $N/N_1 = H_1(N, \mathbf{F}_\ell)$, whose dimension $1 + |D|(r-1)$ is not 1 modulo ℓ. So D acts faithfully on N/N_1, the centre of Π/N_1 has trivial image in D, and the kernel of the finite quotient $A \to \mathrm{Aut}(\Pi/N_1)$ meets Π inside N.

The modular group, frames and ρ_2

Let $B = B_3/\langle (uv)^3 \rangle$, $s = uvu$, $r = uv$, $\tau = sr$. Then $s^2 = r^3 = 1$, $B = \langle s \rangle * \langle r \rangle$ and $\tau = u^{-1}$. The elements $x = u^2$, $y = v^2$ freely generate the kernel F of $B \to \mathfrak{S}_3$, and $sxs^{-1} = y$. We identify $\widehat{F}$ with $\widehat{F}_2$; by (B1) every $F \in \widehat{\mathrm{GT}_0}$ acts on $\widehat{B} = \widehat{C_2} \amalg \widehat{C_3}$, by the identity on $\mathfrak{S}_3$ and by Φ_F on $\widehat{F}$. Let $\vartheta = \mathrm{Ad}(s)|_{\widehat{F}}$, which exchanges x and y.

We use the theorem of Herfort and Ribes [15]: in a free profinite product $G_1 \amalg G_2$, every finite subgroup is conjugate into a factor, and the centralizer of a nontrivial element of a factor lies in that factor. In particular $C_{\widehat{B}}(s) = \langle s \rangle$, $C_{\widehat{B}}(r) = \langle r \rangle$, and $C_{\widehat{F}}(y) = \overline{\langle y \rangle}$.

Lemma 5. *Let $F \in \widehat{\mathrm{GT}_0}$. There is a unique $\beta_F \in \widehat{F}$ with $\Phi_F(s) = \beta_F s \beta_F^{-1}$. If $\lambda_F = 1$ there is a unique $A_F \in \widehat{F}$ with $\Phi_F(r) = A_F r A_F^{-1}$. Moreover $\beta_{FG} = \Phi_F(\beta_G)\beta_F$, and $A_{FG} = \Phi_F(A_G)A_F$ when $\lambda_F = \lambda_G = 1$.*

Proof. $\Phi_F(s)$ has order two, so it is gsg^{-1} for some $g \in \widehat{B}$. Since Φ_F is the identity on $\mathfrak{S}_3$, the image of g commutes with that of s, so g or gs lies in $\widehat{F}$. Uniqueness holds because $C_{\widehat{B}}(s) \cap \widehat{F} = 1$. If $\lambda = 1$, Φ_F is the identity on $B^{\mathrm{ab}} = \mathbf{Z}/6$, where $r \neq r^{-1}$; so the element $\Phi_F(r)$ of order three is conjugate to r, and the same argument applies. The product formulas follow from uniqueness. □

Definition 6. For $F \in \widehat{\mathrm{GT}}_0$, $\rho_2(F) \in \widehat{\mathbf{Z}}$ is defined by $[\beta_F^{-1}] = (\rho_2(F), \rho_2(F))$ in $\widehat{F}^{\mathrm{ab}} = \widehat{\mathbf{Z}}x \oplus \widehat{\mathbf{Z}}y$.

Lemma 7. (a) *$f = \vartheta(\beta_F)\beta_F^{-1}$; so $g_F = \beta_F^{-1}$ satisfies $f(x,y) = g_F(y,x)^{-1}g_F(x,y)$, the class of β_F^{-1} is of the form (ρ,ρ), and ρ_2 is the coefficient of [23, Definition 5.1].* (b) *$\rho_2(FG) = \rho_2(F) + \lambda_F\rho_2(G)$.* (c) *If $\lambda_F = 1$ then $\rho_2(F) \in 2\widehat{\mathbf{Z}}$.* (d) *For $\sigma \in G_{\mathbf{Q}}$, $\rho_2(\sigma)$ is the Kummer cocycle κ_2 of the positive roots of 2: $\sigma(2^{1/n}) = \zeta_n^{\kappa_2(\sigma)}2^{1/n}$.*

Proof. (a) From $y = sxs^{-1}$ we get $\Phi_F(y) = \beta s\beta^{-1}x^\lambda\beta s^{-1}\beta^{-1} = \beta\vartheta(\beta)^{-1}y^\lambda\vartheta(\beta)\beta^{-1}$. Comparing with $f^{-1}y^\lambda f$, the element $f\beta\vartheta(\beta)^{-1}$ centralizes y^λ, so it is y^e for some $e \in \widehat{\mathbf{Z}}$. Abelianizing, with $[\beta] = (b_1, b_2)$ and $[f] = 0$, gives $(b_1 - b_2, b_2 - b_1) = (0, e)$; so $b_1 = b_2$ and $e = 0$. (b) Φ_F acts on $\widehat{F}^{\mathrm{ab}}$ by λ_F; apply Lemma 5. (c) In the quotient of $\widehat{F}$ of nilpotency class two, relation (II) with $\lambda = 1$ gives $[x,y]^{3e} = 1$ for $f = [x,y]^e$, so $f = 1$ there. Writing $g_F = x^\rho y^\rho [x,y]^d$ one finds $g_F(y,x)^{-1}g_F(x,y) = [x,y]^{\rho^2+2d}$, so $\rho^2 \in 2\widehat{\mathbf{Z}}$. (d) This is [23, §5, after Definition 5.1]. □

4 Relation (IV)

The coloured four-point group

Let $R_0 = \langle d, z \mid (zd)^2 = 1\rangle$ and $R = \widehat{R_0}$. With $a = zd$ we have $R_0 = \langle a\rangle * \langle d\rangle \cong C_2 * \mathbf{Z}$. The map $\iota_0 : d \mapsto u$, $z \mapsto v^2$ identifies R_0 with the subgroup of index three of B that is the image of $J = \langle u, v^2\rangle \subset B_3$; one has $(uv^2)^2 = (uv)^3$, and J has the presentation $\langle u, z \mid (uz)^2 = (zu)^2\rangle$. Geometrically R_0 is the orbifold fundamental group of the moduli $\mathcal{R}$ of four points on the sphere in two unordered pairs: for the quartet $\{0,\infty\}, \{t, 1/t\}$ the coordinate is $w = t^2 + t^{-2}$, and $\mathcal{R}$ is the w-line with a cusp at $w = \infty$, where points of different pairs meet, a cusp at $w = 2$, where the two points of a pair meet, and a point of order two at $w = -2$. The loops z and d go around ∞ and 2. In terms of mapping classes, R_0 is the group of classes of the four-pointed sphere preserving each pair, modulo the class that flips both pairs; it is the image of that group in $\Gamma_{0,[4]}/V = B$, where V is the normal four-group.

By (B1), $F \in \widehat{\mathrm{GT}}_0$ acts on R through ι_0:

$$\Phi_R(d) = d^\lambda, \qquad \Phi_R(z) = h\, z^\lambda h^{-1}, \qquad h = f(z, d^2).$$

For $F \in \widehat{\mathrm{GT}}$ this is the action induced by A_F on $\widehat{\Gamma}_{0,[4]}$, because the projection of the sphere braid group on four strands to B sends $(\sigma_1, \sigma_2, \sigma_3)$ to (u, v, u) and $f(vu^2v, u^2) = 1$ in $\widehat{B}$. Let S be the automorphism of R_0 exchanging d and z.

Proposition 8. *For every $F \in \widehat{\mathrm{GT}}$ there is $g_0 \in R$ with $S\Phi_R S = \mathrm{Ad}(g_0)\Phi_R$.*

The proof occupies the next three subsections.

Finite p-subgroups

Lemma 9. *Let Γ be a subgroup of finite index of a mapping class group $\Gamma_{g,[n]}$ with $g \le 2$ and $2g - 2 + n > 0$, let P be a normal subgroup of finite index of Γ, and let $H \subset \Gamma$ be a finite p-group with $H \cap P = 1$.*

(a) Let α be a continuous automorphism of $\widehat{\Gamma}$ that preserves $\widehat{P}$ and induces the identity on Γ/P. Then there are $e \in \widehat{P}$ and a homomorphism $h \mapsto h'$ from H to Γ with $h' \in Ph$ and $\alpha(h) = eh'e^{-1}$ for all $h \in H$.

(b) $N_{\widehat{\Gamma}}(H) = \overline{N_\Gamma(H)}$ and $C_{\widehat{\Gamma}}(H) = \overline{C_\Gamma(H)}$.

The same holds for a group $\Gamma = K \rtimes H$ when K has a subgroup of finite index that is an iterated extension of finitely generated free groups.

Proof. Mapping class groups of genus at most two, and hence their subgroups of finite index, are good and of finite virtual cohomological type, so the theorem of Boggi and Zalesskii applies to them: every finite p-subgroup of the completion is conjugate to a subgroup of the discrete group, and normalizers of finite p-subgroups of the discrete group are the closures of the discrete normalizers [7, Theorem A and the theorem on hyperelliptic mapping class groups of the last section]. The groups $K \rtimes H$ of the last sentence satisfy the same hypotheses. (a) Apply this to the group $PH = P \rtimes H$, whose completion is $\widehat{P} \rtimes H$ and contains $\alpha(H)$: there are $g = eh_0$, with $e \in \widehat{P}$ and $h_0 \in H$, and a finite $H'' \subset PH$ with $\alpha(H) = gH''g^{-1}$. Put $h' = e^{-1}\alpha(h)e \in h_0 H'' h_0^{-1} \subset \Gamma$; modulo $\widehat{P}$ it is h. (b) The first equality is the cited statement. The centralizer is the kernel of $N_{\widehat{\Gamma}}(H) \to \mathrm{Aut}(H)$, hence open in the closure of $N_\Gamma(H)$; an open subgroup of the closure of a subgroup N is the closure of its intersection with N, which here is $C_\Gamma(H)$. □

The Klein configuration

Let $H = \{1, a, b, ab\}$ act on $\mathbf{P}^1$ by $a(t) = -t$, $b(t) = 1/t$, and let $Z = \mathbf{P}^1 \setminus \{0, \infty, \pm 1, \pm i\}$. For $t \in Z$ let $x(t)$ be the configuration of ten labelled points

$$A_\pm = (0, \infty), \quad B_\pm = (1, -1), \quad C_\pm = (i, -i), \quad D_h = h(t) \ (h \in H).$$

It is invariant under H, which permutes the labels. Fix $t_0 = 1 + \sqrt{2}$ and use $x(t_0)$ as reference surface. Then H is a subgroup of $M = \Gamma_{0,[10]}$. Let $G \subset M$ be the group of classes whose permutation of the labels commutes with those of H, $P = \Gamma_{0,10}$, and $N = C_G(H)$.

