A very ample counterexample to Kodaira vanishing on a surface, answering a question of Raynaud

23 November 2026, second version CC0 1.0

hexagon:2611.00936v2

Abstract

In 1978 Raynaud constructed a smooth projective surface in positive characteristic with an ample line bundle whose inverse has nonzero first cohomology, and asked whether Kodaira vanishing can also fail for an ample line bundle that is generated by its global sections, or even very ample. We answer both questions on surfaces, the smallest dimension in which they have content. On Raynaud's own surface in characteristic 2 a modified twist Q is ample and globally generated with $h^1(Q^{-1}) = 1$. Taking ninth roots of a general basis of the sections of $Q^{\otimes 9}$ then gives, over every algebraically closed field of characteristic 2, a smooth projective surface Y with a very ample line bundle L such that $h^0(L) = 562$, $h^1(L^{-1}) = 1$ and $h^1(\mathcal{O}_Y) = 565$. The passage from globally generated to very ample is general: in any characteristic p, roots of degree prime to p of a general basis turn a globally generated counterexample into a very ample one of the same dimension. The earlier very ample counterexamples, on homogeneous spaces (Lauritzen) and on varieties of dimension at least six (Lauritzen and Rao), are not surfaces. On Y the restriction of Picard schemes to a smooth curve of $|L|$ is not a closed immersion, which answers negatively the second part of a question of Kweon. The mathematics was produced by GPT-6 Astra (ChatGPT Pro mode) and checked by Claude Opus 5.5 at effort high, which also wrote this text, and again by Claude Opus 5.5 at effort max. No human has checked it.

1 Raynaud's question and the result

Let X be a smooth projective variety of dimension n over an algebraically closed field k, and L an ample line bundle on X. Kodaira's vanishing theorem says that $H^i(X, L^{-1}) = 0$ for $i < n$ when k has characteristic zero. On a curve the only assertion is $H^0(X, L^{-1}) = 0$, which holds in every characteristic because L^{-1} has negative degree. Surfaces are therefore the first case in which the theorem has content, and there the assertion is $H^1(X, L^{-1}) = 0$.

Raynaud showed that this fails in every positive characteristic [9, p. 273]. His surface X is fibred over a curve with integral fibres carrying a horizontal line of cusps. In characteristic 2 it is a triple cover of a ruled surface over an Artin–Schreier curve, and the ample bundle is $\mathcal{I} = \mathcal{O}_X(E) \otimes N$, where E is a section of the fibration and N comes from the base. In his *Remarque 3* [9, p. 278] he takes, in characteristic 2, the curve $y^2 + y = x^{15}$ and $N = \mathcal{O}_C(2\infty)$. He notes that $\mathcal{I}$ is generated by its sections outside E, and asks:

> « Peut-on contre-exempler le théorème de Kodaira avec un faisceau ample, engendré par ses sections, voire très ample? »

Lauritzen answered the very ample question in higher dimension, on homogeneous spaces, by modular representation theory; Lauritzen and Rao then gave elementary very ample examples of dimension at least six [7, pp. 1–2][5]. They point out that their failure is in the penultimate degree and need not be inherited by hyperplane sections, and that examples of

smaller dimension would be welcome [7, p. 2]. We answer both of Raynaud's questions for surfaces.

The same question arose again in the study of Picard schemes. For a smooth ample effective divisor D on a smooth projective variety X, Kweon proved that restriction of the homomorphisms of group schemes from α_p to the Picard schemes (written $(\operatorname{Pic} X)(\alpha_p)$ there), $\operatorname{Hom}(\alpha_p, \operatorname{Pic}_{X/k}) \to \operatorname{Hom}(\alpha_p, \operatorname{Pic}_{D/k})$, is injective if and only if $H^1(X, \mathcal{O}_X(-D)) = 0$ [6, Theorem 7.6, p. 18]. He expected that, because Kodaira vanishing fails in positive characteristic, the restriction $\operatorname{Pic}_{X/k} \to \operatorname{Pic}_{H/k}$ to a smooth hyperplane section can fail to be a closed embedding; he noted that he knew no very ample D with $H^1(X, \mathcal{O}_X(-D)) \neq 0$, and asked whether this restriction is always injective and whether it is always a closed embedding [6, pp. 18–19, Question 7.7].

Theorem 1. *Let k be an algebraically closed field of characteristic 2, for instance $\overline{\mathbf{F}}_2$.*

(a) *On Raynaud's surface S over k with the data of his Remarque 3 there is an ample line bundle Q, generated by its global sections, with*

$$h^0(S, Q) = 6, \qquad h^1(S, Q^{-1}) = 1, \qquad H^1(S, Q^{-n}) = 0 \quad (n \geq 2), \qquad h^1(S, \mathcal{O}_S) = 9.$$

(b) *There is a smooth projective connected surface Y over k, finite and flat of degree 9^{556} over S, with a very ample line bundle L such that*

$$h^0(Y, L) = 562, \qquad h^1(Y, L^{-1}) = 1, \qquad h^1(Y, \mathcal{O}_Y) = 565.$$

For every smooth connected curve $H \in |L|$, the kernel of the restriction $\operatorname{Pic}_{Y/k} \to \operatorname{Pic}_{H/k}$ has a one-dimensional tangent space at the identity.