The discrete centralizer. A class in N is represented by a homeomorphism commuting with H, because the fixed locus of H in Teichmüller space is connected [18]. It descends to the quotient sphere $\mathbf{P}^1/H$, with coordinate $w = t^2 + t^{-2}$, and fixes there the images $\infty, 2, -2$ of the pairs A, B, C, whose stabilizers in H are different, and the image of D. Hence N/H is the pure mapping class group of a sphere with four marked points, that is,

$$N/H \ \cong \ \pi_1(\mathbf{P}^1_w \setminus \{\infty, 2, -2\}, w_0), \qquad w_0 = 6,$$

a loop being the motion of the point D.

The correction. Let $F \in \widehat{\mathrm{GT}}$ and $\alpha = A_F$ on $\widehat{M}$. By Lemma 9(a), applied to $\Gamma = G$, there are $e \in \widehat{P}$ and $h \mapsto h'$ with $\alpha(h) = eh'e^{-1}$, where h' and h permute the labels in the same way. The finite group $\{h'\}$ acts by Möbius transformations on a sphere with ten marked points [18]. Put A_+, A_-, B_+ at $0, \infty, 1$. Then a', which fixes $A_\pm$ and has order two, is $t \mapsto -t$; b' exchanges $0, \infty$ and fixes 1, so it is $t \mapsto 1/t$; hence $B_- = -1$, the fixed points $C_\pm$ of $a'b'$ are $\pm i$ in one of two orders, and $D_h = h(t_1)$ for some $t_1 \in Z$. The configuration is $x(t_1)$, or $x(t_1)$ with the labels C_+, C_- exchanged. Since Z is connected, a path from t_0 to t_1 gives $p \in P$ with $h' = php^{-1}$ in the first case and $h' = p\theta h \theta^{-1} p^{-1}$ in the second, where $\theta \in G$ is a half twist exchanging C_+ and C_- in a disc containing no other marked point. So there is $e_1 \in \widehat{P} \cup \widehat{P}\theta$ such that

$$\alpha' = \mathrm{Ad}(e_1^{-1}) \circ A_F \quad \text{fixes } H \text{ pointwise.}$$

By Lemma 9(b), α' preserves $C_{\widehat{G}}(H) = \overline{N}$ and induces an automorphism α_N of $\overline{N}/H$.

The two forgetting maps

Let $M_b \subset M$ be the stabilizer of the sets $\{A_+, A_-\}$ and $\{D_1, D_b\}$. Then $N_b = N \cap M_b$ has index two in N, $N = HN_b$ and $H \cap N_b = \langle b \rangle$. Forgetting the six other points maps M_b to the group of classes of a four-pointed sphere preserving two pairs, and b to the class flipping both. So we get $q_b : N/H \to R_0$. On the locus Z the retained quartet is $\{0, \infty\}, \{t, 1/t\}$, with coordinate w. Hence q_b is induced by the inclusion of $\mathbf{P}^1_w \setminus \{\infty, 2, -2\}$ in $\mathcal{R}$; it is surjective.

In the same way the stabilizer M_a of $\{B_+, B_-\}$ and $\{D_1, D_a\}$ gives $q_a : N/H \to R_0$. Its quartet $\{1,-1\},\{t,-t\}$ is carried by the map $\xi \mapsto (\xi-1)/(\xi+1)$ to $\{0,\infty\},\{r,1/r\}$ with $r=(t-1)/(t+1)$, and

$$r^2 + r^{-2} = \frac{2(w+6)}{w-2} =: T(w).$$

T is an involution of $\mathcal{R}$ that fixes $w=-2$ and $w_0=6$ and exchanges the two cusps. So $q_a = T_* \circ q_b$, where T_* maps the conjugacy class of d to that of z and conversely. We claim that $T_* = \mathrm{Ad}(g_1)S$ with $g_1 \in R_0$. By the generators of Fouxe-Rabinovitch for the automorphism group of a free product [11], $\mathrm{Aut}(R_0)$ is generated by the inner automorphisms, the partial conjugations, which here are inner, and the automorphisms $\iota : d \mapsto d^{-1}$, $t : d \mapsto da$ and $t' : d \mapsto ad$, each fixing a. The order of $\mathrm{Out}(R_0)$ is not stated in [11]; we derive it. One has $t' = \mathrm{Ad}(a)t$, $t^2 = 1$ and $\iota t \iota = \mathrm{Ad}(a)t$, so $\mathrm{Out}(R_0)$ consists of the classes of $1, \iota, t, t\iota$. They send the image $(1,0)$ of d in $R_0^{\mathrm{ab}} = \mathbf{Z}\times\mathbf{Z}/2$ to $(1,0)$, $(-1,0)$, $(1,1)$, $(-1,1)$, so they are distinct. The image of $z = ad^{-1}$ is $(-1,1)$. Hence a class that exchanges the conjugacy classes of d and z is that of $t\iota$; both T_* and S are such classes.

Proof of Proposition 8. Both maps extend to $\overline{N}/H \to R$, surjectively. The element e_1 lies in the completion of $M_b \cap M_a$, and its images under the two forgetting maps are pure, because θ is supported in a disc that contains only forgotten points. By (B3) and the description of Φ_R,

$$q_b\alpha_N = \mathrm{Ad}(g_b)\Phi_R q_b, \qquad q_a\alpha_N = \mathrm{Ad}(g_a)\Phi_R q_a$$

for some $g_a, g_b \in R$. Substituting $q_a = \mathrm{Ad}(g_1)Sq_b$ in the second equation and using the first gives $\mathrm{Ad}(g_1)S\,\mathrm{Ad}(g_b)\Phi_R q_b = \mathrm{Ad}(g_a)\Phi_R\,\mathrm{Ad}(g_1)Sq_b$. As q_b is surjective, $S\Phi_R$ and $\Phi_R S$ differ by an inner automorphism. □

From the proposition to the relation

Let $E \subset \widehat{F}_2$ be the kernel of $x\mapsto 1$, $y\mapsto 0$ in $\mathbf{Z}/2$. It is free on x^2, y, xyx^{-1}, and f, which has exponent sums zero, has class $(0,c,-c)$ in E^{ab} for a unique $c = c(f) \in \widehat{\mathbf{Z}}$.

Proposition 10. *For every $F \in \widehat{\mathrm{GT}}$: $f(u,v^4) = v^{8c}f(u^2,v^2)u^{4c}(uv)^{-6c}$ in $\widehat{B}_3$, with $c = c(f)$.*

Proof. Put $k = S(h) = f(d,z^2)$. Then $S\Phi_R S(z) = z^\lambda$ and $S\Phi_R S(d) = kd^\lambda k^{-1}$. Evaluate $S\Phi_R S = \mathrm{Ad}(g_0)\Phi_R$ on z and on d: $g_0 h$ centralizes z^λ and $k^{-1}g_0$ centralizes d^λ. By [15], applied to $R = \langle a\rangle \amalg \overline{\langle d\rangle}$ and to its image under S, these centralizers are $\overline{\langle z\rangle}$ and $\overline{\langle d\rangle}$. So $g_0 = z^s h^{-1} = kd^t$ with $s,t \in \widehat{\mathbf{Z}}$, that is,

$$z^s = k\,d^t\,h.$$

The kernel K_0 of $R_0 \to \mathbf{Z}/2$, $d\mapsto 1$, $z\mapsto 0$, is free on z and d^2, since $dzd^{-1} = z^{-1}d^{-2}$. The elements $h = f(z,d^2)$ and k lie in $\widehat{K_0}$, so $t \in 2\widehat{\mathbf{Z}}$. In K_0^{ab}, with basis z, d^2, the class of h is 0. The substitution $x\mapsto d$, $y\mapsto z^2$ sends x^2, y, xyx^{-1} to $d^2, z^2, (z^{-1}d^{-2})^2$, of classes $(0,1),(2,0),(-2,-2)$; so the class of k is $c(2,0) - c(-2,-2) = (4c, 2c)$. Abelianizing the displayed equation gives $(s,0) = (4c, 2c + t/2)$: $s = 4c$, $t = -4c$, and $f(d,z^2) = z^{4c}f(d^2,z)d^{4c}$ in R. The preimage of K_0 in J is $K_0 \times \langle (uv)^3\rangle$, so the kernel of $\widehat{J} \to R$ is $\overline{\langle (uv)^3\rangle} \cong \widehat{\mathbf{Z}}$, and $f(u,v^4) = v^{8c}f(u^2,v^2)u^{4c}(uv)^{3e}$ for some $e \in \widehat{\mathbf{Z}}$. The character $u\mapsto 1$, $v^2\mapsto 0$ of J takes the value 2 on $(uv)^3 = (uv^2)^2$; applied to this equation it gives $0 = 4c + 2e$. □

Proposition 11. *$c(f) = \rho_2(F)$ for every $F \in \widehat{\mathrm{GT}}$.*

Proof. Let $j = \mathrm{Ad}(u^{-1}s)\circ\iota_0 : R \to \widehat{B}$. Then $j(z) = x$, $j(d) = x^{-1}s$, $j(a) = s$, $j(d^2) = x^{-1}y^{-1}$, and the image of j contains $\widehat{F}$. Let $\varphi = \mathrm{Ad}(h^{-1})\Phi_R$, so $\varphi(z) = z^\lambda$, $\varphi(d) = h^{-1}d^\lambda h$. Since $\iota_0\Phi_R = \Phi_F\iota_0$, we have $j\Phi_R j^{-1} = \mathrm{Ad}(c_0\Phi_F(c_0)^{-1})\Phi_F$ with $c_0 = u^{-1}s$, and $c_0\Phi_F(c_0)^{-1} =$

$u^{-1}s\beta s^{-1}\beta^{-1}u^\lambda = u^{-1}fu^\lambda$ by Lemma 7(a). Also $j(h) = f(x, x^{-1}y^{-1}) = u^{-1}fu$, because $u^{-1}yu = x^{-1}y^{-1}$ in B. Hence

$$j\varphi j^{-1} = \mathrm{Ad}(u^{\lambda-1}) \circ \Phi_F.$$

Applying this to $j(a) = s$ gives $\varphi(a) = bab^{-1}$ with $b = z^m j^{-1}(\beta_F)$, $m = (\lambda-1)/2$. From the proof of Proposition 10, $S\Phi_R S = \mathrm{Ad}(kd^{-4c})\Phi_R$; as $\Phi_R = \mathrm{Ad}(h)\varphi$ and $S(h) = k$, this is $S\varphi S = \mathrm{Ad}(g)\varphi$ with $g = d^{-4c}h$. Evaluate both sides on a, using $S(a) = dad^{-1}$ and $S(\varphi(d)) = k^{-1}z^\lambda k$: the left side is the conjugate of a by $k^{-1}z^\lambda k \cdot S(b) \cdot d$, the right side its conjugate by gb. The quotient of the two conjugators centralizes a, so it lies in $\langle a \rangle$ [15]. Apply the character $L : R \to \widehat{\mathbf{Z}}$ with $L(z) = 1$, $L(d) = -1$, which kills a and satisfies $L \circ S = -L$. Since $L(j^{-1}x) = L(j^{-1}y) = 1$, we have $L(b) = m - 2\rho_2$, and $L(h) = L(k) = 0$. Therefore

$$4c + m - 2\rho_2 = L(gb) = \lambda - (m - 2\rho_2) - 1,$$

that is, $4(c - \rho_2) = \lambda - 1 - 2m = 0$. □

Propositions 10 and 11 prove Theorem 1. In the notation of [23, (5.2)], where $f \equiv A_F \cdot [x, y]$ modulo the second derived group, one has $c(f) = -A_F(-1, 1)$: on E^{ab} conjugation by x exchanges the last two coordinates, y acts trivially, and $[x, y]$ has class $(0, -1, 1)$. So Proposition 11 is the equality $\rho_2 = -A_F(-1, 1)$ on all of $\widehat{\mathrm{GT}}$, which Nakamura and Schneps prove under (III′) [23, Proposition 5.3].