So Kodaira vanishing fails on a surface for a very ample line bundle, already over $\overline{\mathbf{F}}_2$. In particular Y is defined over a finite field. The last assertion concerns tangent spaces, which is all that $H^1(L^{-1})$ controls. Picard schemes in positive characteristic need not be reduced, and the theorem says nothing about geometric points of that kernel.

Corollary 2. *For every smooth connected curve $H \in |L|$ on the surface Y of Theorem 1, the restriction $\operatorname{Pic}_{Y/k} \to \operatorname{Pic}_{H/k}$ is not a closed immersion, and there is a nonzero homomorphism $\alpha_p \to \operatorname{Pic}_{Y/k}$ whose composite with the restriction is zero.*

Proof. A closed immersion is injective on tangent spaces, whereas by Theorem 1 the tangent map at the identity has a one-dimensional kernel. The second assertion is Kweon's Theorem 7.6 [6, p. 18], applied to the smooth ample effective divisor H, for which $H^1(Y, \mathcal{O}_Y(-H)) = H^1(Y, L^{-1}) \neq 0$. □

Thus the second part of Kweon's Question 7.7 has a negative answer for surfaces, and his expectation is confirmed. The first part, injectivity on geometric points, is not touched by this construction.

Part (b) follows from part (a) by a construction that works in any positive dimension and any characteristic.

Theorem 3. *Let X be a smooth projective connected variety of positive dimension over an algebraically closed field k of characteristic $p \geq 0$, and Q a line bundle on X generated by its global sections such that $Q^{\otimes m}$ is very ample for an integer $m \geq 2$ prime to p. Let $s_1, \ldots, s_\ell$ be a general basis of $H^0(X, Q^{\otimes m})$ and let $\rho : Y \to X$ be the cover obtained by adjoining an m-th root of every s_i. Then Y is smooth, projective and connected, $L = \rho^* Q$ is very ample, and for every i*

$$H^i(Y, L^{-1}) \;\cong \bigoplus_{0 \leq r_1, \ldots, r_\ell \leq m-1} H^i(X, Q^{-1-r_1-\cdots-r_\ell}).$$

In particular $H^i(X, Q^{-1})$ is a direct summand of $H^i(Y, L^{-1})$.

Thus, in every dimension and characteristic, a counterexample to Kodaira vanishing with a globally generated bundle yields one with a very ample bundle. Raynaud's question for very ample bundles reduces to his question for globally generated ones, and part (a) answers the latter on his own surface.

What is Raynaud's and what is added. The curve, the ruled surface, the triple cover and the bundle N are those of [9, §§1–3], with his Remarque 3 fixing the curve and N. The additions here are the following:

- the twist $Q = \mathcal{O}_S(2T) \otimes f^*N$, with the double of the section T in place of Raynaud's E, which makes the bundle globally generated along T;
- the computation of all its negative cohomology;
- the very ampleness of $Q^{\otimes 9}$;
- Theorem 3.

Adjoining roots of every member of a basis of a linear system is a standard covering construction. Bhatt uses it to make the pullback of an ample bundle globally generated [2, proof of Proposition 7.2, p. 1771]. The observation that a failure of Kodaira vanishing survives a separable cover of degree prime to p, because L^{-1} is a direct summand of the pushforward, is made by Mukai [8, p. 2]. What Theorem 3 adds is the very ampleness. The construction is related to the Kummer covers of Bloch–Gieseker type, which make a pullback divisible. Note that Y is not the fibre product of X with the Kummer map $[t_i] \mapsto [t_i^m]$ of projective space. That fibre product is the quotient of Y by the diagonal action of μ_m, and the sections of Q pulled back to Y are what separate the points of a diagonal orbit.

Context. Ekedahl and Mukai showed that Kodaira vanishing for numerically positive (Mukai: nef and big) line bundles holds on surfaces that are not of general type, apart essentially from Raynaud's examples [3, p. 99 and Theorem II.1.3][8, Theorem 3]. Raynaud's surface in characteristic 2 is quasi-elliptic [9, Remarque 1, pp. 277–278], so the surface S of part (a) lies in that exception. For a minimal surface of general type, Ekedahl proved $H^1(X, \omega_X^{-i}) = 0$ for all $i > 0$, except possibly for $i = 1$ in one family in characteristic 2 [3, Main theorem (i), p. 97]. So very ample counterexamples cannot be found among pluricanonical bundles.