5 Normalization and the group Λ

Proof of Theorem 2. Let $F \in \widehat{\mathrm{GT}}$. Choose $\sigma_0 \in G_{\mathbf{Q}}$ with cyclotomic character λ_F; then $F_0 = F\sigma_0^{-1}$ has $\lambda = 1$, and $\rho_2(F_0) \in 2\widehat{\mathbf{Z}}$ by Lemma 7(c). On the Galois group of $\mathbf{Q}^{\mathrm{ab}}$ the Kummer cocycle κ_2 is a homomorphism with image $2\widehat{\mathbf{Z}}$: the image is a closed subgroup, it lies in $n\widehat{\mathbf{Z}}$ exactly when $2^{1/n} \in \mathbf{Q}^{\mathrm{ab}}$, and this happens for $n = 1, 2$ only, because for $n > 2$ the field $\mathbf{Q}(2^{1/n})$ is not normal over $\mathbf{Q}$. So there is σ_1 fixing $\mathbf{Q}^{\mathrm{ab}}$ with $\kappa_2(\sigma_1) = \rho_2(F_0)$, and $F_* = F_0\sigma_1^{-1}$ is normalized by Lemma 7(b),(d).

Let $F = (1, f)$ be normalized. By Theorem 1, $f(u, v^4) = f(u^2, v^2)$ in $\widehat{B}_3$; exchanging u and v and using (I), also $f(u^4, v) = f(u^2, v^2)$. Relation (R) of [14, p. 4] is the equation $\Pi = 1$ in $\widehat{\Gamma}_1^2$, the completed mapping class group of the torus with two boundary circles, where

$$\Pi = f(e_3, a)f(b^2, c^2)f(e_2, e_3)f(e_1, e_2)f(a^2, b^2)f(c, e_1),$$

a, b, c are the twists along a chain of three curves, $e_1 = (ab)^6$, $e_3 = (bc)^6$ and $e_2 = cbab^{-1}c^{-1}$ [24, p. 659]. Read Π in $\widehat{B}_4$. Schneps's computation [24, p. 670] shows that its image in $\widehat{B}_4$ modulo the centre is trivial: the two forms of (IV) just written turn the first, third, fourth and sixth factors into values of f on pure braids, the two middle factors combine by (II), and the result is (III). So Π is a power Ω^n of the central element $\Omega = (abc)^4$. But Π lies in the closed commutator subgroup, and Ω has degree 12 in $\widehat{B}_4^{\mathrm{ab}} = \widehat{\mathbf{Z}}$; so $n = 0$ and $\Pi = 1$ in $\widehat{B}_4$. The homomorphism $\widehat{B}_4 \to \widehat{\Gamma}_1^2$ sending a, b, c to the three twists gives (R). Hence $f \in \Lambda$. □

6 The character δ

Level twelve

Let $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $L = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, $S_0 = ULU = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $R_+ = UL$ and $R_0 = -R_+$; so $R_+^3 = -I$. For a ring A, $u \mapsto U$, $v \mapsto L$ defines $\tilde{q}_A : \widehat{B}_3 \to \mathrm{SL}_2(A)$ when $\mathrm{SL}_2(A)$ is profinite. We write $\tilde{q}$ for $A = \mathbf{Z}/12$ and $q : \widehat{B} \to \mathrm{PSL}_2(\mathbf{Z}/12) = \mathrm{SL}_2(\mathbf{Z}/12)/\{\pm I\}$ for the induced map. On the free group $\widehat{F}$ let $q_{12} : \widehat{F} \to \mathrm{SL}_2(\mathbf{Z}/12)$ be $x \mapsto U^2$, $y \mapsto L^2$, and E_{12} its image.

Lemma 12. (a) *Under* $\mathbf{Z}/12 = \mathbf{Z}/4 \times \mathbf{Z}/3$, $E_{12} = V_4 \times \mathrm{SL}_2(\mathbf{F}_3)$, *where* $V_4 = \{I, U^2, L^2, U^2L^2\}$ *modulo* 4 *is a four-group;* E_{12} *does not contain* $-I$, *so it maps injectively to* $\mathrm{PSL}_2(\mathbf{Z}/12)$. (b) *The elements of* E_{12} *whose class commutes with that of* R_+ *form* $\{I\} \times (\langle -I\rangle \times \langle R_0\rangle)$. (c) *Those whose class commutes with that of* S_0 *form* $\{I, U^2L^2\} \times \langle S_0\rangle$.

Proof. (a) Modulo 4 the matrices U^2, L^2 commute and have order two; modulo 3 they are U^{-1}, L^{-1}, which generate $\mathrm{SL}_2(\mathbf{F}_3)$. The cube of (U^2, U^{-1}) is (U^2, I), so E_{12} is the full product. No element of V_4 is $-I$. (b) Let (g_4, g_3) conjugate R_+ to $\pm R_+$. The sign is $+$, because R_+ has trace 1, which is not -1 modulo 4. Conjugation by R_+ permutes the three nontrivial elements of V_4 cyclically up to sign, so $g_4 = I$; and the centralizer of R_+ in $\mathrm{SL}_2(\mathbf{F}_3)$ is $\langle R_+\rangle$. (c) Conjugation by S_0 exchanges U^2 and L^2, so $g_4 S_0 g_4^{-1}$ is S_0 or $U^2L^2S_0$; the latter is not $-S_0$. So the sign is $+$, $g_4 \in \{I, U^2L^2\}$, and g_3 lies in the centralizer $\langle S_0\rangle$ of S_0 in $\mathrm{SL}_2(\mathbf{F}_3)$. □

Lemma 13. *Let* $F \in \widehat{\mathrm{GT}}_0$ *with* $\lambda_F = 1$. *There is a unique* $t_F \in \mathbf{Z}/12$ *with* $\tilde{q} \circ \Phi_F = \mathrm{Ad}(U^{t_F}) \circ \tilde{q}$ *on* $\widehat{B}_3$. *The map* $F \mapsto t_F$ *is a continuous homomorphism,* $t_F \equiv 0 \bmod 4$, *and* $t_F \equiv -\rho_2(F) \bmod 3$.

Proof. Let $\ell \in \{2, 3\}$ and $M = \tilde{q}_{\mathbf{Z}_\ell}(\Phi_F(v))$. Then M is conjugate to L, so it has trace 2, and $UMU = MUM$. Writing $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, the lower left entries give $\gamma(1+\gamma) = 0$. If $\gamma = 0$ then M is a power of U and the image of $\tilde{q}_{\mathbf{Z}_\ell} \circ \Phi_F$ is abelian, which is false since Φ_F is surjective. So $\gamma = -1$, the upper left entries give $\beta = (\alpha - 1)^2$, and $M = U^t L U^{-t}$ with $t = 1 - \alpha$. As U^t commutes with U, this proves the existence of t_F; it is unique because a power of U commuting with L is trivial; additivity follows. Modulo 4 the element $q_{12}(f)$ lies in the commutator subgroup of V_4, which is trivial; so $M \equiv L$ and $t \equiv 0$. Modulo 3: apply q to $\Phi_F(s) = \beta s \beta^{-1}$. In $\mathrm{PSL}_2(\mathbf{F}_3) = A_4$ the element $U^{-t} q(\beta)$ centralizes the involution $q(s)$, so it lies in the four-group, the image of the quaternion group $Q_8 \subset \mathrm{SL}_2(\mathbf{F}_3)$. In $\mathrm{SL}_2(\mathbf{F}_3)/Q_8 = \mathbf{Z}/3$ the classes of U and $L = S_0 U S_0^{-1}$ are equal, say to 1; so $q_{12}(\beta)$, with $[\beta] = (-\rho_2, -\rho_2)$, has class $-4\rho_2 = -\rho_2$, and $-t - \rho_2 = 0$. □

Definition 14. $H_0 = \{F \in \widehat{\mathrm{GT}}_0 : \lambda_F = 1,\ t_F = 0\}$. For $F \in H_0$ we have $q \circ \Phi_F = q$ on $\widehat{B}$ and $q_{12} \circ \Phi_F = q_{12}$ on $\widehat{F}$ (Lemma 12(a)). So $q_{12}(A_F)$ and $q_{12}(\beta_F)$ lie in the groups of Lemma 12(b),(c). Define $a_3(F), a_2(F) \in \mathbf{F}_2$ by

$$q_{12}(A_F) \bmod 3 \in (-I)^{a_3(F)} \langle R_0\rangle, \qquad q_{12}(\beta_F) \bmod 3 \in S_0^{a_2(F)} \langle -I\rangle,$$

and $\delta = a_2 + a_3$.

By Lemma 5 and the invariance of q_{12}, the maps $F \mapsto q_{12}(A_F)$ and $F \mapsto q_{12}(\beta_F)$ are homomorphisms from H_0 to abelian groups. So a_2, a_3, δ are continuous characters of H_0, a closed subgroup of index 1 or 3 in $\{\lambda = 1\}$. By Lemma 13, $\Lambda_0 \subset H_0$.

The lattice of the group $(2, 3, 6)$

Let $\omega = e^{2\pi i/3}$, $\zeta = e^{2\pi i/6}$, $\mathcal{L} = \mathbf{Z}[\omega]$, and let $\mathcal{L} \rtimes \mu_6$ be the group of the maps $(t, \xi) : w \mapsto \xi w + t$ of $\mathbf{C}$. The assignment $u \mapsto (0, \zeta)$, $v \mapsto (1, \zeta)$ defines a surjection $\pi_6 : B \to \mathcal{L} \rtimes \mu_6$ whose kernel is the second derived group B''; it maps B' onto $\mathcal{L}$. One computes $\pi_6(s) = (\zeta, -1)$, $\pi_6(x) = (0, \omega)$, $\pi_6(y) = (1 + \zeta, \omega)$; so $\pi_6(F) \subset (1 - \omega)\mathcal{L} \rtimes \langle \omega\rangle$ and $\pi_6(F') \subset 3\mathcal{L}$.

Lemma 15. *Let* $F \in H_0$. *Then* Φ_F *acts on* $\widehat{B'}^{\mathrm{ab}} = \widehat{\mathcal{L}}$ *as multiplication by a unit* ν_F *of* $\widehat{\mathbf{Z}}[\omega]$ *with*

$$\nu_F \equiv 1 \bmod 3, \qquad \nu_F \equiv (-1)^{a_2(F)} \bmod 4.$$

Proof. Φ_F preserves the closure of B'' and fixes u; the induced automorphism of $\widehat{\mathcal{L}} \rtimes \mu_6$ fixes $(0,\zeta)$, so it is $(t,\xi) \mapsto (\nu t, \xi)$ for a unit ν. With $\pi_6(f) = (\phi, 1)$, $\phi \in 3\widehat{\mathcal{L}}$, the image of $\Phi_F(v) = f^{-1}vf$ is $(1 + (\zeta - 1)\phi, \zeta)$; so $\nu = 1 + \omega\phi \equiv 1 \bmod 3$. Let $\pi_6(\beta_F) = (b, \omega^j)$. Conjugating $(\zeta, -1)$ by it gives $(2b + \omega^j\zeta, -1)$, which must be $(\nu\zeta, -1)$: $2b = (\nu - \omega^j)\zeta$. The map $q \bmod 3 : \widehat{B} \to \mathrm{PSL}_2(\mathbf{F}_3) = A_4$ kills B'', so it factors through a surjection $\kappa : \widehat{\mathcal{L}} \rtimes \mu_6 \to A_4$; it maps $\widehat{\mathcal{L}}$ onto the normal four-group of A_4, so it kills $2\widehat{\mathcal{L}}$, and $\kappa(0,-1) = q(u^3) = 1$. As $2\widehat{\mathcal{L}} \rtimes \{\pm 1\}$ has index 12, it is the kernel of κ. By definition of a_2, $q(\beta_F) = q(s)^{a_2}$ modulo 3, that is, $\kappa(b, \omega^j) = \kappa(\zeta, -1)^{a_2} = \kappa(a_2\zeta, 1)$. Hence $j = 0$ and $b \equiv a_2\zeta \bmod 2$, and $\nu = 1 + 2b/\zeta \equiv 1 + 2a_2 \bmod 4$. □

7 Normalized elements and mapping class groups of genus one

Let $f \in \Lambda$. For a surface Σ with marked points and two pants decompositions P, Q, choose a path of A-moves and S-moves from P to Q, the i-th replacing a curve with twist b_i by one with twist c_i, and let

$$W_f(P,Q) = \prod_i f(b_i^{\epsilon_i}, c_i^{\epsilon_i})^{-1} \in \widehat{\Gamma}(\Sigma)_{\mathrm{pure}},$$

the product taken in the order of the path, with $\epsilon_i = 1$ for an A-move and 2 for an S-move. By [14, Theorem 2 and Lemma 6] this is independent of the path: the cycles that generate the relations give the unit element, by (I), (II), (III) and (R). Their automorphism is $F_P(b) = WbW^{-1}$ for a twist b along a curve of Q, with $W = W_f(P,Q)$ [14, formula (3)]. Clearly $W_f(P,Q)W_f(Q,Q') = W_f(P,Q')$ and $W_f(gP, gQ) = gW_f(P,Q)g^{-1}$ for every mapping class g, pure or not.

Lemma 16. *Let Γ be the mapping class group of Σ, with all permutations of the marked points allowed.*

(a) $\alpha_P(g) = W_f(P, gP)\, g$ defines a homomorphism $\Gamma \to \widehat{\Gamma}$, which extends to an automorphism α_P of $\widehat{\Gamma}$. It induces the identity on permutations, restricts to F_P on the pure subgroup, and $\alpha_Q = \mathrm{Ad}(W_f(P,Q)^{-1})\alpha_P$.

(b) Let p be the map forgetting a marked point, defined on its stabilizer, with stable target. Then $p \circ \alpha_P = \alpha_{pP} \circ p$, where pP is the image of P.

(c) In genus one with $n \geq 1$ marked points, let $\mathrm{per} : \widehat{\Gamma} \to \mathrm{SL}_2(\widehat{\mathbf{Z}})$ be the action on the homology of the closed torus. There is $\bar{\eta} \in \mathrm{GL}_2(\widehat{\mathbf{Z}})$ with $\mathrm{per}(\alpha_P(g)) = \bar{\eta}\ \mathrm{per}(g)\ \bar{\eta}^{-1}$ for all g. Moreover α_P, extended by (b) *to the group with one more marked point, maps each loop around a puncture, in the fundamental group $\widehat{\Pi}$ of the punctured torus, to a $\widehat{\Pi}$-conjugate of itself.*

Proof. (a) The two properties of W_f give $\alpha_P(g)\alpha_P(g') = \alpha_P(gg')$. A homomorphism to a profinite group extends to the completion. On a twist T along a curve of Q, $W_f(P, TP) = W_f(P,Q)\, TW_f(Q,P)T^{-1}$, so $\alpha_P(T) = W_f(P,Q)TW_f(P,Q)^{-1} = F_P(T)$. The extension is an automorphism because it is one on the open pure subgroup [14, Theorem 4] and the identity on the quotient. (b) The image of a pants decomposition, after deleting inessential curves and identifying parallel ones, is a pants decomposition. A move not involving the forgotten point keeps its type and its weight. An A-move in a four-holed sphere one of whose holes is the forgotten point has both curves mapped to boundary curves of one pair of pants; their twists commute, the weight dies because f is a commutator word, and the two decompositions have the same image. An S-move is supported in a one-holed torus, which survives since the target is stable. So $p(W_f(P,Q)) = W_f(pP, pQ)$. (c) Add a marked point $*$. The kernel of forgetting $*$ is $\Pi = \pi_1(\Sigma, *)$, a free group, and it has its full profinite topology in the larger group, by

(F). By (b), α_P lifts to an automorphism of the larger completed group that preserves $\widehat{\Pi}$; let η be its restriction. Then $\alpha_P(g)$ acts on $\widehat{\Pi}$ by $\eta \circ g \circ \eta^{-1}$, up to inner automorphisms. The loop C_i around the i-th puncture is, as a mapping class, the twist along the curve surrounding that puncture and $*$; by (a), $\eta(C_i) = WC_iW^{-1}$ with W pure. Discrete pure classes map C_i to Π-conjugates of itself; the set of $\widehat{\Pi}$-conjugates is closed, so the same holds for W. Hence η preserves the kernel of $\widehat{\Pi} \to H_1(T, \widehat{\mathbf{Z}})$, and $\bar{\eta}$ is the induced map. □

Lemma 17. *Let $F = (1, f)$ be normalized.* (a) *On $\widehat{\Gamma}_{1,1} = \widehat{B}_3/\overline{\langle (uv)^6 \rangle}$, with P the curve of u, $\alpha_P = \Phi_F$.* (b) *On $\widehat{\Gamma}_{1,2} = \widehat{B}_4/\overline{\langle \Omega \rangle}$, with a, b, c the twists along a chain and $P = \{a, c\}$, $\alpha_P = \mathrm{Ad}(g) \circ A_F$ with $g = f((ab)^3, c^2)$.*

Proof. (a) The curve of v is obtained by one S-move, of weight $f(u^2, v^2)^{-1}$. (b) Put $\omega_3 = (ab)^3$ and $e_1 = \omega_3^2$, the twist along the boundary of a neighbourhood of the first two curves. The path $\{a, c\} \to \{a, e_1\} \to \{b, e_1\}$ consists of an A-move and an S-move, so α_P fixes a, c and conjugates b by $W = f(c, e_1)^{-1} f(a^2, b^2)^{-1}$. In B_4 one has $(\omega_3 c)^2 = \Omega a^2 = (c\omega_3)^2$: indeed $\Omega = \omega_3 \cdot cba^2bc$ and $\omega_3 = ba^2ba^2$. So $u \mapsto c$, $v^2 \mapsto \omega_3$ defines a homomorphism $J \to B_4$, and the normalized relation (IV) gives $f(c, e_1) = f(c^2, \omega_3)$. On the other hand $A_F(c) = g'^{-1} c g'$ with $g' = f(ba^2b, c^2)$; since $ba^2b = \omega_3 a^{-2}$ and a commutes with ω_3 and c, $g' = f(\omega_3, c^2) = g$, and g commutes with a. Thus $g = f(c, e_1)^{-1}$, and $\mathrm{Ad}(g) A_F$ fixes a and c and conjugates b by $gf(a^2, b^2)^{-1} = W$. □

8 The hexagonal self-similarity

Let $q : \widehat{\Gamma}_{1,2} \to \widehat{\Gamma}_{1,1}$ forget the second point; on generators $(a, b, c) \mapsto (u, v, u)$. Its kernel is $\widehat{K}$, where $K = \pi_1(E \setminus O)$ is free on $\xi_1 = ca^{-1}$ and $\xi_2 = abac^{-1}b^{-1}a^{-1}$, with its full topology, by (F); on $K^{\mathrm{ab}} = \mathbf{Z}^2$ the elements a, c act by U and b by L. Let $G_3 = q^{-1}(\langle uv \rangle)$; the image of uv in $\widehat{\Gamma}_{1,1}$ has order six.