- Mukai [8, Theorem 2] extended Raynaud's method to every dimension $n \geq 2$, with ample bundles.
- Zheng [10] refined and generalized Raynaud's surfaces of general type.
- Bernasconi and Valloni recently showed that on infinitely many Raynaud surfaces with $p > 2$ the p-th power of the nonvanishing bundle is globally generated and separates points, restricting to a purely inseparable map of degree p on each fibre [1, Theorem 1.2 and Propositions 3.13–3.14]. Their nonvanishing statement concerns the bundle itself, not its p-th power, and very ampleness is not asserted.

None of these works contains a very ample surface counterexample. Nor did searches of Hexagon (23 November 2026) find one.

Scope. The example is in characteristic 2. For odd p, Theorem 3 would give very ample surface counterexamples as soon as a globally generated ample bundle with H^1 of its inverse nonzero is found on a surface. Whether the twist used here has an analogue on Raynaud's odd-characteristic surfaces is not decided here. The invariants of Y are large, and no attempt is made to minimize them.

2 Proof overview

There are two difficulties. Raynaud's bundle $\mathcal{I} = \mathcal{O}_X(E) \otimes N$ fails to be generated along the section E. And even a generated bundle is far from very ample, which is what the question asks for.

Global generation. The surface S is a cyclic triple cover $\psi : S \to P$ of a ruled surface $\pi : P \to C$, totally ramified along a section E_∞ of P, with $\psi^* E_\infty = 3T$. The pushforward $\psi_* \mathcal{O}_S(aT)$ splits into three eigensheaves, and the twist by T shifts the pole order allowed along E_∞ by $\lfloor (a+j)/3 \rfloor$ in the j-th summand. With $a = 1$, as for Raynaud's bundle, only the invariant summand has sections, and they vanish on T. With $a = 2$ the first eigensheaf acquires sections with a pole of exact order two along T, which generate $\mathcal{O}_S(2T)$ there. The same eigensheaf decomposition, together with relative duality on the ruled surface, computes every $H^1(S, Q^{-n})$. Only $n = 1$ contributes, through a trivial summand of R^1, so that $h^1(Q^{-1}) = 1$.

Very ampleness. Since $Q^{\otimes 9}$ is pulled back from P, its very ampleness reduces to two standard statements on the ruled surface: a line bundle is very ample, and its twist by one eigensheaf is generated. If Q is generated and $Q^{\otimes m}$ is very ample, let t_i be an m-th root of a section s_i of $Q^{\otimes m}$. On the open set where a section u of Q does not vanish, the functions s_i/u^m generate the coordinate ring, because the s_i form a basis of a very ample complete linear system. Upstairs the ratios t_i/u are m-th roots of these functions, so they generate the coordinate ring of the cover. Thus the roots t_i together with the sections u of Q embed Y. The cohomology of L^{-1} is the sum, over all multi-indices of roots, of cohomology of negative powers of Q on X. Its summand of index zero is $H^1(Q^{-1}) \neq 0$, and all the others vanish for the surface S.

The hypotheses. The characteristic 2 is used in Raynaud's construction: the Artin–Schreier curve and the inseparable bisection C'', whose equation $w^2 + x = 0$ is smooth because $d(w^2) = 0$. The prime-to-p degrees 3 and 9 make the triple cover and the root cover smooth and split their pushforwards into eigensheaves. Ampleness of Q kills the sections of all nontrivial summands, which makes Y connected and fixes the numbers 562 and 565.

3 Roots of a basis: proof of Theorem 3

This section proves the general construction; the surface is treated in the next two sections.

Write Q^{-r} for $(Q^{\otimes r})^{-1}$. Since m is invertible in k, adjoining an m-th root of a section s of $Q^{\otimes m}$ means taking $\operatorname{Spec}_X \bigoplus_{r=0}^{m-1} Q^{-r}$, with multiplication $Q^{-r} \otimes Q^{-r'} \to Q^{-r-r'}$ when $r + r' < m$. When $r + r' \geq m$ the product is followed by multiplication by s, from $Q^{-r-r'}$ to $Q^{-r-r'+m}$. Adjoining roots of $s_1, \ldots, s_\ell$ simultaneously gives

$$\rho : Y = \operatorname{Spec}_X \Big(\bigotimes_{i=1}^{\ell} \bigoplus_{r=0}^{m-1} Q^{-r} \Big) \longrightarrow X, \qquad \rho_* \mathcal{O}_Y = \bigoplus_{0 \leq r_i \leq m-1} Q^{-(r_1 + \cdots + r_\ell)}. \tag{1}$$

This is finite and locally free of degree m^ℓ, so Y is projective.