Proposition 18. *Let $F = (1, f)$ be normalized, and let $\tilde{A} = A_F(a^2, b^2) \in \widehat{\Gamma}_{1,2}$ be the frame of Lemma 5 evaluated at a^2, b^2. Then $\Psi = \mathrm{Ad}(\tilde{A}^{-1}) \circ A_F$ preserves G_3, and there are an isomorphism $\varphi : \widehat{B} \to G_3$ with $\varphi(\overline{B'}) = \widehat{K}$, induced by an isomorphism of discrete groups, and an element $h \in G_3$ such that*

$$\Psi \circ \varphi = \mathrm{Ad}(h) \circ \varphi \circ \Phi_F.$$

Proof. Step 1: the points of order three. Let $T = \mathbf{R}^2/\mathbf{Z}^2$ with the nine points of $T[3]$ marked, $\Gamma = \Gamma_{1,[9]}$, $P_1 = \Gamma_{1,9}$, and $H = T[3] \subset \Gamma$ acting by translations. Fix a pants decomposition and let $\alpha = \alpha_P$ on $\widehat{\Gamma}$ (Lemma 16). By Lemma 9(a) with $p = 3$ there are $e \in \widehat{P_1}$ and $h \mapsto h'$ with $\alpha(h) = eh'e^{-1}$, $h' \in P_1 h$. By Lemma 16(c), $\mathrm{per}(h')$ is conjugate to $\mathrm{per}(h) = 1$. The group $\{h'\}$ acts conformally on a complex torus T' with nine marked points [18], trivially on homology, hence by translations, and simply transitively on the marked points; so, moving the origin, the point labelled h is $\psi(h)$ for an isomorphism $\psi : H \to T'[3]$, and h' is the translation by $\psi(h)$, transported by the marking $m : T \to T'$. Choose an oriented linear isomorphism $\ell : T \to T'$; then $\ell^{-1}\psi$ is given by a matrix $D \in \mathrm{GL}_2(\mathbf{F}_3)$. Let $A \in \mathrm{GL}_2(\mathbf{Z})$ lift D, with $\det A = \varepsilon = \pm 1$. Then $g = m^{-1}\ell A$ fixes every marked point, has orientation ε, and $h' = ghg^{-1}$.

We claim $\varepsilon = 1$. Let $\widehat{\Pi}$ be the completed fundamental group of $T \setminus T[3]$ and η as in Lemma 16(c). The outer automorphism θ of $\widehat{\Pi}$ induced by $\eta^{-1} e g$ commutes with the outer action of each $h \in H$. The quotient of $T \setminus T[3]$ by H is a once-punctured torus, with fundamental group free on two generators A_1, B_1 and Π the kernel of its abelianization modulo 3. Conjugation embeds its completion in $\mathrm{Aut}(\widehat{\Pi})$ as the preimage of the outer action of H, so θ extends to an automorphism Θ of this free profinite group of rank two, inducing the identity on H. A loop around the puncture lies in Π and is one of the nine puncture loops of $T \setminus T[3]$; by Lemma 16(c) and the orientation of g, Θ maps it to a conjugate of its ε-th

power. In the quotient of class two this loop is $[A_1, B_1]$, which Θ raises to the power $\det \Theta^{\mathrm{ab}}$. So $\varepsilon = \det \Theta^{\mathrm{ab}} \equiv 1 \bmod 3$, and $\varepsilon = 1$.

Hence $c_1 = (eg)^{-1}$ is pure and $\alpha' = \mathrm{Ad}(c_1) \circ \alpha$ fixes H pointwise. The normalizer of H in Γ is $H \rtimes \mathrm{SL}_2(\mathbf{Z})$, the affine maps; its subgroup $\mathcal{N}_0 = \mathrm{SL}_2(\mathbf{Z})$ fixing the origin is mapped isomorphically to $\Gamma_{1,1}$ by forgetting the eight other points, so its closure $\overline{\mathcal{N}_0}$ is $\widehat{\Gamma}_{1,1}$. By Lemma 9(b), α' preserves $N_{\widehat{\Gamma}}(H) = H\overline{\mathcal{N}_0}$, and it preserves the stabilizer of the origin; so it preserves $\overline{\mathcal{N}_0}$. By Lemmas 16(b) and 17(a),

$$\alpha'|_{\overline{\mathcal{N}_0}} = \mathrm{Ad}(c_2) \circ \Phi_F \quad \text{on } \widehat{\Gamma}_{1,1}, \qquad c_2 \in \widehat{\Gamma}_{1,1}.$$

In particular α' fixes the central involution $\iota = -1$ of $\overline{\mathcal{N}_0}$.

Step 2: the quotient by ι. Let $C = C_\Gamma(\iota)$. The quotient $S = T/\iota$ is a sphere with four branch points $b_0, \dots, b_3$, the images of $T[2]$, with b_0 the image of the origin, and four more marked points $t_1, \dots, t_4$, the images of the pairs $\pm t$ of points of order three. By Birman and Hilden [4], $C/\langle\iota\rangle$ is the group $\mathcal{D}$ of mapping classes of S that fix b_0 and preserve $\{b_1, b_2, b_3\}$ and $\{t_1, \dots, t_4\}$. We claim

$$C_{\widehat{\Gamma}}(\iota) = \overline{C} \cong \widehat{C}, \qquad \widehat{C}/\langle\iota\rangle = \widehat{\mathcal{D}}.$$

The first equality is Lemma 9(b). For the second, mark also the three points of $T[2] \setminus 0$: let $\Gamma^+ = \Gamma_{1,[9]+[3]}$ and $C^+ = C_{\Gamma^+}(\iota)$. Forgetting the three points maps C^+ isomorphically to C, since a homeomorphism commuting with ι preserves its fixed points [4]. Now T minus the twelve points is an unramified double cover of S minus eight points; let $\Pi^+ \subset \Phi$ be their fundamental groups. The completion of $\Gamma_{0,[8]}$ acts faithfully on $\widehat{\Phi}$ by outer automorphisms: for the pure group this is Asada's theorem in genus zero [2], and an element with a nontrivial permutation moves the conjugacy classes of the puncture loops. Hence so does $\widehat{\mathcal{D}}$. If $z \in \widehat{C^+}$ acts trivially on $\widehat{\Pi^+}$, its image in $\widehat{\mathcal{D}}$ is represented by an automorphism of $\widehat{\Phi}$ that is the identity on the open subgroup $\widehat{\Pi^+}$; such an automorphism is the identity, because the centralizer of an open subgroup of a nonabelian free profinite group is trivial. So $z \in \langle\iota\rangle$, and $z = 1$ because ι permutes the punctures. Thus $\widehat{C^+}$ embeds in $\widehat{\Gamma^+}$. Finally the kernel $\widehat{K_3}$ of $\widehat{\Gamma^+} \to \widehat{\Gamma}$ is the completion of the fundamental group K_3 of the space of three unordered points of $T \setminus T[3]$, by (F) applied to the three successive point-pushing kernels for ordered points and passage to a subgroup of finite index; the discrete centralizer of ι in K_3 is trivial, so by Lemma 9(b), applied to $K_3 \rtimes \langle\iota\rangle$, its centralizer in $\widehat{K_3}$ is trivial. Hence $\widehat{C^+} \to \overline{C}$ is injective, which proves the claim; the last equality is right exactness of completion.

So α' induces an automorphism δ_8 of $\widehat{\mathcal{D}}$. The pure subgroup $\Gamma_{0,8}$ of $\mathcal{D}$ is the image of the classes in C that fix each pair $\pm t$ and act trivially on $T[2]$. By Lemma 16(a),(c), α' preserves the closure of this group, and δ_8 acts on $\mathcal{D}/\Gamma_{0,8} = \mathfrak{S}_{\{b_1,b_2,b_3\}} \times \mathfrak{S}_{\{t_j\}}$ by conjugation by a permutation σ_2 of b_1, b_2, b_3.

Step 3: the parameter. By Hoshi, Minamide and Mochizuki [16, Corollary C; Corollary 2.8 in the body], for the curve of type $(g, r) = (0, 3)$ and $n \geq 2$ the outer automorphism group of the profinite group $\Pi_n = \widehat{\Gamma}_{0,n+3}$ is $\widehat{\mathrm{GT}} \times \mathfrak{S}_{n+3}$. With $n = 5$: $\mathrm{Out}(\widehat{\Gamma}_{0,8}) = \widehat{\mathrm{GT}} \times \mathfrak{S}_8$, the first factor acting by $A_{F'}$ and the second by permutations of the marked points. That this decomposition is compatible with forgetting points is not part of their statement; it follows from (B3). Let (F', σ) be the class of $\delta_8|\widehat{\Gamma}_{0,8}$. Conjugation by an element of $\widehat{\mathcal{D}}$ with permutation π has class $(1, \pi)$; comparing with its image under δ_8 gives $\sigma\pi\sigma^{-1} = \sigma_2\pi\sigma_2^{-1}$ for all π in $\mathfrak{S}_3 \times \mathfrak{S}_4$, whose centralizer in $\mathfrak{S}_8$ is trivial. So $\sigma = \sigma_2$. To find F', fix j and forget the points t_i, $i \neq j$. On the torus this is forgetting all marked points except the pair $\pm t_j$, which lands in $\Gamma_{1,[2]} = \Gamma_{1,2} \times \langle\iota\rangle$; and $\Gamma_{1,[2]}/\langle\iota\rangle = \Gamma_{1,2} = B_4/\langle\Omega\rangle$ is the group of classes of S fixing t_j and preserving the set of branch points, with pure subgroup $\Gamma_{0,5}$ [4]. By Lemmas 16(b) and 17(b), α' induces on $\widehat{\Gamma}_{1,2}$ the automorphism A_F up to an inner automorphism; so the class of the induced automorphism of $\widehat{\Gamma}_{0,5}$ is (F, π) for some π. It is also (F', σ_2). So $F' = F$.

Step 4: forgetting the branch points and taking the double cover. Let $\mathcal{D}_5$ be the group of classes of S fixing b_0 and preserving $\{t_1,\dots,t_4\}$; it is $B_4/\langle\Omega\rangle$, the braids of the t_j in the disc $S\setminus b_0$, and its pure subgroup is $\Gamma_{0,5}$. By Step 3, δ_8 preserves the kernel of $\widehat{\Gamma}_{0,8}\to\widehat{\Gamma}_{0,5}$ and induces there the class $(F,1)$. An element of $\widehat{\mathcal{D}_5}$ that centralizes $\widehat{\Gamma}_{0,5}$ is trivial: its permutation is trivial by [16], and $\widehat{\Gamma}_{0,5}$ has trivial centre. It follows that δ_8 maps the kernel of $\widehat{\mathcal{D}}\to\widehat{\mathcal{D}_5}$ into itself (the image of an element k of the kernel commutes with the image of $\widehat{\Gamma}_{0,8}$), that it induces an automorphism δ_5 of $\widehat{\mathcal{D}_5}$, and that $\delta_5=\mathrm{Ad}(c_3)\circ A_F$: two automorphisms that agree on $\widehat{\Gamma}_{0,5}$ agree. Let $E'\to S$ be the double cover branched at $t_1,\dots,t_4$, and O',P' the points above b_0. By [4], lifting classes so as to fix O' and P' identifies $\mathcal{D}_5$ with $\Gamma_{1,2}$, the generators of B_4 becoming the twists a,b,c along a chain.