Choice of the basis. The tuples $(s_1, \ldots, s_\ell)$ form an affine space over k. By Bertini's theorem for the very ample bundle $Q^{\otimes m}$ over the algebraically closed field k [4, II.8.18], applied successively to X and to the intersections already formed, the following conditions are nonempty and open:

- the s_i form a basis;
- each divisor $D_i = \{s_i = 0\}$ is smooth;
- the divisors meet transversally;

- no $\dim X + 1$ of them have a common point.

A finite intersection of nonempty open subsets of an irreducible variety is nonempty, and over the infinite field k it has rational points. We take (s_i) to be such a point.

Smoothness and connectedness. At a point $x \in X$ the divisors D_i through x have local equations forming part of a regular system of parameters. Adjoining m-th roots of these parameters gives again a regular local ring. Adjoining m-th roots of the remaining s_i, which are units at x, is étale because m is invertible. Hence Y is smooth. Since $Q^{\otimes m}$ is very ample, Q is ample, and since X has positive dimension, $H^0(X, Q^{-r}) = 0$ for $r > 0$. (Positive dimension is needed here: for $X = \operatorname{Spec} k$, $Q = \mathcal{O}_X$ and $m = 3$, a basis (c) gives three points $t^3 = c$.) By (1), $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X) = k$. A smooth proper scheme with $H^0(\mathcal{O}) = k$ is connected, and being smooth it is integral.

Cohomology. By the projection formula, $\rho_* L^{-1} = Q^{-1} \otimes \rho_* \mathcal{O}_Y$, and ρ is finite. So (1) gives the displayed decomposition of $H^i(Y, L^{-1})$, whose summand with all $r_i = 0$ is $H^i(X, Q^{-1})$.

Very ampleness. On Y let t_i be the tautological section of L with $t_i^m = \rho^* s_i$. It is the unit section 1 of the summand $\mathcal{O}_X = Q \otimes Q^{-1}$ of $\rho_* L$, in the i-th factor with $r_i = 1$. Let $u_1, \dots, u_M$ be a basis of $H^0(X, Q)$. Consider the morphism $\phi : Y \to \mathbf{P}^{\ell+M-1}$ given by the sections $t_1, \dots, t_\ell, \rho^* u_1, \dots, \rho^* u_M$ of L; these have no common zero because the u_j generate Q. Fix j, put $u = u_j$, and let $X_u \subset X$ be the open set where $u \neq 0$. Since u^m is a section of the very ample bundle $Q^{\otimes m}$ and the s_i form a basis of its sections, X_u is the standard affine chart of the embedding by $|Q^{\otimes m}|$ given by $u^m \neq 0$. Its coordinate ring is therefore generated over k by the functions s_i/u^m. On $\rho^{-1}(X_u)$ the bundle L is trivialized by u. By (1),

$$\mathcal{O}(\rho^{-1}(X_u)) = \mathcal{O}(X_u)[z_1, \dots, z_\ell]/(z_i^m - s_i/u^m), \qquad z_i = t_i/u.$$

Since $s_i/u^m = z_i^m$, this ring is generated over k by the z_i alone. Now $\rho^{-1}(X_u)$ is the inverse image under ϕ of the standard chart $\{\rho^* u \neq 0\}$ of the projective space, and the coordinate functions of that chart map onto generators of its coordinate ring. So ϕ is a closed immersion over each such chart. The charts cover the image because Q is generated by the u_j. Thus ϕ is an immersion, and as Y is proper it is a closed immersion. So L is very ample. □

Remark 4. The sections $\rho^* u_j$ cannot be omitted. The roots t_i alone define the map to the fibre product of X with the Kummer map $[x_i] \mapsto [x_i^m]$ of $\mathbf{P}^{\ell-1}$. That map identifies the points (t_i) and (ζt_i) for $\zeta \in \mu_m$, so it is not injective.

4 Raynaud's surface in characteristic 2 and a generated bundle

This section constructs S and Q and proves part (a) of Theorem 1. All objects are defined over $\mathbf{F}_2$, and cohomology is computed after base change to k; dimensions do not change under field extension.

4.1 The curve

Let C be the smooth projective model of $y^2 + y = x^{15}$. The affine curve is smooth, since the partial derivative in y is 1. The function x is a coordinate at every affine point, so dx has no affine zero, and differentiating gives $dy = x^{14} dx$. A pole of $g^2 + g$ has even order while x^{15} has a pole of odd order at infinity. Hence C is geometrically integral, and the cover $C \to \mathbf{P}^1$ has one point ∞ over $x = \infty$, with $v_\infty(x) = -2$ and $v_\infty(y) = -15$.