Step 5: the composite map. The composite $\mathcal{N}_0\subset C\to\mathcal{D}\to\mathcal{D}_5=\Gamma_{1,2}$ kills ι and defines $\varphi: B\to\Gamma_{1,2}$, and by construction $\delta_5\circ\varphi=\varphi\circ(\alpha'|_{\overline{\mathcal{N}_0}})$ on $\widehat{B}$. It is the map on orbifold fundamental groups induced by the map of moduli $(E,O)\mapsto(E';O',P')$. For $E: y^2=x^3+a_4x+a_6$, the points t_j are the roots of the 3-division polynomial $3x^4+6a_4x^2+12a_6x-a_4^2$, the curve E' is $t^2=x^4+2a_4x^2+4a_6x-a_4^2/3$, and with $X=(t+x^2+a_4)/2$, $Z=X-a_4/3$, $Y=Xx+a_6/2$ one finds

$$Y^2=Z^3+\tfrac{1}{108}(4a_4^3+27a_6^2),\qquad O'=\infty,\quad P'=(-a_4/3,\,-a_6/2).$$

So E' is always the curve $E_0: Y^2=Z^3+1$ up to isomorphism, and $(E,O)\mapsto P'$ identifies the moduli of elliptic curves up to ±1 with the quotient of $E_0\setminus O'$ by $\mu_6=\mathrm{Aut}(E_0)$. This quotient is the fibre of the moduli of two-pointed curves over the point E_0. Therefore φ is injective, its image is the preimage under q of a cyclic subgroup of order six of $\Gamma_{1,1}$, which after conjugation is $\langle uv\rangle$, and $\varphi(B')=K$. Passing to completions, $\varphi:\widehat{B}\to G_3$ is an isomorphism.

Step 6: the conjugator. By Steps 1, 4 and 5, $A_F\circ\varphi=\mathrm{Ad}(h_0)\circ\varphi\circ\Phi_F$ for some $h_0\in\widehat{\Gamma}_{1,2}$. In $\widehat{B}_3$ we have $\Phi_F(uv)=\tilde{A}_3(uv)\tilde{A}_3^{-1}$ with $\tilde{A}_3=A_F(u^2,v^2)$, because both sides agree modulo the centre and in $\widehat{B}_3^{\mathrm{ab}}$. The restriction of A_F to $\widehat{\langle a,b\rangle}=\widehat{B}_3$ is Φ_F, so Ψ fixes ab. Also $q\circ A_F=\Phi_F\circ q$, since $q(f(ba^2b,c^2))=f((uv)^3u^{-2},u^2)=1$. So Ψ preserves $\widehat{K}$ and G_3, and $\Psi\varphi=\mathrm{Ad}(h)\varphi\Phi_F$ with $h=\tilde{A}^{-1}h_0$. Apply q: since Φ_F fixes τ, the element $q(h)$ centralizes $q(\varphi(\widehat{B}))=\langle uv\rangle$. Modulo the central involution $(uv)^3$, the group $\widehat{\Gamma}_{1,1}$ is $\widehat{B}=\widehat{C_2}\amalg\widehat{C_3}$ and uv generates the factor of order three, whose centralizer is that factor [15]; the preimage of this factor in $\widehat{\Gamma}_{1,1}$ is $\langle uv\rangle$. So $q(h)\in\langle uv\rangle$ and $h\in G_3$. □

9 Proof of Theorem 3

Let F be normalized; then $F\in H_0$. We compute the action of the two sides of Proposition 18 on abelianizations modulo 12.

The right side. φ induces an isomorphism $C:\widehat{\mathcal{L}}=\widehat{B'}^{\mathrm{ab}}\to\widehat{K}^{\mathrm{ab}}$, and $C\zeta C^{-1}$ is the action of $\varphi(\tau)$, which is $R_+^{\pm1}$. By Lemma 15, $\varphi\Phi_F\varphi^{-1}$ acts on $\widehat{K}^{\mathrm{ab}}/12$ by $C\nu_FC^{-1}$, which is $(-1)^{a_2(F)}I$ modulo 4 and I modulo 3. The element $h\in G_3$ acts by R_+^k for some $k\in\mathbf{Z}/6$.

The left side. For every $g\in\widehat{\Gamma}_{1,2}$ the action of $A_F(g)$ on $\widehat{K}^{\mathrm{ab}}/12$ is $\tilde{q}(\Phi_F(q(g)))=\tilde{q}(q(g))$, because $t_F=0$. So the action N of A_F on $\widehat{K}^{\mathrm{ab}}/12$ commutes with $\mathrm{SL}_2(\mathbf{Z}/12)$ and is a scalar. Since $A_F(ca^{-1})=g^{-1}(ca^{-1})g$ with $g=f(ba^2b,c^2)\in\widehat{K}$, N fixes the class of ξ_1, and $N=I$. The element $\tilde{A}$ acts by $A_F(U^2,L^2)=q_{12}(A_F)$. So Ψ acts by $q_{12}(A_F)^{-1}$, which is I modulo 4 and lies in $(-I)^{a_3(F)}\langle R_0\rangle$ modulo 3 (Lemma 12(b)).

Comparison. Modulo 4: $I=R_+^k\cdot(-1)^{a_2}$. As R_+ has order six modulo 4 and $R_+^3=-I$, this gives $k\equiv 3a_2 \bmod 6$. Modulo 3: $q_{12}(A_F)^{-1}=R_+^k=(-1)^kR_0^k$. On the line fixed by R_0 the left side acts by $(-1)^{a_3}$ and the right side by $(-1)^k=(-1)^{a_2}$. So $a_2(F)=a_3(F)$. □

Remark 19. *Where the two bits are read* (added in version 2). Let $F\in H_0$. By Lemma 15 the bit $a_2(F)$ is determined by the unit ν_F, that is, by the action of F on $\widehat{\mathcal{L}}\rtimes\mu_6=\widehat{B}/\overline{B''}$, the

profinite triangle group $(2,3,6)$, which is metabelian. The bit $a_3(F)$ is not read there, in the following sense.

(a) $q_{12}(A_F) = (5I)^{a_3(F)}$, where $5I$ is I modulo 4 and $-I$ modulo 3. Indeed $\pi_6(r) = (\zeta, \omega)$, and the equation $A\,\pi_6(r)\,A^{-1} = (\nu_F\zeta, \omega)$ has in $(1-\omega)\widehat{\mathcal{L}} \rtimes \langle\omega\rangle$ the one solution $A = ((\nu_F - 1)\zeta/(1-\omega), 1)$, which is therefore $\pi_6(A_F)$. As $\nu_F \equiv 1 \bmod 2$, its image under the map κ of the proof of Lemma 15 is trivial, so $q_{12}(A_F)$ is $\pm I$ modulo 3; and it is I modulo 4 by Lemma 12(b).

(b) The second derived group of $E_{12} = V_4 \times \mathrm{SL}_2(\mathbf{F}_3)$ is $\{I, 5I\}$, because the derived group of $\mathrm{SL}_2(\mathbf{F}_3)$ is the quaternion group, whose derived group is $\{\pm I\}$. So $a_3(F) = 1$ exactly when $q_{12}(A_F)$ is the nontrivial element of the second derived group of E_{12}.

(c) In $\mathrm{PSL}_2(\mathbf{Z}/12)$ the normal subgroup generated by $q(u^6)$ has order 8 and contains the class of $5I$, and the quotient is $\mathrm{PSL}_2(\mathbf{Z}/6)$. Indeed U^6 is U^2 modulo 4 and I modulo 3, and the conjugates of U^2 generate the kernel of $\mathrm{SL}_2(\mathbf{Z}/4) \to \mathrm{SL}_2(\mathbf{Z}/2)$, which has order 8 and contains $-I$. Now π_6 identifies the triangle group $\langle s, r \mid s^2, r^3, (sr)^6\rangle$ with $\mathcal{L} \rtimes \mu_6$, and $sr = u^{-1}$; so $\overline{B''}$ is the closed normal subgroup of $\widehat{B}$ generated by u^6, and every quotient of $\widehat{B}$ through which both q and π_6 factor is a quotient of $\mathrm{PSL}_2(\mathbf{Z}/6)$, where the class of $5I$ is trivial.

So Theorem 3 compares a bit read in the metabelian quotient $\widehat{B}/\overline{B''}$ with a bit read, by (a) and (b), in a second derived group, at a level that by (c) the group $(2,3,6)$ does not reach; that level is reached by the triangle group $(2,3,12)$, the quotient of $\widehat{B}$ by the closed normal subgroup generated by u^{12}, through which q factors. This is why the proof of Theorem 3 does not stay in the group $(2,3,6)$ and passes through genus one. Statements (a) and (b) are derived here from Lemmas 12 and 15; the orders in (b) and (c) were also computed by machine.

10 Relation (III′) and the questions of Lochak

This section is new in version 2.

Relation (III′)

Place the five marked points on a circle, in their cyclic order. Let $\tau_1, \ldots, \tau_4$ be the standard generators of $\Gamma_{0,[5]}$, where τ_i exchanges the points i and $i+1$ by a half twist along the arc of the circle between them, and for $i \in \mathbf{Z}/5$ let $x_{i,i+1} \in \Gamma_{0,5}$ be the twist along the boundary of a neighbourhood of the arc from i to $i+1$; so $x_{i,i+1} = \tau_i^2$ for $i \le 4$. In these terms the pentagon relation (III) is

$$f(x_{34}, x_{45})\, f(x_{51}, x_{12})\, f(x_{23}, x_{34})\, f(x_{45}, x_{51})\, f(x_{12}, x_{23}) = 1 \qquad \text{in } \widehat{\Gamma}_{0,5} \tag{1}$$

[23, §1]. Let $M \subset \Gamma_{0,[5]}$ be the stabilizer of the fifth point. It is generated by τ_1, τ_2, τ_3 and is the quotient of B_4 by its centre, as in (B2), and in M

$$x_{45} = (\tau_1\tau_2)^3, \qquad x_{51} = (\tau_2\tau_3)^3,$$

the twists along the curves surrounding the points $1,2,3$ and $2,3,4$ [23, §1]. For $F = (\lambda, f) \in \widehat{\mathrm{GT}}_0$ let $g = g_F = \beta_F^{-1} \in \widehat{F}_2$. By Lemma 7(a), $f(x,y) = g(y,x)^{-1} g(x,y)$, and g is the only element of $\widehat{F}_2$ with this property: if g' is another one, then $c = g'g^{-1}$ satisfies $c(y,x) = c(x,y)$, so it is fixed by $\vartheta = \mathrm{Ad}(s)$, and $C_{\widehat{B}}(s) \cap \widehat{F} = 1$.

Theorem 20. *For every $F = (\lambda, f) \in \widehat{\mathrm{GT}}$,*

$$f(\tau_1\tau_3, \tau_2^2) = g(x_{45}, x_{51})\, f(x_{12}, x_{23})\, f(x_{34}, x_{45}) \qquad \text{in } \widehat{\Gamma}_{0,[5]}. \tag{III$'$}$$

This is the relation of [23, Theorem 1.1], in their notation; they prove it for Galois elements. Furusho's survey writes the same relation in $\widehat{B}_5$ [12, p. 191]; we prove it in $\widehat{\Gamma}_{0,[5]}$, where [23] state it and define $\Pi\Gamma$.

Proof. Let $\varepsilon \in M$ be the image of $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 \in B_4$. Since $\Delta\sigma_i\Delta^{-1} = \sigma_{4-i}$ and Δ^2 generates the centre of B_4, the element ε has order two, and $\mathrm{Ad}(\varepsilon)$ exchanges τ_1 and τ_3 and fixes τ_2. So it exchanges x_{12} and x_{34}, fixes x_{23}, and exchanges $x_{45} = (\tau_1\tau_2)^3$ and $x_{51} = (\tau_2\tau_3)^3 = (\tau_3\tau_2)^3$. Put the first four points at $-a, -b, b, a$ on the real line, with $0 < b < a$, and the fifth at ∞; then ε is the class of the rotation $\iota : t \mapsto -t$.

Step 1: the right side commutes with ε. Let R be the right side of (III$'$). Then $\varepsilon R\varepsilon^{-1} = g(x_{51}, x_{45})\, f(x_{34}, x_{23})\, f(x_{12}, x_{51})$. By relation (I) and the property of g,

$$(\varepsilon R\varepsilon^{-1})^{-1}R = f(x_{51}, x_{12})\, f(x_{23}, x_{34})\, f(x_{45}, x_{51})\, f(x_{12}, x_{23})\, f(x_{34}, x_{45}),$$

which is conjugate to the left side of (1), hence equal to 1. This is the computation by which [23, §1] deduce (III) from (III$'$), read backwards.

Step 2: the centralizer. Let N be the centralizer of ε in $\Gamma_{0,[5]}$ and $N_1 = N \cap \Gamma_{0,5}$. We use the theorem of Boggi and Zalesskii [7, Theorem A (iii)]: let p be a prime and G a group of finite virtual p-cohomological type, that is, of finite virtual p-cohomological dimension and with finite cohomology in every finite $\mathbf{F}_p[G]$-module; if the map from G to its profinite completion $\widehat{G}$ induces isomorphisms on the cohomology in every discrete $\mathbf{F}_p[[\widehat{G}]]$-module in all large degrees, then for every finite p-subgroup H of G the centralizer of H in $\widehat{G}$ is the closure of the centralizer of H in G. Take $p = 2$, $G = \Gamma_{0,[5]}$ and $H = \langle\varepsilon\rangle$, of order two. The hypotheses hold: G is good, and its subgroup $\Gamma_{0,5}$ of finite index is torsion-free and an iterated extension of free groups of finite rank, the kernel of forgetting a point being free of rank three and $\Gamma_{0,4}$ free of rank two. (This is the case $\Gamma = \Gamma_{0,[5]}$, $P = \Gamma_{0,5}$ of Lemma 9(b); here $H \cap P = 1$ because ε is not pure.) So the centralizer of ε in $\widehat{\Gamma}_{0,[5]}$ is $\overline{N}$. Its intersection with the open subgroup $\widehat{\Gamma}_{0,5}$ is $\overline{N_1}$. So $R \in \overline{N_1}$. The left side $L = f(\tau_1\tau_3, \tau_2^2)$ of (III$'$) lies in $\overline{N}$, because $\tau_1\tau_3$ and τ_2 commute with ε. It is pure, because f lies in the closed commutator subgroup of $\widehat{F}_2$ and the permutations of $\tau_1\tau_3$ and τ_2^2 commute. So $L \in \overline{N_1}$ as well.

Step 3: forgetting the fifth point is injective on $\overline{N_1}$. Let $p : M \to B$ be the map that forgets the fifth point, followed by the quotient of $\Gamma_{0,[4]}$ by its normal four-group V; as in Section 4, $p(\tau_1) = p(\tau_3) = u$ and $p(\tau_2) = v$. Let $\gamma \in N_1$ with $p(\gamma) = 1$. The image of γ in $\Gamma_{0,[4]}$ is pure and lies in V, so it is trivial. Hence γ lies in the kernel of the forgetting map, which consists of the classes $\mathrm{Push}(c)$ that push the fifth point along a loop $c \in \Pi = \pi_1(\mathbf{P}^1 \setminus \{\pm a, \pm b\}, \infty)$, and $c \mapsto \mathrm{Push}(c)$ is injective [5]. Write $\gamma = \mathrm{Push}(c)$. Then $\gamma = \varepsilon\gamma\varepsilon^{-1} = \mathrm{Push}(\iota_* c)$, so $\iota_* c = c$. The group $\Pi \rtimes \langle\iota_*\rangle$ is the orbifold fundamental group of the quotient of $\mathbf{P}^1 \setminus \{\pm a, \pm b\}$ by ι, a sphere with two punctures and two cone points of order two; it is a free product $C_2 * C_2 * \mathbf{Z}$. By the subgroup theorem of Kurosh [19], a subgroup of a free product is a free product of a free group and of conjugates of subgroups of the factors. The centralizer of ι_* has ι_* in its centre, and a free product of two nontrivial groups has trivial centre; so the centralizer is a single one of these groups, and as it contains an element of order two it is a conjugate of a factor of order two. So this centralizer is $\{1, \iota_*\}$, which meets Π trivially. Hence $c = 1$ and $\gamma = 1$: the map p is injective on N_1. Its image contains $p(\tau_1^2\tau_3^2) = x^2$, $p(\tau_2^2) = y$ and $p(\tau_1\tau_3\tau_2^2\tau_3^{-1}\tau_1^{-1}) = xyx^{-1}$, which generate a subgroup of index two of F; so $p(N_1)$ has finite index in B. Consequently every subgroup K of finite index of N_1 is the preimage in N_1 of the subgroup $p(K)$ of finite index of B; as p extends to a continuous map of the completions, K is open in the topology induced by $\widehat{\Gamma}_{0,[5]}$. Hence $\overline{N_1}$ is the profinite completion of N_1, the closure

of $p(N_1)$ in $\widehat{B}$ is its profinite completion, and the map $\overline{N_1} \to \widehat{B}$ induced by p is the completion of an isomorphism: it is injective.

Step 4: comparison. In B we have $p(x_{45}) = (uv)^3 = 1$ and $p(x_{51}) = (vu)^3 = 1$. So $p(R) = g(1,1)\, f(u^2, v^2)\, f(u^2, 1) = f(u^2, v^2)$, and $p(L) = f(u^2, v^2)$. By Step 3, $L = R$. □

Corollary 21. *The subgroup* $\mathrm{I}\Gamma$ *of Nakamura and Schneps [23, Definition 1.1] is all of* $\widehat{\mathrm{GT}}$. *Consequently their theorems on* $\mathrm{I}\Gamma$ *hold for* $\widehat{\mathrm{GT}}$*: for every surface* Σ *with a quilt decomposition* Q/P *there is a representation* $\widehat{\mathrm{GT}} \to \mathrm{Aut}\,\widehat{\Gamma}(\Sigma)$*, whose class in* $\mathrm{Out}\,\widehat{\Gamma}(\Sigma)$ *does not depend on the quilt [23, Theorem 1.3 and the text after it]; and these representations are compatible with the homomorphisms* $\widehat{\Gamma}(\Sigma') \to \widehat{\Gamma}(\Sigma)$ *for the connected subsurfaces* Σ' *made of pairs of pants of* P *[23, Theorem 1.4].*

Proof. $\mathrm{I}\Gamma$ is the set of the elements of $\widehat{\mathrm{GT}}$ that satisfy (III′) and (IV). Apply Theorems 20 and 1. □

The questions of Lochak

Lochak [20] writes τ_1, τ_2 for the generators u, v of B_3. He calls two elements $F_0 = (\lambda_0, f_0)$ and $F_1 = (\lambda_1, f_1)$ of $\widehat{\mathrm{GT}}$ a *compatible pair* if $\lambda_0 = \lambda_1$ and there are $a, b, c \in \widehat{\mathbf{Z}}$ with

$$f_1(u^2, v^2) = v^{2a}\, f_0(u, v^4)\, u^{2b}\, (uv^2)^{2c} \qquad \text{in } \widehat{B}_3 \tag{2}$$

[20, Definition 5.4]. The compatible pairs form a group, which the map $(F_0, F_1) \mapsto F_0$ identifies with a subgroup of $\widehat{\mathrm{GT}}$ [20, Proposition–Definition 5.6]. He writes $\mathrm{I}\Gamma$ for it; we write $\mathrm{I}\Gamma_{\mathrm{L}}$, to keep it apart from the group of [23]. Its subgroup $\mathrm{I}\Gamma_{\mathrm{L},s}$ of the pairs with $F_0 = F_1$ consists of the $F \in \widehat{\mathrm{GT}}$ that satisfy

$$f(u^2, v^2) = v^{-8\rho}\, f(u, v^4)\, u^{-4\rho}\, (uv^2)^{4\rho}, \qquad \rho = \rho_2(F), \tag{3}$$

which is the form in which he writes relation (IV) [20, pp. 58 and 60]. For a surface S of type (g, n) let $\Gamma(S) = \Gamma_{g,[n]}$, let $\check{\Gamma}(S)$ be its procongruence completion and $\check{C}(S)$ the procongruence completion of the complex of curves of S, and let Aut^* and Out^* refer to the automorphisms that preserve inertia, that is, the set of conjugacy classes of the procyclic subgroups generated by twists [20, pp. 2 and 6]. The surface S is *of generic type* if $g \geq 1$, $3g - 3 + n > 2$ and $(g, n) \neq (2, 0)$. Lochak proves that for S of generic type

$$\mathrm{Out}^*(\check{\Gamma}(S)) = \mathrm{Out}(\check{C}(S)) = \mathrm{I}\Gamma_{\mathrm{L}} \tag{4}$$

[20, Theorem 5.10 ii)]. In his closing remarks he asks whether $\mathrm{I}\Gamma_{\mathrm{L},s} = \mathrm{I}\Gamma_{\mathrm{L}}$; whether $\mathrm{I}\Gamma_{\mathrm{L}} = \widehat{\mathrm{GT}}$, in which case everything would be decided in genus zero; and whether $\mathrm{I}\Gamma_{\mathrm{L},s} = \widehat{\mathrm{GT}}$, "that is, showing that relation (IV) is actually redundant", so that in every genus only pentagons would be needed [20, p. 62].