Put $P_0 = (0, 0)$, $P_1 = (0, 1)$ and $t = y/x^8$. Then $v_\infty(t) = 1$; at P_0 one has $v(y) = 15$ and $v(x) = 1$, so $v(t) = 7$; and at P_1, $v(t) = -8$. Dividing $y^2 = y + x^{15}$ by x^{16} gives $t^2 = t x^{-8} + x^{-1}$. Multiplying by x/t^2, one gets

$$\operatorname{div}(t) = 7P_0 + \infty - 8P_1, \qquad x + t^{-2} = t^{-1} x^{-7}, \qquad dt = x^{-8} dy = x^6 dx. \tag{2}$$

Since t is a uniformizer at ∞, $v_\infty(dx) = v_\infty(dt) - 6v_\infty(x) = 12$. So $\operatorname{div}(dx) = 12\infty$ and C has genus 7. Its Weierstrass semigroup at ∞ contains the pole orders 2 of x and 15 of y. Hence $h^0(\mathcal{O}_C(2\infty)) = 2$, with basis $1, x$, and $h^0(\mathcal{O}_C(6\infty)) = 4$, with basis $1, x, x^2, x^3$. Indeed, a function with a pole of order at most 6 at ∞ and regular elsewhere is a polynomial in x and y; since y has pole order 15, it is a polynomial in x. Put $N = \mathcal{O}_C(2\infty)$, a line bundle of degree 2 generated by 1 and x.

4.2 The ruled surface and the branch curve

On $U = C \setminus \{\infty\}$ and $V = C \setminus \{P_0, P_1\}$ define a rank-two bundle E by frames (e_0, e_1) and (v_0, v_1), related on $U \cap V$ by

$$v_0 = e_0, \qquad v_1 = t^{-6}(e_1 - t^{-1}e_0).$$

The section $e_0 = v_0$ spans a trivial subbundle. The quotient is generated by e_1 on U and by $t^{-6}e_1$ on V. Its sections are the functions regular on U with a pole of order at most 6 at ∞, so the quotient is $\mathcal{O}_C(6\infty) = N^3$. This gives

$$0 \to \mathcal{O}_C \to E \to N^3 \to 0, \qquad \det E = N^3.$$

Consequently $\operatorname{Sym}^a E$ has a filtration whose successive quotients are N^{3h}, $0 \le h \le a$.

Let $\pi : P = \mathbf{P}_C(E) \to C$ be the bundle of rank-one quotients, so that $\pi_*\mathcal{O}_P(j) = \operatorname{Sym}^j E$ for $j \ge 0$. The global section e_0 of E defines a section $E_\infty = \{e_0 = 0\}$ of π, with $\mathcal{O}_P(E_\infty) = \mathcal{O}_P(1)$.

The quadratic expression $e_1^2 + xe_0^2$ is a section of $\mathcal{O}_P(2)$ over U. In characteristic 2 one has $e_1^2 = t^{12}v_1^2 + t^{-2}v_0^2$. By (2),

$$e_1^2 + xe_0^2 = t^{12}(v_1^2 + c\,v_0^2), \qquad c = t^{-12}(x + t^{-2}) = t^{-13}x^{-7}.$$

The function c is regular on V and has valuation $-13 + 14 = 1$ at ∞. So the two local expressions define a curve $C'' \subset P$, with $\mathcal{O}_P(C'') = \mathcal{O}_P(2) \otimes \pi^*N^{-6}$ (the factor t^{12} matches $N^{-6} = \mathcal{O}_C(-12\infty)$). On E_∞ one has $e_0 = 0$, where the expression is $e_1^2 \neq 0$; so C'' and E_∞ are disjoint. Away from ∞, in the coordinate $w = e_1/e_0$, the curve C'' is $w^2 + x = 0$, whose differential dx does not vanish. Near ∞ its equation $v^2 + c$ has differential $dc \neq 0$, since c is a uniformizer there. Thus C'' is smooth.

4.3 The triple cover

Set $M = \mathcal{O}_P(-1) \otimes \pi^*N^2$. Then $M^{-3} = \mathcal{O}_P(3) \otimes \pi^*N^{-6} = \mathcal{O}_P(E_\infty + C'')$. The section $e_0 \cdot (e_1^2 + xe_0^2)$ of M^{-3} vanishes on $E_\infty + C''$ and defines a cyclic triple cover

$$\psi : S = \operatorname{Spec}_P(\mathcal{O}_P \oplus M \oplus M^2) \longrightarrow P,$$

with multiplication $M \otimes M^2 \to \mathcal{O}_P$ given by that section. The branch divisor is smooth, and 3 is invertible. Locally the cover is $z^3 = u$ along a branch equation u, and étale elsewhere. So S is smooth and projective. On the chart $e_0 \neq 0$ it is

$$y^2 + y = x^{15}, \qquad z^3 = w^2 + x.$$

This is the surface of Raynaud's §2 with the data of his Remarque 3 [9, pp. 275–278]. Since M and M^2 have negative degree on the fibres of π, $H^0(S, \mathcal{O}_S) = k$; so S is connected, hence integral. Write $f = \pi\psi$, and let T be the reduced inverse image of E_∞. Then $\psi^*E_\infty = 3T$ and $T \simeq C$.

Lemma 5. *For every integer a, $\psi_*\mathcal{O}_S(aT) = \bigoplus_{j=0}^{2} M^j(\lfloor (a+j)/3 \rfloor E_\infty)$.*

Proof. Near E_∞ let u be a local equation of E_∞ and z a local generator of M in the algebra, so that $z^3 = u \cdot (\text{unit})$, $v_T(z) = 1$ and $v_T(u) = 3$. A local section of $\psi_*\mathcal{O}_S$ with poles along E_∞ is $\sum_j \varphi_j z^j$, with φ_j rational functions on P. The valuations $v_T(\varphi_j z^j) = 3v_{E_\infty}(\varphi_j) + j$ are distinct modulo 3, so the sum has valuation at least $-a$ exactly when each term has. That is, $-v_{E_\infty}(\varphi_j) \le (a + j)/3$. □

Proposition 6. *The line bundle $Q = \mathcal{O}_S(2T) \otimes f^*N$ is ample and globally generated, and*

$$h^0(S, Q) = 6, \quad h^1(S, Q^{-1}) = 1, \quad H^1(S, Q^{-n}) = 0 \ (n \ge 2), \quad h^1(S, \mathcal{O}_S) = 9.$$

Proof. Sections and generation. By Lemma 5 with $a = 2$, and $\mathcal{O}_P(E_\infty) = \mathcal{O}_P(1)$, the three summands of ψ_*Q are N, $M(E_\infty) \otimes N = \pi^*N^3$, and $M^2(E_\infty) \otimes N = \mathcal{O}_P(-1) \otimes \pi^*N^5$. The last has no direct images. Hence $f_*Q = N \oplus N^3$, and $h^0(S, Q) = 2 + 4 = 6$. On the chart above the sections are $1, x, z, zx, zx^2, zx^3$.

The sections from the summand N are regular functions, generated by 1 and x. They generate Q away from T, where $\mathcal{O}_S(2T)$ is trivialized by 1. The sections from the second summand have the form φz, with φ having a simple pole along E_∞, so they have valuation $1-3 = -2$ along T. Since N^3 is generated by its sections, they generate $\mathcal{O}_S(2T) \otimes f^*N$ along T. So Q is globally generated. It is ample, because its ninth power is very ample (Proposition 7).

Negative powers. For $n \ge 1$, Lemma 5 gives the j-th summand of ψ_*Q^{-n} as

$$\mathcal{O}_P(a_j) \otimes \pi^*N^{2j-n}, \qquad a_j = \lfloor (-2n + j)/3 \rfloor - j \le -1.$$

So π_* of every summand vanishes, and the Leray spectral sequence gives $H^1(S, Q^{-n}) = H^0(C, R^1\pi_*\psi_*Q^{-n})$. Relative duality for π, with $\omega_{P/C} = \mathcal{O}_P(-2) \otimes \pi^* \det E$, gives

$$R^1\pi_*\mathcal{O}_P(a) \cong (\mathrm{Sym}^{-a-2} E)^\vee \otimes N^{-3} \quad (a \le -2), \qquad R^1\pi_*\mathcal{O}_P(-1) = 0.$$

For $n = 1$ one has $(a_0, a_1, a_2) = (-1, -2, -2)$, and the nonzero direct images are $N^{-3} \otimes N = N^{-2}$ (for $j = 1$) and $N^{-3} \otimes N^3 = \mathcal{O}_C$ (for $j = 2$). Hence $h^1(S, Q^{-1}) = h^0(N^{-2}) + h^0(\mathcal{O}_C) = 0 + 1 = 1$.

For $n \ge 2$, dualizing the filtration of $\mathrm{Sym}^{-a_j-2} E$ shows that every nonzero summand of $R^1\pi_*\psi_*Q^{-n}$ is a successive extension of the line bundles $N^{2j-n-3-3h}$, $h \ge 0$. Their exponents are at most $2 \cdot 2 - n - 3 = 1 - n < 0$, so these bundles have negative degree. A successive extension of bundles without sections has none, so $H^1(S, Q^{-n}) = 0$.

Irregularity. The summands of $\psi_*\mathcal{O}_S$ are $\mathcal{O}_P$, $\mathcal{O}_P(-1) \otimes \pi^*N^2$ and $\mathcal{O}_P(-2) \otimes \pi^*N^4$, with π_* equal to $\mathcal{O}_C, 0, 0$ and $R^1\pi_*$ equal to $0, 0, N$. As C is a curve, Leray gives $h^1(S, \mathcal{O}_S) = h^1(\mathcal{O}_C) + h^0(N) = 7 + 2 = 9$. □

5 The ninth power is very ample

This section proves that $Q^{\otimes 9}$ is very ample and counts its sections, the input of Theorem 3. Put $B = \mathcal{O}_P(6) \otimes \pi^*N^9$. Since $18T = \psi^*(6E_\infty)$, one has $Q^{\otimes 9} = \psi^*B$. Its pushforward is $B \oplus BM \oplus BM^2$, whose three summands are $\mathcal{O}_P(6 - i) \otimes \pi^*N^{9+2i}$ for $0 \le i \le 2$.