Corollary 22. (a) $\mathrm{I}\Gamma_{\mathrm{L},s} = \mathrm{I}\Gamma_{\mathrm{L}} = \widehat{\mathrm{GT}}$*: for every* $F \in \widehat{\mathrm{GT}}$ *the pair* (F, F) *is compatible, and every compatible pair is of this form.*

(b) *For every surface* S *of generic type,* $\mathrm{Out}^*(\check{\Gamma}(S)) = \mathrm{Out}(\check{C}(S)) \cong \widehat{\mathrm{GT}}$*, and every inertia-preserving automorphism of* $\check{\Gamma}(S)$ *is of strict lego type in the sense of [20, Definition 5.8].*

Proof. (a) In B_3 one has $(uv^2)^2 = (uv)^3$ (Section 4), and this element is central. So relation (IV) of Theorem 1 can be written $f(u^2, v^2) = v^{-8\rho} f(u, v^4) u^{-4\rho} (uv)^{6\rho}$ with $(uv)^{6\rho} = (uv^2)^{4\rho}$. This is (3), and it is (2) for $f_0 = f_1 = f$ with $a = -4\rho$, $b = -2\rho$, $c = 2\rho$. So (F, F) is a

compatible pair for every $F \in \widehat{\mathrm{GT}}$. Now let (F_0, F_1) be any compatible pair. Comparing (2) for (F_0, F_1) and for (F_0, F_0), and using that $(uv^2)^2$ is central, we get

$$f_1(x,y) = y^{\alpha}\, f_0(x,y)\, x^{\beta}\, (uv)^{3\gamma} \qquad \text{in } \widehat{B}_3,$$

with $x = u^2$, $y = v^2$ and $\alpha, \beta, \gamma \in \widehat{\mathbf{Z}}$. Modulo the centre this is an equation in $\widehat{F}$, where f_0 and f_1 lie in the closed commutator subgroup; in $\widehat{F}^{\mathrm{ab}}$ it gives $\alpha = \beta = 0$, and then $f_1 = f_0$. (That F_1 is determined by F_0 is also in the proof of [20, Proposition–Definition 5.6], through [20, Proposition 5.5].)

(b) This is (4) with (a); by [20, Definition 5.8] an automorphism is of strict lego type when its pair has $F_0 = F_1$. □

For $g \leq 2$ the procongruence completion is the profinite completion [20, p. 60]; so for the generic types of genus one and two, Corollary 22(b) is a statement about $\widehat{\Gamma}_{g,[n]}$.

Remark 23. *What is used, and neighbouring statements.* (a) Corollary 22(b) uses Theorem 5.10 of [20] as published. We have read its statement and the definitions and propositions cited above, and we have not checked its proof, which rests on work of Boggi and Funar on procongruence curve complexes. Corollary 21 uses Theorems 1.3 and 1.4 of [23] in the same way. (b) Boggi's preprint [6] states a result of the same kind as Corollary 22(b) for pure mapping class groups: for $\chi(S) < g - 2$ and $(g,n) \neq (1,2)$, the group of the outer automorphisms of the procongruence completion of the pure mapping class group that preserve the conjugacy class of the procyclic subgroup generated by a nonseparating twist is naturally isomorphic to $\mathfrak{S}_n \times \widehat{\mathrm{GT}}$; and for $\chi(S) < g - 2$ the group $\widehat{\mathrm{GT}}$ embeds in the corresponding group for the profinite completion [6, abstract of version 6]. We have read only that abstract, and we have not compared the statements or the actions. (c) Lochak writes that $\mathrm{I}\Gamma_{\mathrm{L},s}$ "is contained in the group" that [23] denote by $\mathrm{I}\Gamma$ [20, p. 61]. The inclusion that follows from the definitions is the opposite one, because the group of [23] is defined by (IV) and (III′) together. The inclusion as he states it says that (IV) implies (III′) on $\widehat{\mathrm{GT}}$; his paper does not mention (III′) and gives no argument for it. By Theorem 20 it holds. (d) Lochak, Nakamura and Schneps name as a subject of future work the question whether the Lego action of $\mathrm{I}\Gamma$ extends to the full group $\widehat{\mathrm{GT}}$, acting on the groups with one marked point as in their theorem [21, §8]. By Corollary 21 the group $\mathrm{I}\Gamma$ of that question is $\widehat{\mathrm{GT}}$. We have not compared the action of [23] with theirs.

11 What remains open

Version 1 listed first the question whether (III′) holds on $\widehat{\mathrm{GT}}$; Theorem 20 answers it. It listed second the question whether the action of Λ_0 and the Galois action fit into one action of $\widehat{\mathrm{GT}}$ on the tower with marked points; by Corollary 21 the action of Nakamura and Schneps is such an action, subject to the comparison in item 1.

1. The comparison of the actions. For a normalized element the formulas of [14] and the representations of [23] should agree, since for $\lambda = 1$ and $\rho_2 = 0$ the relation (R′) of [23, §1] is (R); for the groups with at most one marked point the action of [21] should agree with that of [23]; and on procongruence completions both should induce the outer action of [20]. We have checked none of these.

2. Whether $G_{\mathbf{Q}} = \widehat{\mathrm{GT}}$. By Corollary 22(b) this is now the question whether every inertia-preserving outer automorphism of the procongruence mapping class group of a surface of generic type is Galois.

3. The statements quoted in Section 1 about the character on all of $\widehat{\mathrm{GT}}_0$: its extension, its vanishing on $G_{\mathbf{Q}}$ and its nonvanishing, and with them the strictness of the inclusion $\widehat{\mathrm{GT}} \subset \widehat{\mathrm{GT}}_0$. They need their own check.

4. A check by a human reader. Theorem 20 and Corollaries 21 and 22 have been checked by a second reader, Claude Opus 5.5 at effort max, and Remark 19 by its author only.

Provenance and checks

The mathematics of Sections 4 to 9 is the work of GPT-6 Astra (ChatGPT Pro mode), in research runs of November 2026 on notes of Grothendieck's *La longue marche à travers la théorie de Galois*. The runs contain two proofs of Theorem 3; this note follows the one through Λ. They also contain two proofs of Theorem 1. The earlier one, of 16 November 2026, is in Lochak's formulation: it proves, by the method of Section 4, that every element of $\widehat{\mathrm{GT}}$ forms a compatible pair with itself in the sense of [20, Definition 5.4] (statements E.43 and E.44 of that run's text, in the records). Version 1 did not say so.

Claude Fable 5.2 at effort max checked every step of the runs' proofs of Theorems 1 and 2 (18 November), re-proved the computation of Section 9 independently (21 November), and checked the chain behind Proposition 18, re-deriving Lemma 17(b) (23 November). Claude Opus 5.5 at effort max checked the parity computation, exhaustively over $\mathbf{Z}/12$ with its own program, and recomputed the identity of Step 5 (22 November). As second reader of this text (23 November) it followed every step of Sections 4 and 8 and of the proofs rewritten by the editor, re-derived the matrix and braid identities, opened [16] at the source, and found no mathematical error; its corrections of citation and precision are incorporated. No fault was found in the statements. Before deposit Claude Opus 5.5 at effort max (23 November) read the text, recomputed the abelianizations and the character argument of Propositions 10 and 11, and checked the Galois decomposition in the proof of Theorem 2.

This text was written by Claude (Claude Fable 5.2 at effort max) as editor. It differs from the runs' texts in these places, which the second reader checked: the finite-subgroup steps cite [7] (Lemma 9) where one of the runs proves them by Smith theory; the shear t_F is defined by Lemma 13; Lemma 15 and the first half of the proof of Proposition 11 are new derivations; in Step 2 the twelve points are used only for the topology of the centralizer.

Read at the source for this text: [23] (§§1, 5, 6, author's version), [14] (pp. 3–5, 18–19), [24] (pp. 659, 670), [21] (§§1, 2, 8), [7] (statements and the proof of Theorem A). [16] was opened by the second reader (Corollary C and Corollary 2.8). Of [3], [9] and [8] only the abstracts were read; [3] also in its first section. The bibliographic data of [2] and [11] were confirmed; [1] was not opened, and what is taken from it is proved in (F). The other references are cited from memory or with the locators of the runs. No human has checked the proofs or this text.

Version 2 (24 November 2026). The statements and proofs of version 1 are unchanged. The changes are these.

(i) A correction in Section 1: version 1 said that the strictness of the inclusion $\widehat{\mathrm{GT}} \subset \widehat{\mathrm{GT}}_0$ is known. No published proof was found, and the sentence now says so.

(ii) The paper of Lochak [20], which asks the question that Theorem 1 answers, is cited, and the earlier proof of Theorem 1 in his formulation is recorded above.

(iii) Relation (III′), which version 1 listed as open, is proved (Theorem 20), with Corollaries 21 and 22 and Remark 23.

(iv) Remark 19 is new. The abstract, the scope, the proof overview and the list of open questions are brought into line.

(v) The title names both relations. The title of version 1 began "Relation (IV) of Nakamura and Schneps holds".

Points (i) and (ii), and the observation that a sentence of [20, p. 61] bears on (III′), come from a survey of connections made on 24 November by Claude Opus 5.5 at effort max, which also proved Corollary 22 in substance. The editor, Claude Fable 5.2 at effort max, wrote the additions on 24 November with the sources open; it re-derived Corollary 22 from [20], found and proved Theorem 20 and Corollary 21, and took Remark 19 from its own report of 21 November, whose finite computations it repeated by machine on 24 November.

Theorem 20 and Corollaries 21 and 22 were then checked by a second reader, Claude Opus 5.5 at effort max, on 24 November 2026, in a run separate from the survey: every step was followed, and the braid and abelianization identities were recomputed; [23] (Theorem 1.1, Definition 1.1, Theorems 1.3 and 1.4), [7, Theorem A (iii)], [12, p. 191] and [20] (Definitions 5.4 and 5.8, Proposition–Definition 5.6, Theorem 5.10, pp. 61–62) were opened at the statements used. It found no gap. What it asked for is in the text: the hypotheses of the theorem of Boggi and Zalesskii in Step 2, a reference in Step 3, and the comparison of [20, Theorem 5.10 ii)] with the printed page, which bears the star as quoted in (4). Remark 19 has been checked by its author only.

Read at the source for version 2: [20] in the author's version (pp. 2, 6, 48, 50–52 and 58–62; the statement of Theorem 5.10 and the labels of his relations were read on the page images at high magnification, because extracted text loses the stars and the primes); [23] in the author's version (§1, the end of §3, and §4 to Lemma 4.2); [12] (pp. 189–191); [9] (abstract and introduction); [21] (§1 and §8); [7] (Theorem A with its definitions, and Corollary B, again); and the abstract of [6]. [5] and [19] are cited from memory. No human has checked version 2.

The records of the checks are at the permanent page of this work, WB-0007:

`https://github.com/KokunoYumeto/mathematics-workbenches/blob/main/works/WB-0007.md`.

This text is dedicated to the public domain (CC0 1.0).

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