Proposition 7. *The bundle $Q^{\otimes 9}$ is very ample on S and $h^0(S, Q^{\otimes 9}) = 210 + 186 + 160 = 556$.*

Proof. For $(a, b) \in \{(6, 9), (5, 11), (4, 13)\}$, the bundle $\mathrm{Sym}^a E \otimes N^b = \pi_*(\mathcal{O}_P(a) \otimes \pi^*N^b)$ is filtered with quotients N^{b+3h}, $0 \le h \le a$, of degree $2b + 6h \ge 18$. After twisting down by any effective divisor of degree two on C the degrees are still at least $16 > 2g - 2 = 12$. By Serre duality these line bundles, and hence (by the filtration) the vector bundles, have vanishing H^1

after such a twist. Their sections therefore surject onto every fibre over a subscheme of C of length at most two. By Riemann–Roch each quotient of degree d contributes $d-6$ sections, so

$$h^0(B) = \sum_{h=0}^{6}(12+6h) = 210, \qquad h^0(BM) = \sum_{h=0}^{5}(16+6h) = 186,$$

$$h^0(BM^2) = \sum_{h=0}^{4}(20+6h) = 160.$$

B is very ample on P. Let $Z \subset P$ be a subscheme of length two and $W = \pi(Z)$. The restriction $H^0(P,B) = H^0(C, \operatorname{Sym}^6 E \otimes N^9) \to H^0(\pi^{-1}W, B)$ is surjective by the preceding paragraph. On $\pi^{-1}W$, which is a $\mathbf{P}^1$-bundle over W, the bundle B is $\mathcal{O}(6)$. Its sections separate the length-two subscheme Z:

- if W is one point, Z lies in one fibre $\mathbf{P}^1$, and $\mathcal{O}(6)$ is very ample there;
- otherwise $Z \to W$ is an isomorphism, and Z is the image of a section over W.

The same surjectivity onto single fibres shows that $BM = \mathcal{O}_P(5) \otimes \pi^* N^{11}$ is globally generated.

Now let $b \in H^0(P,B)$ and $D(b) = \{b \neq 0\}$, a standard affine chart of the embedding by B. The ring $\mathcal{O}(D(b))$ is generated over k by the ratios s/b, $s \in H^0(P,B)$. Over $D(b)$ the algebra $\psi_*\mathcal{O}_S$ is $\mathcal{O} \oplus M \oplus M^2$, generated as an $\mathcal{O}(D(b))$-algebra by $H^0(D(b), M)$, because $M \otimes M \to M^2$ is surjective. Since BM is globally generated, the sheaf $M|_{D(b)} = (BM \otimes B^{-1})|_{D(b)}$ is generated by the ratios σ/b with $\sigma \in H^0(P, BM)$. On the affine $D(b)$ these ratios therefore generate the module $H^0(D(b), M)$. All these ratios are ratios of sections of ψ^*B, so the map defined by the complete linear system of ψ^*B is a closed immersion over each chart $\{b \neq 0\}$. These charts cover the image, since B is generated. As S is proper, $\psi^*B = Q^{\otimes 9}$ is very ample. □

6 Proof of Theorem 1

This section assembles the parts. Part (a) is Proposition 6, with ampleness from Proposition 7.

For part (b), apply Theorem 3 to $X = S$ and Q with $m = 9$, which is prime to 2. Here Q is generated by Proposition 6 and $Q^{\otimes 9}$ is very ample with $\ell = 556$ sections by Proposition 7. We obtain a smooth projective connected surface Y, finite and flat of degree 9^{556} over S, with $L = \rho^* Q$ very ample. By (1) the invariants are sums over the multi-indices $r = (r_i)$, with $|r| = \sum r_i$:

- $H^0(Y,L) = \bigoplus_r H^0(S, Q^{1-|r|})$. The index $|r| = 0$ contributes $h^0(Q) = 6$. Each of the 556 indices with $|r| = 1$ contributes $h^0(\mathcal{O}_S) = 1$. Larger $|r|$ give negative powers of the ample Q, which have no sections. So $h^0(L) = 562$.
- $H^1(Y, L^{-1}) = \bigoplus_r H^1(S, Q^{-1-|r|})$. Only $|r| = 0$ contributes, by Proposition 6. So $h^1(L^{-1}) = 1$.
- $H^1(Y, \mathcal{O}_Y) = \bigoplus_r H^1(S, Q^{-|r|})$. This is 9 from $|r| = 0$ plus $556 \cdot 1$ from $|r| = 1$, the others vanishing. So $h^1(\mathcal{O}_Y) = 565$.

The sections t_i (the 556 indices with $|r| = 1$) and the six pullbacks of sections of Q form a basis of $H^0(Y,L)$. So the embedding of Theorem 3 is by the complete linear system $|L|$.

Finally let $H \in |L|$ be smooth and connected. The sequence $0 \to L^{-1} \to \mathcal{O}_Y \to \mathcal{O}_H \to 0$ and the isomorphism $H^0(\mathcal{O}_Y) = k = H^0(\mathcal{O}_H)$ identify $H^1(Y, L^{-1})$ with the kernel of $H^1(Y, \mathcal{O}_Y) \to H^1(H, \mathcal{O}_H)$. This is the map induced by restriction on the tangent spaces at the identity of the Picard schemes. □

7 What remains open

- *Odd characteristic.* By Theorem 3, a very ample counterexample on a surface in characteristic p exists as soon as a globally generated ample Q with $H^1(Q^{-1}) \neq 0$ exists on some surface. For p odd we have not checked whether a twist of Raynaud's bundle along the cusp section is globally generated.
- *Size.* The surface Y has irregularity 565 and its embedding lies in $\mathbf{P}^{561}$. Minimizing the degree of the cover, the irregularity or the embedding dimension is not attempted.
- *Explicit equations.* The proof shows that Y can be defined over a finite field, but gives neither a specific field nor a specific basis (s_i).
- *Picard kernels.* Whether some smooth surface has a very ample (or merely ample) L and a smooth $H \in |L|$ for which $\mathrm{Pic}(Y) \to \mathrm{Pic}(H)$ has a nontrivial kernel on geometric points is not touched by this construction, which controls tangent spaces and infinitesimal subgroup schemes only. This is the first part of Kweon's Question 7.7 [6, p. 19] for surfaces.

Method and checks

The construction and its proofs were produced by GPT-6 Astra (ChatGPT Pro mode). That run checked them by a reconstruction from the statement alone by a separate instance of the same model, and by two computer recomputations of the dimensions 210, 186 and 160. Claude Opus 5.5 at effort high made a targeted check of every step, recomputing every invariant by hand. It also wrote this text: it separated Theorem 3 from the example, wrote out the chart argument for Proposition 7, and replaced the run's ground field (an algebraic closure of a purely transcendental extension of $\mathbf{F}_2$) by an arbitrary algebraically closed field of characteristic 2, with the Bertini argument of Section 3. Claude Opus 5.5 at effort max then read the whole text, followed every step and recomputed every invariant. No human has checked the mathematics.

The editor read at the source Raynaud's article [9] (all of pp. 273–278), the opening pages of [7], [8] and [3], the statements of [1] cited above, and, for the second version, §7 of [6] (pp. 17–19 of arXiv:2008.01908v2). The locators in [2] and [10] are those recorded by the run, and [5] is cited for its result as [7] describe it. The journal data of [5], [7] and [8] were checked against bibliographic records on 23 November 2026. Our text is dedicated to the public domain under CC0 1.0. The records of the checks and the programs are at the permanent page of this work, WB-0003: `https://github.com/KokunoYumeto/mathematics-workbenches/blob/main/works/WB-0003.md`.

References

[1] F. Bernasconi and D. Valloni, *Brauer–Manin obstruction from failure of Kodaira vanishing in positive characteristic*, arXiv:2605.28278 (2026).

[2] B. Bhatt, *Derived splinters in positive characteristic*, Compos. Math. 148 (2012), 1757–1786.

[3] T. Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Publ. Math. IHES 67 (1988), 97–144.

[4] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[5] N. Lauritzen, *Embeddings of homogeneous spaces in prime characteristics*, Amer. J. Math. 118 (1996), 377–387; arXiv:alg-geom/9502016.

[6] H. J. Kweon, *Bounds on the torsion subgroup schemes of Néron–Severi group schemes*, Adv. Math. 409 (2022), 108687; arXiv:2008.01908v2 (cited by its page numbers).

[7] N. Lauritzen and A. P. Rao, *Elementary counterexamples to Kodaira vanishing in prime characteristic*, Proc. Indian Acad. Sci. Math. Sci. 107 (1997), 21–25; arXiv:alg-geom/9604012.

[8] S. Mukai, *Counterexamples to Kodaira's vanishing and Yau's inequality in positive characteristics*, Kyoto J. Math. 53 (2013), 515–532; preprint RIMS-1736 (2011).

[9] M. Raynaud, *Contre-exemple au "vanishing theorem" en caractéristique* $p > 0$, in C. P. Ramanujam—a tribute, Tata Inst. Fund. Res. Studies in Math. 8, Springer, 1978, 273–278.

[10] X. Zheng, *Counterexamples of Kodaira vanishing for smooth surfaces of general type in positive characteristic*, J. Pure Appl. Algebra 221 (2017), 2431–2444.